Smoothing long memory time series using R

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Abstract

This paper provides first a brief summary of the SEMIFAR (semiparametric fractional autoregressiove) and ESEMIFAR (exponential SEMIFAR) models. Those models are extended slightly to include the moving average part. Under common distribution condition it is shown that the long memory parameter is not affected by the log-transformation. A simple data-driven algorithm is proposed, by which the selected bandwidth and the selected orders of the ARMA model are all consistent. An R package is developed for practical implementation. The application of the proposals are illustrated by different kind of time series.

Keywords: Nonparametric regression with long memory, SEMIFAR, ESEMIFAR, bandwidth selection, model selection, implementation in R,

JEL Codes: C14, C51

1 Introduction

Literature research and model research required.

In many areas of research data are observed spatially, depending on two separate dimensions in a lattice. In recent years one can observe more frequently some sort of apparent memory in the decay of spatial correlations to depend and change over its direction within the spatial process. For instance, long-memory in the sense of slowly decaying autocorrelations in (high frequency) financial data across trading time and trading day produces a random field on a lattice in both dimensions simultaneously. Beran, Feng, and Ghosh (2015) state that daily average trade duration data has often shown long memory with a clear non zero mode. Therefore a log-normal conditional distribution is suggested. The simplest approach to model long range dependence in a positive valued time series is to take the exponential of a linear long memory process such as FARIMA leading to stochastic volatility models. Due to the long range dependence there is an unobservable latent process which makes the estimation and interpretation of the fitted parameters very challenging.

The SEMIFAR and ESEMIFAR models introduced by Beran and Feng (2002c) and Beran, Feng, and Ghosh (2015) are designed for simultaneous modeling of stochastic trends, deterministic trends and stationary short- and long-memory components in a time series such that the trend generating mechanisms can be distinguished.

2 Local polynomial regression with long memory

A well-established model for analysing financial time series data is the multiplicative error model (MEM) (Engle, 2002) which is given by

$$X_t = s\lambda_t\eta_t, \tag{1}$$

where the scale parameter is denoted by $s > 0$, $\lambda_t > 0$ denotes the conditional mean of $X^* = X_t/s$, and η_t are i.i.d. random variables with zero mean and unit variance.

Following Feng and Zhou (2015) we can rewrite (1) as a semiparametric MEM given by

$$X_t = s(\tau_t)\lambda_t\eta_t, \tag{2}$$

where $\tau_t = t/n$ denotes the rescaled time and where the scale parameter s in (1) is replaced with a nonparametric scale function denoted by $s(\tau_t)$. By taking the logs of (2) we have

$$Y_t = g(\tau_t) + Z_t, \tag{3}$$

where $Y_t = \ln(X_t)$, $g(\tau_t) = \ln[s(\tau_t)]$, $Z_t = \ln(\lambda_t) + \epsilon_t$ and $\epsilon_t = \ln(\eta_t)$. Following Beran and Feng (2002c) we assume that Z_t follows a zero mean FARIMA (p, d, q) process

$$(1 - B)^d\phi(B)Z_t = \psi(B)\epsilon_t, \tag{4}$$

where $d \in (0, 0.5)$ is the long-memory parameter, B is the backshift operator, $\phi(z) = 1 - \sum_{i=1}^{p}\phi_i z^i$ and $\psi(z) = 1 + \sum_{i=1}^{q}\psi_i z^i$ are AR- and MA-polynomials with all roots outside the unit circle. Equation (4) defines a stationary and invertible FARIMA process with $E(\epsilon_t) = 0$ and $var(\epsilon_t) = \sigma_\epsilon^2$. Model (3) is equivalent to a SEMIFAR process (Beran and Feng, 2002c) with no integer differencing ($m = 0$) and an additional MA-part. We have $X_t^* = \exp(Z_t)$. Subsequently, model (2) is an extended version of an ESEMIFAR introduced by Beran, Feng, and Ghosh (2015). However, the authors assumed that X_t^* is log-normally distributed whereas in this paper we relax this assumption and suppose that X_t^* satisfies condition **A1** of Feng et al. (2020).

