Logical Aspects of Artificial Intelligence Temporal Logics for Multi-agent Systems

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October 5th, 2022 – Lecture 4

Plan of the lecture

- Concurrent game structures.
- Introduction to ATL.
- Exercises session.

Breaking news

- Exam on Wednesday November 9th, 2pm-5pm/2p-6pm
- Room 1E14
- Lecture notes and exercises sheets with correction authorised.

Temporal Logics for Multi-Agent Systems

Introduction to multi-agent systems

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- In this lecture, we present the basic ingredients in the logic ATL and variants.

Other (online) ressources

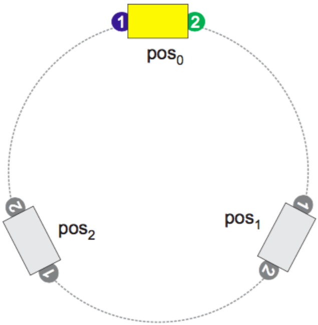
- Valentin Goranko's slides (ESSLLI'18)
- See also the proceedings of the international conferences:
  - International Conference on Autonomous Agents and Multi-Agent Systems. (AAMAS)
  - European Conference on Artificial Intelligence. (ECAI)
  - International Conference on Principles of Knowledge Representation and Reasoning. (KR)
- Book "Logical Methods for Specification and Verification of Multi-Agent Systems" by W. Jamroga, 2020.

  https://home.ipipan.waw.pl/w.jamroga/papers/jamroga15specifmas-20200411.pdf
- Book "Temporal Logics in Computer Science" by S. Demri, V. Goranko, M. Lange, Cambridge University Press, 2016.

Concurrent Game Structures

The two-robot example

- Two robots Robot_1 and Robot_2, and a carriage.
- Robot_1 can only push the carriage in clockwise direction, Robot_2 can only push it in anti-clockwise direction.



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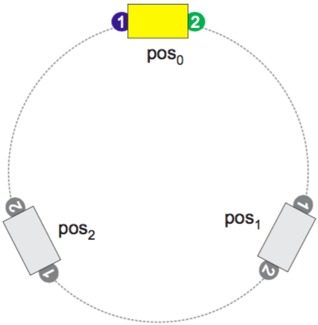

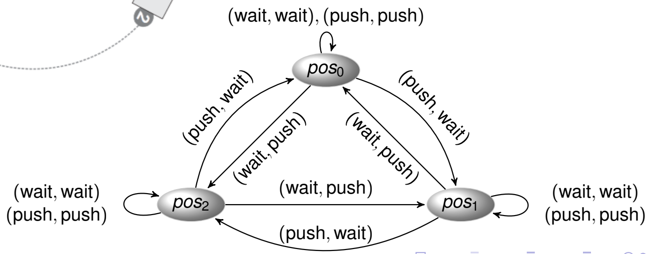

Concurrent game structure: definition

$$\mathfrak{M} = (\mathit{Agt}, S, \mathit{Act}, \texttt{act}, \delta, L)$$

- *Agt* is a non-empty set of k **agents.**
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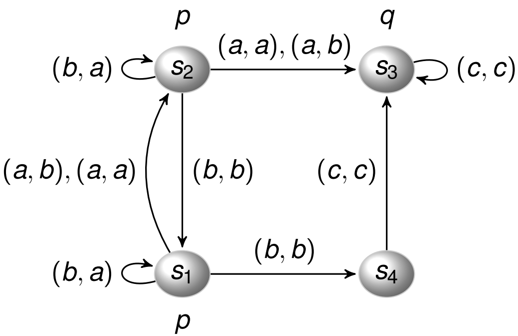
Difference

An example



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$S = \{s_1, s_2, s_3, s_4\}$

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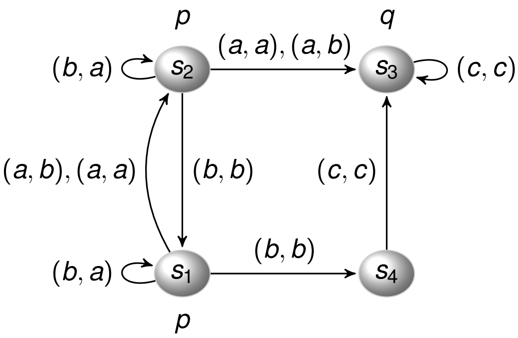

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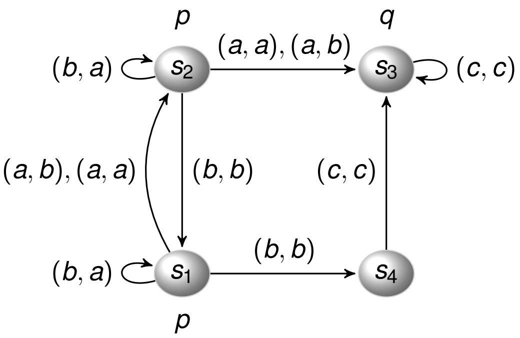

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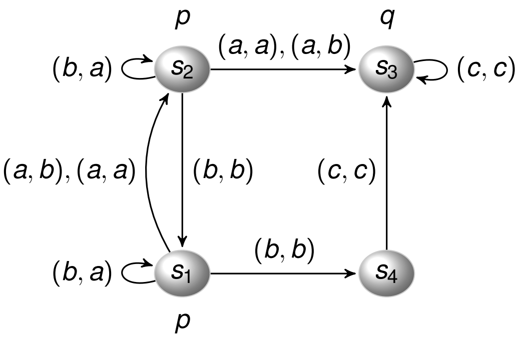

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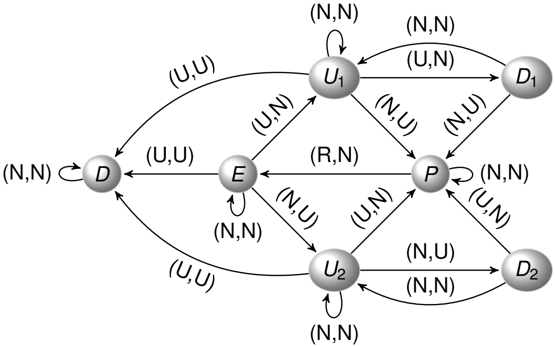
- Labelling $L : S \to \mathcal{P}(\mathrm{PROP})$. $L(s_1) = \{p\}$.

