Mining Data Streams (Part 2)

CS246: Mining Massive Datasets
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Today's Lecture

- **More algorithms for streams:**
  - **(1) Filtering a data stream: Bloom filters**
    - Select elements with property **x** from stream
  - **(2) Counting distinct elements: Flajolet-Martin**
    - Number of distinct elements in the last ***k*** elements of the stream
  - **(3) Estimating moments: AMS method**
    - Estimate std. dev. of last ***k*** elements
  - **(4) Counting frequent items**

(1) Filtering Data Streams

Filtering Data Streams

- **Each element of data stream is a tuple**
- Given a list of keys **S**
- **Determine which tuples of stream are in *S***

- **Obvious solution: Hash table**
  - But suppose we **do not have enough memory** to store all of ***S*** in a hash table
    - E.g., we might be processing millions of filters on the same stream

Applications

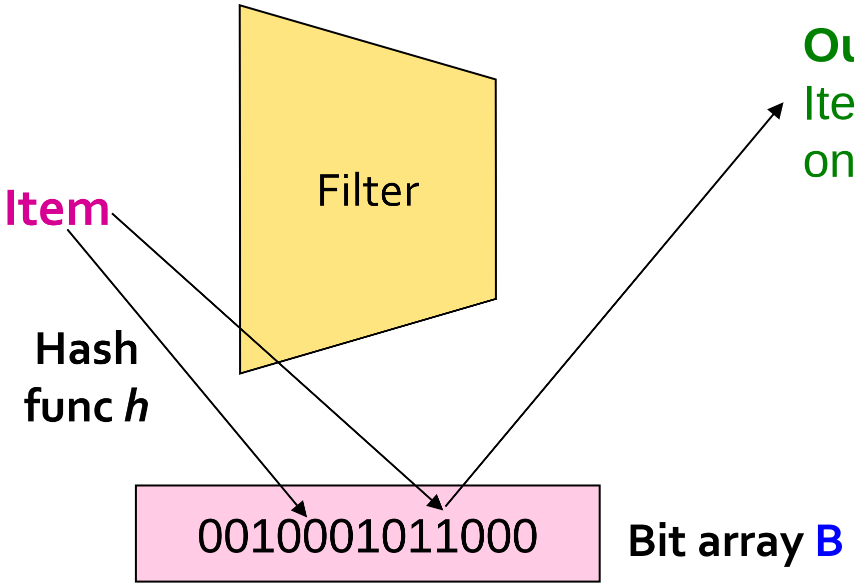
- **Example: Email spam filtering**
  - We know 1 billion "good" email addresses
  - If an email comes from one of these, it is **NOT** spam
- **Publish-subscribe systems**
  - You are collecting lots of messages (news articles)
  - People express interest in certain sets of keywords
  - Determine whether each message matches user's interest

First Cut Solution (1)

Given a set of keys S that we want to filter

- Create a **bit array B** of n bits, initially all 0s
- Choose a **hash function h** with range $[0,n)$
- Hash each member of $s \in S$ to one of n buckets, and set that bit to **1**, i.e., $B[h(s)]=1$
- Hash each element a of the stream and output only those that hash to bit that was set to **1**
  - **Output a if B[h(a)] == 1**

First Cut Solution (2)



Output the item since it may be in *S*.
Item hashes to a bucket that at least one of the items in **S** hashed to.

Drop the item.
It hashes to a bucket set to **0** so it is surely not in ***S***.

- **Creates false positives but no false negatives**
  - If the item is in ***S*** we surely output it, if not we may still output it

First Cut Solution (3)

- **|S| = 1 billion email addresses**
  |B|= 1GB = 8 billion bits
- If the email address is in ***S***, then it surely hashes to a bucket that has the big set to **1**, so it always gets through (***no false negatives***)
- Approximately **1/8** of the bits are set to **1**, so about **1/8th** of the addresses not in ***S*** get through to the output (***false positives***)
  - Actually, less than **1/8th**, because more than one address might hash to the same bit

Analysis: Throwing Darts (1)

- **More accurate analysis for the number of false positives**

- **Consider:** If we throw ***m*** darts into ***n*** equally likely targets, **what is the probability that a target gets at least one dart?**

