Set and Groups: Part 1

At the end of this topic you should be able to:

- give a definition of a set and provide examples of sets;
- recognise and utilise the notation that forms the body of set theory;
- specify a set using set comprehension;
- explain the meaning of the *universal set* and the *empty set*;
- determine the cardinality of a set;
- explain the meaning of the terms *subset* and *proper subset*;
- perform standard set operations and recognise the various set operators;
- use Venn diagrams to represent sets;
- provide a statement of De Morgan’s law;
- find the power set of a given set.

What is a set?

- a **set** is an unordered collection of distinct objects;
- these objects are called the **elements** or **members** of the set;
- the objects can be anything – people, words, numbers, animals or even other sets.

Examples of possible sets could be:

  - songs on my phone;
  - students at UEL;
  - positive integers less than ten;
  - countries in the European Union;
  - words in the English language.

Notation

We often use upper-case letters to refer to sets.

When we list the members of a set we enclose them in curly brackets, separating each element of the set with a comma.

Examples

If A is the set of all the countries in the UK, we could write:

A = {WALES, SCOTLAND, ENGLAND, NORTHERN IRELAND}

If B is the set of positive even numbers greater than zero and less than ten, then:

$B = \{ 2, 4, 6, 8 \}$

When reasoning about sets in mathematics, we often use lower case letters to represent the elements of a set. For example, we might have:

$C = \{ a, d, f, e, x, g \}$

Membership of a set

The following notation is used to indicate that an object is a member of a set:

$x \in A$ means x is an element of A

$x \notin A$ means x is not an element of A

Two sets are said to be equal if and only if they contain exactly the same elements (no more and no fewer).

Ordering in a set

A set is an *unordered* collection of elements.

Therefore the order in which the elements is listed in a set is irrelevant. There is no notion of "first element" or "second element" etc.

Example

$$\{ a, b, c, d, e\} = \{e, d, b, c, a\}$$

These two sets are identical – they contain exactly the same elements. We can list them in any order we choose.

Repetition in a set

Elements of a set are named only once.

If A is the set {1, 2, 3}, we could add 3 to this set as many times as we wanted, and the set would remain unchanged.

Number sets

It is common to refer to the sets of numbers by the following letters:

$\mathbb{N}$ is the set of natural numbers (positive whole numbers, including zero)

$\mathbb{Z}$ is the set of integers (positive and negative whole numbers, including zero)

$\mathbb{R}$ is the set of real numbers (decimal numbers)

$\mathbb{Q}$ is the set of rational numbers (will be explained later in the term)

$\mathbb{C}$ is the set of complex numbers (will be explained later in the term)

Specifying a set

There are three ways to specify a set:

1. By listing the elements

A = {CAT, DOG, HORSE, MOUSE}

$B = \{a, c, g, m, z\}$

2. By describing the elements

S = {*Students at UEL*}

E ={*even numbers greater than 0 and less than 10*}

3. By comprehension

$$Y = \{x \in \mathbb{N} | x < 10\}$$

This reads: the set of x's which are elements of $\mathbb{N}$ such that x is less than 10.

So in this case: $Y = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Worked examples

1. Express the following specification of a set M in words:

$$M = \{x \in \mathbb{R} | x \geq 100\}$$

<u>Solution</u>

M is the set of real numbers greater than or equal to 100.

2. Using set comprehension, specify a set D that contains all the negative integers.

<u>Solution</u>

$$D = \{x \in \mathbb{Z} | x < 0\}$$

The universal set

Consider the following set:

{ROSE, LILY, IRIS, HYACINTH, VIOLET}

What kind of objects are these?

Flowers?
Names?
Words in the English language?

It is important to know what sort of objects a set contains. When we specify a set, we should always be aware of a bigger set from which our set takes its objects.

