Matrices

At the end of this lecture you should be able to:

- provide a definition of a matrix;
- use appropriate terms to describe different types of matrix (such as square matrix, column vector and row vector);
- transpose a matrix;
- add and subtract matrices;
- apply scalar and matrix multiplication;
- define the identity matrix and explain its significance;
- calculate the determinant and inverse of a 2 x 2 matrix;
- use the inverse to solve simple matrix equations in the form of $A \times X = B$.
- briefly explain the importance of matrices in computer graphics.

Definition and examples

A matrix is a grid of numbers, consisting of rows and columns. The numbers are enclosed in round or square brackets.

$$\begin{pmatrix} 3 & 1 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

A 2 x 3 matrix

$$\begin{pmatrix} 2 & 4 \\ -1 & 0 \\ 9 & 10 \\ 1 & -3 \end{pmatrix}$$

A 4 x 2 matrix

$$\begin{pmatrix} 3 & 7 & 0 \\ 20 & 3 & -5 \\ -2 & 1 & 45 \end{pmatrix}$$

A 3 x 3 matrix
(An example of a square matrix)

$$\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

A column matrix
(or column vector)

$$\begin{pmatrix} -11 & 2 & 33 \end{pmatrix}$$

A row matrix
(or row vector)

We often use an upper case letter to denote a particular matrix. For example:

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 6 \end{pmatrix}$$

Matrix operations

There are a number of operations that can be performed on matrices. We will be studying the following:

- transposition;
- addition/subtraction;
- scalar multiplication;
- matrix multiplication.

Transposition

To transpose a matrix we take each row of the original matrix and make it a column of the transposed matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$A^{\mathsf{T}} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Addition and subtraction of matrices

To add (or subtract) two matrices they must be the same shape.

Then just add (or subtract) the corresponding elements.

Example

$$\begin{pmatrix} 3 & 10 & 1 \\ -2 & 4 & 6 \\ 3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 11 & 21 \\ 2 & 5 & 8 \\ 0 & 1 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 3+5 & 10+11 & 1+21 \\ -2+2 & 4+5 & 6+8 \\ 3+0 & 1+1 & 2+12 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 21 & 22 \\ 0 & 9 & 14 \\ 3 & 2 & 14 \end{pmatrix}$$

Scalar multiplication

To multiply a matrix by a scalar quantity (a simple number), just multiply each element by the number.

Example

$$7\begin{pmatrix} 3 & 10 & 1 \\ -2 & 4 & 6 \\ 3 & 1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 3\times 7 & 10\times 7 & 1\times 7 \\ -2\times 7 & 4\times 7 & 6\times 7 \\ 3\times 7 & 1\times 7 & 2\times 7 \end{pmatrix}$$

$$=\begin{pmatrix} 21 & 70 & 7 \\ -14 & 28 & 42 \\ 21 & 7 & 14 \end{pmatrix}$$

Matrix multiplication

To multiply two matrices together, the number of columns of the first matrix must be equal the number of rows of the second.

Then we multiply the elements of each row of the first matrix by the elements of each column of the second and add them.

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$$

The process is explained on the next slide.

Matrix multiplication – the process

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & \\ & \end{pmatrix}$$

1x7 + 2x9 + 3x11

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ & \end{pmatrix}$$

1x8 + 2x10 + 3x12

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 139 & \end{pmatrix}$$

4x7 + 5x9 + 6x11

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}$$

4x8 + 5x10 + 6x12

Note: Matrix multiplication is not commutative: $A \times B \neq B \times A$

Worked examples

1. Consider the following matrices:

$$A = \begin{pmatrix} -2 & 2 & 3 \\ 1 & 8 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 6 & 7 \\ 9 & 8 & 5 \end{pmatrix}$$

Find the value of:

a) $A + B$ b) $A - B$ c) $3A + 2B$ d) A^{T}

<u>Solution</u>

a) $$A+B = \begin{pmatrix} -2+3 & 2+6 & 3+7 \\ 1+9 & 8+8 & 4+5 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 10 \\ 10 & 16 & 9 \end{pmatrix}$$

b) $$A-B = \begin{pmatrix} -2-3 & 2-6 & 3-7 \\ 1-9 & 8-8 & 4-5 \end{pmatrix} = \begin{pmatrix} -5 & -4 & -4 \\ -8 & 0 & -1 \end{pmatrix}$$

c) $$3A+2B = \begin{pmatrix} -6 & 6 & 9 \\ 3 & 24 & 12 \end{pmatrix} + \begin{pmatrix} 6 & 12 & 14 \\ 18 & 16 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 18 & 23 \\ 21 & 40 & 22 \end{pmatrix}$$

d) $$A^{\mathsf{T}} = \begin{pmatrix} -2 & 1 \\ 2 & 8 \\ 3 & 4 \end{pmatrix}$$

2. Consider the following matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

Calculate A x B.