- **In our case:**
  - **Targets** = bits/buckets
  - **Darts** = hash values of items

Analysis: Throwing Darts (2)

- We have ***m*** darts, ***n*** targets
- **What is the probability that a target gets at least one dart?**

Equals **1/e** as **n** →∞

Equivalent

$$1 - (1 - 1/n)^{n(m/n)}$$

$$\mathbf{1 - e^{-m/n}}$$

Probability some target **X** not hit by a dart

Probability at least one dart hits target **X**

Analysis: Throwing Darts (3)

- **Fraction of 1s in the array B =**
  = probability of false positive = $1 - e^{-m/n}$

- **Example:** **10^9** darts, **$8 \cdot 10^9$** targets
  - Fraction of **1s** in **B = $1 - e^{-1/8}$ = 0.1175**
    - Compare with our earlier estimate: **1/8 = 0.125**

Bloom Filter

- Consider: $|S| = m$, $|B| = n$
- Use k independent hash functions $h_1, \ldots, h_k$
- **Initialization:**
  - Set **B** to all **0s**
  - Hash each element $s \in S$ using each hash function h_i, set $B[h_i(s)] = 1$ (for each $i = 1, \ldots, k$)

(**note:** we have a single array B!)

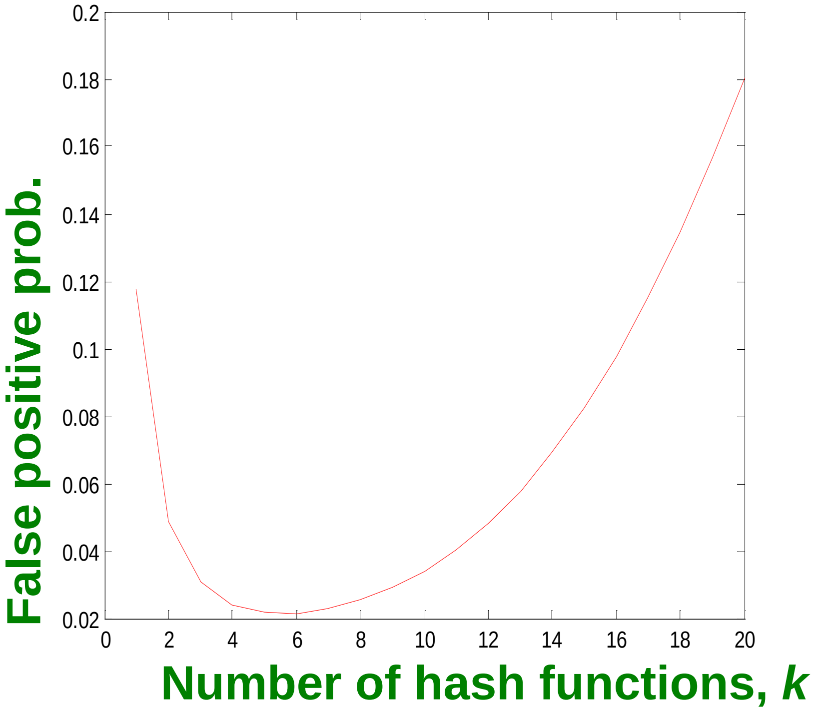
- **Run-time:**
  - When a stream element with key x arrives
    - If $B[h_i(x)] = 1$ for all $i = 1, \ldots, k$ then declare that x is in S
      - That is, x hashes to a bucket set to **1** for every hash function $h_i(x)$
    - Otherwise discard the element x

Bloom Filter -- Analysis

- **What fraction of the bit vector B are 1s?**
  - Throwing $k \cdot m$ darts at n targets
  - So fraction of **1**s is $(1 - e^{-km/n})$
- But we have k independent hash functions and we only let the element x through **if all** k hash element x to a bucket of value **1**
- So, false **positive probability = $(1 - e^{-km/n})^k$**

Bloom Filter – Analysis (2)

- ***m* = 1 billion, *n* = 8 billion**
  - **k = 1**: $(1 - e^{-1/8})$ = **0.1175**
  - **k = 2**: $(1 - e^{-1/4})^2$ = **0.0493**



