Permutations and Combinations
(Combinatorics)

At the end of this lecture you should be able to:

- explain the concept of a **factorial** and perform calculations on factorials;
- distinguish between a **permutation** and a **combination**;
- make calculations involving permutations and combinations, with and without repetition;
- analyse a particular scenario to determine the correct formula to use;
- explain how **Pascal's Triangle** is formed;
- describe how Pascal's Triangle is related to the subject of combinations;
- explain the importance of combinatorics to computer science.

Placing items in order

Imagine 6 runners in a race.

How many possible outcomes could there be?

In other words how many different ways are there of ordering the 6 competitors?

Position	Ways of Choosing	Comments
1	6	There are 6 different possible ways of choosing the winner.
2	6 x 5	For each winner there are 5 different choices for 2nd place. So there are 6 x 5 (or 30) ways of choosing the first 2 people.
3	6 x 5 x 4	There are 4 choices for 3rd place. So the number of ways of choosing the first 3 is 6 x 5 x 4 = 120.
4	6 x 5 x 4 x 3	We continue in the same way. So the number of ways of choosing the first 4 is 6 x 5 x 4 x 3 = 360.
5	6 x 5 x 4 x 3 x 2	The number of ways of choosing the first 5 is 6 x 5 x 4 x 3 x 2 = 720.
6	6 x 5 x 4 x 3 x 2 x 1	Once the first 5 are chosen, there is only one choice for last place. So again there are 720 (6 x 5 x 4 x 3 x 2 x 1) possibilities.

We see that the number of ways of arranging 6 items in order is:

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Factorials

The sequence that we have just seen (6 x 5 x 4 x 3 x 2 x 1) is given a special name. It is called 6 factorial and is written with an exclamation mark, like this: 6!

So: 6! = 6 x 5 x 4 x 3 x 2 x 1 = 720

7! = 7 x 6 x 5 x 4 x 3 x 2 x 1 = 5040

4! = 4 x 3 x 2 x 1 = 24

And so on.

In general: $n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots\ldots. \times 1$

Worked examples

1. In how many different ways can the letters a, b, c, d be arranged?

   Solution

   There are 4 letters, so the number of ways of arranging them is 4! = 4 x 3 x 2 x 1 = 24.

2. Find the value of $\frac{7!\times 4!}{5!}$

   Solution

$$\frac{7!\times 4!}{5!} = \frac{7\times 6\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1} \times 4\times 3\times 2\times 1}{\cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}$$

$$= 7\times 6\times 4\times 3\times 2\times 1$$

$$= 1008$$

Choosing just some of the items

Consider again the 6 runners. Imagine we are interested only in the first two places.

The number of ways we can select some or all items from a list, when (like in this example) *the order is important* is called a **permutation.**

A permutation of 2 items from 6 is written like this: $P(6,2)$.

Anther common notation is ${}^{6}P_{2}$

Look again at the table.

Position	Ways of Choosing
1	6
2	6 x 5
3	6 x 5 x 4
4	6 x 5 x 4 x 3
5	6 x 5 x 4 x 3 x 2
6	6 x 5 x 4 x 3 x 2 x 1

You can see that if we are only interested in the first two places (the first and second rows of the table), then instead of calculating the whole sequence, (6 factorial) we use only the first two and discard the last four.

Effectively we divided 6 factorial by 4 factorial.

$$P(6,2) = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}} = 6\times5 = 30$$

Similarly, if we wanted to know the possibilities for the first 4 places:

$$P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times\cancel{2}\times\cancel{1}}{\cancel{2}\times\cancel{1}} = 6\times5\times4\times3 = 360$$

The formula for permutations

In general, if we want to know how many ways we can choose k items from n items when the order in which we select them **is** important, we use the following formula:

$$P(n,k)=\frac{n!}{(n-k)!}$$

Combinations

Imagine a situation, such as a lottery, where we select a number of items, but the order in which they are selected doesn't matter.

The number of ways we can select several items from a list, when *the order is <u>not</u> important,* is called a **combination.**

The notation for combinations is similar to permutations, but uses a C instead of a P. So we can write $C(6,2)$ or ${}^{6}C_{2}$.

Another common notation for combinations is $\binom{n}{k}$ - for example $\binom{6}{2}$.

Suppose we have a small lottery consisting of 12 balls, and we have to choose 3.

There will be fewer choices now. When we were dealing with permutations we could choose three items - for example the numbers 5, 6, 7 – and arrange them in 3! ways.

