Equations and graphs

At the end of this lecture you should be able to:

- express a function as an equation;
- draw a graphs of linear, quadratic and cubic functions;
- determine the gradient of a linear graph;
- find a function from a linear graph;
- describe the properties of a **parabola**;
- find the solutions of a quadratic equation by drawing a graph;
- explain the terms **critical point**;
- explain the terms **maximum** and **minimum**, and determine the coordinates of these points from a graph;
- describe the properties of a cubic function.

Functions as equations

In mathematics we often designate a letter (such as y) to a function, and write the function as an equation.

Consider the function: $f(x) = 3x + 1$

If we let $f(x) = y$, then we have the equation: $y = 3x + 1$

Note

From now on, we will assume that unless otherwise stated, we are dealing with real numbers, so the function maps from $\mathbb{R}$ to $\mathbb{R}$.

Graphs of Functions

In the last slide we saw that we can express a function as an equation.

For example:

$$y = 3x + 1$$

This equation contains two variables, y and x. The value of y depends on the value of x.

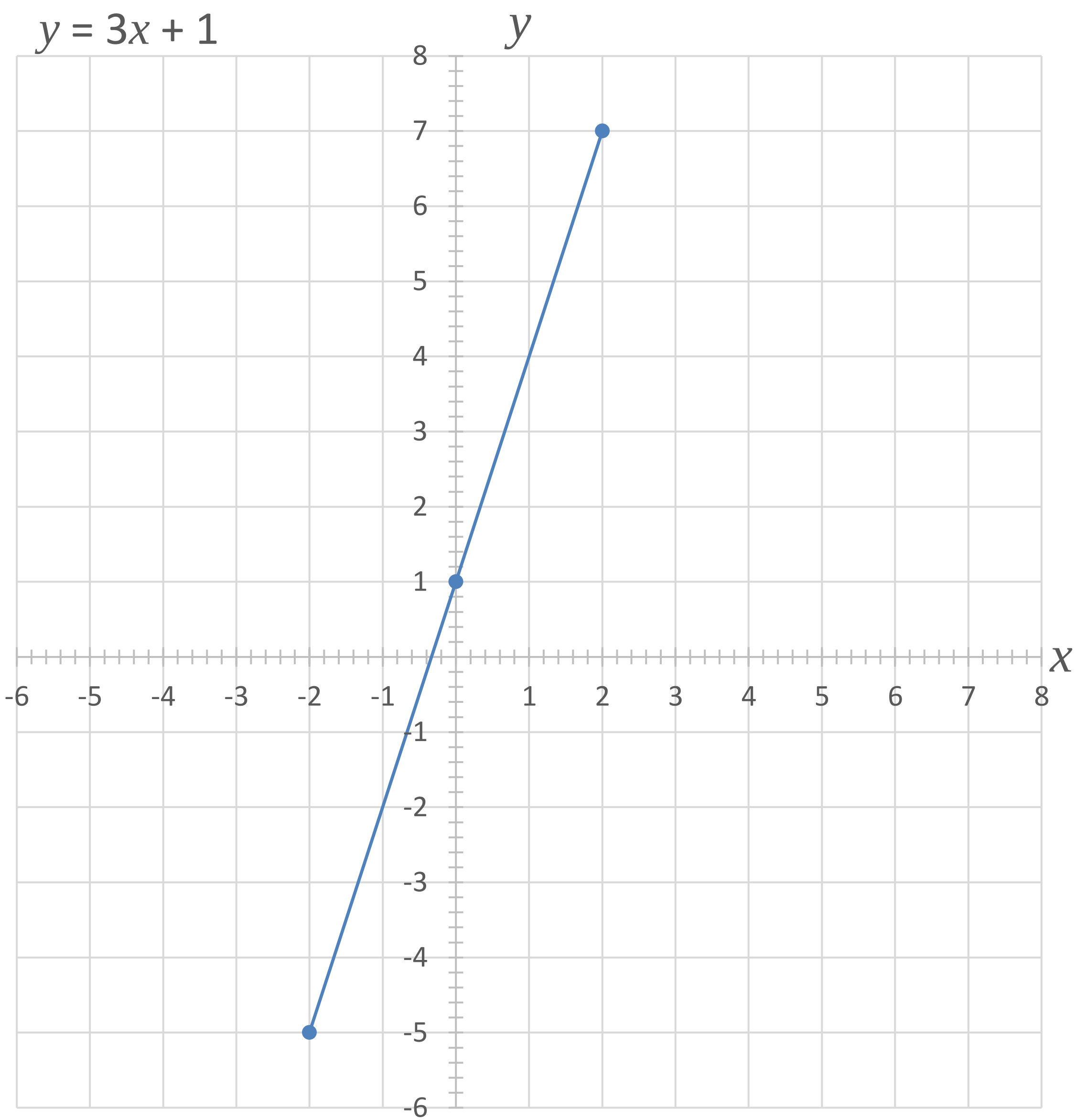
One way to find the value of y for a particular value of x is to draw a graph.

Drawing Graphs on Paper

First we construct a table to find values of y for a few values of x.

$y = 3x + 1$

x	y
-2	-5
0	1
2	7

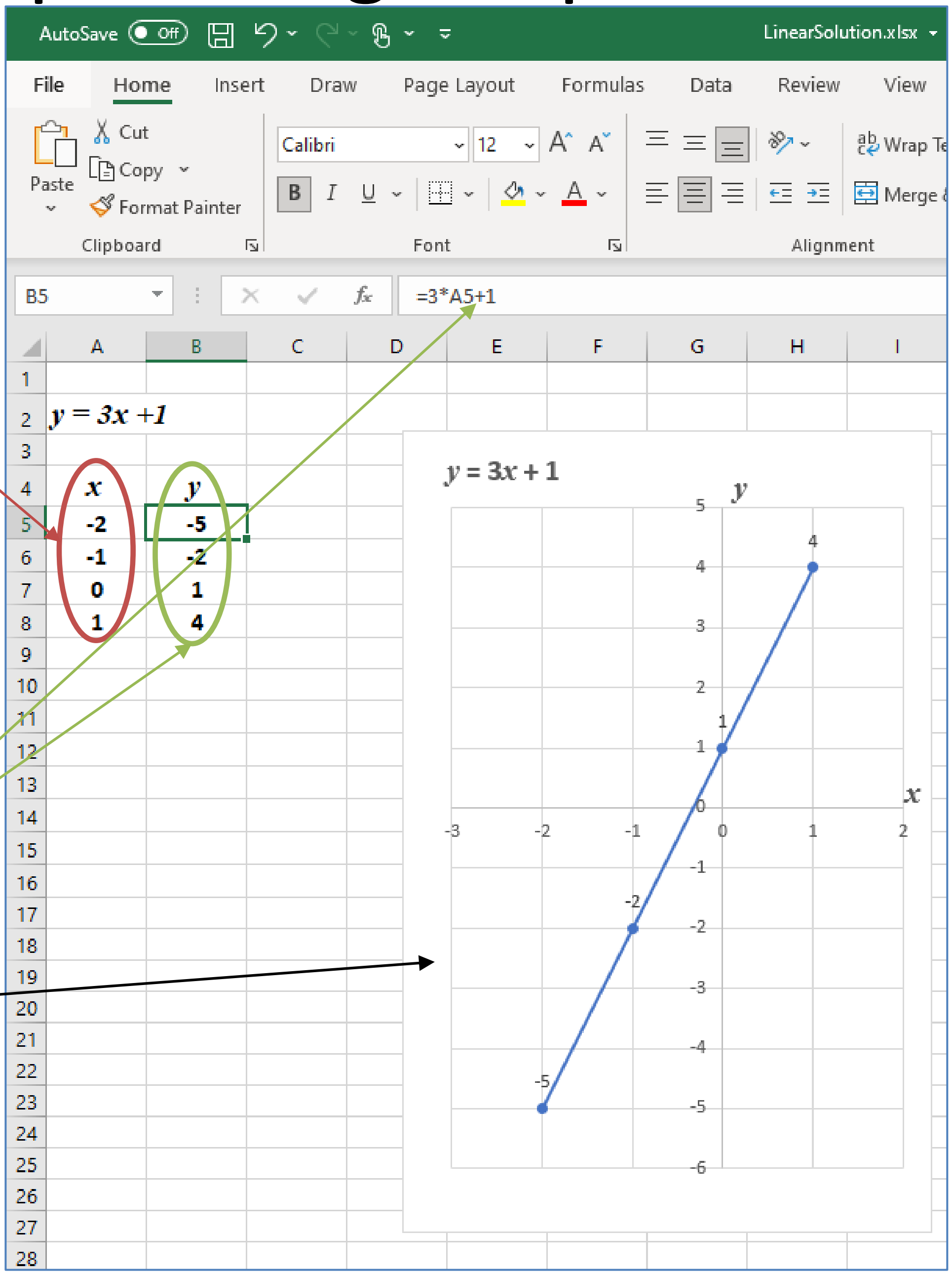


In our graph, for both the x and the y axis, we have used a scale of 1cm to represent a value of 1.

When drawing graphs you should choose a sensible scale that allows the graph to fit into the space available.