- **What happens as we keep increasing *k*?**

- "Optimal" value of ***k***: ***n/m* ln(2)**
  - **In our case:** Optimal **k = 8 ln(2) = 5.54 ≈ 6**
    - **Error at k = 6**: $(1 - e^{-1/6})^2$ = **0.0235**

Bloom Filter: Wrap-up

- **Bloom filters guarantee no false negatives, and use limited memory**
  - Great for pre-processing before more expensive checks
- **Suitable for hardware implementation**
  - Hash function computations can be parallelized
- Is it better to have **1** big **B** or ***k*** small **B**s?
  - **It is the same:** $(1 - e^{-km/n})^k$ vs. $(1 - e^{-m/(n/k)})^k$
  - But keeping **1 big B** is simpler

(2) Counting Distinct Elements

Counting Distinct Elements

- **Problem:**
  - Data stream consists of a universe of elements chosen from a set of size ***N***
  - Maintain a count of the number of distinct elements seen so far
- **Obvious approach:**
  Maintain the set of elements seen so far
  - That is, keep a hash table of all the distinct elements seen so far

Applications

- **How many different words are found among the Web pages being crawled at a site?**
  - Unusually low or high numbers could indicate artificial pages (spam?)
- **How many different Web pages does each customer request in a week?**
- **How many distinct products have we sold in the last week?**

Using Small Storage

- **Real problem: What if we do not have space to maintain the set of elements seen so far?**
- **Estimate the count in an unbiased way**
- **Accept that the count may have a little error, but limit the probability that the error is large**

Flajolet-Martin Approach

- Pick a hash function h that maps each of the N elements to at least $\log_2 N$ bits
- For each stream element a, let $r(a)$ be the number of trailing **0s** in $h(a)$
  - **r(a)** = position of first 1 counting from the right
    - E.g., say $h(a) = 12$, then 12 is 1100 in binary, so $r(a) = 2$
- Record R = **the maximum** $r(a)$ **seen**
  - $R = \max_a r(a)$, over all the items a seen so far
- **Estimated number of distinct elements =** 2^R

Why It Works: Intuition

- Very very rough and heuristic intuition why Flajolet-Martin works:
  - ***h(a)*** hashes ***a*** with **equal prob.** to any of ***N*** values
  - Then ***h(a)*** is a sequence of $\log_2 N$ bits, where 2^{-r} fraction of all ***a***s have a tail of ***r*** zeros
    - About 50% of ***a***s hash to ****0**
    - About 25% of ***a***s hash to ****00**
    - So, if we saw the longest tail of ***r=2*** (i.e., item hash ending ***100**) then we have probably seen **about *4*** distinct items so far
  - **So, it takes to hash about 2^r items before we see one with zero-suffix of length *r***

Why It Works: More formally

- **Now we show why Flajolet-Martin works**
- **Formally, we will show that probability of finding a tail of *r* zeros:**
  - **Goes to 1 if $m \gg 2^r$**
  - **Goes to 0 if $m \ll 2^r$**

  where m is the number of distinct elements seen so far in the stream
- **Thus, 2^R will almost always be around *m*!**

Why It Works: More formally

- **What is the probability that a given *h*(*a*) ends in at least *r* zeros is 2^{-r}**
  - **h(a)** hashes elements uniformly at random
  - Probability that a random number ends in at least ***r*** zeros is $\mathbf{2^{-r}}$
- Then, the probability of **NOT** seeing a tail of length ***r*** among ***m*** elements:

$$\boxed{\left(\boxed{1-2^{-r}}\right)^{m}}$$

Prob. all end in fewer than ***r*** zeros.

