Mathematical Logic – Part 2

At the end of this lecture you should be able to:

- define the term **predicate**;
- explain the notion of a predicate as a Boolean function;
- give values to the variables in predicates by **substitution** and **quantification**;
- negate quantified predicates;
- state the following laws of natural deduction: **modus ponens**, **modus tollens** and the **chain rule**;
- apply the above laws to very simple proofs;
- explain how mathematical logic is used in formal methods of software engineering.

Predicate Logic

We have learnt that a *proposition* is an expression that has a value of TRUE or FALSE.

A **predicate** is also an expression that has a value of TRUE or FALSE.

However a predicate contains *variables*; we don't know if the predicate is TRUE or FALSE until we know the value of the variables.

Examples of predicates

$P(x)$:	x is a prime number	(usually pronounced P of x)
$T(x, y)$	x is taller than y	
$D(x)$:	x is a duck	

As you can see, we don't know whether $P(x)$, for example, is TRUE or FALSE until we know the value of x.

For example: $P(5)$ is TRUE

but: $P(10)$ is FALSE

Predicates are functions

A predicate, just like a function, accepts variables and produces an output.

In this case the output is a truth value – either TRUE or FALSE.

So a predicate is *a truth-valued function*.

If A is the set of people, then the predicate $T(x, y)$ from the previous slide would have the following signature:

$$T: A \times A \to \mathbb{B}$$

$\mathbb{B}$ is a special, pre-defined set known as the **Boolean** set.

$$\mathbb{B} = \{\text{TRUE}, \text{FALSE}\}$$

The Domain of Discourse

When we have a predicate such as $P(x)$, it is always important to know the set of values from which is x drawn.

This set is know as the **domain of discourse**.

For a predicate like "x is an even number", our domain of discourse might be natural numbers.

For a predicate such as "studies physics", our domain might be people or students.

We can make our domain of discourse clear in one of two ways.

- we can simply state it in advance;
- we can incorporate it into the definition, as we shall in the coming slides.

Giving values to the variables

Once we give a value to the variables, we know whether the predicate is TRUE or FALSE. This is also known as **binding** the variables.

There are two ways of doing this.

1. **Substitution**

   As we saw previously, we can substitute specific values for the variables.

   Using the examples form the earlier slide:

$P(7)$:	7 is a prime number
T(MARY, AHMED):	Mary is taller than Ahmed
D(BASIL):	Basil is a duck

2. **Quantification**

   A **quantifier** is a device for making a statement about a *set* of values, not just one value.

   There are three quantifiers that we can use, each with its own symbol.

The Universal Quantifier, $\forall$

This quantifier enables us to make a statement about *all* the elements in a particular set:

If $P(x)$ is a predicate:

$\forall x \bullet P(x)$ reads: For *all* x, $P(x)$ is true.

This assumes we have already stated our domain of discourse in advance.

If we had not, we would have to include it in the statement – for example if the domain of discourse was the set A, we would write:

$$\forall x \in A \bullet P(x)$$

<u>Example</u>

If $M(x)$ is the predicate x *chases mice* (defined over the set of cats), then

$\forall x \bullet M\ (x)$ means: All cats chase mice.

The Existential Quantifier $\exists$

This quantifier makes a statement about *at least one* of the elements in a particular set:

If $P(x)$ is a predicate:

$\exists x \bullet P(x)$ reads: There exists *at least one* x, for which $P(x)$ is true.

As before, this assumes we have already stated our domain of discourse in advance – otherwise (if our domain of discourse was the set A) we would write:

$$\exists x \in A \bullet P(x)$$

<u>Example</u>

If $M(x)$ is the predicate x *chases mice* (defined over the set of cats), then

$\exists x \bullet M(x)$ means: There is at least one cat that chases mice.

The Unique Existential Quantifier $\exists!$

This quantifier makes a statement about *one and only one* of the elements in a particular set:

If $P(x)$ is a predicate:

$\exists! x \bullet P(x)$ reads: There exists *one and only one* x, for which $P(x)$ is true.

<u>Example</u>

Our domain of discourse is the set of cats.