Drawing Graphs using as Spreadsheet

Fill in some x-values



Create corresponding y-values using a formula

Generate the graph

Full instructions are provided separately

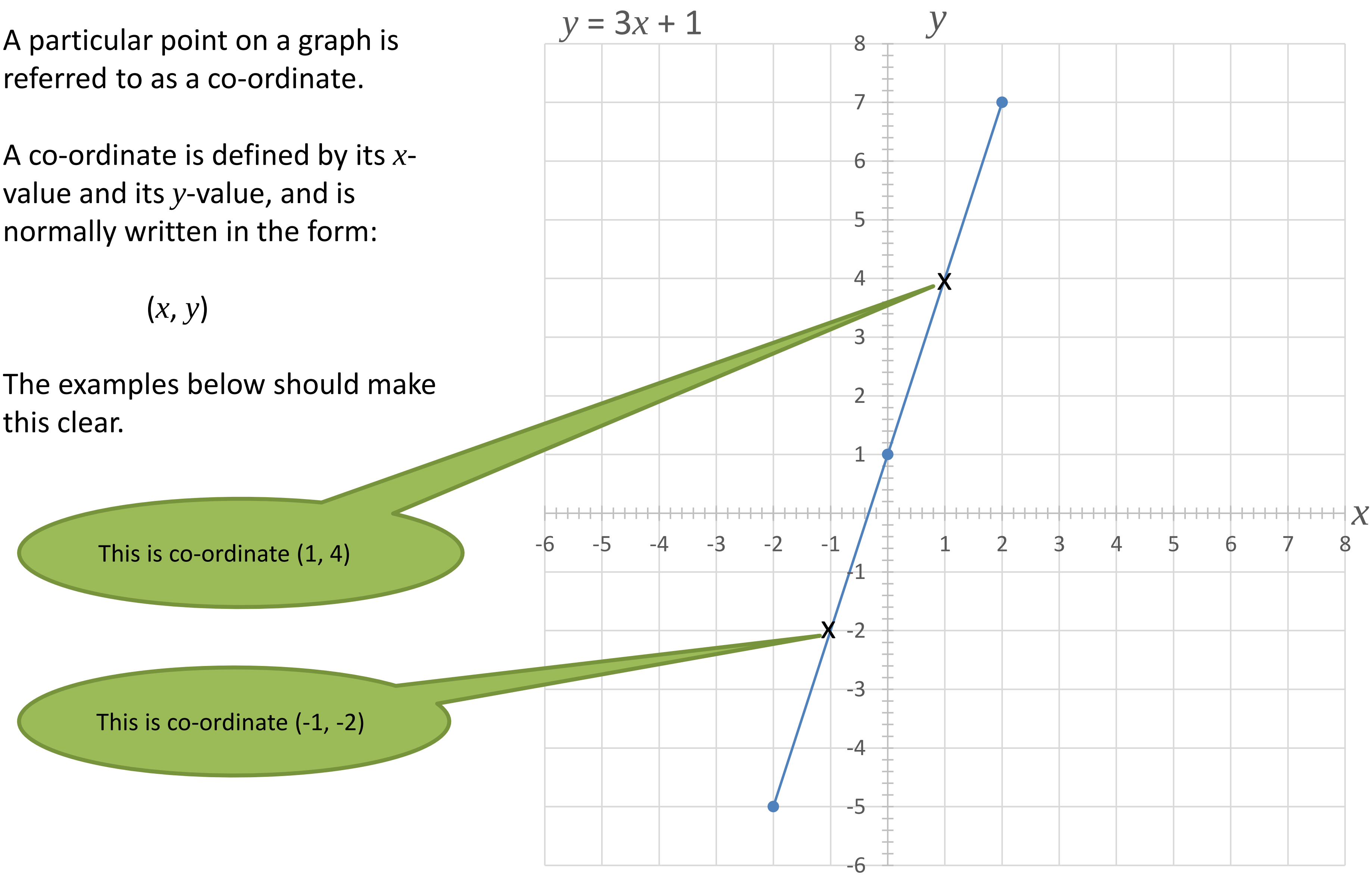
Co-ordinates

A particular point on a graph is referred to as a co-ordinate.

A co-ordinate is defined by its x-value and its y-value, and is normally written in the form:

$$(x, y)$$

The examples below should make this clear.



Using our graph

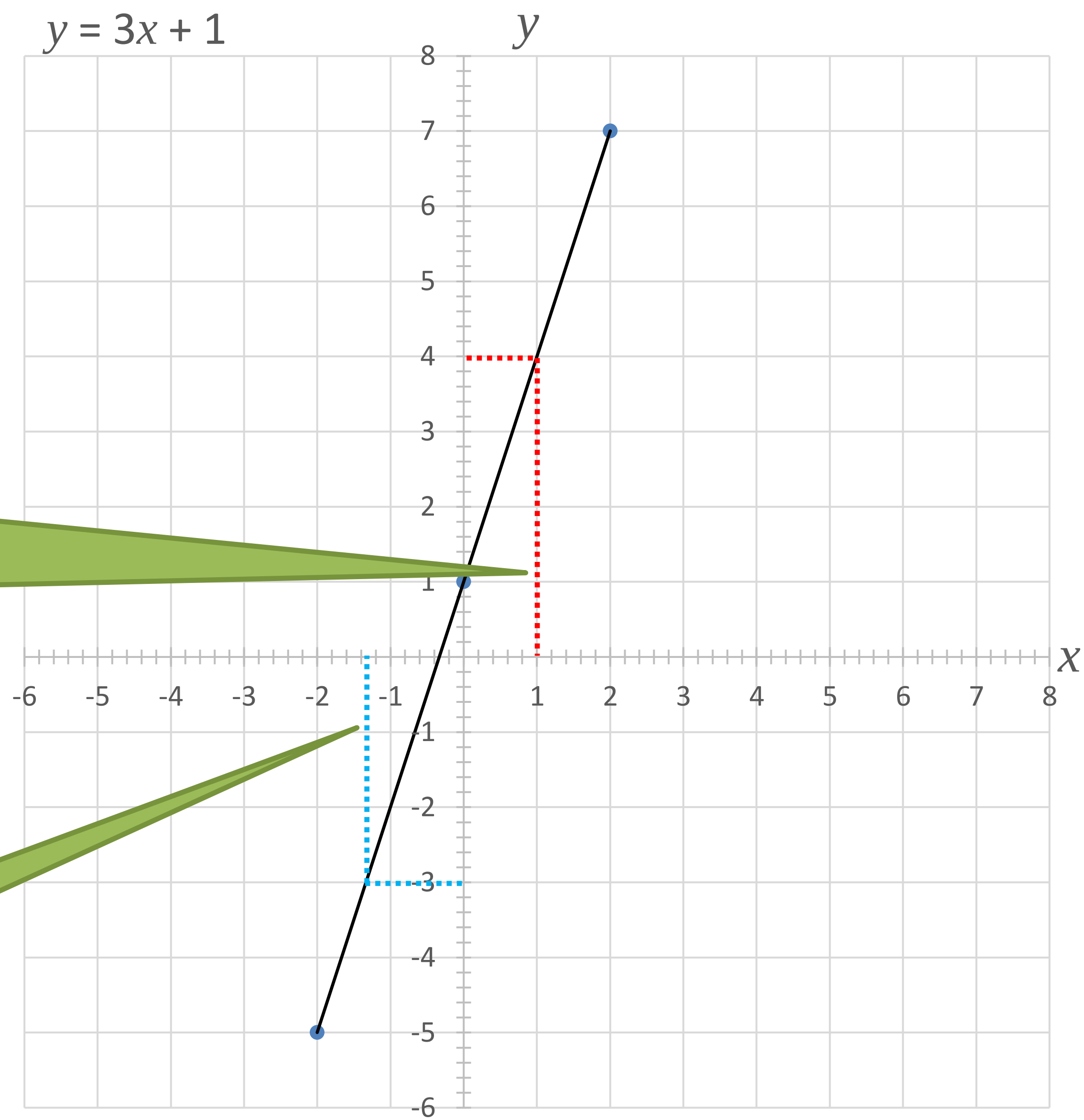
We can use our graph to find a value of y for a corresponding value of x, and vice versa.

Example 1

What is the value of y when $x = 1$?

Draw a line parallel to the y-axis at the point where $x = 1$, and see where it crosses the graph; then read off the corresponding value of y.

We see that $y = 4$



Example 2

What is the value of x when $y = -3$?

$x = -1.3$ (approximately)

Worked example

Consider the equation:

$$y = -2x + 3$$

a) Complete the table below:

x	y
-1	
0	
1	
2	
3	

b) Use this table to plot a graph for the above equation.

c) Use your graph to find the value of y when $x = 0.5$

d) Use your graph to find the value of x when $y = 0.5$

e) What do you notice about the slope of the graph compared to the previous example?

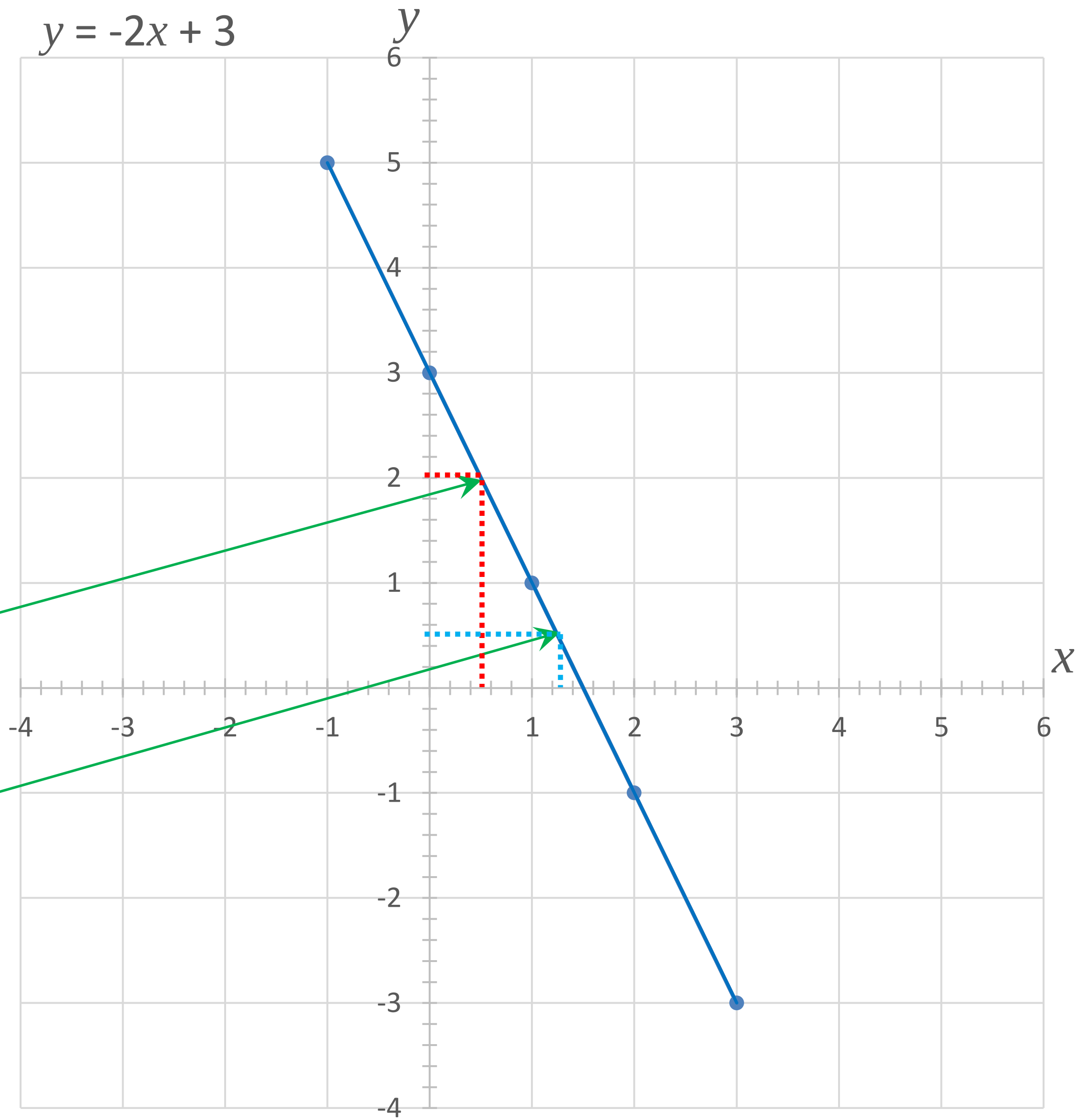
Solution

$y = -2x + 3$

a)

x	y
-1	5
0	3
1	1
2	-1
3	-3

b)



c) When $x = 0.5$, $y = 2$

d) When $y = 0.5$, $x = 1.3$ (estimated)

e) The slope of the graph is in the opposite direction to the previous example.

Linear Functions

So far we have drawn graphs of the following two functions:

$$y = 3x + 1$$
$$y = -2x + 3$$

Both of these gave us graphs which were straight lines.

Any function that contains a simple variable (such as x), which is not raised to a power (such as x^2 or x^3) produces a straight line graph.

Such functions are therefore called **linear** functions – and their corresponding graphs are called linear graphs.

A linear function takes the form: $y = mx + c$

x and y are the variables.

m and c are *constants*: their values are fixed for that particular function.

Finding a Linear Function from a Graph

Look at the graph on the right. We know it has the general form:

$y = mx + c$

When $x = 0$, $y = c$.

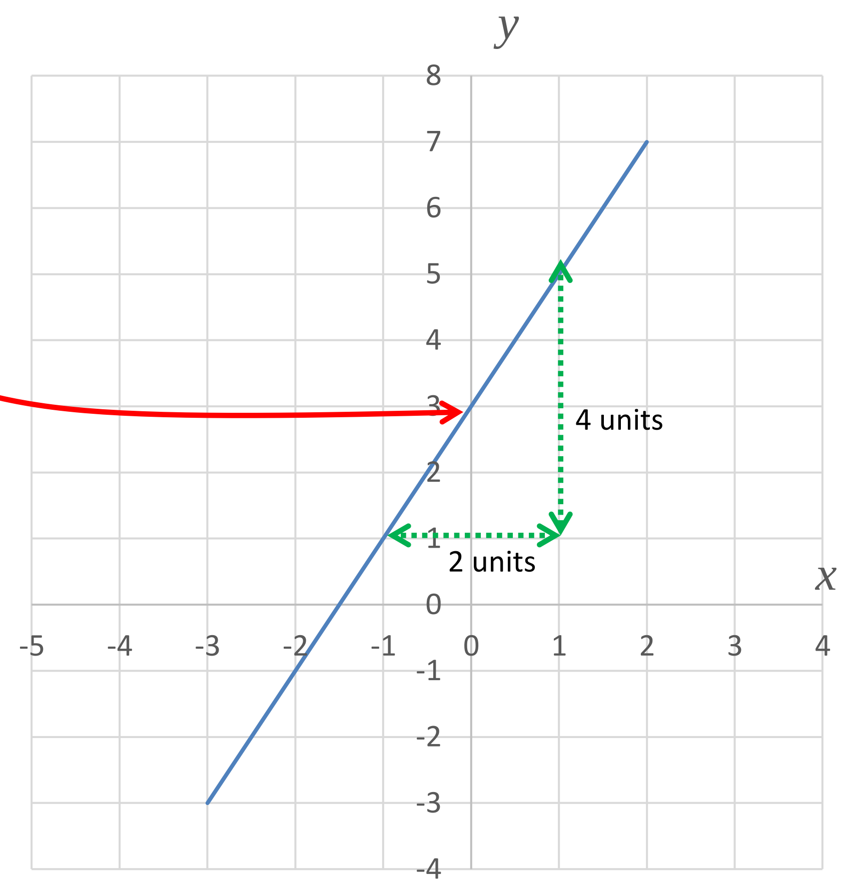
This is the point at which the graph crosses the y-axis.

This tells us the value of c in that particular function.

In our function, $c = 3$

To find the value of m by looking at a graph, we have to work out the **gradient** or **slope** of the graph.

We find this by dividing any distance on the y-axis by the corresponding distance on the x-axis.



In our graph, the green lines show us that $m = \frac{4}{2} = 2$

If we substitute these values into the general form $y = mx + c$, we get $y = 2x + 3$.

Negative Gradients

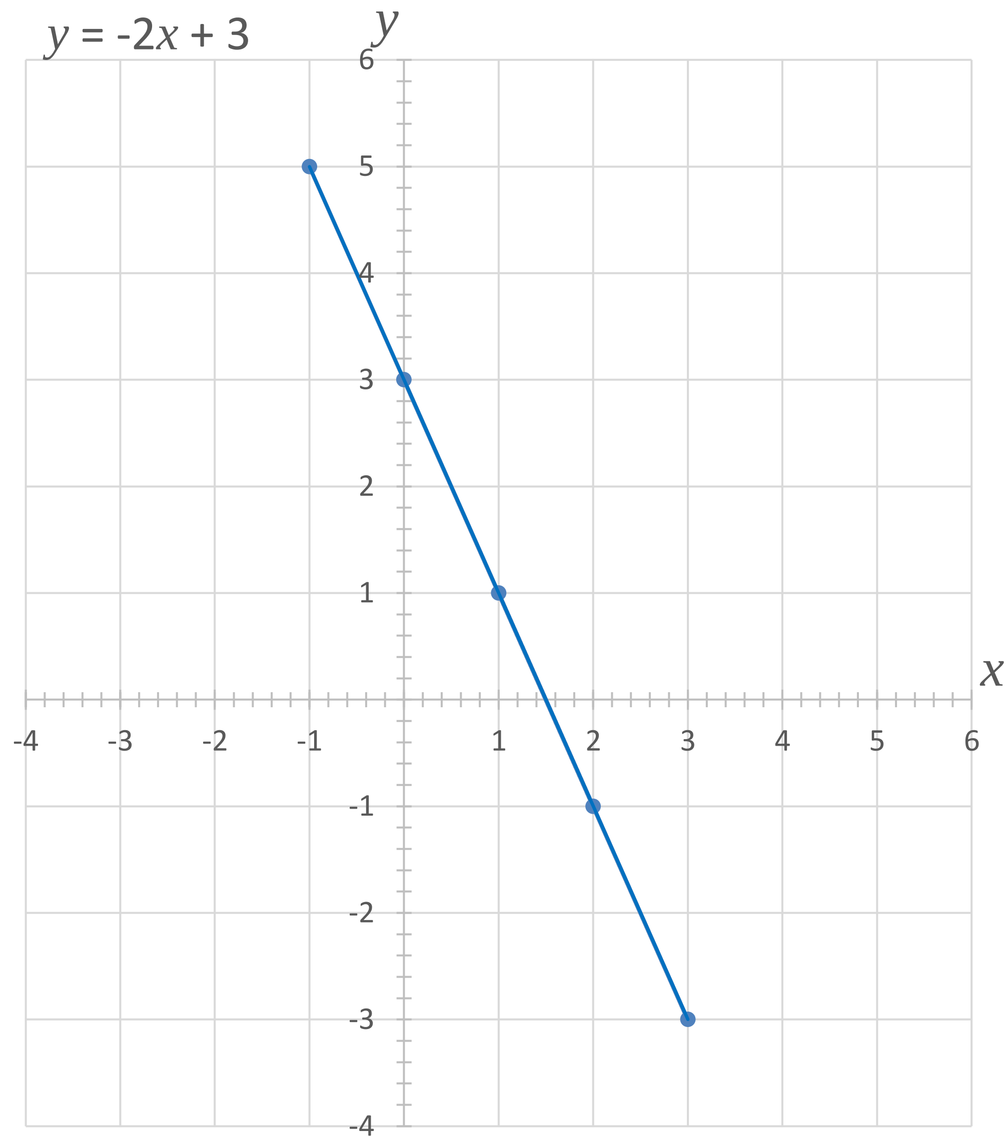
In the previous worked example we saw the graph shown on the right.

This graph slopes from top left to bottom right.

Our previous graph had sloped from bottom left to top right.

