Heaps

Algorithmic Design

Alberto Casagrande
Email: acasagrande@units.it

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Heaps

Abstract Data Types which store totally ordered values w.r.t. $\preceq$

They (efficiently) support the following tasks:

- building a heap from a set of data
- finding the minimum w.r.t. $\preceq$
- extracting the minimum w.r.t. $\preceq$
- decreasing the one of the values w.r.t. $\preceq$
- inserting a new value

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A min-heap is a heap s.t. $\preceq$ is $\leq$

A max-heap is a heap s.t. $\preceq$ is $\geq$

Heaps

They can be used to implement priority queues

The next element to be extracted minimizes a priority *criterion*

E.g., In emergencies, more serious patients must be served first

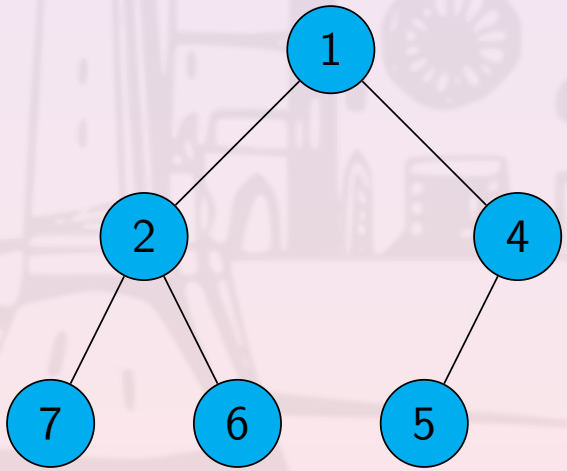
Their conditions may change and become more and more serious

Binary Heaps

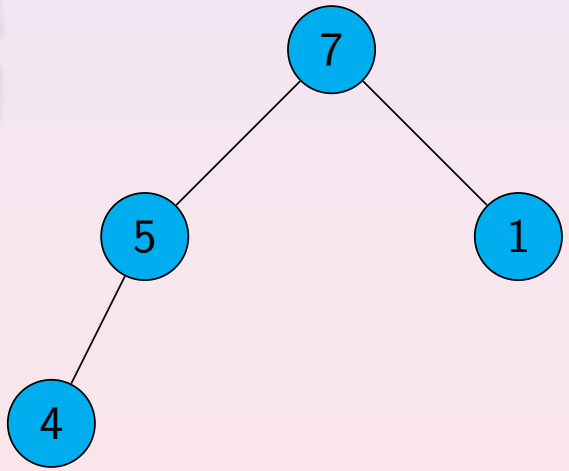
Binary Heaps

Are nearly complete binary trees: it is complete up to the second-last level and all leaves of the last level are on the left

The relation $parent(p) \preceq p$ holds for any node (heap property)



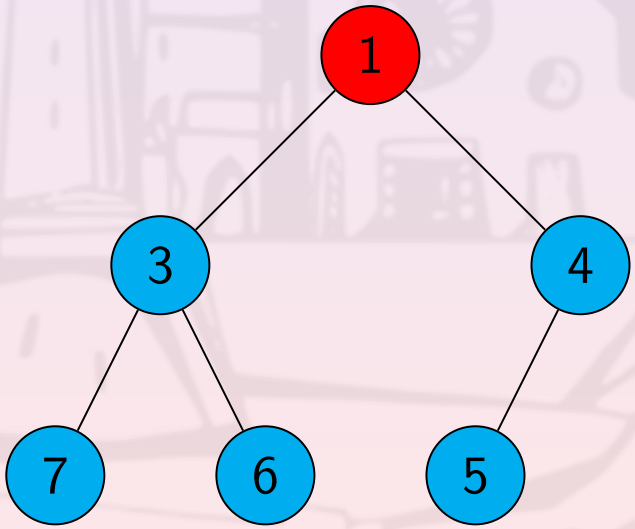
Min-heap



Max-heap

A Possible Representation (Not the Only One)

Use an **array**: the first position stores the root key



Min-heap

1	2	3	4	5	6
1	3	4	7	6	5

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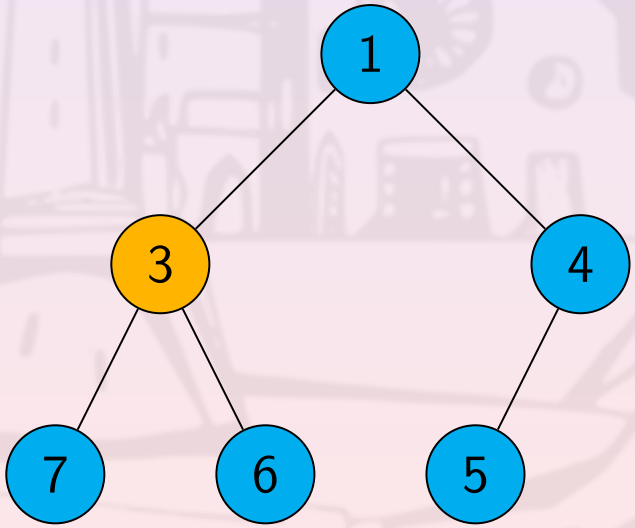

