Heaps

Algorithmic Design

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Heaps

Abstract Data Types which store totally ordered values w.r.t. $\preceq$

They (efficiently) support the following tasks:

- building a heap from a set of data
- finding the minimum w.r.t. $\preceq$
- extracting the minimum w.r.t. $\preceq$
- decreasing the one of the values w.r.t. $\preceq$
- inserting a new value

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A min-heap is a heap s.t. $\preceq$ is $\leq$

A max-heap is a heap s.t. $\preceq$ is $\geq$

Heaps

They can be used to implement priority queues

The next element to be extracted minimizes a priority *criterion*

E.g., In emergencies, more serious patients must be served first

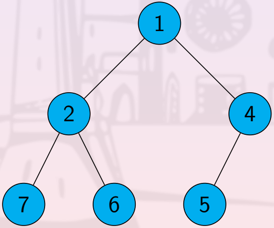
Their conditions may change and become more and more serious

Binary Heaps

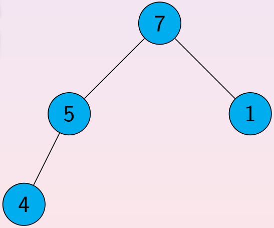
Binary Heaps

Are nearly complete binary trees: it is complete up to the second-last level and all leaves of the last level are on the left

The relation $parent(p) \preceq p$ holds for any node (heap property)



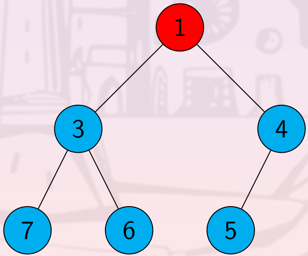
Min-heap



Max-heap

A Possible Representation (Not the Only One)

Use an **array**: the first position stores the root key



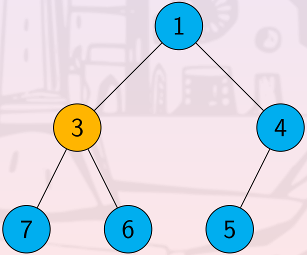
Min-heap

1	2	3	4	5	6
1	3	4	7	6	5

A Possible Representation (Not the Only One)

Use an **array**: the first position stores the root key

The *i*-th position of the array represents a node whose:



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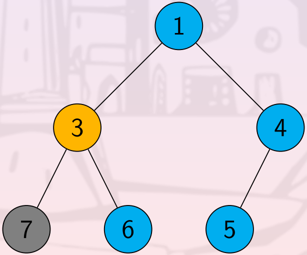

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A Possible Representation (Not the Only One)

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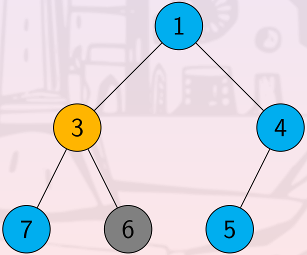
The i-th position of the array represents a node whose:

- left child has index $2 * i$
- right child has index $2 * i + 1$



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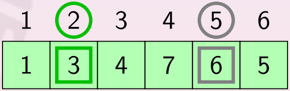

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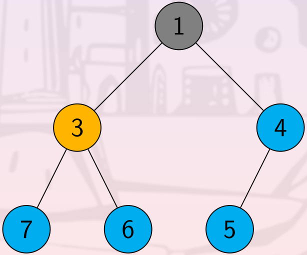
A Possible Representation (Not the Only One)

Use an **array**: the first position stores the root key

The i-th position of the array represents a node whose:

- left child has index $2 * i$
- right child has index $2 * i + 1$
- parent has index $\lfloor i/2 \rfloor$



Min-heap

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Array-based Representation: Few Useful Functions

H.size will denote the heap size

```
def LEFT(i):
  return 2*i
enddef

def RIGHT(i):
  return 2*i+1
enddef

def PARENT(i):
  return floor(i/2)
enddef
```

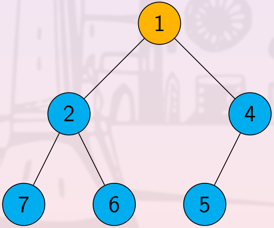
```
def GET_ROOT():
  return 1
enddef

def IS_ROOT(i):
  return i = 1
enddef

def IS_VALID_NODE(H, i):
  return H.size ≥ i
enddef
```