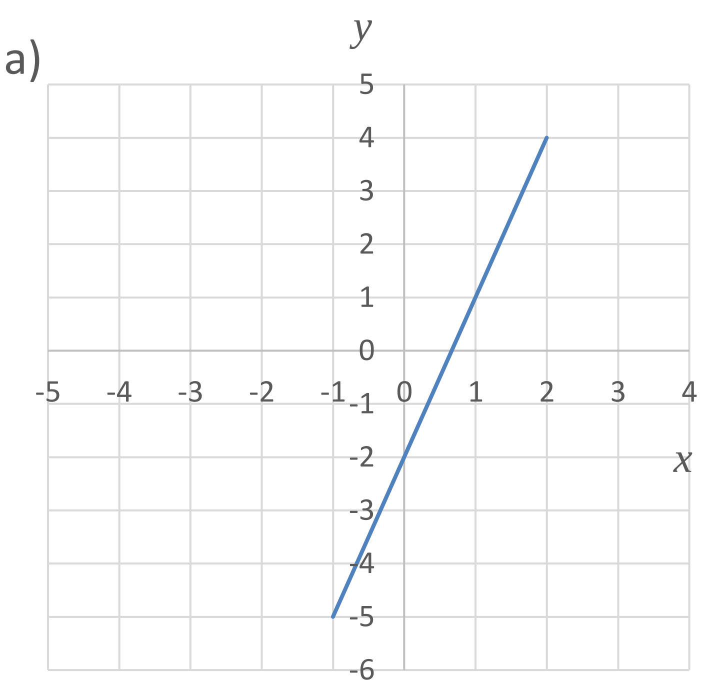
In the function on the right, the coefficient of x is negative. This will always produce a graph sloping from top left to bottom right.

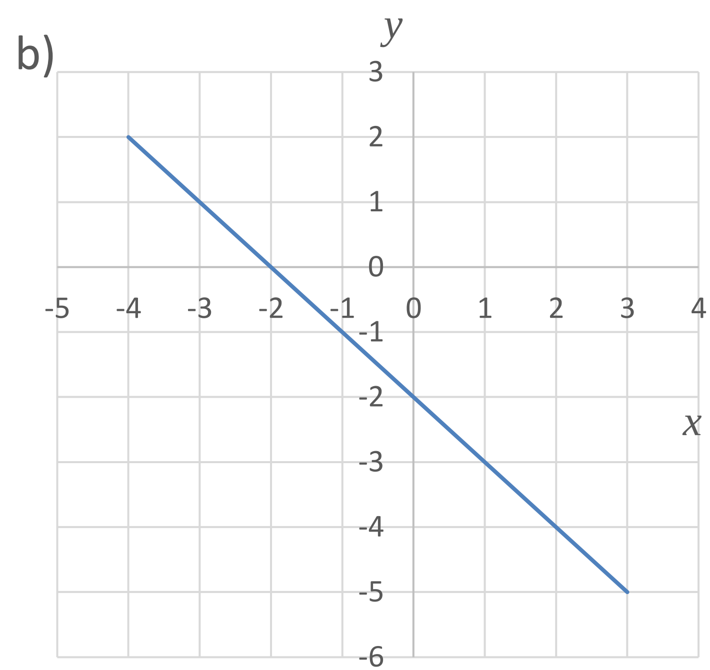
A positive x-coefficient will produce a graph sloping from bottom left to top right.



Worked example

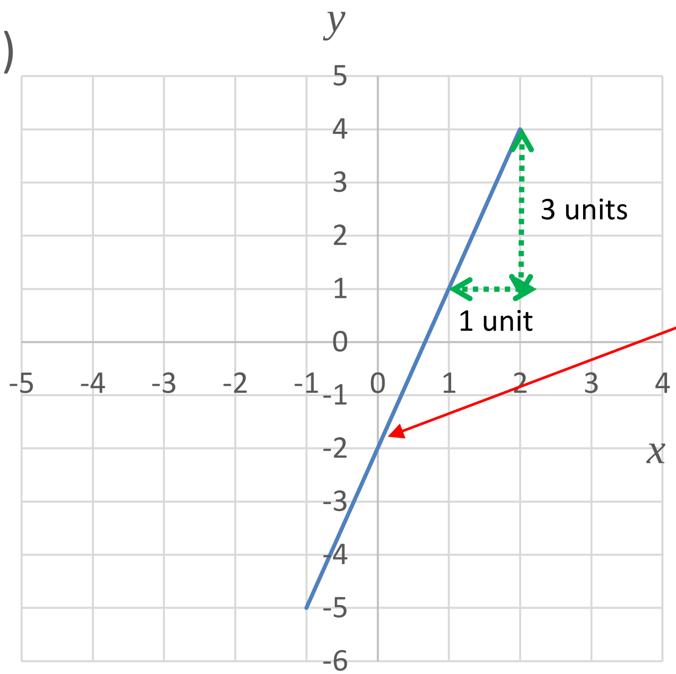
For the graphs below, determine which functions they represent.





Solution

a)



The general form of the function is:

$y = mx + c$

The graph crosses the y-axis at the point (0, -2).

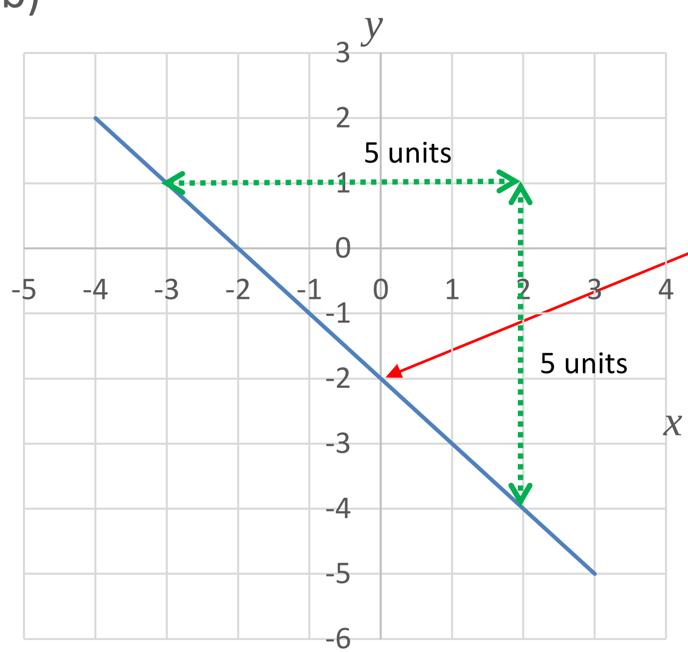
Therefore c = -2

The gradient of the graph (see the green lines) is $\frac{3}{1} = 3$

Therefore $m = 3$

The function is therefore: $y = 3x - 2$

b)



The general form of the function is:

$y = mx + c$

The graph crosses the y-axis at the point (0, -2).

Therefore c = -2

The gradient of the graph (see the green lines) is:

$$\frac{5}{5} = 1$$

But it is negative, so:

$$m = -1$$

The function is therefore: $y = -x - 2$

Quadratic Functions

The graph on the right is the graph of the function:

$$y = x^2 + 2x - 3$$

This function, unlike our previous examples, has a term that contains a power of x, namely x^2.

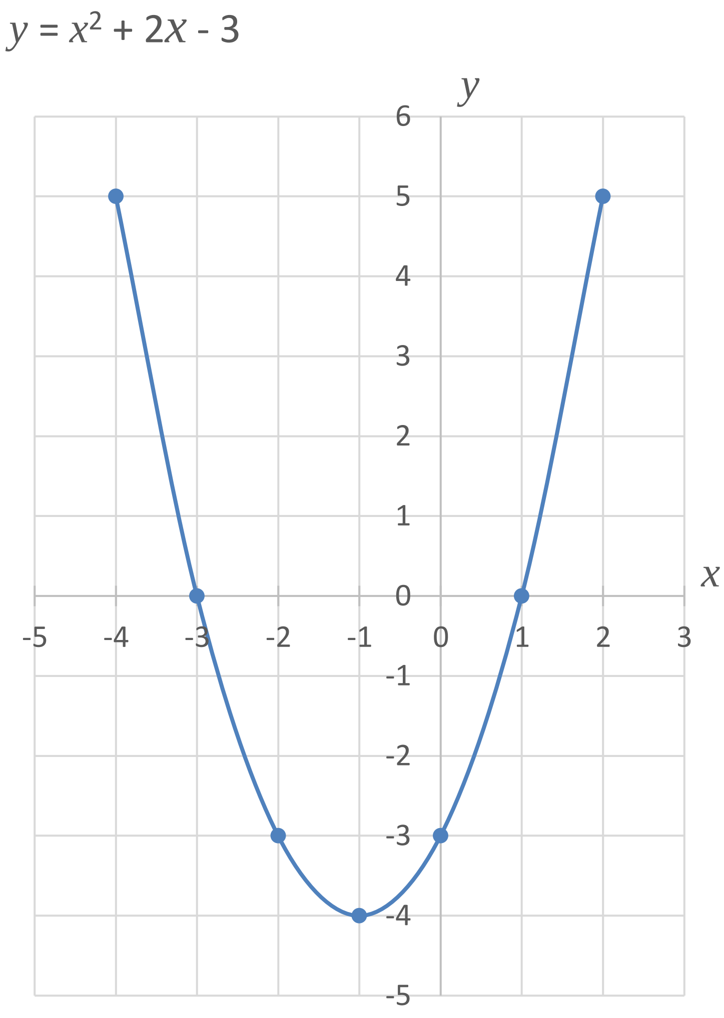
Functions that contain squares of numbers are called **quadratic** functions.

They produce a distinctively shaped graph called a **parabola**.

