Analysis I (Tao)

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Exercises 2.1

Ex. 2.2.1 Prove the associativity of addition.

We begin with two natural numbers a and b. We consider the base case where $c = 0$, and prove that $(a + b) + c = a + (b + c)$. Because it has been proved that $0 + m = m + 0 = m$ for any natural m, $(a + b) + c = a + b$ and $a + (b + c) = a + b$, the base case is correct.

Now we check whether $(a+b)+(c{+}{+}) = a+(b+(c{+}{+}))$. By the definition of addition (for which it has been proved that $(n{+}{+})+m = (n+m){+}{+} = n+(m{+}{+})$), the LHS is $((a+b)+c))++$, while the RHS is $a+(b+c)++ = (a+(b+c))++$, from which it follows from the Peano axioms that $a+(b+c) = (a+(b+c))$. We know this is true from the base case, so $(a + b) + (c + +)$ and $a + (b + (c + +))$ are indeed equal to each other. Our inductive hypothesis is vindicated, and so associativity is true.

Ex. 2.2.2 $\forall a$ where $a \in \mathbb{P}$, there exists a single $b \in \mathbb{N}$ such that $b + + = a$.

0, the first natural number, has successor 1. There exist no positive numbers before 1, so $0 + + = 1$ is the base case. We now consider $((b + +)) + + = a + +$. By the Peano axioms $((b + +)) + + = a + + \implies b + + = a$, which is true for the base case, the inductive hypothesis is justified.

Ex. 2.2.3 This exercise asks us to prove the order properties of the naturals.

a) (Order is Reflexive) By definition, $a \geq a$ iff $a = a + m$. We suppose that $m = 0$ and use induction to prove that the equality holds. Our base case assumes that $a = 0$, from which it follows that $0 = 0 + 0$, and so we have our base case. We increment a to get $a + + = (a + +) + m$. By addition, $(a + +) + m = (a + m) + + = a + +$ (Since $m = 0$), and so the inductive hypothesis holds.

b) (Order is Transitive) e.g. $(a \geq b) \wedge (b \geq c) \implies a \geq c$ $((a = b) \vee (b = c)) \implies a \geq c$, and $a = c$ is impossible unless $a = b = c$, so the only remaining case to consider is $a \neq b \neq c$. In this case $a = b + m$ and $b = c + k$, so

$a = (c + k) + m$, and by associativity $a = c + (k + m) \implies a \geq c$.

c) (Order is antisymmetric) If $a \geq b \implies a = b + m$ and $b \geq a \implies b = a + k$, then $a = (a + k) + m$ and $b = (b + m) + k$, both of which are contradictions if $k \vee m \in \mathbb{P}$, as that would imply $(b = b + m = c) \vee (b = b + k = d)$, where c and d are positive, since the sum of positive and a natural is positive. But $b \neq c$, since at the minimum $c = b + +$

We show that m and k must both be 0, and show this directly. $b = b + (m + k)$ and $a = a + (k + m)$ by applying associativity to the above. Since $b + 0 = b = b + (m + k) \implies 0 = (m + k)$ by the cancellation law (Prop. 2.2.6 in Tao). By the axiom of substitution, the above also applies for a, and so $b = a + 0$ and $a = b + 0$, which must both be true if $a \geq b$ and $b \geq a$, from which it follows that $a = b$.

d) $a \geq b$ iff $a + c \geq b + c$ If $a + c \geq b + c$, then $a + c = b + c + k = (b + k) + c \implies a = (b + k) \implies a \geq b$ by associativity, commutativity, and cancelation.

If $a \geq b$, then $a = b + m$ and $a + c = b + m + c = b + c + m$, which implies $a = c \geq b + c$.

Having proven the conditional in both directions, we can conclude that $a \geq b$ iff $a + c \geq b + c$.

e) $b > a$ iff $b \geq a + +$ First things first, $b \geq a + +$ can be written as $b = (a + +) + k = a + (k + +)$.

If $b = a + m, m \neq 0 \iff b = a + (k + +)$, then $a + m = a + (k + +) \implies m = k + +$. We have proven $a = b + +$ for any positive a and its accompanying natural b, and since m must be positive and k must be natural, the axiom of substitution $m = k + +$ holds, so for ever m in the LHS we can find a k in the RHS that satisfies the equality, and the biconditional is proven.