In the following local polynomial estimation of the scale function $g^{(\nu)}$, the ν-th derivative of g, is exemplified briefly (see e.g. Beran and Feng, 2002a, Beran and Feng, 2002b, Beran and Feng, 2002c, and Beran et al., 2013). Under the assumption that g is at least $(l+1)$-times differentiable at a point t_0, $g(\tau_t)$ can be approximated by a local polynomial of order l for τ_t in a neighbourhood of τ_0. Following Gasser and Müller (1979), the weight function is determined to be a second order kernel with compact support $[-1, 1]$ having the polynomial form $K(u) = \sum_{i=0}^{r} a_i u^{2i}$, for ($|u| \leq 1$),where $K(u) = 0$ if $|u| > 1$ and a_i are such that $\int_{-1}^{1} K(u)du = 1$ holds. Here, $r \in \{0, 1, 2, 3\}$ denotes the kernel used for estimating $g^{(\nu)}$, corresponding to the uniform, epanechnikov, bisquare and triweight kernel. $\hat{g}^{(\nu)}$ ($\nu \leq l$) can now be obtained by solving the locally weighted least squares problem

$$Q = \sum_{i=1}^{t} \left[Y_t - \sum_{j=0}^{l} b_j (\tau_i - \tau_0)^j \right]^2 K\Big(\frac{\tau_i - \tau_0}{h}\Big), \tag{5}$$

where h denotes the bandwidth and $K[(\tau_i - \tau_0)/h]$ are the weights ensuring that only observations in the neighbourhood of τ_0 are used. Consider the case where $l - \nu$ is odd. Define $m = l + 1$, then we have $m \geq \nu + 2$ and $m - \nu$ is even. A point τ is said to be in the interior for each $\tau_t \in [h, 1-h]$, at the left boundary if $\tau_t \in [0, h)]$ and at the right boundary if $\tau_t \in (1-h, 1]$. Following Beran and Feng (2002b) a common definition for an interior point is $\tau = ch$ with $c = 1$ and for a boundary point we have $c \in [0, 1)$. Beran and Feng (2002a) and Beran and Feng (2002b) Asymptotic expressions for the bias, variance and mean integrated squared error (MISE) are presented in Theorem 1 and 2 by Beran and Feng (2002b). The asymptotic mean integrated squared error (AMISE) is given by

$$\text{AMISE}(h) = h^{2(m-\nu)} \frac{I[g^{(m)}]\beta^2}{m!} + \frac{(nh)^{2d-1} V(1)}{h^{2\nu}}, \tag{6}$$

where $I[g^{(m)}] = \int_{c_b}^{1-d_b} [g^m(\tau)]^2 d\tau$ with $0 \leq c_b < d_b \leq 1$ in order to reduce the so-called boundary effect. Moreover, $\beta = \int_{-1}^{1} u^m K(u) du$ and for $d > 0$ we have $V(1) = 2c_f \Gamma(1-2d) \sin(\pi d) \int_{-1}^{1} \int_{-1}^{1} K(x) K(y) |x - y|^{2d-1} dx dy$. For $d = 0$, V reduces to $V(1) = 2\pi c_f \int_{-1}^{1} K^2(x) dx$. c_f stands for the spectral density of the ARMA part of (4) at frequency zero and is given by

$$c_f = f(0) = \frac{\sigma_\epsilon^2}{2\pi} \frac{(1 + \psi_1 + \cdots + \psi_q)^2}{(1 - \phi_1 - \cdots - \phi_p)^2}. \tag{7}$$

The asymptotically optimal bandwidth, denoted by h_A, that minimizes the AMISE is given by

$$h_A = C n^{(2d-1)/(2m+1-2d)}, \tag{8}$$

with

$$C = \left(\frac{[m!]^2}{2(m-\nu)} \frac{(2\nu + 1 - 2d)}{\beta^2} \frac{(d_b - c_b) V(1)}{I[g^{(m)}]} \right)^{1/(2m+1-2d)}. \tag{9}$$

Based on these results Beran and Feng (2002a) proposed two iterative plug-in algorithms for automatic bandwidth selection, namely Algorithm **A** and **B**. In this paper we only consider a strongly adapted version of Algorithm **B** which is presented in the following.

3 Data-driven estimation

In order to obtain a selected bandwidth the unknown constants $I[g^{(m)}]$, d and V in (8) have to be replaced with consistent estimators. Please note, that the estimation of V relates to that of cf. $I[g^{(m)}]$ is estimated by means of local polynomial regression and numerical integration. The remaining two quantities d and V can be obtained via maximum likelihood. Inserting those estimates into (8) yields a plug-in estimator for the bandwidth, which minimises the MISE.