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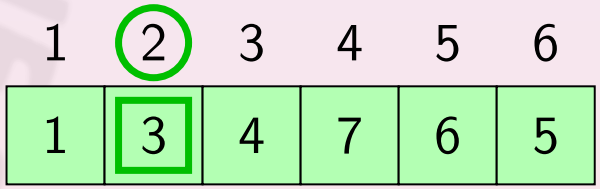

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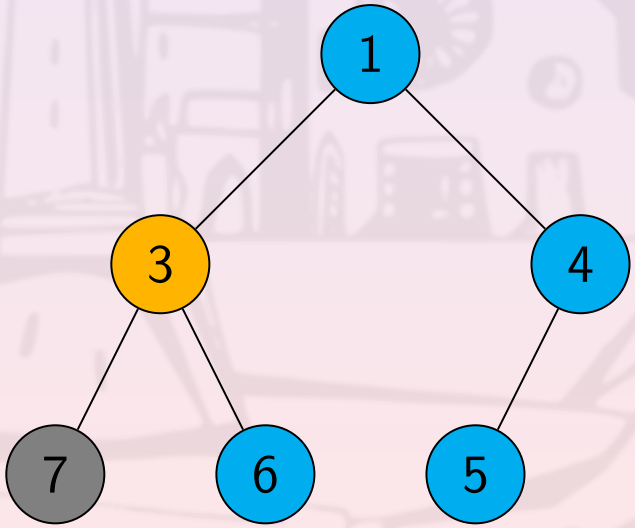

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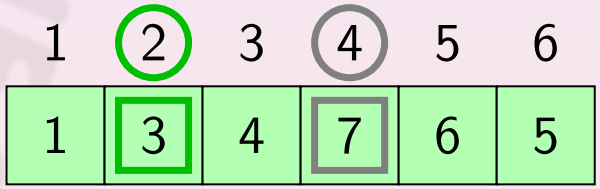

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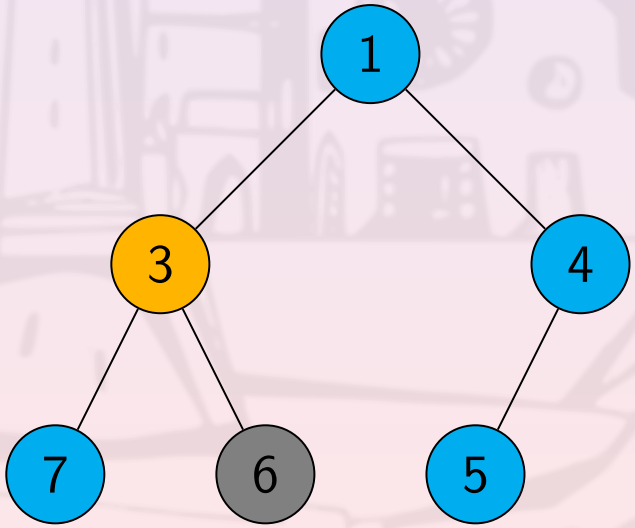
- right child has index $2 * i + 1$



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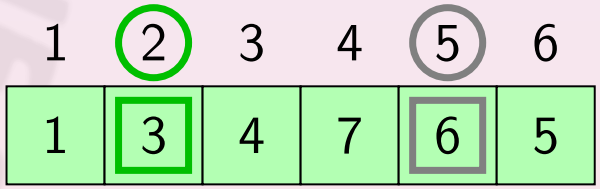
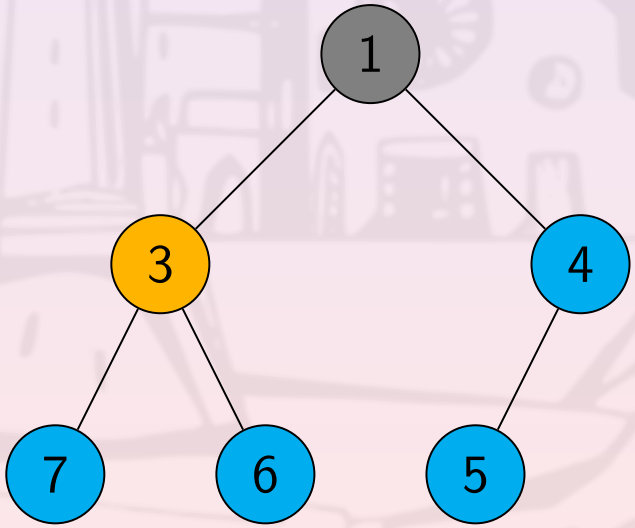


A Possible Representation (Not the Only One)

Use an **array**: the first position stores the root key

The i-th position of the array represents a node whose:

- left child has index $2 * i$
- right child has index $2 * i + 1$
- parent has index $\lfloor i/2 \rfloor$



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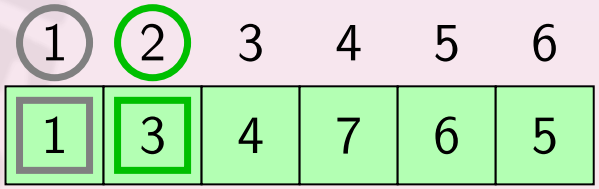


Array-based Representation: Few Useful Functions

H.size will denote the heap size

```
def LEFT(i):
  return 2*i
enddef
```

```
def RIGHT(i):
  return 2*i+1
enddef
```

```
def PARENT(i):
  return floor(i/2)
enddef
```

```
def GET_ROOT():
  return 1
enddef
```

```
def IS_ROOT(i):
  return i = 1
enddef
```

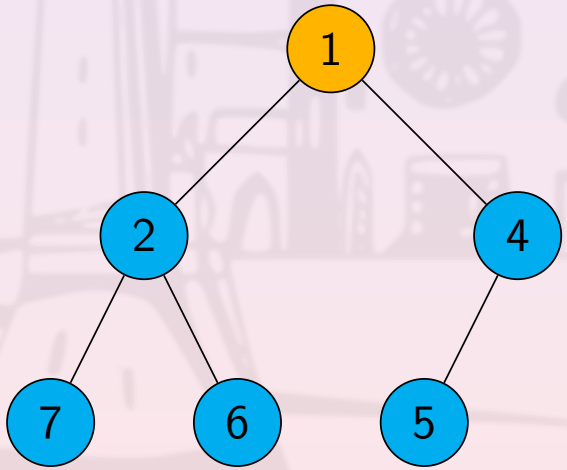
```
def IS_VALID_NODE(H,i):
  return H.size ≥ i
enddef
```

Finding the Minimum

Finding the Minimum

The minimum w.r.t. $\preceq$ is in the root of the heap

If this was not the case, the heap property did not hold



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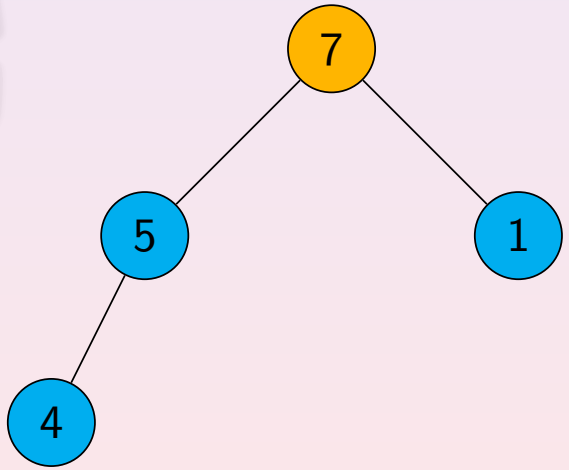

Max-heap

Finding the Minimum: Pseudo-Code

The minimum w.r.t. $\preceq$ is the root's key

```
def HEAP_MIN(H):
  return H.root
enddef
```

For array-based representation, we can rephrase it as...

```
def HEAP_MIN(H):
  return H[1]
enddef
```

In both the cases, the complexity is $\Theta(1)$

Removing the Minimum

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We must preserve both:

- heap topological structure
- heap property

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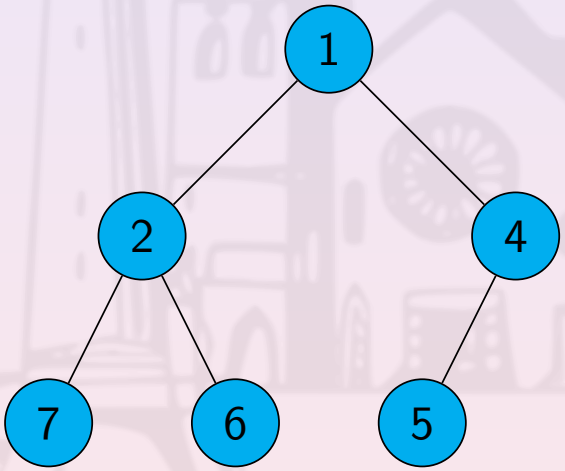
Removing the Minimum and Preserving Topology

Replace the root's key by that of the rightmost leaf of the last level



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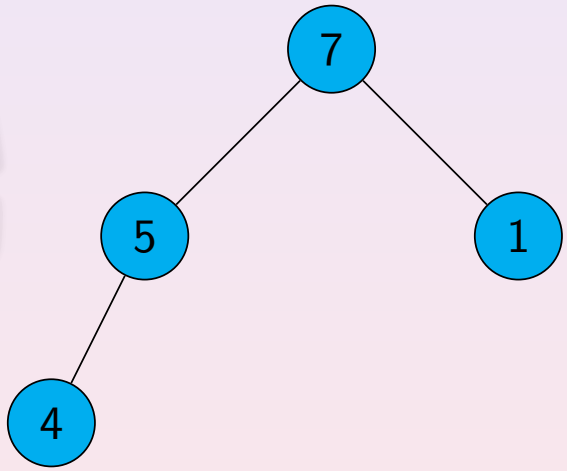

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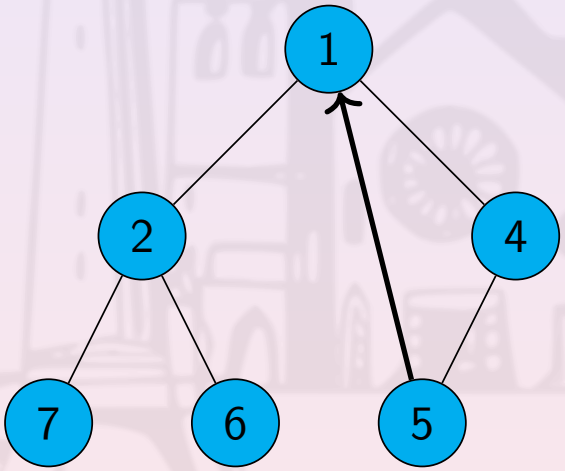

