VE281

Data Structures and Algorithms

Graph Search; Topological Sorting

Outline

- Graph Search
- Topological Sorting

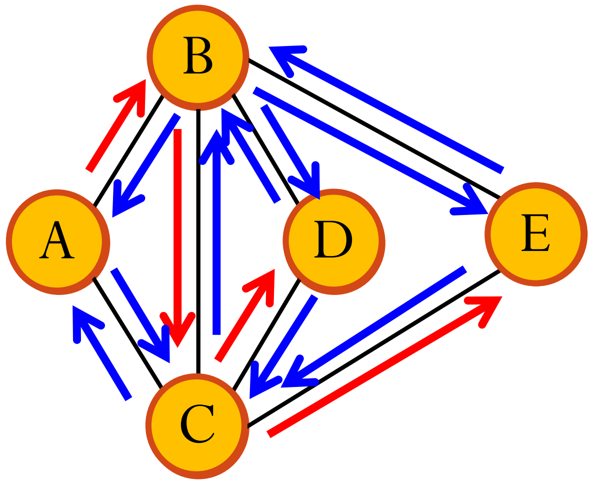
Graph Search

- A node u is **reachable** from a node v if and only if there is a path from v to u.
- A graph search method starts at a given node v and visits **every** node that is **reachable** from v.
- Many graph problems are solved using a search method.
  - Find a path from one node to another.
  - Find if the graph is connected.
- Commonly used search methods:
  - Depth-first search.
  - Breadth-first search.

Depth-First Search (DFS)

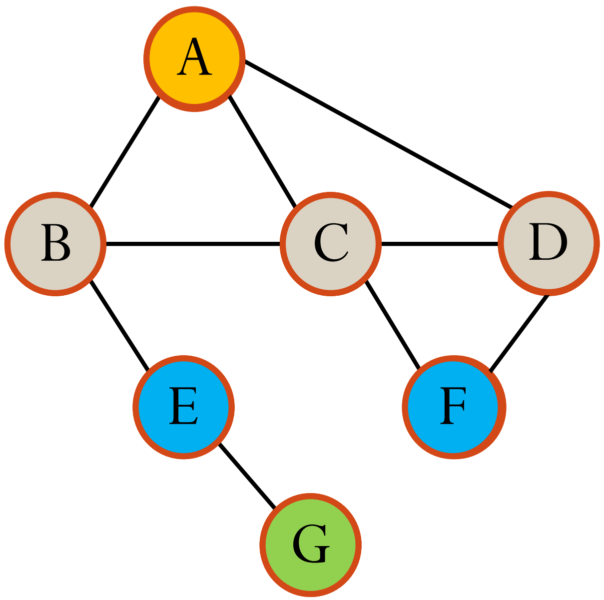
```
DFS(v) {
  visit v;
  mark v as visited;
  for(each node u adjacent to v)
    if(u is not visited) DFS(u);
}
```

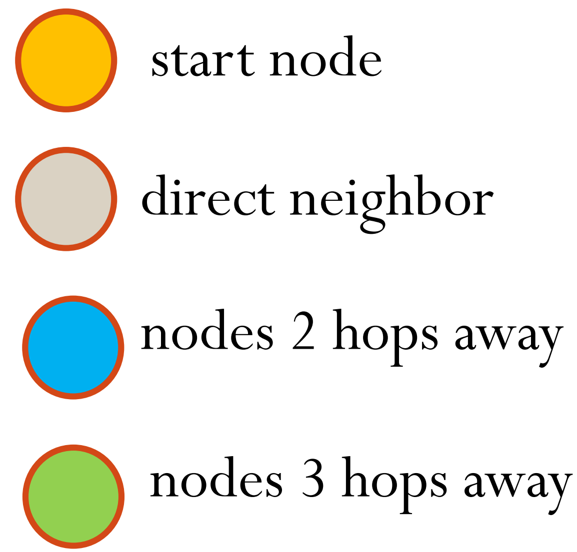
- How to mark a node "visited"?
  - Keep a "visited" field in the node



Breadth-First Search (BFS)

- Given a start node, visit all directly connected neighbors first, then nodes 2 hops away, 3 hops away, and so on.





