TD: Logical Aspects of Artificial Intelligence
Introduction to DLs & Properties (21/09/2022)

$\star\star$ Exercises related to the previous session $\star\star$

Exercise 1. Let $\mathcal{K} = (\mathcal{T} \cup \{A \sqsubseteq C\}, \mathcal{A})$ be a knowledge base such that A is a concept name and B is a concept name that does not occur in $\mathcal{K}$ (B is "new"). Show that $\mathcal{K}$ is consistent iff $\mathcal{K}' = (\mathcal{T} \cup \{A \equiv B \sqcap C\}, \mathcal{A})$ is consistent.

Exercise 2. Let $\mathcal{T}^\star = \{A_1 \equiv C_1, \ldots, A_m \equiv C_m\}$ be an $\mathcal{ALC}$ TBox satisfying the following properties.

- Every A_i is a concept name, and $A_i \equiv C_i$ is an abbreviation for $A_i \sqsubseteq C_i$ and $C_i \sqsubseteq A_i$.
- For all $i, j \in [1, m]$, if A_j occurs in C_i, then $j > i$.
- If $i \neq j \in [1, m]$, then A_i and A_j are syntactically distinct.

Such a TBox $\mathcal{T}^\star$ is called **acyclic**.

1. Briefly define an acyclic graph from $\mathcal{T}^\star$, which would justify the terminology "$\mathcal{T}^\star$ is acyclic".
2. Given an interpretation $\mathcal{I}$, show that there exists an interpretation $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{T}^\star$, the interpretations of the role names and concept names different from $\{A_1, \ldots, A_m\}$ are identical in $\mathcal{I}$ and $\mathcal{J}$.
3. Design an algorithm that takes as input a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with acyclic $\mathcal{T}$ and returns an ABox $\mathcal{A}'$ such that $\mathcal{K}$ is consistent iff $(\emptyset, \mathcal{A}')$ is consistent, and $\mathcal{A}'$ contains no A_i's. The proof for the soundness of the algorithm is not requested.
4. Explain why your algorithm terminates and analyse its computational complexity.

Exercise 3. (Exponential-size interpretations) Define a family of concepts $(C_n)_{n\geq 1}$ such that each C_n is of polynomial size in n (for a fixed polynomial), C_n is satisfiable, and the interpretations satisfying C_n have at least 2^n individuals in its domains.

Exercise 4. (Infinite models) Let $\mathcal{ALCIN}$ be the extension of $\mathcal{ALC}$ with unqualified number restrictions and inverse roles. Let $C = \neg A \sqcap \exists r.A$ and $\mathcal{T} = \{A \sqsubseteq \exists r.A, \top \sqsubseteq (\leq 1\ r^-)\}$. Show that for all interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $C^{\mathcal{I}} \neq \emptyset$ and $\mathcal{I} \models \mathcal{T}$, $\Delta^{\mathcal{I}}$ is infinite.

$\star\star$ Exercises related to today session$\star\star$

Exercise 5. Let us consider the translation map $\mathfrak{t}$ into first-order logic. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation.

1. Let C be a complex concept in $\mathcal{ALC}$. Show that for all $\mathfrak{a} \in \Delta^{\mathcal{I}}$, we have $\mathfrak{a} \in C^{\mathcal{I}}$ iff $\mathcal{I}, \rho[\mathtt{x} \leftarrow \mathfrak{a}] \models \mathfrak{t}(C, \mathtt{x})$ where ρ is a first-order assignment.

2. Show that $\mathcal{I} \models \mathcal{K}$ iff $\mathcal{I} \models \mathfrak{t}(\mathcal{K})$.

Exercise 6. (Model-checking in PTIME) Let $\mathcal{I}$ be an interpretation with finite domain and C be an $\mathcal{ALC}$ concept. Recapitulate the main arguments to show that the algorithm seen in the lecture to compute $C^{\mathcal{I}}$ indeed runs in polynomial time.

Exercise 7. (from exam 2021/2022) Let X be a finite set of $\mathcal{ALC}$ concepts closed under subconcepts and $\mathcal{K}$ (resp. C) be a knowledge base (resp. a concept) such that $\mathsf{sub}(\mathcal{K}) \cup \mathsf{sub}(C) \subseteq X$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that

- $\mathcal{I} \models \mathcal{K}$ and $C^{\mathcal{I}} \neq \emptyset$,
- for all role names r occurring in X, $r^{\mathcal{I}}$ is reflexive and transitive.