The general form of a quadratic function is:

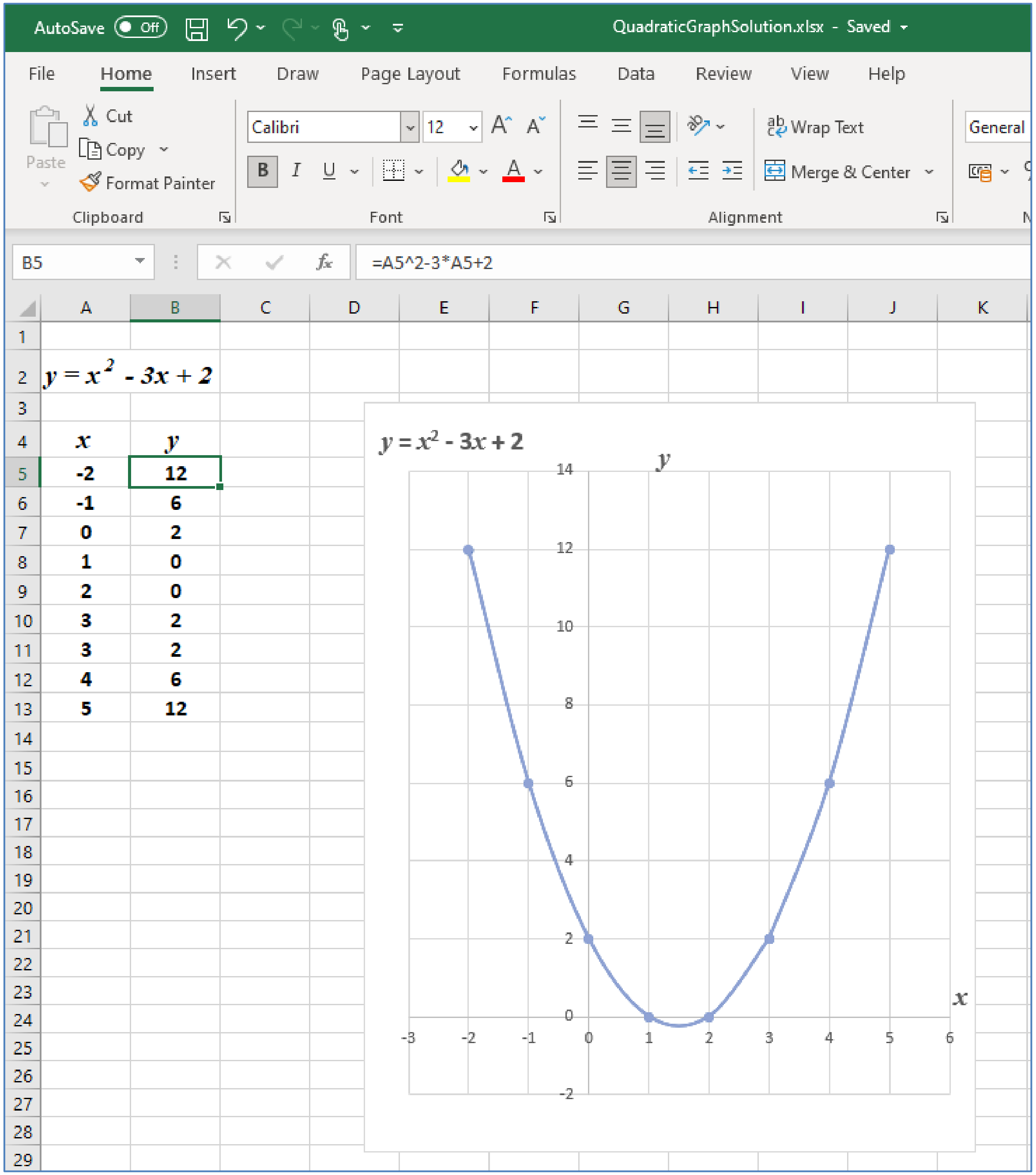
$$y = ax^2 + bx + c$$

Notice that the table we have used to construct the graph needs more values than those we used for the straight line graphs.



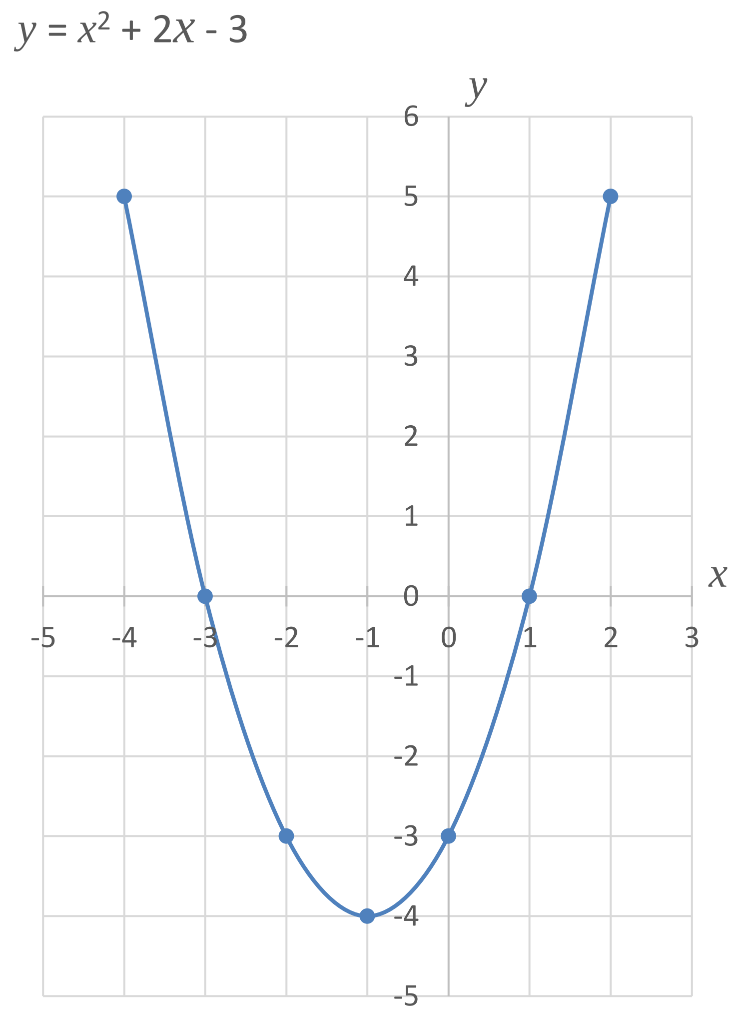
x	y
-4	5
-3	0
-2	-3
-1	-4
0	-3
1	0
2	5

A graph of a quadratic function created with Microsoft Excel



Some properties of Quadratic Functions

- Our graph has a minimum point, (-1, -4)
- There are no values for y below this point.
- For every value of y above the minimum there are 2 corresponding values of x. For example, when $y = 0$, x can be equal to -3 or 1.
- As we will see in the next slide, if the coefficient of x^2 is negative, the parabola is the other way up and has a maximum instead of a minimum.

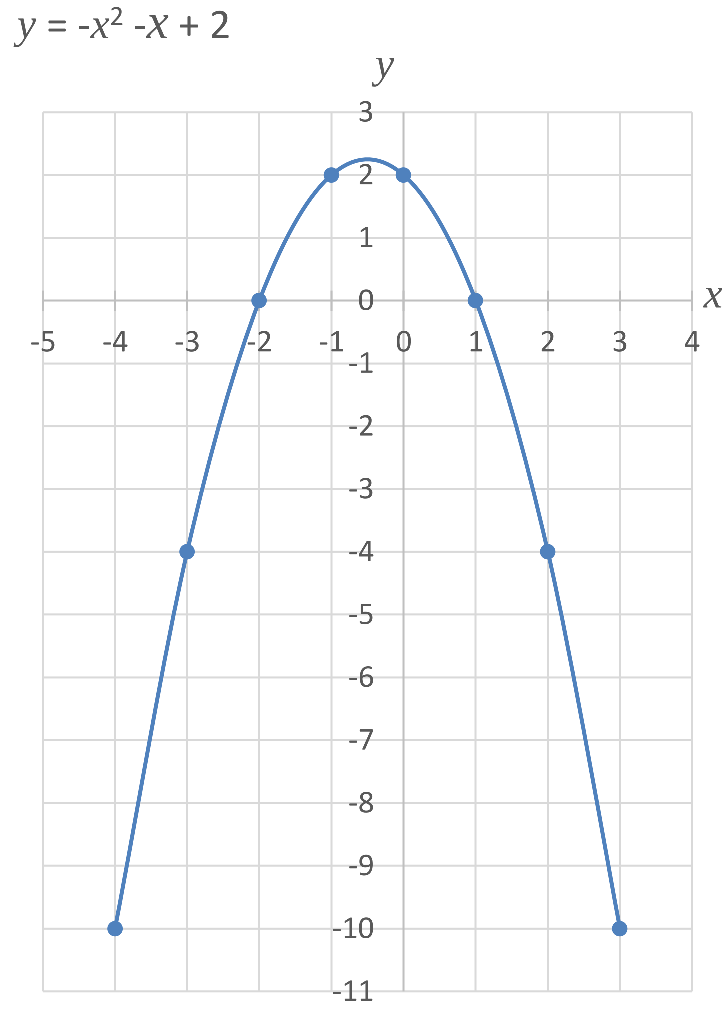


x	y
-4	5
-3	0
-2	-3
-1	-4
0	-3
1	0
2	5

A parabola with a maximum point

Opposite is the graph of $y = -x^2 - x + 2$

As there is a negative coefficient of x^2, the parabola has a maximum rather than a minimum.



Worked example

Consider the function:

$$y = 2x^2 - 1$$

a) Complete the table below and sketch a graph of the function.

x	y
-2	
-1	
0	
1	
2	

b) What are the co-ordinates of the minimum point?

