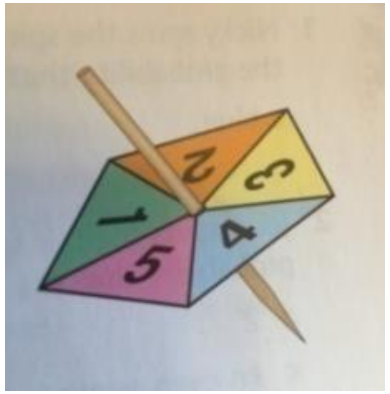
Probability

At the end of this lecture you should be able to:

- define the terms **experiment**, **outcome**, **sample space** and **event**;
- determine distinct sets for a sample space and event from a given scenario;
- provide a definition of **probability**;
- calculate the probability of a particular event being successful;
- represent a probability distribution using a **discrete random variable**;
- calculate the **expected value** of an experiment;
- distinguish between **mutually exclusive** and **non-mutually exclusive** events;
- use the **addition rule** to determine the probability of two alternative events happening;
- distinguish between independent and conditional events, and calculate the **probability** in both cases;
- construct and interpret **tree diagrams** for conditional events.

Probability

Terminology and definitions



Consider the 5-sided spinner that is shown in the diagram.

If the spinner is not weighted in any way, and there is an equal chance of it landing on any one of the five numbers, it is called a **fair** spinner.

If we spin the spinner once, this is called an **experiment**.

The number that it lands on is called an **outcome**.

The set of all possible outcomes is called the **sample space**.

A set of possible outcomes is called an **event**.

The **probability** of an event is a measure of how likely it is that that event will happen.

Outcomes, sample space and events

For our spinner there are five possible outcomes:

The spinner can land on 1.
The spinner can land on 2.
The spinner can land on 3.
The spinner can land on 4.
The spinner can land on 5.

If we call the sample space for the spinner S, then: $S = \{1, 2, 3, 4, 5\}$

Let's define a few events:

A: the event that the spinner lands on a 5
B: the event that the spinner lands on a 3 or 4
C: the event that the spinner lands on an odd number
D: the event that the spinner lands on a number less than 5

We see that: $A = \{5\}$ $B = \{3, 4\}$ $C = \{1, 3, 5\}$ $D = \{1, 2, 3, 4\}$

Calculating probability

The probability that an event will happen is given by the following formula:

$$probability = \frac{number\ of\ outcomes\ in\ the\ event}{number\ of\ outcomes\ in\ the\ sample\ space}$$

We use the notation $P(E)$ to mean the probability that E will happen. We can also use the notation for cardinality that we learnt when studied set theory:

If E is an event and S is the sample space, then: $$P(E) = \frac{n(E)}{n(S)}$$

For the examples on the previous slide:

$S = \{1, 2, 3, 4, 5\}$ Therefore $n(S) = 5$

Event	$A = \{5\}$	$B = \{3, 4\}$	$C = \{1, 3, 5\}$	$D = \{1, 2, 3, 4\}$
Cardinality	$n(A) = 1$	$n(B) = 2$	$n(C) = 3$	$n(D) = 4$
Probability	$P(A) = 1/5$	$P(B) = 2/5$	$P(C) = 3/5$	$P(D) = 4/5$

We can express the probability as a fraction, a decimal or a percentage. So we could have written, for example, $P(B) = 0.4$ or $P(B) = 40\%$.

Note that:

- If an event is certain to happen as a result of a particular experiment the probability is 1.

    For example, the probability of our spinner landing on either a 1, 2, 3, 4 or 5 is 1.

- If an event is impossible the probability is 0.

    For example, the probability of our spinner landing on a 6 is 0.

- All other probabilities will lie between 0 and 1.

Worked examples

1. A 6-sided dice is thrown.

   a) What is the sample space, S?
   b) Give the value of E, the event that the dice lands on a number less than 3.
   c) What is the cardinality of the set S and the set E?
   d) Calculate the probability that E happens.

   <u>Solution</u>

   a) $S = \{1, 2, 3, 4, 5, 6\}$

   b) $E = \{1, 2\}$

   c) $n(S) = 6$ $\qquad$ $n(E) = 2$

   d) $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$

2. Two three sided spinners (see below) are spun together.

a) What is the sample space, S?
b) Give the value of E, the event one spinner lands on blue and the other on yellow.
c) What is the cardinality of the set S and the set E?
d) Calculate the probability that E happens.

<u>Solution</u>

a) S = {(RED, RED), (RED, BLUE), (RED, YELLOW), (BLUE, RED), (BLUE, BLUE), (BLUE, YELLOW), (YELLOW, RED), (YELLOW, BLUE), (YELLOW, YELLOW)}

b) E = {(BLUE, YELLOW), (YELLOW, BLUE)}

c) $n(S) = 9$ $n(E) = 2$

d) $P(E) = \frac{n(E)}{n(S)} = \frac{2}{9}$

Probability that one OR another event happens

Consider the 5-sided spinner again.

Let A be the event of getting a 2 and B be the event of getting a 5: A = { 2 } B = { 5 }

The event of getting a 2 *or* a 5 is: {2, 5}

We know that this is the union of A and B.

So the event of getting a 2 *or* a 5 is $A \cup B$.

So we see that the probability of event A *or* B happening is: $P(A \cup B)$.

Mutually exclusive events

The examples we have just seen are **mutually exclusive** events. They cannot both happen at the same time; a spinner cannot land on a 2 and a 5 at the same time. Of course, events containing a single outcome are always mutually exclusive.

When the events contain more than one outcome, they may or may not be mutually exclusive.

For example, if you throw a dice, the event of getting an even number and the event of getting a 5 *are* mutually exclusive.

For mutually exclusive events, there are no common elements.

In other words, for two mutually exclusive events, A and B: $A \cap B = \varnothing$

Non-mutually exclusive events

Consider throwing a 6-sided dice:

Let A be the event of throwing an odd number.
Let B be the event of throwing a number less than 4.

$A = \{1, 3, 5\}$ $B = \{1, 2, 3\}$

These events are *not* mutually exclusive. It is possible to throw a number which is odd *and* is less than 4 (the number 1 or the number 3).

The intersection of the two sets is not empty: $A \cap B = \{1, 3\}$

The addition rule

We have seen that the probability of A or B happening is given by $P(A \cup B)$.

In the previous example of the 6-sided dice we had: A: The event of throwing an odd number.
B: The event of throwing a number less than 4.

$A = \{1, 3, 5\}$ $\qquad B = \{1, 2, 3\}$ $\qquad A \cup B = \{1, 2, 3, 5\}$

so $\quad P(A \cup B) = {}^{4}/_{6} = {}^{2}/_{3}$

There is another way we can calculate the probability of one or another event happening.

When we studied sets we saw that: $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

If the sample space is S, then

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

So we have: $\quad \boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$ This is called the addition rule.

In this example: $\quad A \cap B = \{1, 3\}$

So: $\quad P(A) = {}^{3}/_{6} = {}^{1}/_{2} \qquad P(B) = {}^{3}/_{6} = {}^{1}/_{2} \qquad P(A \cap B) = {}^{2}/_{6} = {}^{1}/_{3}$

$$P(A \cup B) = {}^{1}/_{2} + {}^{1}/_{2} - {}^{1}/_{3} = {}^{2}/_{3}$$

This is, of course, the same answer that we got before.

The addition rule for mutually exclusive events

For mutually exclusive events there are no common elements: $A \cap B = \varnothing$

So: $P(A \cap B) = 0$

So: *For mutually exclusive events:* $P(A \cup B) = P(A) + P(B)$.

For mutually exclusive events, it is also true that:

- The sum of the individual probabilities must add up to 1.

- The probability of *not* getting a successful event is found by subtracting the probability from 1. For example, the probability of *not* throwing a number below 5 on the spinner is $(1 - {}^{1}/_{5})$ or ${}^{4}/_{5}$.