For all $\mathfrak{a}, \mathfrak{a}' \in \Delta^{\mathcal{I}}$, we write $\mathfrak{a} \sim \mathfrak{a}'$ iff for all concepts $D \in X$, we have $\mathfrak{a} \in D^{\mathcal{I}}$ iff $\mathfrak{a}' \in D^{\mathcal{I}}$. As $\sim$ is an equivalence relation, equivalence classes of $\sim$ are written $[\mathfrak{a}]$ to denote the class of $\mathfrak{a}$. Let us define the interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$:

- $\Delta^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid \mathfrak{a} \in \Delta^{\mathcal{I}}\}$.
- $A^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid \text{ there is } \mathfrak{a}' \in [\mathfrak{a}] \text{ such that } \mathfrak{a}' \in A^{\mathcal{I}}\}$ for all $A \in X$.
- $A^{\mathcal{J}} \stackrel{\text{def}}{=} \emptyset$ for all concept names $A \notin X$ (arbitrary value).
- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \{([\mathfrak{a}], [\mathfrak{b}]) \mid \text{ there are } \mathfrak{a}' \in [\mathfrak{a}], \mathfrak{b}' \in [\mathfrak{b}] \text{ such that for all } \forall r.D \in X,\ \mathfrak{a}' \in (\forall r.D)^{\mathcal{I}} \text{ implies } \mathfrak{b}' \in (\forall r.D)^{\mathcal{I}}\}$ for all role names r occurring in X.
- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \emptyset$ for all role names r not occurring in X (arbitrary value).
- $a^{\mathcal{J}} \stackrel{\text{def}}{=} [\mathfrak{a}]$ with $a^{\mathcal{I}} = \mathfrak{a}$, for all individual names a.

1. Show that for all role names r occurring in X, $r^{\mathcal{J}}$ is reflexive and transitive.
2. Show that $(\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}$ implies $([\mathfrak{a}], [\mathfrak{b}]) \in r^{\mathcal{J}}$, for all role names r occurring in X.
3. Assuming that the concept constructors occurring in X are among $\forall r$ for some r, $\sqcap$ and $\neg$, show that for all $D \in X$ and $\mathfrak{a} \in \Delta^{\mathcal{I}}$, we have $\mathfrak{a} \in D^{\mathcal{I}}$ iff $[\mathfrak{a}] \in D^{\mathcal{J}}$. (This restriction on the concept constructors allows us to reduce the number of cases in the induction step).
4. Conclude that there is a finite interpretation $\mathcal{I}^\star$ such that $\mathcal{I}^\star \models \mathcal{K}$ and $(C)^{\mathcal{I}^\star} \neq \emptyset$ and for all role names r occurring in X, $(r)^{\mathcal{I}^\star}$ is reflexive and transitive.

Correction: Exercise 1

Let us show that $\mathcal{K}$ is consistent iff $\mathcal{K}' = (\mathcal{T} \cup \{A \equiv B \sqcap C\}, \mathcal{A})$ is consistent.

First, suppose that $\mathcal{K}$ is consistent. There is an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $\mathcal{I} \models \mathcal{T} \cup \{A \sqsubseteq C\}$ and $\mathcal{I} \models \mathcal{A}$. In particular, we have $A^{\mathcal{I}} \subseteq C^{\mathcal{I}}$. Let $\mathcal{I}^\star = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}^\star})$ be the interpretation obtained from $\mathcal{I}$ by only modifying the interpretation of B with $\mathcal{I}^\star(B) = A^{\mathcal{I}}$. Let us enumerate properties satisfied by $\mathcal{I}^\star$.

- $A^{\mathcal{I}^\star} = A^{\mathcal{I}} = B^{\mathcal{I}^\star} \cap C^{\mathcal{I}^\star}$. Indeed, B does not occur in $\mathcal{K}$ and therefore we get $C^{\mathcal{I}^\star} = C^{\mathcal{I}}$.
- Due to "freshness" of B, we can also conclude that $\mathcal{I}^\star \models \mathcal{T}$ and $\mathcal{I}^\star \models \mathcal{A}$.

Consequently, $\mathcal{K}'$ is consistent.

Secondly, suppose that $\mathcal{K}'$ is consistent. There is an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $\mathcal{I} \models \mathcal{T} \cup \{A \equiv B \sqcap C\}$ and $\mathcal{I} \models \mathcal{A}$. In particular, we have $A^{\mathcal{I}} = B^{\mathcal{I}} \cap C^{\mathcal{I}}$. This entails that $A^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ and therefore $\mathcal{I} \models A \sqsubseteq C$. Hence, $\mathcal{I} \models \mathcal{K}$ and therefore $\mathcal{K}$ is consistent.

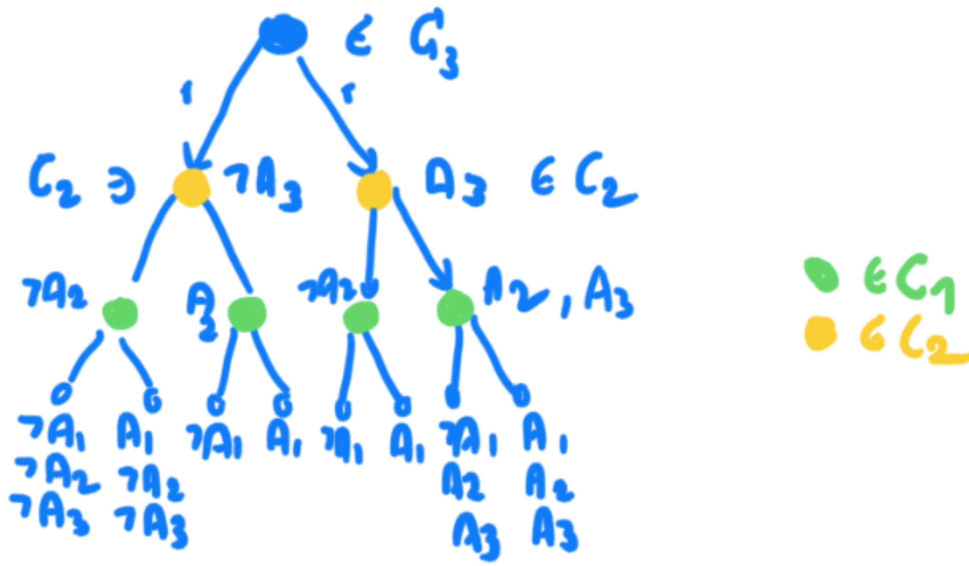
Correction: Exercise 3

We use the following notation: $(\forall r)^i.D$ with $i \geq 0$ such that $(\forall r)^0.D \stackrel{\text{def}}{=} D$ and for all $i \geq 0$, $(\forall r)^{i+1}.D \stackrel{\text{def}}{=} (\forall r)^i.(\forall r.D)$.

1. Let us define the family $(C_n)_{n\geq 1}$ inductively.
   - $C_1 \stackrel{\text{def}}{=} A_0 \sqcap \exists r.((\neg A_0) \sqcap A_1) \sqcap \exists r.((\neg A_0) \sqcap \neg A_1)$.
   - For all $n \geq 2$,

$$C_n \stackrel{\text{def}}{=} (\exists r.A_n) \sqcap (\exists r.\neg A_n) \sqcap (\forall r.C_{n-1}) \sqcap \bigsqcap_{i=1}^{n-1} (\forall r)^i.\Big((A_n \Rightarrow \forall r.A_n) \sqcap ((\neg A_n) \Rightarrow \forall r.\neg A_n)\Big)$$

   Observe that the size of C_n is quadratic in n. Here is an interpretation $\mathcal{I}$ in which $C_3^{\mathcal{I}}$ is not empty.



2. Let us show that the only interpretations satisfying C_n have at least 2^n elements. The proof is by induction on n. Let us show a stronger property, namely if $\mathfrak{a} \in C_n^{\mathcal{I}}$, then one can extract from the element $\mathfrak{a}$, a complete binary tree of depth n (with the edge-relation $r^{\mathcal{I}}$) for which all the elements in the left subtree are in $A_n^{\mathcal{I}}$ and all the elements in the right subtree are not in $A_n^{\mathcal{I}}$.