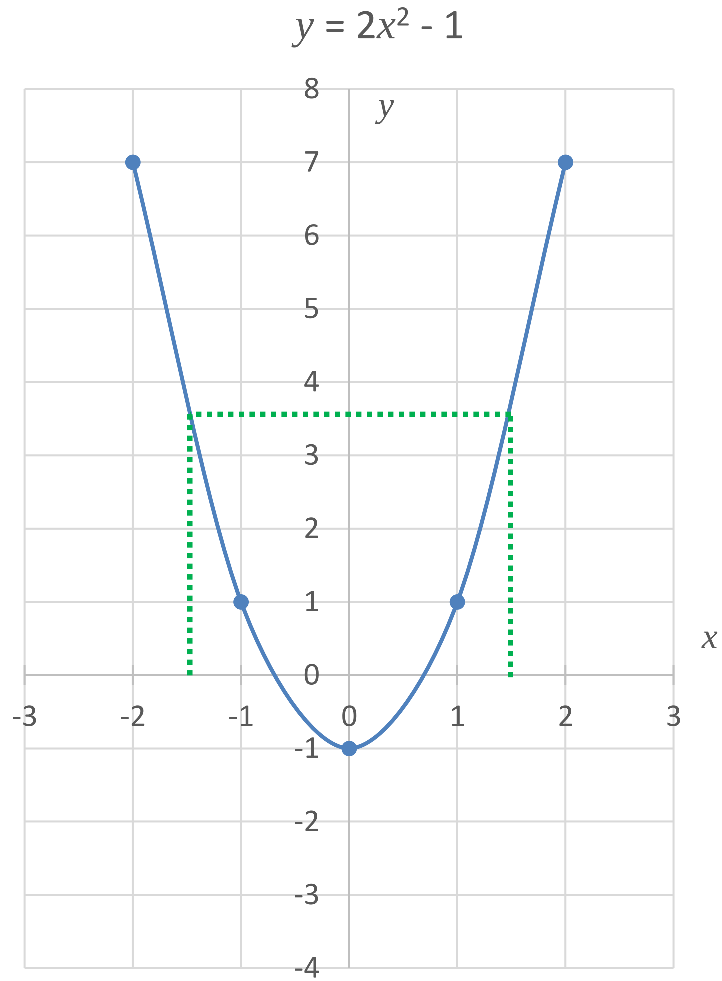
c) Estimate the values of x when $y = 3.5$

Solution

a)

$y = 2x^2 - 1$

x	y
-2	7
-1	1
0	-1
1	1
2	7



b) The minimum point is (0, -1).

c) when $y = 3.5$:

$x = -1.5$ and $x = 1.5$

(See green lines)

Using a graph to find the solutions to a quadratic equation

Imagine that we are asked to solve the following equation:

$$3x^2 + 2x - 3 = 0$$

One way to do this is by means of algebra, using the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

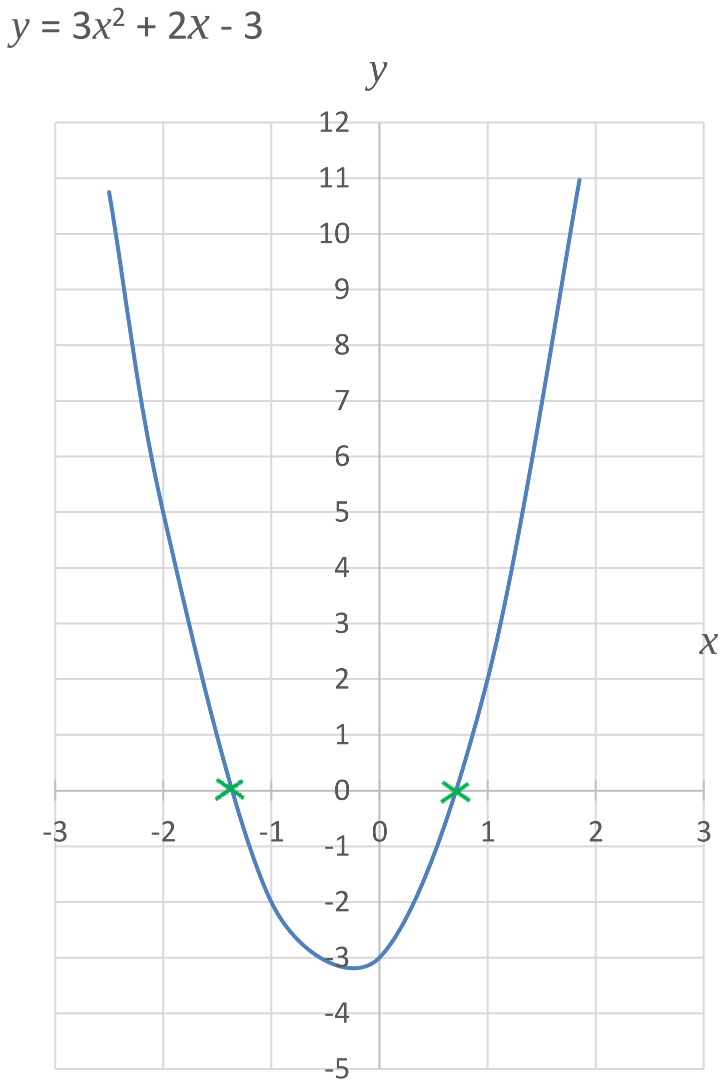
Another method is to plot the graph of

$$y = 3x^2 + 2x - 3$$

This has been done on the right.

We can now read off the values of x when $y = 0$, as marked with the green cross. This gives the solutions to the equation.

We see that these are approximately -1.4 and 0.7.



So the solutions to the equation are: $x \approx -1.4$ and $x \approx 0.7$

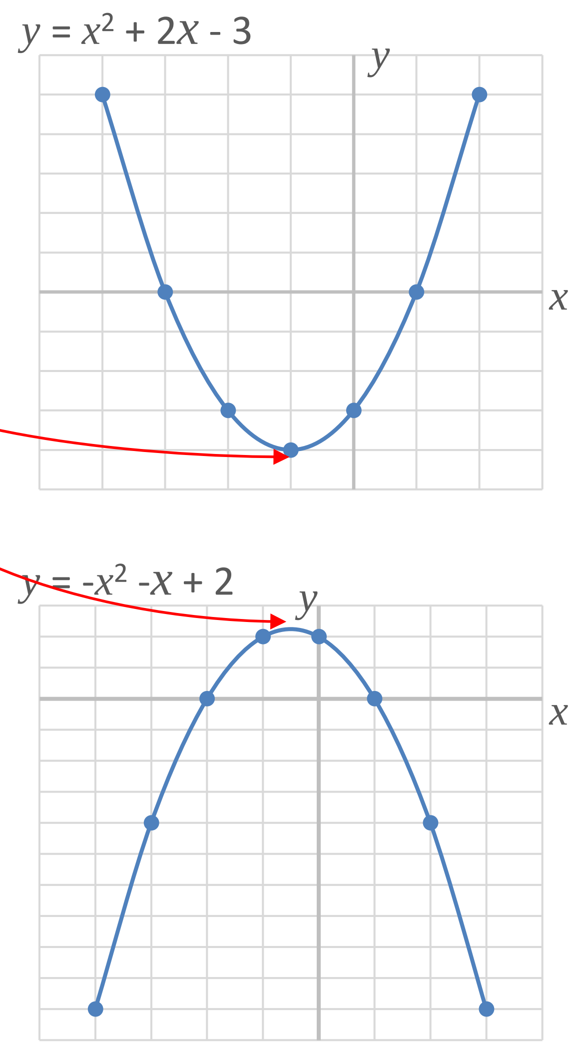
Critical points

The two examples we saw earlier are shown on the right.

In each case there is a point at which the slope of the graph changes from negative to positive or vice versa.

Such a point is called a critical point. A parabola has one critical point.

In the first example, the critical point is called the **minimum**, in the second case it is called a **maximum**.



Cubic functions

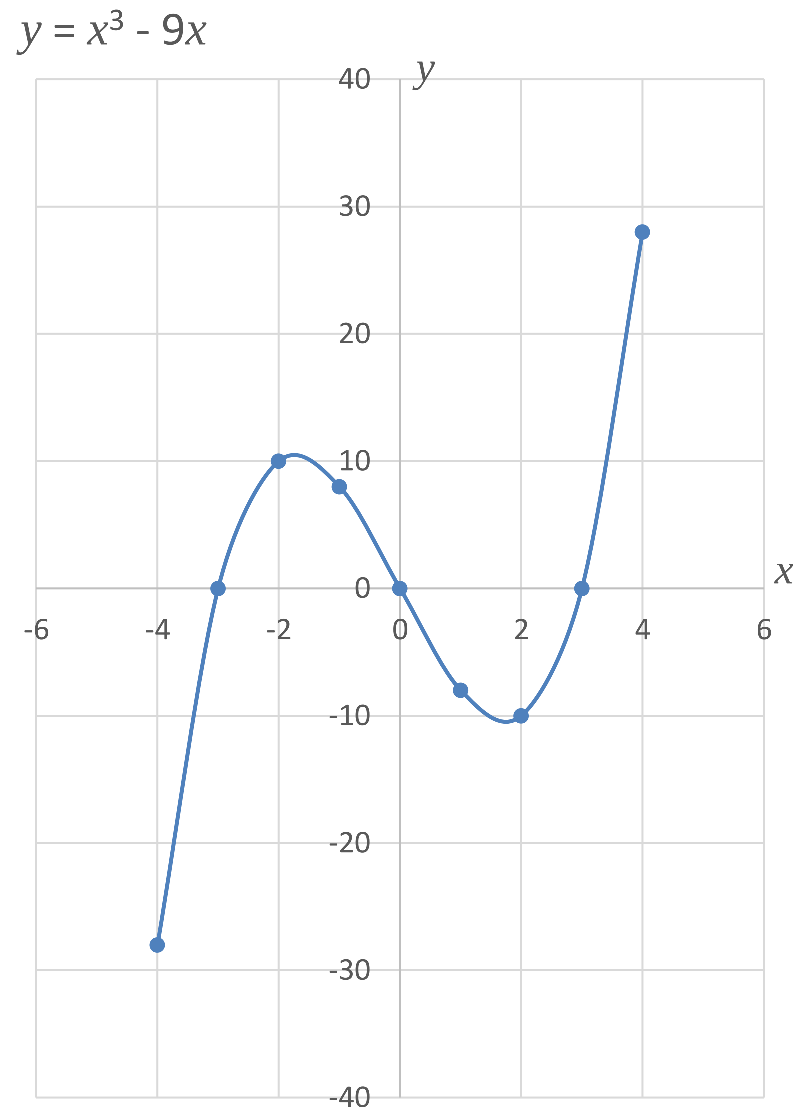
Consider the following function, which is a **cubic** function:

$$y = x^3 - 9x$$

The graph is shown on the right.