Worked examples

1. A 6-sided dice is thrown, and the following events are defined: A is the event of throwing a number less then 4.
B is the event of throwing an even number.
C is the event of throwing a number starting with ‘F”.

a) What is the sample space, S?
b) Give the value of A, B and C.
c) Give the values of: $A \cap B$ $A \cap C$ $B \cap C$
d) Which pairs of events are mutually exclusive?
e) Calculate the probability of throwing an even number or a number less than 4.
f) Calculate the probability of throwing an even number or a number starting with ‘F”.
g) Calculate the probability of throwing a number less than 4 or a number starting with ‘F”.

Solution

a) $S = \{1, 2, 3, 4, 5, 6\}$
b) $A = \{1, 2, 3\}$ $B = \{2, 4, 6\}$ $C = \{4, 5\}$
c) $A \cap B = \{2\}$ $A \cap C = \varnothing$ $B \cap C = \{4\}$
d) Only A and C (the intersection is empty).
e) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = {}^3/_6 + {}^3/_6 - {}^1/_6 = {}^5/_6$
f) $P(B \cup C) = P(B) + P(C) - P(B \cap C) = {}^3/_6 + {}^2/_6 - {}^1/_6 = {}^2/_3$
g) $P(A \cup C) = P(A) + P(C) - P(A \cap C) = {}^3/_6 + {}^2/_6 - 0 = {}^5/_6$

2. Consider the following reduced pack of cards consisting of just the following*:

2♣, Q♣
4♦, 10♦, Q♦
2♥, Q♥
2♠, 4♠, K♠

What is the probability of drawing a card that is red or is a queen?

<u>Solution</u>

Let A be the event of drawing a queen.
Let B be the event of drawing a red card.

These events are *not* mutually exclusive. You can draw a card that is a queen *and* a red card.

There are 10 cards altogether; there are 5 red cards; there are 3 queens; there are 2 cards that are red and are queens.

$$P(A) = {}^{3}/_{10} \qquad P(B) = {}^{5}/_{10} \qquad P(A \cap B) = {}^{2}/_{10}$$

$$P(A \cup B) = {}^{3}/_{10} + {}^{5}/_{10} - {}^{2}/_{10} = {}^{6}/_{10} = 0.6$$

* If you are not familiar with playing-cards, the make-up of a standard deck is explained in the appendix at the end of this document.

Probability distribution

When we throw a dice, there are 6 possible outcomes. If the dice is fair, then the probability of throwing any of the numbers from 1 to 6 is $^1/_6$.

The notation we use for the probability of an event can also be used for the probability of outcomes. So for example P(1) is the probability of throwing a 1.

In general, if A is the event of getting outcome a, then $P(A)$ = $P(a)$.

We write the **probability distribution** for the dice as follows:

P(1) = $^1/_6$
P(2) = $^1/_6$
P(3) = $^1/_6$
P(4) = $^1/_6$
P(5) = $^1/_6$
P(6) = $^1/_6$

Since individual outcomes are mutually exclusive, the total of all the probabilities must be 1.

And again because they are mutually exclusive we can add the probabilities to find the probability of one or the other event happening.

The probability of throwing a 2 or a 5 is $^1/_6 + {}^1/_6 = {}^1/_3$.

Non-uniform probability distribution

Now imagine we toss a coin. This time the coin isn't fair – it has been weighted so that it is 3 times more likely to land on tails than on heads.

How can we find the probability distribution?

We let x represent the probability of getting heads.

Therefore the probability of getting tails is $3x$.

So: $P(\text{HEADS}) = x$
$P(\text{TAILS}) = 3x$

We know that the probabilities must add up to 1.

Therefore: $x + 3x = 1$
$4x = 1$
$x = 0.25$

So our probability distribution is: $P(\text{HEADS}) = 0.25$
$P(\text{TAILS}) = 0.75$

Worked example

An unfair 6-sided dice is weighted so that the chances of throwing a 4, 5 or 6 are equal. The chance of throwing a 2 or a 3 is twice as likely as throwing a 4, 5 or 6. The chance of throwing a 1 is three times as likely as throwing a 4, 5 or 6.

a) Find the probability distribution of the dice.
b) What is the probability of throwing a 3 or 4?

