Fundations

Algorithmic Design

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Algorithms and Computational Models

What is an Algorithm?

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Is a sequence of well-defined steps that transforms a set of inputs into a set of outputs in a finite amount of time

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Definition (Computational Model)

Is a mathematical tool to perform computations.

A function described by an algorithm is calculable.

A function *implementable* in a computational model is computable.

Functions, Computability and Calcolability

Are all the functions computable in any specific model?

If this is not the case

- are there calculable functions that are not computable?
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Why Is This Relevant For Us?

What if calculability will not be related to computability?

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What if calculability will not be related to computability?

Algorithms would not guarantee implementability!

Halting Problem

Let h be the function that establish whether any program p eventually ends its execution ($\downarrow$) on an input i or runs forever($\uparrow$)

$$h(p,i) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } p(i) \text{ never ends} \\ 1 & \text{otherwise} \end{cases}$$

Definition (Halting problem)

Can we implement h?

Computability of Halting Problem

For any computable function $f(a, b)$, define

$$g_f(i) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } f(i,i) = 0 \\ \uparrow & \text{otherwise} \end{cases}$$

Since f is computable, so it is g_f. Let G_f implement it.

Can h be one of the f's?

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- If $f(\text{G\_f}, \text{G\_f}) = 0 \implies g_f(\text{G\_f}) = 0$ and $h(\text{G\_f}, \text{G\_f}) = 1$
- If $f(\text{G\_f}, \text{G\_f}) \neq 0 \implies g_f(\text{G\_f}) \uparrow$ and $h(\text{G\_f}, \text{G\_f}) = 0$

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Thus, $h \neq f$ for all computable f's and h is not computable.

Church-Turing Thesis

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Every *effectively* calculable function is a computable function.

$$\text{calculability} \Longrightarrow \text{computability}$$

If we have an algorithm for f, then f can be formally computed

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It also means that:

- all the "reasonable" computational models are equivalent
- we can avoid "hard-to-be-programmed" models (e.g., Turing machine)

Random-Access Machine (RAM)

- variables to store data (no types)
- arrays
- integer and floating point constants
- algebraic functions: $+$, $-$, $/$, $*$, $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$
- assignments
- pointers (no pointer arithmetic)
- conditional and loop statements
- procedure definitions and recursion
- simple “reasonable” functions, e.g., the length of an array

Algorithms are defined as programs on RAM.

A Simple Algorithm

Input: An array A of numbers $< a_1, \ldots, a_n >$.
Output: The maximum among $a_1, \ldots, a_n$.

```
def find_max(A):
    max_value ← A[1]
    for i ← 2..|A|:
        if A[i] > max_value:
            max_value ← A[i]
        endif
    endfor

    return max_value
enddef
```

RAM is not Real Hardware!!!

RAM models real hardware, but it lacks

- real HW limitations s.a. finiteness
- memory hierarchy
- instruction execution time

Time Complexity

How to Measure Algorithm Efficiency?

What about execution time?

How to Measure Algorithm Efficiency?

What about execution time? (for what input?)

Algorithms are not programs

Assuming 1 time unit per instruction is not realistic because execution time depends on:

- CPU instruction sets
- CPU/Memory/Bus Clock
- language and compiler
- OS memory handling
- ...

How to Measure Algorithm Efficiency?

~~What about execution time~~?

Any other ideas?

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Any other ideas?

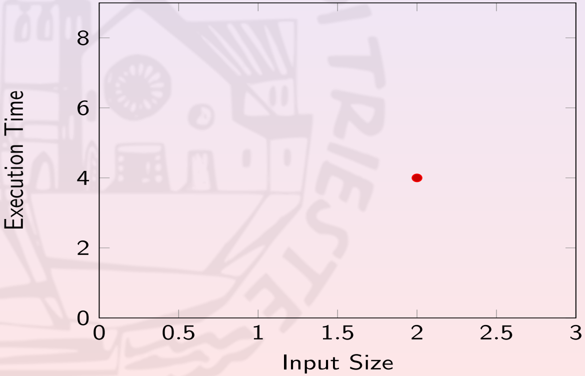
What about scalability?

Definition (Scalability)

Effectiveness of a system in handling input growth.