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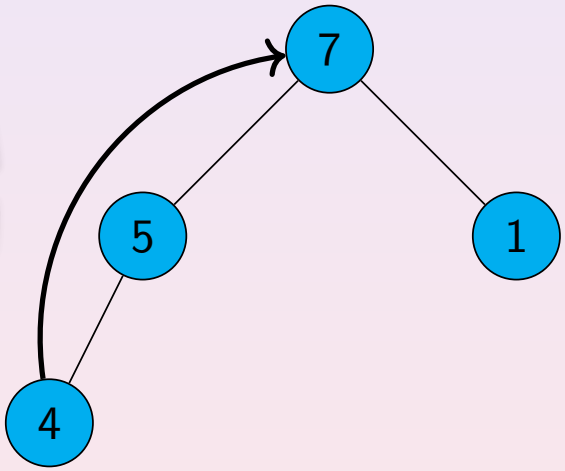

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Removing the Minimum and Preserving Topology

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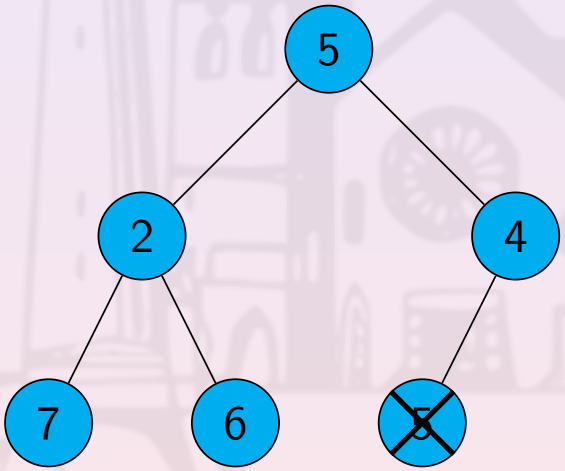
Delete the the rightmost leaf of the last-level



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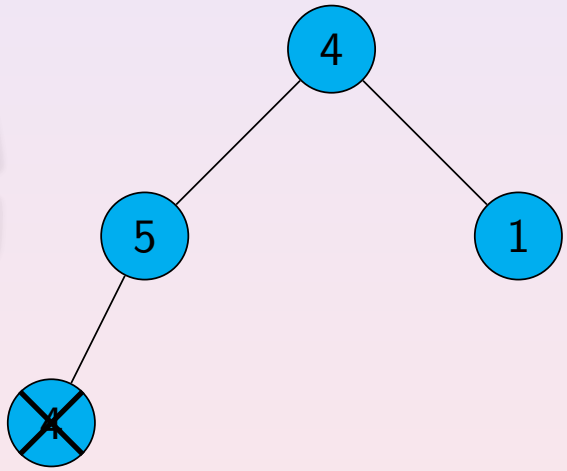

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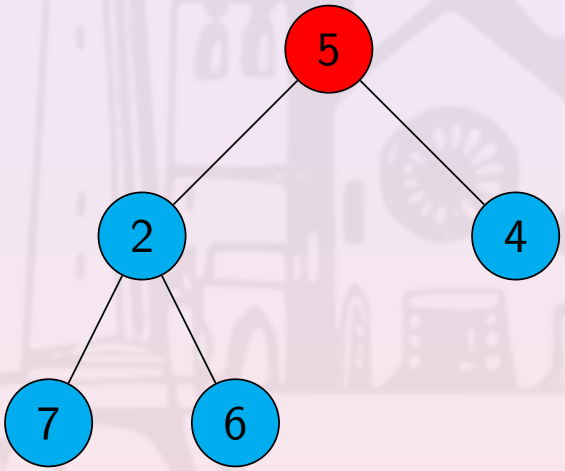

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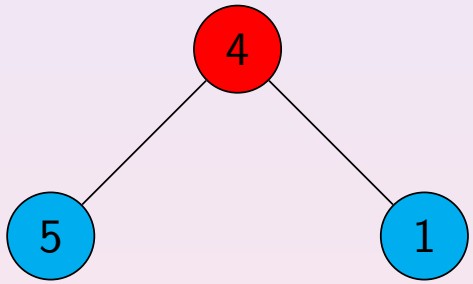
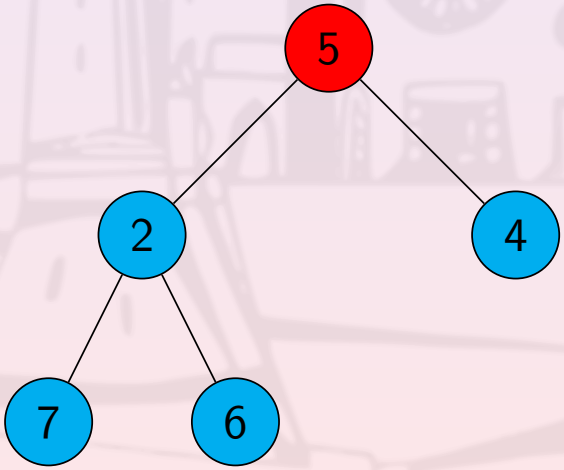


Max-heap

The heap property may be lost (only in one point)!