- This bigger set is called the universal set. It is usually given the letter U.
- If we take the set of students at UEL, the universal set could be the set of people.
- If we take the set {ELEPHANT, LION, DOG}, the universal set could be the set of animals.
- If we have a set of positive whole numbers, the universal set could be natural numbers. But it could also be integers, or real numbers. We need to be clear which one it is.

The empty set

It is possible for a set to contain no members.

This is called **the empty set** and is referred to by this symbol: $\varnothing$

Note

A set containing only one member is often referred to as a **unitary** set.

Cardinality

The number of elements that a set contains is called the **cardinality** of that set.

There are two ways of referring to the cardinality of a set A.

Either: $n(A)$

Or: $|A|$

In this course we will use the first one.

<u>Examples</u>

$A = \{a, b, c, d, e, f\}$ $n(A) = 6$

$B = \{1, 4, 6\}$ $n(B) = 3$

$C = \varnothing$ $n(C) = 0$

$D = \{0\}$ $n(D) = 1$

Finite sets and infinite sets

In the previous examples we were able to state the cardinality of the sets easily.

This is because when we count the members, the counting comes to an end and we have a final result.

Such sets are called **finite** sets.

However there are certain sets where this is not so straightforward, because we never finish counting. Such sets are called **infinite** sets.

All of the following sets are infinite sets:

$\mathbb{N}$ (the set of natural numbers)

$\mathbb{Z}$ (the set of integers)

$\mathbb{R}$ (the set of real numbers)

Worked example

State whether each of the following sets is finite or infinite:

a) The set of positive odd integers greater than 10.

b) The set of positive odd integers less than 10.

c) The set of all the people in the world.

<u>Solution</u>

a) Infinite – it will range from 11 up to infinity.

b) Finite – it is the set {1, 3, 5, 7, 9}

c) Finite.

Subsets

Consider a set A and a larger set B.

If all the elements contained in A are also elements of B, then A is a **subset** of B.

The following notation is used to mean A is a subset of B : $A \subset B$

The following means A is *not* a subset of B: $A \not\subset B$

<u>Example</u>

Consider the following sets:

C = {RED, BLUE, YELLOW, GREEN, PURPLE}

A = {RED, BLUE, GREEN}

B = {RED, BLUE, YELLOW, PINK}

In this case $A \subset C$ but $B \not\subset C$

Proper subsets

Strictly speaking the expression:

$$A \subset B$$

means A is a *proper* subset of B.

This means that we exclude the possibility that A could be equal to B. In other words, there is always at least one more element in B than there is in A.

If we want to include the possibility that the two sets are equal, then we use the symbol $\subseteq$.

So: $A \subseteq B$ means that A is a subset of B or is equal to B.

Worked example

Consider the following sets of natural numbers:

$A = \{ 1, 4, 6, 7, 9, 10 \}$ $B = \{ 6, 7, 9, 10 \}$ $C = \{ 3, 7, 9, 10\}$ $D = \{ 10, 7, 6, 9 \}$

For each of the following, state whether the expression is true or false:

a) $B \subset A$ b) $A \subset B$ c) $B = D$ d) $C \not\subset A$ e) $D \subset B$ f) $B \subseteq A$

<u>Solution</u>

a) True

b) False

c) True

d) True

e) False

f) True

Set Operations

In arithmetic, the basic operations we can perform are:

- addition
- subtraction
- multiplication
- division

We are now going to study the following operations that we can perform on sets:

- union
- intersection
- difference
- complement
- Cartesian product

Union

The **union** of two sets, A and B is the set which contains all the elements of A and all the elements of B.

We use the following notation:

$$A \cup B \text{ means } A \text{ union } B$$

<u>Example</u>

if A = { JOHN, DELROY, ADEWALE, MOHAMMED }

and B = { JOHN, SHEILA, DELROY, ZELDA }

then $A \cup B$ = {JOHN, SHEILA, ADEWALE, MOHAMMED, DELROY, ZELDA }

<u>Note</u>

Although John and Delroy appear in both sets, they only appear once in the final set: as we know, we do not record duplicates in a set.

