Ep. 3: Convergence in a Metric Space

LetsSolveMathProblems: Navigating Metric Space Topology

Problem 1. *Let $\mathbb{R}$ be given the discrete metric. If $S \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, what can we say about S?*

Problem 2. *Let $a_1, a_2, a_3, \ldots$ be a sequence in a metric space. A* subsequence *of $a_1, a_2, a_3, \ldots$ is a sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ where $1 \leq n_1 < n_2 < n_3 < \cdots$. For example, one possible subsequence is $a_2, a_4, a_6, \ldots$ obtained by taking terms with even indices, i.e., by selecting $n_1 = 2, n_2 = 4, n_3 = 6, \cdots$.*

Prove that if $a_1, a_2, \ldots$ converges to a, then every subsequence $a_{n_1}, a_{n_2}, \ldots$ also converges to a.

Problem 3. *In the episode, we claimed without proof that the Euclidean metric for $\mathbb{R}^n$ satisfies the triangle inequality. Either try to prove this (especially if you know the Cauchy-Schwarz inequality for $\mathbb{R}^n$), or read a proof of it online or in a real analysis textbook.*

Problem 4. *Let $a_1, a_2, \ldots.$ and $b_1, b_2, \ldots$ be real sequences converging to a and b, respectively.*

(a) Prove that the sequence $a_1 + b_1, a_2 + b_2, \ldots$ converges to $a + b$.

(b) If $a \neq 0$ and $a_n \neq 0$ for all $n \geq 1$, then prove that the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \ldots$ converges to $\frac{1}{a}$.

Problem 5. *Define a distance function on $\mathbb{R}^2$ by letting*

$$d((a_1, a_2), (b_1, b_2)) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

for all $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$. Prove that this makes $\mathbb{R}^2$ into a metric space, i.e., check that this distance function satisfies all three axioms for a metric.

Problem 6. *Let X be a metric space and $S \subseteq X$. Show that S is dense in X if and only if for all $x \in X$, there exists a sequence $s_1, s_2, \ldots$ converging to x with $s_n \in S$ for all n.*