But if order is unimportant there is only one way. So we have to divide the permutation by 3 factorial.

$$C(12,3) = \frac{P(12,3)}{3!} = \frac{12!}{9!\times 3!} = \frac{12\times 11\times 10\times \cancel{9}\times \cancel{8}\times \cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{\cancel{9}\times \cancel{8}\times \cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}\times 3\times 2\times 1}$$

$$= \frac{12\times 11\times 10}{3\times 2\times 1} = 220$$

The formula for combinations

In general, if we want to know how many ways we can choose k items from n items when the order in which we select them is <u>not</u> important, we use the following formula:

$$C(n,k)=\frac{n!}{(n-k)!k!}$$

What is the value of 0! ?

There are times when a calculation results in 0! , so this needs to have a value.

Imagine we have n objects. In how many different ways can we choose zero of these objects? The answer has to be 1. There is only one way of choosing zero items!

So logically: C(n,0) = 1

Therefore, using our formula for combinations: $$\frac{n!}{n! \times 0!} = 1$$

You can see that the only way this can work is if 0! is equal to 1.

Allowing repetition (when order is important)

Imagine you have to choose a 4-digit code for an intruder alarm.

- There are 10 digits available, from 0 to 9.
- Order is important.
- However, this time you are allowed to repeat digits – for example a code of 5508 or 4142.
  - There are 10 ways of choosing the first digit.
  - There are 10 ways of choosing the second digit – so the number of ways of choosing the first 2 digits is: 10 x 10.
  - There are 10 ways of choosing the third digit – so the number of ways of choosing the first 3 digits is: 10 x 10 x 10.
  - There are 10 ways of choosing the final digit – so the number of ways of choosing all 4 digits is: 10 x 10 x 10 x 10.

The number of different codes is: $10^4 = 10000$

The formula to use when order IS important and repetition IS allowed

In general, if we want to know how many ways we can choose k items from n items when the order in which we select them **is** important **and repetitions are allowed**, we use the following formula:

$$n^k$$

Allowing repetition (when order is not important)

In certain restaurants you can choose dishes by means of a colour code. Each colour represents a particular price.

- Imagine there are four colours available – red, blue, green yellow.
- You are going to choose 3 dishes.
- You want to know how many different combinations of dishes you could have - each colour can be chosen more than once.

All the possible combinations are listed below:

(R, R, R)	(R, B, B)	(R, G, Y)	(B, B, Y)	(G, G, G)
(R, R, B)	(R, B, Y)	(R, Y, Y)	(B, G, G)	(G, G, Y)
(R, R, G)	(R, G, Y)	(B, B, B)	(B, G, Y)	(G, Y, Y)
(R, R, Y)	(R, G, G)	(B, B, G)	(B, Y, Y)	(Y, Y, Y)

There are 20 different possibilities. The formula to use in this situation is given on the next slide. It is an easy formula to use, but a difficult one to derive. For those who are interested, the explanation is given later.

The formula to use when order IS NOT important and repetition IS allowed

In general, if we want to know how many ways we can choose k items from n items when the order in which we select them **is not** important **and repetitions are allowed**, we use the following formula:

$$\frac{(n+k-1)!}{k!(n-1)!}$$

Summary of formulae

Selecting k items from n items.

	Repetitions NOT allowed	Repetitions allowed
Order IS important	$P(n,k)=\dfrac{n!}{(n-k)!}$ *Permutation*	n^k
Order IS NOT important	$C(n,k)=\dfrac{n!}{(n-k)!k!}$ *Combination*	$\dfrac{(n+k-1)!}{k!(n-1)!}$

For interest only

Deriving the formula for choosing items when order is not important and repetition is allowed

We will refer to the previous example of the restaurant with colour-coded dishes. This time we will have 5 colours – red, blue, green, yellow and pink. We will again select 3 dishes.

Since order is not important, we have to find a way of eliminating duplicates such as (R, R, G) and (R, G, R). Because we are allowing repetitions, it is more complicated than simply dividing the permutation by 3! like we did before.

One way to do it is to go through the colours in order, place a marker to show that an item has been selected, and then place a different marker between each type of item.

We will list the items in this order: R, B, G, Y, P and we will use 1 to show an item is selected and 0 to separate the items.

An example will make this clear.

(R, R, G) would be (1 1 0 0 1 0 0)

Here we have put a 1 for red, another 1 for red, then a zero to show we have moved to blue. There are no blues, so another 0 shows we have moved to green. We have placed a 1 for green, then another zero to show we have moved to yellow. Finally another zero to show we have moved to pink.

(R, G, R) would be the same. So we have found a way to have only one string for each combination.