Restoring the Heap Property (HEAPIFY)

- find the node n, among the root and its children, whose key is minimum w.r.t. $\preceq$



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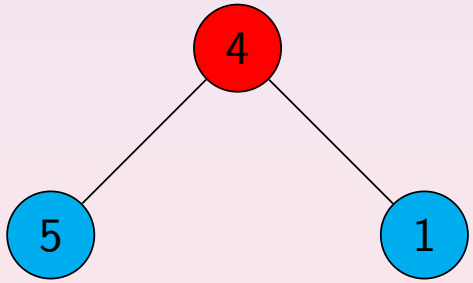

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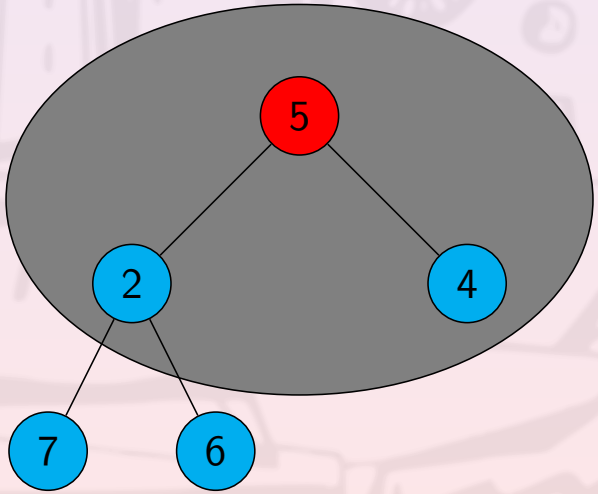

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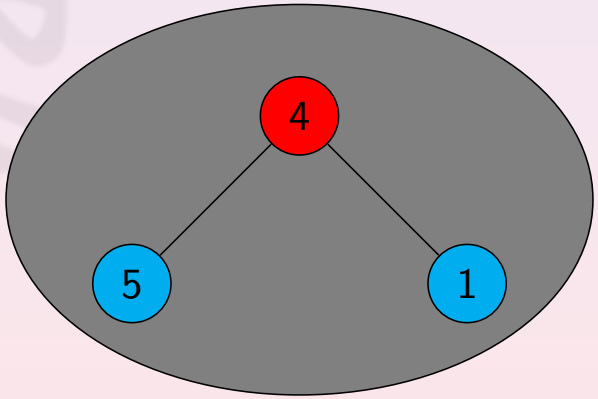

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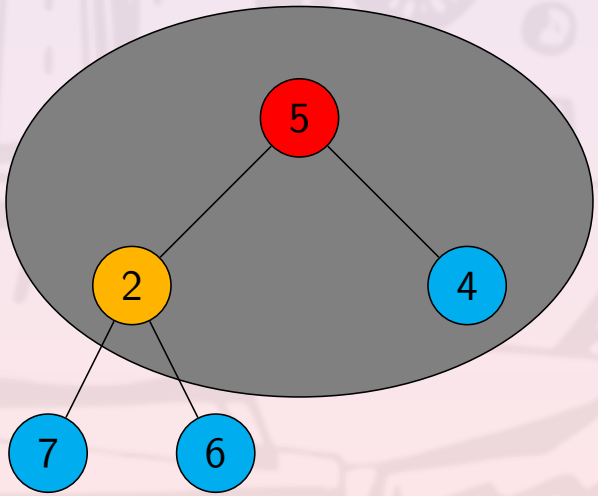

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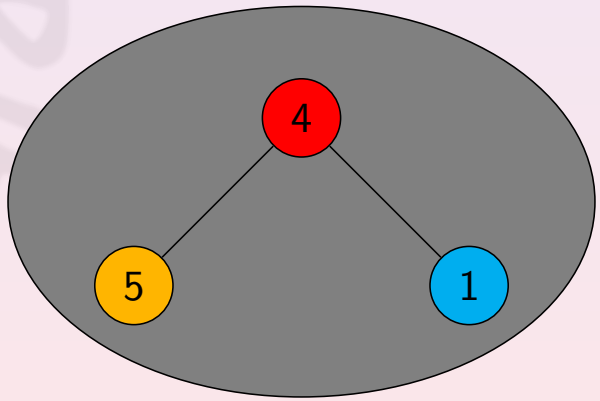

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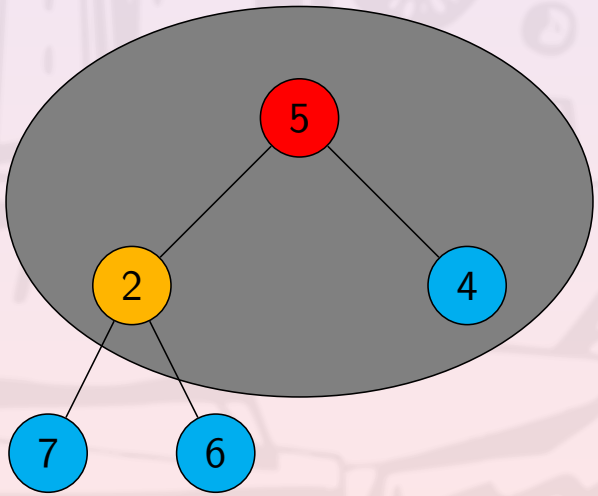

