Ep. 8: Matrix Invertibility and Similarity

LetsSolveMathProblems: Navigating Linear Algebra

In what follows, if an $n \times n$ matrix A is deemed to be a *real* (or *complex*) matrix, then assume that A is an operator on $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Problem 1. *Find the inverse of* $\begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$ *using the row-reduction method from this episode.*

Note: You can show that for an invertible 2×2 *matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *its inverse is equal to* $\frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. *Use this to check your answer!*

Problem 2. *Let A be an invertible, complex matrix that contains at least one nonreal entry.*

(a) Is it possible for A^{-1} to contain only real entries?

(b) Is it possible for A to be similar to a complex matrix B with only real entries?

Problem 3. *(a) Find a 2×2 invertible real matrix that is not similar to its inverse.*

(b) Find a 2×2 real matrix that is not diagonalizable, not nilpotent, and not invertible.

Problem 4. *(a) If A and B are 2×2 real matrices of rank 1 with the same trace, are they necessarily similar?*

(b) If A and B are 2×2 real matrices of full rank (i.e., rank 2) with the same determinant and trace, are they necessarily similar?

Problem 5. *Let A and S be $n \times n$ real matrices, where S is invertible. Recall that $(SAS^{-1})^m = SA^mS^{-1}$ for any $m \geq 1$. Explain how this observation allows us to "quickly" calculate B^m for any $m \geq 1$ if we can write $B = SAS^{-1}$ for some diagonal matrix A.*

Problem 6. *Prove that if A and B are 2×2 nilpotent real matrices of rank 1, then A and B are similar.*