Logical Aspects of Artificial Intelligence
Tableaux for DLs & Undecidability

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Plan of the lecture

- Tableaux calculus for checking $\mathcal{ALC}$ concept satisfiability.
- Tableaux calculus for checking $\mathcal{ALC}$ knowledge base consistency.
- Undecidability result with role axioms.
- Exercises session.

$\boxed{\textbf{Recapitulation of the previous lecture(s)}}$

$\mathcal{ALC}$ in a nutshell

$$C ::= \top \mid \bot \mid A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.C \mid \forall r.C$$

- Interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$.
- TBox $\mathcal{T} = \{C \sqsubseteq D, \ldots\}$.
- ABox $\mathcal{A} = \{a : C, (b, b') : r, \ldots\}$.
- Knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. (a.k.a. ontology)
- Decision problems include concept satisfiability, knowledge base consistency, and other problems for classification.

$$\top^{\mathcal{I}} \stackrel{\text{def}}{=} \Delta^{\mathcal{I}}$$

$$\bot^{\mathcal{I}} \stackrel{\text{def}}{=} \emptyset$$

$$(\neg C)^{\mathcal{I}} \stackrel{\text{def}}{=} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$$

$$(C_1 \sqcup C_2)^{\mathcal{I}} \stackrel{\text{def}}{=} C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$$

$$(C_1 \sqcap C_2)^{\mathcal{I}} \stackrel{\text{def}}{=} C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$$

$$(\exists r.C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{\mathfrak{a} \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(\mathfrak{a}) \cap C^{\mathcal{I}} \neq \emptyset\}$$

$$(\forall r.C)^{\mathcal{I}} \stackrel{\text{def}}{=} \{\mathfrak{a} \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(\mathfrak{a}) \subseteq C^{\mathcal{I}}\}$$

A few properties about $\mathcal{ALC}$

- Concept satisfiability problem is PSPACE-complete.
- Knowledge base consistency problem is EXPTIME-complete.
- $\mathcal{ALC}$ has many well-known fragments and extensions, some of them to deal with
  - inverse roles,
  - number restrictions,
  - properties on the role interpretations,
  - inclusions between the composition of roles,
  - etc..
- Reduction of decision problems for DLs to first-order logic.

  (to modal logic

- Filtration construction leading to an NEXPTIME upper bound for the $\mathcal{ALC}$ knowledge base consistency problem.

Expansion rules for $\mathcal{ALC}$ ABox consistency

$\sqcap$-rule: If $a : C \sqcap D \in \mathcal{A}$ and $\{a : C, a : D\} \not\subseteq \mathcal{A}$ then

$$\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : C, a : D\}$$

$\sqcup$-rule: If $a : C \sqcup D \in \mathcal{A}$ and $\{a : C, a : D\} \cap \mathcal{A} = \emptyset$ then

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$\forall$-rule: If $\{(a, b) : r, a : \forall r.C\} \subseteq \mathcal{A}$ and $b : C \notin \mathcal{A}$, then

$$\mathcal{A} \longrightarrow \mathcal{A} \cup \{b : C\}$$

Today's objectives

- Termination, soundness, completeness, blocking technique.
- Equivalences between
  - $(\mathcal{T}, \mathcal{A})$ is consistent (for $\mathcal{ALC}$)
  - $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$ ($\rightarrow$ depends on $\mathcal{T}$)
  - $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$ derivable in at most $\mathfrak{f}(\text{size}(\mathcal{T}, \mathcal{A}))$ steps.

Example

$$\mathcal{A} = \{(a, b) : s, (a, c) : r\} \cup$$
$$\{a : A_1 \sqcap \exists s.A_5, a : \forall s.(\neg A_5 \sqcup \neg A_2), b : A_2, c : A_3 \sqcap \exists s.A_4\}$$