Finding the Minimum

Finding the Minimum

The minimum w.r.t. $\preceq$ is in the root of the heap

If this was not the case, the heap property did not hold



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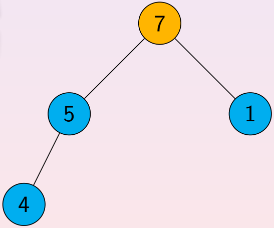

Max-heap

Finding the Minimum: Pseudo-Code

The minimum w.r.t. $\preceq$ is the root's key

```
def HEAP_MIN(H):
  return H.root.key
enddef
```

For array-based representation, we can rephrase it as...

```
def HEAP_MIN(H):
  return H[1]
enddef
```

In both the cases, the complexity is $\Theta(1)$

Removing the Minimum

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We must preserve both:

- heap topological structure
- heap property

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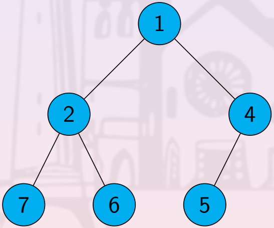
Removing the Minimum and Preserving Topology

Replace the root's key by that of the rightmost leaf of the last level



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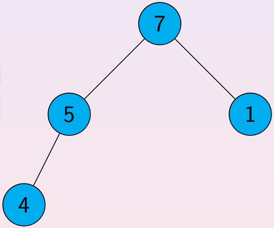

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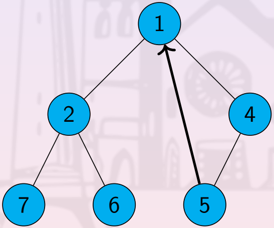

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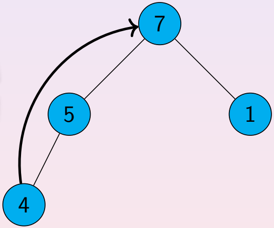

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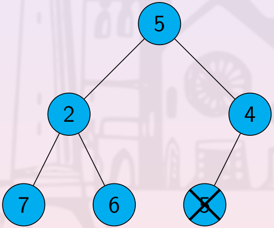
Removing the Minimum and Preserving Topology

Replace the root's key by that of the rightmost leaf of the last level

Delete the the rightmost leaf of the last-level



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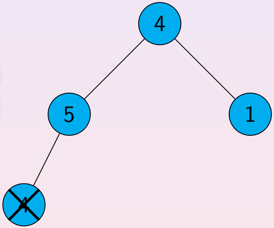

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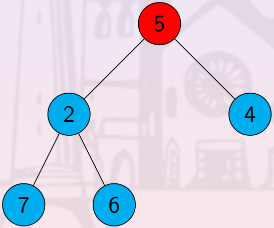

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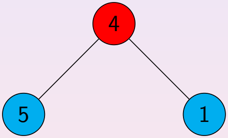
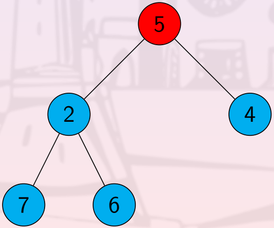


Max-heap

The heap property may be lost (only in one point)!

Restoring the Heap Property (HEAPIFY)

- find the node n, among the root and its children, whose key is minimum w.r.t. $\preceq$



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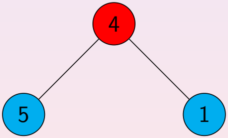

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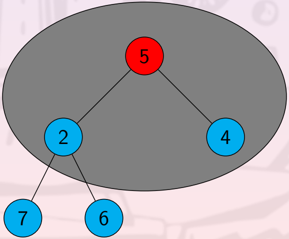

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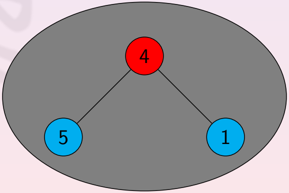

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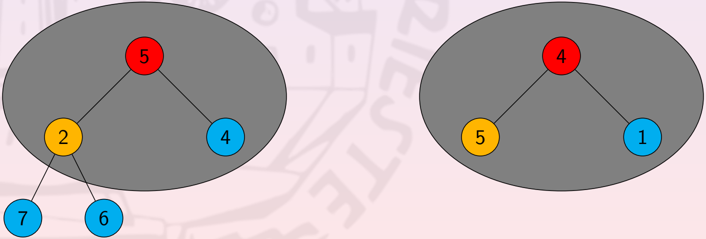

