Sets and Groups Part 2 – Answers to Tutorial Questions

1. Use the laws of set algebra to simplify the following expression:

$$A \cup (\overline{A} \cap B)$$

<u>Solution</u>

$A \cup (\overline{A} \cap B) = (A \cup \overline{A}) \cap (A \cup B)$ Distributive Law

$= U \cap (A \cup B)$ Complement Law

$= A \cup B$ Identity Law

2. Use set algebra to show that: $B \cap (\overline{A \cap B}) = B \cap \bar{A}$

<u>Solution</u>

$B \cap (\overline{A \cap B})$	$= B \cap (\bar{A} \cup \bar{B})$	De Morgan's Law
	$= (B \cap \overline{A}) \cup (B \cap \bar{B})$	Distributive Law
	$= (B \cap \bar{A}) \cup \emptyset$	Complement Law
	$= B \cap \bar{A}$	Identity Law

3. In the last tutorial you drew Venn diagrams to prove the following: $A \backslash B = A \cap \bar{B}$

a) Bearing this in mind, use set algebra to show that:

$$B \cup (A \backslash B) = B \cup A$$

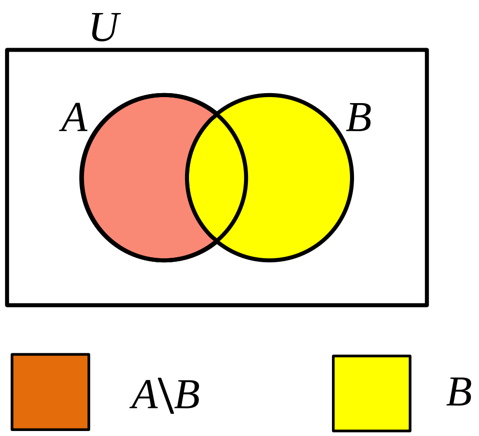
b) Verify that this is true by drawing a Venn diagram.

<u>Solution</u>

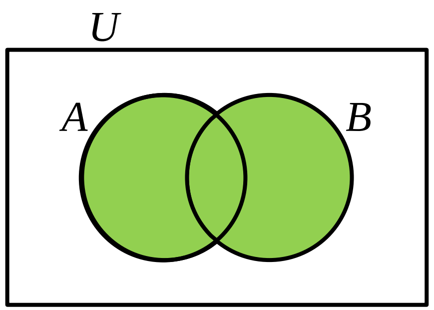
a)

$B \cup (A \backslash B) = B \cup (A \cap \bar{B})$	From previous tutorial
$= (B \cup A) \cap (B \cup \bar{B})$	Distributive Law
$= (B \cup A) \cap U$	Complement Law
$= B \cup A$	Identity Law

b)



Shaded area: $B \cup (A \backslash B)$



Shaded area: $B \cup A$

4. Consider the following numbers: 2.35 -335 2 -3.75 0

Which of these numbers are:

a) Real numbers b) Integers c) Natural numbers?

Solution

a) All of them b) -335, 2, 0 c) 2, 0

5. Which of the following is a rational number? Give a reason for your answer.

7.5 e (Euler's constant) $\sqrt{7}$ $\sqrt{25}$

Solution

7.5 because it can be represented as $\frac{15}{2}$

$\sqrt{25}$ because it can be represented as $\frac{5}{1}$

6. Simplify the following expressions containing complex numbers:

a) $i(3 + i)$ b) $(2 - 3i)(4 + i)$ c) $(3 - i)^2$

d) $(1 + i)^3$ e) $\frac{3+4i}{1+i}$

Solution

In each case, remember that $i^2 = -1$.

a) $i(3 + i) = 3i + i^2$

$= -1 + 3i$

b) $(2 - 3i)(4 + i) = 8 + 2i - 12i - 3i^2$

$= 11 - 10i$

c) $(3 - i)^2 = 9 - 6i + i^2$

$= 8 - 6i$

d) $(1 + i)^3 = (1 + i)(1 + i)^2 = (1 + i)(1 + 2i + i^2)$

$= (1 + i)(1 + 2i - 1)$

$= 2i(1 + i)$

$= 2i + 2i^2$

$= -2 + 2i$

e) Multiply numerator and denominator by $(1 - i)$

$$\frac{3 + 4i}{1 + i} = \frac{(3 + 4i)(1 - i)}{(1 + i)(1 - i)}$$

$$= \frac{7 + i}{1 - i^2} \quad = \frac{7 + i}{2}$$

7. Consider the following sets:

A = the set of integers less than 10

B = the set of natural numbers less than 10

S = the set of people living in London

T = the set of real numbers greater than 1000

In each case, state whether the set is: (a) countable or non-countable
(b) finite or infinite.

Solution

A is countable and infinite.
B is countable and finite.
S is countable and finite.
T is non-countable and infinite.

8. Consider the following combinations of a set and an operation, and state, giving reasons, which of them constitute a formal group:

   a) Natural numbers under subtraction

   b) The set {0} under addition

   c) The set {1} under multiplication

<u>Solution</u>

a) We know that subtraction is not an associative operation, so natural numbers under subtraction is not a group.

b) The set {0} under addition

Is there an identity element?

The identity element is 0 because 0 + 0 = 0.

Are there inverses?

0 + 0 = 0, so there is an inverse for each element (there is in fact only one element).

Is the operation associative?

We already know that addition is associative with integers

Is there closure?

It is closed because the result of 0 + 0 is in the group.

Therefore the set {0} under addition is a group.

c) The set {1} under multiplication

Is there an identity element?

The identity element is 1 because 1 x 1 = 1.

Are there inverses?

1 x 1 = 1, so there is an inverse for each element (there is in fact only one element).

Is the operation associative?

We already know that addition is associative with integers

Is there closure?

It is closed because the result of 1 x 1 is in the group.

Therefore the set {1} under multiplication is a group.

9. Show that the set of natural numbers starting from 0 ($\mathbb{N}_0$) under addition is a monoid but not a group.

Solution

Is there an identity element?

The identity element is 0.

Are there inverses?

There are no inverses apart from zero.

Is the operation associative?

We already know that addition is associative with integers

Is there closure?

It is closed because the result of adding any two natural numbers is always a natural number

Therefore $\mathbb{N}_0$ under addition has an identity, closure and associativity so is a monoid but not a group.

10. Show that the set of natural numbers starting from 1 ($\mathbb{N}_1$) under addition is a semigroup but not a group.

Solution

Is there an identity element?

There is no identity element (0 is not in the group).

Are there inverses?

There are no inverses because there is no identity element

Is the operation associative?

We already know that addition is associative with integers

Is there closure?

It is closed because the result of adding any two natural numbers is always a natural number

Therefore $\mathbb{N}_1$ under addition has associativity and closure and is a semigroup but not a group.

11. Is the set of rational numbers countable? Can you provide a proof for your answer?

<u>Solution</u>

Yes, it is countable. It can be counted as follows: