# Ep. 7: Determinant

LetsSolveMathProblems: Navigating Linear Algebra

**Problem 1.** *Find the determinant of the following real matrices, and try to come up with a geometric reason for why the determinant equals the value you found.*

*(a)* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *for some fixed* $\theta \in \mathbb{R}$.

*(Hint: Show that* $A$ *represents the rotation about the origin by* $\theta$ *radians counter-clockwise.)*

*(b)* $B = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

*(c)* $C = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$

*(d)* $D = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

**Problem 2.** *Given a* $4 \times 4$ *real matrix* $A$*, Bob applied the following elementary row operations to* $A$ *(in order from (i) to (v)) to obtain* $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 5 & -10 \\ 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 3 \end{bmatrix}$*:*

*(i) Scale the first row by* $2$

*(ii) Add the first row to the second row*

*(iii) Switch the first and the fourth rows*

*(iv) Add* $-15$ *times the first row to the third row*

*(v) Scale the third row by* $15$

*Find* $\det B$ *and* $\det A$. *(Hint: To find* $\det B$, *try finding* $\det B^T$ *by expanding along its first row.)*

**Problem 3.** *Let $A$ and $B$ be $n \times n$ matrices (whose entries are elements of a field $F$). Prove that* $\det(AB) = \det(A)\det(B)$.

*Hint: Review this episode's proof that $A$ is invertible if and only if* $\det(A) \neq 0$.

**Problem 4.** *Preferably using determinant, compute the area of the parallelogram with vertices at* $(-2,-2), (-3,3), (-1,2),$ *and* $(0,-3)$.

**Problem 5.** *Let* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

(a) *(Expanding along second row.) Prove the following equality:*

$$\det A = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

(b) *(Expanding along third column.) Prove the following equality:*

$$\det A = a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

(c) *Generalize parts (a) and (b) for the expansion along any $k^{th}$ row or column.*

**Problem 6.** *Let $V = F^n$ (where $F$ is a field), where $n < \infty$. Consider a pair $(\bigwedge^n V,\ \pi)$, in which $\bigwedge^n V$ is a vector space over $F$ and $\pi : V^n \to \bigwedge^n V$ is an alternating multilinear map, such that the following is met:*

*For any vector space $U$ over $F$ and any alternating multilinear map $\phi : V^n \to U$, there exists the unique linear map $\psi : \bigwedge^n V \to U$ such that $\phi = \psi \circ \pi$. See the following diagram:*

$$\begin{tikzcd} V^n \arrow[r, "\pi"] \arrow[dr, "\phi"'] & \bigwedge^n V \arrow[d, "\psi"] \\ & U \end{tikzcd}$$

Show that $(F,\ \det)$ satisfies the above universal property. Moreover, show that if a pair $(\bigwedge^n V,\ \pi)$ satisfies the above universal property, then $\bigwedge^n V$ and $F$ are isomorphic.