Intersection

The **intersection** of two sets, A and B is the set which contains all the elements that are common to both A and B.

We use the following notation:

$$A \cap B \text{ means } A \text{ intersection } B$$

<u>Example</u>

if $A = \{$ JOHN, DELROY, ADEWALE, MOHAMMED $\}$

and $B = \{$ JOHN, SHEILA, DELROY, ZELDA $\}$

then $A \cap B = \{$JOHN, DELROY$\}$

<u>Note</u>

If two sets have no common elements, then the intersection is the empty set.

So: if $X = \{a, b, d, e, g\}$

and $Y = \{m, n\}$

then $X \cap Y = \varnothing$

The exclusion principle

When we find the union of two sets, we don't include duplicates.

So the cardinality of the union is found be adding the cardinality of the two sets and subtracting the cardinality of the intersection (otherwise we would be counting the common elements twice).

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

For example: if $A = \{a, b, d, e, g\}$ $n(A) = 5$

and $B = \{a, b, c, f\}$ $n(B) = 4$

$A \cap B = \{a, b\}$ $n(A \cap B) = 2$

$$n(A \cup B) = 5 + 4 – 2 = 7$$

This is correct because: $A \cup B = \{a, b, d, e, g, c, f\}$ so $n(A \cup B) = 7$

Difference

The **difference** of two sets, A and B, is the set which contains the elements that belong to A but do not belong to B.

We use the following notation:

$$A \backslash B \text{ means } A \text{ difference } B$$

<u>Example</u>

if $A = \{$ JOHN, DELROY, ADEWALE, MOHAMMED $\}$

and $B = \{$ JOHN, SHEILA, DELROY, ZELDA $\}$

then $A \backslash B = \{$ ADEWALE, MOHAMMED $\}$

Complement

The **complement** of a set A is the set which contains all the elements in the universal set except for those that belong to A.

There are several different notations for the complement of A:

$$\overline{A} \quad \textit{or} \quad A' \quad \text{or} \quad A^c$$

We will use the first one.

<u>Example</u>

If the universal set, U, is all UEL students

and A = {UEL students who study computing}

then $\overline{A}$ = {UEL students who do not study computing}

Another way of describing the complement is: $\overline{A} = U \setminus A$

You should be able to see that: $A \cup \overline{A} = U$ $\qquad A \cap \overline{A} = \varnothing$

and: $\overline{\varnothing} = U$ $\qquad \overline{U} = \varnothing$

<u>Note</u>
In some texts you will see this described as the *absolute complement*, while the difference is referred to as the *relative complement.*

Cartesian product

The Cartesian product of a set A and a set B is written as:

$$A \times B$$

It is best explained by an example:

if $A = \{a, b, c\}$ and $B = \{d, e\}$

Then $A \times B = \{(a,d) , (a,e) , (b,d) , (b,e) , (c,d) , (c,e)\}$

The Cartesian product results in a set of **ordered pairs**, made up of one element from the first set, followed be one element from the second.

Worked example

Consider the following sets:

$A = \{ b, d, f, g, h, x, y \}$ $\quad B = \{ f, g \}$ $\quad C = \{ g, x, z \}$ $\quad D = \{ z \}$

a) Evaluate the following:

i) $A \cap C$ ii) $B \cup C$ iii) $B \setminus C$ iv) $B \cap D$ v) $B \times C$

b) If the universal set is $\{ b, d, f, g, h, x, y, z, w \}$, what is the value of $\overline{A}$?