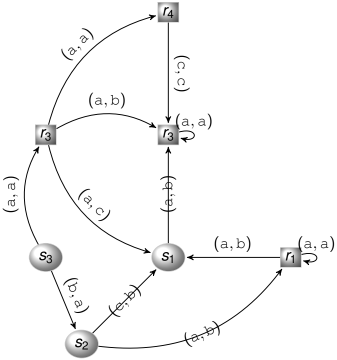
Another concurrent game structure

- Two agents share a file in a cyberspace,
- Each agent can apply the action Update (U) if she is enabled to do so, or Skip (N).
- State P is reached when both agents have processed the file.
- Action Reset (R) allows to move to the initial state E.



Turn-based CGS

- Turn-based CGS: only one agent at a time is executing an action.



- **Turn-based CGS** $\mathfrak{M}$: for all $s \in S$, there is at most one agent $a \in Agt$ such that $\text{card}(\texttt{act}(a, s)) > 1$.

The Logic ATL and Variants

Basic concepts: joint action

- **Coalition** $A \subseteq \mathit{Agt}$ with **opponent coalition** $\bar{A} = \mathit{Agt} \setminus A$.

- $\mathfrak{g} \colon A \to \mathit{Act}$: **joint action** by $A \subseteq \mathit{Agt}$ in s.
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- $\mathfrak{g} \colon A \to \mathit{Act} \sqsubseteq \mathfrak{g}' \colon A' \to \mathit{Act} \overset{\text{def}}{\Leftrightarrow} A \subseteq A'$ and $\mathfrak{g}$ is the restriction of $\mathfrak{g}'$ to A.

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- $D_A(s)$: set of joint actions by A in s.

Basic concepts: outcome set

- Joint action $\mathfrak{g} : A \rightarrow Act$ in s.
- $\texttt{out}(s, \mathfrak{g})$ defined as the set of states reachable from s in one step when the actions performed by the agents in A are determined by $\mathfrak{g}$.

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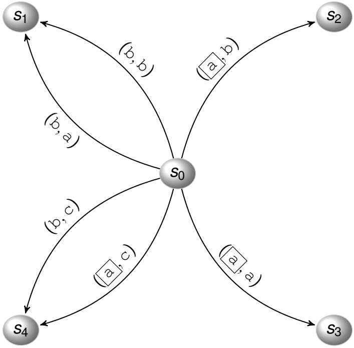
- Set of **outcomes**:

$$\mathtt{out}(s, \mathfrak{g}) \stackrel{\text{def}}{=} \{s' \in S \mid \exists\, \mathfrak{f} \in D_{Agt}(s) \text{ s.t. } \mathfrak{g} \sqsubseteq \mathfrak{f} \text{ and } s' = \delta(s, \mathfrak{f})\}$$



$$\mathtt{out}(s_0, [1 \mapsto \mathtt{a}]) = \{s_2, s_3, s_4\}$$
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- **Computation** $\lambda = s_0 \xrightarrow{\mathfrak{f}_0} s_1 \xrightarrow{\mathfrak{f}_1} s_2 \ldots$ such that for all i, we have $s_{i+1} \in \delta(s_i, \mathfrak{f}_i)$. (**history** = finite computation)
- Herein, computations can be also written $s_0 s_1 s_2 \ldots$ (without joint actions).
- Linear model $L(s_0) \to L(s_1) \to L(s_2) \cdots$ (sequence of propositional valuations)

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- Linear model $L(s_0) \to L(s_1) \to L(s_2) \cdots$ (sequence of propositional valuations)
- **Strategy** σ_A for A is a map from the set of finite computations (histories) to the set of joint actions by A such that

$$\sigma_A(s_0 \xrightarrow{\mathfrak{f}_0} s_1 \cdots \xrightarrow{\mathfrak{f}_{n-1}} s_n) \in D_A(s_n)$$

Positional strategies

- Memory-based strategies vs. positional strategies.

- σ_A is a **positional strategy** $\overset{\text{def}}{\Leftrightarrow}$ for all $s_0 \xrightarrow{\mathfrak{f}_0} s_1 \cdots \xrightarrow{\mathfrak{f}_{n-1}} s_n$ and $s'_0 \xrightarrow{\mathfrak{f}'_0} s'_1 \cdots \xrightarrow{\mathfrak{f}'_{m-1}} s'_m$ with $s_n = s'_m$, we have

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- **Memoryless strategy** $\overset{\text{def}}{=}$ positional strategy.

$$\sigma_A : s \in S \mapsto \mathfrak{f} \in D_A(s)$$

Computations respecting a strategy

- $\lambda = s_0 \xrightarrow{\mathfrak{f}_0} s_1 \xrightarrow{\mathfrak{f}_1} s_2 \cdots$ **respects** $\sigma_A \overset{\text{def}}{\Leftrightarrow} \forall\, i < |\lambda|,$

$$s_{i+1} \in \mathtt{out}(s_i, \sigma_A(s_0 \underset{\underset{D_A(s_i)}{\Cap}}{\xrightarrow{\mathfrak{f}_0}} s_1 \ldots \xrightarrow{\mathfrak{f}_{i-1}} s_i))$$

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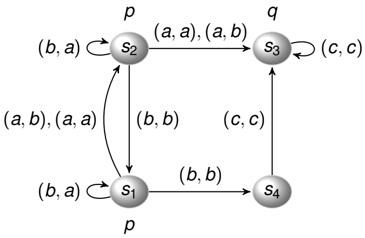
- λ respecting σ_A is **maximal** whenever λ cannot be extended further while respecting the strategy.

- $\mathtt{Comp}(s, \sigma_A)$: set of maximal computations from s respecting the strategy σ_A.