A→B→C→D→E→F→G

Breadth-First Search (BFS)

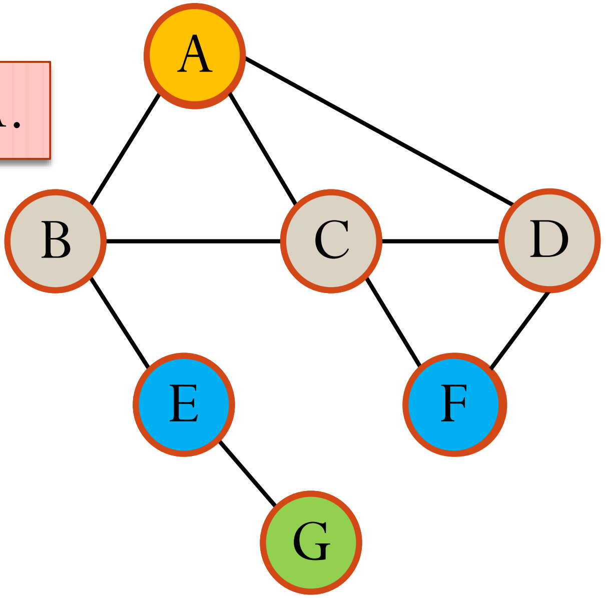
Implementation

- BFS can be implemented using a queue.

```
BFS(s) {
  queue q; // An empty queue
  visit s and mark s as visited;
  q.enqueue(s);
  while(!q.isEmpty()) {
    v = q.dequeue();
    for(each node u adjacent to v) {
      if(u is not visited) {
        visit u and mark u as visited;
        q.enqueue(u);
      }
    }
  }
}
```

Breadth-First Search (BFS)
Example

Start node is node A.




```
BFS(s) {
  queue q; // An empty queue
  visit s and mark s as visited;
  q.enqueue(s);
  while(!q.isEmpty()) {
    v = q.dequeue();
    for(each node u adjacent to v) {
      if(u is not visited) {
        visit u and mark u as visited;
        q.enqueue(u);
      }
    }
  }
}
```

Queue: A B C D E F G

Visit Order: A B C D E F G

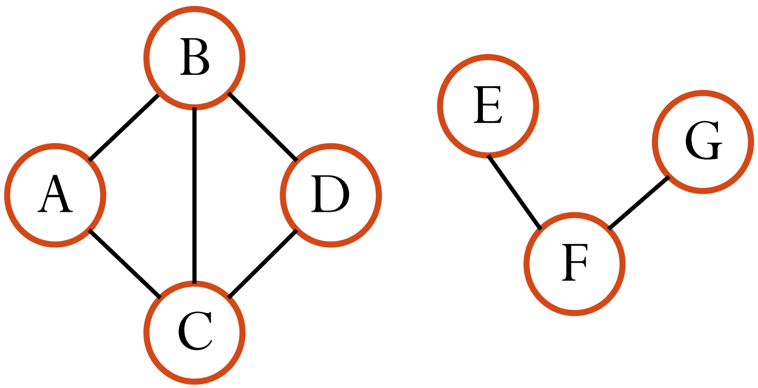
Breadth-First Search (BFS)

Time Complexity

- If graph is implemented as **adjacency matrix**:
  - Visit each node exactly once: $O(V)$.
  - The row of each node in the adjacency matrix is scanned once: $O(|V|)$ for each node.
  - Total running time: $O(|V|^2)$.
- If graph is implemented as **adjacency list**:
  - Visit each node exactly once: $O(|V|)$.
  - Adjacency list of each node is scanned once.
  - Size of entire adjacency list is $2|E|$ for undirected graph and $|E|$ for directed graph.
  - Total running time: $O(|V| + |E|)$.

Traverse All the Nodes in a Graph

- The graph may not be connected. How can we traverse all the nodes in the graph?



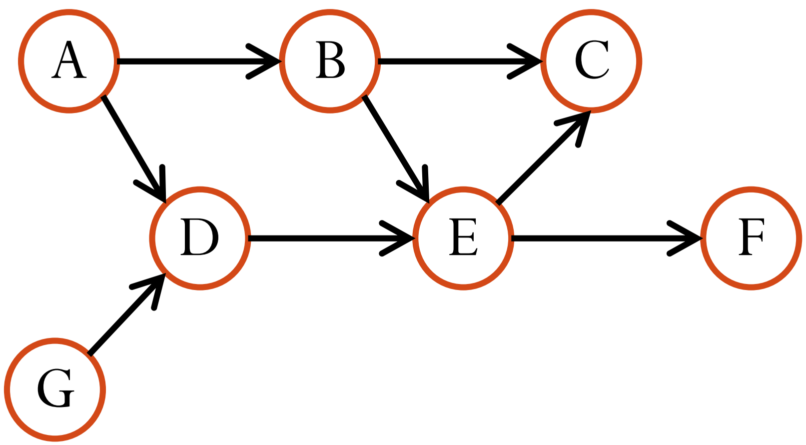

```
for(each node v in the graph)
   if(v is not visited)
      DFS(v);
```

Outline

- Graph Search
- Topological Sorting

Topological Sorting

- **Topological sorting** : an ordering on nodes of a **directed graph** so that **<u>for each</u>** edge (v_i, v_j) (means: an edge **from** v_i **to** v_j) in the graph, v_i is before v_j in the ordering.
  - Also known as **topological ordering**.

