Mathematical Logic, Part 2 – Answers to Tutorial Questions

1. $D(x)$ is the predicate x *is a duck*, defined over the domain of animals.

   a) Write the following statements in words:

      i) $D(\textsc{Basil})$
      ii) $\forall x \bullet \neg D(x)$
      iii) $\exists x \bullet D(x)$
      iv) $\exists! x \bullet \neg D(x)$
      v) $\neg \forall x \bullet D(x)$

   b) If we had not stated the domain of discourse in advance, how would we have written part (ii), using A to represent the set of animals?

   <u>Solution</u>

   a) i) Basil is a duck.
      ii) No animal is a duck.
      iii) There is at least one animal that is a duck.
      iv) There exists one and only one animal that is not a duck.
      v) Not all animals are ducks.

   b) $\forall x \in A \bullet \neg D(x)$

2. $B(x)$ is the predicate x *is a bird*, and $F(x)$ is the predicate x *can fly*, both defined over the domain of animals.

a) Write the following statements in words:

i) $\forall x \bullet (B(x) \Rightarrow F(x))$
ii) $\exists x \bullet (B(x) \wedge F(x))$
iii) $B(\text{JACK}) \vee \exists x \bullet F(x)$

Solution

i) All birds can fly (for every animal, if x is a bird then x can fly).
ii) There is at least one bird that can fly (there exists an animal that is a bird and can fly).
iii) Jack is a bird or there is at least one animal that can fly.

b) Write the following statements in symbols:

i) If Mary is a bird, then no animal can fly.
ii) Only birds can fly (every animal is a bird or it cannot fly).
iii) There is one and only one bird that cannot fly.

Solution

i) $B(\text{MARY}) \Rightarrow \forall x \bullet \neg F(x)$
ii) $\forall x \bullet (B(x) \vee \neg F(x))$
iii) $\exists! x \bullet (B(x) \wedge \neg F(x))$

3. Negate the predicate $\exists x \bullet (\neg P(x) \wedge \neg Q(x))$ using the universal quantifier ($\forall$) instead of the existential quantifier ($\exists$).

Solution

$$\neg\exists x \bullet (\neg P(x) \wedge \neg Q(x)) \quad \equiv \quad \forall x \bullet \neg(\neg P(x) \wedge \neg Q(x))$$

4. Simplify your answer to question 3 by using De Morgan's law.

Solution

$$\forall x \bullet (P(x) \vee Q(x))$$

5. Show that if the following statements are true:

- if I do the ironing I have a cup of tea in the afternoon;
- if I have a cup of tea in the afternoon it is Thursday.
- I do the ironing.

then it follows that:

- it is Thursday.

<u>Solution</u>

First assign letters to the propositions:

D: I do the ironing
C: I have a cup of tea in the afternoon
T: It is Thursday

We have to show that:

$$\frac{D \Rightarrow C; C \Rightarrow T; D}{T}$$

<u>Proof</u>

1.	$D \Rightarrow C$	Premise
2.	$C \Rightarrow T$	Premise
3.	D	Premise
4.	$D \Rightarrow T$	Chain rule on 1 and 2
5.	T	Modus Ponens on 3 and 4

6. Show that if the following statements are true:

Bernard is a cat and Susan is a cat;
Bernard likes watching television.
If there is at least one cat that likes watching television then the moon is made of cheese.

then it follows that:

The moon is made of cheese.

Solution on next slide

Solution

Define the following predicates: $C(x)$: x is a cat.
$T(x)$: x likes watching television.

the following proposition: P: the moon is made of cheese.

and the following constants: b: Bernard
s: Susan

We must prove that:

$$\frac{C(b) \wedge C(s);\ \ T(b);\ \exists x \bullet (C(x) \wedge T(x)) \Rightarrow P}{P}$$

Proof

1. $C(b) \wedge C(s)$: Premise
2. $T(b)$ Premise
3. $\exists x \bullet (C(x) \wedge T(x)) \Rightarrow P$ Premise
4. $C(b)$ AND-Elimination on 1
5. $C(b) \wedge T(b)$ AND-Introduction on 2 and 4
6. $\exists x \bullet (C(x) \wedge T(x))$ Existential Generalisation on 5
7. P Modus Ponens on 3, 6

7. Show that if the following statements are true:

- Sam is a snake;
- If Sam can bite then Paris is in France;
- All snakes can bite;

then it follows that: Paris is in France.

Solution

Define the following predicates:

$S(x)$: x is a snake.
$B(x)$: x can bite.

and the following proposition: P: Paris is in France.

We must prove that:

$$\frac{S(SAM);\quad B(SAM) \Longrightarrow P;\quad \forall x \bullet (S(x) \Longrightarrow B(x))}{P}$$

Proof

1. $S(SAM)$	Premise
2. $B(SAM) \Longrightarrow P$	Premise
3. $\forall x \bullet (S(x) \Longrightarrow B(x))$	Premise
4. $S(SAM) \Longrightarrow B(SAM)$	Universal Instantiation on 3
5. $B(SAM)$	Modus Ponens on 1, 4
6. P	Modus Ponens on 2, 5

8. Prove that the following statement holds for all $n \geq 1$:

$$2 + 2^2 + 2^3 + 2^4 + \ldots + 2^n = 2^{n+1} - 2$$

Solution

Base step

Show that it holds when $n = 1$: $2 = 2^2 - 2$

Inductive step

Assume the statement is true for some value $n = k$:

$$2 + 2^2 + 2^3 + 2^4 + \ldots + 2^k = 2^{k+1} - 2 \qquad \text{Equation 1}$$

Now take the sequence up to $k + 1$:

$$2 + 2^2 + 2^3 + 2^4 + \ldots + 2^k + 2^{k+1}$$

Substituting from equation 1 this becomes:

$$2^{k+1} - 2 + 2^{k+1} = 2 \times 2^{k+1} - 2$$

$$= 2^1 \times 2^{k+1} - 2 = 2^{(k+1)+1} - 2$$

Thus if it true for $n = k$, it is true for $n = k + 1$, and since it is true for $n = 1$, it is true for all $n \geq 1$.