Ep. 9: Eigenvectors and Generalized Eigenvectors

LetsSolveMathProblems: Navigating Linear Algebra

Problem 1. *Determine whether each of the following complex matrices is diagonalizable. For each diagonalizable matrix, find an eigenbasis.*

(a) $\begin{bmatrix} 5 & 4 \\ 2 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 20 & 9 \\ -25 & -10 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 & 5 \\ 5 & 0 & 0 \\ 2 & -2 & 5 \end{bmatrix}$ *Hint: The characteristic polynomial of this matrix is* $-(\lambda-5)^2(\lambda+2)$.

Problem 2. *If A and B are $n \times n$ diagonalizable real matrices with the same characteristic polynomial, are they necessarily similar?*

Problem 3. *Suppose that $\phi \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$.*

(a) *If $\dim \ker(\phi - \lambda) = 1$ for some $\lambda \in \mathbb{C}$, then is ϕ necessarily diagonalizable?*

(b) *If $\dim \ker(\phi - \lambda)^2 = 1$ for some $\lambda \in \mathbb{C}$, then is ϕ necessarily diagonalizable?*

(c) *If $\phi \in \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3)$, do the answers to parts (a) or (b) change?*

Problem 4. *Suppose that the set of all eigenvalues of $\phi \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ consists of $\lambda_1, \ldots, \lambda_k$. What is the dimension of $\bigcap_{i=1}^k \mathrm{im}(\phi - \lambda_i)^n$?*

Note: In case you aren't familiar with the set intersection notation, $\bigcap_{i=1}^k \mathrm{im}(\phi - \lambda_i)^n$ denotes the set of vectors that belong to $\mathrm{im}(\phi - \lambda_i)^n$ for every $i = 1, 2, \ldots, k$.

Problem 5. *Does there exist a 2×2 real matrix that is not invertible, not nilpotent, and not diagonalizable?*