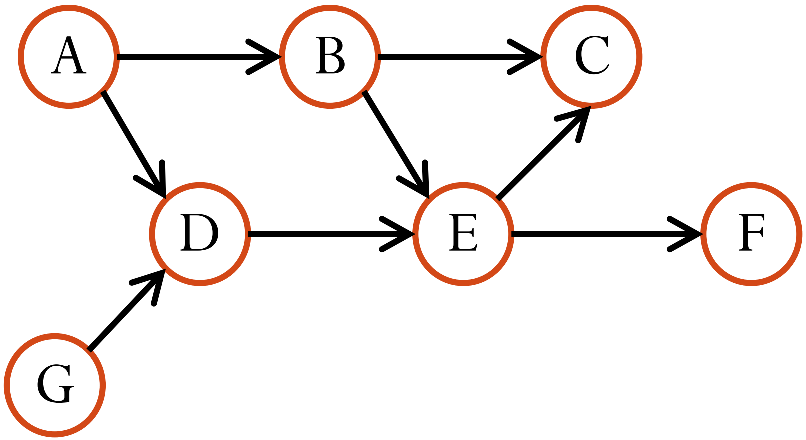


A topological sorting is: A, G, D, B, E, C, F

Which Graph Has Topological Sorting?

- Is there any "topological sorting" for directed graph **with cycles**?
  - In other words, can we order the nodes so that for each edge (v_i, v_j), v_i is before v_j in the ordering?
  - **Answer**: **No!** (Why?)
- How about **directed acyclic graph (DAG)**?
  - Yes! Guarantee to have a topological ordering.
  - Why? There is always a **source node** S in a DAG. Put S first. For the graph without S, again, there is a source node. Put it next …
- Next, we will focus on topological sorting on **DAG**.