If $M(x)$ is the predicate x *chases mice*, then

$\exists! x \bullet M(x)$ means: There is one and only one cat who chases mice.

Worked examples

1. $B(x)$ is the predicate x *can bark*, defined over the domain of dogs.

   a) Write the following statements in words:

      i) $B(\text{ROVER})$
      ii) $\forall x \bullet B(x)$
      iii) $\exists x \bullet B(x)$
      iv) $\exists! x \bullet B(x)$
      v) $\neg B(\text{ROVER})$
      vi) $\neg \forall x \bullet B(x)$
      vii) $\neg \exists x \bullet B(x)$

   b) If we had not stated the domain of discourse in advance, how would we have written part (ii), using D to represent the set of dogs?

<u>Solution</u>

a) i) Rover can bark.
   ii) All dogs can bark.
   iii) There is at least one dog that can bark.
   iv) There is one and only one dog that can bark.
   v) Rover cannot bark.
   vi) It is not true that all dogs can bark.
   vii) There does not exist a dog that can bark.

b) $\forall x \in D \bullet B(x)$

2. $B(x)$ is the predicate x *can bark*, and $T(x)$ is the predicate x *has a tail*, both defined over the domain of dogs.

Write the following in words:

a) $\forall x \bullet (B(x) \wedge T(x))$

b) $\exists x \bullet (B(x) \wedge \neg T(x))$

c) $\forall x \bullet B(x) \wedge \exists x \bullet \neg T(x))$

Solution

a) All dogs can bark and have tails.

b) There is at least one dog that can bark and that has no tail.

c) All dogs can bark, and there is at least one dog that has no tail.

3. $D(x)$ is the predicate x *is a duck*, and $S(x)$ is the predicate x *can swim*, both defined over the domain of animals.

a) Write the following statements in words:

i) $\forall x \bullet (D(x) \Rightarrow S(x))$
ii) $\exists x \bullet (D(x) \wedge \neg S(x))$
iii) $D(\text{B}\small{\text{ASIL}}) \wedge \exists! x \bullet S(x)$

b) Write the following statements in symbols:

i) If Basil is a duck, then all animals can swim.
ii) Only ducks can swim (every animal is a duck or it cannot swim).
iii) There is one and only one duck that can swim.

<u>Solution</u>

a) i) All ducks can swim (if x is a duck then x can swim).
ii) There exists an animal that is a duck and cannot swim (there is at least one duck that cannot swim)
iii) Basil is a duck and there is one and only one animal that can swim.

b) i) $D(\text{B}\small{\text{ASIL}}) \Rightarrow \forall x \bullet S(x)$
ii) $\forall x \bullet (D(x) \vee \neg S(x))$
iii) $\exists! x \bullet (D(x) \wedge S(x))$

Negating quantified predicates

Consider the following predicate, defined over the domain of students at a particular university.

$$M(x)\text{: } x \text{ is good at maths}$$

The following statement means that it is not true that all students are good at maths:

$$\neg\forall x \bullet M(x)$$

This means that there must be at least one student that is not good at maths:

$$\exists x \bullet \neg M(x)$$

Therefore: $\neg\forall x \bullet M(x) \equiv \exists x \bullet \neg M(x)$

Similarly, the following statement means that there does not exist a single student who is good at maths:

$$\neg\exists x \bullet M(x)$$

This is the same as saying that all students are poor at maths:

$$\forall x \bullet \neg M(x)$$

Therefore: $\neg\exists x \bullet M(x) \equiv \forall x \bullet \neg M(x)$

Worked example

Negate the predicate $\forall x \bullet (\neg P(x))$

<u>Solution</u>

$$\neg \forall x \bullet (\neg P(x))$$

$$\equiv \exists x \bullet \neg(\neg P(x))$$

$$\equiv \exists x \bullet P(x))$$

Proof by Natural Deduction

Mathematical logic gives us the ability to make **logical arguments**.

The way we do this is to make a number of premises, and to show that if these **premises** are correct, a certain **conclusion** follows.

The notation for this is to place the premises above a horizontal line, and the conclusion below it.

For example:

$$\frac{P;\, Q;\, R}{S}$$

This means that if P, Q and R are true, then it follows that S is also true.