<u>Solution</u>

a) i) $A \cap C = \{ g, x \}$

ii) $B \cup C = \{ f, g, x, z \}$

iii) $B \setminus C = \{ f \}$

iv) $B \cap D = \varnothing$

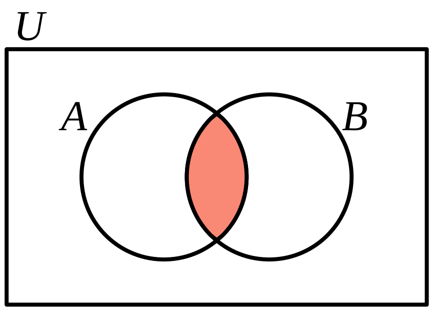
v) $B \times C = \{ (f, g), (f, x), (f, z), (g, g), (g, x), (g, z) \}$

b) $\overline{A} = \{ z, w \}$

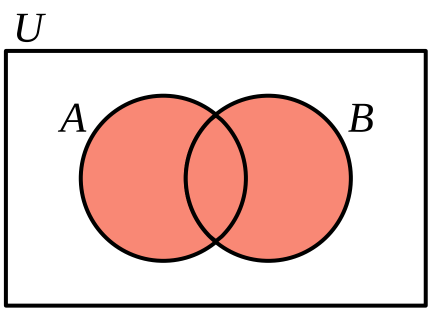
Venn Diagrams

A Venn diagram is a way to represent sets pictorially.

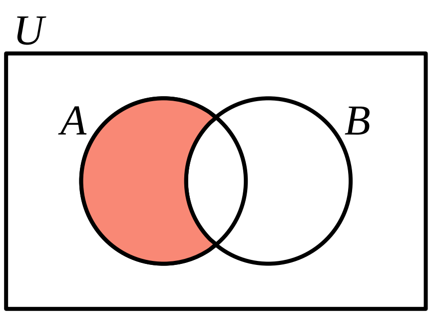
The universal set is represented by a rectangle, and a set is represented by a circle.



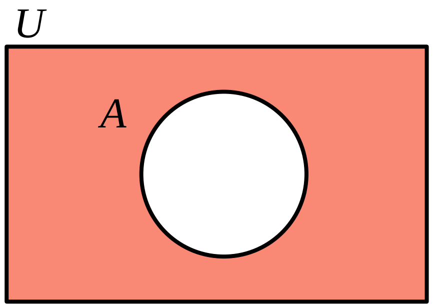
The shaded area represents the intersection of A and B, $A \cap B$.



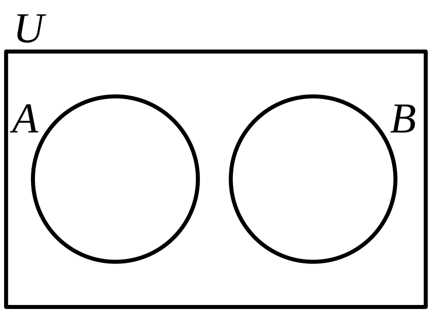
The shaded area represents the union of A and B, $A \cup B$.



The shaded area represents A difference B, $A \setminus B$.

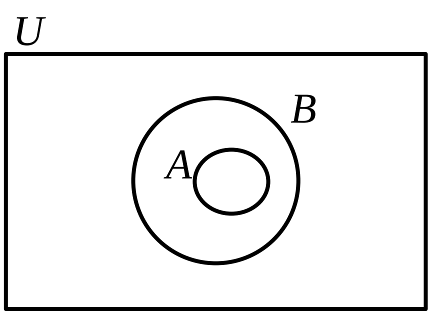


The shaded area represents the complement of A, $\overline{A}$.



In this diagram, A and B, have no common elements. They are said to be **disjoint**.