Ex. 2.2.4 Justify the three statements marked (why?) in the proof of Proposition 2.2.13, which assist in proving the trichotomy of order.

i) $b \geq 0$ for all b. $b \geq 0 \implies b = 0 + m = m$. This is true for our base case where $m = 0$, since b is assumed to be any natural number. $b + + = 0 + m + + = m + + \implies b = m$, proving the inductive hypothesis.

ii) If $a > b \implies a + + > b$. $a > b \implies a = b + m, m \neq 0$, and so $a + + = (b + m) + + = b + (m + +)$. Now $a + + > b \implies a + + = b + k, k \neq 0$. The implication then holds if we can ensure that $b + (m + +) = b + k$, for some positive m and k. Our base case is that $m + +$, and k are positive if $m = 0$, such that $0 + + = k$. Then $(m + +) + + = k + + \implies m + + = k$, which is true from our base case, so we can always find our necessary m and k, and the proof is done.

iii) If $a = b$ then $a++ > b$ $a++ > b \implies a++ = b+m, m \neq 0$. We induct on a and b starting with the base case $a = b = 0$, so that $a++ > b$ becomes $0++ = 0+m = m$, so the base case holds, sice 0++ is positive. We consider $(a++)++ = (b++)+m = (b+m)++ \implies a++ = b+m$, so the induction hypothesis is true.

Ex. 2.3.1 I have divided the solution into three parts:

i) We prove first that $n \times 0 = 0$. Our base case, where $n = 0$, is identical to the statement $0 \times m = 0$, so the abse case is cleared. Now we write $(n++) \times 0 = (n \times 0) + 0 = 0$, so we have proven the inductive hypothesis.

ii) Now prove that $n \times (m++) = (n \times m) + n$. Our base case presumes that $n = 0$, such that $0 \times (m++) = (0 \times m) + 0$. By the definition of multiplication, both the LHS and RHS equal 0, so the base case is established. Now we write $(n++) \times (m++) = ((n++) \times m) + (n++)$. The LHS is, by the definition of multiplication and by the induction hypothesis, $(n \times (m{+}{+})) + (m{+}{+}) = (n \times m) + n + (m{+}{+})$. The RHS is $(n \times m) + m + (n{+}{+})$. Now $(n \times m) + n + (m++) = (n \times m) + (m++) + n = (n \times m) + m + (n++)$, so LHS and RHS are equal and the induction hypothesis is done

iii) Finally we show that $n \times m = m \times n$ If $n = 0$, we know from *i*) that $n \times m = m \times n$, so we are left showing that $(n++) \times m = m \times (n++)$, i.e. $(n \times m) + m = (m \times n) + m$. By the Cancellation Law and the base case, the equality holds, and the proof is done.

Ex. 2.3.2 If $n, m \in \mathbb{N}$, then $n \times m = 0$ iff $(n = 0) \vee (m = 0)$, and if $n, m \in \mathbb{P} \implies nm \in P$.

$(n = 0) \vee (m = 0) \implies nm = 0$. If $nm = 0$, we only need to consider the case were $(n \neq 0) \wedge (m \neq 0)$ to prove the biconditional, as we have considered the others. Since $nm = 0$ only if $(n = 0) \vee (m = 0)$ by the definition of multiplication, it follows that nm cannot equal 0, so it must be positive, as nm is not of undefined type.

Ex. 2.3.3 Prove that $(ab)c = a(bc)$ We hold a and b fixed and set c to 0, in which case we get $(ab)0 = 0$ and $a(bc) = a \times 0 = 0$, so we have our base case. Now we generalise with $(ab)(c++) = a(b(c++))$. The LHS is $((a \times b) \times c) + (a \times b)$, the RHS is $a \times ((b \times c) + b) = (a \times (b \times c)) + (a \times b)$, and they equal each other because $((a \times b) \times c) + (a \times b) = (a \times (b \times c)) + (a \times b)$, which is true because of our base case

Ex. 2.3.4 Prove that $(a+b)^2 = a^2 + 2ab + b^2, \forall a, b \in \mathbb{N}$ $m^{n++} = m^n \times m$ by the definition of exponentiation, ergo $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b)$, since $(a+b) = c$, where c is some natural number, so we can write $(a+b)^2 =$

$c(a+b) = ca + cb$, and by commutativity we get $a(a+b) + b(a+b)$, and also $a(a+b) + b(a+b) = aa + ab + ba + bb$, and by the defintion of exponentiation $bb = b^1 b = b^{1++} = b^2$, which also applies to a by substitution.