Prob. that given **h(a)** ends in fewer than ***r*** zeros

Why It Works: More formally

- **Note:** $(1-2^{-r})^m = (1-2^{-r})^{2^r(m2^{-r})} \approx e^{-m2^{-r}}$
- **Prob. of NOT finding a tail of length *r* is:**
  - If ***m << 2^r***, then prob. tends to **1**
    - $(1-2^{-r})^m \approx e^{-m2^{-r}} = 1$ as **m/2^r → 0**
    - So, the probability of finding a tail of length ***r*** tends to **0**
  - If ***m >> 2^r***, then prob. tends to **0**
    - $(1-2^{-r})^m \approx e^{-m2^{-r}} = 0$ as **m/2^r → ∞**
    - So, the probability of finding a tail of length ***r*** tends to **1**
- **Thus, 2^R will almost always be around *m*!**

Why It Doesn't Work

- **E[2^R] is actually infinite**
  - Probability halves when $R \to R+1$, but value doubles
- **Workaround involves using many hash functions h_i and getting many samples of R_i**
- **How are samples R_i combined?**
  - **Average?** What if one very large value 2^{R_i}?
  - **Median?** All estimates are a power of **2**
  - **Solution:**
    - Partition your samples into small groups
    - Take the median of groups
    - Then take the average of the medians

(3) Computing Moments

Generalization: Moments

- **Suppose a stream has elements chosen from a set *A* of *N* values**

- **Let m_i be the number of times value *i* occurs in the stream**

- **The k^{th} *moment* is**

$$\sum_{i \in A} (m_i)^k$$

Special Cases

$$\sum_{i \in A} (m_i)^k$$

- **0^{th}moment =** number of distinct elements
  - The problem just considered
- **1^{st} moment =** count of the numbers of elements = length of the stream
  - Easy to compute
- **2^{nd} moment =** ***surprise number S*** **=** a measure of how uneven the distribution is

Example: Surprise Number

- **Stream of length 100**
- **11 distinct values**

- Item counts: **10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9**
  Surprise *S* = 910

- Item counts: **90, 1, 1, 1, 1, 1, 1, 1 ,1, 1, 1**
  Surprise *S* = 8,110

[Alon, Matias, and Szegedy]

AMS Method

- **AMS method works for all moments**
- **Gives an unbiased estimate**
- **We will just concentrate on the 2nd moment *S***
- **We pick and keep track of many variables *X*:**
  - For each variable ***X*** we store ***X.el*** and ***X.val***
    - ***X.el*** corresponds to the item ***i***
    - ***X.val*** corresponds to the **count** of item ***i***
  - Note this requires a count in main memory, so number of ***X***s is limited
- **Our goal is to compute** $S = \sum_i m_i^2$

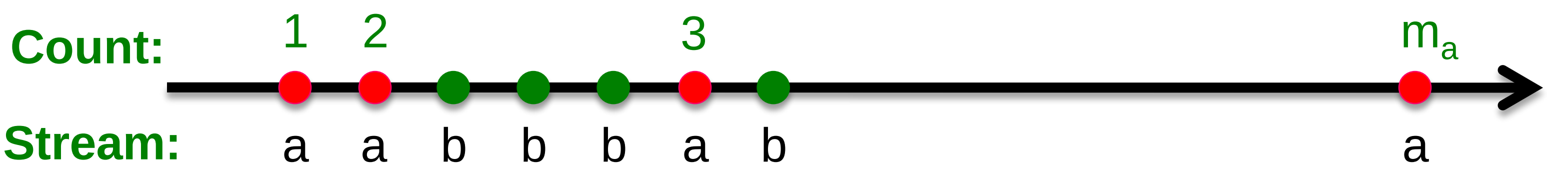
One Random Variable (X)

- **How to set *X.val* and *X.el*?**
  - Assume stream has length ***n*** (we relax this later)
  - Pick some random time ***t*** (***t<n***) to start, so that any time is equally likely
  - Let at time ***t*** the stream have item ***i***. ***We set X.el = i***
  - Then we maintain count ***c*** (***X.val = c***) of the number of *is* in the stream starting from the chosen time ***t***
- **Then the estimate of the 2nd moment ($\sum_i m_i^2$) is:**

$$S = f(X) = n\,(2 \cdot c - 1)$$

  - Note, we will keep track of multiple **X**s, (**X_1, X_2,… X_k**) and our final estimate will be $S = 1/k \sum_j^k f(X_j)$

Expectation Analysis



- **2nd moment is $S = \sum_i m_i^2$**
- c_t … number of times item at time t appears from time **t** onwards ($c_1=m_a$, $c_2=m_a-1$, $c_3=m_b$)
- $E[f(X)] = \frac{1}{n}\sum_{t=1}^{n} n(2c_t - 1)$

$$= \frac{1}{n}\sum_i n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$$

m_i … total count of item i in the stream (we are assuming stream has length **n**)