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Restoring the Heap Property (HEAPIFY)

- find the node n, among the root and its children, whose key is minimum w.r.t. $\preceq$
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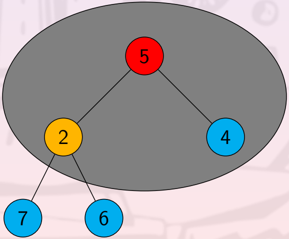

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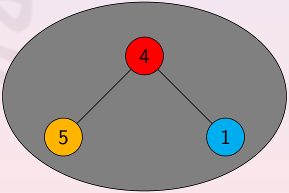

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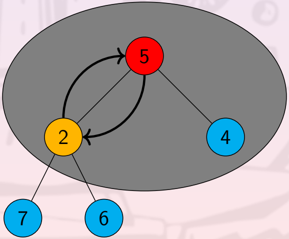

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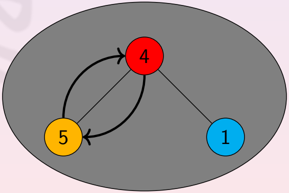

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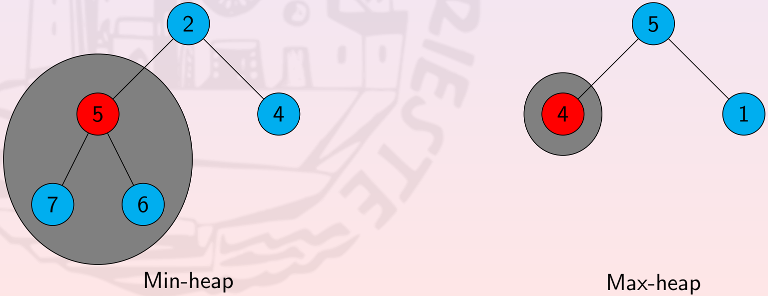

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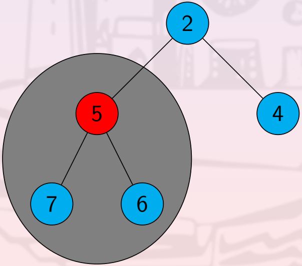

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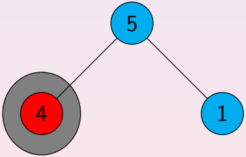

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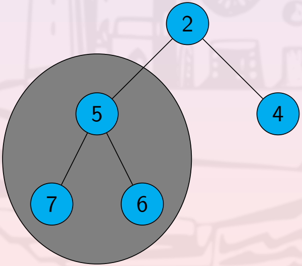

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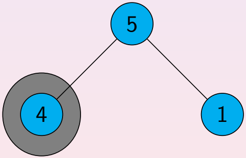
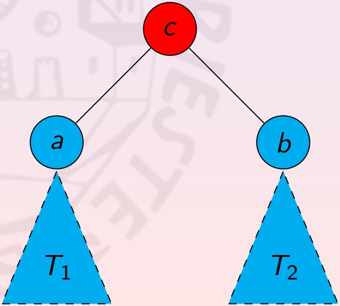


Max-heap

Correctness of HEAPIFY

Before the iteration: the heap property holds in T_1 and T_2



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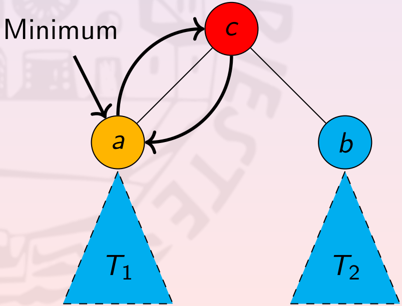

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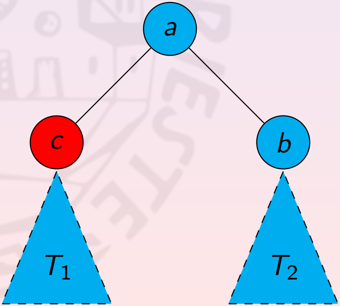
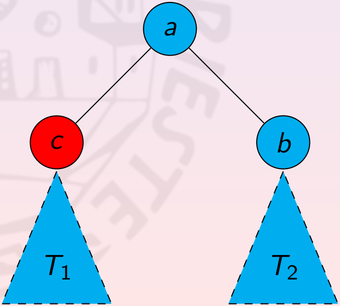