<u>Solution</u>

2 x 1 + 3 x 0

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & \\ & \end{pmatrix}$$

2 x 2 + 3 x 3

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 13 \\ & \end{pmatrix}$$

4 x 1 + 5 x 0

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 13 \\ 4 & \end{pmatrix}$$

4 x 2 + 5 x 3

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 13 \\ 4 & 23 \end{pmatrix}}}$$

The Determinant of a matrix

Square matrices have a special property called the determinant, which can be very useful in matrix algebra.

The determinant of a matrix A is written as det(A).

To find the determinant of a 2 x 2 matrix we multiply each pair of opposite corners and subtract the result (we do the left-to-right diagonal first , and then the right-to-left diagonal)

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

det(A) = (4 x 1) – (3 x 2)
= -2

Note: You will also see the following notation for the determinant:

$$|A| \qquad \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$$

Worked example:

Find the determinant of the following matrix, A:

$$A = \begin{pmatrix} -3 & 3 \\ -1 & 4 \end{pmatrix}$$

Solution

det(A) = (-3 x 4) - (-1 x 3)
= -12 + 3
= -9

Calculating the determinant of a 3 x 3 matrix

If:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

then: det(A) = $a(ei - fh) - b(di - fg) + c(dh - eg)$

There is a pattern here:

$$\begin{pmatrix} a & \\ \times & \\ & \begin{vmatrix} e & f \\ h & i \end{vmatrix} \end{pmatrix} - \begin{pmatrix} & b & \\ & \times & \\ \begin{vmatrix} d \\ g \end{vmatrix} & & \begin{vmatrix} f \\ i \end{vmatrix} \end{pmatrix} + \begin{pmatrix} & c \\ & \times \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \end{pmatrix}$$

1. Multiply a by the determinant of the 2×2 matrix that is not in a's row or column.
2. Do the same for b then give it a negative sign;
3. Do the same for c, but leave it positive.
4. Add them up.

We could also write this as (using the alternative notation for the determinant):

$$|A| = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Finding the determinant of 3 x 3 and larger matrices can be very fiddly, but it is possible to use an online calculator:

www.matrixcalc.org/en

For interest only

Identity matrices

An identity matrix is a square matrix that has 1s along the left-to-right diagonal and 0s everywhere else.

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In calculations we refer to the identity matrix as I.

When we multiply a matrix by the identity matrix, the original is unchanged:

$$\mathrm{A} \times \mathrm{I} = \mathrm{A} \qquad \mathrm{I} \times \mathrm{A} = \mathrm{A}$$

(It's like multiplying by 1 in arithmetic).

The inverse of a matrix

The inverse of a matrix, A, is a matrix that, when multiplied by A, will give you the identity matrix.

The inverse of A is written as A^{-1}.

$$A \times A^{-1} = \mathrm{I}$$

Unusually for matrix multiplication, this operation is commutative, so:

$$A^{-1} \times A = \mathrm{I}$$

We find that for three matrices, A, B and X:

if: $A \times X = B$ then: $X = A^{-1} \times B$

Compare this to arithmetic:

if: $ax = b$ then: $x = \frac{1}{a}b$

So the inverse acts like the reciprocal in arithmetic.

<u>Why does this work?</u>

Consider the expression: $A \times X = B$

Multiply both sides by A^{-1}: $A^{-1} \times A \times X = A^{-1} \times B$

But: $A^{-1} \times A = \mathrm{I}$

So: $\mathrm{I} \times X = A^{-1} \times B$

But: $\mathrm{I} \times X = X$

So: $X = A^{-1} \times B$

Finding the inverse

For a 2 x 2 matrix we calculate the inverse by the following formula:

if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

For higher order matrices it is more complicated, and we will not deal with that in this module.

Example:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

det(A) =4x3 – 2x1 = 10

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 0.3 & -0.1 \\ -0.2 & 0.4 \end{pmatrix}$$

Note:

- Only square matrices have an inverse.

- There is no inverse if the determinant is 0.

Worked examples

1. Where possible, find the inverse of the following matrices:

a) $A = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ b) $B = \begin{pmatrix} 3 & 8 \\ 2 & 6 \end{pmatrix}$ c) $C = \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix}$

Solution

a) A is not invertible because it is not a square matrix.

b) det(B) = 6x3 – 8x2 = 2.

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 6 & -8 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -1 & 3/2 \end{pmatrix}$$

c) det(C) = 0. Therefore C is not invertible.

2. Consider two of the matrices from the previous section:

$$B = \begin{pmatrix} 3 & 8 \\ 2 & 6 \end{pmatrix} \qquad A = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

If B x X = A, find the value of X.