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$$\mathcal{A} \xrightarrow{*} \mathcal{A} \cup \{a : A_1, a : \exists s.A_5, a_{\text{new}} : A_5, (a, a_{\text{new}}) : s$$
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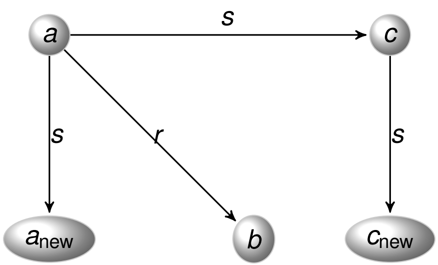
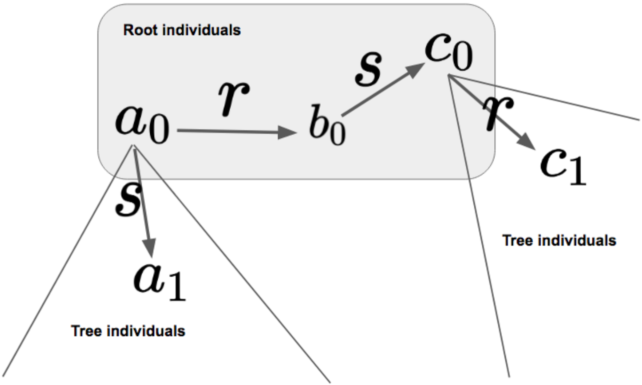
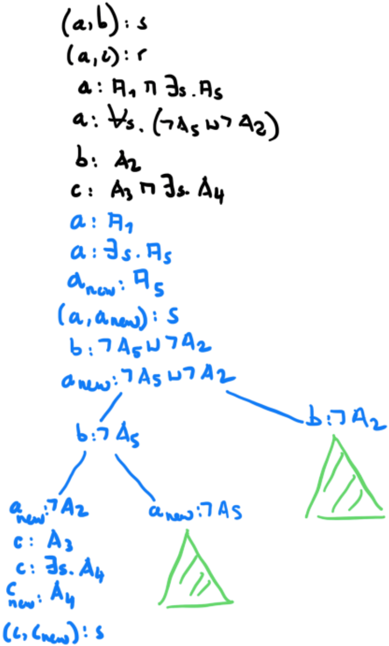


Terminology: root vs. tree individuals

- **Tree individuals** are generated by application of the ∃-rule.
- If $(a, b) : r$ is added by application of the ∃-rule, b is an *r*-**successor** of *a*.
- Root individuals have no predecessors or ancestors.



Why “Tableaux”?



Termination

- The $\exists$**-weight** of C is the number of its subconcepts of the form $\exists r.D$.

$$\mathsf{w}_\exists(C) \stackrel{\text{def}}{=} \mathsf{card}(\{\exists r.D \mid \exists r.D \in \mathsf{sub}(C)\})$$

⚠ The definition assumes that C is in NNF.

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- The **∀∃-depth** of C, written $\mathrm{d}_{\forall\exists}(C)$, is the maximal number of imbrications of $\exists r.$ and $\forall s.$ in C.

(a.k.a. quantifier de

- $\mathrm{d}_{\forall\exists}(\exists r.\top \sqcup \forall r.\exists s.A) = 2$

- $\mathrm{d}_{\forall\exists}(\mathcal{A}) = \max\{\mathrm{d}_{\forall\exists}(C) \mid a : C \in \mathcal{A}\}$.

Decorating individual names

- Let $\mathcal{A}$ be an ABox with $\mathbf{W} = \mathrm{w}_{\exists}(\mathcal{A})$, $\mathbf{D} = \mathrm{d}_{\forall\exists}(\mathcal{A})$ and $\mathbf{N}$ is the number of distinct individual names in $\mathcal{A}$.

- Let $\mathcal{A}^0$ be the variant of $\mathcal{A}$ where $a : C$ is replaced by $a^0 : C$.

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$\forall$-rule: If $\{(a^i, b^j) : r, a^i : \forall r.C\} \subseteq \mathcal{A}$ and $b^j : C \notin \mathcal{A}$, then

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Quantities about $\mathcal{A}^0 \xrightarrow{*} \mathcal{A}'$

- If $a^i : C \in \mathcal{A}'$, then $i + \mathsf{d}_{\forall\exists}(C) \leq \mathbf{D}$.
  Trees from individual names labelled by zero have depth at most **D**.

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- $a^i : C \in \mathcal{A}'$ implies
  $\mathrm{card}(\{(a^i, b^j) \mid (a^i, b^j) : r \in \mathcal{A}'\}) \leq \mathbf{N} + \mathbf{W}$
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- $a^i : C \in \mathcal{A}'$ implies $C \in \mathrm{sub}(\mathcal{A})$.

- The length of the derivation $\mathcal{A}^0 \xrightarrow{*} \mathcal{A}'$ is at most

$$\mathbf{N} \times (\mathbf{N} + \mathbf{W})^{\mathbf{D}+1} \times \mathrm{card}(\mathrm{sub}(\mathcal{A}))$$

(why?)

Main algorithm

- We shall show that $\mathcal{A}$ is consistent iff $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

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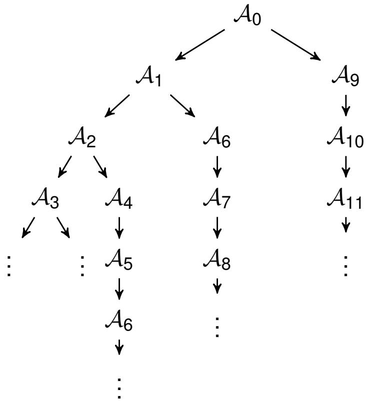