Topological Sorting

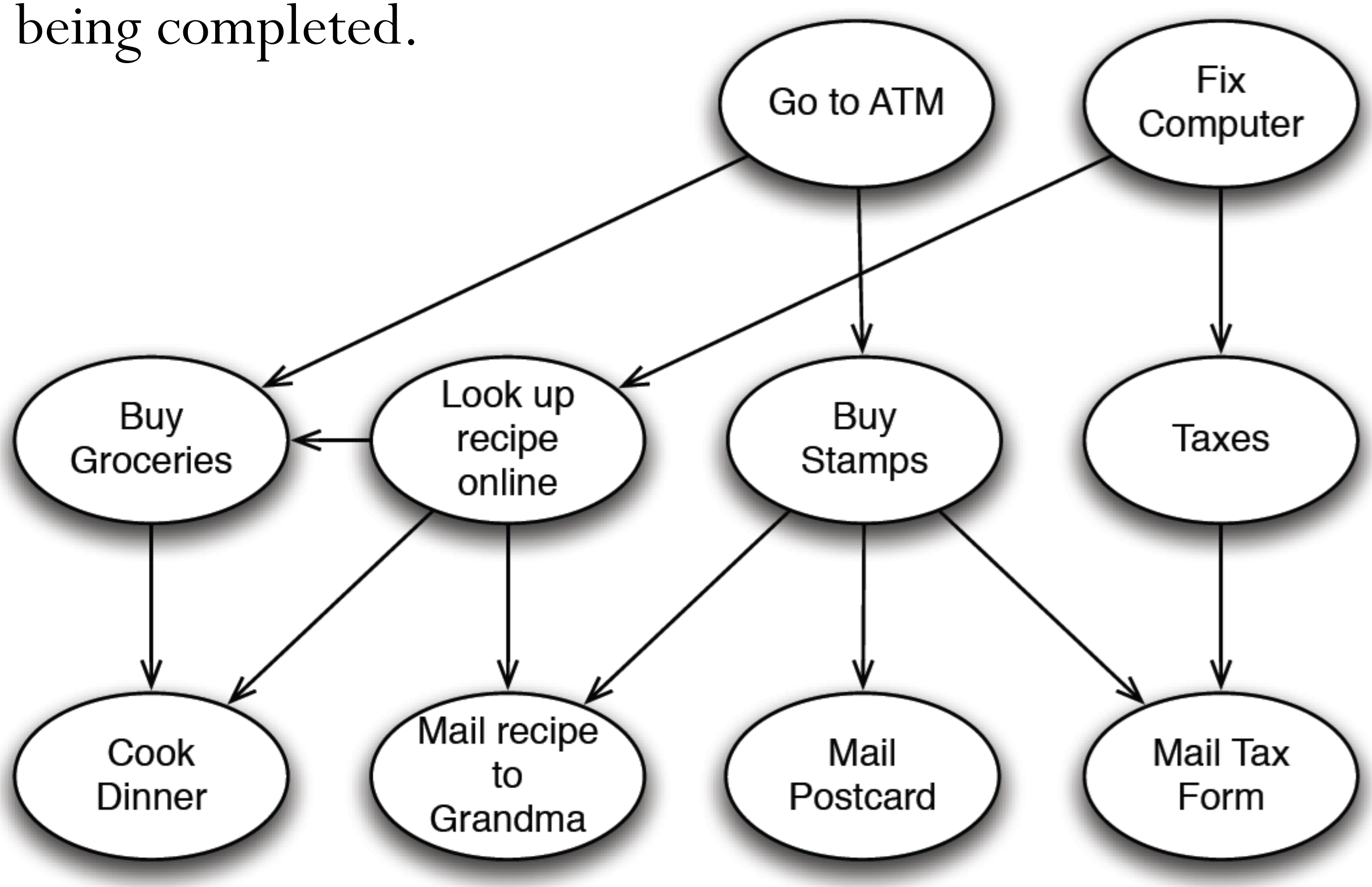


- Topological sorting is not necessarily **unique**:
  - A, G, D, B, E, C, F and A, B, G, D, E, F, C are both topological sorting.
- Are the following orderings topological sorting?
  - A, B, E, G, D, C, F
  - A, G, B, D, E, F, C

Topological Sorting

Applications

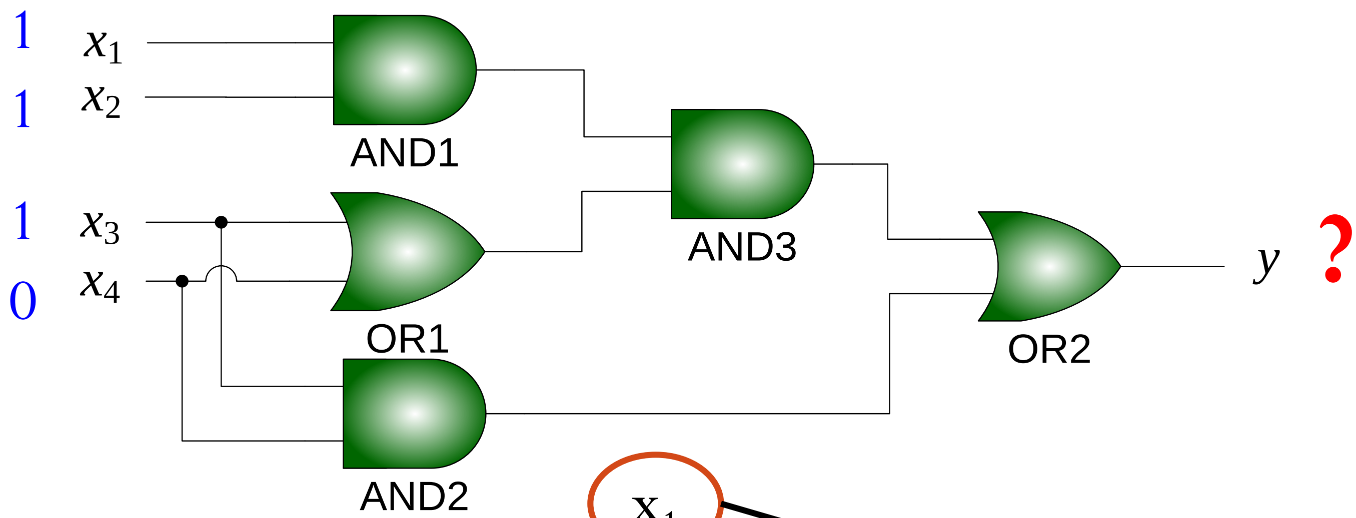
- Scheduling tasks when some tasks depend on other tasks being completed.

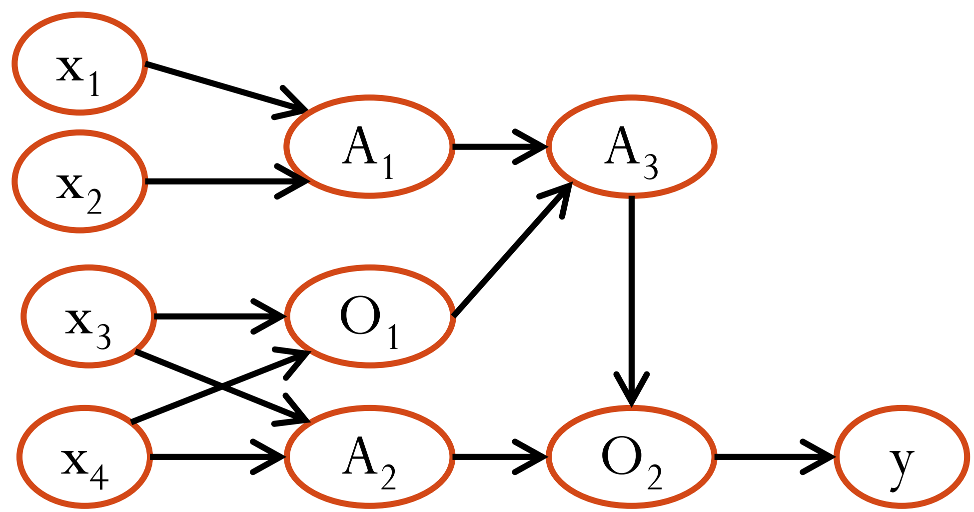


Topological Sorting

Applications

- Evaluating a combination logic circuit given a set of inputs.