Growth Complexity

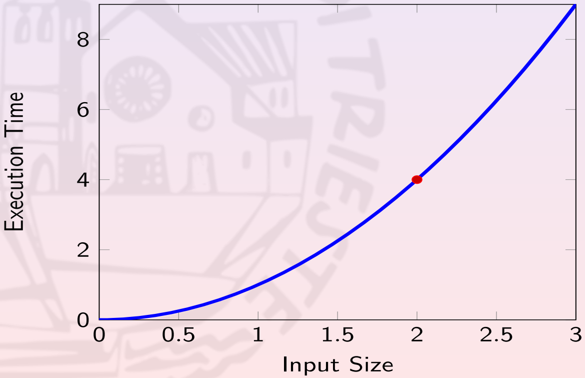
We do **not** measure the execution time for a given input



Growth Complexity

We do **not** measure the execution time for a given input

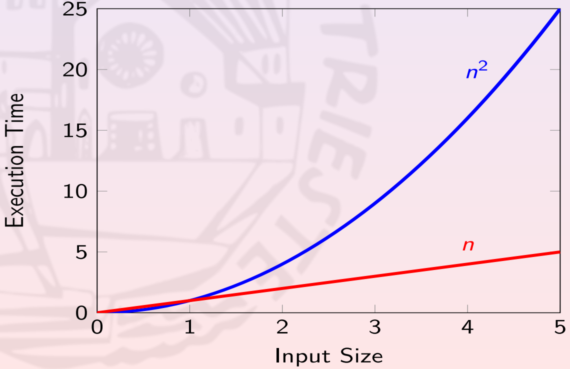
We **estimate** the relation between input size and execution time



Complexity Quiz!

Which growth is preferable between:

- n^2 and n?



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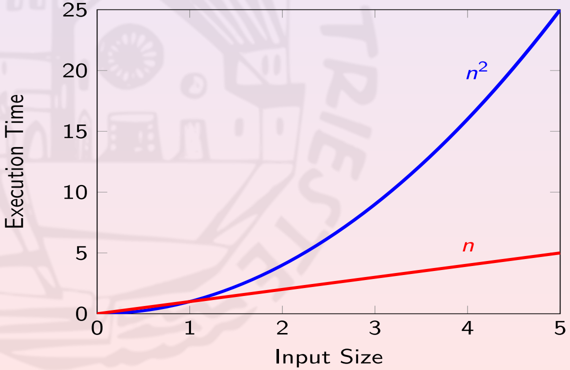

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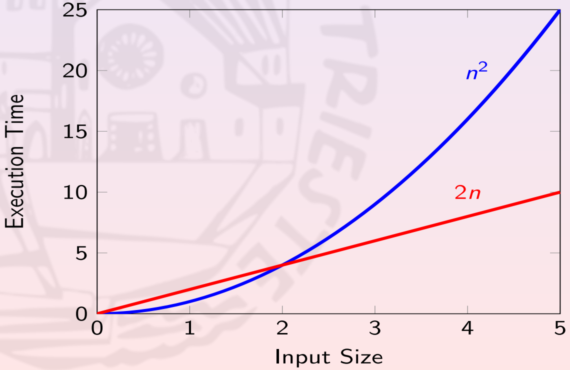

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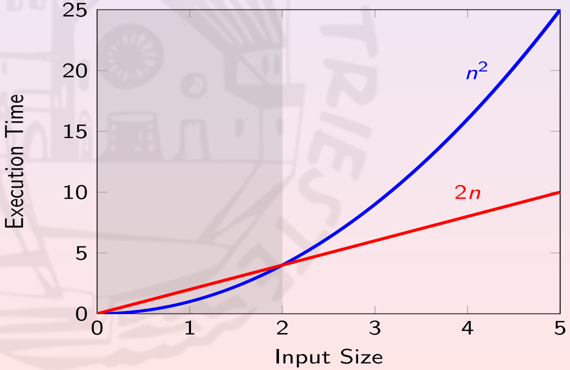

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Which growth is preferable between:

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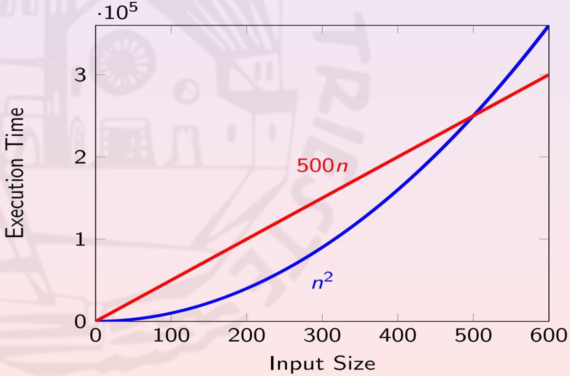
- n^2 and $500 * n$?