Computation tree given a strategy

- Positional $\sigma_{\{1\}}$: select a on s_1, b on s_2, otherwise c.



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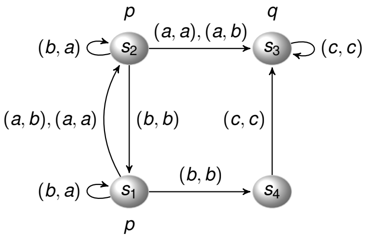
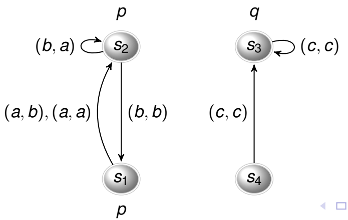


- $\sigma_{\{1\}}$ generates a set of computations whose linear models can be defined by a Büchi automaton (BA).

(tree $\sigma_{\{1\}}$)



Trimming a CGS

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- Underlying transition system (S, R, L) such that for all $s, s' \in S$, we have

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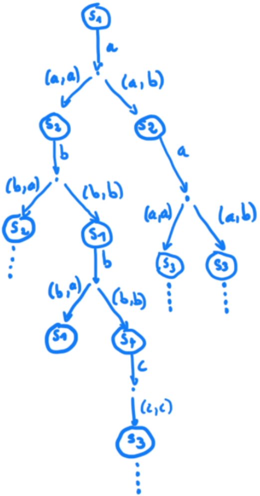
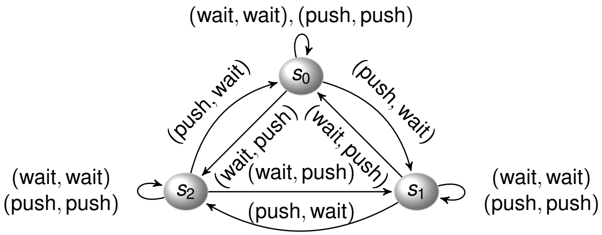
- R represents the set of moves allowed by the opponent coalition $(Agt \setminus A)$ when A has the positional strategy σ.

Strategies as infinite trees

- For non-positional strategies, computations organised as a tree not necessarily generated from a BA.
  $Agt = \{1, 2\}$; Strategy for $\{1\}$



Examples of strategies



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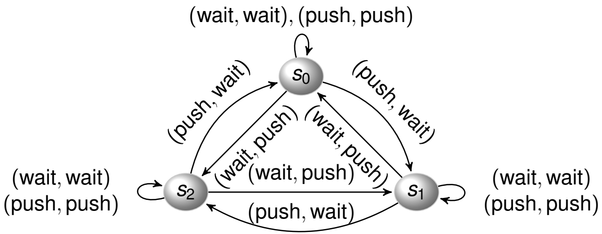

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- The set of maximal computations respecting σ from s_0 (projected on S only):

$$\{s_0^\omega\} \cup s_0^+\left((s_1^+ s_2^+)^\omega \cup (s_1^+ s_2^+)^* s_1^\omega \cup (s_1^+ s_2^+)^* s_2^\omega\right)$$

- Which temporal properties are satisfied by such computations respecting σ?

Specifying properties on ω-sequences

- LTL: linear-time temporal logic.
- LTL formulae:

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \mathsf{X}\varphi \mid \varphi \mathsf{U} \psi$$

- Atomic formulae are propositional variables.
- LTL models λ are ω-sequences of propositional valuations of the form $\lambda : \mathbb{N} \to \mathcal{P}(\mathrm{PROP})$.

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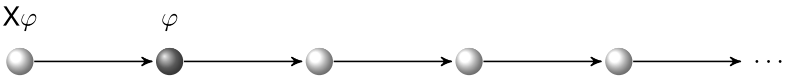
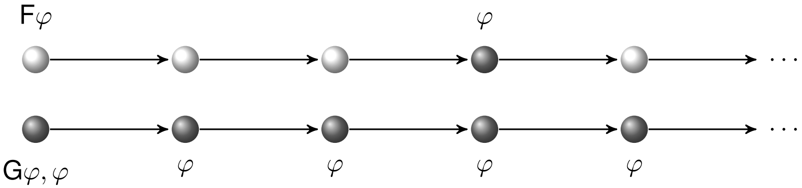
- $\mathsf{X}\varphi$ states that the next state satisfies φ:



Semantics of the linear-time temporal operators

- Fφ states that some future (or possibly, the current) state satisfies φ without specifying explicitly which one that is.



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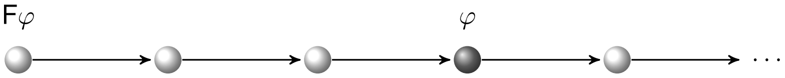

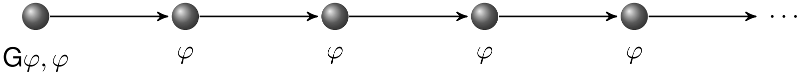

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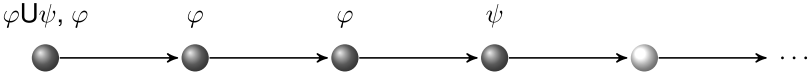
- φUψ states that φ is true until ψ is true.



Satisfaction relation

- $\lambda, i \models p \stackrel{\text{def}}{\Leftrightarrow} p \in \lambda(i)$,
- $\lambda, i \models \neg\varphi \stackrel{\text{def}}{\Leftrightarrow} \lambda, i \not\models \varphi$,
- $\lambda, i \models \varphi_1 \wedge \varphi_2 \stackrel{\text{def}}{\Leftrightarrow} \lambda, i \models \varphi_1$ and $\lambda, i \models \varphi_2$,
- $\lambda, i \models \mathsf{X}\varphi \stackrel{\text{def}}{\Leftrightarrow} \lambda, i+1 \models \varphi$,
- $\lambda, i \models \varphi_1 \mathsf{U} \varphi_2 \stackrel{\text{def}}{\Leftrightarrow}$ there is $j \geq i$ such that $\lambda, j \models \varphi_2$ and $\lambda, k \models \varphi_1$ for all $i \leq k < j$.

$$\mathsf{F}\varphi \stackrel{\text{def}}{=} \top \mathsf{U} \varphi \qquad \mathsf{G}\varphi \stackrel{\text{def}}{=} \neg \mathsf{F} \neg \varphi \qquad \varphi \Rightarrow \psi \stackrel{\text{def}}{=} \neg\varphi \vee \psi \ldots$$

About LTL

- Models(φ): set of models λ such that $\lambda, 0 \models \varphi$.
- Models can be viewed as ω-words over the alphabet $\mathcal{P}(\mathrm{PROP})$.
- Models(φ) can be effectively represented by a Büchi automaton $\mathbb{A}_\varphi$. (automata-based approach)
- LTL satisfiability problem is PSPACE-complete.

