Ep. 6: Direct Sum, Quotient Space, and Universal Properties

LetsSolveMathProblems: Navigating Linear Algebra

Let V, W, U be vector spaces over a field F.

Problem 1. *Do there exist vector spaces V and W such that V is finite-dimensional and* $\dim W > 0$*, yet V is isomorphic to the external direct sum $V \oplus W$? Can the answer change if V may be infinite-dimensional?*

Problem 2. *For $(a,b),(c,d) \in \mathbb{R}^2$, let $(a,b)\cdot(c,d)$ denote the dot product of the two vectors, defined to be $ac+bd \in \mathbb{R}$. Prove that the map $\phi : \mathbb{R}^2 \to (\mathbb{R}^2)^*$ given by $v \mapsto (w \mapsto v \cdot w)$ is an isomorphism.*

Problem 3. *Suppose that W is a subspace of V and that U is a subspace of W. Prove that $(V/U)/(W/U)$ is isomorphic to V/W.*

Problem 4. *Suppose that W is a subspace of V such that $W \neq V$. If V/W is finite-dimensional, then is it necessary that V is finite-dimensional?*

Problem 5. *Suppose that W is a subspace of V. We can define the quotient space to be the pair $(V/W,\ \pi)$, where V/W is a vector space over F, and $\pi : V \to V/W$ is a linear map such that* $\ker \pi = W$ *and the following is met:*

For any space U over the field F and any linear map $\phi : V \to U$ such that $W \subseteq \ker \phi$, there exists the unique linear map $\psi : V/W \to U$ such that $\psi \circ \pi = \phi$. See the following diagram:

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "\phi"'] & V/W \arrow[d, "\psi"] \\ & U \end{tikzcd}$$

(a) *Show that for the quotient space V/W defined in this episode, there exists some map $\pi : V \to V/W$ such that $(V/W,\ \pi)$ satisfies the above universal property.*

(b) *Conversely, show that if some (U', π') satisfies the above universal property, then there is an isomorphism from U' to V/W defined in this episode.*

Problem 6. *(**Warning**: This problem is one of the hardest problems in this series.)*

In this problem, $V \oplus W$ denotes the external direct sum of V and W.

*We say that a map $\phi : V \oplus W \to U$ is bilinear if for any fixed $v' \in V$, the map $w \mapsto \phi(v', w)$ (with domain W and codomain U) is linear **and** for any fixed $w' \in W$, the map $v \mapsto \phi(v, w')$ (with domain V and codomain U) is linear.*

Given spaces V and W, their tensor product can be defined to be the pair $(V \otimes W,\ \tau)$, where $V \otimes W$ is a vector space over F, and $\tau : V \oplus W \to V \otimes W$ is a bilinear map, such that the following is met:

For any bilinear map $\phi : V \oplus W \to U$, there exists the unique linear map $\psi : V \otimes W \to U$ such that $\phi = \psi \circ \tau$. See the following diagram:

$$\begin{tikzcd} V \oplus W \arrow[r, "\tau"] \arrow[dr, "\phi"'] & V \otimes W \arrow[d, "\psi"] \\ & U \end{tikzcd}$$

For any $(v, w) \in V \oplus W$, let $v \otimes w = \tau((v, w))$. Throughout this problem, $\dim V = n < \infty$.

(a) Suppose that $\dim W = m < \infty$ (and recall that $\dim V = n$). Show that if V_1 is a vector space over F of dimension mn, then there exists some bilinear map $\tau_1 : V \oplus W \to V_1$ such that (V_1, τ_1) satisfies the above universal property. Conversely, show that if (V_2, τ_2) satisfies the above universal property, then V_2 is isomorphic to V_1, so that V_2 has dimension mn.

(b) Show that there exists the unique linear map $f : V \otimes V^* \to \mathrm{Hom}(V, V)$ satisfying $v \otimes g \mapsto (v' \mapsto vg(v'))$ for any $(v, g) \in V \oplus V^*$, and show that f is an isomorphism.

(c) Let V be a vector space over $\mathbb{R}$ (again of dimension $n < \infty$) and treat $\mathbb{C}$ as a vector space over $\mathbb{R}$ (as in Problem 1 of Ep. 2). Consider $V \otimes \mathbb{C}$, which is a vector space over $\mathbb{R}$.

We will now turn $V \otimes \mathbb{C}$ into a vector space over $\mathbb{C}$. First, show that any element of $V \otimes \mathbb{C}$ equals $v = \sum_{i=1}^{N} a_i(v_i \otimes z_i)$ for some $N \geq 1$, $a_i \in \mathbb{R}, v_i \in V, z_i \in \mathbb{C}$. Then, define $w \cdot v$ (where $w \in \mathbb{C}$) to be $\sum_{i=1}^{N} a_i(v_i \otimes (w \cdot z_i))$.

Now find the dimension of $V \otimes \mathbb{C}$ over the field $\mathbb{C}$.