Tarea 2-U2

1) Encuentre el costo asintótico en terminos de O (ó grande) de la siguiente recurrencia considerando que $T(n)=1$ si $n==0$. Puede utilizar cualquier método conocido para encontrarlo. (4p)

$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log^2(n)}$ si $n>0$

$T(n)=1$ si $n==0$

$T\left(\frac{n}{2}\right) = 2\cdot T\left(\frac{n}{4}\right) + \frac{n}{2}\cdot\frac{1}{\log^2\left(\frac{n}{2}\right)}$

$T\left(\frac{n}{2}\right) = 2\cdot T\left(\frac{n}{4}\right) + \frac{n}{2}\cdot\frac{1}{[\log(n)-\log(2)]^2} = 2T\left(\frac{n}{2^2}\right) + \frac{n}{2^1}\cdot\frac{1}{[\log(n)-1.\log(2)]^2}$

$$\boxed{\log_a(b) = \frac{\log_c(b)}{\log_c(a)}}$$

$T\left(\frac{n}{4}\right) = 2\cdot T\left(\frac{\frac{n}{4}}{2}\right) + \frac{n}{4}\cdot\frac{1}{\log^2\left(\frac{n}{4}\right)}$

$T\left(\frac{n}{4}\right) = 2\cdot T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\cdot\frac{1}{[\log(n)-\log(2^2)]^2} = 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\cdot\frac{1}{[\log(n)-2.\log(2)]^2}$

$T\left(\frac{n}{8}\right) = 2T\left(\frac{\frac{n}{8}}{2}\right) + \frac{n}{2^3}\cdot\frac{1}{\log^2\left(\frac{n}{8}\right)}$

$T\left(\frac{n}{8}\right) = 2T\left(\frac{n}{2^4}\right) + \frac{n}{2^3}\cdot\frac{1}{[\log(n)-\log(2^3)]} = 2T\left(\frac{n}{2^4}\right) + \frac{n}{2^3}\cdot\frac{1}{[\log(n)-3.\log(2)]^2}$

$T(n) = 2\cdot\left[2T\left(\frac{n}{2^2}\right) + \frac{n}{2^1}\cdot\left[\log\left(\frac{n}{2^1}\right)\right]^{-2}\right] + \frac{n}{2^0}\cdot\left[\log\left(\frac{n}{2^0}\right)\right]^{-2}$

$T(n) = 2^2 T\left(\frac{n}{2^2}\right) + n\cdot\left[\log\left(\frac{n}{2^1}\right)\right]^{-2} + n\cdot\left[\log\left(\frac{n}{2^0}\right)\right]^{-2}$

$T(n) = 2^2\cdot\left[2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\left[\log\left(\frac{n}{2^2}\right)\right]^{-2}\right] + n\cdot\left[\log\left(\frac{n}{2^1}\right)\right]^{-2} + n\cdot\left[\log\left(\frac{n}{2^0}\right)\right]^{-2}$

$T(n) = 2^3\cdot T\left(\frac{n}{2^3}\right) + n\cdot\log\left(\frac{n}{2^2}\right)^{-2} + n\left[\log\left(\frac{n}{2^1}\right)\right]^{-2} + n\left[\log\left(\frac{n}{2^0}\right)\right]^{-2}$

$T(n) = 2^k\cdot T\left(\frac{n}{2^k}\right) + n\cdot\left[\sum_{i=0}^{k-1}\left[\log\left(\frac{n}{2^i}\right)\right]^{-2}\right]$

$T(n) = 2^{\log_2(n)}\cdot T\left(\frac{n}{2^{\log_2(n)}}\right) + n\cdot\left(\sum_{i=0}^{\log_2(n)-1}\left[\log\left(\frac{n}{2^i}\right)\right]^{-2}\right)$

$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$

$T(0)=1$

$T(1) = 2T\left(\frac{1}{2}\right) + f(1)$

$T(2) = 2T\left(\frac{2}{2}\right) + f(2)$

$\frac{n}{2^k} = 1$; $n = 2^k$

$\log_2(n) = K.1$

$$\sum_{i=0}^{\log_2(n)-1}\frac{1}{[\log(n)-i\cdot\log(2)]^2} = \sum_{i=0}^{\log_2(n)-1}\frac{1}{(L-C.i)^2} \approx \frac{1}{C}\cdot\int_{L-CK}^{L}\frac{1}{x^2}\,dx$$