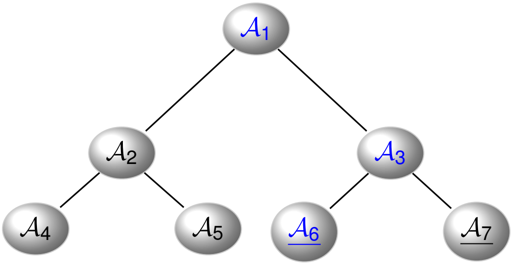
The auxiliary function exp

- **Expansion function** $\exp(\mathcal{A}, \mathbf{R}, X)$ taking as arguments
  - an ABox $\mathcal{A}$,
  - an expansion rule **R**,
  - a subset X of $\mathcal{A}$ (with one or two elements) allowing the application of **R**
- ...and returning the set of ABoxes obtained from $\mathcal{A}$ by applying the rule **R** with main assertions in X.
- $\exp(\{a : E, a : C \sqcup D\}, \sqcup\text{-rule}, \{a : C \sqcup D\})$ is equal to

$$\{\{a : E, a : C \sqcup D, a : C\}, \{a : E, a : C \sqcup D, a : D\}\}$$

Algorithm for depth-first visit

procedure EXPAND($\mathcal{A}$)
    if $\mathcal{A}$ has a clash **then** return $\emptyset$
    end if
    if $\mathcal{A}$ is clash-free and complete **then** return $\mathcal{A}$
    end if
    for applicable $\mathbf{R}, X$ on $\mathcal{A}$ and $\mathcal{A}' \in \exp(\mathcal{A}, \mathbf{R}, X)$ **do**
        if EXPAND($\mathcal{A}'$) $\neq \emptyset$ **then** return EXPAND($\mathcal{A}'$)
        end if
    end for
    return $\emptyset$
end procedure



Soundness

- Let $\mathcal{A}$ be a finite ABox with at least one concept assertion, complete, clash-free and all the concepts in NNF. Then, $\mathcal{A}$ is consistent.

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- Let us define $\mathcal{I} \stackrel{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ as follows.
  - $\Delta^{\mathcal{I}} \stackrel{\text{def}}{=} \{a \mid a : C \in \mathcal{A}\}$.
  - $a^{\mathcal{I}} \stackrel{\text{def}}{=} a$ for all individual names a in $\mathcal{A}$.
  - $A^{\mathcal{I}} \stackrel{\text{def}}{=} \{a \mid A \in \mathsf{con}_{\mathcal{A}}(a)\}$ for all concept names $A \in \mathsf{sub}(\mathcal{A})$.
  - $r^{\mathcal{I}} \stackrel{\text{def}}{=} \{(a, b) \mid (a, b) : r \in \mathcal{A}\}$.

- Let us show that for all $a : C \in \mathcal{A}$, we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

Proof by structural induction

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- Case $a : C \sqcup D$ in the induction step.
  - As $\mathcal{A}$ is complete, $a : C \in \mathcal{A}$ or $a : D \in \mathcal{A}$.
  - W.l.o.g., suppose $a : C \in \mathcal{A}$. By (IH), $a^{\mathcal{I}} \in C^{\mathcal{I}}$.
  - By definition of $\cdot^{\mathcal{I}}$, we conclude $a^{\mathcal{I}} \in (C \sqcup D)^{\mathcal{I}}$.

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- Case $a : \exists r.C$ in the induction step.
  - As $\mathcal{A}$ is complete, $\{(a, b) : r, b : C\} \subseteq \mathcal{A}$ for some b.
  - By definition of $r^{\mathcal{I}}$, $(a, b) \in r^{\mathcal{I}}$.
  - By (IH), $b^{\mathcal{I}} \in C^{\mathcal{I}}$.
  - By definition of $\cdot^{\mathcal{I}}$, we conclude $a^{\mathcal{I}} \in (\exists r.C)^{\mathcal{I}}$.

Concluding the soundness

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- The cases in the induction step for $\sqcap$-concept assertions and $\forall$-concept assertions are similar.
- If expand($\mathcal{A}$) $\neq \emptyset$, then $\mathcal{A}$ is consistent.
- Indeed, expand($\mathcal{A}$) $\neq \emptyset$ if there is some $\mathcal{A}'$ with $\mathcal{A} \subseteq \mathcal{A}'$ such that $\mathcal{A}'$ is complete and clash-free.
- Consistency of $\mathcal{A}'$ leads to the consistency of $\mathcal{A}$.

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- Let $\mathcal{I} \stackrel{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be such that $\mathcal{I} \models \mathcal{A}$.
- If $\mathcal{A}$ is complete, we are done.
- Otherwise, if $\mathcal{A}$ is not complete, we show that there is $\mathcal{A}'$ such that $\mathcal{A} \longrightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.

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- Otherwise, if $\mathcal{A}$ is not complete, we show that there is $\mathcal{A}'$ such that $\mathcal{A} \longrightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.
- As the length of a derivation from $\mathcal{A}$ is bounded by an exponential in the size of $\mathcal{A}$, there is $\mathcal{A}'$ such that $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ and $\mathcal{A}'$ is complete, clash-free (and consistent).