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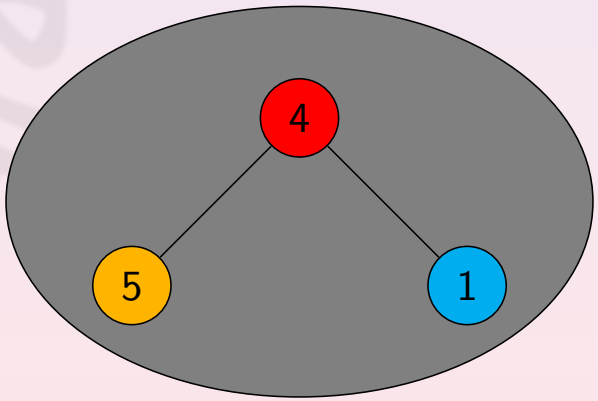

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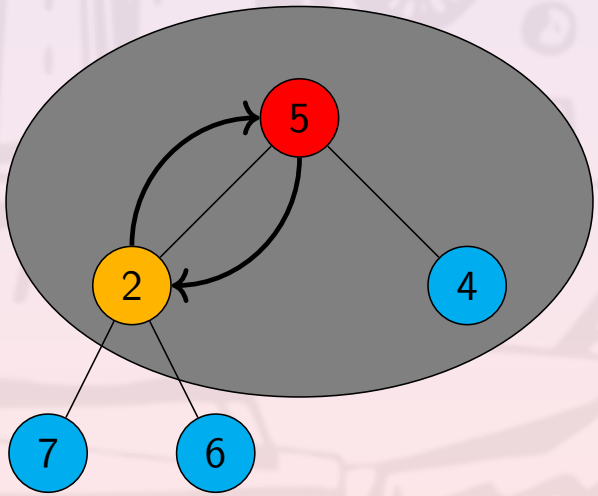

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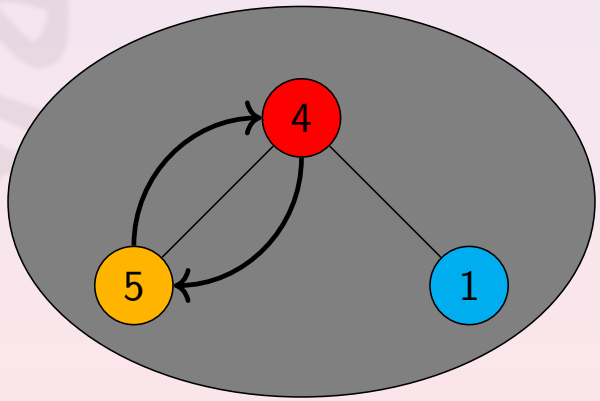

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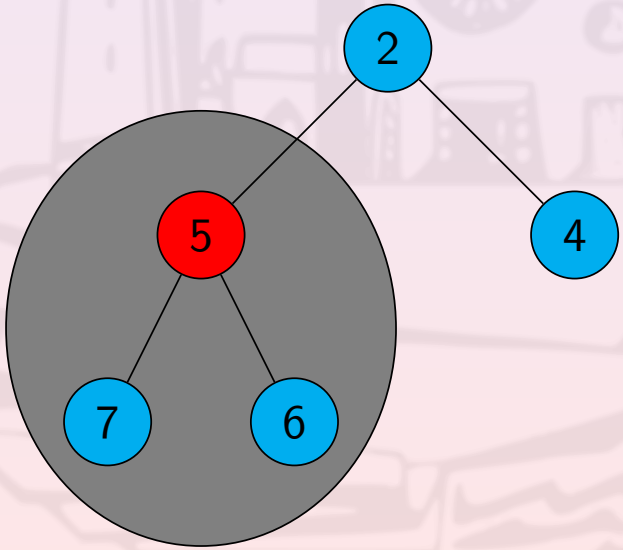

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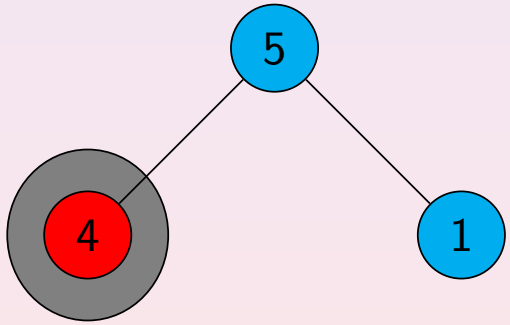

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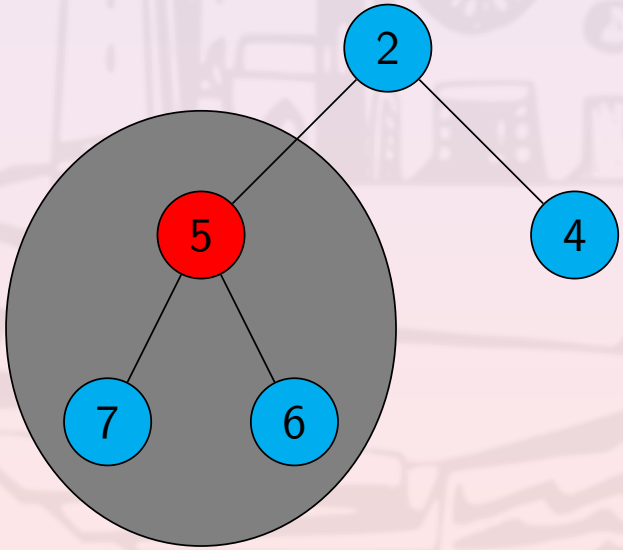

