TD: Logical Aspects of Artificial Intelligence
Introduction to Description Logics (14/09/2022)

Exercise 1. (© Ivan Varzinczak 2018) Let $\mathsf{N_C} = \{\mathtt{Man}, \mathtt{Woman}, \mathtt{Parent}, \mathtt{Grandparent}\}$ and $\mathsf{N_R} = \{\mathtt{marriedTo}, \mathtt{hasChild}\}$. Formalize in $\mathcal{ALC}$ the following concepts expressed in natural language.

1. Married women.
2. Fathers married to married women.
3. Men who are single or have unmarried daughters.
4. Men who are single or have unmarried daughters that do not have married sons.
5. Men who have only unmarried daughters.
6. Mothers married to married men or single men who are not parents.
7. Men married to a woman who has only married daughters.
8. Men who are fathers of women who are not married but are mothers.
9. Men or women who are granparents whose children are married men.

Exercise 2. Propose concept names, role names and individual names to express in some knowledge base the properties below. Any feature for description logics is allowed but try to minimize what is out of $\mathcal{ALC}$.

1. Employed students are students and employees.
2. Students are not taxpayers.
3. Employed students are taxpayers.
4. Employed students who are parents are not taxpayers.
5. To work for is to be employed by.
6. John is an employed student, John works for IBM.

Exercise 3. Show that if the concept C in $\mathcal{ALC}$ is valid, then $\forall r.C$ is valid for all role names r.

Exercise 4. Determine which concepts below are satisfiable.

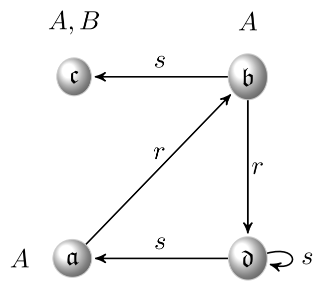
1. $(\neg\exists r.(A \sqcup \neg A)) \sqcup \exists s.\ \bot$.

2. $(\neg\forall r.A) \sqcup \exists r.A$.

3. $(\exists r.A) \sqcap (\exists s.\neg A)$.

4. $(\exists r.C) \sqcap (\forall r.\neg C)$.

Exercise 5. Consider the interpretation $\mathcal{I}$ below.



For the concepts C below, compute $C^{\mathcal{I}}$.

$\neg\exists r.\neg A \sqcap \neg B \quad \forall s.A \quad \exists s.\neg A \quad \exists s.\exists s.\exists s.\exists s.A$

$\forall s^{-}.\exists s.\exists s.\exists s.A \quad (\exists s.(A \sqcap \neg\forall s.\neg B)) \sqcap (\neg\forall r.(\exists r.(A \sqcup \neg A)))$

Exercise 6. Show that $(\mathcal{T}, \mathcal{A}) \models a : C$ iff $(\mathcal{T}, \mathcal{A} \cup \{a : \neg C\})$ is not consistent.

Exercise 7. Consider the $\mathcal{ALC}$ knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with

- $\mathcal{T} = \{A_0 \sqsubseteq \forall r.A_1,\ A_1 \sqsubseteq \neg A_4,\ A_0 \sqsubseteq A_2 \sqcup A_3,\ A_2 \sqsubseteq \exists r.A_4,\ \exists r.\neg A_1 \sqsubseteq A_5,\ A_3 \sqsubseteq A_5\}$,

- $\mathcal{A} = \{a : A_0, (a, b) : r, b : A_4\}$.

(the A_i's are concept names)

1. Do we have $\mathcal{T} \models A_0 \sqsubseteq \exists r.A_1$?

2. Is $\mathcal{K}$ consistent?

Exercise 8. (Tree interpretation property) A tree is understood below as a directed graph (V, E) such that there is a unique root $\mathfrak{r}$ such that there is no $v \in V$ with $(v, \mathfrak{r}) \in E$ and for every node $v \in V \setminus \{\mathfrak{r}\}$, there is a unique node $v' \in V$ such that $(v', v) \in E$.