<u>Solution</u>

a) Let the probability of throwing a 4, 5, 6 be x. The probability of throwing a 2 or 3 is $2x$. The probability of throwing a 1 is $3x$.

The probability distribution is:

$P(1) = 3x$
$P(2) = 2x$
$P(3) = 2x$
$P(4) = x$
$P(5) = x$
$P(6) = x$

The events are mutually exclusive, so the probabilities must add up to 1: $3x + 2x + 2x + x + x + x = 1$

$10x = 1$ $\qquad x = 0.1$

The probability distribution is:

$P(1) = 0.3$
$P(2) = 0.2$
$P(3) = 0.2$
$P(4) = 0.1$
$P(5) = 0.1$
$P(6) = 0.1$

b) The events are mutually exclusive, therefore the probability of throwing a 3 or a 4 is: $P(3) + P(4) = 0.3$

Independent events

Consider the spinner and the coin shown on the right.

Any events associated with the spinner and the coin are **independent events**. Spinning the spinner has no effect on tossing the coin, and vice versa.

Imagine we want to calculate the probability of getting yellow on the spinner and tails on the coin.

Because they are independent events, it doesn't matter if we spin the spinner, then toss the coin, or toss the coin and then spin the spinner – or do them both at the same time.

The probability distributions for each are shown below:

Spinner
$P(\text{RED}) = {}^{1}/_{3}$
$P(\text{BLUE}) = {}^{1}/_{3}$
$P(\text{YELLOW}) = {}^{1}/_{3}$

Coin
$P(\text{HEADS}) = {}^{1}/_{2}$
$P(\text{TAILS}) = {}^{1}/_{2}$

When dealing with independent events we find the probability of both events happening simply by multiplying the two individual probabilities together.

$$P(\text{YELLOW}, \text{HEADS}) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

Worked Example

Three fair coins are tossed. What is the probability of getting exactly two heads?

<u>Solution</u>

The table shows that there are 8 possible outcomes:

Coin 1	Coin 2	Coin 3	Number of heads
HEADS	HEADS	HEADS	3
HEADS	HEADS	TAILS	2
HEADS	TAILS	HEADS	2
HEADS	TAILS	TAILS	1
TAILS	HEADS	HEADS	2
TAILS	HEADS	TAILS	1
TAILS	TAILS	HEADS	1
TAILS	TAILS	TAILS	0

If S is the sample space, then $n(S)$ = 8

There are 3 outcomes (outcomes 2, 3 and 5) in which there are exactly two heads.

If E is the event of getting exactly 2 heads, then $n(E)$ = 3

Therefore: $P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

Random variables

A **random variable** is a set of possible values from a random experiment.

When the values are confined to discrete values, we refer to the random variable as a **discrete random variable**.

A random variable is not quite the same as the sample space, because we are able to *assign* a value to a particular outcome.

A random variable provides a useful way of expressing a probability distribution.

<u>Example</u>

We saw that a five sided-spinner can have five following possible outcomes: The spinner can land on 1, 2, 3, 4, or 5.

We know that the sample space for the spinner is: $S = \{1, 2, 3, 4, 5\}$

In this case, as the outcomes are numbers, it makes sense to define our random variable X as: $X = \{1, 2, 3, 4, 5\}$

We write the probability for each outcome as $P(X = x)$

In the case of the spinner we can write:

$P(X = 1) = \frac{1}{5}$ $P(X = 2) = \frac{1}{5}$ $P(X = 3) = \frac{1}{5}$ $P(X = 4) = \frac{1}{5}$ $P(X = 5) = \frac{1}{5}$

The sum of the probabilities is, of course, 1

Random variables (continued)

Now consider the experiment that we looked at in the last worked example, namely tossing three coins.