The logic ATL (Alternating-time Temporal Logic)

- $\langle\!\langle A \rangle\!\rangle \Phi$: the agents are divided into proponents in A and opponents in $Agt \setminus A$.
- Φ: property on computations (“objective”).
- $\mathfrak{M}, s \models \langle\!\langle A \rangle\!\rangle \Phi$ equivalent to solving a game with winning condition Φ. (A versus $Agt \setminus A$)

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- $\mathfrak{M}, s \models \langle\langle A \rangle\rangle \Phi$ equivalent to solving a game with winning condition Φ. (A versus $Agt \setminus A$)

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\langle A \rangle\rangle \, \mathsf{X}\varphi \mid \langle\langle A \rangle\rangle \, \mathsf{G}\varphi \mid \langle\langle A \rangle\rangle \, \varphi \mathsf{U} \varphi$$

$$p \in \mathrm{PROP} \quad A \subseteq Agt$$

ATL modalities, informally

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- $\langle\langle A \rangle\rangle \psi \mathrm{U} \varphi$: "The coalition A has a collective strategy to eventually reach an outcome satisfying φ, while maintaining in the meantime the truth of ψ, on every computation respecting that strategy".

Satisfaction relation, formally

$\mathfrak{M}, s \models p \quad \overset{\text{def}}{\Leftrightarrow} \quad p \in L(s)$

$\mathfrak{M}, s \models \langle\!\langle A \rangle\!\rangle \mathrm{X}\varphi \quad \overset{\text{def}}{\Leftrightarrow}$ there is a strategy σ_A s.t.
for all $s_0 \xrightarrow{\mathfrak{f}_0} s_1 \ldots \in \mathtt{Comp}(s, \sigma_A)$,
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$\mathfrak{M}, s \models \langle\langle A \rangle\rangle \mathrm{G} \varphi \quad \overset{\text{def}}{\Leftrightarrow}$ there is a strategy σ_A s.t. for all $\lambda = s_0 \xrightarrow{\mathfrak{f}_0} s_1 \ldots \in \mathtt{Comp}(s, \sigma_A)$, for all i, we have $\mathfrak{M}, s_i \models \varphi$.

Getting acquainted with ATL

- The semantics for “$\langle\langle A \rangle\rangle$G” involves an existential quantification followed by two universal quantifications.

(why?)

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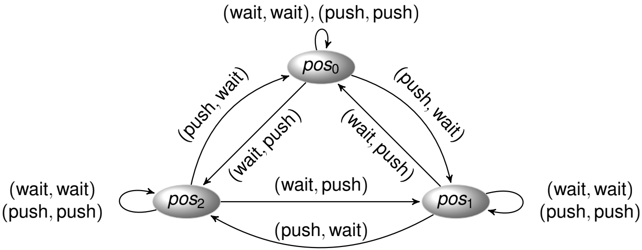
(why?)

- $\langle\langle A\rangle\rangle\ \mathsf{F}\varphi \stackrel{\text{def}}{=} \langle\langle A\rangle\rangle\ (\top \mathsf{U} \varphi)$.
  The coalition A has a joint strategy to eventually reach an outcome satisfying φ.

- $[\![\varphi]\!]^{\mathfrak{M}} \stackrel{\text{def}}{=} \{s \in S \mid \mathfrak{M}, s \models \varphi\}$.

Playing with formulae



- $\mathfrak{M}, pos_0 \not\models \langle\langle 1\rangle\rangle \mathrm{X}\, \mathtt{pos}_1$ and $\mathfrak{M}, pos_0 \not\models \langle\langle 2\rangle\rangle \mathrm{X}\, \mathtt{pos}_1$.
- $\mathfrak{M}, pos_0 \models \langle\langle 1,2\rangle\rangle \mathrm{X}\, \mathtt{pos}_0 \wedge \langle\langle 1,2\rangle\rangle \mathrm{X}\, \mathtt{pos}_1 \wedge \langle\langle 1,2\rangle\rangle \mathrm{X}\, \mathtt{pos}_2$.

$\mathfrak{M}, pos_0 \not\models \langle\langle 1\rangle\rangle \mathrm{F}\, \mathtt{pos}_1$ and $\mathfrak{M}, pos_1 \models \langle\langle 1\rangle\rangle \mathrm{F}\, (\mathtt{pos}_1 \vee \mathtt{pos}_2)$

$\mathfrak{M}, pos_0 \models \langle\langle 1\rangle\rangle \mathrm{G}\, \neg \mathtt{pos}_1$ and $\mathfrak{M} \models \langle\langle 1,2\rangle\rangle \mathrm{X} \langle\langle 1\rangle\rangle (\mathtt{pos}_0\, \mathrm{U}\, \mathtt{pos}_2)$

Decision problems

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  Input: φ in ATL, a finite CGS $\mathfrak{M}$ and a state s,

  Question: $\mathfrak{M}, s \models \varphi$?

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- **Satisfiability problem for** ATL:

  Input: φ in ATL,
  Question: Is there a CGS $\mathfrak{M}$ and s in $\mathfrak{M}$ such that $\mathfrak{M}, s \models \varphi$?

