Ep. 3: Linear Maps

LetsSolveMathProblems: Navigating Linear Algebra

Let V, W be vector spaces over a field F.

Problem 1. *Are there functions* $\phi, \psi : \mathbb{R} \to \mathbb{R}$ *(which may be nonlinear) such that* ϕ *is injective,* ψ *is surjective, and* $\phi \circ \psi$ *is neither injective nor surjective? Does the answer change if* ϕ, ψ *are assumed to be linear?*

Problem 2. *Suppose that* $\phi : V \to W$ *is an isomorphism. Show that* $\phi^{-1} : W \to V$ *is linear.*

Problem 3. *Match each linear map* $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ *to its description.*

Linear Maps:

(a) $\phi(e_1) = e_2,\ \phi(e_2) = e_1$

(b) $\phi(e_1) = e_1,\ \phi(e_2) = -e_2$

(c) $\phi(e_1) = -e_1,\ \phi(e_2) = -e_2$

Descriptions:

(i) Reflection about the x*-axis*

(ii) Reflection about the line $y = x$

(iii) Rotation of 180 *degrees about the origin*

Problem 4. *Check that* $Hom(V, F)$ *is a vector space over* F*, where vector addition and scalar multiplication are function addition and function scalar multiplication. This is the algebraic dual space of* V*.*

Problem 5. *For any* $\phi \in \mathrm{Hom}(V, V)$*, let* $\phi^2 := \phi \circ \phi$*. Does there exist* $\phi \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ *such that* $\phi(e_1) \neq 0$ *and* $\phi(e_2) \neq 0$*, yet* $\phi^2(e_1) = \phi^2(e_2) = 0$*?*