Aproximacion Integral

$$\sum_{i=0}^{K}\frac{1}{(a-b.i)^2} \approx \frac{1}{b}\cdot\int_{a-b.K}^{a}\frac{1}{x^2}\,dx$$

$\log(n)=L$
$\log(2)=C$
$K=\log_2(n)$

$$\frac{1}{\log(2)}\cdot\int_{\log(n)-\log(2)\cdot\underbrace{\frac{\log(n)}{\log(2)}}_{\text{cambio de base}}}^{\log(n)}\frac{1}{x^2}\,dx = \frac{1}{\log(2)}\cdot\int_{\log(n)-\log(n)}^{\log(n)}\frac{1}{x^2}\,dx = \frac{1}{\log(2)}\cdot\int_{0}^{\log(n)}\frac{1}{x^2} = \int_{\varepsilon}^{\log(n)}\frac{1}{x^2}\,dx$$

$$\boxed{\int x^a\,dx = \frac{x^{a+1}}{a+1} ; a \neq -1}$$

Nunca $\left[\log\left(\frac{n}{2^i}\right)\right]$ sera 0, por lo que es un numero cercano a cero, $\varepsilon>0$; ε es un número muy pequeño

$$\int_{\varepsilon}^{\log(n)}\frac{1}{x^2}\,dx = \left[\frac{x^{-2+1}}{-2+1}\right]_{\varepsilon}^{\log(n)} = \frac{x^{-1}}{-1}\Big|_{\varepsilon}^{\log(n)} = -\frac{1}{x}\Big|_{\varepsilon}^{\log(n)} = -\frac{1}{\log(n)} - \left(-\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon} - \frac{1}{\log(n)}$$

$$\therefore \sum_{i=0}^{\log_2(n)-1}\frac{1}{\left[\log\left(\frac{n}{2^i}\right)\right]^2} \approx \frac{1}{\varepsilon} - \frac{1}{\log(n)} \longrightarrow O\left(\frac{1}{\log(n)}\right) ; \text{ O de la sumatoria}$$

$$\boxed{O\left(\frac{1}{\log(n)}\right) < O(n)}$$

$T(n) = 2^{\log_2(n)}\cdot T\left(\cancel{\frac{n}{2^{\log_2(n)}}}^{1}\right) + n\cdot\left[\frac{1}{\varepsilon} - \frac{1}{\log(n)}\right]$

$T(n) = n\cdot 1 + \frac{n}{\varepsilon} - \frac{n}{\log(n)}$

$T(n) = O(n) \quad O(1) \quad O(n)\cdot O\left(\frac{1}{\log(n)}\right)$

$\rightarrow$ Prevalece $O(n)$; $\therefore$ El Orden de crecimiento solicitado es $O(n)$

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2) Encuentre el orden de crecimiento en término de O $(O(g(n)))$ de las siguientes sumas. (Utilice la función $g(n)$ mas simple y precisa posible). Justifica (2p cada uno)

a) $\sum_{i=1}^{\log_2(n)} 4^i = 4^1+4^2+4^3+\dots+4^{\log_2(n)} = \frac{4^{\log_2(n)}\cdot 4-4^1}{4-1} = \frac{n^2\cdot 4-4}{3} = \frac{4n^2-4}{3}$ ↓ $O(n^2)$

$g(n) = n^2$

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$(2^2)^{\log_2(n)} = 2^{2\cdot\log_2(n)} = 2^{\log_2(n^2)} = n^2$

b) $\sum_{i=1}^{n} (i+1)\cdot 3^{i-1} = (1+1)\cdot 3^{1-1} + (2+1)\cdot 3^{2-1} + \dots + (n+1)\cdot 3^{n-1}$

$2\cdot 3^0 + 3\cdot 3^1$

$\sum_{i=1}^{n} i\cdot r^{i-1} = \frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}$

$\sum_{i=1}^{n} i\cdot 3^{i-1} + \sum_{i=1}^{n} 3^{i-1}$ PG

→ $\frac{3^{n-1}\cdot 3-3^0}{3-1} = \frac{3^n-1}{2}$

→ $\frac{3-3^n(n-1)+3^{n+1}\cdot n}{(3-1)^2}$ = // Obtenido por Symbolab

→ $\sum_{i=1}^{n}(i+1)\cdot 3^{i-1} = \frac{3-3^n(n-1)+3^{n+1}\cdot n}{4} + \frac{(3^n-1)2}{2\cdot 2} = \frac{3-n\cdot 3^n+3^n+3n\cdot 3^n}{4} + \frac{6n-3}{4} = \frac{\cancel{3}-n\cdot 3^n+3^n+3n\cdot 3^n+6n-\cancel{3}}{4}$