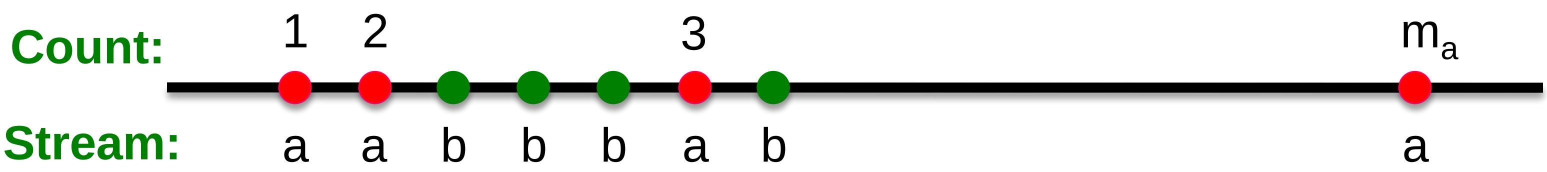
Group times by the value seen

Time t when the last i is seen ($c_t=1$)

Time t when the penultimate i is seen ($c_t=2$)

Time t when the first i is seen ($c_t=m_i$)

Expectation Analysis



- $E[f(X)] = \frac{1}{n} \sum_i n\,(1 + 3 + 5 + \cdots + 2m_i - 1)$
  - Little side calculation: $(1 + 3 + 5 + \cdots + 2m_i - 1) = \sum_{i=1}^{m_i} (2i - 1) = 2\frac{m_i(m_i+1)}{2} - m_i = (m_i)^2$
- **Then $E[f(X)] = \frac{1}{n} \sum_i n\,(m_i)^2$**
- **So, $\mathbf{E[f(X)]} = \sum_i (m_i)^2 = S$**
- **We have the second moment (in expectation)!**

Higher-Order Moments

- **For estimating k^{th} moment we essentially use the same algorithm but change the estimate:**
  - For **k=2** we used ***n*** **(2·c – 1)**
  - For **k=3** we use: ***n*** **(3·c² – 3c + 1)** (where **c=X.val**)
- **Why?**
  - **For k=2:** Remember we had $(1+3+5+\cdots+2m_i-1)$ and we showed terms ***2c-1*** (for **c=1,…,m**) sum to ***m²***
    - $\sum_{c=1}^{m} 2c-1=\sum_{c=1}^{m} c^2-\sum_{c=1}^{m}(c-1)^2=m^2$
    - **So:** $\mathbf{2c-1=c^2-(c-1)^2}$
  - **For k=3:** **c³ - (c-1)³ = 3c² - 3c + 1**
- **Generally:** Estimate $= n\left(c^k-(c-1)^k\right)$

Combining Samples

- **In practice:**
  - Compute $f(X) = n(2c - 1)$ for as many variables ***X*** as you can fit in memory
  - Average them in groups
  - Take median of averages
- **Problem: Streams never end**
  - We assumed there was a number ***n***, the number of positions in the stream
  - But real streams go on forever, so ***n*** is a variable – the number of inputs seen so far