   - Suppose that $C_1^{\mathcal{I}} \neq \emptyset$. So, there is $\mathfrak{a} \in \Delta^{\mathcal{I}}$ such that $\mathfrak{a} \in (\exists r.((\neg A_0) \sqcap A_1)^{\mathcal{I}}$ and $\mathfrak{a} \in (\exists r.(\neg A_0) \sqcap \neg A_1)^{\mathcal{I}}$. Consequently, there is $\mathfrak{b}_1$ such that $(\mathfrak{a}, \mathfrak{b}_1) \in r^{\mathcal{I}}$ and $\mathfrak{b}_1 \in A_1^{\mathcal{I}}$, and there is $\mathfrak{b}_2$ tel que $(\mathfrak{a}, \mathfrak{b}_2) \in r^{\mathcal{I}}$ and $\mathfrak{b}_2 \notin A_1^{\mathcal{I}}$. Obviously $\mathfrak{b}_1 \neq \mathfrak{b}_2$ and therefore $\mathsf{card}(\Delta^{\mathcal{I}}) \geq 2$. Similarly, $\mathfrak{a} \neq \mathfrak{b}_1$ and $\mathfrak{a} \neq \mathfrak{b}_2$ thanks to the properties on A_0. It is easy to build $\mathcal{I}$ such that $C_1^{\mathcal{I}} \neq \emptyset$. The tree rooted at $\mathfrak{a}$ contains $\mathfrak{b}_1$ and $\mathfrak{b}_2$.
   - As induction hypothesis, suppose that the property holds for $n - 1 \geq 1$. Suppose that $C_n^{\mathcal{I}} \neq \emptyset$. There is $\mathfrak{a} \in \Delta^{\mathcal{I}}$ such that $\mathfrak{a} \in (\exists r.A_n)^{\mathcal{I}}$ and $\mathfrak{a} \in (\exists r.\neg A_n)^{\mathcal{I}}$. Consequently, there is $\mathfrak{b}_1$ such that $(\mathfrak{a}, \mathfrak{b}_1) \in r^{\mathcal{I}}$ and $\mathfrak{b}_1 \in A_n^{\mathcal{I}}$, and there is $\mathfrak{b}_2$ such that $(\mathfrak{a}, \mathfrak{b}_2) \in r^{\mathcal{I}}$ and $\mathfrak{b}_2 \notin A_n^{\mathcal{I}}$. So, $\mathfrak{b}_1 \neq \mathfrak{b}_2$. Moreover, as $\mathfrak{a} \in (\forall r.C_{n-1})^{\mathcal{I}}$, $\mathfrak{b}_1 \in C_{n-1}^{\mathcal{I}}$ and $\mathfrak{b}_2 \in C_{n-1}^{\mathcal{I}}$. By (IH), there is a complete binary tree $\mathcal{T}_1$ rooted

at $\mathfrak{b}_1$ of depth $n-1$ and there is a complete binary tree $\mathcal{T}_2$ rooted at $\mathfrak{b}_2$ of depth $n-1$. In order to extract a complete binary tree of depth n from $\mathfrak{a}$, it is sufficient to show that the nodes in $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint. Actually, as

$$\mathfrak{a} \in (\prod_{i=1}^{n-1} (\forall r)^i . \Big((A_n \Rightarrow \forall r.A_n) \sqcap ((\neg A_n) \Rightarrow \forall r.\neg A_n) \Big))^{\mathcal{I}},$$

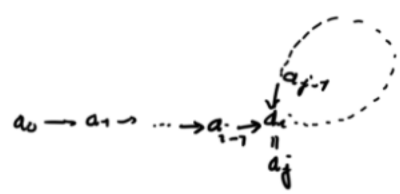
$\mathcal{T}_1$ is disjoint from $\mathcal{T}_2$. Similarly, $\mathfrak{a}$ is necessarily different from $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

Correction: Exercise 4

Let $C = \neg A \sqcap \exists r.A$, $\mathcal{T} = \{A \sqsubseteq \exists r.A, \top \sqsubseteq (\leq 1\ r^-)\}$ and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that $C^{\mathcal{I}} \neq \emptyset$ and $\mathcal{I} \models \mathcal{T}$. Below, we show that $\Delta^{\mathcal{I}}$ is infinite. The proof can be found also on page 62 in [1].

Ad absurdum, suppose that $\Delta^{\mathcal{I}}$ is finite and let $\mathfrak{a}_0 \in C^{\mathcal{I}}$. As $\mathfrak{a}_0 \in (\exists r.A)^{\mathcal{I}}$, there is $\mathfrak{a}_1 \in \Delta^{\mathcal{I}}$ such that $\mathfrak{a}_1 \in A^{\mathcal{I}}$ and $(\mathfrak{a}_0, \mathfrak{a}_1) \in r^{\mathcal{I}}$. As $\mathcal{I} \models A \sqsubseteq \exists r.A$ and $\mathfrak{a}_1 \in A^{\mathcal{I}}$, there is $\mathfrak{a}_2 \in \Delta^{\mathcal{I}}$ such that $\mathfrak{a}_2 \in A^{\mathcal{I}}$ and $(\mathfrak{a}_1, \mathfrak{a}_2) \in r^{\mathcal{I}}$. By using the same argument, we can show that there is a sequence $\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots$ such that $\mathfrak{a}_0 \notin A^{\mathcal{I}}$, $\{\mathfrak{a}_1, \mathfrak{a}_2, \ldots\} \subseteq A^{\mathcal{I}}$ and for all $i \in \mathbb{N}$, we have $(\mathfrak{a}_i, \mathfrak{a}_{i+1}) \in r^{\mathcal{I}}$.