$\sum_{i=1}^{n}(i+1)\cdot 3^{i-1} = \frac{3^n+3n\cdot 3^n-n\cdot 3^n+6n}{4}$; $O(3n\cdot 3^n) \longrightarrow O(n\cdot 3^n)$

c) $\sum_{i=2}^{n-1} \log_2 i^2 \approx \int_2^n \log_2(x)\,dx$; $\log_2(x) = \frac{\ln(x)}{\ln(2)}$

$\int \ln(x)\,dx = x\cdot\ln(x) - x$

$\int_2^n \log_2(x)\,dx = \frac{1}{\ln(2)}\cdot\int_2^n \ln(x)\,dx$

$\frac{1}{\ln(2)}\cdot[n\cdot\ln(n)-n-(2\ln(2)-2)] = \frac{1}{\ln(2)}\cdot[n\cdot\ln(n)-n+2\ln(2)+2] = \frac{n\cdot\ln(n)}{\ln(2)} - \frac{n}{\ln(2)} + 2 + \frac{2}{\ln(2)}$

$\sum_{i=2}^{n-1} \log_2 i^2 \approx n\cdot\log_2(n) - \frac{n}{\ln(2)} + C \longrightarrow O(n\cdot\log(n))$

$\boxed{\sum_{K=1}^{N} K = \frac{N(N+1)}{2}}$

d) $\sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1}(i+j+k)$

$\sum_{K=1}^{j-1}(i+j+K) = \sum_{K=1}^{j-1}(i+j) + \sum_{K=1}^{j-1}K = (i+j)(j-1) + \frac{(j-1)(j-1+1)}{2} = (ij-i+j^2-j) + \frac{j^2}{2} - \frac{j}{2} = ij - i + \frac{3j^2}{2} - \frac{3}{2}j$

$\sum_{j=1}^{i-1}(ij-i+\frac{3}{2}j^2-\frac{3}{2}j) = i\sum_{j=1}^{i-1} j - \sum_{j=1}^{i-1} i + \frac{3}{2}\sum_{j=1}^{i-1} j^2 - \frac{3}{2}\sum_{j=1}^{i-1} j = i\cdot\frac{(i-1)(i-1+1)}{2} - (i-1)(i-1+1) + \frac{3}{2}\left(\frac{(i-1)\cdot i(2i-1)}{6}\right) - \frac{3}{2}\left(\frac{i(i-1)}{2}\right)$

→ $\frac{2\cdot i^2(i-1)}{2\cdot 2} - \frac{4i(i-1)}{4} + \frac{(i^2-i)(2i-1)}{4} - \frac{3i(i-1)}{4} = \frac{2i^3-2i^2-4i^2+4i+2i^3-i^2-2i^2+i-3i^2+3i}{4}$

→ $\frac{4i^3-12i^2+8i}{4} = i^3-3i^2+2i = i(i^2-3i+2) = i(i-2)(i-1)$

$\boxed{\sum_{K=1}^{n} K^3 = \frac{n^2\cdot(n+1)^2}{4}}$

$\sum_{i=1}^{n}(i^3-3i^2+2i) = \sum_{i=1}^{n} i^3 - 3\cdot\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i = \frac{n^2\cdot(n+1)^2}{4} - 3\cdot\frac{1}{6}\cdot n(n+1)(2n+1) + 2\cdot\left(\frac{n(n+1)}{2}\right) = \frac{n^2(n+1)^2}{4} - \frac{1}{2}n(n+1)(2n+1)\cdot\frac{2}{2} + \frac{4}{4}(n^2+n)$

→ $\frac{n^2(n^2+2n+1)-(2n^2+2n)(2n+1)+4n^2+4n}{4} = \frac{n^4+2n^3+n^2-(4n^3+2n^2+4n^2+2n)+4n^2+4n}{4} = \frac{n^4+2n^3+n^2-4n^3-2n^2-4n^2-2n+4n^2+4n}{4}$

→ $n^4-2n^3-n^2-n$

$\sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{K=1}^{j-1}(i+j+K) = n^4-2n^3-n^2-n$