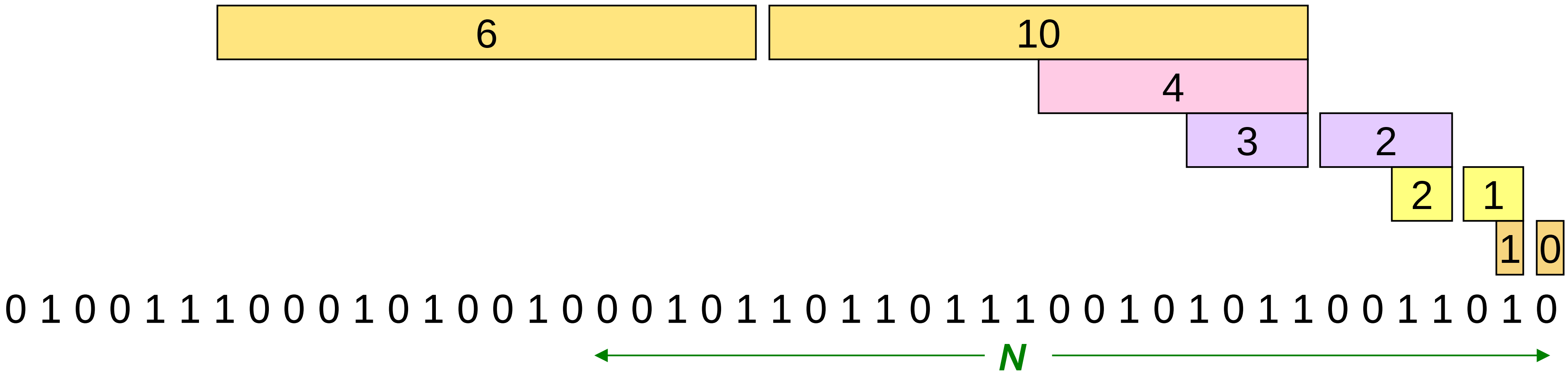
Streams Never End: Fixups

- **(1)** The variables ***X*** have ***n*** as a factor – keep ***n*** separately; just hold the count in ***X***
- **(2)** Suppose we can only store ***k*** counts. We must throw some ***X***s out as time goes on:
  - **Objective:** Each starting time ***t*** is selected with probability ***k/n***
  - **Solution: (fixed-size sampling!)**
    - Choose the first ***k*** times for ***k*** variables
    - When the n^{th} element arrives ($n > k$), choose it with probability ***k/n***
    - If you choose it, throw one of the previously stored variables **X** out, with equal probability

Counting Itemsets

Counting Itemsets

- **New Problem:** **Given a stream, which items appear more than *s* times in the window?**
- **Possible solution:** Think of the stream of baskets as one binary stream per item
  - **1** = item present; **0** = not present
  - Use **DGIM** to estimate counts of **1**s for all items



Extensions

- **In principle, you could count frequent pairs or even larger sets the same way**
  - **One stream per itemset**
- **Drawbacks:**
  - Only approximate
  - **Number of itemsets is way too big**