3.1 The IPI-algorithm for estimating g

We introduce an IPI-procedure for SEMIFARIMA models by translating and adapting the main features of the IPI for SEMIFAR models introduced by Beran and Feng (2002a) from the programming language `S` to `R`. The algorithm processes as follows:

i) In the first iteration start with an initial bandwidth h_0 set beforehand and select p and q denoting the AR- and MA-order, respectively.

ii) Estimate g from Y_t employing h_{j-1} and calculate the residuals $\tilde{Z}_t = Y_t - \hat{g}(\tau_t)$. Estimate d and V by fitting a FARIMA (with predefined AR- and MA-order in Step i) to $\hat{Z}_t$.

iii) Set $h_{d,j} = (h_{j-1})^{\alpha}$, where α denotes an inflation factor. Estimate $I[g^{(m)}]$ via a local polynomial of order $l^* = l + 2$ and with h_d, j. Now, we obtain h_{j-1} by

$$h_j = \left(\frac{[m!]^2}{2m} \frac{(1-2\hat{d})}{\beta^2} \frac{(d_b - c_b)\hat{V}(1)}{I[\hat{g}^{(m)}]} \right)^{1/(2m+1-2\hat{d})} \cdot n^{(2\hat{d}-1)/(2m+1-2\hat{d})}. \tag{10}$$

iv) Repeat steps ii) and iii) until convergence or a given number of iterations has been reached and set $\hat{h}_{opt} = h_j$.

We propose to set the initial bandwidths to $h_0 = 0.1$ for $l = 1$ and $h_0 = 0.2$ for $l = 3$. Moreover, for $p = 3$ it is recommended to employ $c_b = 1 - d_b$ such that only 90% of all observations are used for estimating an interior point in order to reduce the boundary effect. For $l = 1$ all observations are used and hence $c_b = 1 - d_b = 0$. The bandwidth $h_{d.j}$ used for estimating $g^{(m)}$ is enlarged by means of an exponential inflation factor denoted by

α. We have $\alpha = \alpha_{\text{opt}} = (2m+1-2d)/(2m+3-2d)$, $\alpha = \alpha_{\text{nai}} = (2m+1-2d)/(2m+5-2d)$ and $\alpha = \alpha_{var} = \frac{1}{2}$. Using α_{opt} results in bandwidth $h_{d,j}$ that minimizes the MSE of $\hat{I}[g^{(m)}]$ and consequently the rate of convergence of $\hat{h}_j$ is optimal. Whereas for α_{nai} the optimal rate of convergence is achieved for $\hat{m}^{(m)}$ and α_{var} ensures a stable selection of the bandwidth. Moreover, we have $\alpha_{\text{var}} > \alpha_{\text{nai}} > \alpha_{\text{opt}}$ and $\alpha_{\text{nai}} \to \alpha_{\text{var}}$ as $d \to 0.5$. The choice of α depends on the underlying data, which is to be analysed. For a more detailed insight on inflation methods we refer the reader to Beran and Feng (2002a) among others.

3.2 Data-driven estimation of g' and g''

The IPI for SEMIFARIMA models can also be applied to bandwidth selection for estimating $g^{(\nu)}$ with $\nu > 0$. In this paper, only the cases for $\nu = 1$ and $\nu = 2$ are discussed. The proposed IPI is now employed as a data-driven pilot method to obtain estimates for d, c_f and $h_{\nu,0}$ with order l_d, say. Estimation of $g^{(\nu)}$ is then carried out with $l = \nu + 1$ and $m = \nu + 2$. As previously, $g^{(m)})$, which is required for calculating $I[g^{(m)}]$ is estimated with order $l^* = l + 2$. The following two-stage procedure is proposed.

i) In the first stage $\hat{d}$, $\hat{c}_f$, and $\hat{h}_{\text{opt}}$ are obtained by means of the main IPI-algorithm for estimating g with order $l_d = 1$ or $l_d = 3$.

ii) Set $h_{\nu,0} = \hat{h}_{\text{opt}}$. Carry out an IPI-procedure as proposed above with fixed $\hat{c}_f$ and $\hat{d}$ in order to select a bandwidth for estimating $g^{(\nu)}$. Please note that (8) should be used.

Explicit formulas of the equivalent kernels for estimating $g^{(\nu)}$ at an interior point τ_t can be found in Müller (1988). The corresponding inflation factors are defined as previously and are determined by m and d.