Let C be an $\mathcal{ALC}$ concept and $\mathcal{T}$ be a TBox. An interpretation $\mathcal{I}$ is a **tree model** for C with respect to $\mathcal{T}$ iff the conditions below hold:

- $\mathfrak{t}_{\mathcal{I}} = (\Delta^{\mathcal{I}}, \bigcup_r r^{\mathcal{I}})$ is a tree,

- the root of $\mathbf{t}_{\mathcal{I}}$ belongs to $C^{\mathcal{I}}$,
- $\mathcal{I} \models \mathcal{T}$.

Given an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, a **path** in $\mathcal{I}$ is a finite sequence $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in (\Delta^{\mathcal{I}})^+$ such that for all $i \in [1, n-1]$, we have $(\mathfrak{a}_i, \mathfrak{a}_{i+1}) \in \bigcup_r r^{\mathcal{I}}$. An $\mathfrak{a}$**-path** is a path such that $\mathfrak{a}_1 = \mathfrak{a}$. Given an interpretation $\mathcal{I}$, its **unravelling** at $\mathfrak{a} \in \Delta^{\mathcal{I}}$ is the interpretation $\mathcal{U} = (\Delta^{\mathcal{U}}, \cdot^{\mathcal{U}})$ such that

- $\Delta^{\mathcal{U}}$ is the set of $\mathfrak{a}$-paths in $\mathcal{I}$.
- For all concept names A, we have $A^{\mathcal{U}} \stackrel{\text{def}}{=} \{(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in \Delta^{\mathcal{U}} \mid \mathfrak{a}_n \in A^{\mathcal{I}}\}$,
- For all role names r, we have $r^{\mathcal{U}} \stackrel{\text{def}}{=} \{((\mathfrak{a}_1, \ldots, \mathfrak{a}_n), (\mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{a}_{n+1})) \mid (\mathfrak{a}_n, \mathfrak{a}_{n+1}) \in r^{\mathcal{I}}\}$.

1. Show that $\mathcal{U}$ is a tree model for $\top$ with respect to the empty TBox.
2. Show that for all concepts C and all $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in \Delta^{\mathcal{U}}$, we have $(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in C^{\mathcal{U}}$ iff $\mathfrak{a}_n \in C^{\mathcal{I}}$.
3. Conclude that if C is satisfiable with respect to the TBox $\mathcal{T}$, then C has a tree interpretation with respect to $\mathcal{T}$.
4. Determine whether if C is satisfiable with respect to the TBox $\mathcal{T}$, then C has always a *finite tree* interpretation with respect to $\mathcal{T}$.

Exercise 9. Let X be a non-empty set with a distinguished element $x_0 \in X$ and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation for $\mathcal{ALC}$. Let $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be the interpretation defined as follows.

- $\Delta^{\mathcal{J}} \stackrel{\text{def}}{=} X \times \Delta^{\mathcal{I}}$.
- $A^{\mathcal{J}} \stackrel{\text{def}}{=} X \times A^{\mathcal{I}}$ for every concept name A.
- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \{((x, \mathfrak{a}), (x, \mathfrak{b})) \mid x \in X, (\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}\}$ for every role name r.
- $a^{\mathcal{J}} \stackrel{\text{def}}{=} (x_0, \mathfrak{a})$ with $a^{\mathcal{I}} = \mathfrak{a}$, for every individual name a.

1. For all $\mathcal{ALC}$ concepts C, show that $C^{\mathcal{J}} = X \times C^{\mathcal{I}}$.
2. Given a knowledge base $\mathcal{K}$, show that $\mathcal{I} \models \mathcal{K}$ implies $\mathcal{J} \models \mathcal{K}$.
3. Conclude that there is no consistent $\mathcal{ALC}$ knowledge base $\mathcal{K}$ such that for all interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, $\mathcal{I} \models \mathcal{K}$ implies $\Delta^{\mathcal{I}}$ is finite.