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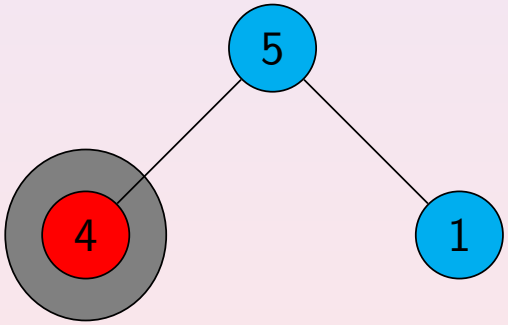

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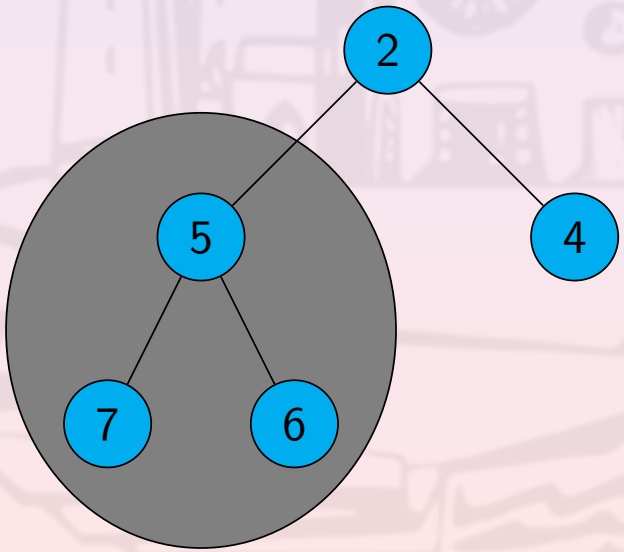

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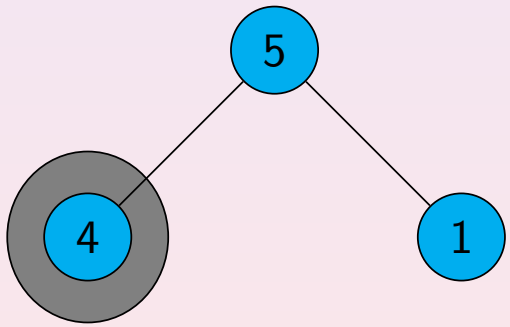
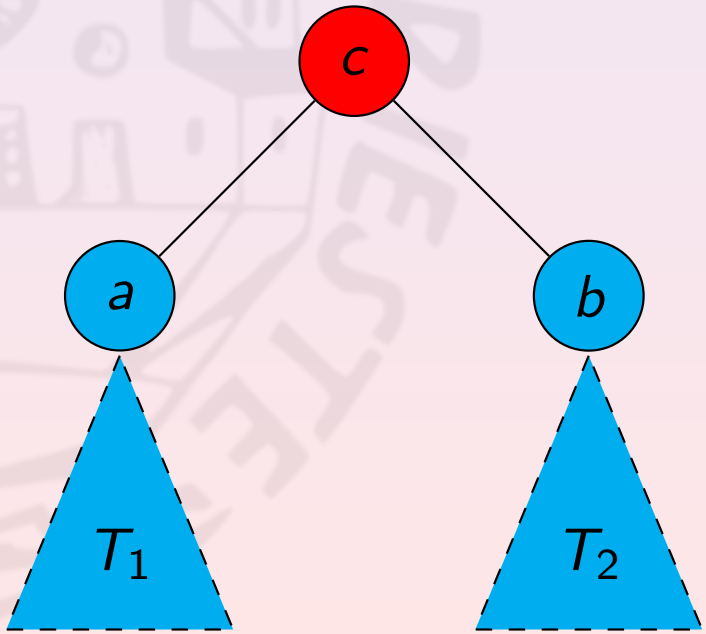