Single steps in the completeness proof

- It remains to prove that non-completeness implies the existence of one expansion preserving consistency.
- Guidance from the interpretations to choose disjuncts and tree individuals.
- If the $\sqcup$-rule is applicable on $a : C \sqcup D$, then there is $E \in \{C, D\}$ such that $\mathcal{I} \models \mathcal{A} \cup \{a : E\}$.
- $\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : E\}$ and $\mathcal{I} \models \mathcal{A} \cup \{a : E\}$.

Single steps in the completeness proof (II)

- If the $\exists$-rule is applicable on $a : \exists r.C$, then we use the fact that $a^{\mathcal{I}} \in (\exists r.C)^{\mathcal{I}}$.
- There is $\mathfrak{a} \in \Delta^{\mathcal{I}}$ such that $\mathfrak{a} \in C^{\mathcal{I}}$ and $(a^{\mathcal{I}}, \mathfrak{a}) \in r^{\mathcal{I}}$.
- Let $\mathcal{I}'$ be equal to $\mathcal{I}$ except that $\mathcal{I}'(c) = \mathfrak{a}$ for some fresh c.
- Then, $\mathcal{A} \longrightarrow \mathcal{A} \cup \{c : C, (a, c) : r\}$ and

$$\mathcal{I}' \models \mathcal{A} \cup \{c : C, (a, c) : r\}$$

Decision procedure of ABox consistency

- $\mathcal{A}$ is consistent iff $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.
- Derivations $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ have length bounded by an exponential in size($\mathcal{A}$).
- Existence of $\mathcal{A}'$ amounts to explore a tree of bounded depth and bounded degree.

Adding a TBox – First properties

- $\mathcal{I} \models C \sqsubseteq D$ iff $\mathcal{I} \models \top \sqsubseteq \neg C \sqcup D$.
- $\mathcal{I} \models C \equiv D$ iff $\mathcal{I} \models \top \sqsubseteq (\neg C \sqcup D) \sqcap (\neg D \sqcup C)$.
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$\sqsubseteq$-rule: If $a : C \in \mathcal{A}$, $\top \sqsubseteq D \in \mathcal{T}$ and $a : D \notin \mathcal{A}$, then

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- The termination argument for ABox consistency does not work anymore.

(Why?)

Termination with the blocking technique

- Given $\mathcal{A} \xrightarrow{*} \mathcal{A}'$, a is an **ancestor** of b in $\mathcal{A}'$ iff

$$\{(a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k\} \subseteq \mathcal{A}'$$

  with $a_1 = a$, $a_{k+1} = b$ and b is a tree individual.

⚠ The notion of ancestor assumes that one can distinguish the **root individuals** (individual names from $\mathcal{A}$) from the **tree individuals** (those introduced by applying the $\exists$-rule).

Termination with the blocking technique

- Given $\mathcal{A} \xrightarrow{*} \mathcal{A}'$, a is an **ancestor** of b in $\mathcal{A}'$ iff

$$\{(a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k\} \subseteq \mathcal{A}'$$

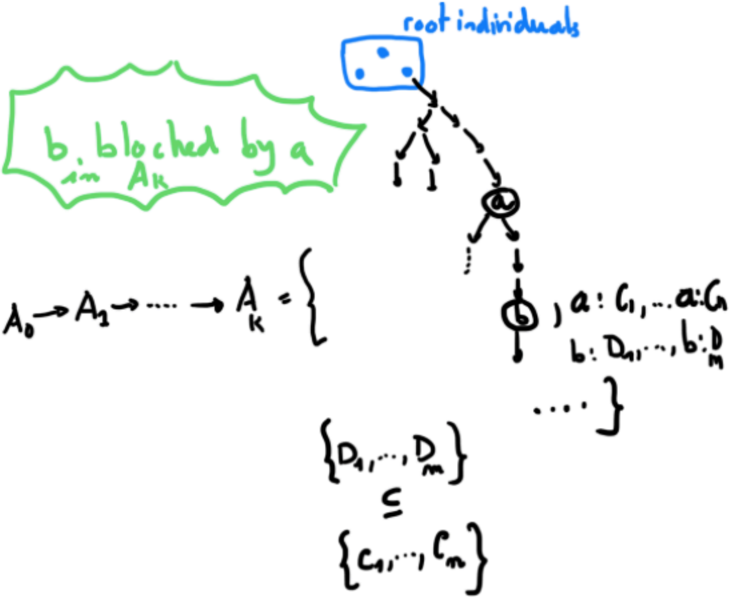
  with $a_1 = a$, $a_{k+1} = b$ and b is a tree individual.

⚠ The notion of ancestor assumes that one can distinguish the **root individuals** (individual names from $\mathcal{A}$) from the **tree individuals** (those introduced by applying the $\exists$-rule).

- An individual name b in $\mathcal{A}'$ is **blocked by** a if
  - a is an ancestor of b,
  - $\mathsf{con}_{\mathcal{A}'}(b) \subseteq \mathsf{con}_{\mathcal{A}'}(a)$.