4 Implementation in R

Based on the algorithms introduced in the previous section a R-package is currently developed, which is an extension of the already published *smoots* package. Hence, this package will be coined *smootslm*. The main functions are called *tsmoothlm* and *dsmoothlm* for estimating the trend and its derivatives, respectively, under presence of long-memory errors. Local polynomial estimation of $g^{(\nu)}$ and kernel smoothing of g are carried out by means of the functions *gsmooth* and *knsmooth*, which are implemented in the *smoots* package (see Feng et al., 2020).

In the following the function *tsmoothlm* is explained in more detail. The first argument y denotes the input time series. The second and third argument (*pmin* and *pmax*) are the minimum and maximum AR-order of the stochastic part Z_t in (3), respectively. Accordingly, the fourth and fifth argument (*qmin* and *qmax*) stand for the the minimum and maximum MA-order of Z_t. All four arguments can take the value 0, 1, 2, 3, 4 or 5 while $p_{\min} \leq p_{\max}$ and $q_{\min} \leq q_{\max}$. The optimal order is determined via BIC. The default setting is $p_{\min} = q_{\min} = p_{\max} = q_{\max} = 0$. The order of the polynomial for trend estimation is set via the argument p and the user can choose between 1 and 3, where $p = 1$ is the predefined option. The argument *mu* controls for the smoothness of the weight function. We have $\mu \in \{0, 1, 2, 3\}$, with $\mu = 1$ for the Epanechnikov kernel as default. Furthermore, the inflation factor α can be selected by the argument *InfR* with three different options, i.e. *"Opt"*, *"Nai"* and *"Var"*, which corresponds to $\alpha_{\text{opt}} = (2m+1-2d)/(2m+3-2d)$, $\alpha_{\text{nai}} = (2m+1-2d)/(2m+5-2d)$ and $\alpha_{var} = \frac{1}{2}$, respectively. Moreover, the starting bandwidth h_0 can be set beforehand by the argument *bStart* with default $h_0 = 0.1$ for $p = 1$ and $h_0 = 0.2$ for $p = 3$. However, the choice of *bStart* should not affect the finally selected bandwidth if the IPI converges. Argument *bb* controls for boundary bandwidth. The default is $bb = 1$ meaning that the k-nearest neighbour method is applied, which results in a total bandwidth of $2\hat{h}$ at each observation point τ_t. For $bb = 0$ however, the total bandwidth is shortened at boundary points. By the argument *cb*, which is set to $cb = 0.05$ per default, the percentage of observations omitted for calculating $I[g^{(m)}] = \int_{c_b}^{1-d_b} [g^m(\tau)]^2 d\tau$ in (8) can be controlled. Additionally, the smoothing method with $\hat{h}_{\text{opt}}$ can be selected via the argument *method*. The user may choose between local polynomial regression ("*lpr*") and kernel regression ("*kr*"). However, originally kernel regression has been only incorporated in the *smoots* package as a comparison to local polynomial regression.

For estimating $g^{(\nu)}$ the function *dsmoothlm* is applied. Please recall that here *tsmoothlm* is employed as a pilot method with minimum and maximum AR- and MA-order *pmin.p*, *pmax.p*, *qmin.p* as well as *qmax.p*, a local polynomial estimator of order *pp*, the inflation rate *InfR.p*, the kernel *mu.p* and a starting bandwidth *bStart.p* in order to obtain reasonable estimates for $\hat{d}$, $\hat{c_f}$ and $\hat{h}_{\nu,0}$ (see section 3.2). The options and default settings for these arguments are the same as for *tsmoothlm*. In addition to that, the order of the derivative to be estimated is set via the argument *nu* and can take the value 1 or 2 for the first and second derivative, respectively. Moreover, the argument *mu* controls the kernel used for bandwidth selection after the pilot stage. Please note that the S3 methods (*print* and *plot*) implemented in the *smoots* package can be employed to the estimation results of the functions above. The output objects of *tsmoothlm* and *dsmoothlm* are basically lists containing input parameters and estimation results. Furhter detailed information on the functions are to be published in the users guideline of the *smootslm* package.