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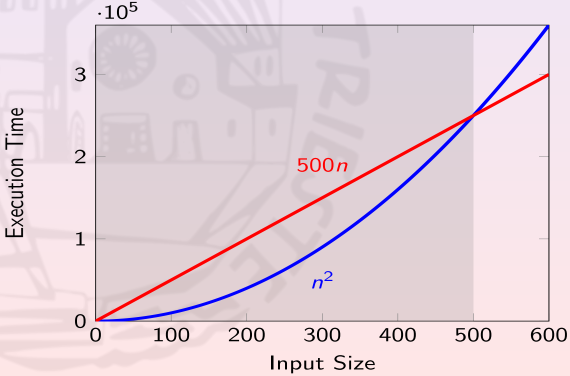

Asymptotic Time Complexity

Constants are not useful. We are looking at asymptotic behaviour.

We can abstract the single instruction execution time !!!

This intuition is also supported by linear time speedup theorem.

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We can abstract the single instruction execution time !!!

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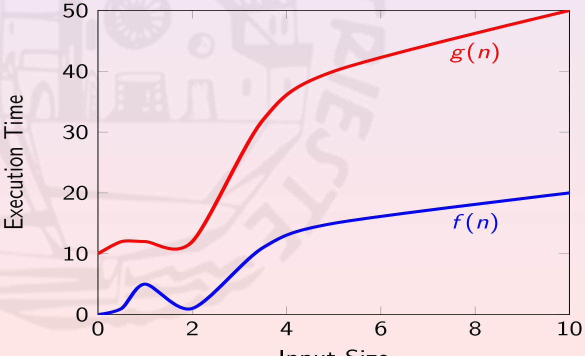
How to group all the functions that asymptotically are the same?

big O notation

$$O(f(n)) \stackrel{def}{=} \{g(n) | \exists c > 0 \exists n_0 > 0 \; m \geq n_0 \implies g(m) \leq c * f(m)\}$$

So, $g(n) \in O(f(n))$ iff



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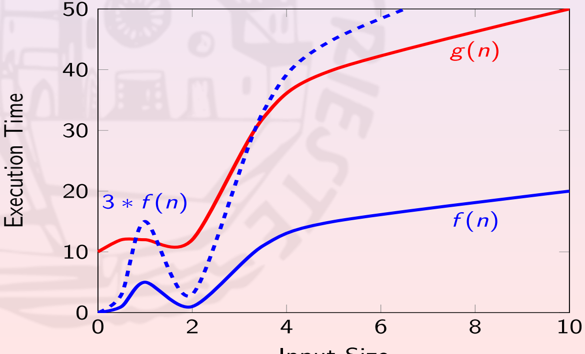

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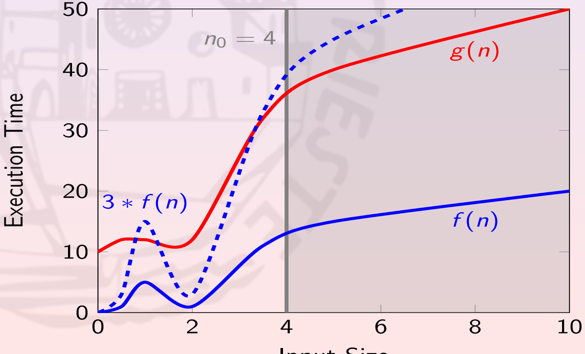