- **Validity problem for** ATL:

  Input: φ in ATL,
  Question: Is it true that for all CGS $\mathfrak{M}$ and s in $\mathfrak{M}$, we have $\mathfrak{M}, s \models \varphi$?

Computational complexity

- Model-checking problem for ATL is PTIME-complete.
  Labeling algorithm presented during the next lecture.
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- Model-checking problem for ATL is PTIME-complete.
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- Satisfiability and validity problems are EXPTIME-complete.

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- $\models_{pos}$: variant of $\models$ in which only positional strategies are legitimate.

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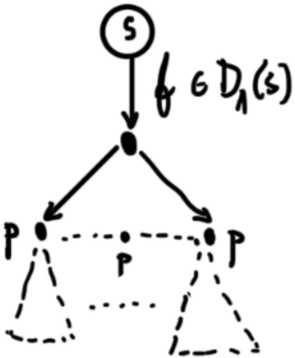
- Positional strategies are sufficient for ATL:
$$\mathfrak{M}, s \models \varphi \text{ iff } \mathfrak{M}, s \models_{pos} \varphi$$
- Positional strategies amount to remove transitions in the CGS (and keep only the ones related to the positional strategy of A).
- This property does not hold for the extension ATL*. (see next lecture)

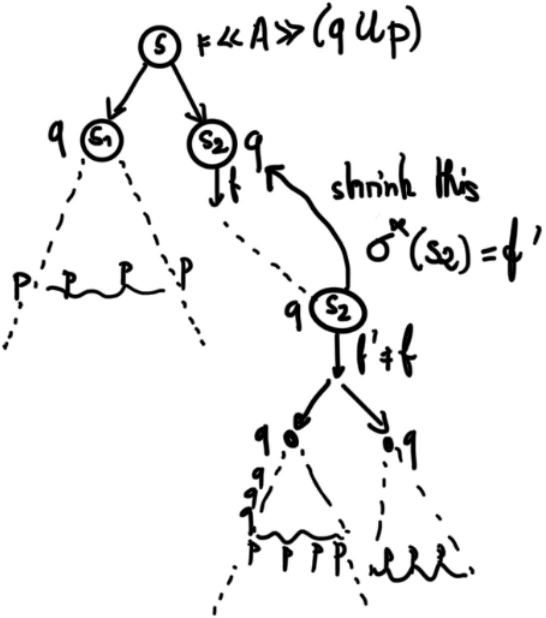
“Proof”: positional strategies are sufficient for ATL



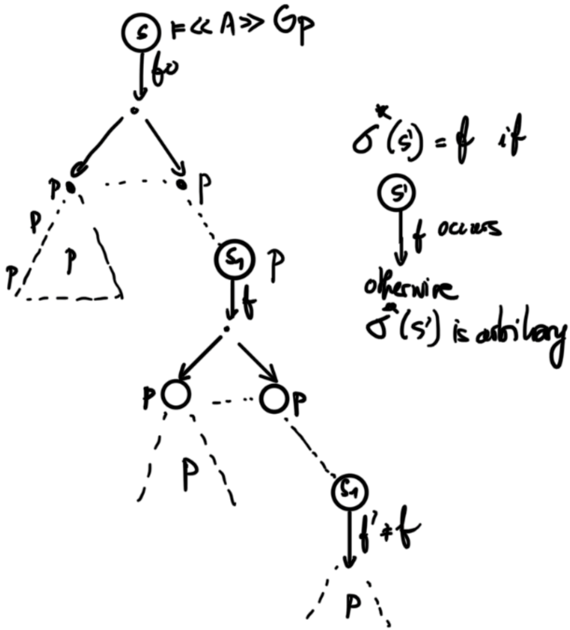
strategy σ
witnessing $s \models \langle\langle A \rangle\rangle X p$

positional σ^*
$\sigma^*(s) = f$
$\forall s' \neq s$
$\sigma^*(s')$ is arbitrary

Formula $\langle\langle A \rangle\rangle(q \mathcal{U} p)$

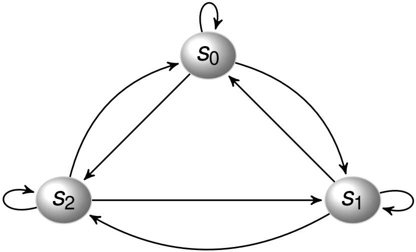


Formulae $\langle\langle A\rangle\rangle Gp$



Relationships between ATL and CTL

- Computation Tree Logic CTL: branching-time temporal logic well-known to perform model-checking.
- A CGS without transitions labelled by action tuples defines a model for CTL (or with 1 agent and 1 action).



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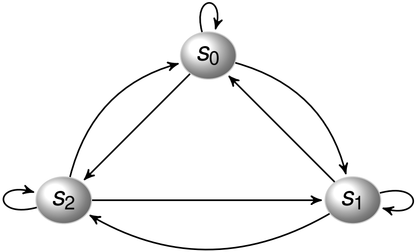
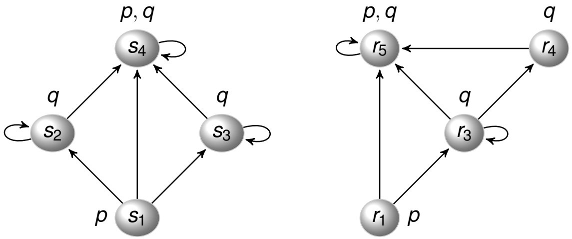


- Existential path quantifier E in CTL corresponds to $\langle\langle Agt \rangle\rangle$.
- Universal path quantifier A in CTL corresponds to $\langle\langle \emptyset \rangle\rangle$.