5 Application to different kinds of time series

In this and the following sections the SEMIFARIMA and ESEMIFARIMA are applied to four real data examples: *tempNH* (mean monthly temperature changes), *gdpGER* (GER GDP), *dax* (German stock index) and *vix* (CBOES volatility index). The *tempNH* data set has already been subject to an application example in Feng (2007), where the author employed the original version of the IPI for SEMIFAR models. The remaining three data sets have been already used by Feng et al. (forthcoming) and are implemented in the *smoots* package, which was recently published on the *CRAN* network.

5.1 Application to environmental data

The SEMIFARIMA model defined by (3) and (4) is applied to the time series of mean monthly Northern Hemisphere temperature changes (NHTM) from 1880 to 2018. The data is available at the website of the National Aeronautics and Space Administration (NASA). Bandwidth selection is carried out by means of the IPI for SEMIFARIMA models introduced in section 3. For model- and bandwidth selection the *tsmoothlm* function is applied, with $p = 1$, $pmin = qmin = 0$, $pmax = qmax = 3$ and $InfR$ = *"Opt"*. The remaining arguments are set on their default.

In Figure 1a) the fitted trend together with the observations is illustrated. The optimal bandwidth is $\hat{h}_{\text{opt}} = 0.165$ and a FARIMA $(0, \hat{d}, 0)$ has been selected following the BIC with $\hat{d} = 0.405$ implying strong long-range dependence in the temperature data. Apparently, the SEMIFARIMA captures the trend quite well. A clear upward trend can be observed approximately after 1970, which could be interpreted as an indicator for global warming. The trend-adjusted residuals are shown in Figure 1b) and first and second derivative are depicted in Figures 1c) and 1d), respectively. Please note that for the estimation of derivatives the dependence structure has been estimated by pilot smoothing with order $pp = 3$. The derivatives match the features of the trend shown in Figure 1a) and provide further information about global temperature changes. For instance the slope of the first derivative indicates how strong the trend is increasing or decreasing. The intersections of the second derivative with the x-axis indicate a shift in the slope of $\hat{g}$.

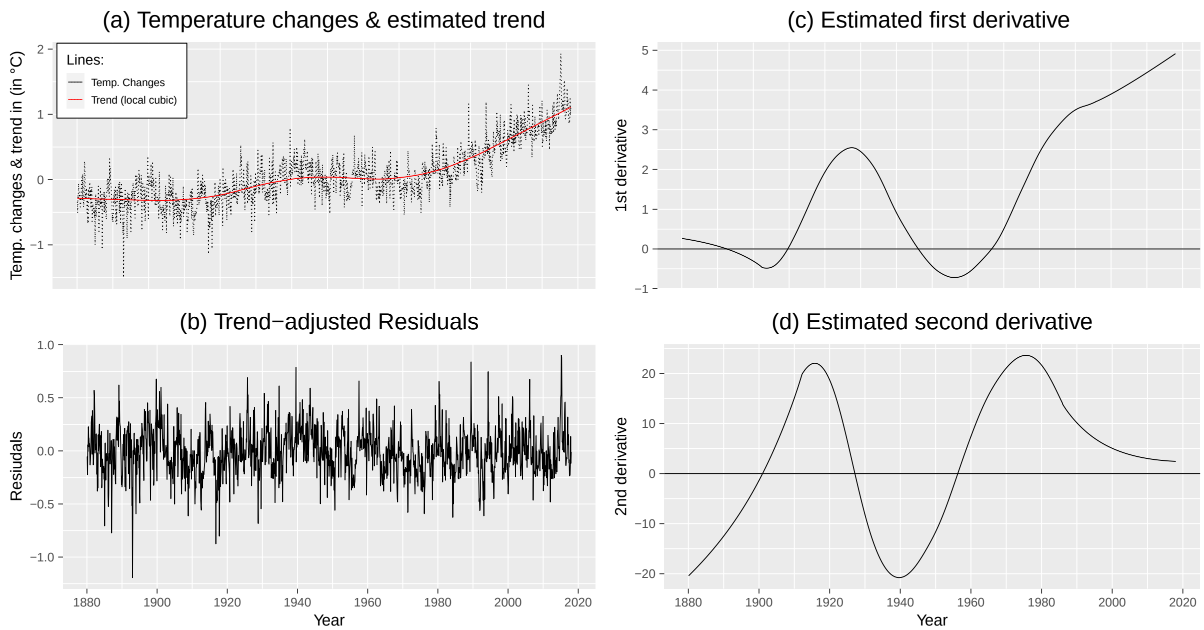

Figure 1: Estimated trend, residuals and the trend‘s derivatives for the NHTM series