$$A \cap B = \varnothing$$

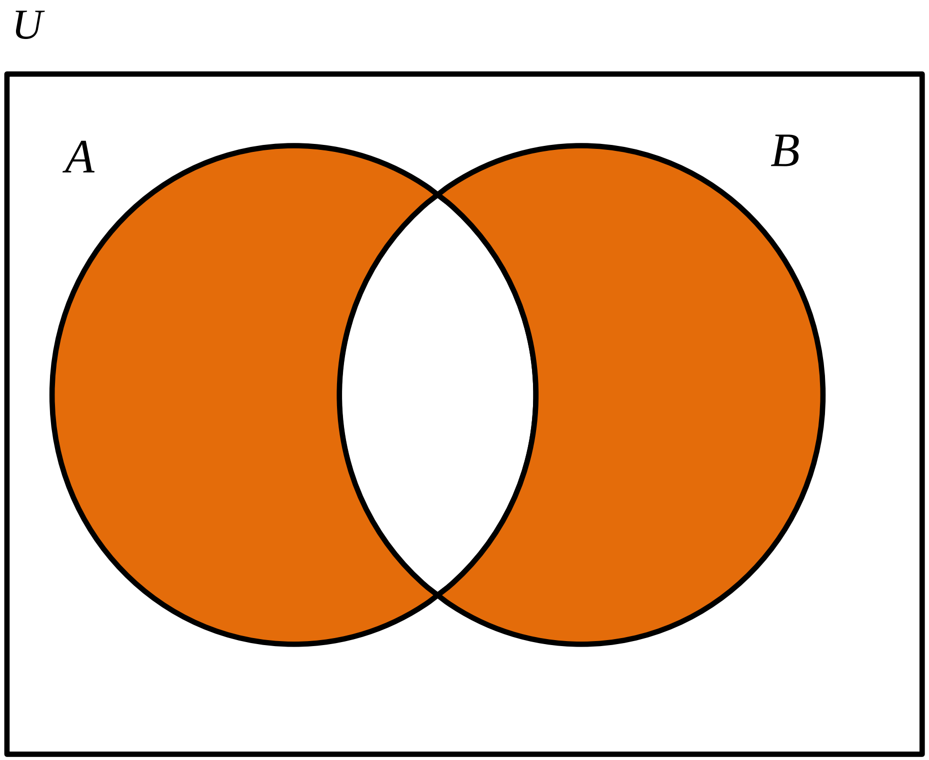


In this diagram, A is a proper subset of B.
$A \subset B$

Symmetric difference

Symmetric difference, represented by the symbol Δ (or sometimes ⊕).

It is defined as: $A \, \Delta \, B = (A \backslash \mathrm{B}) \cup (B \backslash \mathrm{A})$



The shaded area in the Venn diagram below represents the symmetric difference:

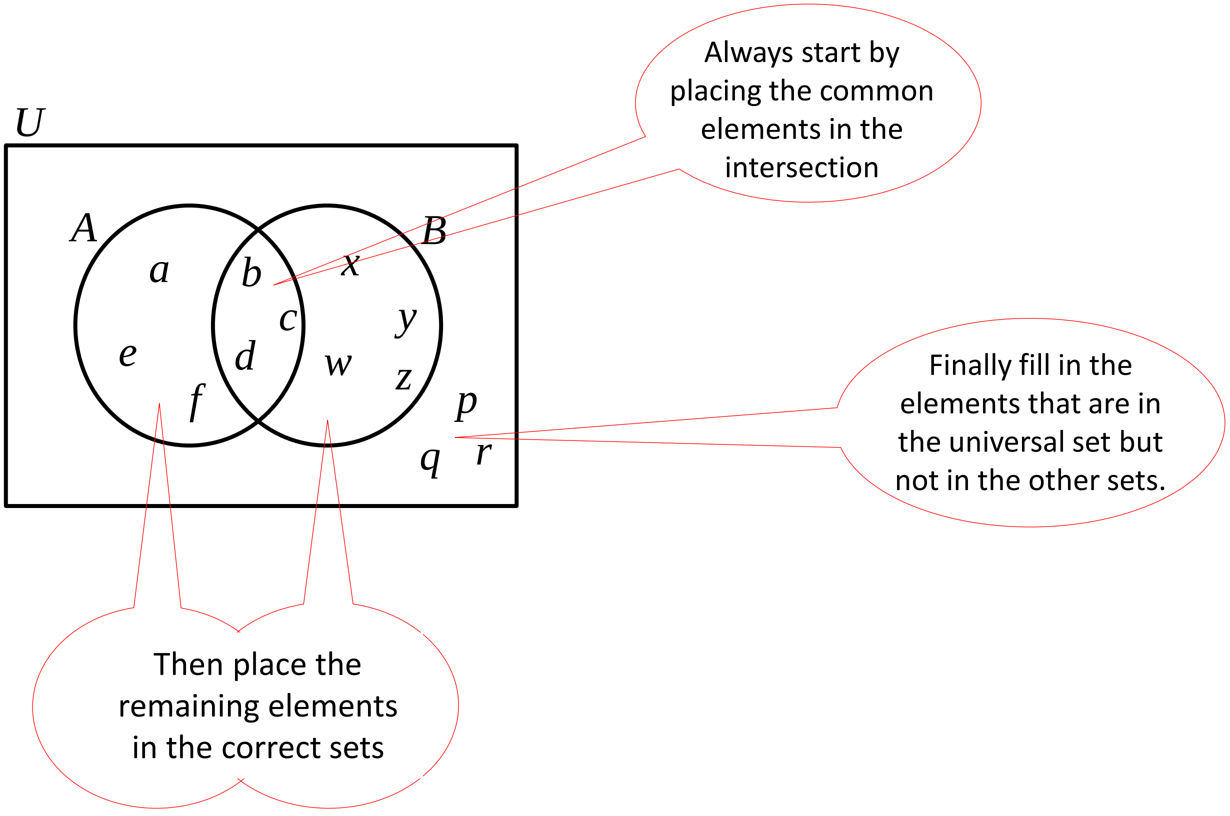
Worked examples

1. $A = \{ a, b, c, d, e, f \}$ $B = \{ x, b, c, d, y, w, z \}$

   The universal set $U = \{a, b, c, d, e, f, x, y, w, z, p, q, r \}$

   Represent this information on a Venn diagram.

   <u>Solution</u>



2. This question refers to students at a particular sixth-form college, where there is a total of 100 students.

   P is the set of students who take physics, and C is the set of students who take chemistry.

   30 students take physics, and 25 take chemistry. 20 students take both.

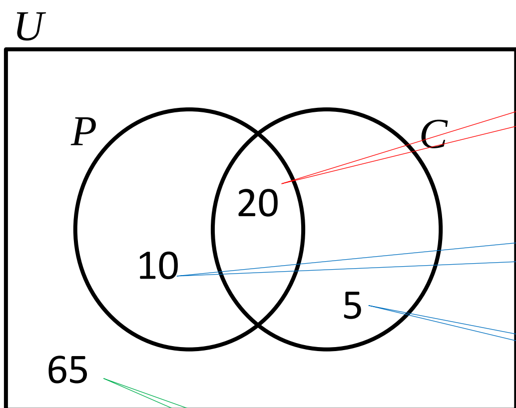
   a) Represent this information on a Venn diagram.

   b) Give values for the following:

      i) n($P \cap C$)   ii) n($P \cup C$)   iii) n($P \setminus C$)   iv) n($\overline{P \cup C}$)

Solution

a)



1. Start with the number of elements common to both sets, in this case 20.

2. Then fill in the remaining numbers in the other sets. 30 take physics in total, so 10 remain.

25 take chemistry in total, so 5 remain.

3. Finally, fill in the number in the universal set that are not contained in the other sets. In this case 35 take physics and/or chemistry, so 65 take neither.

b) i) $n(P \cap C) = 20$

ii) $n(P \cup C) = 35$

iii) $n(P \setminus C) = 10$

iv) $n(\overline{P \cup C}) = 65$

3. This question refers the same college as the previous question, where there are 100 students.

   P is the set of students who take physics, and C is the set of students who take chemistry, and M is the set of students who take mathematics.