- CTL formulae

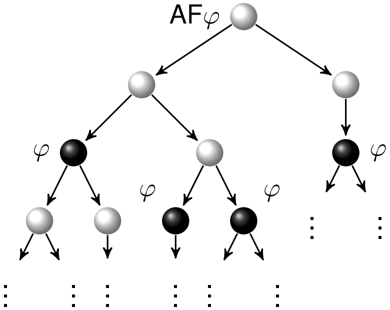
$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{EX}\varphi \mid \mathsf{E}(\varphi \mathsf{U} \varphi) \mid \mathsf{A}(\varphi \mathsf{U} \varphi).$$

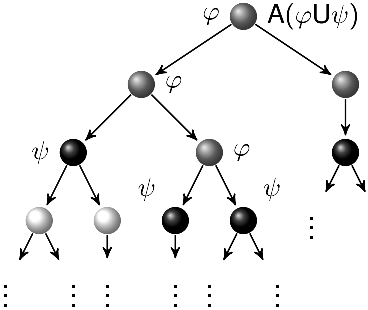
- CTL models of the form $\mathcal{T} = (S, R, L)$.



Informal semantics for $\mathsf{A}(\varphi\mathsf{U}\psi)$

$$\mathsf{AF}\varphi \stackrel{\text{def}}{=} \mathsf{A}\top\mathsf{U}\varphi \qquad \mathsf{EG}\varphi \stackrel{\text{def}}{=} \neg\mathsf{AF}\neg\varphi$$





CTL semantics

- Path π in $\mathcal{T}$: sequence of states in the graph (S, R).
- A path is maximal if it is either infinite, or is finite and ends in a state with no successors.
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$\mathcal{T}, s \models \mathrm{A}(\varphi_1 \mathrm{U} \varphi_2)$	iff	for all paths π such that $\pi(0) = s$, there is $i \geq 0$ such that $\mathcal{T}, \pi(i) \models \varphi_2$ and for every $j \in [0, i-1]$, we have $\mathcal{T}, \pi(j) \models \varphi_1$

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- $\rightarrow$ Reduction from CTL satisfiability (resp. model-checking) to ATL satisfiability (resp. model-checking).

  (E corresponds to $\langle\langle Agt \rangle\rangle$ and A corresponds to $\langle\langle \emptyset \rangle\rangle$.)

$\boxed{\textbf{Fixpoints and Operators}}$

Introducing a predecessor operator $\mathtt{pre}$

- CGS $\mathfrak{M} = (Agt, S, Act, \mathtt{act}, \delta, L)$, $A \subseteq Agt$, and $Z \subseteq S$.
- $\mathtt{pre}(\mathfrak{M}, A, Z)$: set of states from which A has a collective move that guarantees that the outcome to be in Z.

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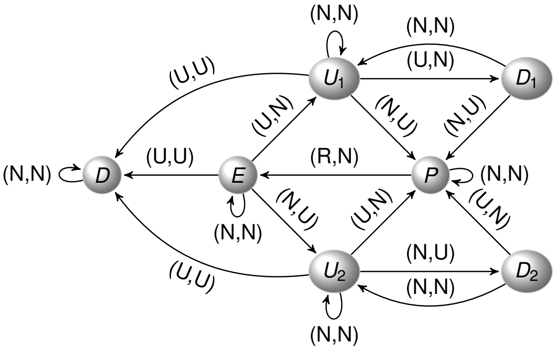
- $\mathtt{pre}(\mathfrak{M}, A, Z)$: set of states from which A has a collective move that guarantees that the outcome to be in Z.
- Definition of $\mathtt{pre}(\mathfrak{M}, A, \cdot) : \mathcal{P}(S) \to \mathcal{P}(S)$

$$\mathtt{pre}(\mathfrak{M}, A, Z) \stackrel{\text{def}}{=}$$

$$\{s \in S \mid \text{there is } \mathfrak{f} \in D_A(s) \text{ such that } \mathtt{out}(s, \mathfrak{f}) \subseteq Z\}$$

Example



$$\texttt{pre}(\mathfrak{M}, \{1\}, \{D, U_1, P\}) = \quad ??$$

Proof of $[\![\langle\!\langle A\rangle\!\rangle \mathsf{X}\varphi]\!]^{\mathfrak{M}} = \mathtt{pre}(\mathfrak{M}, A, [\![\varphi]\!]^{\mathfrak{M}})$

- By definition, $\mathtt{pre}(\mathfrak{M}, A, [\![\varphi]\!]^{\mathfrak{M}})$ is equal to

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- Let σ be a strategy such that $\sigma(s) = \mathfrak{f}$.
- The strategy σ witnesses satisfaction of $\mathfrak{M}, s \models \langle\!\langle A \rangle\!\rangle \mathsf{X}\varphi$.

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- Let σ be a strategy such that $\sigma(s) = \mathfrak{f}$.
- The strategy σ witnesses satisfaction of $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle X\varphi$.
- Conversely, if $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle X\varphi$ witnessed by σ, then
  $s \in \mathtt{pre}(\mathfrak{M}, A, [\![\varphi]\!]^{\mathfrak{M}})$ as $\mathtt{out}(s, \sigma(s)) \subseteq [\![\varphi]\!]^{\mathfrak{M}}$

Equivalences based on fixpoint characterisations

$$\boxed{\langle\langle A \rangle\rangle \mathsf{G}\varphi} \Leftrightarrow \varphi \wedge \langle\langle A \rangle\rangle \mathsf{X} \boxed{\langle\langle A \rangle\rangle \mathsf{G}\varphi}$$

$$\boxed{\langle\langle A \rangle\rangle (\varphi \mathsf{U} \psi)} \Leftrightarrow (\psi \vee (\varphi \wedge \langle\langle A \rangle\rangle \mathsf{X} \boxed{\langle\langle A \rangle\rangle (\varphi \mathsf{U} \psi)}))$$

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- $[\![\langle\langle A\rangle\rangle \mathsf{G}\varphi]\!]^{\mathfrak{M}}$ and $[\![\langle\langle A\rangle\rangle(\varphi \mathsf{U} \psi)]\!]^{\mathfrak{M}}$ are fixpoints.

(but in which sense?)

