Ep. 2: Least Upper Bound Property and Uncountability of Reals

LetsSolveMathProblems: Navigating Metric Space Topology

Note: If you choose to do only one problem, I would recommend Problem 4 the most.

Problem 1. *In this episode, we showed that $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable. Show that the set of irrationals $\mathbb{R} - \mathbb{Q}$ (i.e., the set of real numbers that are not rational) is uncountable. Thus there are "way more" irrationals than rationals on the real line!*

Problem 2. *Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Suppose that $\max A$ exists. Prove that $\sup A = \max A$.*

Problem 3. *Let us change the definition of countable as follows: a set A is countable if there exists a sequence in A containing every element of A, where we do not allow any terms of the sequence to be repeated. Recall that the definition in the episode allowed repeated terms. Explain why this "new" definition is in fact exactly the same as the definition allowing repeats.*

Problem 4. *Sketch a picture of each of the following subsets of $\mathbb{R}$ on the real line (except maybe part (e)). Afterwards, find the supremum and infimum of each subset and specify whether they agree with maximum or minimum of the set (where applicable). If the given subset A is not bounded above or not bounded below, then say $\sup A = \infty$ or $\inf A = -\infty$, respectively.*

(a) $\{5 + \frac{1}{n} : n = 1, 2, 3, \dots\}$

(b) $\{5 + \frac{1}{n} : n = 1, 2, 3, 4, 5\}$

(c) $\{\sum_{i=1}^{n} \frac{1}{2^i} : n = 1, 2, 3, \dots\}$

(d) $\{n : n = 1, 2, 3, \dots\}$

(e) $\mathbb{Q}$

Problem 5. *In this problem, we give two different ways of deducing the greatest lower bound property of $\mathbb{R}$ from its least upper bound property. Suppose that $A \subseteq \mathbb{R}$ is nonempty and bounded below.*

(a) *Let $A' = \{-s : s \in S\}$. Explain why $\sup A'$ exists. Then, show that $\inf A$ exists and equals $-\sup A'$.*

(b) *Consider the set L consisting of all lower bounds of A. Note that L is nonempty and bounded above (why?), so it has a well-defined supremum, say $M = \sup L$. Show that $M = \inf A$, i.e., M is the greatest lower bound of A.*