- An individual name b is **blocked in** $\mathcal{A}'$ iff it is blocked by some individual name or, one or more of its ancestors is blocked in $\mathcal{A}'$.

b blocked by *a*



Expansion rules with blocking

$\sqcap$-rule: If $a : C \sqcap D \in \mathcal{A}$, $\boxed{a \text{ is not blocked}}$ and $\{a : C, a : D\} \not\subseteq \mathcal{A}$ then $\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : C, a : D\}$.

$\sqcup$-rule: If $a : C \sqcup D \in \mathcal{A}$, $\boxed{a \text{ is not blocked}}$ and $\{a : C, a : D\} \cap \mathcal{A} = \emptyset$ then $\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : E\}$ for some $E \in \{C, D\}$.

$\exists$-rule: If $a : \exists r.C \in \mathcal{A}$, $\boxed{a \text{ is not blocked}}$ and there is no b such that $\{(a, b) : r, b : C\} \subseteq \mathcal{A}$ then

$$\mathcal{A} \longrightarrow \mathcal{A} \cup \{(a, c) : r, c : C\} \qquad \text{where } c \text{ is fresh}$$

$\forall$-rule: If $\{(a, b) : r, a : \forall r.C\} \subseteq \mathcal{A}$, $\boxed{a \text{ is not blocked}}$ and $b : C \notin \mathcal{A}$, then $\mathcal{A} \longrightarrow \mathcal{A} \cup \{b : C\}$.

$\sqsubseteq$-rule: If $a : C \in \mathcal{A}$, $\top \sqsubseteq D \in \mathcal{T}$, $\boxed{a \text{ is not blocked}}$ and $a : D \notin \mathcal{A}$, then $\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : D\}$.

Termination

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIs of the form $\top \sqsubseteq D$.
- **N**: number of root individuals in $\mathcal{A}$, $\mathbf{M} = \text{card}(\text{sub}(\mathcal{K}))$, $\mathbf{W} = \text{w}_\exists(\mathcal{K})$.

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$$\text{card}(\{(a, b) \mid (a, b) : r \in \mathcal{A}'\}) \leq \mathbf{N} + \mathbf{W}$$
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- The length of the derivation $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ is at most

$$\mathbf{N} \times (\mathbf{N} + \mathbf{W})^{(2^{\mathbf{M}}+1)} \times \mathbf{M}$$

Soundness

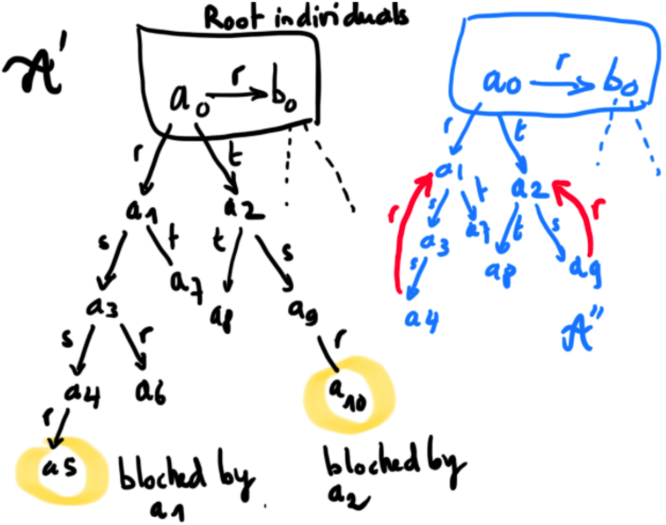
- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIs of the form $\top \sqsubseteq D$.
- $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ with $\mathcal{A}'$ complete and clash-free.
- We construct $\mathcal{A}''$ as the ABox made of the following assertions

$$\{a : C \mid a : C \in \mathcal{A}',\ a \text{ is not blocked}\} \cup$$

$$\{(a, b) : r \mid (a, b) : r \in \mathcal{A}',\ b \text{ is not blocked}\} \cup$$

$$\{(a, b') : r \mid (a, b) : r \in \mathcal{A}',\ a \text{ is not blocked and } b \text{ is blocked by } b'\}$$

Construction of $\mathcal{A}''$



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- $\mathcal{A}''$ is complete and clash-free.

Proof: $\mathcal{A}''$ is complete

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- Suppose $a : C \sqcap D \in \mathcal{A}''$.
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  As $\mathcal{A}'$ is complete, $\{a : C, a : D\} \subseteq \mathcal{A}'$.
  By $(\star)$, $\{a : C, a : D\} \subseteq \mathcal{A}''$.

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- Suppose that $a : C \in \mathcal{A}''$ and $\top \sqsubseteq D \in \mathcal{T}$.
  By $(\star)$, $a : C \in \mathcal{A}'$.
  As $\mathcal{A}'$ is complete, $a : D \in \mathcal{A}'$.
  By $(\star)$, $a : D \in \mathcal{A}''$.

