2. A *rotary chain system* is a set of links connected by smoothly rotating joints. Figure 2 shows an example of such a system with 3 links (AB, BC and CD) connected by unpowered rotary joints in a single plane. If an impulse is applied to the system ($\mathbf{J}_0$) then the effect is for equal and opposite impulses to be applied across all of the joints ($\mathbf{J}_i$ and $-\mathbf{J}_i$) that stop the joints from breaking apart. (All the $\mathbf{J}_i$ are in the same plane as the rotary chain system.)

   (a) In the general case a rotary chain system may be made up of n links, with the vector along link i being given by $\mathbf{l}_i$. (So in Figure 2(a) the vector along the first link $\mathbf{l}_1$ is the vector from A to B.) The mass of each link i is m_i and its moment of inertia is $\frac{1}{12}m_i\mathbf{l}_i^2$. If the system starts from rest and then a single known impulse $\mathbf{J}_0$ is applied to the end of the first link, let the linear velocity of the centres of mass just after the impulse be $\mathbf{v}_i$ and the angular velocities ω_i. Write down three sets of equations relating

   - linear velocity and momentum
   - angular velocity and momentum
   - joint constraints

   and hence explain why it should be possible to solve these equations in the general case. (You are *not* expected to show that the system is always invertible. Recall that the torque τ about the origin generated by a force $\mathbf{f}$ which is applied through a point $\mathbf{x}$ is given by $\tau = \mathbf{x} \times \mathbf{f}$ where $\times$ is the vector cross-product operator.) [*5 marks*]

   (b) Now consider the specific case of a two-link rotary chain system shown in Figure 2(b), where the coordinates of the three points shown are as follows: $O = (0, 0, 0)$, $E = (0, 0, 6)$ and $F = (0, 3\sqrt{2}, 6 + 3\sqrt{2})$. The masses of the 2 links are both 1kg, and the system starts from rest when a single impulse $\mathbf{J}_0 = (0, 0, J)$ is applied to O where $J = 41$Ns. (1Ns is the total impulse generated by a force of 1N applied consistently over 1 second; a 1N force causes a 1kg mass to accelerate at 1ms^{-2}.) Solve the system of equations, and hence show that the point F has a velocity of (0,18,-2)ms^{-1} immediately after the impact. [*7 marks*]

   (c) In (b) you solved the system for a specific value of $\mathbf{J}_0$. Show that if you have generated solutions for the system for two specific values of $\mathbf{J}_o$ — say $\mathbf{J}_o = \mathbf{J}^*$ and $\mathbf{J}_o = \mathbf{J}^\dagger$ — then it is possible to write down the solution for another specific value of $\mathbf{J}_0 = \alpha\mathbf{J}^* + \beta\mathbf{J}^\dagger$ for arbitrary scalars α and β. [*3 marks*]

(d) Outline an algorithm for the general case of an n-link rotary chain system that can solve the system dynamics in time linear in n after an arbitrary impulse $\mathbf{J}_0$ is applied to it. [5 marks]

(e) On another occasion the rotary chain system OEF is falling freely under gravity when the end point O hits a solid table whose top surface is in the plane $z = 0$, and undergoes an elastic collision with coefficient of restitution $e = 0.5$. Just before the moment of impact the coordinates of the 3 points O, E and F are as given above, and both links have zero angular velocity and linear velocity (0,0,-7)ms^{-1}. What is the linear velocity of the point F immediately after impact? What can you say about the value of the linear acceleration of F immediately after the impact? [5 marks]

In the above gravity is acting in the negative z-direction, and you may take its value to be 10ms^{-2}. The numerical examples should be simple enough to be solved by hand with the aid of a calculator, and if you do so you should show enough of your working to make it clear in which order you performed the calculations. It is permissible to use a computer program instead; in this case you should provide a commented listing of any computer code that you write yourself, and explain which software packages you used if you did so.