Correctness of HEAPIFY

Before the iteration: the heap property holds in T_1 and T_2

After the iteration:

- the heap property still holds in T_2 and between a and b
- T_1 has been messed up, but it is shorter than the original tree and all the keys in T_1 are greater than a



Removing the Minimum: Complexity

Replacing the root's key costs $\Theta(1)$

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For each iteration of HEAPIFY:

- 2 comparisons to find the minimum
- 1 swap at most

The distance from a leaf is decreased by one at each iteration

The total cost of HEAPIFY is the height of the heap: $O(\log n)$

HEAPIFY: Array-Based Pseudo-Code

```
def HEAPIFY(H, i):
  m ← i

   for j in [LEFT(i), RIGHT(i)]:
      if IS_VALID_NODE(H, j) and H[j] ⪯ H[m]:
        m ← j
      endif
   endfor

   if i != m:
     swap(H, i ,m)
     HEAPIFY(H,m)
   endif
enddef
```

Removing the Minimum: Array-Based Pseudo-Code

```
def REMOVE_MIN(H):
  H[1] ← H[H.size]

  H.size ← H.size −1

  HEAPIFY(H, 1)
enddef
```

Builing a Heap

Building a Binary Heap (BUILD_HEAP)

Building a tree satisfying heap topology is easy

What about heap property?

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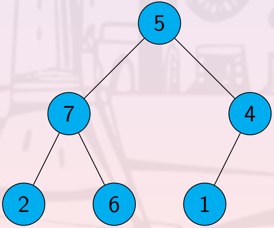

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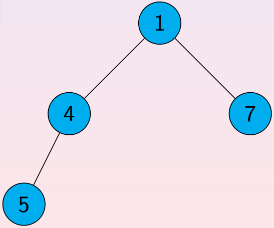

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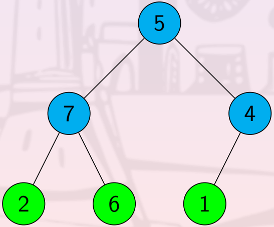

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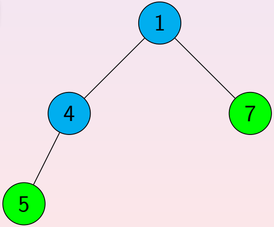

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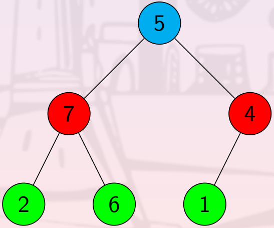
Building a tree satisfying heap topology is easy

What about heap property? Fix it bottom-up by using HEAPIFY

- fix the heaps rooted on the second-last level with children



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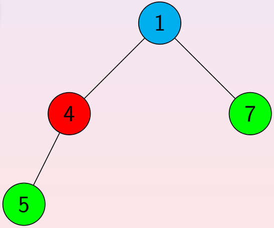

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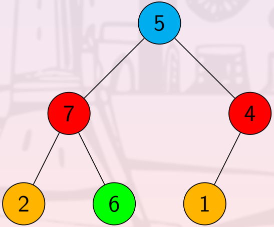

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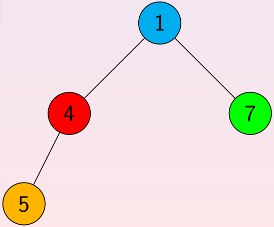

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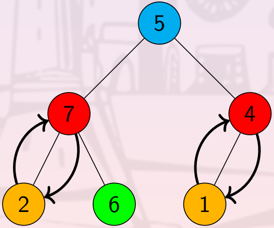

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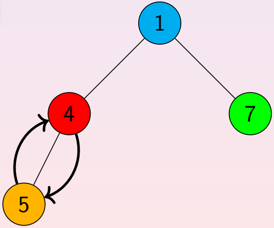

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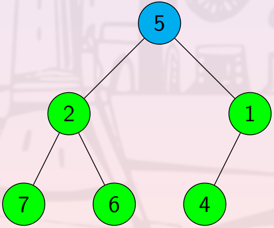

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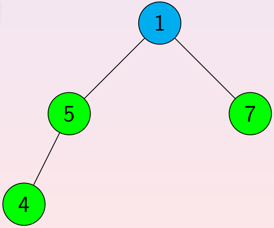