5.2 Application to GDP data

A model which is commonly used in the field of macroeconomic research is the well-known log-linear growth model. Feng et al. (2020) have achieved a semiparametric local-linear extensions of this model by applying a Semi-ARMA to log-transformed GDP series. We follow this approach and additionally incorporate long memory by assuming that the log-transformed annually GER-GDP series from 1850 to 2016 follows a SEMI-FARIMA defined by (3) and (4). For this purpose the *tsmoothlm* function is employed, with $p = 1$, $pmin = qmin = 0$, $pmax = qmax = 3$ and $InfR = $ *"Opt"*. The remaining arguments are set on their default. We obtained an optimal bandwidth of 0.161 and a FARIMA $(2, \hat{d}, 2)$ model given by

$$Z_t = -0.134 Z_{t-1} + 0.542 Z_{t-2} + \frac{(1.363\epsilon_{t-1} + 0.275\epsilon_{t-2} + \epsilon_t)}{(1 - B)^{0.191}}. \tag{11}$$

As a benchmark a kernel regression is carried out using the same bandwidth. Estimated trends together with log-gdp series are shown in Figure 2(a). At the interior both estimators are approximately equal. However, we can see that the kernel estimator clearly shows poor estimation quality at the boundaries which indicates that the local-linear estimator is to be preferred. The trend-adjusted residuals obtained by the local-linear approach are depicted in Figure 2(b). Moreover, the corresponding derivatives are illustrated in Figures 2(c) and 2(d), which reveal further information on the course of the German economy.

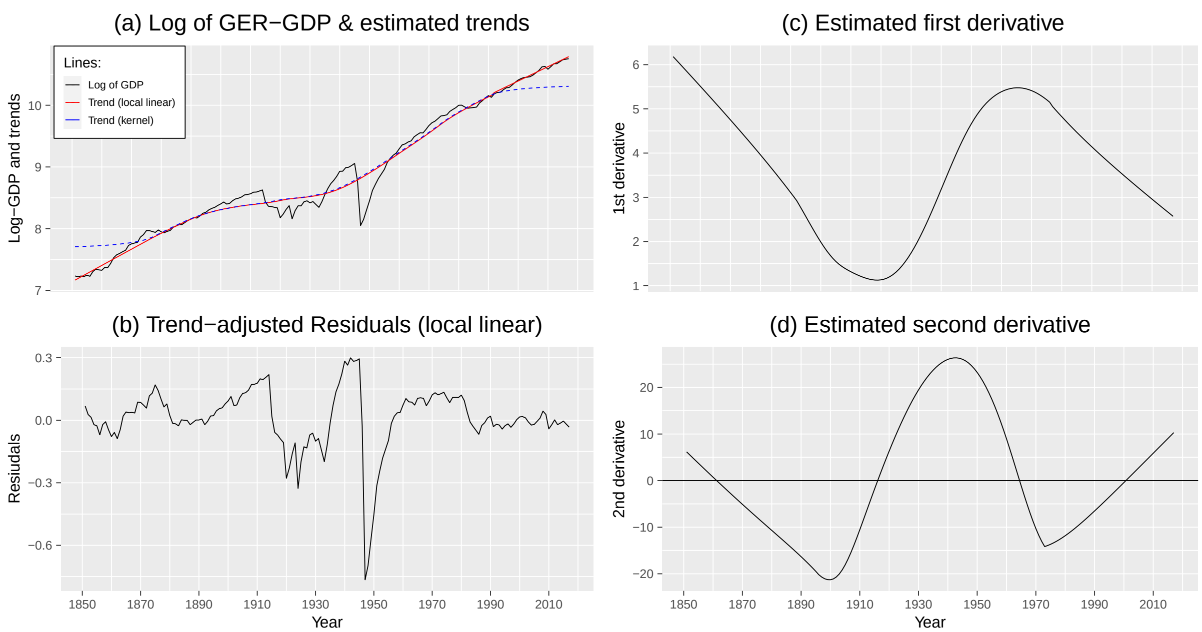

Figure 2: Estimated trend, residuals and the trend‘s derivatives for the GER-GDP series