∴ $O(n^4)$

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3) Resuelva las siguientes recurrencias, por el método que usted conozca. Justificar las respuestas. Asumir que $T(n)$ es constante para una n suficientemente pequeña n, por ejemplo $T(n)=c$, si $n==1$. (2p cada uno)

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a) $T(n) = T\left(\frac{9n}{10}\right) + n^2$

$T\left(\frac{9n}{10}\right) = T\left(\frac{9^2}{10^2}n\right) + \frac{9}{10}n^2$

$T\left(\frac{9^2}{10^2}n\right) = T\left(\frac{9^3}{10^3}n\right) + \frac{9^2}{10^2}n^2$

$T(n) = \left[T\left(\frac{9^2}{10^2}n\right) + \frac{9}{10}n^2\right] + n^2$

$T(n) = T\left(\frac{9^2}{10^2}n\right) + \frac{9}{10}n^2 + n^2$

$T(n) = \left[T\left(\frac{9^3}{10^3}n\right) + \frac{9^2}{10^2}n^2\right] + \frac{9}{10}n^2 + n^2$

$T(n) = T\left(\frac{9^3}{10^3}n\right) + \frac{9^2}{10^2}n^2 + \frac{9}{10}n^2 + n^2$

$T(n) = T\left(\frac{9^k}{10^k}n\right) + \sum_{i=0}^{k-1}\left(\frac{9}{10}\right)^k \cdot n^2$

$\frac{9^k}{10^k}n = 1$

$\frac{9^k \cdot n}{10^k} = 1$

$n = \frac{10^k}{9^k} = \left(\frac{10}{9}\right)^k$

$\log_{\frac{10}{9}}(n) = \log_{\frac{10}{9}}\left(\left[\frac{10}{9}\right]^k\right)$

$\log_{\frac{10}{9}}(n) = k$

$\sum_{i=0}^{k-1}\left(\frac{9}{10}\right)^k = \left(\frac{9}{10}\right)^0 + \left(\frac{9}{10}\right)^1 + \left(\frac{9}{10}\right)^2 + \dots + \left(\frac{9}{10}\right)^{k-1}$

$T(n) = \overset{1}{\cancel{T\left(\left[\frac{9}{10}\right]^{\log_{\frac{10}{9}}(n)} \cdot n\right)}} + \sum_{i=0}^{k-1}\left(\frac{9}{10}\right)^k \cdot n^2$

Suma P.G $\rightarrow \frac{a_n \cdot r - a_1}{r-1}$

P.G $\rightarrow \frac{\left(\frac{9}{10}\right)^{k-1} \cdot \frac{9}{10} - \left(\frac{9}{10}\right)^0}{\frac{9}{10} - 1}$

P.G $\rightarrow \frac{\left(\frac{9}{10}\right)^k - 1}{-\frac{1}{10}} = \frac{\left(\frac{9}{10}\right)^k - 1}{\frac{-1}{10}} = 10 - 10 \cdot \left(\frac{9}{10}\right)^k$

$T(n) = c + 10 - 10 \cdot \left(\frac{9}{10}\right)^{\log_{\frac{10}{9}}(n)}$

$T(n) = c + \left[10 - 10 \cdot \left(\frac{9}{10}\right)^{\log_{\frac{10}{9}}(n)}\right] \cdot n^2$

$T(n) = c + 10n^2 - 10 \cdot \left(\frac{9}{10}\right)^{\log_{\frac{10}{9}}(n)} \longrightarrow O(n^2)$

$O(n^2 \cdot \log(n))$

$-10\left(\frac{10}{9}\right)^{-1 \cdot \log_{\frac{10}{9}}(n)}$

$-10\left(\frac{10}{9}\right)^{\log_{\frac{10}{9}}(n^{-1})}$

$b^{\log_b a} = a$

$-10\, n^{-1} \rightarrow \frac{10}{n}$

b) $T(n) = T(n-2) + \log_2(n^2)$

$T(n) = T(n-2) + 2 \cdot \log_2(n)$

$T(n-2) = T(n-4) + 2 \cdot \log(n-2)$

$T(n-4) = T(n-6) + 2 \cdot \log(n-4)$

$T(n) = T(n-2) + 2\log_2(n)$

$T(n) = [T(n-4) + 2\log(n-2)] + 2\log_2(n)$

$T(n) = [T(n-2k)] + \sum_{i}^{k-1} 2\log(n-2i)$

$n - 2k = 1 \;;\; k = n/2$

$T(n) \approx \sum_{i=0}^{n/2} 2\log(n-2i) = \sum \log(2j) \approx \int_1^{n/2} \log(x)\,dx \rightarrow \frac{n}{2} \cdot \log\left(\frac{n}{2}\right)$