We are interested in the number of heads thrown:

Coin 1	Coin 2	Coin 3	Number of heads
HEADS	HEADS	HEADS	3
HEADS	HEADS	TAILS	2
HEADS	TAILS	HEADS	2
HEADS	TAILS	TAILS	1
TAILS	HEADS	HEADS	2
TAILS	HEADS	TAILS	1
TAILS	TAILS	HEADS	1
TAILS	TAILS	TAILS	0

The number of heads thrown can be 0, 1, 2, or 3

So the random variable representing the number of heads thrown is as follows: X = {0, 1, 2, 3}

This enables us to express the probability distribution as follows:

No of heads thrown, x	0	1	2	3
Number of occurrences	1	3	3	1
$P(X = x)$	$^1/_8$	$^3/_8$	$^3/_8$	$^1/_8$

Random variables (continued)

Look again at the probability distribution from the last slide

No of heads thrown, x	0	1	2	3
Number of occurrences	1	3	3	1
$P(X = x)$	$^1/_8$	$^3/_8$	$^3/_8$	$^1/_8$

Consider the following events:

1. The event of throwing *exactly* two heads:

   Here we have $P(X = 2) = \frac{3}{8}$

2. The event of throwing *at least* two heads:

   In this case $P(X \geq 2) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

Expected value (mean value)

An **expected value** refers to the average value of the result of performing an experiment a very large number of times.

We express the expected value as $E(x)$, where x is the value of a random variable defined for our experiment.

Rather than actually throwing a dice, say, ten thousand times we can use a formula to do this calculation.

The calculation involves taking each value (x), multiplying it by its probability (p) and adding them all together.

This is expressed as: $E(x) = \sum xp$

<u>Example</u>

The following table represents the probability distribution for a fair 5-sided spinner:

Number thrown, x	1	2	3	4	5
$P(X = x)$	$^1/_5$	$^1/_5$	$^1/_5$	$^1/_5$	$^1/_5$

$$E(x) = \frac{1}{5} \times 1 + \frac{1}{5} \times 2 + \frac{1}{5} \times 3 + \frac{1}{5} \times 4 + \frac{1}{5} \times 5$$

$$= 3$$

For a fair spinner, this is exactly the result we would expect to find.

Expected value (continued)

Now let's consider the probability for a dice which is weighted as follows:

Number thrown, x	1	2	3	4	5	6
$P(X = x)$	0.3	0.2	0.2	0.1	0.1	0.1

$$E(x) = 0.3 \times 1 + 0.2 \times 2 + 0.2 \times 3 + 0.1 \times 4 + 0.1 \times 5 + 0.1 \times 6$$

$$= 2.8$$

In this case we see that the expected value is not in our sample set – this is to be expected, because it is an average.

Worked example

Consider the previous example of tossing a coin three times, where X was the random variable representing the number of heads thrown.

What is the expected value?

<u>Solution</u>

We saw that the probability distribution was as follows

No of heads thrown, *x*	0	1	2	3
Number of occurrences	1	3	3	1
***P*(*X* = *x*)**	$^1/_8$	$^3/_8$	$^3/_8$	$^1/_8$

Therefore: $E(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$

Conditional Probability

Sometimes the outcome of an event is dependent on another event.

As an example, imagine that a bag contains 3 green balls and 4 red balls.

What is the probability of picking a green ball, followed by a red ball?

The probability of picking a green ball is $\frac{3}{7}$

If we have picked a green ball, then there are 2 green balls left and 4 red balls (6 in all).