Some Useful Properties

For any $c_1, c_2 \in \mathbb{N}$ and for any $k \in \mathbb{Z}$

- $f(n) \in O(f(n))$
- $O(f(n)) = O(c_1 * f(n) + k)$
- if $c_1 \geq c_2$, then $O(f(n)^{c_1} + k * f(n)^{c_2}) = O(f(n)^{c_1})$
- $O(f(n)^{c_1}) \subseteq O(f(n)^{c_1+c_2})$ es. $n \in O(n^2)$
- if $h(n) \in O(f(n))$ and $h'(n) \in O(g(n))$, then
  - $h(n) + h'(n) \in O(g(n) + f(n))$
  - $h(n) * h'(n) \in O(g(n) * f(n))$

big Ω notation

$$\Omega(f(n)) \stackrel{def}{=} \{g(n) | \exists c > 0 \exists n_0 > 0 \; m \geq n_0 \implies c * f(m) \leq g(m)\}$$

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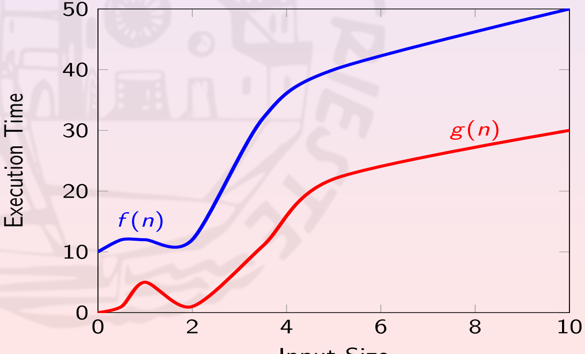

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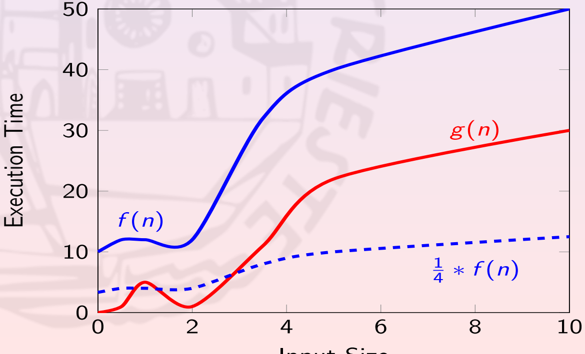

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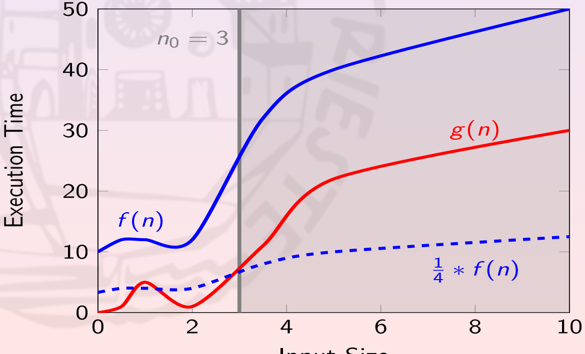

big Θ notation

$$\Theta(f(n)) \overset{def}{=} \{g(n) | \exists c_1, c_2 > 0 \exists n_0 > 0$$
$$m \geq n_0 \Longrightarrow c_1 * f(m) \leq g(m) \leq c_2 * f(m)\}$$

Theorem

$$f(n) \in \Theta(g(n)) \Longleftrightarrow f(n) \in O(g(n)) \cap \Omega(g(n))$$

Cost Criteria

How to Get Cost Function from the Code?

We are not interested in the execution time of the single instruction

We are interested in how many time instructions are executed

How to Get Cost Function from the Code?

We are not interested in the execution time of the single instruction

We are interested in how many time instructions are executed

But does the instructions always cost the same?

Are $1+1$ and $2^{7^{11}} + 3^{5^{13}}$ equivalent?

Uniform Cost Criterion

Time cost is:

- 1 for any Boolean and algebraic expression evaluation
- 0 for assignments and control instructions

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- 1 for any Boolean and algebraic expression evaluation
- 0 for assignments and control instructions

We are not trying to evaluate execution time
(as instead done in Slide 13)!

The instruction time cost means "*that instruction is time-relevant*" and not "*the execution time of that instruction is...*"

Uniform Cost Criterion: an Example

```
def test(n):
  Z ← 2                          // costs 1
  for i ← 1..n:                  // loop n time
    Z ← Z * Z                    // costs 1

  return Z
```

Cost function is $T(n) = 1 + n * 1 \in \Theta(n)$

Uniform Cost Criterion: an Example

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Cost function is $T(n) = 1 + n * 1 \in \Theta(n)$

But $test(n) = 2^{2^n}$... in linear time we used 2^n space!!!