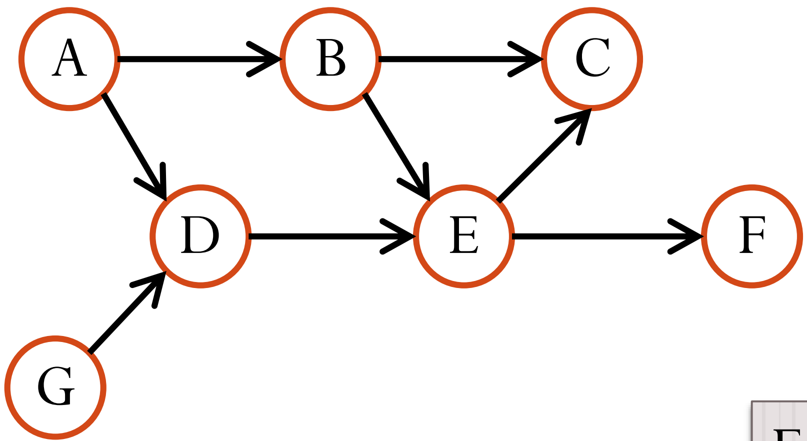


Topological Sorting: Algorithm

- Based on a **queue**.
- Algorithm:
  1. Compute the in-degrees of all nodes. (**in-degree**: number of **incoming** edges of a node.)
  2. **Enqueue** all in-degree 0 nodes into a queue.
  3. While queue is not empty
     1. **Dequeue** a node v from the queue and visit it.
     2. Decrement in-degrees of node v's neighbors.
     3. If any neighbor's in-degree becomes 0, **enqueue** it into the queue.

Topological Sorting Algorithm

Example



Queue

Enqueue A and G

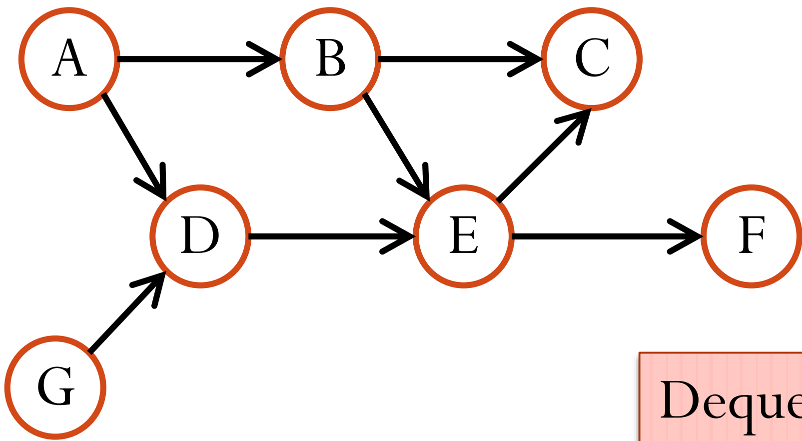
In-degrees

A	B	C	D	E	F	G
0	1	2	2	2	1	0

Order

Topological Sorting Algorithm

Example



Queue

A
G

Dequeue A, visit A, and decrement in-degrees of A's neighbors.