Correction: Exercise 1

1. Married women: $\texttt{Woman} \sqcap \exists \texttt{marriedTo}.\top$.
2. Fathers married to married women.

   $$(\exists \texttt{hasChild}.\top) \sqcap \texttt{Man} \sqcap (\exists \texttt{marriedTo}.(\texttt{Woman} \sqcap \exists \texttt{marriedTo}.\top)).$$

3. Men who are single or have unmarried daughters.

   $$\texttt{Man} \sqcap (\neg \exists \texttt{marriedTo}.\top \sqcup \exists \texttt{hasChild}.(\texttt{Woman} \sqcap \neg \exists \texttt{marriedTo}.\top).)$$

4. Men who are single or have unmarried daughters that do not have married sons.

   $$\texttt{Man} \sqcap (\neg \exists \texttt{marriedTo}.\top \sqcup$$

   $$\exists \texttt{hasChild}.(\texttt{Woman} \sqcap \neg \exists \texttt{marriedTo}.\top \sqcap \forall \texttt{hasChild}.(\texttt{Man} \Rightarrow \neg \exists \texttt{marriedTo}.\top)).)$$

5. Men who have only unmarried daughters.

   $$\texttt{Man} \sqcap \forall \texttt{hasChild}.(\texttt{Woman} \Rightarrow \neg \exists \texttt{marriedTo}.\top)$$

6. Mothers married to married men or single men who are not parents.

   $$(\texttt{Woman} \sqcap \exists \texttt{hasChild}.\top) \sqcap \exists \texttt{marriedTo}.(\texttt{Man} \sqcap (\neg \exists \texttt{marriedTo}.\top \Rightarrow \forall \texttt{hasChild}.\bot))$$

7. Men married to a woman who has only married daughters.

   $$\texttt{Man} \sqcap \exists \texttt{marriedTo}.(\texttt{Woman} \sqcap \forall \texttt{hasChild}.(\texttt{Woman} \Rightarrow \exists \texttt{marriedTo}.\top))$$

8. Men who are fathers of women who are not married but are mothers.

   $$\texttt{Man} \sqcap \exists \texttt{hasChild}.(\texttt{Woman} \sqcap \neg \exists \texttt{marriedTo}.\top \sqcap \exists \texttt{hasChild}.\top)$$

9. Men or women who are granparents whose children are married men.

   $$(\texttt{Men} \sqcup \texttt{Women}) \sqcap \exists \texttt{hasChild}.(\exists \texttt{hasChild}.\top) \sqcap \forall \texttt{hasChild}.(\texttt{Man} \sqcap \exists \texttt{marriedTo}.\top)$$

Correction: Exercise 2

- $\mathsf{N_C} = \{\texttt{Student}, \texttt{Tax}, \texttt{EmployedStudent}\}$.

- $\mathsf{N_R} = \{\texttt{Pays}, \texttt{HasChild}, \texttt{WorksFor}, \texttt{EmployedBy}\}$.
- $\mathsf{N_I} = \{\texttt{John}, \texttt{IBM}\}$.

1. Employed students are students and employees.

$$\texttt{EmployedStudent} \equiv \texttt{Student} \sqcap \exists\texttt{WorksFor}.\top$$

2. Students are not taxpayers: $\texttt{Student} \sqsubseteq \neg\exists\texttt{Pays}.\texttt{Tax}$.
3. Employed students are taxpayers: $\texttt{EmployedStudent} \sqsubseteq \exists\texttt{Pays}.\texttt{Tax}$.
4. Employed students who are parents are not taxpayers.

$$\texttt{EmployedStudent} \sqcap \exists\texttt{HasChild}.\top \sqsubseteq \neg\exists\texttt{Pays}.\texttt{Tax}$$

5. To work for is to be employed by. We can use here role inclusion axioms (RIA), most probably to be seen in Lecture 2.

$$\texttt{WorksFor} \sqsubseteq \texttt{EmployedBy} \text{ and } \texttt{EmployedBy} \sqsubseteq \texttt{WorksFor}$$

6. John is an employed student, John works for IBM.

$$\texttt{John} : \texttt{EmployedStudent} \text{ and } (\texttt{John}, \texttt{IBM}) : \texttt{WorksFor}$$

Correction: Exercise 3

Let us show that if the concept C in $\mathcal{ALC}$ is valid, then $\forall r.C$ is valid for all role names r.

Assume that C is valid. *Ad absurdum*, let us suppose that $\forall r.C$ is not valid. So, there is an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $(\forall r.C)^{\mathcal{I}} \neq \Delta^{\mathcal{I}}$. Let $\mathfrak{a} \in (\Delta^{\mathcal{I}} \setminus (\forall r.C)^{\mathcal{I}})$ and therefore $\mathfrak{a} \in (\neg\forall r.C)^{\mathcal{I}}$. By duality between $\exists$ and $\forall$, we have $\mathfrak{a} \in (\exists r.\neg C)^{\mathcal{I}}$. Consequently, there is $\mathfrak{b} \in r^{\mathcal{I}}(\mathfrak{a})$ such that $\mathfrak{b} \in (\neg C)^{\mathcal{I}}$. As $\mathfrak{b} \notin C^{\mathcal{I}}$, this leads to a contradiction with the validity of C.