Logarithmic Cost Criterion

Time cost is:

- $\max_{i \in [0,c]}(\log a_i)$ for any Boolean and algebraic expression involving $a_0, \ldots a_c$ as operands
- 0 for assignments and control instructions

Logarithmic Cost Criterion: the Previous Example

```
def test(n):
  Z ← 2                        // costs 2
  for i ← 1..n:                // loop n time
    Z ← Z * Z                  // costs log Z

  return Z
```

The total cost function is

$$T(n) = \sum_{i=1}^{n} \log 2^{2^i} = \sum_{i=1}^{n} 2^i = 2 * (2^n - 1) \in \Theta(2^n)$$

Uniform vs Logarithmic Cost Criteria

Logarithmic cost better represents big-number algorithms.

If value representation space is bounded (as for CPU arithmetic), uniform cost is enough.

Where not otherwise declared, we will use uniform cost criterion.

Some Useful Notions

Arrays and Lists (Abstract Data Types)

Arrays Are indexed collections of values fixed in length.

1	2	3	4	5	6	7	8	9	10
-4	0	1	2	5	6	7	11	12	13

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Single-Linked Lists Are sequences of values supporting `head` and `next` operations

Head $\xrightarrow{\text{next}}$ $\boxed{-4}$ $\xrightarrow{\text{next}}$ $\boxed{0}$ $\xrightarrow{\text{next}}$ $\boxed{1}$ $\dashrightarrow^{\text{next}}$

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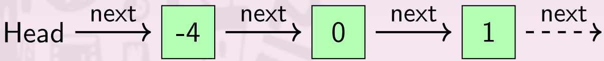
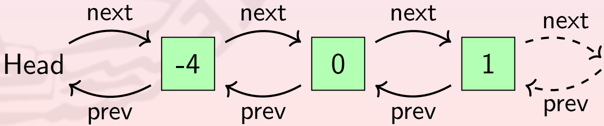


Double-Linked Lists Are sequences of values supporting `head`, `next` and `previous` operations



Queue and Stacks (Abstract Data Types)

Queues Are collections of values ruled according the FIFO policy. They support `head`, `is_empty`, `insert_back`, `extract_head` operations

Stacks Are collections of values ruled according the LIFO policy. They support `top`, `is_empty`, `insert_top`, `extract_top` operations

Graphs (Graph Theory)

Are pairs (V, E) where:

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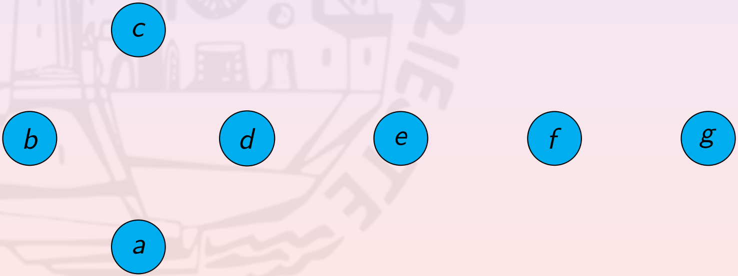
V is a set of nodes



Graphs (Graph Theory)

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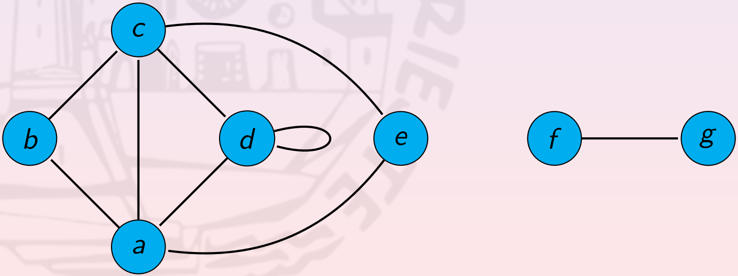
E is a set of edges



Graphs (Graph Theory)

Are pairs (V, E) where:

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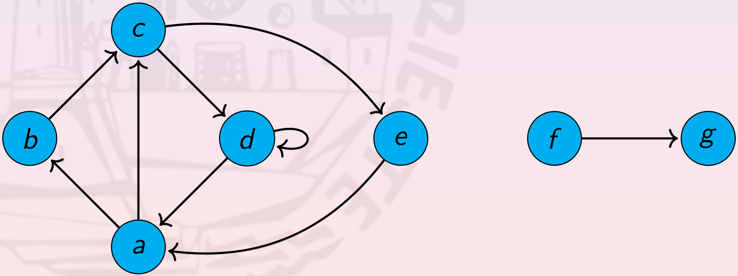
E is a set of edges

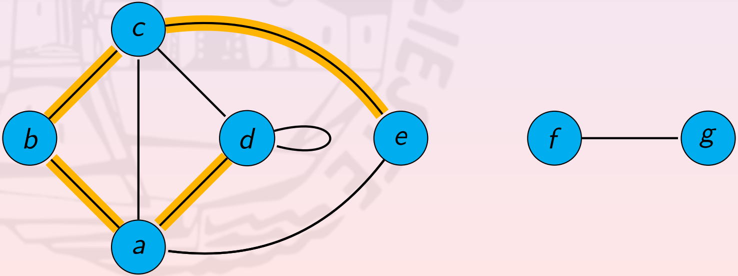
If the edges are (un)directed, the graph is (un)directed



Paths and Cycles

A path of length n between $a, b \in V$ is a sequence $e_1, \ldots, e_n$ s.t.

- e_1 involves a
- e_n involves b
- e_i and e_{i+1} involve a common node n_i



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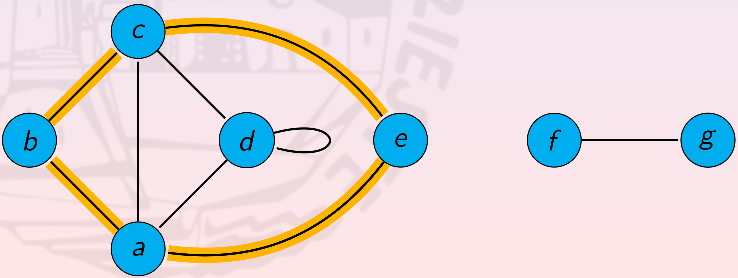
A cycle is a path whose initial and final node coincide.



Connected and Acyclic Graphs (Graph Theory)

A graph is connected if there is a path between every pairs of nodes

A graph is acyclic if it does not contains cycles

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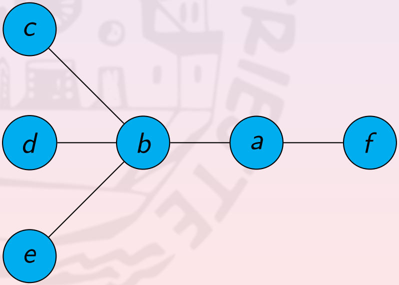
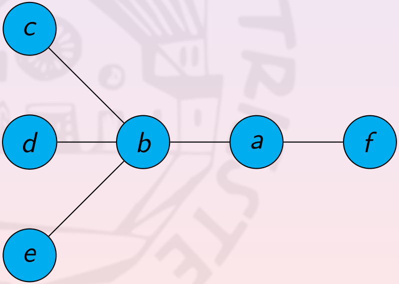
A tree is an connected and acyclic undirected graphs



Trees (Abstract Data Types)

Organize data in a hierarchical finite tree (graph theory)



Trees (Abstract Data Types)

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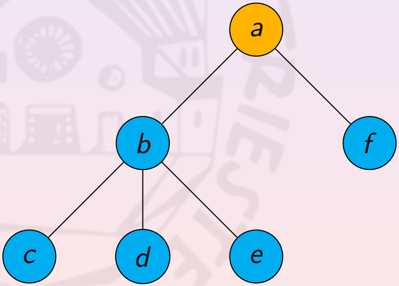
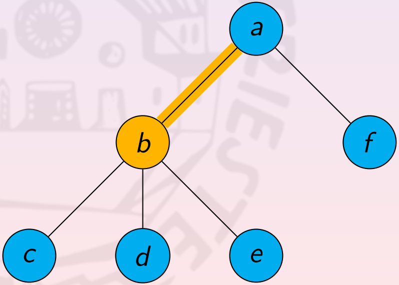
One of the nodes is the root of the graph