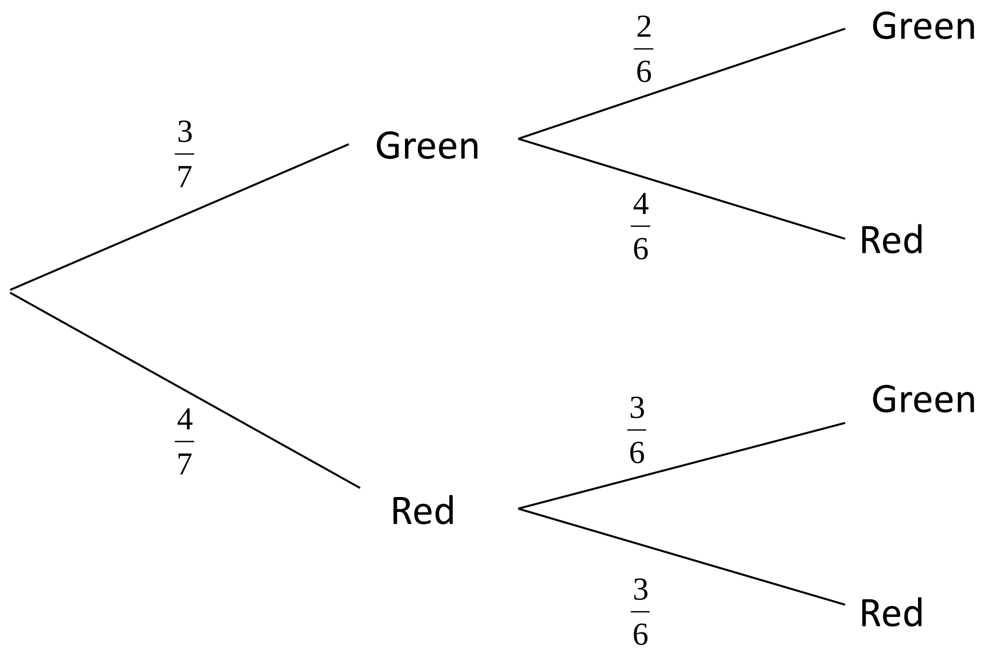
So the probability of picking a red ball is now $\frac{4}{6}$

Therefore, the probability of picking a green ball followed by a red ball is: $\frac{3}{7} \times \frac{4}{6} = \frac{2}{7}$

Tree diagrams

Tree diagrams provide a useful way of representing conditional probability. The diagram below represents the previous example of randomly selecting balls from a bag that contains 3 green balls and 4 red balls, as in the last example.

It can be seen that the diagram shows the probability of all possible events, obtained by multiplying the individual probabilities.



$$P(\text{GREEN, GREEN}) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$

$$P(\text{GREEN, RED}) = \frac{3}{7} \times \frac{4}{6} = \frac{2}{7}$$

$$P(\text{RED, GREEN}) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

$$P(\text{RED, RED}) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

Because the events are mutually exclusive we can easily calculate the probability of alternative events happening by adding vertically.

For example: $P(\text{GREEN, GREEN}) \text{ or } P(\text{RED, GREEN}) = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$

Worked example

A bag contains 3 green balls, 3 yellow balls and 4 red balls.

a) A ball is picked randomly from the bag, and is then returned to the bag. Another ball is then picked. What is the probability of picking a green ball, followed by a red ball?

b) A ball is picked randomly from the bag, but is not returned. Another ball is then picked. What is the probability of picking a green ball, followed by another green ball?

c) i) Represent the probabilities described in part b) on a tree diagram.

ii) Use this diagram to find the probability of picking a red ball followed by a yellow ball OR a yellow ball followed by a green ball.

Solution

a) In this case the events are independent.