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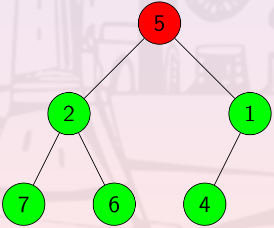
Building a Binary Heap (BUILD_HEAP)

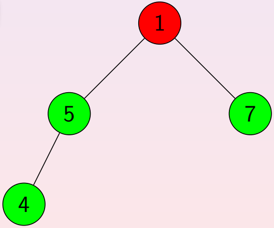
Building a tree satisfying heap topology is easy

What about heap property? Fix it bottom-up by using HEAPIFY

- fix the heaps rooted on the second-last level with children
- fix the heaps rooted on the third-last level



Min-heap



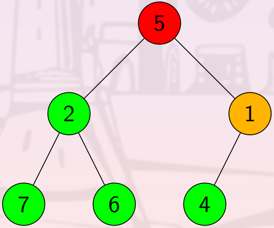
Max-heap

Building a Binary Heap (BUILD_HEAP)

Building a tree satisfying heap topology is easy

What about heap property? Fix it bottom-up by using HEAPIFY

- fix the heaps rooted on the second-last level with children
- fix the heaps rooted on the third-last level



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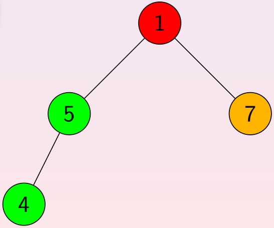

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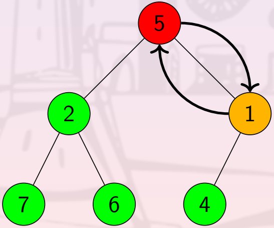

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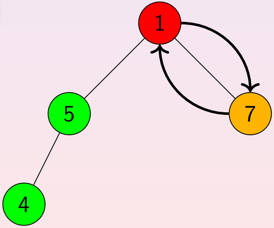

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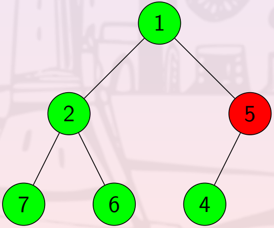

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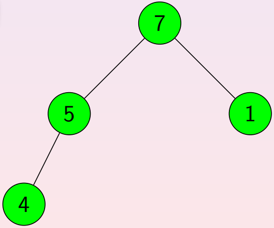

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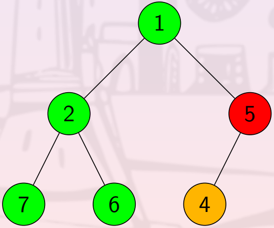

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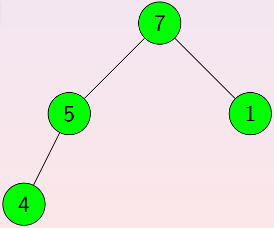

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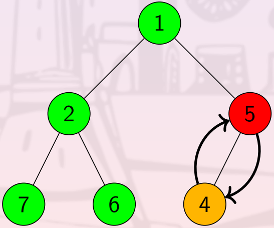

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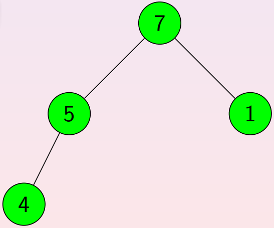

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Min-heap

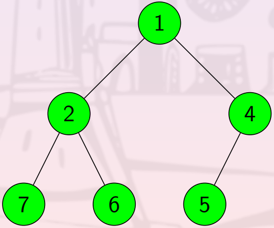
Max-heap

Building a Binary Heap (BUILD_HEAP)

Building a tree satisfying heap topology is easy

What about heap property? Fix it bottom-up by using HEAPIFY

- fix the heaps rooted on the second-last level with children
- fix the heaps rooted on the third-last level
- ...