Note that the property holds for extensions of $\mathcal{ALC}$ too, as the proof above does not use too many features from $\mathcal{ALC}$.

Correction: Exercise 4

1. Let us show that $(\neg\exists r.(A \sqcup \neg A)) \sqcup \exists s.\ \bot.$ is satisfiable. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{\mathfrak{a}\}$ and $\cdot^{\mathcal{I}}$ returns an empty set for all concept names and role names. We have

- $\mathfrak{a} \in (A \sqcup \neg A)^{\mathcal{I}}$ ($A \sqcup \neg A$ is logically equivalent to $\top$).
- $\mathfrak{a} \in (\neg\exists r.(A \sqcup \neg A))^{\mathcal{I}}$ because $r^{\mathcal{I}} = \emptyset$.
- Consequently, $\mathfrak{a} \in ((\neg\exists r.(A \sqcup \neg A)) \sqcup \exists s.\perp)^{\mathcal{I}}$.

2. Let us show that $(\neg\forall r.A) \sqcup \exists r.A$ is satisfiable.

   Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{\mathfrak{a}, \mathfrak{b}\}$, $r^{\mathcal{I}} = \{(\mathfrak{a}, \mathfrak{b})\}$ and $A^{\mathcal{I}} = \{\mathfrak{b}\}$. We have

   - $\mathfrak{a} \in (\exists r.A)^{\mathcal{I}}$ as $r^{\mathcal{I}} = \{(\mathfrak{a}, \mathfrak{b})\}$ and $\mathfrak{b} \in A^{\mathcal{I}}$.
   - Consequently, $\mathfrak{a} \in ((\neg\forall r.A) \sqcup \exists r.A)^{\mathcal{I}}$.

3. Let us show that $(\exists r.A) \sqcap (\exists s.\neg A)$ is satisfiable.

   Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2\}$, $r^{\mathcal{I}} = \{(\mathfrak{a}_0, \mathfrak{a}_1)\}$, $s^{\mathcal{I}} = \{(\mathfrak{a}_0, \mathfrak{a}_2)\}$, and $A^{\mathcal{I}} = \{\mathfrak{a}_1\}$. We have

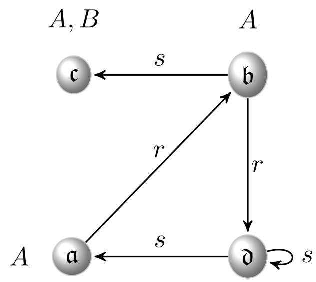
   - $\mathfrak{a}_0 \in (\exists r.A)^{\mathcal{I}}$ as $r^{\mathcal{I}} = \{(\mathfrak{a}_0, \mathfrak{a}_1)\}$ and $\mathfrak{a}_1 \in A^{\mathcal{I}}$.
   - $\mathfrak{a}_0 \in (\exists s.\neg A)^{\mathcal{I}}$ as $s^{\mathcal{I}} = \{(\mathfrak{a}_0, \mathfrak{a}_2)\}$ and $\mathfrak{a}_2 \notin A^{\mathcal{I}}$.
   - Consequently, $\mathfrak{a}_0 \in ((\exists r.A) \sqcap (\exists s.\neg A))^{\mathcal{I}}$ since

     $$((\exists r.A) \sqcap (\exists s.\neg A))^{\mathcal{I}} = (\exists r.A)^{\mathcal{I}} \cap (\exists s.\neg A)^{\mathcal{I}}.$$

4. Let us show that $(\exists r.C) \sqcap (\forall r.\neg C)$ is *not* satisfiable.

   Ad absurdum, suppose that $\mathfrak{a} \in ((\exists r.C) \sqcap (\forall r.\neg C))^{\mathcal{I}}$ for some interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. As $\mathfrak{a} \in (\exists r.C)^{\mathcal{I}}$, there is $\mathfrak{b} \in \Delta^{\mathcal{I}}$ such that $\mathfrak{b} \in C^{\mathcal{I}}$ and $(\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}$. As $\mathfrak{a} \in (\forall r.\neg C)^{\mathcal{I}}$, for all $\mathfrak{b}'$ such that $(\mathfrak{a}, \mathfrak{b}') \in r^{\mathcal{I}}$, we have $\mathfrak{b}' \notin C^{\mathcal{I}}$. In particular, $\mathfrak{b} \notin C^{\mathcal{I}}$, which leads to a contradiction.

