Ep. 6: Direct Sum, Quotient Space, and Universal Properties

LetsSolveMathProblems: Navigating Linear Algebra

Let V, W, U be vector spaces over a field F.

Problem 1. Do there exist vector spaces V and W such that V is finite-dimensional and dim W > 0, yet V is isomorphic to the external direct sum $V \oplus W$? Can the answer change if V may be infinite-dimensional?

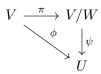
Problem 2. For $(a,b),(c,d) \in \mathbb{R}^2$, let $(a,b)\cdot(c,d)$ denote the dot product of the two vectors, defined to be $ac+bd \in \mathbb{R}$. Prove that the map $\phi: \mathbb{R}^2 \to (\mathbb{R}^2)^*$ given by $v \mapsto (w \mapsto v \cdot w)$ is an isomorphism.

Problem 3. Suppose that W is a subspace of V and that U is a subspace of W. Prove that (V/U)/(W/U) is isomorphic to V/W.

Problem 4. Suppose that W is a subspace of V such that $W \neq V$. If V/W is finite-dimensional, then is it necessary that V is finite-dimensional?

Problem 5. Suppose that W is a subspace of V. We can define the quotient space to be the pair $(V/W, \pi)$, where V/W is a vector space over F, and $\pi : V \to V/W$ is a linear map such that $\ker \pi = W$ and the following is met:

For any space U over the field F and any linear map $\phi: V \to U$ such that $W \subseteq \ker \phi$, there exists the unique linear map $\psi: V/W \to U$ such that $\psi \circ \pi = \phi$. See the following diagram:



- (a) Show that for the quotient space V/W defined in this episode, there exists some map $\pi: V \to V/W$ such that $(V/W, \pi)$ satisfies the above universal property.
- (b) Conversely, show that if some (U', π') satisfies the above universal property, then there is an isomorphism from U' to V/W defined in this episode.

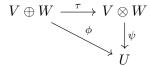
Problem 6. (Warning: This problem is one of the hardest problems in this series.)

In this problem, $V \oplus W$ denotes the external direct sum of V and W.

We say that a map $\phi: V \oplus W \to U$ is bilinear if for any fixed $v' \in V$, the map $w \mapsto \phi(v', w)$ (with domain W and codomain U) is linear **and** for any fixed $w' \in W$, the map $v \mapsto \phi(v, w')$ (with domain V and codomain U) is linear.

Given spaces V and W, their tensor product can be defined to be the pair $(V \otimes W, \tau)$, where $V \otimes W$ is a vector space over F, and $\tau : V \oplus W \to V \otimes W$ is a bilinear map, such that the following is met:

For any bilinear map $\phi: V \oplus W \to U$, there exists the unique linear map $\psi: V \otimes W \to U$ such that $\phi = \psi \circ \tau$. See the following diagram:



For any $(v, w) \in V \oplus W$, let $v \otimes w = \tau((v, w))$. Throughout this problem, dim $V = n < \infty$.

- (a) Suppose that dim $W=m<\infty$ (and recall that dim V=n). Show that if V_1 is a vector space over F of dimension mn, then there exists some bilinear map $\tau_1:V\oplus W\to V_1$ such that (V_1,τ_1) satisfies the above universal property. Conversely, show that if (V_2,τ_2) satisfies the above universal property, then V_2 is isomorphic to V_1 , so that V_2 has dimension mn.
- (b) Show that there exists the unique linear map $f: V \otimes V^* \to \operatorname{Hom}(V, V)$ satisfying $v \otimes g \mapsto (v' \mapsto vg(v'))$ for any $(v,g) \in V \oplus V^*$, and show that f is an isomorphism.
- (c) Let V be a vector space over \mathbb{R} (again of dimension $n < \infty$) and treat \mathbb{C} as a vector space over \mathbb{R} (as in Problem 1 of Ep. 2). Consider $V \otimes \mathbb{C}$, which is a vector space over \mathbb{R} .

We will now turn $V \otimes \mathbb{C}$ into a vector space over \mathbb{C} . First, show that any element of $V \otimes \mathbb{C}$ equals $v = \sum_{i=1}^{N} a_i(v_i \otimes z_i)$ for some $N \geq 1$, $a_i \in \mathbb{R}$, $v_i \in V$, $z_i \in \mathbb{C}$. Then, define $w \cdot v$ (where $w \in \mathbb{C}$) to be $\sum_{i=1}^{N} a_i(v_i \otimes (w \cdot z_i))$.

Now find the dimension of $V \otimes \mathbb{C}$ over the field \mathbb{C} .