Tree Levels, Parents, Children and Siblings

The depth of a node is its distance from the root



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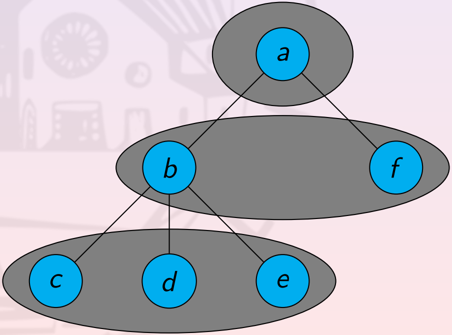
A level is a set of nodes having the same depth



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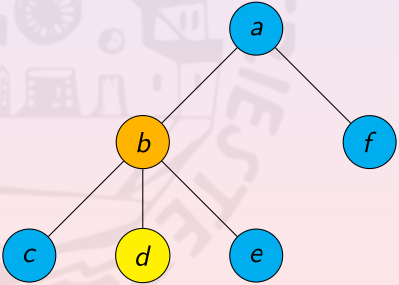

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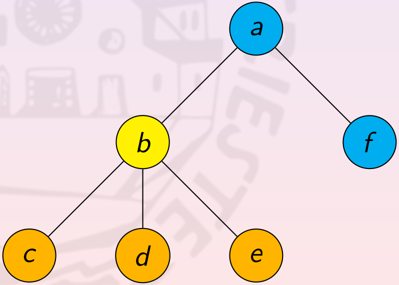
The children of a node have it as parent



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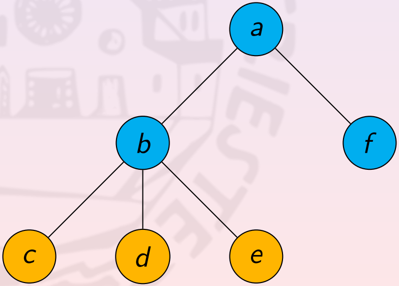
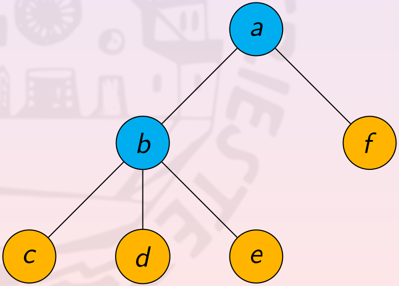
Two nodes are siblings if they have the same parent



Tree Leaves and Height

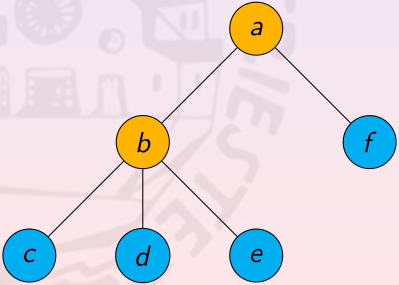
The leaves are nodes without children



Tree Leaves and Height

The leaves are nodes without children

The internal nodes have children

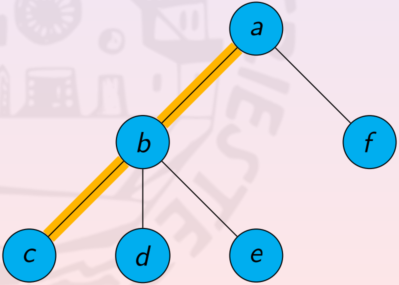


Tree Leaves and Height

The leaves are nodes without children

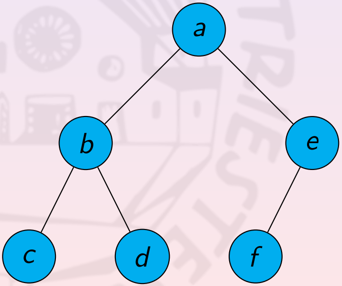
The internal nodes have children

The height of a tree is the max depth among those of its leaves



n-ary Tree and Completeness

Every node of a n-ary tree can have up to n children



n-ary Tree and Completeness

Every node of a *n*-ary tree can have up to *n* children

A *n*-ary tree is complete if the nodes in all the levels but the last one have *n* children