In this case, there are two critical points – a maximum and a minimum.

x	y
-4	-28
-3	0
-2	10
-1	8
0	0
1	-8
2	-10
3	0
4	28

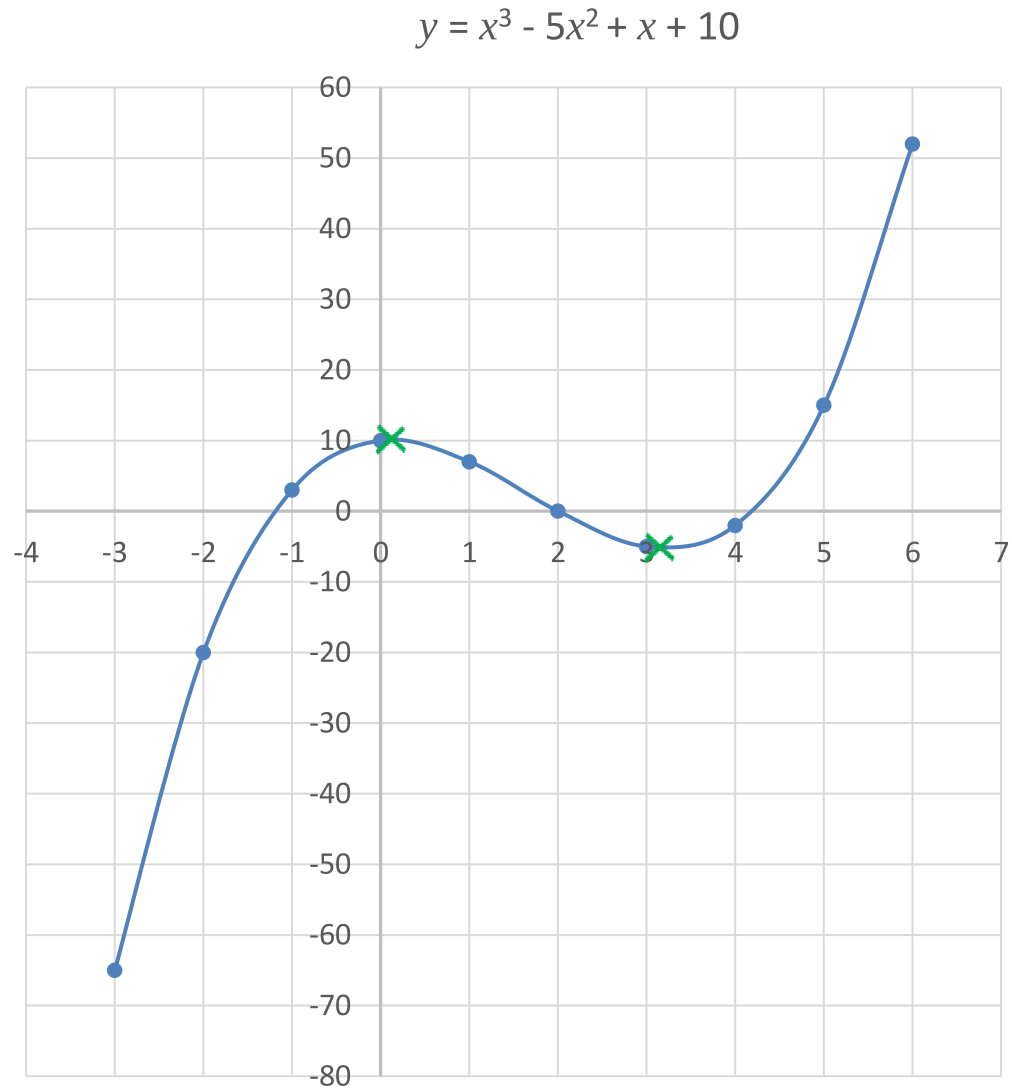


Worked example

Plot the graph of $y = x^3 - 5x^2 + x + 10$, using values for x from -3 to 6. Use your graph to estimate the coordinates of the maximum and minimum points.

Solution

x	y
-3	-65
-2	-20
-1	3
0	10
1	7
2	0
3	-5
4	-2
5	15
6	52

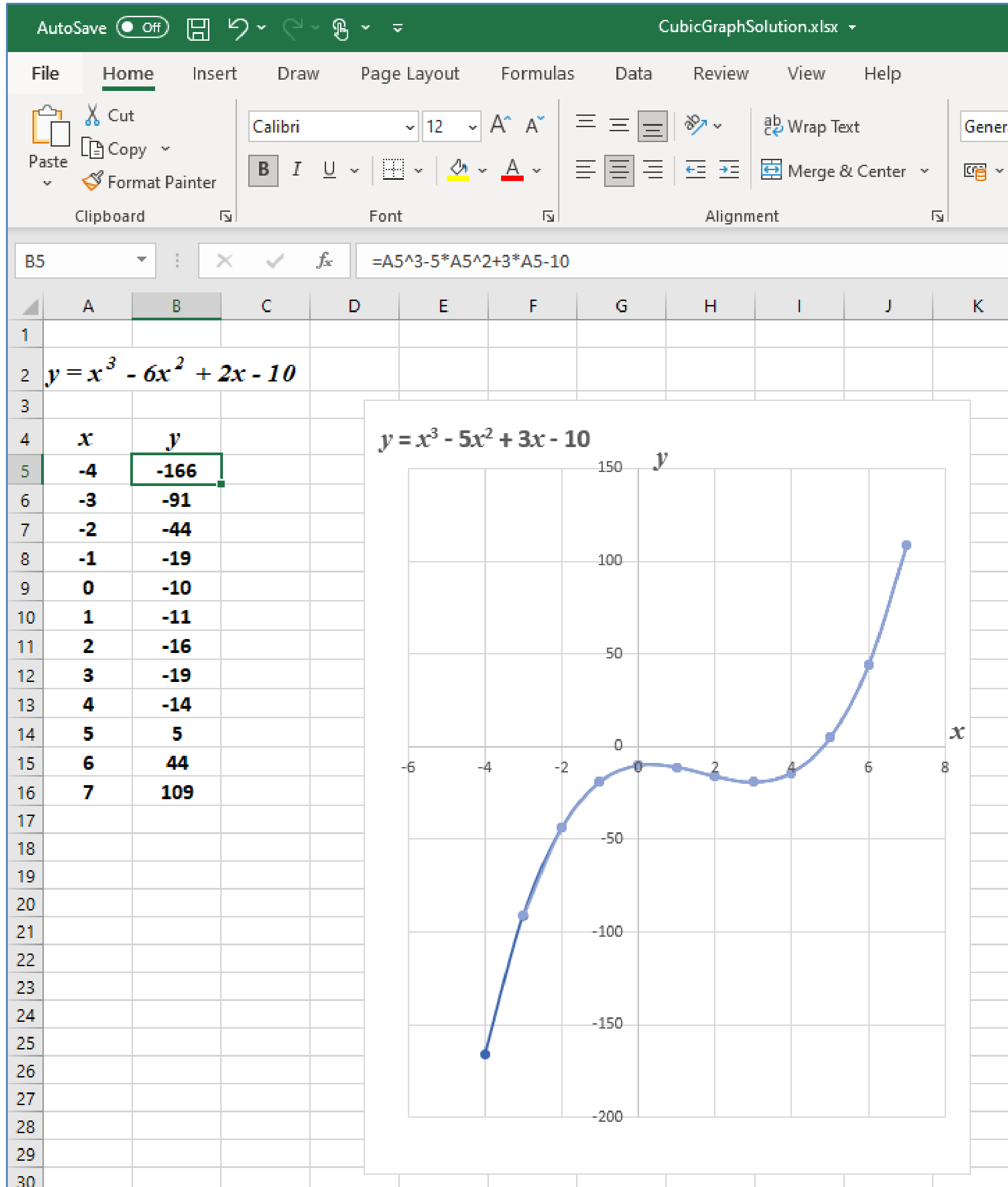


The critical points have been marked with a green cross.