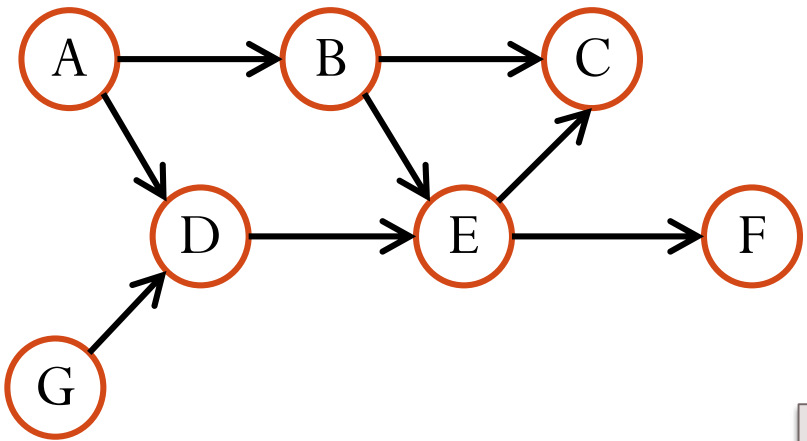
In-degrees

A	B	C	D	E	F	G
0	1	2	2	2	1	0

Order

Topological Sorting Algorithm

Example



Queue

G

Enqueue B

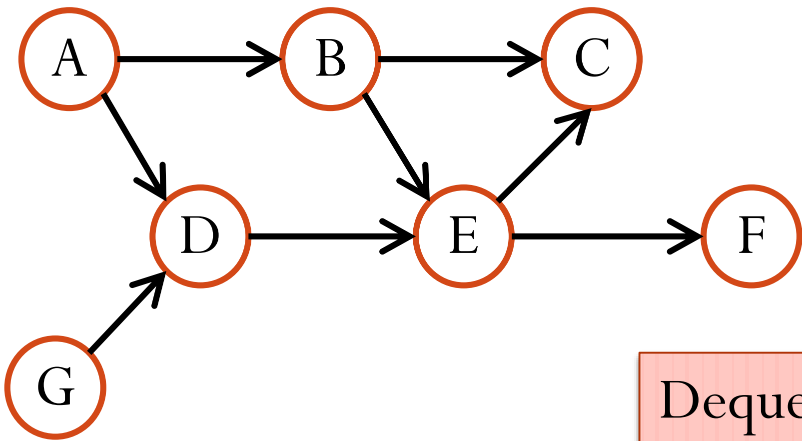
In-degrees

A	B	C	D	E	F	G
0	~~1~~ 0	2	~~2~~ 1	2	1	0

Order

A						

Topological Sorting Algorithm
Example



Queue

G
B

Dequeue G, visit G, and decrement in-degrees of G's neighbors.

In-degrees

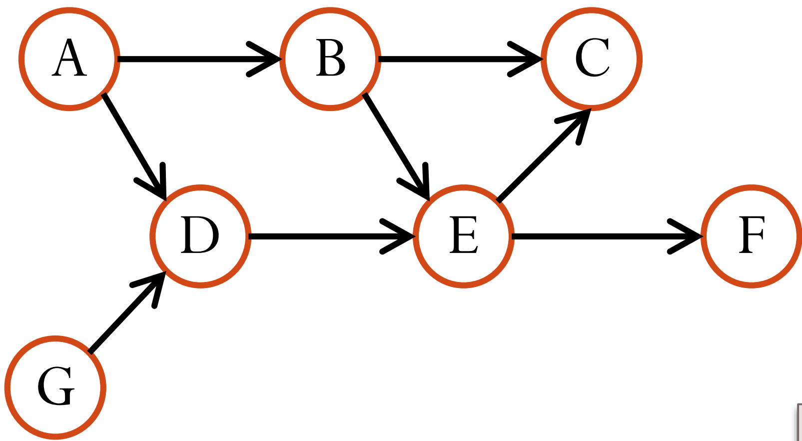
A	B	C	D	E	F	G
0	0	2	1	2	1	0

Order

A						

Topological Sorting Algorithm

Example



Queue

B

Enqueue D

In-degrees

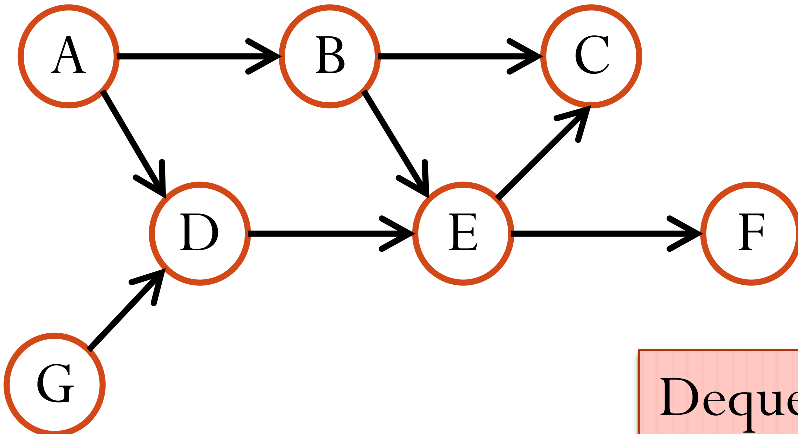
A	B	C	D	E	F	G
0	0	2	~~1~~ 0	2	1	0

Order

A	G					

Topological Sorting Algorithm

Example



Queue

B
D

Dequeue B, visit B, and decrement in-degrees of B's neighbors.

In-degrees

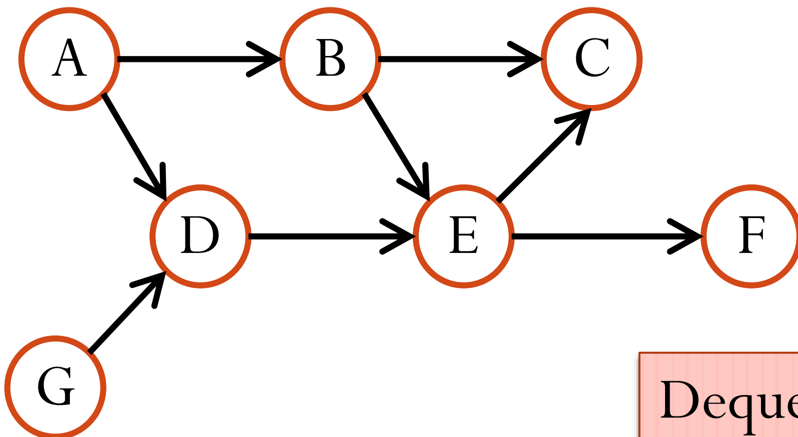
A	B	C	D	E	F	G
0	0	2	0	2	1	0

Order

A	G					

Topological Sorting Algorithm

Example



Queue

D

Dequeue D, visit D, and decrement in-degrees of D's neighbors.