Case with the $\exists$-rule

- Suppose that $a : \exists r.C \in \mathcal{A}''$.
  By $(\star)$, $a : \exists r.C \in \mathcal{A}'$ and a not blocked.
  By completeness of $\mathcal{A}'$, there is b such that
  $\{(a,b) : r, b : C\} \subseteq \mathcal{A}'$.

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  So, $b' : C \in \mathcal{A}''$.
- *Case with the $\forall$-rule left as an exercise.*

More about the soundness proof

- $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ with $\mathcal{A}'$ complete and clash-free and $\mathcal{A}''$ computed as above.

- Let us define $\mathcal{I} \stackrel{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ from $\mathcal{A}''$ as follows.
  - $\Delta^{\mathcal{I}} \stackrel{\text{def}}{=} \{a \mid a : C \in \mathcal{A}''\}$.
  - $a^{\mathcal{I}} \stackrel{\text{def}}{=} a$ for all individual names a in $\mathcal{A}''$.
  - $A^{\mathcal{I}} \stackrel{\text{def}}{=} \{a \mid A \in \text{con}_{\mathcal{A}''}(a)\}$ for all concept names $A \in \text{sub}(\mathcal{A}'')$.
  - $r^{\mathcal{I}} \stackrel{\text{def}}{=} \{(a, b) \mid (a, b) : r \in \mathcal{A}''\}$.

(previous construction with $\mathcal{A}''$ instead)

- One can show that for all $a : C \in \mathcal{A}''$, we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

(left as an exercise.)

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- One can show that for all $a : C \in \mathcal{A}''$, we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.
- Consequently, $\mathcal{I} \models \mathcal{A}$ as $\mathcal{A} \subseteq \mathcal{A}''$.
- Moreover, $\mathcal{I} \models \top \sqsubseteq C$ for all $\top \sqsubseteq C \in \mathcal{T}$.
- $a \in \Delta^{\mathcal{I}}$
  $\rightarrow a : C \in \mathcal{A}''$ ($\mathcal{A}''$ is complete)
  $\rightarrow a \in C^{\mathcal{I}}$ (see above)

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- One can prove that non-completeness implies the existence of one expansion preserving consistency.

Complexity issues

- $\mathcal{ALC}$ concept satisfiability in PSPACE, knowledge base consistency in EXPTIME.
- The algorithm for ABox consistency runs in exponential space:
  - Because of the nondeterministic $\sqcup$-rule, exponentially many ABoxes may be generated.
  - Complete ABoxes may be exponentially large.
- PSPACE bound for ABox consistency can be regained by exploring the tree-like interpretations in a depth-first manner having only one path at a time.

Recapitulation: Tableaux for $\mathcal{ALC}$ knowledge base consistency

- Tableaux-based algorithm to decide $\mathcal{ALC}$ knowledge base consistency.
- All other standard decision problems can be handled too.
- Termination is guaranteed thanks to the blocking technique.
- In the worst-case, exponential space is used but optimisations exist to meet the optimal upper bound EXPTIME.
- Tableaux can be extended to richer variants of $\mathcal{ALC}$ (with inverses, nominals, number restrictions, etc.)

$$\boxed{\textbf{Undecidability with Role Inclusion Axioms}}$$

DLs: a playground to study extensions and fragments

- Many developments to extend $\mathcal{ALC}$ while preserving the decidability status / complexity of the main decision problems.
- Many developments to study fragments of $\mathcal{ALC}$ (or variants) to identify tractable fragments.
- It is also important to identify undecidable extensions.

Tiling system

- **Tiling system**: (T, H, V, t_0) where
  - T is a finite set of **tile types** and $t_0 \in T$,
  - $H, V \subseteq T \times T$ are two relations referred to as the **horizontal**, resp. **vertical matching relation**.

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- A set of tile types (a.k.a. **tiles**)

$t_1 =$ 

$t_2 =$ 

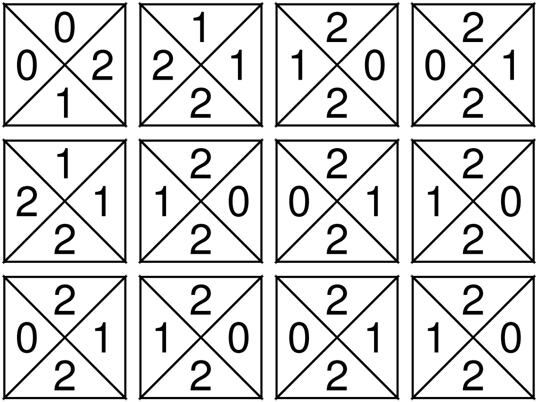
$t_3 =$ 

$t_4 =$ 

- ... with its matching relations
  - $H = \{(t_1, t_3), (t_1, t_4), (t_2, t_1), (t_3, t_2), (t_4, t_1)\}$
  - $V = \{(t_1, t_2), (t_1, t_4), (t_2, t_3), (t_4, t_1), (t_4, t_2)\}$

A tiling for the ($[0,3] \times [0,2]$)-arena

tiling $\tau : [0,3] \times [0,2] \to T$



An undecidable tiling problem

- The $(\infty \times \infty)$-tiling problem.