The probability of picking a green ball is: $^3/_{10}$

The probability of picking a red ball is: $^4/_{10}$

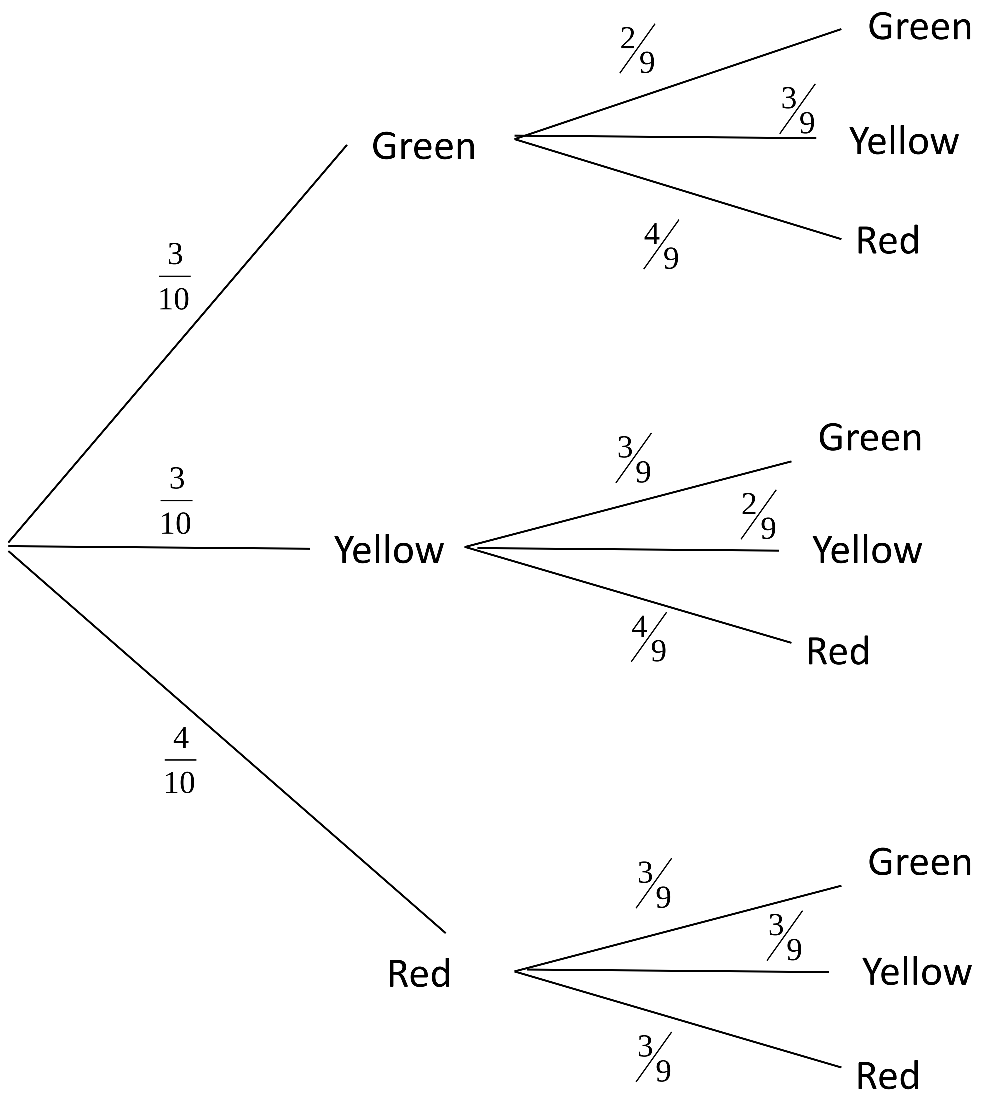
The probability of picking a green ball followed by a red ball is: $\frac{3}{10}\times\frac{4}{10}=\frac{12}{100}=\frac{3}{25}$

b) In this case, the events are not independent. The probability of picking a green ball is: $^3/_{10}$

There are 9 balls left, one green ball has gone, so the probability of another green ball is: $^2/_9$.

So the probability of picking a green ball followed by another green ball is: $\frac{3}{10}\times\frac{2}{9}=\frac{1}{15}$

c) i)



$$P(\text{GREEN, GREEN}) = \frac{3}{10} \times \frac{2}{9} = \frac{1}{15}$$

$$P(\text{GREEN, YELLOW}) = \frac{3}{10} \times \frac{3}{9} = \frac{1}{10}$$

$$P(\text{GREEN, RED}) = \frac{3}{10} \times \frac{4}{9} = \frac{2}{15}$$

$$P(\text{YELLOW, GREEN}) = \frac{3}{10} \times \frac{3}{9} = \frac{1}{10}$$

$$P(\text{YELLOW, YELLOW}) = \frac{3}{10} \times \frac{2}{9} = \frac{1}{15}$$

$$P(\text{YELLOW, RED}) = \frac{3}{10} \times \frac{4}{9} = \frac{2}{15}$$

$$P(\text{RED, GREEN}) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

$$P(\text{RED, YELLOW}) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

$$P(\text{RED, RED}) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

ii) The probability of picking a red ball followed by a yellow ball OR a yellow ball followed by a green ball = ${}^{2}/_{15} + {}^{1}/_{10} = {}^{7}/_{30}$

Binomial probability

Imagine an experiment with exactly two mutually exclusive outcomes, such as flipping a coin, and imagine we repeat this experiment several times.