Correction: Exercise 5



- $(\neg\exists r.(\neg A \sqcap \neg B))^{\mathcal{I}} = \{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}$.
- $(\forall s.A)^{\mathcal{I}} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$.
- $(\exists s.\neg A)^{\mathcal{I}} = \{\mathfrak{d}\}$.
- $(\exists s.\exists s.\exists s.\exists s.A)^{\mathcal{I}} = \{\mathfrak{d}\}$.
- $(\forall s^{-}.\exists s.\exists s.\exists s.A)^{\mathcal{I}} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}\}$.
- $(\exists s.(A \sqcap \neg\forall s.\neg B)) \ \sqcap \ (\neg\forall r.(\exists r.(A \sqcup \neg A)))^{\mathcal{I}} = \emptyset$ since already

$$(\exists s.(A \sqcap \neg\forall s.\neg B))^{\mathcal{I}} = \emptyset.$$

Correction: Exercise 6

Let us show that $(\mathcal{T}, \mathcal{A}) \models a : C$ iff $(\mathcal{T}, \mathcal{A} \cup \{a : \neg C\})$ is not consistent.

First assume that $(\mathcal{T}, \mathcal{A}) \models a : C$. *Ad absurdum*, suppose that $(\mathcal{T}, \mathcal{A} \cup \{a : \neg C\})$ is consistent. This means that there is an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $\mathcal{I} \models \mathcal{T}$, $\mathcal{I} \models \mathcal{A}$ and $\mathcal{I} \not\models a : C$. This leads to a contradiction as $(\mathcal{T}, \mathcal{A}) \models a : C$.

Now suppose that $(\mathcal{T}, \mathcal{A} \cup \{a : \neg C\})$ is not consistent. *Ad absurdum*, suppose that not $(\mathcal{T}, \mathcal{A}) \models a : C$. So, there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$ and $\mathcal{I} \models a : \neg C$. Consequently, $(\mathcal{T}, \mathcal{A} \cup \{a : \neg C\})$ is consistent, which leads to a contradiction.

Correction: Exercise 7

- $\mathcal{T} = \{A_0 \sqsubseteq \forall r.A_1,\ A_1 \sqsubseteq \neg A_4,\ A_0 \sqsubseteq A_2 \sqcup A_3,\ A_2 \sqsubseteq \exists r.A_4,\ \exists r.\neg A_1 \sqsubseteq A_5,\ A_3 \sqsubseteq A_5\}$,
- $\mathcal{A} = \{a : A_0, (a, b) : r, b : A_4\}$.

(the A_i's are concept names)

1. We have $\mathcal{T} \not\models A_0 \sqsubseteq \exists r.A_1$ iff there is some interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $\mathcal{I} \models \mathcal{T}$ and there is $\mathfrak{a} \in \Delta^{\mathcal{I}}$ such that $\mathfrak{a} \in A_0^{\mathcal{I}}$ and $\mathfrak{a} \notin (\exists r.A_1)^{\mathcal{I}}$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{0\}$, $r^{\mathcal{I}} = \emptyset$ for all role names r, $A_0^{\mathcal{I}} \stackrel{\text{def}}{=} \{0\}$, $A_3^{\mathcal{I}} \stackrel{\text{def}}{=} \{0\}$, $A_5^{\mathcal{I}} \stackrel{\text{def}}{=} \{0\}$, and $B^{\mathcal{I}} = \emptyset$ for all other concept names B.
    - We have $0 \in A_0^{\mathcal{I}}$ and $0 \notin \exists r.A_1^{\mathcal{I}}$.
    - $\mathcal{I} \models A_0 \sqsubseteq \forall r.A_1$ because $A_0^{\mathcal{I}} = (\forall r.A_1)^{\mathcal{I}} = \{0\}$.
    - $\mathcal{I} \models A_1 \sqsubseteq \neg A_4$ because $A_1^{\mathcal{I}} = \emptyset$.
    - $\mathcal{I} \models A_0 \sqsubseteq A_2 \sqcup A_3$ because $A_0^{\mathcal{I}} = (A_2 \sqcup A_3)^{\mathcal{I}} = \{0\}$.
    - $\mathcal{I} \models A_2 \sqsubseteq \exists r.A_4$ because $A_2^{\mathcal{I}} = \emptyset$.
    - $\mathcal{I} \models \exists r.\neg A_1 \sqsubseteq A_5$ because $(\exists r.\neg A_1)^{\mathcal{I}} = \emptyset$ and $A_5^{\mathcal{I}} = \{0\}$.
    - $\mathcal{I} \models A_3 \sqsubseteq A_5$ because $A_3^{\mathcal{I}} = A_5^{\mathcal{I}} = \{0\}$.