Input: A tiling system (T, H, V, t_0).

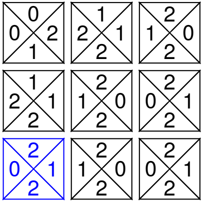
Question: Is there a **tiling** $\tau : \mathbb{N} \times \mathbb{N} \to T$ such that for all $i, j \in \mathbb{N}$,

(hori) if $\tau(i,j) = t$ and $\tau(i+1,j) = t'$, then $(t,t') \in H$,
(verti) if $\tau(i,j) = t$ and $\tau(i,j+1) = t'$, then $(t,t') \in V$

- The $(\infty \times \infty)$-tiling problem is undecidable.

Complexity about $\mathcal{ALC}$ problems

- Concept satisfiability problem is PSPACE-complete.
- PSPACE-hardness by reduction from $(n \times n)$-tiling game problem.



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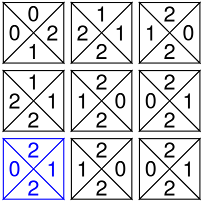

- EXPTIME-complete knowledge base consistency problem. EXPTIME-hardness from $(n \times \infty)$-tiling game problem.

A standard undecidability result

- $\mathcal{ALC}$ + role axioms $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ had undecidable knowledge base consistency problem.

  (actually TBox consistency is undecidable)

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- Reduction from $(\infty \times \infty)$-tiling problem.

- $\mathcal{ALC}$ + local role value maps $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ has undecidable *concept satisfiability problem.*

  (not pre

An undecidable extension of $\mathcal{ALC}$

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$$r \circ s \sqsubseteq q \qquad q \sqsubseteq r \circ s,$$

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- Reduction from the $(\infty \times \infty)$-tiling problem to knowledge base consistency for such an $\mathcal{ALC}$ extension.

The reduction

- Given a tiling system $\mathtt{T} = (T, H, V, t_0)$, we introduce two role names r_x and r_y.
- We build a TBox $\mathcal{T}_{\mathtt{T}}$ such that $\mathtt{T}$ is a positive instance of the $(\infty \times \infty)$-tiling problem iff $\mathcal{T}_{\mathtt{T}}$ is consistent.

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$$\top \sqsubseteq \exists r_x.\top \sqcap \exists r_y.\top$$

- Every individual belongs to a unique tile type.

$$\top \sqsubseteq \bigsqcup_{t \in T} (t \sqcap \bigsqcap_{t' \neq t} \neg t')$$

The reduction

- Given a tiling system $\mathbb{T} = (T, H, V, t_0)$, we introduce two role names r_x and r_y.
- We build a TBox $\mathcal{T}_\mathbb{T}$ such that $\mathbb{T}$ is a positive instance of the $(\infty \times \infty)$-tiling problem iff $\mathcal{T}_\mathbb{T}$ is consistent.
- Every individual has a horizontal and a vertical successor:

$$\top \sqsubseteq \exists r_x.\top \sqcap \exists r_y.\top$$

- Every individual belongs to a unique tile type.

$$\top \sqsubseteq \bigsqcup_{t \in T} (t \sqcap \bigsqcap_{t' \neq t} \neg t')$$

- Tile types of adjacent individuals satisfy the matching relations:

$$\top \sqsubseteq \bigsqcup_{(t,t') \in H} (t \sqcap \forall r_x.t') \sqcap \bigsqcup_{(t,t') \in V} (t \sqcap \forall r_y.t')$$

The properties

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- $\mathcal{T}_{\mathtt{T}}$ is made of the above GCIs and role axioms.
- $\mathcal{T}_{\mathtt{T}}$ is consistent iff $\mathtt{T}$ is a positive instance.
- TBox consistency problem for $\mathcal{ALC}$ augmented with role axioms of the form $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ is undecidable.

Correctness proof (or how to extract a grid)

- Let $\mathcal{I}$ be an interpretation satisfying the TBox $\mathcal{T}_{\mathbb{T}}$.

- We define a map $\mathfrak{f} : \mathbb{N} \times \mathbb{N} \to \Delta^{\mathcal{I}}$ such that for all i, j
  - $(\mathfrak{f}(i,j), \mathfrak{f}(i+1,j)) \in r_x^{\mathcal{I}}$
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- Then, we define $\tau : \mathbb{N} \times \mathbb{N} \to T$ from $\mathfrak{f}$ as follows:

$$\tau(i,j) \stackrel{\text{def}}{=} \text{unique } t \text{ such that } \mathfrak{f}(i,j) \in t^{\mathcal{I}}$$

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- Unicity of t guaranteed by $\mathcal{I} \models \top \sqsubseteq \bigsqcup_{t \in T}(t \sqcap \bigsqcap_{t' \neq t} \neg t')$.