Each experiment is called a Bernoulli trial (after the Swiss mathematician, Jacob Bernoulli).

In a series of Bernoulli trials, the repeated actions must all be independent.

In fact, we don't have to restrict the experiment to ones with only two "natural" outcomes, because when there are multiple outcomes, we can define one outcome as *success* and all others as *failure*.

So the following are all examples of Bernoulli trials:

- throwing a dice, where getting a 6 is a success, all other outcomes are failures;

- asking a question in an opinion poll, where "yes" is considered success, but "no", "don't know" and "won't vote" are failures;

- answering a question in a multiple choice exam, where one answer is correct (success) all other answers are incorrect (failure).

The binomial probability formula

There is a formula we can use to calculate the number of successful outcomes in a series of Bernoulli trials:

$$P(k \text{ successes in } n \text{ trials}) = {}^{n}C_{k}\, p^{k} q^{n-k}$$

n = total number of trials
k = number of successes
n - k = number of failures
p = probability of success in one trial
q = 1 – p = probability of failure in one trial

The examples that follow will make this clear.

Worked examples

1. A multiple choice paper consists of 10 questions, each with 5 possible answers. Only one answer is correct.

   If the questions are answered completely randomly, what is the probability of getting *exactly* 4 correct answers?

<u>Solution</u>

We use the formula:

$$P(k \text{ successes in } n \text{ trials}) = {}^{n}C_{k}\, p^{k} q^{n-k}$$

In this case:

n = 10
k = 4
n - k = 6
p = 0.2 (There are 5 correct answers)
q = 0.8

$$P(4 \text{ successes in } 10 \text{ trials}) = {}^{10}C_{4} \times 0.2^{4} \times 0.8^{6}$$

$$= 210 \times 0.0016 \times 0.262144$$

$$= 0.088$$

Note: If you had been asked the probability of getting *at least* 4 correct answers, you would have to add all the probabilities from 4 correct answers up to 10 correct answers.

2. A coin is tossed 5 times. What is the probability of getting exactly 3 heads if:

a) the coin is a fair coin;
b) the coin is weighted so that it is three times more likely to land on tails than on heads?

Solution

In both cases we use the formula: $$P(k \text{ successes in } n \text{ trials}) = {}^{n}C_{k}\, p^{k} q^{n-k}$$

In each case:

$n = 5$
$k = 3$
$n - k = 2$

a) In this case: $p = 0.5$, $q = 0.5$

$$P(3 \text{ successes in } 5 \text{ trials}) = {}^{5}C_{3} \times 0.5^{3} \times 0.5^{2}$$
$$= 10 \times 0.125 \times 0.25$$
$$= 0.3125$$

b) In this case: $p = 0.25$, $q = 0.75$

$$P(3 \text{ successes in } 5 \text{ trials}) = {}^{5}C_{3} \times 0.25^{3} \times 0.75^{2}$$
$$= 10 \times 0.15625 \times 0.5625$$
$$= 0.0879$$

Appendix – a standard pack of cards

A standard deck consists of 52 cards.

There are 4 suits – clubs (♣), diamonds (♦), hearts (♥) and spades(♠).

Clubs and spades are coloured black, diamonds and hearts red.

In each suit there are 13 cards: 2 to 10, jack, queen, king and ace.

2♣, 3♣, 4♣, 5♣, 6♣, 7♣, 8♣, 9♣, 10♣, J♣, Q♣, K♣, A♣
2♦, 3♦, 4♦, 5♦, 6♦, 7♦, 8♦, 9♦, 10♦, J♦, Q♦, K♦, A♦
2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥, J♥, Q♥, K♥, A♥
2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠, A♠