Combinatorics – Answers to Tutorial Questions

1. In how many different ways can the letters x, y, z, w, v be arranged?

   Solution

   5! = 5 x 4 x 3 x 2 x 1 = 120

2. Find the value of: $\frac{8! \times 5!}{4! \times 3!}$

   Solution

$$\frac{8! \times 5!}{4! \times 3!} = \frac{8 \times 7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1} \times 5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$$

= 8 x 7 x 6 x 5 x 5 x 4

= 33600

3. Find the value of: a) $P(10, 3)$ b) $C(9, 6)$.

Solution

a)

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!}$$

$$= \frac{10\times 9\times 8\times \cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{\cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}$$

$$= 10 \text{ x } 9 \text{ x } 8 = 720$$

b)

$$C(n,k) = \frac{n!}{(n-k)!k!}$$

$$C(9,6) = \frac{9!}{(9-6)!\times 6!} = \frac{9!}{3!\times 6!}$$

$$= \frac{9\times 8\times 7\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{3\times 2\times 1\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}$$

$$= \frac{9\times 8\times 7}{3\times 2\times 1} = 84$$

4. A committee of 20 people has to elect a chair, a vice-chair, a secretary and a treasurer. How many different ways are there of choosing these posts?

Solution

This is the same as selecting 4 from 20 where order is significant and there is no repetition, so this is a permutation.

The correct formula is: $P(n,k) = \dfrac{n!}{(n-k)!}$ where n = 20 and k = 4.

$$P(20,4) = \frac{20!}{16!}$$

$$= 20 \times 19 \times 18 \times 17 = 116280$$

5. The winner of a children's competition is allowed to draw three prizes from a bag of 10 unique items. The runner up is then allowed to draw two items.

How many different sets of prizes can be chosen by:

a) the winner;
b) the runner-up?

Solution

In both cases, order is not significant and there is no repetition, so this is a combination.

The correct formula is: $$C(n,k)=\frac{n!}{(n-k)!k!}$$

a) In this case n = 10 and k = 3.

$$C(10,3)=\frac{10!}{7!\times 3!}$$

$$=\frac{10\times 9\times 8\times \cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{\cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}\times\ 3\times 2\times 1}$$

$$=\frac{10\times 9\times 8}{3\times 2\times 1}=120$$

b) In this case n = 7 and k = 2.

$$C(7,2)=\frac{7!}{5!\times 2!}$$

$$=\frac{7\times 6\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{\cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}\times\ 2\times 1}$$

$$=\frac{7\times 6}{2\times 1}=21$$

6. A gift shop sells 10 different colours of wrapping paper. Customers can get a discount if they buy three rolls of paper. They can choose three of the same colour, or two of one colour and one of another colour, or 3 different colours.

How many different combinations can a customer choose from?

Solution

Order is unimportant, but repetition is allowed.

The correct formula is: $\dfrac{(n+k-1)!}{k!(n-1)!}$ where n = 10 and k = 3.

$$\frac{(10+3-1)!}{3!(10-1)!}=\frac{12!}{3!\times 9!}$$

$$=\frac{12\times 11\times 10\times \cancel{9}\times \cancel{8}\times \cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{3\times 2\times 1\ \times \cancel{9}\times \cancel{8}\times \cancel{7}\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}$$

$$=\frac{12\times 11\times 10}{3\times 2\times 1}=220$$

7. Imagine an alien alphabet consisting of the following symbols:

€ % @ π * £

How many different three letter "words" can be made from these symbols? (symbols can be repeated).

<u>Solution</u>

Order is important, and repetition is allowed.

The correct formula is: n^k where n = 6 and k = 3.

$$6^3 = 216$$

8. How many 3-digit numbers can be made from the digits 1-6 if:

a) you are allowed to repeat digits;
b) you are not allowed to repeat digits;
c) you are not allowed to repeat digits and the number must end in 3;
d) you are not allowed to repeat digits and the number must end in 1 or 4?

<u>Solution</u>

a) Order is important, and repetition is allowed, so the correct formula to use is n^k, where n = 6 and k = 3.

$$n^k = 6^3 = 216$$

b) In this case we must calculate $P(6, 3)$.

$$P(6,3) = \frac{6!}{3!}$$

$$= \frac{6 \times 5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3} \times \cancel{2} \times \cancel{1}}$$

$$= 120$$

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c) This is the same as putting the 3 at then end and arranging the other 5 digits in the first two slots:

$$P(5,2) = \frac{5!}{3!}$$

$$= \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3} \times \cancel{2} \times \cancel{1}}$$

= 20

d) The number can end in 1 or 4.

From part c) we see that there are 20 possibilities when one number is fixed at the end. So there are 20 numbers ending with 1 and 20 ending in 4.

So the total number is 40.

9. Four friends go on a fairground ride. They must sit in a row. Tracey does not want to sit at the end of the row. In how many different ways can the four be arranged?

Solution

If there are no restrictions, then the number of ways of arranging the four friends is:

$4! = 4 \times 3 \times 2 \times 1 = 24$

If Tracey were to sit at one end, then we just have to arrange the other three. The number of ways of doing this is:

$3! = 3 \times 2 \times 1 = 6$

If she sits at the other end there are also 6 ways of arranging the others.

So there are 12 ways of arranging people if Tracey sits at the one or other end.

So the number of ways of arranging them if Tracey does *not* sit at the end is:

$24 – 12 = 12$

10. Use Pascal's triangle to find the value of C(4, 2); verify your answer by using the correct formula.

<u>Solution</u>

Remember to count rows and columns starting from 0:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$C(4,2) = 6$$

$$C(4,2) = \frac{4!}{2! \times 2!}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \ \times \ 2 \times 1} = 6$$

11. Use the binomial theorem to expand the expression $(2x - y)^4$

<u>Solution</u>

$$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ . \ . \ . \ + {}^nC_{n-1} a^1 b^{n-1} + {}^nC_n a^0 b^n$$

In this case: $n = 4$, $a = 2x$ and $b = -y$

$$(2x - y)^4 = {}^4C_0(2x)^4(-y)^0 + {}^4C_1(2x)^3(-y)^1 + {}^4C_2(2x)^2(-y)^2 + {}^4C_3(2x)^1(-y)^3 + {}^4C_4(2x)^0(-y)^4$$

$$= 16x^4 + 4(8x^3)(-y) + 6(4x^2)(y^2) + 4(2x)(-y^3) + (y^4)$$

$$= 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$$

12. Use the binomial theorem to find the 3rd term in the expansion of the expression $(x + 2y)^6$

Solution

$$(a+b)^n = \sum_{k=0}^{n} {}^{n}\mathrm{C}_{k} a^{n-k} b^k$$

In this case, $a = x$ and $b = 2y$

$n = 6$, and because we start at zero, the third term will be given when $k = 2$.

The third term is therefore: ${}^{6}\mathrm{C}_{2}\,(x^4)\,(2y)^2$

$= 15\,(x^4)\,(4y^2)$

$= 60x^4y^2$