Ep. 4: Limit Points, LimSup, and Completeness of $\mathbb{R}^n$

LetsSolveMathProblems: Navigating Metric Space Topology

Problem 1. *If $\mathbb{R}^n$ is given the discrete metric, is it complete?*

Problem 2. *Let $a_1, a_2, \ldots$ be a bounded real sequence. Give a definition of $\liminf_{n\to\infty} a_n$ by considering the infimum of $\{a_k, a_{k+1}, \ldots\}$ for each k, similarly as in the approach used to define $\limsup_{n\to\infty} a_n$ in the episode. By recalling the proof that $\limsup_{n\to\infty} a_n$ is the maximum limit point of $a_1, a_2, \ldots$, convince yourself that $\liminf_{n\to\infty} a_n$ is the minimum limit point of $a_1, a_2, \ldots$.*

Problem 3. *Let $a_1, a_2, \ldots$ be a real sequence and $a \in \mathbb{R}$. Prove that if $\lim_{n\to\infty} a_n = a$, then $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$. (Hint: Problem 2 of Episode 3 trivializes this.)*

Recall that the converse also holds, as we proved in the episode.

Problem 4. *Let $a_1, a_2, \ldots$ be a real sequence and $a \in \mathbb{R}$. We say that $\sum_{n=1}^{\infty} a_n = a$ if the sequence $a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots$ converges to a. Prove that if $\sum_{n=1}^{\infty} a_n = a$, then $\lim_{n\to\infty} a_n = 0$.*

Note: It can be shown that the converse is not true! For example, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, even though $\lim_{n\to\infty} \frac{1}{n} = 0$. If you are not familiar with this fact, try to prove this or look up "divergence of harmonic series."

Problem 5. *Let $f : X \to Y$, where X and Y are metric spaces. Let $a \in X$ and $b \in Y$. Show that $\lim_{x\to a} f(x) = b$ if and only if the following holds: For all $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(a)) < \epsilon$ whenever $0 < d(x, a) < \delta$. Explain what goes wrong if we change $0 < d(x, a) < \delta$ to $d(x, a) < \delta$.*

Problem 6. *Consider $A = \mathbb{Q} \cap (0, 1) \subseteq \mathbb{R}$. (In case you are not familiar with the set intersection $\cap$ notation, $\mathbb{Q} \cap (0, 1)$ is the set of elements in both $\mathbb{Q}$ and $(0, 1)$.)*

(a) Show that every subset of a countable set is countable, and deduce that A is countable.

(b) By the previous part, there exists a real sequence $a_1, a_2, \ldots$ such that $a_n \in A$ for all n and every element of A shows up at at least once in this sequence. Find all limit points (in $\mathbb{R}$) of this sequence. What are its $\limsup$ and $\liminf$?