- Afterwards, easy to check τ is a tiling as $\mathcal{I} \models \mathcal{T}_{\mathbb{T}}$.

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More cases for defining $\mathfrak{f}$

- As $\mathcal{I} \models r_x \circ r_y \equiv r_y \circ r_x$, when

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$$\begin{tikzcd} \mathfrak{a} \arrow[r, dotted, "r_x"] & \mathfrak{f}(i+1,j+1) \\ \mathfrak{f}(i,j) \arrow[u, dotted, "r_y"] \arrow[r, "r_x"] & \mathfrak{f}(i+1,j) \arrow[u, "r_y"'] \end{tikzcd}$$

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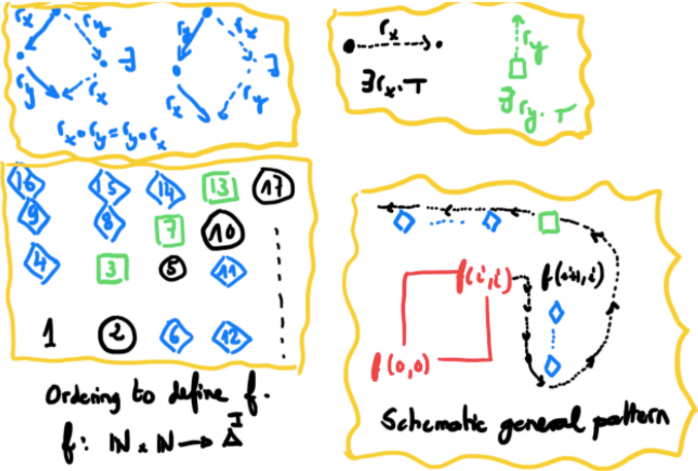
\end{tikzcd}$$

- With these four cases, how to build $\mathfrak{f}$ on $\mathbb{N} \times \mathbb{N}$?

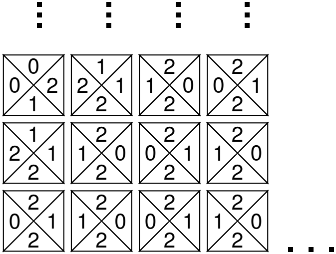
Construction of the map $\mathfrak{f}$: a bit of organisation



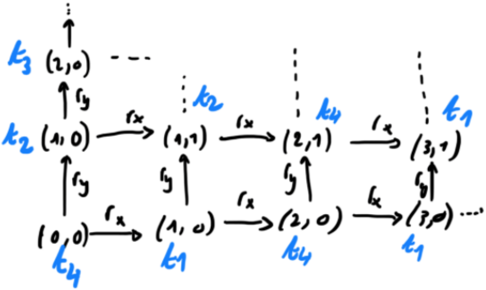
The other direction (easy)

- Let $\mathbb{T} = (T, H, V, t_0)$ be a tiling system and $\tau : \mathbb{N} \times \mathbb{N} \to T$ be a tiling.
- Interpretation $\mathcal{I} \stackrel{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:
  - $\Delta^{\mathcal{I}} \stackrel{\text{def}}{=} \mathbb{N} \times \mathbb{N}$
  - $r_x^{\mathcal{I}} \stackrel{\text{def}}{=} \{((i, j), (i + 1, j)) \mid i, j \in \mathbb{N}\}$
  - $r_y^{\mathcal{I}} \stackrel{\text{def}}{=} \{((i, j), (i, j + 1)) \mid i, j \in \mathbb{N}\}$
  - $t^{\mathcal{I}} \stackrel{\text{def}}{=} \{(n, m) \mid \tau(n, m) = t\}$ for every $t \in T$
- It is easy to check $\mathcal{I}$ satisfies all the GCIs and the role axioms from $\mathcal{T}_{\mathbb{T}}$.

Tiling τ



Interpretation $\mathcal{I}$



Conclusion

- Today lecture: tableaux for DLs.
  - Rules for checking concept satisfiability.
  - Rules for checking knowledge base consistency.
  - Termination, soundness, completeness.
  - Undecidability result with role axioms.
- Next week lecture: reasoning about multiagent systems with ATL.

Other topics related to DLs

- More tableaux-style systems and complexity results for $\mathcal{ALC}$ extensions (e.g. for $\mathcal{SROIQ}$, $\mathcal{ALCIQ}$, $\mathcal{ALCOI}$, etc.)
- More fragments with nice computational properties while retaining sufficient expressivity (e.g. $\mathcal{EL}$, $\mathcal{FL}_0$, DL-Lite, etc.)
- Playing with ontologies, ontology editors, etc.....
- Query answering with respect to ontologies for large data sets.