   30 students take physics, and 25 take chemistry and 22 take mathematics.

   20 students take both physics and chemistry. 18 students take physics and mathematics. 15 students take chemistry and mathematics.

   12 students take physics, chemistry and mathematics.

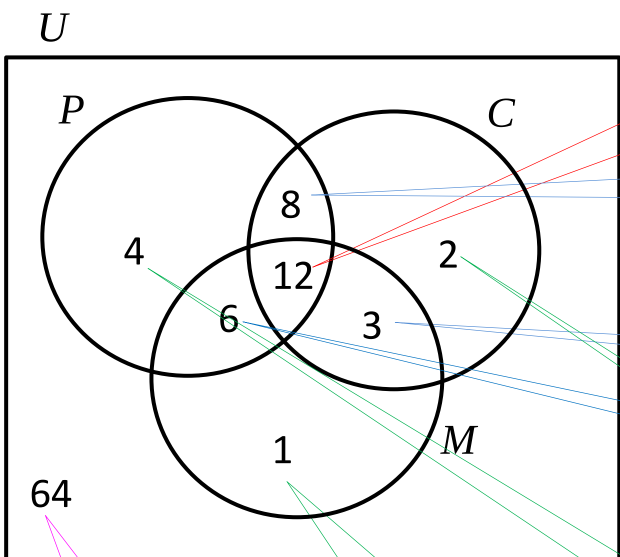
   a) Represent this information on a Venn diagram.

   b) Give values for the following:

      i) $n(P \cap C \cap M)$ ii) $n(P \cup C \cup M)$ iii) $n(P \setminus M)$ iv) $n(\overline{P \cap C \cap M})$

<u>Solution</u>

a)



1. Start with the number of elements common to all 3 sets, in this case 12.

2. Next, look at the number of elements common to two sets. In the case of physics and chemistry there are 20 taking both these subjects, so 8 remain.

There are 15 who study both chemistry and mathematics, so 3 remain.

There are 18 who study both physics and mathematics, so 6 remain.

3. Now consider those elements that are contained in only one set. 22 people study mathematics, so 1 remains.

30 people study physics, so 4 remain.

25 people study chemistry, so 2 remain.

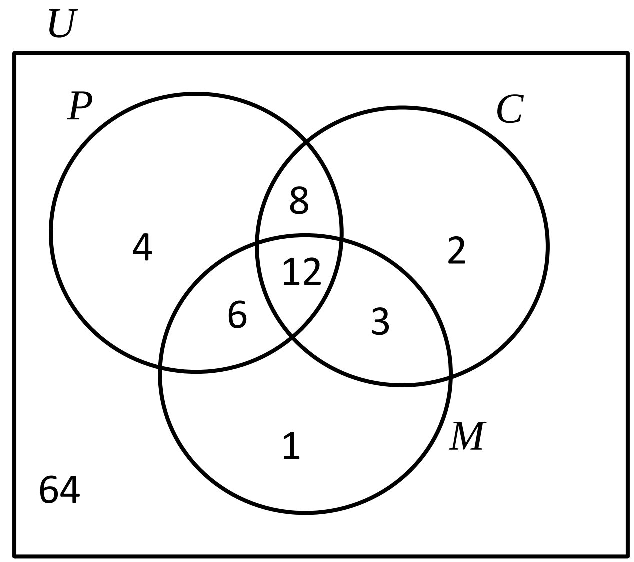
4. Finally, fill in the number in the universal set that are not contained in the other sets. In this case 36 take physics and/or chemistry and/or mathematics, so 64 take neither.

b) i) $n(P \cap C \cap M) = 12$

ii) $n(P \cup C \cup M) = 36$

iii) $n(P \setminus M) = 12$

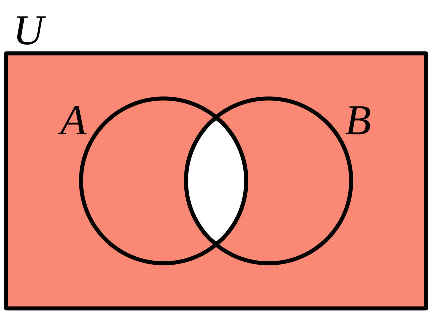
iv) $n(\overline{P \cap C \cap M}) = 88$