As $\Delta^{\mathcal{I}}$ is supposed to be finite, there are $0 \leq i < j$ such that $\mathfrak{a}_i = \mathfrak{a}_j$. Without any loss of generality, let us assume that i is minimal.



Moreover, as $j > 0$ and $\mathfrak{a}_0 \notin A^{\mathcal{I}}$, we have $i > 0$ and $j > 0$. This means that $\mathfrak{a}_{i-1}$ is an r-predecessor of $\mathfrak{a}_i$ and $\mathfrak{a}_{j-1}$ is also an r-predecessor of $\mathfrak{a}_j = \mathfrak{a}_i$. By minimality of i, $\mathfrak{a}_{i-1}$ and $\mathfrak{a}_{j-1}$ are distinct and therefore $\mathfrak{a}_i$ has at least two predecessors, which is in contradiction with $\mathcal{I} \models \top \sqsubseteq (\leq 1\ r^-)$. Indeed, $\mathcal{I} \models \top \sqsubseteq (\leq 1\ r^-)$ enforces that every individual in $\Delta^{\mathcal{I}}$ has at most

[1]Introduction to Description Logics by Baader, Horrocks Lutz, Sattler, 2017.

one r-predecessor.

Correction: Exercise 6

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation with finite domain $\Delta^{\mathcal{I}}$ and C be an $\mathcal{ALC}$ concept. Below, we show that $C^{\mathcal{I}}$ can be computed in polynomial time in $\mathsf{size}(\mathcal{I}) + \mathsf{size}(C)$. To do so, we take the algorithm seen during the lecture and we annotate it so that the complexity analysis is straighforward.

Let $C_1, \ldots, C_k$ be the subconcepts of C ordered by increasing size. This means that $\mathsf{sub}(C) = \{C_1, \ldots, C_k\}$ and $\mathsf{size}(C) \leq k$. Without loss of generality, we can assume that $\mathcal{I}$ interprets only the concept names and role names that occur in C (and there are at most k such names). The size $\mathsf{size}(\mathcal{I})$ can be defined from a reasonably succinct encoding whose values is in $\mathcal{O}(k \times \mathsf{card}(\Delta^{\mathcal{I}})^2)$. Indeed, each concept name $A \in \mathsf{sub}(C)$, $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and for each role name r occurring in $\mathsf{sub}(C)$, $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

For each $\mathfrak{a} \in \Delta^{\mathcal{I}}$, the algorithm builds a set of concepts $l(\mathfrak{a})$ such that

1. for every $i \in [1, k]$, either $C_i \in l(\mathfrak{a})$, or $\neg C_i \in l(\mathfrak{a})$, but not both at the same time,
2. for every $D \in \{C_1, \ldots, C_k, \neg C_1, \ldots, \neg C_k\}$, $D \in l(\mathfrak{a})$ iff $\mathfrak{a} \in D^{\mathcal{I}}$. This is the property that guarantees correctness but its proofs is out of the scope of this exercise.

The algorithm works as follows for each $i \in [1, k]$ (i from 1 to k) and for each $\mathfrak{a} \in \Delta^{\mathcal{I}}$, we insert either C_i in $l(\mathfrak{a})$ or $\neg C_i$ dans $l(\mathfrak{a})$. The total number of insertions is in $\mathcal{O}(k \times \mathsf{card}(\Delta^{\mathcal{I}}))$, which is polynomial in $\mathsf{size}(\mathcal{I}) + \mathsf{size}(C)$. It remains to show that each single insertion can be done in polynomial-time too. Actually, we shall check that each insertion requires linear-time in $\mathsf{size}(\mathcal{I})$ in the worst-case.

Note that for all $\mathfrak{a} \in \Delta^{\mathcal{I}}$, $l(\mathfrak{a})$ is initialized to the empty set.

We run the algorithm with increasing $i \in [1, k]$ and below $\mathfrak{a} \in \Delta^{\mathcal{I}}$. Let us consider three significant cases (other cases for other concept constructors are handled in a similar fashion).