In-degrees

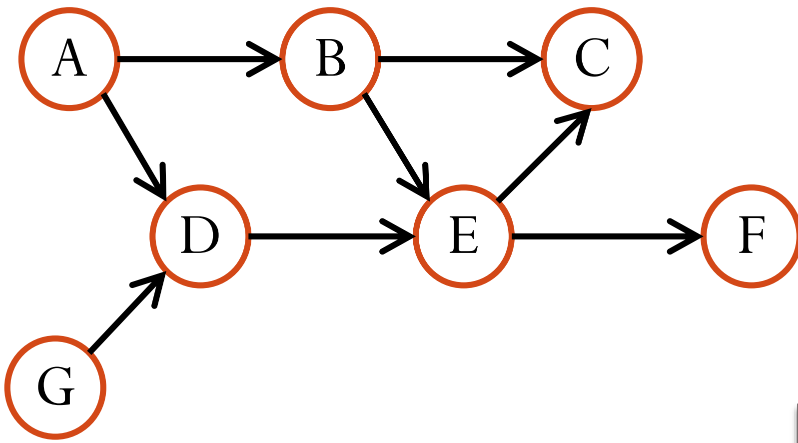
A	B	C	D	E	F	G
0	0	~~2~~ 1	0	~~2~~ 1	1	0

Order

A	G	**B**				

Topological Sorting Algorithm

Example



Queue

Enqueue E

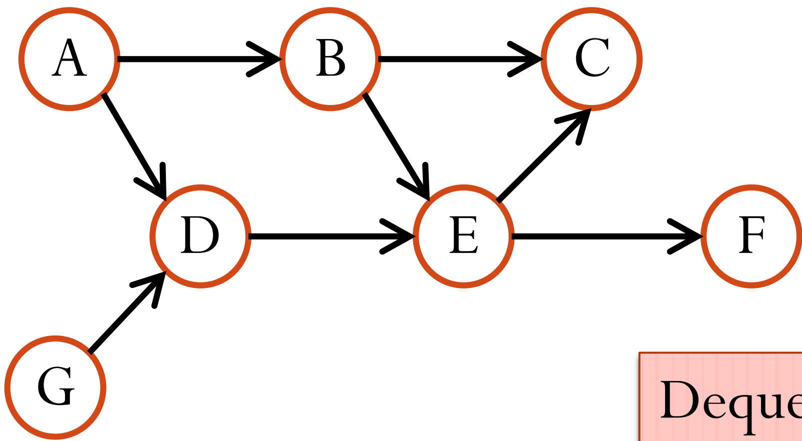
In-degrees

A	B	C	D	E	F	G
0	0	1	0	~~1~~ 0	1	0

Order

A	G	B	**D**			

Topological Sorting Algorithm
Example



Queue

E

Dequeue E, visit E, and decrement in-degrees of E's neighbors.