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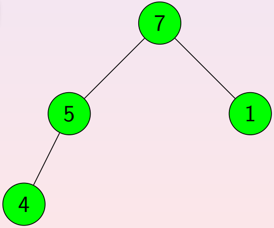


Max-heap

Complexity of `BUILD_HEAP`

HEAPIFY costs $O(h)$ (i.e., $\leq c * h$) on a tree having height h

If the considered tree has n nodes:

- its height is $\lfloor \log_2 n \rfloor$
- it contains at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ at height h

Complexity of BUILD_HEAP

The costs $T_{\mathtt{bh}}(n)$ of executing BUILD_HEAP on a n-sized tree is:

$$
\begin{aligned}
T_{\mathtt{bh}}(n) &\leq \sum_{h=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil * (c * h) \leq \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h} * (c * h) \\
&\leq c * n * \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{h}{2^h} \leq c * n * \sum_{h=0}^{\infty} \frac{h}{2^h} \\
&\leq c * n * \frac{1/2}{(1 - 1/2)^2} = 2 * c * n \in O(n)
\end{aligned}
$$

BUILD_HEAP: Pseudo-Code

```
def BUILD_HEAP_AUX(H, node):
  if IS_VALID_NODE(H, node):
    BUILD_HEAP_AUX(H, LEFT(node))
    BUILD_HEAP_AUX(H, RIGHT(node))

    HEAPIFY(H, node)
  endif
enddef

def BUILD_HEAP(A):
  H ← BUILD_HEAP_TREE(A)
  BUILD_HEAP_AUX(H, GET_ROOT(H))

  return H
enddef
```

BUILD_HEAP: Array-Based Pseudo-Code

The array-based representation helps in avoiding recursion

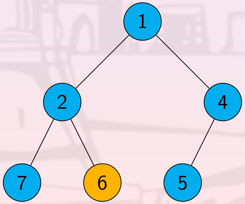
Finding the nodes of the i-th level is easy ...

... they are represented by elements in positions $[2^i, 2^{i+1} - 1]$

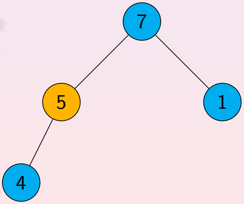
```
def BUILD_HEAP(A):
  A.size = |A|

  for i ← PARENT(A.size) downto 1:
    HEAPIFY(A, i)
  endfor

  return A
enddef
```

Decreasing a Key

Decreasing a Node's Key w.r.t. $\preceq$



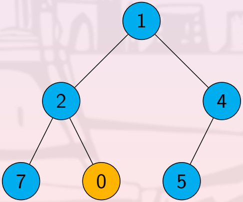
Min-heap



Max-heap

Decreasing a Node's Key w.r.t. $\preceq$

Preserves the heap property on the sub-tree rooted on the node



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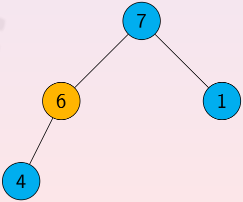
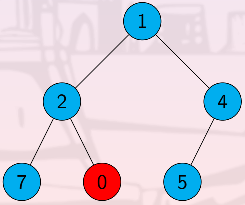


Max-heap

Decreasing a Node's Key w.r.t. $\preceq$

Preserves the heap property on the sub-tree rooted on the node, but it may broke the property w.r.t. its parent



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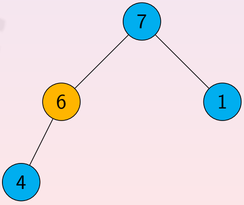
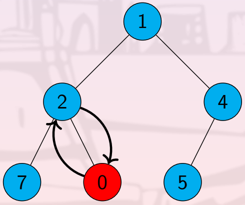


Max-heap

Decreasing a Node's Key w.r.t. $\preceq$

Preserves the heap property on the sub-tree rooted on the node, but it may broke the property w.r.t. its parent

Swapping the keys of the node and its parent solves the problem on the subtree rooted on the parent



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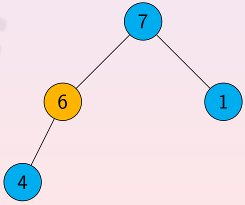

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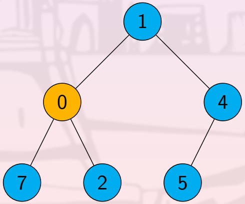

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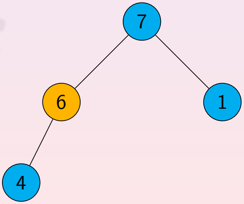

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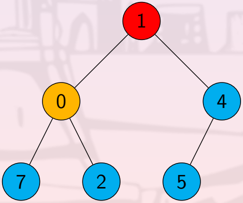

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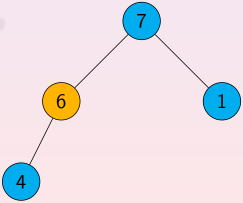

