Ep. 5: Open Sets and Closed Sets

LetsSolveMathProblems: Navigating Metric Space Topology

Problem 1. *Given a metric space X and $A \subseteq X$, we define the topological boundary of A to be $\partial A = \overline{A} - \operatorname{int} A$, i.e., ∂A is the set of points in $\overline{A}$ but not in $\operatorname{int} A$.*

(a) Consider the half-open interval $A = [0,1) \subseteq \mathbb{R}$ (where, as almost always, it is implicit that $\mathbb{R}$ has the Euclidean metric). Find ∂A and $\partial(\partial A)$.

(b) For $A = \mathbb{Q} \cap (0,1) \subseteq \mathbb{R}$, compute ∂A and $\partial(\partial A)$. This should convince you that the topological boundary can behave pretty unintuitively[1].

(c) If $\mathbb{R}$ is given the discrete metric in part (a), does the answer change? (Hint: Show that every subset of a discrete metric space is both open and closed.)

Problem 2. *(a) Let X be a metric space. Show that any finite union of closed sets in X is closed in X, and show that any arbitrary intersection of closed sets in X is closed in X.*

Note: Probably the quickest solution is to use what are called De Morgan's Laws, which you are encouraged to look up. Be aware that the notation $\overline{A}$, which denoted topological closure in the episode, is sometimes used to denote the set complement A^c, as well.

(b) Show that an infinite intersection of open sets need not be open, and show that an infinite union of closed sets need not be closed.

Problem 3. *Let $a_1, a_2, a_3, \ldots$ be a sequence in a metric space X and $a \in X$. Here is a "topological definition" of $\lim_{n\to\infty} a_n = a$: For every open set $U \subseteq X$ such that $a \in U$, there exists a positive integer N such that $a_n \in U$ for all $n \geq N$. Prove that this topological definition is the same as the definition of sequential convergence given in Episode 3.*

Problem 4. *(a) Show that the set of irrational numbers $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$.*

(b) Let $A = \mathbb{R} - \mathbb{Q} \subseteq \mathbb{R}$. Find $\overline{A}$ and $\operatorname{int}(A)$.

Problem 5. *Let X be a metric space. Let $U \subseteq X$ be open in X.*

(a) Suppose $A \subseteq U$ is closed in U. (Note that the metric space structures on A and U are inherited from X.) Show that A need not be open in X. Similarly, show that A need not be closed in X.

(b) If $B \subseteq U$ is open in U, show that B is open in X.

Problem 6. *Let $A \subseteq X$, where X is a metric space.*

[1] Fascinatingly enough, it turns out $\partial(\partial(\partial A))$ does always equal $\partial(\partial A)$, a fact you can try to prove. I believe I first learned this fact from the real analysis text of [Pugh] (see references in the description of the first episode), which also contains many other interesting topological facts.

(a) Explain why $\overline{A}$ *is the smallest closed set containing* A*. That is, if* B *is a closed set in* X *such that* $A \subseteq B$*, then* $\overline{A} \subseteq B$*.*

(b) Similarly, show that $\operatorname{int}(A)$ *is the largest open set contained in* A*.*

(c) Show that A *is closed if and only if* $\overline{A} = A$*. Moreover, show that* A *is open if and only if* $\operatorname{int}(A) = A$*.*

Problem 7. *Let* $A \subseteq X$*, where* X *is a metric space.*

(a) Show that $x \in X$ *is a limit point of* A *if and only if there exists a sequence* $a_1, a_2, a_3, \ldots$ *with pairwise distinct*[2] *elements in* A *converging to* x*.*

(b) Show that $x \in X$ *is a limit point of* A *if and only if there exists a sequence* $a_1, a_2, a_3, \ldots$ *of elements in* $A - \{x\}$ *satisfying* $d(x, a_n) < \frac{1}{n}$*. Recall that in the episode, we roughly used this fact in showing that* $\overline{A}$ *is closed.*

Problem 8. *If* $A \subseteq \mathbb{R}$ *is uncountable, is it possible for* A *to not have any limit points in* $\mathbb{R}$*?*

[2]That is, $a_n \neq a_m$ whenever $n \neq m$.