Fixpoint theory

- $\mathcal{G} : \mathcal{P}(X) \to \mathcal{P}(X)$ is **monotone** if for all $Y_1, Y_2 \subseteq X$, $Y_1 \subseteq Y_2$ implies $\mathcal{G}(Y_1) \subseteq \mathcal{G}(Y_2)$.
- Given $\mathcal{G} : \mathcal{P}(X) \to \mathcal{P}(X)$, a set $Y \subseteq X$ is
  - a **fixpoint** of $\mathcal{G}$ if $\mathcal{G}(Y) = Y$,
  - a **least fixpoint** if Y is a fixpoint and $Y \subseteq Z$ for every fixpoint Z,
  - a **greatest fixpoint** if Y is a fixpoint and $Y \supseteq Z$ for every fixpoint Z.

Knaster-Tarski Theorem: a restricted form

- Knaster-Tarski Theorem (a restricted form).
  Let $\mathcal{G} : \mathcal{P}(X) \to \mathcal{P}(X)$ be a monotone operator. Then $\mathcal{G}$ has
  - a least fixpoint $\mu\mathcal{G}$ and,
  - a greatest fixpoint $\nu\mathcal{G}$.
- Moreover, $\mu\mathcal{G}$ obtained by applying the successive iterations of $\mathcal{G}$ beginning with $\emptyset$ until a fixpoint is reached.

$$\emptyset \subseteq \mathcal{G}(\emptyset) \subseteq \mathcal{G}^2(\emptyset) \subseteq \mathcal{G}^3(\emptyset) \cdots$$

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- $\nu\mathcal{G}$ obtained by applying the successive iterations of $\mathcal{G}$, beginning with X, until a fixpoint is reached.

$$X \supseteq \mathcal{G}(X) \supseteq \mathcal{G}^2(X) \supseteq \mathcal{G}^3(X) \cdots$$

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$$X \supseteq \mathcal{G}(X) \supseteq \mathcal{G}^2(X) \supseteq \mathcal{G}^3(X) \cdots$$

- If X is finite, the fixpoints $\mu\mathcal{G}$ and $\nu\mathcal{G}$ can be obtained in a number of steps bounded by card(X).

$[\![\langle\!\langle A\rangle\!\rangle \mathrm{G}\varphi]\!]^{\mathfrak{M}}$ is a greatest fixpoint

- Given $A \subseteq Agt$, a formula φ, and a CGS $\mathfrak{M}$, we define $\mathcal{G}_{A,\varphi} : \mathcal{P}(S) \to \mathcal{P}(S)$:

$$\mathcal{G}_{A,\varphi}(Z) \stackrel{\text{def}}{=} [\![\varphi]\!]^{\mathfrak{M}} \cap \texttt{pre}(\mathfrak{M}, A, Z).$$

- $\mathcal{G}_{A,\varphi}(S)$ contains all the states satisfying φ.
($\texttt{pre}(\mathfrak{M}, A, S) = S$)

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- $\mathcal{G}_{A,\varphi}(S)$ contains all the states satisfying φ. ($\texttt{pre}(\mathfrak{M}, A, S) = S$)
- $\mathcal{G}_{A,\varphi}(\mathcal{G}_{A,\varphi}(S))$ contains all the states satisfying φ and A has a strategy such that in one step all the states satisfy φ.
- $\mathcal{G}^{n}_{A,\varphi}(S)$ contains all the states satisfying φ and A has a strategy such that in the steps $0, \ldots, n-1$ all the states satisfy φ.

$[\![\langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi]\!]^{\mathfrak{M}}$ is a greatest fixpoint

- Given $A \subseteq Agt$, a formula φ, and a CGS $\mathfrak{M}$, we define $\mathcal{G}_{A,\varphi} \colon \mathcal{P}(S) \to \mathcal{P}(S)$:

$$\mathcal{G}_{A,\varphi}(Z) \stackrel{\text{def}}{=} [\![\varphi]\!]^{\mathfrak{M}} \cap \mathtt{pre}(\mathfrak{M}, A, Z).$$

- $\mathcal{G}_{A,\varphi}(S)$ contains all the states satisfying φ.
  $(\mathtt{pre}(\mathfrak{M}, A, S) = S)$

- $\mathcal{G}_{A,\varphi}(\mathcal{G}_{A,\varphi}(S))$ contains all the states satisfying φ and A has a strategy such that in one step all the states satisfy φ.

- $\mathcal{G}^{n}_{A,\varphi}(S)$ contains all the states satisfying φ and A has a strategy such that in the steps $0, \ldots, n-1$ all the states satisfy φ. $(\mathcal{G}^{n}_{A,\varphi}(S) \subseteq \mathcal{G}^{n-1}_{A,\varphi}(S) \subseteq \cdots \subseteq \mathcal{G}^{1}_{A,\varphi}(S))$

$[\![\langle\!\langle A\rangle\!\rangle \mathrm{G}\varphi]\!]^{\mathfrak{M}}$ is a greatest fixpoint

- Given $A \subseteq Agt$, a formula φ, and a CGS $\mathfrak{M}$, we define $\mathcal{G}_{A,\varphi} \colon \mathcal{P}(S) \to \mathcal{P}(S)$:

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- $[\![\langle\!\langle A\rangle\!\rangle \mathrm{G}\varphi]\!]^{\mathfrak{M}} = \nu Z.([\![\varphi]\!]^{\mathfrak{M}} \cap \texttt{pre}(\mathfrak{M}, A, Z))$ (greatest fixpoint)

About $\mathcal{G}_{A,\varphi}$

- $\mathcal{G}_{A,\varphi}$ is monotone as $\texttt{pre}$ is monotone.
- Computing $\nu Z.(\llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \texttt{pre}(\mathfrak{M}, A, Z))$.
  - $X_0 = S$.
  - $X_1 = \llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \texttt{pre}(\mathfrak{M}, A, X_0)$.
  - $X_2 = \llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \texttt{pre}(\mathfrak{M}, A, X_1)$.
  - $\ldots$
  - $X_{i+1} = \llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \texttt{pre}(\mathfrak{M}, A, X_i)$.
  - $\ldots$
- For all i, $X_{i+1} \subseteq X_i$. (proof left as an exercise)
- There is $N \leq \mathrm{card}(S)$ such that $X_N = X_{N+1} = X_{N+2} = \cdots$.