<u>Solution</u> B x X = A

Therefore $X = B^{-1}$ x A

In the last question we found that:

$$B^{-1} = \begin{pmatrix} 3 & -4 \\ -1 & 3/2 \end{pmatrix}$$

Therefore

$$X = \begin{pmatrix} 3 & -4 \\ -1 & 3/2 \end{pmatrix} \times \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} (3\times 6)+(-4\times 2) \\ (-1\times 6)+(3/2\times 2) \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ -3 \end{pmatrix}$$

Application to Computing

Computer graphics

A digital image is essentially a matrix.

The rows and columns of the matrix correspond to rows and columns of pixels.

The numerical value of each entry corresponds to the colour value of the pixel.

Matrix algebra is extremely important in performing the operations necessary to manipulate digital images.

Decoding digital video requires numerous operations such as matrix multiplication, transformations and so on.

Using matrices to solve linear equations

Matrices provide a convenient way of solving linear equations.

Consider the following three equations:

$$\begin{aligned} x + y + z &= 6 \\ 2y + 5z &= -4 \\ 2x + 5y - z &= 27 \end{aligned}$$

We can form a matrix from the coefficients of the variables: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{pmatrix}$

And we can form another matrix from the variables: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

From our knowledge of matrix multiplication we have:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2y+5z \\ 2x+5y-z \end{pmatrix}$$

We will refer to these three matrices as A, X and B respectively.

Our equations (from the last slide) are:

$$\boxed{\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ 2x & + & 5y & - & z & = & 27 \end{array}}$$

If we substitute the values of the right hand side of our equations into B we have:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 27 \end{pmatrix}$$

So any set of linear equations can be written in the form:

$$AX = B$$

In our example:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ 2x & + & 5y & - & z & = & 27 \end{array}$$

Writing these in the form $AX = B$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B = \begin{pmatrix} 6 \\ -4 \\ 27 \end{pmatrix}$$

To solve our equations we need to find X, which will give us the values for x, y, and z.

We use the fact that:

If: $AX = B$ Then: $X = A^{-1}B$

Using a calculator we find that

$$A^{-1} = \begin{pmatrix} \frac{9}{7} & \frac{-2}{7} & \frac{-1}{7} \\ \frac{-10}{21} & \frac{1}{7} & \frac{5}{21} \\ \frac{4}{21} & \frac{1}{7} & \frac{-2}{21} \end{pmatrix}$$

Again using a calculator:

$$X = A^{-1}B = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$$

So the solution to our equation is: $\underline{x = 5,\ y = 3,\ z = -2}$

For interest only

Row operations on a matrix

There are a number of operations that can be performed on the rows of a matrix which prove useful in other calculations.

1. Switch any two rows:

$$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 6 \\ 6 & 7 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 3 & 6 \\ 2 & 1 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

$$R_1 \leftrightarrow R_2$$

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Row operations on a matrix - continued

2. Multiply a row by a nonzero constant:

$$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 6 \\ 6 & 7 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 2 & 10 \\ 4 & 3 & 6 \\ 6 & 7 & 8 \end{pmatrix}$$

$$2R_1 \rightarrow R_1$$

3. Add one row to another and replace one of the rows:

$$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 6 \\ 6 & 7 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 5 \\ 6 & 4 & 11 \\ 6 & 7 & 8 \end{pmatrix}$$

$$R_1 + R_2 \rightarrow R_2$$

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Row operations on a matrix - continued

Operations 2 and 3 can be combined:

$$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 3 & 6 \\ 6 & 7 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 7 & 17 \\ 4 & 3 & 6 \\ 6 & 7 & 8 \end{pmatrix}$$

$$R_1 + 2R_2 \rightarrow R_1$$

Note: Multiplying one row by -1 and adding is equivalent to subtracting one row from another.

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Solving equations using the Gauss-Jordan elimination method

1. Write the **augmented matrix** of the system of equations.

   For example, for the following set of equations:

   $$x + y + z = 5$$

   $$2x + 3y + 5z = 8$$

   $$4x + 5z = 2$$

   the augmented matrix is:

   $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right)$$

2. Perform row operations on the augmented matrix until the matrix on the left hand side of the divider is an identity matrix.

3. Read the solution from the right hand side.

Solving equations using the Gauss-Jordan elimination method – an example

For interest only

$x + y + z = 5$
$2x + 3y + 5z = 8$
$4x + 5z = 2$

1. Write the augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right)$

2. Perform row operations:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right) \quad R_2 - 2R_1 \rightarrow R_2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right)$$

$$R_3 - 4R_1 \rightarrow R_3 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right)$$

$$R_3 + 4R_2 \rightarrow R_3 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right)$$

$$\tfrac{1}{13}R_3 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$R_2 - 3R_3 \rightarrow R_2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$R_1 - R_3 \rightarrow R_1 \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$R_1 - R_2 \rightarrow R_1 \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

3. The solution to the equation is:

$x = 3$ $\quad$ $y = 4$ $\quad$ $z = -2$