    Consequently, $\mathcal{I} \models \mathcal{T}$ and therefore $\mathcal{T} \models A_0 \sqsubseteq \exists r.A_1$ does not hold.

2. *Ad absurdum*, suppose that $\mathcal{K}$ is consistent. So, there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$ and in particular
    (a) $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$,
    (b) $a^{\mathcal{I}} \in A_0^{\mathcal{I}}$,
    (c) $b^{\mathcal{I}} \in A_4^{\mathcal{I}}$,
    (d) $\mathcal{I} \models \{A_0 \sqsubseteq \forall r.A_1,\ A_1 \sqsubseteq \neg A_4\}$.

    As (a) and (b), $\mathcal{I} \models A_0 \sqsubseteq \forall r.A_1$ entails $b^{\mathcal{I}} \in A_1^{\mathcal{I}}$. As $\mathcal{I} \models A_1 \sqsubseteq \neg A_4$, we conclude $b^{\mathcal{I}} \notin A_4^{\mathcal{I}}$, which is in contradiction with (c). Consequently, $\mathcal{K}$ is not consistent.

Correction: Exercise 9

1. The proof is by structural induction. For the base case, we have $A^{\mathcal{J}} = X \times A^{\mathcal{I}}$ by definition. Similarly, $\top^{\mathcal{J}} = X \times \Delta^{\mathcal{I}}$ with $\Delta^{\mathcal{I}} = \top^{\mathcal{I}}$. For the induction step, we only treat the cases with $\neg$, $\sqcap$ and $\exists r.C$ thanks to the duality between $\sqcup$ and $\sqcap$ (resp. between $\exists r\cdot$ and $\forall r\cdot$).

   Case $C = \neg D$.
   - $C^{\mathcal{J}} = (X \times \Delta^{\mathcal{I}}) \setminus D^{\mathcal{J}}$ ($\mathcal{ALC}$ semantics).
   - $C^{\mathcal{J}} = (X \times \Delta^{\mathcal{I}}) \setminus (X \times D^{\mathcal{I}})$ (by induction hypothesis).
   - $C^{\mathcal{J}} = X \times (\Delta^{\mathcal{I}} \setminus D^{\mathcal{I}})$ (by set-theoretical reasoning).
   - $C^{\mathcal{J}} = X \times C^{\mathcal{I}}$ ($\mathcal{ALC}$ semantics).

   Case $C = D_1 \sqcap D_2$.
   - $C^{\mathcal{J}} = D_1^{\mathcal{J}} \cap D_2^{\mathcal{J}}$ ($\mathcal{ALC}$ semantics).
   - $C^{\mathcal{J}} = X \times D_1^{\mathcal{I}} \cap X \times D_2^{\mathcal{I}}$ (by induction hypothesis).
   - $C^{\mathcal{J}} = X \times (D_1^{\mathcal{I}} \cap D_2^{\mathcal{I}})$ (by set-theoretical reasoning).
   - $C^{\mathcal{J}} = X \times (D_1 \sqcap D_2)^{\mathcal{I}}$ ($\mathcal{ALC}$ semantics).