$T(n) = O(n \cdot \log(n))$

$T(n) = T(1) + 2 \cdot \log_2\left(\frac{n!}{(n-k)!}\right)$

$T(n) = c + 2 \cdot \log_2\left(\frac{n!}{\left(n - \left(\frac{n-1}{2}\right)\right)!}\right)$

$T(n) = c + 2 \cdot \log_2\left(\frac{n!}{\left(\frac{n+1}{2}\right)!}\right)$

$\div\; n-2,\, n-4,\, n-6,\, \dots,\, a_n$

$r = n-6-(n-4) = n-n-6+4 = -2$

$a_n = n-2+(k-1)\cdot(-2)$

$a_n = n-\cancel{2}-2k+\cancel{2}$

$a_n = n-2k$

$O\left(\log\left(\frac{n!}{\left(\frac{n+1}{2}\right)!}\right)\right)$

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3) c) $T(n)=3T(n-2)+\sqrt{n}$

$T(n-2)=3T(n-4)+\sqrt{n-2}$

$T(n-4)=3T(n-6)+\sqrt{n-4}$

$T(1)=C$

$\frac{n-1}{2}-1\cdot\frac{2}{2}=\frac{n-1-2}{2}=\frac{n-3}{2}$

$T(n)=3T(n-2)+\sqrt{n}$

$T(n)=3\left[3T(n-4)+\sqrt{n-2}\right]+\sqrt{n}$

$T(n)=3^2T(n-4)+3\sqrt{n-2}+\sqrt{n}$

$T(n)=3^2\left[3T(n-6)+\sqrt{n-4}\right]+3\sqrt{n-2}+\sqrt{n}$

$T(n)=3^3T(n-6)+3^2\sqrt{n-4}+3\sqrt{n-2}+\sqrt{n}$

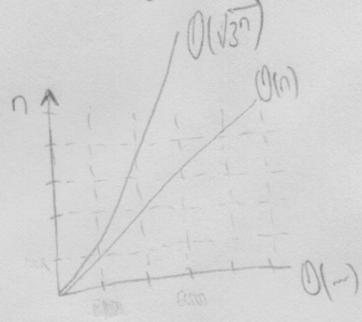
$T(n)=3^kT(n-2k)+\left(\sum_{i=0}^{k-1}3^i\cdot\sqrt{n-2i}\right)$

$T(n)=3^{\frac{n-1}{2}}\cdot T(1)+\left(\sum_{i=0}^{(n-3)/2}3^i\cdot\sqrt{n-2i}\right)$

$n-2K=1$

$n-1=2K$

$K=\frac{n-1}{2}$



$T(n)=c\cdot\sqrt{3^{n-1}}+\left(3^0\cdot\sqrt{n-2\cdot 0}+3^1\cdot\sqrt{n-2\cdot 1}+\dots+3^{\frac{(n-3)}{2}}\cdot\sqrt{n-2\cdot\left(\frac{n-3}{2}\right)}\right)$

$O(\sqrt{3^{n-1}})$ $\quad$ $O(n-2k)$

$\downarrow$

$O(\sqrt{3^n}) > O(n)$

$\therefore$ $O(\sqrt{3^n})$ es la cota superior.

d) $T(n)=\sqrt{n}\cdot T(\sqrt{n})+n$

$n=2^m$

$m=\log_2(n)$

$S(m)=T(2^m)$

$R(m)=\frac{S(m)}{2^m}$

$T(2^m)=2^{m/2}\cdot T(2^{m/2})+2^m$

$S(m)=2^{m/2}\cdot S(m/2)+2^m$

$\frac{S(m)}{2^m}=\frac{S(m/2)}{2^{m/2}}+1$

$R(m)=R(m/2)+1$

$R(m)=O(\log m)$

$\frac{S(m)}{2^m}=O(\log m)$

$T(2^m)=2^m\cdot O(\log m)$

$T(n)=n\cdot O(\log[\log_2(m)])$

$\downarrow$

$O(n\cdot\log[\log(m)])$

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