The maximum point is approximately (0.1, 10.1)

The minimum point is approximately (3.2, -5.2)

A graph of a cubic function created with Microsoft Excel



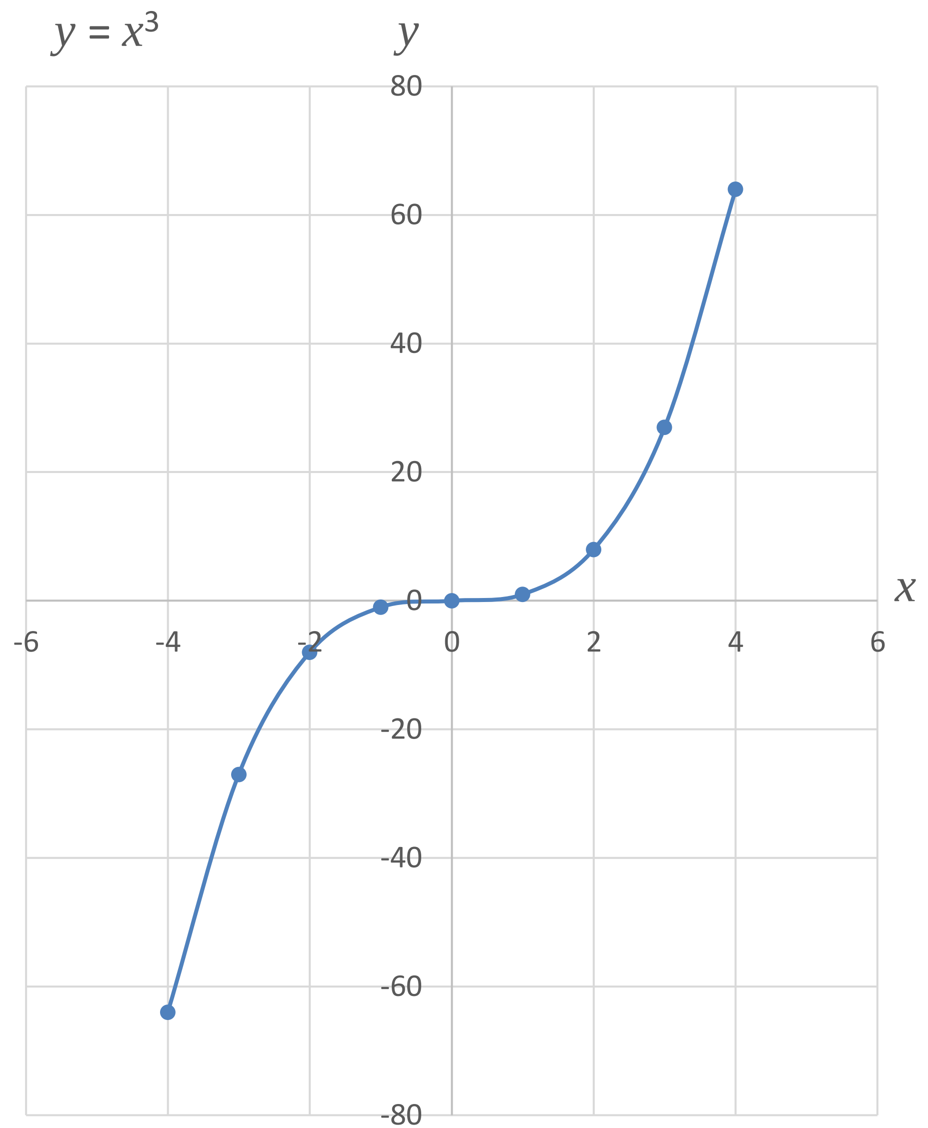
The graph of $y = x^3$

The graph of $y = x^3$ is shown opposite.

You can see that in this case there are no clearly defined maxima or minima.

Rather, the graph tends to flatten out around the origin.

This is the sort of graph to expect when the function does not contain values for x^2 or x.



Exponential functions and logarithmic functions

Two graphs are shown on the right.

The first is the graph of $y = 2^x$. You can see that at first, as the value of x increases, the value of y increases quite slowly. But as x increases further the value of y begins to increase very quickly.

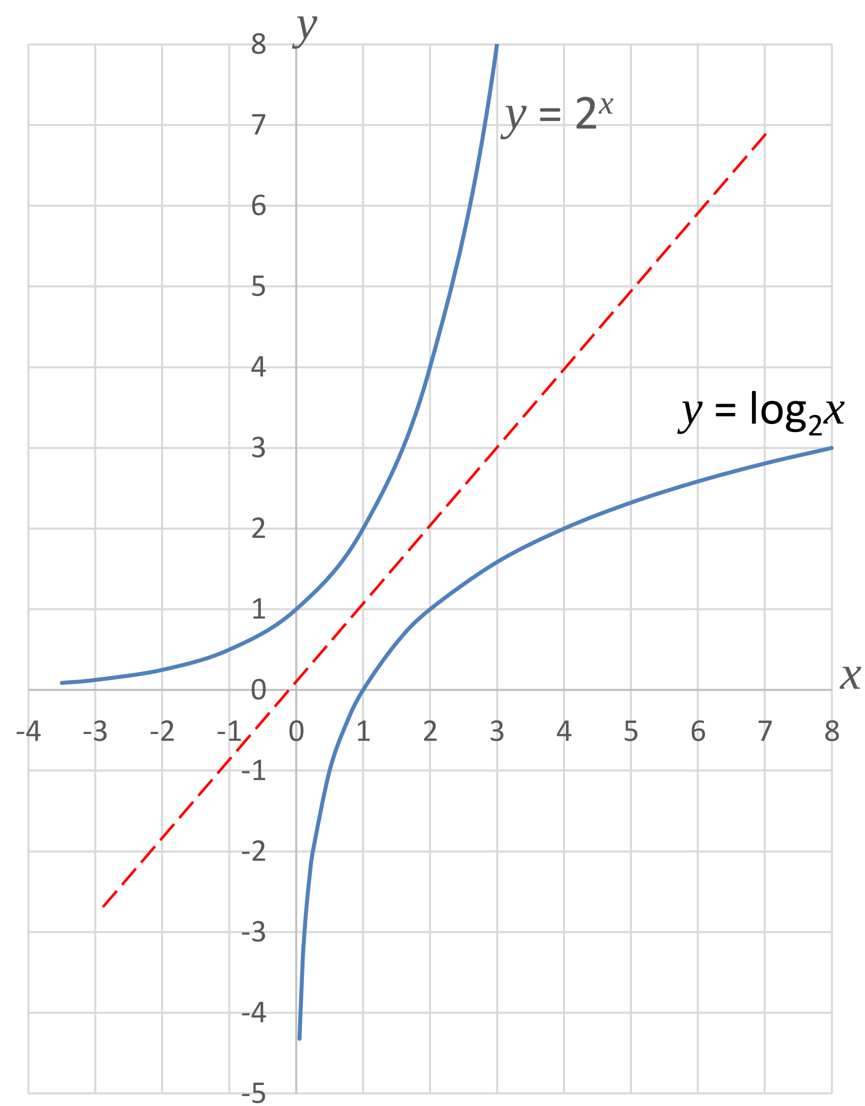
A function that behaves in this way is called an **exponential** function.

It can also be seen that as x decreases, the line gets closer and closer to the x-axis, but never actually meets it (it meets it at minus infinity!).

A line that is approached but is never met (in this case the x-axis) is called an **asymptote**.

We also see the graph of $y = \log_2 x$. You can see that this is a mirror image of the first graph, reflected about the red line. This is understandable when we consider that this is also the graph of $x = 2^y$.

In this case the graph is asymptotic with the y-axis.



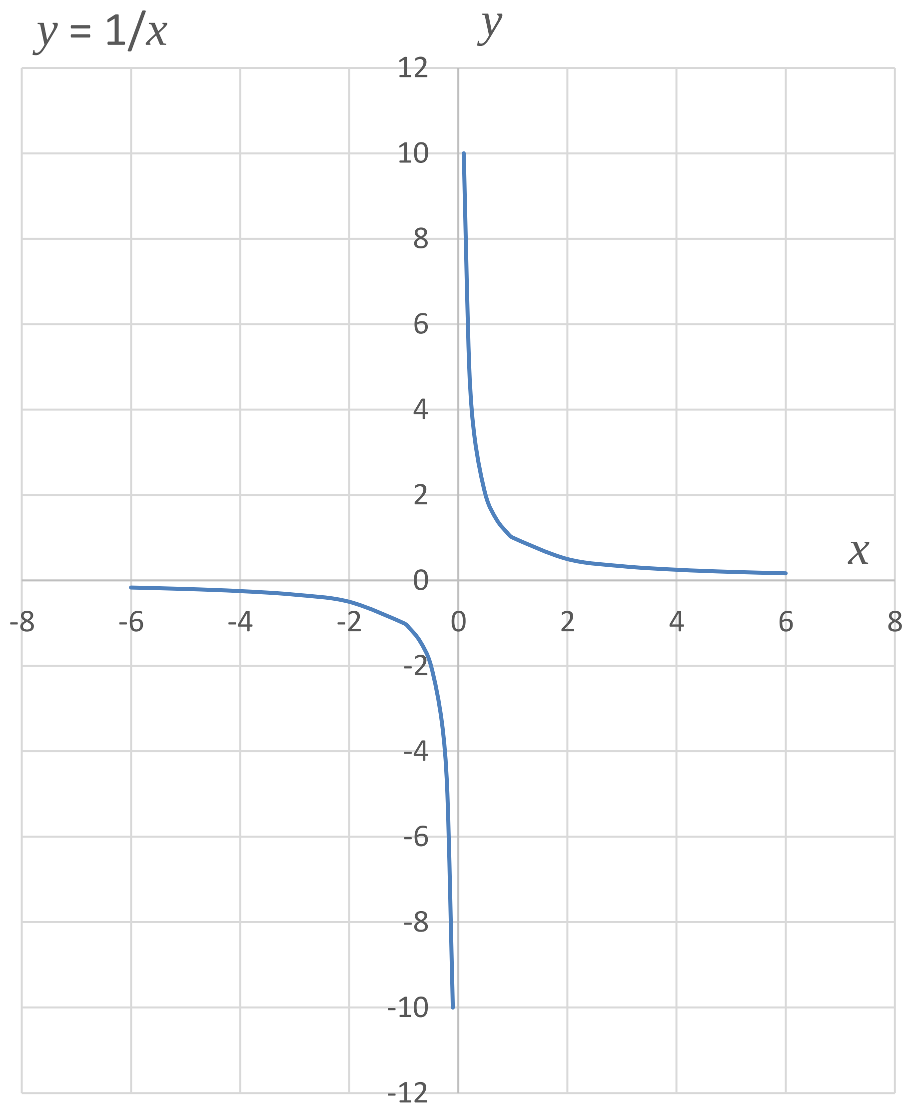
Hyperbolic functions

Opposite we see the graph of $y = 1/x$.

There are two distinct open curves which are mirror images of each other.

Such a graph is called a **hyperbola**.

In this case, each curve is asymptotic with the x- and y-axes.



Trigonometric functions

Below is the graph of $y = sin\ x$. This is known as a sine wave – it is a very important function in physics and electronics, as are other trigonometric functions such as $y = cos\ x$.