Ep. 2: Abstract Vector Space and Series Overview

LetsSolveMathProblems: Navigating Linear Algebra

Problem 1. *Convince yourself that $\mathbb{C}$ can be made into a vector space over $\mathbb{Q}$ by letting the vector addition and the scalar multiplication be given by the field addition in $\mathbb{C}$ and the field multiplication in $\mathbb{C}$, respectively. In this way, $\mathbb{C}$ can be a vector space over $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{Q}$! (If you already know some linear algebra and basic set theory, try to find the dimension of each one.)*

Problem 2. *Harry claims that $\mathbb{R}^2$ is a field if we define the "field addition" and "field multiplication" to be component-wise; for example, he says that $(1,1) \in \mathbb{R}^2$ is the multiplicative identity, etc. Show that Harry is wrong, i.e., his construction does not define a field.*

Problem 3. *Try to make $\mathbb{R}^2$ into a vector space over $\mathbb{Z}/2\mathbb{Z}$ by interpreting 0 and 1 in $\mathbb{Z}/2\mathbb{Z}$ as 0 and 1 in $\mathbb{R}$, and letting the vector addition and scalar multiplication be done component-wise. Show that we do **not** get a valid vector space.*

Problem 4. *Show that if a field has exactly 4 elements, then $1+1=0$ in the field. (As an aside, you may think that if $1+1=0$ in a field, then the field has to be finite; this is not true! Try to find a counterexample.)*