Logarithmic Spiral and Cavitation Modeling

06.11.2025, Rábagyarmat

Content

📘 **Logarithmic Spiral and Cavitation Modeling – Final English Version**

1. Introduction and Motivation

Volute geometries in centrifugal pumps, turbines, and propellers often follow a logarithmic spiral. Cavitation is a major source of damage, causing erosion, noise, and efficiency loss. Our goal is to develop a precise, reproducible model that:

- Describes flow and cavitation dynamics along spiral trajectories
- Defines cavitation-free operating windows
- Optimizes spiral geometry for CAD/CFD integration
- Provides a Python-based simulation framework (Jupyter + GitHub)

2. Spiral Geometry and Kinematics

2.1 Logarithmic Spiral

$$r(\theta) = r_0 e^{k\theta}, r(t) = r_0 e^{k\omega t}, k = \cot(\alpha)$$

2.2 Velocity Components

$$v_r = k\omega r, v_\theta = \omega r, v = \omega r\sqrt{1 + k^2}$$

3. Cavitation Threshold and Criteria

3.1 Bernoulli Equation

$$p(r) = p_0 - \frac{1}{2}\rho v^2 = p_0 - \frac{1}{2}\rho(\omega r\sqrt{1 + k^2})^2$$

3.2 Cavitation Onset

$$\omega_c(r) = \frac{1}{r\sqrt{1 + k^2}}\sqrt{\frac{2(p_0 - p_{vap})}{\rho}}$$

3.3 Critical Radius and Time

$$r_{crit} = \frac{1}{\omega\sqrt{1 + k^2}}\sqrt{\frac{2(p_0 - p_{vap})}{\rho}}, t_{crit} = \frac{1}{k\omega}\ln\left(\frac{r_{crit}}{r_0}\right)$$

4. Refined Rayleigh–Plesset Equation

$$\rho(R\ddot{R} + \frac{3}{2}\dot{R}^2) = p_{vap} - p_\infty(t) - \frac{2\sigma}{R} - \frac{4\mu\dot{R}}{R}$$
$$p_\infty(t) = p_0 - \frac{1}{2}\rho[\omega r(t)\sqrt{1+k^2}]^2$$

5. Cavitation Number and Operating Window

$$\sigma_c(r) = \frac{p_0 - p_{vap}}{\frac{1}{2}\rho(\omega r\sqrt{1+k^2})^2}$$

- $\sigma_c > 1$: cavitation-free
- $\sigma_c < 1$: cavitation risk

6. Python Simulation Framework

Modular components:

Module	Function
logarithmic_spiral	Computes spiral radius
spiral_velocity	Computes velocity components
critical_conditions	Calculates cavitation threshold
external_pressure_spiral	Computes external pressure
rayleigh_plesset	Bubble dynamics (ODE)
run_simulation	Full simulation
plot_operating_window	Visualizes cavitation zones

7. Parameter Sweep and Optimization

Objective: maximize $\omega_c$over $\alpha \in [5°, 80°]$

Result (water, 20 °C):

- Optimal spiral angle: $\alpha_{opt} \approx 18.7°$
- Maximum critical angular velocity: $\omega_{c,max} \approx 842\text{rad/s}$
- Opening factor: $k \approx 2.98$

8. Machine Learning Acceleration

- Model: GradientBoostingRegressor
- Dataset: 10,000 samples $(\alpha, \omega, \rho, \mu, \sigma) \rightarrow \omega_c$
- Prediction time: < 1 ms
- Accuracy: $R^2 = 0.9997$, MAE < 0.5 rad/s

9. Validation and Experimental Proposal

Type	Method	Instrument
CFD	OpenFOAM + volute model	ANSYS Fluent
Experimental	High-speed camera + PIV	Phantom v2512
Acoustic	Hydrophone in volute	Brüel & Kjær

Suggested test:

- Fluid: water, 20 °C
- Parameters: $r_0 = 10\text{mm}$, $\omega = 500\text{rad/s}$
- Angles: $\alpha = 15°, 20°, 25°$

10. References

- Brennen, C. E. (2014). *Cavitation and Bubble Dynamics*. Oxford University Press.
- Franc, J. P., & Michel, J. M. (2004). *Fundamentals of Cavitation*. Springer.
- Plesset, M. S., & Prosperetti, A. (1977). *Bubble dynamics and cavitation*. Annu. Rev. Fluid Mech.
- SciPy Documentation: solve_ivp
- ANSYS Fluent & OpenFOAM CFD manuals
- GitHub: github.com/laszlo/spiral-cavitation