In-degrees

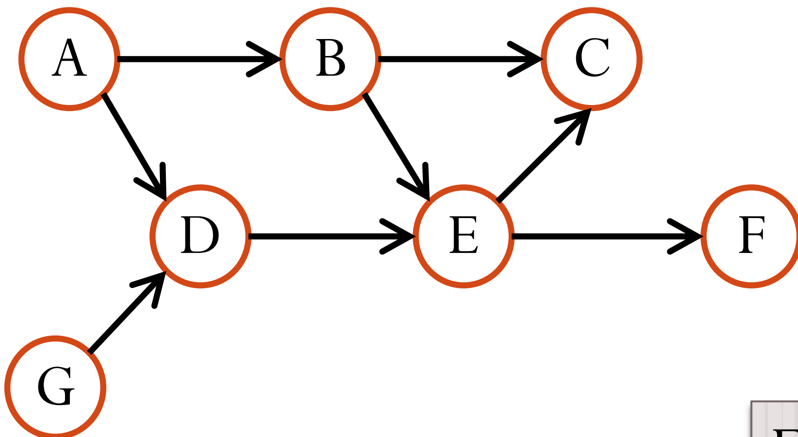
A	B	C	D	E	F	G
0	0	1	0	0	1	0

Order

A	G	B	D			

Topological Sorting Algorithm

Example



Queue

Enqueue C and F

In-degrees

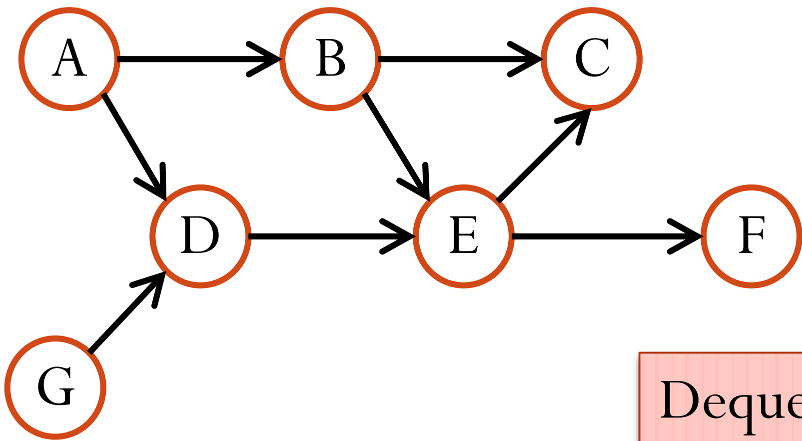
A	B	C	D	E	F	G
0	0	~~1~~ 0	0	0	~~1~~ 0	0

Order

A	G	B	D	**E**		

Topological Sorting Algorithm

Example



Queue

C
F

Dequeue C, visit C, and decrement in-degrees of C's neighbors.

In-degrees

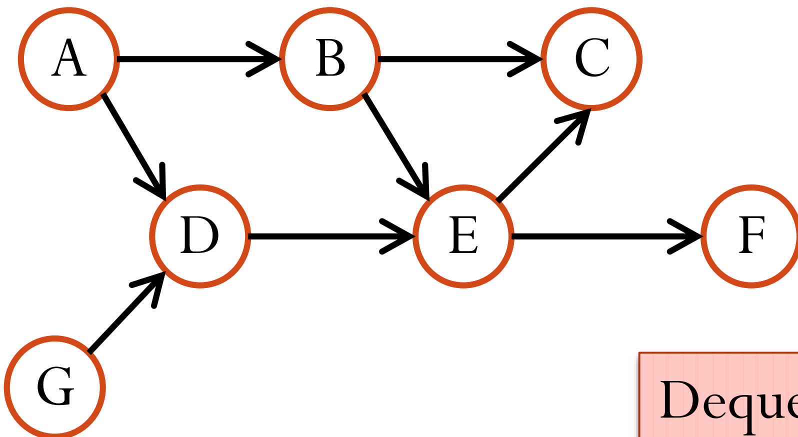
A	B	C	D	E	F	G
0	0	0	0	0	0	0

Order

A	G	B	D	E		

Topological Sorting Algorithm

Example



Queue

F

Dequeue F, visit F, and decrement in-degrees of F's neighbors.

In-degrees

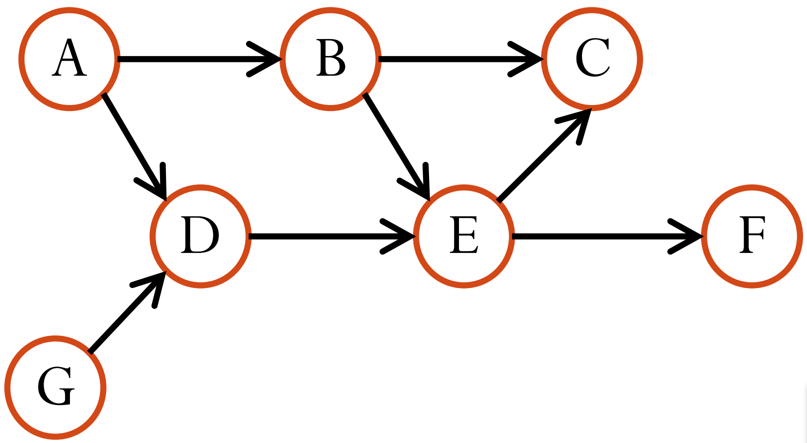
A	B	C	D	E	F	G
0	0	0	0	0	0	0

Order

A	G	B	D	E	C	

Topological Sorting Algorithm

Example



Queue

Queue is now empty. Done!

In-degrees

A	B	C	D	E	F	G
0	0	0	0	0	0	0

Order

A	G	B	D	E	C	**F**

Topological Sorting Algorithm

Time Complexity

Assume adjacency list representation

1. Compute the in-degrees of all nodes. $O(|V| + |E|)$ in total
2. Enqueue all in-degree 0 nodes into a queue. $O(|V|)$ in total
3. While queue is not empty
    1. Dequeue a node v from the queue and visit it. $O(|V|)$ in total
    2. Decrement in-degrees of node v's neighbors. $O(|E|)$ in total
    3. If any neighbor's in-degree becomes 0 … $O(|V|)$ in total
        - … place it in the queue.

Total running time is $O(|V| + |E|)$.