   Case $C = \exists r.D$.
   Let $(x, \mathfrak{a}) \in C^{\mathcal{J}}$. There is $(x', \mathfrak{a}')$ such that $((x, \mathfrak{a}), (x', \mathfrak{a}')) \in r^{\mathcal{J}}$ and $(x', \mathfrak{a}') \in D^{\mathcal{J}}$ (by $\mathcal{ALC}$ semantics). By definition of $r^{\mathcal{J}}$, we have $x = x'$ and $(\mathfrak{a}, \mathfrak{a}') \in r^{\mathcal{I}}$ and by induction hypothesis $\mathfrak{a}' \in D^{\mathcal{I}}$. By $\mathcal{ALC}$ semantics, $\mathfrak{a} \in C^{\mathcal{I}}$.
   Conversely, let x be an arbitrary element of X and suppose that $\mathfrak{a} \in C^{\mathcal{I}}$. So, there is $\mathfrak{a}' \in \Delta^{\mathcal{I}}$ such that $(\mathfrak{a}, \mathfrak{a}') \in r^{\mathcal{I}}$ and $\mathfrak{a}' \in D^{\mathcal{I}}$. By definition of $r^{\mathcal{J}}$, we have $((x, \mathfrak{a}), (x, \mathfrak{a}')) \in r^{\mathcal{J}}$ and $(x, \mathfrak{a}') \in D^{\mathcal{J}}$ by induction hypothesis. By $\mathcal{ALC}$ semantics, we can conclude that $(x, \mathfrak{a}) \in C^{\mathcal{J}}$.

2. Suppose that $\mathcal{I} \models \mathcal{K}$.

   Case $C \sqsubseteq D \in \mathcal{T}$. By assumption, $\mathcal{I} \models C \sqsubseteq D$, i.e. $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. By set-theoretical reasoning, $X \times C^{\mathcal{I}} \subseteq X \times D^{\mathcal{I}}$ and therefore by Question 1, $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ (equivalent to $\mathcal{J} \models C \sqsubseteq D$).

   Case $a : C \in \mathcal{A}$. By assumption, $\mathcal{I} \models a : C$, i.e. $a^{\mathcal{I}} \in C^{\mathcal{I}}$. By definition of $\mathcal{J}$, $a^{\mathcal{J}} = (x_0, a^{\mathcal{I}})$ and $C^{\mathcal{J}} = X \times C^{\mathcal{I}}$ by Question 1. Hence $a^{\mathcal{J}} \in C^{\mathcal{J}}$ (equivalent to $\mathcal{J} \models a : C$).

   Case $(a, b) : r \in \mathcal{A}$. By assumption, $\mathcal{I} \models (a, b) : r$, i.e. $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. By definition of $\mathcal{J}$,
   - $((x_0, a^{\mathcal{I}}), (x_0, b^{\mathcal{I}})) \in r^{\mathcal{J}}$,
   - $a^{\mathcal{J}} = (x_0, a^{\mathcal{I}})$ and $b^{\mathcal{J}} = (x_0, b^{\mathcal{I}})$.

By $\mathcal{ALC}$ semantics, $\mathcal{J} \models (a, b) : r$.

3. *Ad absurdum*, suppose that there is a consistent $\mathcal{ALC}$ knowledge base $\mathcal{K}$ such that for all interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, $\mathcal{I} \models \mathcal{K}$ implies $\Delta^{\mathcal{I}}$ is finite. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models \mathcal{K}$ ($\mathcal{K}$ is assumed to be consistent). Let $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be the interpretation defined as follows with $X = \mathbb{N}$:

   - $\Delta^{\mathcal{J}} \stackrel{\text{def}}{=} \mathbb{N} \times \Delta^{\mathcal{I}}$ (so $\Delta^{\mathcal{J}}$ is infinite).
   - $A^{\mathcal{J}} \stackrel{\text{def}}{=} \mathbb{N} \times A^{\mathcal{I}}$ for every concept name A.
   - $r^{\mathcal{J}} \stackrel{\text{def}}{=} \{((x, \mathfrak{a}), (x, \mathfrak{b})) \mid\ x \in \mathbb{N},\ (\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}\}$ for every role name r.
   - $a^{\mathcal{J}} \stackrel{\text{def}}{=} (0, \mathfrak{a})$ with $a^{\mathcal{I}} = \mathfrak{a}$, for every individual name a.

   By Question 2, we have $\mathcal{J} \models \mathcal{K}$ and $\mathcal{J}$ has an infinite domain, which leads to a contradiction.