$[\![\langle\!\langle A\rangle\!\rangle \varphi \mathsf{U} \psi]\!]^{\mathfrak{M}}$ is a least fixpoint

- Given $A \subseteq Agt$, formulae φ, ψ, and a CGS $\mathfrak{M}$, we define $\mathcal{O}_{A,\varphi,\psi} : \mathcal{P}(S) \to \mathcal{P}(S)$:

$$\mathcal{O}_{A,\varphi,\psi}(Z) \stackrel{\text{def}}{=} [\![\psi]\!]^{\mathfrak{M}} \cup \left([\![\varphi]\!]^{\mathfrak{M}} \cap \mathtt{pre}(\mathfrak{M}, A, Z)\right)$$

- $\mathcal{O}_{A,\varphi,\psi}(\emptyset)$ contains all the states satisfying ψ.
  $(\mathtt{pre}(\mathfrak{M}, A, \emptyset) = \emptyset)$

$[\![\langle\!\langle A \rangle\!\rangle \varphi \mathsf{U} \psi]\!]^{\mathfrak{M}}$ is a least fixpoint

- Given $A \subseteq Agt$, formulae φ, ψ, and a CGS $\mathfrak{M}$, we define $\mathcal{O}_{A,\varphi,\psi} : \mathcal{P}(S) \to \mathcal{P}(S)$:

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- $\mathcal{O}_{A,\varphi,\psi}(\emptyset)$ contains all the states satisfying ψ.
  $(\texttt{pre}(\mathfrak{M}, A, \emptyset) = \emptyset)$
- $\mathcal{O}_{A,\varphi,\psi}(\mathcal{O}_{A,\varphi,,\psi}(\emptyset))$ contains all the states satisfying ψ or those satisfying φ and such that A has a strategy such that in one step all the states satisfy ψ.

$[\![\langle\!\langle A \rangle\!\rangle \varphi \mathsf{U} \psi]\!]^{\mathfrak{M}}$ is a least fixpoint (bis)

- $\mathcal{O}^n_{A,\varphi,\psi}(\emptyset)$ contains all the states satisfying ψ or those satisfying φ and such that A has a strategy such that in at most n steps, a state satisfying ψ is reached and in between all the states satisfy φ.

$[\![\langle\!\langle A \rangle\!\rangle \varphi \mathsf{U} \psi]\!]^{\mathfrak{M}}$ is a least fixpoint (bis)

- $\mathcal{O}^{n}_{A,\varphi,\psi}(\emptyset)$ contains all the states satisfying ψ or those satisfying φ and such that A has a strategy such that in at most n steps, a state satisfying ψ is reached and in between all the states satisfy φ.

- $\mathcal{O}^{1}_{A,\varphi,\psi}(\emptyset) \subseteq \mathcal{O}^{2}_{A,\varphi,\psi}(\emptyset) \subseteq \cdots \subseteq \mathcal{O}^{n}_{A,\varphi,\psi}(\emptyset)$.

$[\![\langle\!\langle A\rangle\!\rangle\varphi \mathsf{U}\psi]\!]^{\mathfrak{M}}$ is a least fixpoint (bis)

- $\mathcal{O}^{n}_{A,\varphi,\psi}(\emptyset)$ contains all the states satisfying ψ or those satisfying φ and such that A has a strategy such that in at most n steps, a state satisfying ψ is reached and in between all the states satisfy φ.

- $\mathcal{O}^{1}_{A,\varphi,\psi}(\emptyset) \subseteq \mathcal{O}^{2}_{A,\varphi,\psi}(\emptyset) \subseteq \cdots \subseteq \mathcal{O}^{n}_{A,\varphi,\psi}(\emptyset)$.

- $[\![\langle\!\langle A\rangle\!\rangle\varphi \mathsf{U}\psi]\!]^{\mathfrak{M}} = \mu Z.([\![\psi]\!]^{\mathfrak{M}} \cup ([\![\varphi]\!]^{\mathfrak{M}} \cap \mathtt{pre}(\mathfrak{M}, A, Z)))$.
  (least fixpoint)

- Valid formula

$$\langle\!\langle A\rangle\!\rangle\varphi \mathsf{U}\psi \Leftrightarrow \psi \vee (\varphi \wedge \langle\!\langle A\rangle\!\rangle \mathsf{X} \langle\!\langle A\rangle\!\rangle\varphi \mathsf{U}\psi)$$

About $\mathcal{O}_{A,\varphi,\psi}$

- $\mathcal{O}_{A,\varphi,\psi}$ is monotone as $\mathtt{pre}$ is monotone.

- Computing $\mu Z.(\llbracket\psi\rrbracket^{\mathfrak{M}} \cup (\llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \mathtt{pre}(\mathfrak{M}, A, Z)))$.
  - $X_0 = \emptyset$.
  - $X_1 = (\llbracket\psi\rrbracket^{\mathfrak{M}} \cup (\llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \mathtt{pre}(\mathfrak{M}, A, X_0))$.
  - $X_2 = (\llbracket\psi\rrbracket^{\mathfrak{M}} \cup (\llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \mathtt{pre}(\mathfrak{M}, A, X_1))$.
  - ...
  - $X_{i+1} = (\llbracket\psi\rrbracket^{\mathfrak{M}} \cup (\llbracket\varphi\rrbracket^{\mathfrak{M}} \cap \mathtt{pre}(\mathfrak{M}, A, X_i))$.
  - ...

- For all i, $X_i \subseteq X_{i+1}$. (proof left as an exercise)

- There is $N \leq \mathrm{card}(S)$ such that $X_N = X_{N+1} = X_{N+2} = \cdots$.

Conclusion

- Today lecture.
  - Concurrent game structures (CGS).
  - Introduction to ATL.
  - Fixpoints and operators.
- Next week lecture.
  - Correction of the exercises.
  - Model-checking problem for ATL in PTIME and other variants from ATL.
  - ATL with incomplete information
  - ATL^+: between ATL and ATL^*, PSPACE-hardness.