Inverse Problem - A Bayesian perspective

许笑颜

University of Science and Technology of China

2024 年 9 月 15 日

问题背景

We have an equation of the form

$$y = \mathcal{G}(\mu) \tag{1}$$

to solve for $\mu \in X$, given $y \in Y$, where X, Y are Banach spaces.

1. μ can be viewed as the initial solution $\mu(t = 0) = \mu_0$ of a pde.
2. y can be viewed as μ_t, which is observable.

Find μ by solving the least-square problem below:

$$\mathrm{argmin}_{u \in X} \frac{1}{2} \|y - \mathcal{G}(u)\|_Y^2$$

问题背景

This problem, too, may be difficult to solve as it may possess minimizing sequences $u^{(n)}$ which do not converge to a limit in X, or it may possess multiple minima and sensitive dependence on the data y.

These issues can be somewhat ameliorated by solving a regularized minimization problem of the form, for some Banach space $(E, \|\cdot\|_E)$ contained in X, and point $m_0 \in E$,

$$\operatorname{argmin}_{u \in E} \left(\frac{1}{2} \|y - \mathcal{G}(u)\|_Y^2 + \frac{1}{2} \|u - m_0\|_E^2 \right). \tag{2}$$

问题背景

To solve the above problem, the Bayesian approach is introduced

1. a posterior distribution of μ^y given y,
2. a prior distribution of $\mu^0 \sim N(m_o, \Gamma)$.

The posterior distribution μ^y will have probability density $\pi^y(\mu)$ of

$$\pi^y(u) \propto \exp\left(-\frac{1}{2}\|y - \mathcal{G}(u)\|_Y^2 - \frac{1}{2}\|u - m_0\|_E^2\right). \tag{3}$$

By solving the least-square problem in (2), we can find the maximum a posterior(MAP) estimate for the posterior distribution.

1. Problem Background
2. Algorithm

Common Algorithm

For common situations:

1. Set the Gaussian distribution of prior μ^0, such as $\mu^0 \sim N(0, \sigma)$.
2. Set the Gaussian distribution of observed noise η, that is $\eta \sim N(0, \Gamma)$
3. Set the linear/nonlinear system $y = \mathcal{G}(\mu) + \eta$. The linear & non-linear of the system depend on the operator mapping $\mathcal{G}$.
4. Set the Gaussian distribution of posterior μ^y, such as $\mu^y \sim N(m, \Sigma)$.
5. Compute the posterior mean and variance m, Σ using Theorem 6.20.

额外推导

Set $y = \mathcal{G}(\mu) + \eta$ and $\eta \sim N(0, \Gamma)$, thus:

$$\eta = y - \mathcal{G}(\mu) \sim N(0, \Gamma), \tag{4}$$

The probability density function of η can also be written as

$$\pi(\eta) = \exp(\frac{(\eta)^2}{2\Gamma}) = \exp(\frac{(y - \mathcal{G}(\mu))^2}{2\Gamma}) \tag{5}$$