Another couple of examples:

(R, Y, P) would be (1 0 0 0 1 0 1)
(B, Y, Y) would be (0 1 0 0 1 1 0)

You can see that there are always four 0s and three 1s – 7 places in all.

Our question can now be put in a different way: If there are 7 places, how can we choose 3 of them to put our 1s in? We are finding C(7, 3).

In general: There will always be one fewer 0's than the total number of items, as there is a 0 between each item. So if there are n items and we are choosing k of them, then there will be $n + k - 1$ places, and we must chose k of them. We need to calculate C($n + k - 1$, k).

So we have: $$\mathrm{C}(n+k-1,k)=\frac{(n+k-1)!}{k!(n+k-1-k)!}=\frac{(n+k-1)!}{k!(n-1)!}$$

Worked examples

1. A teacher has a class of 30 students

    a) She has to chose 3 students to be awarded a prize for good progress. How many different sets of three can be chosen?

    Solution

    As there is no ordering involved here, and no repetition, we use the formula for combinations:

$$C(n,k)=\frac{n!}{(n-k)!k!}$$

In this case n = 30 and k = 3

$$C(30,3)=\frac{30!}{27!\times 3!}=\frac{30\times 29\times 28}{3\times 2\times 1}=4060$$

b) The teacher now has to choose 3 students to be awarded 1st, 2nd and 3rd prize for overall performance. How many different possibilities are there?

<u>Solution</u>

This time there is ordering, but once again no repetition, so we use the formula for permutations:

$$P(n,k) = \frac{n!}{(n-k)!}$$

As in the previous question, n = 30 and k = 3

$$P(30,3) = \frac{30!}{27!} = 30 \times 29 \times 28 = 24360$$

c) A prize is to be awarded for mathematics, science, and languages. No student should receive more than one prize.

How many different possibilities are there?

<u>Solution</u>

In fact this is exactly the same as part b). Just think of it as placing the students in a line with, say, the winners of the mathematics, science and language prizes in first, second and third position.

So the answer is the same as before, 24360

d) Once again there are to be three awards – mathematics, science and languages. But this time the rules are relaxed, and each student is eligible for all three prizes. So any student could receive as many as three prizes.

How many possibilities are there?

<u>Solution</u>

Once again there is ordering, but this time with repetition, so the correct calculation is:

$$n^k = 30^3 = 27000$$

2. A five-a-side football club consists of six different teams.

Three members of the club are to be chosen for a special award. They can be chosen from any team, so that a particular team could end up three awards, two awards, one award or no awards.

How many different ways can the awards be distributed among the teams?

<u>Solution</u>

There is no ordering but there is repetition, because one team could win all three awards, none of the awards, or any number in between.

Because there are 6 teams, n = 6 and k = 3.

The correct calculation for no ordering with repetition allowed is:

$$\frac{(n+k-1)!}{k!(n-1)!} = \frac{8!}{3!\times 5!}$$

$$= \frac{8\times 7\times 6}{3\times 2\times 1}$$

$$= 56$$

3. Five friends are planning a trip to the cinema. They will all sit together in a row.

   Leroy and Calvin want to sit together. In how many different ways can the five friends occupy the row?

Solution

The easiest thing to do is to consider Leroy and Calvin as one item.

We now have to work out how many ways to arrange four items, which is 4! .

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

There are, however, 2! ways in which Leroy and Calvin can be arranged, so for each of the above arrangements there are 2 options.

So the total number of arrangements is: 2 x 24 = 48

4. How many 4-digit numbers can be made from the digits 1-9 if you are not allowed to repeat numbers, and the number must be divisible by 5?

<u>Solution</u>

If the number is divisible by 5, it must end in a 5 (since there is no zero).

So this comes down to arranging 3 numbers out of 8, without repetition.

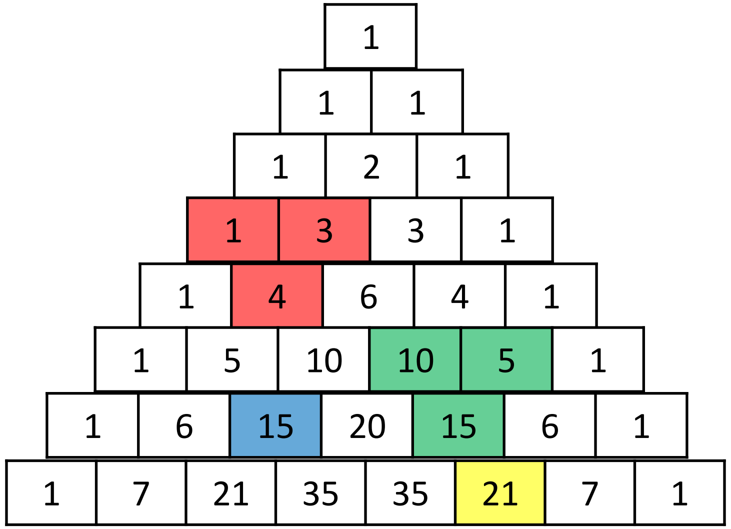
$$P(8,3) = \frac{8!}{5!} = 8 \times 7 \times 6 = 336$$

