Ep. 5: Row Reduction

LetsSolveMathProblems: Navigating Linear Algebra

For Problems 1 and 4, the reduced row echelon forms of relevant matrices are given to you (as *Hints*) to make the problems significantly less computational. However, you should check that at least one of these reduced row echelon forms is indeed correct.[1]

Problem 1. *(a) Are* $v_1 = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 9 \\ -37 \\ 8 \end{bmatrix}, v_3 = \begin{bmatrix} 76 \\ -308 \\ 67 \end{bmatrix} \in \mathbb{R}^3$ *linearly independent? If they are linearly dependent, find a linear combination (i.e.,* $a_1v_1 + a_2v_2 + a_3v_3$*) that adds up to zero despite some* a_i *not equalling zero.*

Hint: The reduced row echelon form of $\begin{bmatrix} 1 & 9 & 76 \\ -5 & -37 & -308 \\ 1 & 8 & 67 \end{bmatrix}$ *is* $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) Let $v_1 = \begin{bmatrix} 4 \\ 27 \\ 13 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 21 \\ 10 \end{bmatrix}, v_3 = \begin{bmatrix} -19 \\ -126 \\ -61 \end{bmatrix} \in \mathbb{R}^3$ *and* $v' = \begin{bmatrix} 15 \\ 106 \\ 50 \end{bmatrix} \in \mathbb{R}^3$*. Is* v' *in the span of* $v_1, v_2,$ *and* v_3*? If so, find a linear combination* $a_1v_1 + a_2v_2 + a_3v_3$ *that equals* v'*.*

Hint: The reduced row echelon form of $\begin{bmatrix} 4 & 3 & -19 & 15 \\ 27 & 21 & -126 & 106 \\ 13 & 10 & -61 & 50 \end{bmatrix}$ *is* $\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Problem 2. *Consider a system of* n *linear equations in* m *variables over a field* F*. If* $n > m$*, is it possible for the system to have exactly one solution? Does the answer change if* $n < m$*?*

Problem 3. *In this problem, the letter* I *denotes the identity matrix of appropriate size.*

(a) If A *and* B *are* $n \times n$ *real matrices and* $AB = I$*, must it be true that* $BA = I$*?*

(b) Does the answer change if A *is an* $n \times m$ *real matrix and* B *is an* $m \times n$ *real matrix, where* $n \neq m$*?*

(c) Finally, if $\phi, \psi : \mathbb{R} \to \mathbb{R}$ *are functions (not necessarily linear) and* $\phi \circ \psi$ *is the identity map on* $\mathbb{R}$*, then must it be true that* $\psi \circ \phi$ *is also the identity map on* $\mathbb{R}$*?*

Problem 4. *Each of the following matrices represents a linear map from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *(for appropriate* n, m*) with respect to the standard bases. Determine which matrices represent injective maps and which matrices represent surjective maps. Moreover, find the rank of each matrix.*

(a) $A = \begin{bmatrix} 20 & 40 & 60 & -57 \\ 5 & 10 & 15 & -14 \\ 1 & 2 & 3 & -3 \end{bmatrix}$

[1] This is a valid request because a matrix's reduced row echelon form is unique! Try to show this.

Hint: The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 1 & -4 & 1 & -4 & -2 \\ 16 & -64 & 16 & 14 & 5 \\ 4 & -16 & 4 & 3 & 1 \end{bmatrix}$

Hint: The reduced row echelon form of B is $\begin{bmatrix} 1 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 1 & 0 \\ 6 & -2 & 2 \\ 3 & -1 & 1 \end{bmatrix}$

Hint: The reduced row echelon form of C is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(d) $D = \begin{bmatrix} -2 & -9 & 3 \\ 1 & 5 & 0 \\ 10 & 45 & -14 \end{bmatrix}$

Hint: The reduced row echelon form of D is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 5. *You may take for granted that $\mathbb{Z}/3\mathbb{Z}$ is a field. Does there exist a system of two linear equations in two variables over the field $\mathbb{Z}/3\mathbb{Z}$ that has exactly 2 solutions? Does there exist such a system with exactly 3 solutions?*

Problem 6. *Let A be an $m \times n$ real matrix. Let A^T denote the transpose of A, which is a matrix obtained by "flipping" rows and columns of A. For example, if* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, *then* $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Show that $\dim \operatorname{im} A = \dim \operatorname{im} A^T$. *(Hint: If B is the reduced row echelon form of A, then show that* $\dim \operatorname{im} A^T = \dim \operatorname{im} B^T$.*)*