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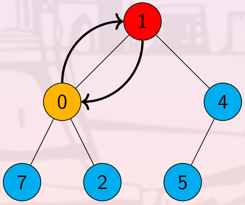
Repeat the process until the heap property is restored



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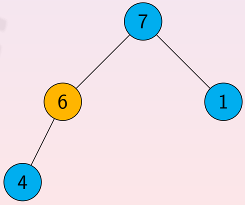

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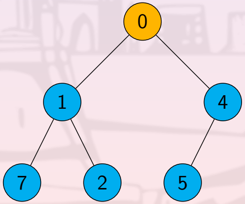

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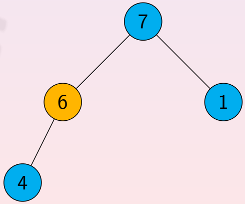


Max-heap

Decreasing a Key w.r.t. $\preceq$: Complexity

Each iteration either:

- ends the computation in time $\Theta(1)$ or
- pushes the problem one step closer to the root in time $\Theta(1)$

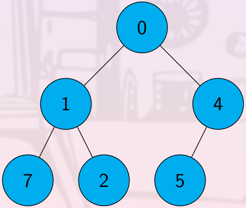
Since the heap height is $\lfloor \log_2 n \rfloor$, the complexity is $O(\log n)$

Decreasing a Key w.r.t. $\preceq$: Pseudo-Code

```
def DECREASE_KEY(H, i, value):
  if H[i] ≼ value:
    error(value+" is not smaller than H["+i+"]")
  endif

  H[i] ← value

  while not (IS_ROOT(i) or H[PARENT(i)] ≼ H[i]):
    swap(H,i,PARENT(i))

    i ← PARENT(i)
  endwhile
enddef
```

Inserting a Value

Inserting a New Value



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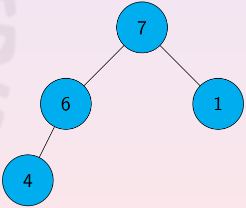
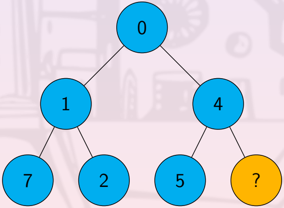


Max-heap

Inserting a New Value

- add a new node N preserving the heap topology



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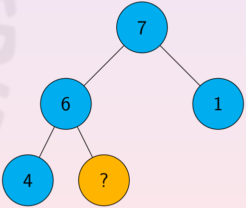
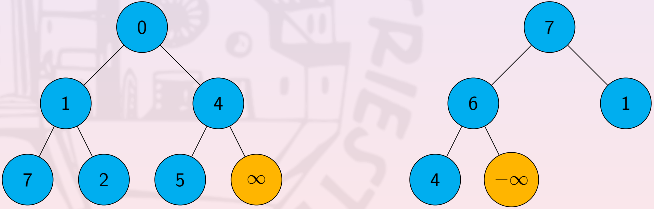


Max-heap

Inserting a New Value

- add a new node N preserving the heap topology
- set the key of N to the maximum value w.r.t. $\preceq$, e.g. ∞ for $\leq$

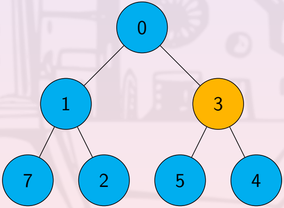


Min-heap

Max-heap

Inserting a New Value

- add a new node N preserving the heap topology
- set the key of N to the maximum value w.r.t. $\preceq$, e.g. ∞ for $\leq$
- decrease the key of N to the desired value



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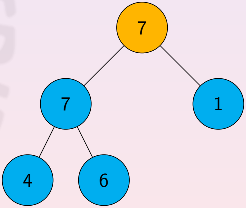

Max-heap

Inserting a New Value: Array-Based Pseudo-Code

```
def INSERT_VALUE(H, value):
   H.size ← H.size+1
   H[H.size] ← ∞⪯

   DECREASE_KEY(H, H.size, value)
enddef
```

Has the same complexity of DECREASE_KEY: $O(\log n)$

Summarizing complexity

DS	Building	Extracting	Inserting	Decreasing
Binary Heap	$\Theta(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
Fibonacci Heap (Amortized)	$\Theta(n)$	$O(\log n)$	$\Theta(1)$	$\Theta(1)$