Case 1: C_i is a concept name.
If $\mathfrak{a} \in C_i^{\mathcal{I}}$ by definition of $\mathcal{I}$, then insert C_i in $l(\mathfrak{a})$ otherwise insert $\neg C_i$ in $l(\mathfrak{a})$. This step takes time in $\mathcal{O}(\mathsf{size}(\mathcal{I}))$.

Case 2: $C_i = C_{i_1} \sqcap C_{i_2}$ for some $i_1, i_2 < i$.
Insert C_i in $l(\mathfrak{a})$ if $\{C_{i_1}, C_{i_2}\} \subseteq l(\mathfrak{a})$ otherwise insert $\neg C_i$ in $l(\mathfrak{a})$. This step takes time in $\mathcal{O}(k)$.

Case 3: $C_i = \exists r.C_{i_1}$ for some $i_1 < i$.
If there is $\mathfrak{a}' \in r^{\mathcal{I}}(\mathfrak{a})$ such that C_{i_1} is in $l(\mathfrak{a}')$, then insert C_i in $l(\mathfrak{a})$, otherwise insert $\neg C_i$ in $l(\mathfrak{a})$. This step takes time in $\mathcal{O}(k \times \mathsf{size}(\mathcal{I}))$.

Correction: Exercise 7

1. Let $[\mathfrak{a}] \in \Delta^{\mathcal{J}}$ and r be a role name occurring in X. Obviously, for all $\forall r.D \in X$, we have $\mathfrak{a} \in (\forall r.D)^{\mathcal{I}}$ implies $\mathfrak{a} \in (\forall r.D)^{\mathcal{I}}$. Hence, $([\mathfrak{a}], [\mathfrak{a}]) \in r^{\mathcal{J}}$ by definition of $r^{\mathcal{J}}$. As above $[\mathfrak{a}]$ is an arbitrary element of $\Delta^{\mathcal{J}}$, $r^{\mathcal{J}}$ is reflexive.

   Now suppose that $([\mathfrak{a}], [\mathfrak{b}]) \in r^{\mathcal{J}}$ and $([\mathfrak{b}], [\mathfrak{c}]) \in r^{\mathcal{J}}$. By definition of $r^{\mathcal{J}}$, there is $\mathfrak{a}' \in [\mathfrak{a}]$ and $\mathfrak{b}' \in [\mathfrak{b}]$ such that for all $\forall r.D \in X$, $\mathfrak{a}' \in (\forall r.D)^{\mathcal{I}}$ implies $\mathfrak{b}' \in (\forall r.D)^{\mathcal{I}}$. Similarly, there is $\mathfrak{b}'' \in [\mathfrak{b}]$ and $\mathfrak{c}' \in [\mathfrak{c}]$ such that for all $\forall r.D \in X$, $\mathfrak{b}'' \in (\forall r.D)^{\mathcal{I}}$ implies $\mathfrak{c}' \in (\forall r.D)^{\mathcal{I}}$.

   Let $\forall r.D \in X$ with $\mathfrak{a}' \in [\mathfrak{a}]$ and $\mathfrak{a}' \in (\forall r.D)^{\mathcal{I}}$. Since $([\mathfrak{a}], [\mathfrak{b}]) \in r^{\mathcal{J}}$, $\mathfrak{b}' \in (\forall r.D)^{\mathcal{I}}$. Since $[\mathfrak{b}] = [\mathfrak{b}'] = [\mathfrak{b}'']$, we have also $\mathfrak{b}'' \in (\forall r.D)^{\mathcal{I}}$. Since $([\mathfrak{b}], [\mathfrak{c}]) \in r^{\mathcal{J}}$, we get $\mathfrak{c}' \in (\forall r.D)^{\mathcal{I}}$. Consequently, $([\mathfrak{a}], [\mathfrak{c}]) \in r^{\mathcal{J}}$ and therefore $r^{\mathcal{J}}$ is transitive.

2. Assume that $(\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}$. As $r^{\mathcal{I}}$ is reflexive and transitive, for all $\mathfrak{b}' \in r^{\mathcal{I}}(\mathfrak{b})$, we have $\mathfrak{b}' \in r^{\mathcal{I}}(\mathfrak{a})$.