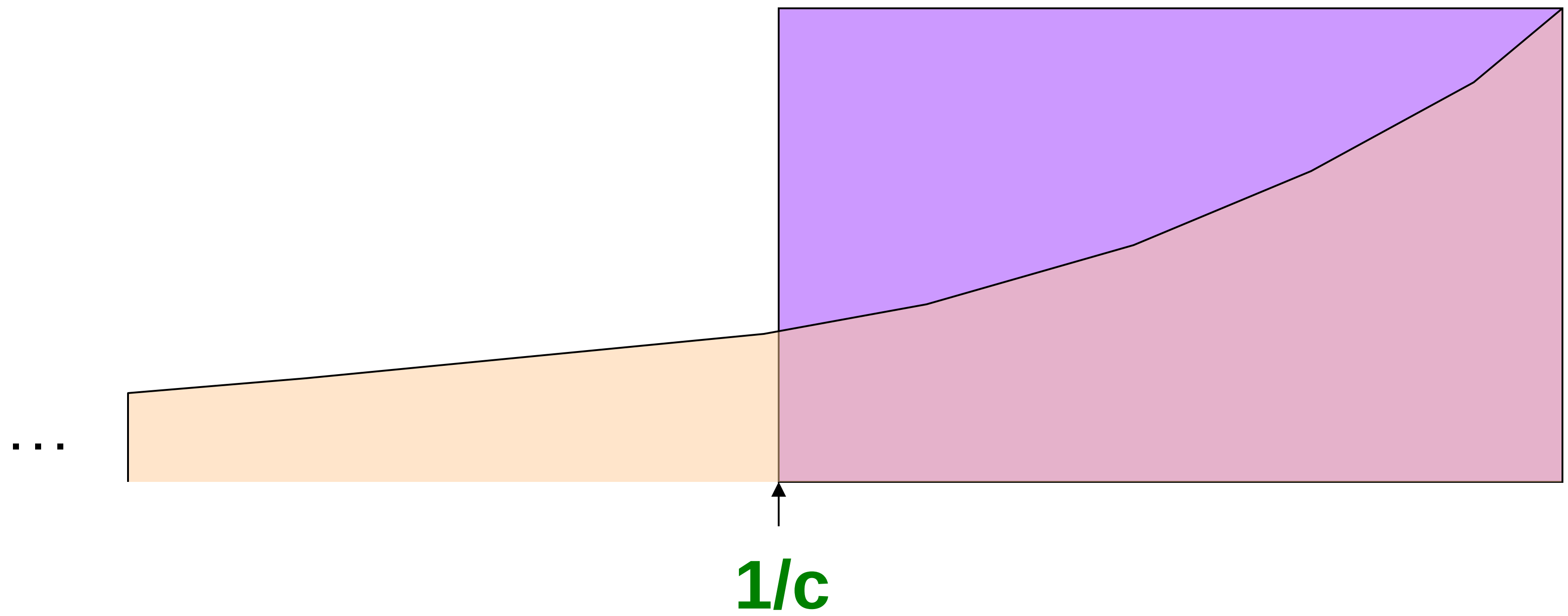
Exponentially Decaying Windows

- **Exponentially decaying windows: A heuristic for selecting likely frequent item(sets)**
  - **What are "currently" most popular movies?**
    - Instead of computing the raw count in last ***N*** elements
    - Compute a **smooth aggregation** over the whole stream
- If stream is $a_1, a_2, \ldots$ and we are taking the sum of the stream, take the answer at time t to be:
$$= \sum_{i=1}^{t} a_i (1-c)^{t-i}$$
    - **c** is a constant, presumably tiny, like 10^{-6} or 10^{-9}
- **When new a_{t+1} arrives:**
Multiply current sum by **(1-c)** and add a_{t+1}

Example: Counting Items

- If each a_i is an "item" we can compute the **characteristic function** of each possible item x as an Exponentially Decaying Window
  - That is: $\sum_{i=1}^{t} \delta_i \cdot (1-c)^{t-i}$
    where $\delta_i = 1$ if $a_i = x$, and **0** otherwise
  - Imagine that for each item x we have a binary stream (**1** if x appears, **0** if x does not appear)
  - New item x arrives:
    - Multiply all counts by **(1-c)**
    - Add **+1** to count for element x
- **Call this sum the "weight" of item x**

Sliding Versus Decaying Windows



- **Important property:** Sum over all weights $\sum_t (1-c)^t$ **is 1/[1 – (1 – c)] = 1/c**

Example: Counting Items

- **What are "currently" most popular movies?**
- **Suppose we want to find movies of weight > ½**
  - **Important property:** Sum over all weights $\sum_t (1-c)^t$ is 1/[1 – (1 – c)] = **1/c**
- **Thus:**
  - There cannot be more than **2/c** movies with weight of ½ or more
- So, **2/c** is a limit on the number of movies being counted at any time

Extension to Itemsets

- **Count (some) itemsets in an E.D.W.**
  - **What are currently "hot" itemsets?**
    - **Problem:** Too many itemsets to keep counts of all of them in memory
- **When a basket B comes in:**
  - Multiply all counts by **(1-c)**
  - For uncounted items in **B**, create new count
  - Add **1** to count of any item in **B** and to any **itemset** contained in **B** that is already being counted
  - **Drop counts < ½**
  - Initiate new counts (next slide)

Initiation of New Counts

- Start a count for an itemset $S \subseteq B$ if every proper subset of ***S*** had a count prior to arrival of basket ***B***
  - **Intuitively:** If all subsets of **S** are being counted this means they are "**frequent/hot**" and thus ***S*** has a potential to be "**hot**"
- **Example:**
  - Start counting ***S*=**{**i, j**} iff both **i** and **j** were counted prior to seeing ***B***
  - Start counting ***S*=**{**i, j, k**} iff {**i, j**}, {**i, k**}, and {**j, k**} were all counted prior to seeing ***B***

How many counts do we need?

- Counts for single items **< (2/c)·(avg. number of items in a basket)**
- **Counts for larger itemsets = ??**
- **But we are conservative about starting counts of large sets**
  - If we counted every set we saw, one basket of **20** items would initiate **1M** counts