To help us with this, there are some basic laws that follow from the definitions we learnt in the previous lecture.

Modus Ponens

Imagine these two statements are true:

> If it’s Wednesday I have eggs for breakfast.
> It is Wednesday.

It follows that I have eggs for breakfast.

This is the law of **modus ponens**. It is stated formally like this:

$$\frac{P \Rightarrow Q; P}{Q}$$

It follows directly from our understanding of the implication operator and its truth table.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

It tells us that if $P \Rightarrow Q$ is true and P is also true, then Q must be true (the first line of the truth table).

Modus Tollens

Imagine these two statements are true:

> If it’s Wednesday I have eggs for breakfast.
> I do not have eggs for breakfast.

It follows that it is not Wednesday.

This is the law of **modus tollens**. It is stated formally like this:

$$\frac{P \Rightarrow Q; \neg Q}{\neg P}$$

Again, it follows directly from our understanding of the implication operator and its truth table.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

It tells us that if $P \Rightarrow Q$ is true and Q is false, then P must be false (the fourth line of the truth table).

The Chain Rule

Imagine these two statements are true:

> If it’s Friday I go shopping.
> If I go shopping I wear my yellow hat.

It follows that if it’s Friday I wear my yellow hat.

This is the **chain rule**. It is stated formally like this:

$$\frac{P \Rightarrow Q; Q \Rightarrow R}{P \Rightarrow R}$$

It arises naturally from our definition of implication.

Worked example

Show that if the following statements are true:

- if the sun is hot then the moon is made of cheese;
- if the moon is made of cheese then grass is blue;
- grass is not blue.

then it follows that:

- the sun is not hot.

Solution

First assign letters to the propositions:

S: the sun is hot
M: the moon is made of cheese
G: grass is blue

We have to show that:

$$\frac{S \Rightarrow M; M \Rightarrow G; \neg G}{\neg S}$$

1.	$S \Rightarrow M$	Premise
2.	$M \Rightarrow G$	Premise
3.	$\neg G$	Premise
4.	$S \Rightarrow G$	Chain rule on 1 and 2
5.	$\neg S$	Modus Tollens on 3 and 4

We have shown that our premises lead to the conclusion $\neg S$: the sun is not hot.

Alternative names

You might come across the following alternative names for the three rules we have just learnt:

- Modus ponens: the rule of detachment
- Modus tollens: the rule of contraposition
- The chain rule: the rule of syllogism

Some other laws

The rules stated below follow directly from the truth tables for AND and OR respectively.

<u>The rule of AND-elimination</u>

$$\frac{P \wedge Q}{P} \qquad \frac{P \wedge Q}{Q}$$

If $P \wedge Q$ is true then it follows that P is true and Q is true.

<u>The rule of AND-introduction</u>

$$\frac{P;Q}{P \wedge Q}$$

If P is true and Q is true, it follows that $P \wedge Q$ is true.

<u>The rule of OR-introduction</u>

$$\frac{P}{P \vee Q}$$

If P is true it follows that $P \vee Q$ is true

Universal Instantiation

$$\frac{\forall x \cdot P(x)}{P(c)} \text{ for any } c \text{ in the domain}$$

If we know something is true for all members of a group, we can conclude it is also true for any arbitrary member of this group.

Existential Generalization

$$\frac{P(c)}{\exists x \cdot P(x)}$$

Suppose we know that $P(c)$is true for some constant c.

Then, there exists an element for which P is true.

Thus, we can conclude $\exists x \cdot P(x)$

Worked examples

1. Show that if the following statements are true:

   - If maths is easy and physics is hard then apples are green;
   - If physics is hard then maths is easy;
   - Physics is hard;

   then it follows that:

   - Apples are green.

<u>Solution</u>

We will define the following propositions: P: Maths is easy
Q: Physics is hard
R: Apples are green

We need to show that: $$\frac{(P \wedge Q) \Rightarrow R; Q \Rightarrow P; Q}{R}$$

1. $(P \wedge Q) \Rightarrow R$	Premise
2. $Q \Rightarrow P$	Premise
3. Q	Premise
4. P	Modus ponens on 2, 3
5. $P \wedge Q$	AND-introduction on 3, 4
6. R	Modus ponens on 5, 1

2. Show that if the following statements are true:

- all dogs can bark;
- Rover is a dog;

then it follows that:

- Rover can bark.