6 Application to high-frequency data

The autoregressive conditional heteroscedasticity (ARCH) model proposed by Engle (1982) and its generalisation, the generalized ARCH (GARCH)) model, introduced by Bollerslev (1986) , is a well-known volatility process approach for modelling non-constant conditional variances. Feng (2004) found that return series often simultaneously exhibit conditional heteroskedasticity and a slowly changing scale. Under regular conditions a process with conditional heteroskedasticity is covariance stationary, but a process with change in volatility is at best locally stationary. However, the majority of GARCH extensions are defined assuming a stationary return series. Based on his findings the author proposed the Semi-GARCH model by adding a smooth scale function to the standard GARCH model. Recently, Feng et al. (forthcoming) introduced the Semi-Log-GARCH model which is an extension of the Log-GARCH introduced by Pantula (1986), Geweke (1986) and Milhøj (1987). Moreover, Feng et al. (2020) and Letmathe et al. (forthcoming) proposed the FI-Log- and Semi-FI-Log-GARCH models, respectively. The FI-Log-GARCH is a special case of the FIAPARCH.

Let r_t^*, $t = 1, ..., n$ denote a return series with $E(r_t^*) = \mu_{r^*}$. We have

$$r_t = \sigma(\tau_t)\sqrt{h_t}\eta_t, \tag{12}$$

where $r_t = r_t^* - \mu_{r^*}$ are the centralized returns, $\sigma^2(x_t) > 0$ stands for a smooth scale function and η_t is defined as in (1). Let $\xi_t^2 = r_t^2/\sigma^2(\tau_t) = h_t\eta_t^2$ and $\alpha_{d,i} = \psi(B) - \phi(B)(1-B)^d$, where $\phi(B) = 1 - \sum_{i=1}^{p^*} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j$ with $p^* = \max(p, q)$ and $\psi(B) = 1 + \sum_{j=1}^{q} \psi_j B^j = 1 - \sum_{j=1}^{q} \beta_j B^j$. Furthermore, α_i and β_j are the ARCH and GARCH coefficients, respectively. $\{\xi_t\}$ is assumed to follow a FI-Log-GARCH process given by

$$\ln h_t = \alpha_0 + \sum_{i=1}^{\infty} \alpha_{d,i} \ln \xi_{t-1}^2 + \sum_{j=1}^{q} \beta_j \ln h_{t-j}, \tag{13}$$

where α_0 is some constant. (12) and (13) together define a SEMI-FI-Log-GARCH. To ensure that our model is well defined we assume that $\xi_t \neq 0$ a.s. and $\mathrm{var}(\xi_t) = 1$. Let $y_t = \ln r_t^2$, $g(\tau_t) = \ln \sigma^2(\tau_t) + \mu_{l\xi^2}$ and $Z_t = \ln \xi_t^2 - \mu_{l\xi^2}$, where $\mu_{l\xi^2} = E(\ln \xi_t^2)$. We can see that the log-transformation of the Semi-FI-Log-GARCH defined by (12) and (13) admits an additive model form $y_t = g(\tau_t) + Z_t$, which is a special case of Model (3). Moreover, let $\epsilon_t = \ln \eta^2 - \mu_{l\epsilon^2}$, with $\mu_{l\epsilon^2} = E(\ln \eta_t^2)$ and then we have $Z_t = \ln h_t + \epsilon_t - (\mu_{l\xi^2} - \mu_{l\epsilon^2})$. It was shown by Feng et al. (2020) that Z_t can be represented as a FARIMA(p^*, d, q) model given by

$$(1-B)^d \phi(B)(Z_t) = \psi(B)\epsilon_t, \tag{14}$$

which is in turn a special case of Model (4). We see that the Semi-FI-Log-GARCH is equivalent to a SEMIFARIMA model with the restriction $p \geq q$. Subsequently, well developed SEMIFARIMA algorithms are applicable for estimating $g(\tau_t)$ and Z_t. $\hat{\sigma}(\tau_t)$ can be obtained by $\hat{\sigma}(\tau_t) = \hat{C}_\sigma \exp[\hat{g}(\tau_t)/2]$, where $C_\sigma^2 = \mathrm{var}\,(r_t/\exp[g(\tau_t)/2])$. C_σ can be estimated consistently by the scale-adjusted returns under the assumption that $E(\xi_1^4) < \infty$.

The Semi-FI-Log-GARCH is applied to

7 The Semi-FI-Log-ACD model* (nachfragen)

8 Concluding remarks

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Appendix