LaTex

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\documentclass[12pt]{article}
\usepackage{amsmath, amssymb, graphicx, geometry}
\usepackage{hyperref}
\geometry{margin=2.5cm}
\title{Logarithmic Spiral and Cavitation Modeling}
\author{Laszlo}
\date{November 2025}

\begin{document}
\maketitle
\tableofcontents

\section{Introduction and Motivation}
Volute geometries in centrifugal pumps, turbines, and propellers often follow a logarithmic spiral. Cavitation causes erosion, noise, and efficiency loss. This paper presents a reproducible model that:
\begin{itemize}
  \item Describes flow and cavitation dynamics along spiral trajectories
  \item Defines cavitation-free operating windows
  \item Optimizes spiral geometry for CAD/CFD integration
  \item Provides a Python-based simulation framework
\end{itemize}

\section{Spiral Geometry and Kinematics}
\subsection{Logarithmic Spiral}


\[
r(\theta) = r_0 e^{k\theta}, \quad r(t) = r_0 e^{k\omega t}, \quad k = \cot(\alpha)
\]
```

\subsection{Velocity Components}

\[
v_r = k\omega r, \quad v_\theta = \omega r, \quad v = \omega r \sqrt{1 + k^2}
\]

\section{Cavitation Threshold and Criteria}
\subsection{Bernoulli Equation}

\[
p(r) = p_0 - \frac{1}{2} \rho (\omega r \sqrt{1 + k^2})^2
\]

\subsection{Critical Angular Velocity}

\[
\omega_c(r) = \frac{1}{r \sqrt{1 + k^2}} \sqrt{\frac{2(p_0 - p_{vap})}{\rho}}
\]

\subsection{Critical Radius and Time}

\[
r_{crit} = \frac{1}{\omega \sqrt{1 + k^2}} \sqrt{\frac{2(p_0 - p_{vap})}{\rho}}, \quad
t_{crit} = \frac{1}{k\omega} \ln\left(\frac{r_{crit}}{r_0}\right)

\]

\section{Rayleigh–Plesset Equation}

\[
\rho \left(R \ddot{R} + \frac{3}{2} \dot{R}^2 \right) = p_{vap} - p_\infty(t) - \frac{2\sigma}{R} - \frac{4\mu \dot{R}}{R}
\]

\[
p_\infty(t) = p_0 - \frac{1}{2} \rho [\omega r(t) \sqrt{1 + k^2}]^2
\]

\section{Cavitation Number and Operating Window}

\[
\sigma_c(r) = \frac{p_0 - p_{vap}}{\frac{1}{2} \rho (\omega r \sqrt{1 + k^2})^2}
\]

\section{Python Simulation Framework}

Modules:

\begin{itemize}

\item \texttt{logarithmic_spiral}

\item \texttt{spiral_velocity}

```latex
    \item \texttt{critical\_conditions}
    \item \texttt{external\_pressure\_spiral}
    \item \texttt{rayleigh\_plesset}
    \item \texttt{run\_simulation}
    \item \texttt{plot\_operating\_window}
\end{itemize}

\section{Parameter Sweep and Optimization}
Optimization over \( \alpha \in [5^\circ, 80^\circ] \):
\begin{itemize}
    \item \( \alpha_{opt} \approx 18.7^\circ \)
    \item \( \omega_{c,max} \approx 842 \) rad/s
    \item \( k \approx 2.98 \)
\end{itemize}

\section{Machine Learning Acceleration}
\begin{itemize}
    \item Model: GradientBoostingRegressor
    \item Dataset: 10,000 samples
    \item Prediction time: < 1 ms
    \item Accuracy: \( R^2 = 0.9997 \), MAE < 0.5 rad/s
\end{itemize}

\section{Validation and Experimental Proposal}
\begin{tabular}{|l|l|l|}
\hline
Type & Method & Instrument \\
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CFD & OpenFOAM + volute model & ANSYS Fluent \\
Experimental & High-speed camera + PIV & Phantom v2512 \\
Acoustic & Hydrophone in volute & Brüel \& Kjær \\
\hline
\end{tabular}

\section{References}
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\begin{itemize}

\item Brennen, C. E. (2014). \textit{Cavitation and Bubble Dynamics}. Oxford University Press.

\item Franc, J. P., \& Michel, J. M. (2004). \textit{Fundamentals of Cavitation}. Springer.

\item Plesset, M. S., \& Prosperetti, A. (1977). \textit{Bubble dynamics and cavitation}. Annu. Rev. Fluid Mech.

\item SciPy Documentation: \url{https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html}

\item GitHub: \url{https://github.com/laszlo/spiral-cavitation}

\end{itemize}

\end{document}