Solution

We will define the following predicates: $P(x)$: x is a dog
$Q(x)$: x can bark

and the following constant: r: Rover

We have to show that:

$$\frac{\forall x \bullet (P(x) \Rightarrow Q(x)); P(r)}{Q(r)}$$

1.	$\forall x \bullet (P(x) \Rightarrow Q(x))$	Premise
2.	$P(r)$	Premise
3.	$P(r) \Rightarrow Q(r)$	Universal instantiation on 1
4.	$Q(r)$	Modus ponens on 2, 3

Application to computing

Formal methods

- certain commercial applications require very high integrity software;
- such situations include safety critical systems and secure financial systems.

- **formal methods** are often used to produce such software;
- with such methods the specification is written not just in a natural language such as English, but in the language of mathematics;
- in this way the specification can be far more precise and unambiguous.

- the mathematics involves the propositional and predicate logic that we have studied here;
- using proofs such as those we have studied, the process can be taken further and used to develop the software itself;
- the applications developed in this way will be of a higher integrity than those developed by non-formal methods.

- well known formal methods include:
    - VDM (Vienna Development Method);
    - Z;
    - B-Method.

Proof by induction

Mathematical induction is a technique that can be used to prove certain statements about natural numbers.

In its simplest form, mathematical induction shows that a statement involving a natural number n holds for all values of n.

The proof involves two steps:

1. The **base step**

   - prove that the statement holds for the first natural number n, usually, $n = 0$ or $n = 1$.

2. The **inductive step**

   - show that if the statement is true for some natural number k, then it is also true for $k + 1$.

   - this shows that if the statement is true when $n = 0$ (for example), it is also true when $n = 1$, $n = 2$, $n = 3$ and so on up to infinity.

Proof by induction – example 1

Prove that the following statement holds for all natural numbers:

$$1 + 2 + 3 + \ ... + n = \frac{n(n+1)}{2}$$

Solution

Base step

Show that it holds when n = 0.

This is easily shown:

$$0 = \frac{0(0+1)}{2}$$

Proof by induction – example 1 (continued)

Inductive step

Assume the statement is true for some value $n = k$:

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \qquad \text{Equation 1}$$

Now take the sequence up to $k + 1$:

$$1 + 2 + 3 + \ldots + k + k + 1$$

Substituting from equation 1 this becomes:

$$\frac{k(k+1)}{2} + k + 1$$

$$= \frac{k^2 + k + 2k + 2}{2}$$

$$= \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

which can be written as: $\dfrac{(k+1)((k+1)+1)}{2}$

Thus if it true for $n = k$, it is true for $n = k + 1$, and since it is true for $n = 0$, it is true for all n.

Proof by induction – example 2

Prove that for all $n \geq 1$, the expression $n^3 + 2n$ is divisible by 3.

<u>Solution</u>

Base step

Show that it holds when $n = 1$:

$1^3 + 2 \times 1 = 3$, which is divisible by 3

Proof by induction – example 2 (continued)

Inductive step

Assume the statement is true for some value $n = k$:

$$k^3 + 2k = 3M \qquad \text{Equation 1}$$

where M is a positive integer.

When $n = k + 1$, the expression becomes:

$$(k + 1)^3 + 2(k + 1)$$

$$= (k + 1)(k + 1)^2 + 2(k + 1)$$

$$= (k + 1)(k^2 + 2k + 1) + 2(k + 1)$$

$$= (k^3 + 3k^2 + 3k + 1) + 2(k + 1)$$

$$= (k^3 + 3k^2 + 5k + 3) = (k^3 + 2k) + (3k + 3k^2 + 3)$$

Substituting from equation 1 this becomes:

$$3M + (3k + 3k^2 + 3) = 3(M + k + k^2 + 1)$$

This is divisible by 3. Thus if it true for $n = k$, it is true for $n = k + 1$, and since it is true for $n = 1$, it is true for all $n \geq 1$.