Pascal's Triangle

Pascal's triangle is a very well known number pattern, used in several areas of mathematics.

The first 8 rows are shown on the right.

Each number is made up by adding the two numbers above it (shown in the diagram by the red and green squares).



Interestingly, each number represents C(n, k), where n is the row number (starting from 0) and k the column number (also starting from 0).

For example, the blue square containing the number 15, represents C(6, 2) and the yellow square, containing the number 21, is C(7, 5).

Pascal's triangle demonstrates clearly that combinations are symmetrical. C(n, k) is the same as C(n, n - k). For example, C(8, 3) is the same as C(8, 5).

This can also be seen from the formula for C(n, k) – and by logic: choosing 2 items from 7 is the same as choosing 5 items to discard.

Worked example

a) Use Pascal's Triangle to find the value of C(5,2).

b) Verify your answer using the formula.

<u>Solution</u>

a) Because we start at zero, we will need to draw 6 rows of the triangle

C(5,2) = 10

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

b) $$C(5,2) = \frac{5!}{3! \times 2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 10$$

Binomial expansion

A binomial expression is an expression that contains two terms. Imagine we had to calculate an expression such as:

$$(a + b)^n$$

where n is a natural number.

Our knowledge of combinatorics can help us with this.

The first few expansions would give us:

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The coefficients form the following pattern:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

which you will recognise as Pascal’s triangle.

Binomial expansion (continued)

Knowing, as we do, that each entry in Pascal's triangle represents a combination, we can rewrite our expansions

For example we saw that: $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

We could write this as : $(a + b)^4 = {}^4C_0a^4b^0 + {}^4C_1a^3b^1 + {}^4C_2a^2b^2 + {}^4C_3a^1b^3 + {}^4C_4a^0b^4$

In general we could write: $(a + b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ . \ . \ . \ + {}^nC_{n-1}a^1b^{n-1} + {}^nC_na^0b^n$

In mathematics, we often use the upper case Greek letter sigma, Σ, to mean *the sum of*.

Using this notation we could write:

$$(a + b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$$

This means that you start of with $k = 0$, than add every term until you reach n.

This is known as the **binomial theorem**.

Worked examples

1. Use the binomial theorem to expand the expression $(x + 2y)^5$

<u>Solution</u>

$$(a + b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ . \ . \ . \ + {}^nC_{n-1}a^1b^{n-1} + {}^nC_na^0b^n$$

The first term in our expression is x, the second is $2y$, and in this case $n = 5$.

$$(x + 2y)^5 = {}^5C_0x^5\,(2y)^0 + {}^5C_1x^4(2y)^1 + {}^5C_2x^3(2y)^2 + {}^5C_3x^2(2y)^3 + {}^5C_4x^1(2y)^4 + {}^5C_5x^0(2y)^5$$

$$= x^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + (2y)^5$$

$$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y + 80xy^4 + 32y^5$$

2. Use the binomial theorem to find the 3rd term in the expansion of the expression $(x - y)^7$

Solution

We saw that

$$(a+b)^n = \sum_{k=0}^{n} {}^{n}\mathrm{C}_{k} a^{n-k} b^k$$

Because we start at 0, the 3rd term will be found when $k = 2$.

Here $a = x$ and $b = -y$, and $n = 7$.

The third term is therefore: ${}^{7}\mathrm{C}_{2} x^{5}(-y)^2$

$= 21x^5y^2$

Application to computing

There are many examples in computer science where we need to compute the number of different ways of doing things.

In computer graphics, for example, colour is made by combining the three primary colours, red, blue and green, in different intensities.

Different systems are used to achieve this, and it is very important to know how many different colours can be obtained using a particular system.

Another very important application is in the area of **algorithm efficiency**.

When writing the code for a particular task, there are always many ways of achieving the same goal.

Some algorithms are a great deal more efficient than others, because they require fewer steps.

Combinatorics plays a very important role in comparing different algorithms by determining the number of possible steps involved in each one.