Ep. 4: Basis

LetsSolveMathProblems: Navigating Linear Algebra

Let V, W be vector spaces over a field F.

Problem 1. *Let $\mathbb{R}$ be a vector space over $\mathbb{Q}$ (as in Problem 1 of Ep. 2). Think about why this space should be infinite-dimensional. In the sketch of a proof provided below, Step 1 is the only part that you are asked to prove. If you know basic set theory, prove Step 2, as well!*

(1) Show that if the space $\mathbb{R}$ over $\mathbb{Q}$ is finite-dimensional, then there would be a bijection from $\mathbb{R}$ to $\mathbb{Q}^n$ for some $n \geq 1$.

(2) Show that there cannot be a bijection from $\mathbb{R}$ to $\mathbb{Q}^n$.

Problem 2. *Suppose that V and W are finite-dimensional and have different dimensions. Show that there cannot be an isomorphism from V to W.*

Problem 3. *Let us modify the definition of a vector space to allow for vector spaces over $\mathbb{Z}$ (even though $\mathbb{Z}$ is not a field). Show that the two definitions of linear dependence we talked about in this episode are **not** equivalent for the "vector space" $\mathbb{Z}^2$ over $\mathbb{Z}$.*

Problem 4. *Let $V^* = \mathrm{Hom}(V, F)$, the algebraic dual space of V from Ep. 3 problems. Find a basis of $(\mathbb{R}^2)^*$.*

Problem 5. *Does there exist a 3×3 real matrix A (with respect to the standard basis $e_1, e_2, e_3 \in \mathbb{R}^3$) such that $\ker A \neq \ker A^2$ and $\ker A^2 \neq \ker A^3$?*