   Now suppose that $\mathfrak{a} \in (\forall r.D)^{\mathcal{I}}$ with $\forall r.D \in X$. By $\mathcal{ALC}$ semantics, for all $\mathfrak{a}' \in r^{\mathcal{I}}(\mathfrak{a})$, we have $\mathfrak{a}' \in D^{\mathcal{I}}$. A fortiori (by the above remark), for all $\mathfrak{b}' \in r^{\mathcal{I}}(\mathfrak{b})$, we have $\mathfrak{b}' \in D^{\mathcal{I}}$, i.e. $\mathfrak{b} \in (\forall r.D)^{\mathcal{I}}$. As $\forall r.D$ is arbitrary, we conclude that for all $\forall r.D \in X$, $\mathfrak{a} \in (\forall r.D)^{\mathcal{I}}$ implies $\mathfrak{b} \in (\forall r.D)^{\mathcal{I}}$. By definition of $r^{\mathcal{J}}$, we get $([\mathfrak{a}], [\mathfrak{b}]) \in r^{\mathcal{J}}$ (take $\mathfrak{a}' = \mathfrak{a}$ and $\mathfrak{b}' = \mathfrak{b}$).

3. The proof is by structural induction. For the base $D = A$, by definition of $\mathcal{J}$, we have $A^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid \text{ there is } \mathfrak{a}' \in [\mathfrak{a}] \text{ such that } \mathfrak{a}' \in A^{\mathcal{I}}\}$. This is equivalent to $A^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid \mathfrak{a} \in A^{\mathcal{I}}\}$ since all the elements in $[\mathfrak{a}]$ agree on the concepts in X. Similarly, $\top^{\mathcal{J}} = \Delta^{\mathcal{J}}$, which is precisely $\{[\mathfrak{a}] \mid \mathfrak{a} \in \top^{\mathcal{I}} = \Delta^{\mathcal{I}}\}$. Let us consider now the induction step with a case analysis depending on the outermost concept constructor.

Case $D = \neg D'$

- $D^{\mathcal{J}} = \Delta^{\mathcal{J}} \setminus (D')^{\mathcal{J}}$ (by $\mathcal{ALC}$ semantics).
- $D^{\mathcal{J}} = \{[\mathfrak{a}] \mid \mathfrak{a} \in \Delta^{\mathcal{I}}\} \setminus \{[\mathfrak{a}] \mid \mathfrak{a} \in (D')^{\mathcal{I}}\}$ (by definition of $\Delta^{\mathcal{J}}$, X is closed under subconcepts and by induction hypothesis).
- $D^{\mathcal{J}} = \{[\mathfrak{a}] \mid \mathfrak{a} \notin (D')^{\mathcal{I}}\}$ (by set-theoretical reasoning).
- $D^{\mathcal{J}} = \{[\mathfrak{a}] \mid \mathfrak{a} \in (\neg D')^{\mathcal{I}}\}$ (by $\mathcal{ALC}$ semantics).

Case $D = D_1 \sqcap D_2$

- $D^{\mathcal{J}} = D_1^{\mathcal{J}} \cap D_2^{\mathcal{J}}$ (by $\mathcal{ALC}$ semantics).
- $D^{\mathcal{J}} = \{[\mathfrak{a}] \mid \mathfrak{a} \in D_1^{\mathcal{I}}\} \cap \{[\mathfrak{a}] \mid \mathfrak{a} \in D_2^{\mathcal{I}}\}$ (X is closed under subconcepts and by induction hypothesis).
- $D^{\mathcal{J}} = \{[\mathfrak{a}] \mid \mathfrak{a} \in D_1^{\mathcal{I}} \cap D_2^{\mathcal{I}}\}$ (by set-theoretical reasoning).
- $D^{\mathcal{J}} = \{[\mathfrak{a}] \mid \mathfrak{a} \in (D_1 \sqcap D_2)^{\mathcal{I}}\}$ (by $\mathcal{ALC}$ semantics).