4. Use Venn diagrams to prove the following:

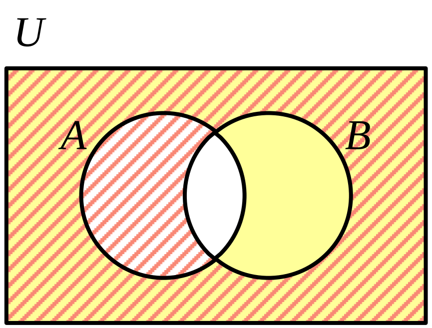
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

<u>Solution</u>



This is the left hand side of the equation

$\overline{A \cap B}$



This is the right hand side of the equation

$\overline{A}$

$\overline{B}$

\+ \+ $\overline{A} \cup \overline{B}$

We can see that in both diagrams the shaded areas are the same.

De Morgan's Law

In the last slide we used Venn Diagrams to show that:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

It can also be shown that:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

These two identities are two versions of a law in set theory known as **De Morgan's Law.**

Classes of sets and Power sets

In set theory we often use the word *class* to describe a collection of sets (or a set of sets).

For example the class C might be defined as follows:

$$C = \{ \{1, 2, 3\}, \{4, 5\}, \{1, 4, 6\} \}$$

The **power set** of a set A is the class of all the subsets of A (including the empty set and A itself).

This is written as: $P(A)$

<u>Example</u>

if: A = { CAT, DOG, PARROT }

Then: $P(A)$ = { ∅, {CAT}, {DOG}, {PARROT}, {CAT, DOG}, {CAT, PARROT}, {PARROT, DOG}, {CAT, DOG, PARROT} }

<u>Note</u>

If a set has a cardinality of n, then the number of elements in the power set is 2^n.

Worked examples

1. If A is the set $\{x, y\}$, what is the power set, $P(A)$?

   Solution

   $$P(A) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$$

2. a) If a set has a cardinality of 5, then how many elements will be in the power set?

   b) How many **proper** subsets does the above set have?

   Solution

   a) Number of subsets = $2^5 = 32$

   b) The number of proper subsets will be one fewer, 31 (because it does not include the set itself).

Application to computing

- The notion of collections is vital in computer science, particularly when it comes to programming.

- Different types of collections such as sequences and sets occur very frequently in everyday situations which we need to model in our programs.

- However, the importance of set theory in computing lies not so much in direct applications, but more in other areas of mathematics that are central to the understanding of computer science.

- We will see in future lectures how set theory is vital to our understanding of functions, of logic and of probability – all of which are of huge importance to computer science.

For interest only

Russell's Paradox.

The British philosopher Bertrand Russell realised that there was a potential problem with the notion of a "set of sets". This is described below:

If T is the set of all the sets which are not elements of themselves, then is T itself an element of T?

If T *is* an element of T then our definition of T no longer stands, therefore T cannot be an element of T.

However, if T is *not* an element of T, then by definition it must be an element of T!

This is known as Russell's Paradox. Using the word *class* in this context instead of *set*, is a neat way of avoiding Russell's Paradox.

Appendix: Summary of Symbols Used in Set Theory

$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
$x \subset A$	x is a (proper) subset of A
$x \not\subset A$	x is not a subset of A
$x \subseteq A$	x is a subset of A or is equal to A
$n(A)$	The cardinality of A
U	The universal set
$\varnothing$	The empty set
$A \cup B$	A union B
$A \cap B$	A intersection B
$A \backslash B$	A difference B
$\overline{A}$	A complement
A x B	The Cartesian product of A and B
$P(A)$	The power set of A