Case $D = \forall r.D'$

First, suppose that $\mathfrak{a} \in (\forall r.D')^{\mathcal{I}}$. *Ad absurdum*, suppose that $[\mathfrak{a}] \notin (\forall r.D')^{\mathcal{J}}$. By $\mathcal{ALC}$ semantics, there is $[\mathfrak{b}]$ such that $([\mathfrak{a}], [\mathfrak{b}]) \in r^{\mathcal{J}}$ and $[\mathfrak{b}] \notin (D')^{\mathcal{J}}$. By definition of $r^{\mathcal{J}}$, there is $\mathfrak{a}' \in [\mathfrak{a}]$ and $\mathfrak{b}' \in [\mathfrak{b}]$ such that for all $\forall r.D'' \in X$, $\mathfrak{a}' \in (\forall r.D'')^{\mathcal{I}}$ implies $\mathfrak{b}' \in (\forall r.D'')^{\mathcal{I}}$. As $[\mathfrak{a}] = [\mathfrak{a}']$, $\mathfrak{a}' \in (\forall r.D')^{\mathcal{I}}$ and therefore $\mathfrak{b}' \in (\forall r.D')^{\mathcal{I}}$. As $r^{\mathcal{I}}$ is reflexive, $\mathfrak{b}' \in (D')^{\mathcal{I}}$ and by the induction hypothesis $[\mathfrak{b}'] \in (D')^{\mathcal{J}}$ (the set X is closed under subconcepts, so we can use the induction hypothesis). However $[\mathfrak{b}] = [\mathfrak{b}']$ and therefore $[\mathfrak{b}] \in (D')^{\mathcal{J}}$, which leads to contradiction.

Second, suppose that $[\mathfrak{a}] \in (\forall r.D')^{\mathcal{J}}$. *Ad absurdum*, suppose that there is $\mathfrak{a}' \in [\mathfrak{a}]$ such that $\mathfrak{a}' \notin (\forall r.D')^{\mathcal{I}}$. By $\mathcal{ALC}$ semantics, there is $\mathfrak{b}'$ such that $(\mathfrak{a}', \mathfrak{b}') \in r^{\mathcal{I}}$ and $\mathfrak{b}' \notin (D')^{\mathcal{I}}$.

By Question 2, $([\mathfrak{a}'], [\mathfrak{b}']) \in r^{\mathcal{J}}$ and therefore $([\mathfrak{a}], [\mathfrak{b}']) \in r^{\mathcal{J}}$. Since X is closed under subconcepts, by the induction hypothesis, we obtain $[\mathfrak{b}'] \notin (D')^{\mathcal{J}}$ too. Hence, $[\mathfrak{a}] \notin (\forall r.D')^{\mathcal{J}}$, which leads to contradiction.

4. First, the concept constructors $\sqcup$ and $\exists r$ may occur in $\mathcal{K}, C$ but these occurrences can be eliminated by using duality properties in order to have X satisfying the assumptions from Question 3. Let us show that $\mathcal{I}^\star = \mathcal{J}$ does the job with $X = \mathsf{sub}(\mathcal{K}) \cup \mathsf{sub}(C)$.

- $\Delta^{\mathcal{J}}$ is finite and $\mathsf{card}(\Delta^{\mathcal{J}}) \leq 2^{\mathsf{card}(X)}$ with finite X.
- By Question 3, for all $D \in X$, we have $D^{\mathcal{J}} = \{[\mathfrak{a}] \mid \mathfrak{a} \in D^{\mathcal{I}}\}$.
- Consequently, $C^{\mathcal{I}} \neq \emptyset$ implies $C^{\mathcal{J}} \neq \emptyset$.
- Furthermore, $D^{\mathcal{I}} \subseteq (D')^{\mathcal{I}}$ implies $D^{\mathcal{J}} \subseteq (D')^{\mathcal{J}}$ and therefore for all GCIs $D \sqsubseteq D' \in \mathcal{T}$, we have $\mathcal{I} \models D \sqsubseteq D'$ implies $\mathcal{J} \models D \sqsubseteq D'$. As $\mathcal{I} \models \mathcal{T}$ we get $\mathcal{J} \models \mathcal{T}$.
- Let $a : D \in \mathcal{A}$. By assumption, we have $\mathcal{I} \models a : D$ and therefore $a^{\mathcal{I}} \in D^{\mathcal{I}}$. By Question 3, $[a^{\mathcal{I}}] \in D^{\mathcal{J}}$. By definition of $\mathcal{J}$, $a^{\mathcal{J}} = [a^{\mathcal{I}}]$ and therefore $\mathcal{J} \models a : D$.
- Let $(a, b) : r \in \mathcal{A}$. By assumption, we have $\mathcal{I} \models (a, b) : r$ and therefore $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. By Question 2, $([a], [b]) \in r^{\mathcal{J}}$. As $a^{\mathcal{J}} = [a]$ and $b^{\mathcal{J}} = [b]$ by definition of $\mathcal{J}$, we get $(a^{\mathcal{J}}, b^{\mathcal{J}}) \in r^{\mathcal{J}}$. Therefore $\mathcal{J} \models (a, b) : r$ (by $\mathcal{ALC}$ semantics).
- For all role names r occurring X, $r^{\mathcal{J}}$ is reflexive and transitive by Question 1.