Max-heap

Correctness of HEAPIFY

Before the iteration: the heap property holds in T_1 and T_2



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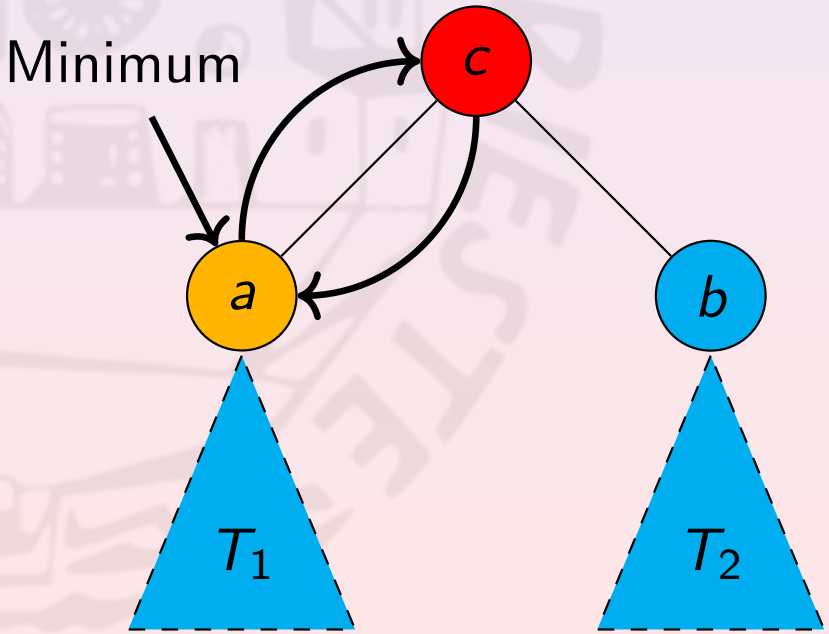

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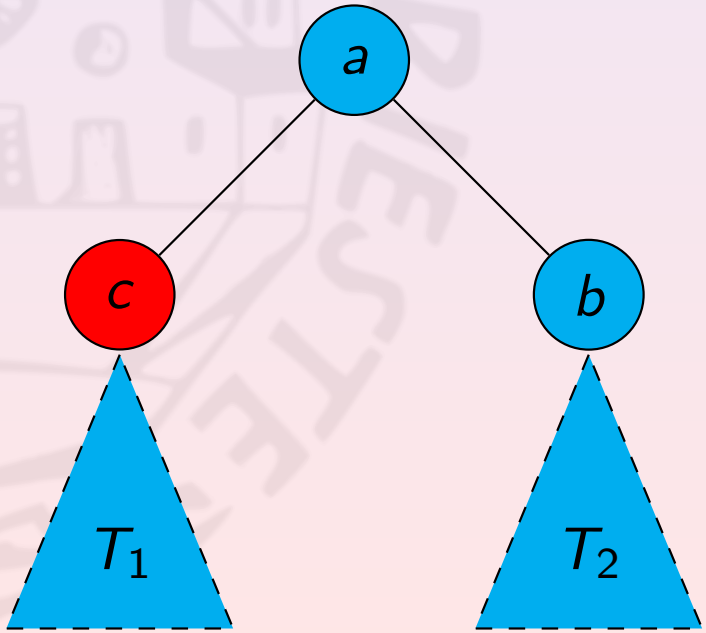
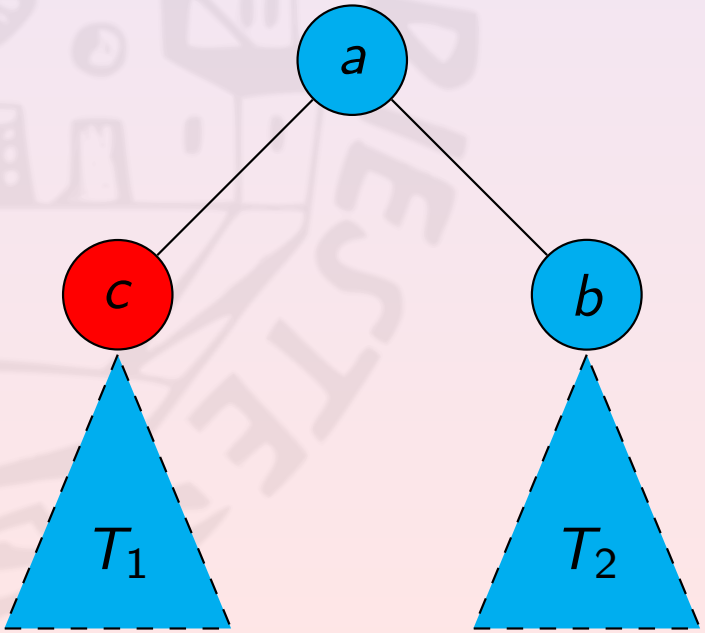


Correctness of HEAPIFY

Before the iteration: the heap property holds in T_1 and T_2

After the iteration:

- the heap property still holds in T_2 and between a and b
- T_1 has been messed up, but it is shorter than the original tree and all the keys in T_1 are greater than a



Removing the Minimum: Complexity

Replacing the root's key costs $\Theta(1)$

Removing the Minimum: Complexity

Replacing the root's key costs $\Theta(1)$

For each iteration of HEAPIFY:

- 2 comparisons to find the minimum
- 1 swap at most

The distance from a leaf is decreased by one at each iteration

The total cost of HEAPIFY is the height of the heap: $O(\log n)$

HEAPIFY: Array-Based Pseudo-Code

```
def HEAPIFY(H, i):
  m ← i

  for j in [LEFT(i), RIGHT(i)]:
      if IS_VALID_NODE(H, j) and H[j] ⪯ H[m]:
        m ← j
      endif
  endfor

  if i != m:
    swap(H, i ,m)
    HEAPIFY(H,m)
  endif
enddef
```

Removing the Minimum: Array-Based Pseudo-Code

```
def REMOVE_MIN(H, i):
  H[1] ← H[H.size]

  H.size ← H.size −1

  HEAPIFY(H, 1)
enddef
```

Builing a Heap

Building a Binary Heap (BUILD_HEAP)

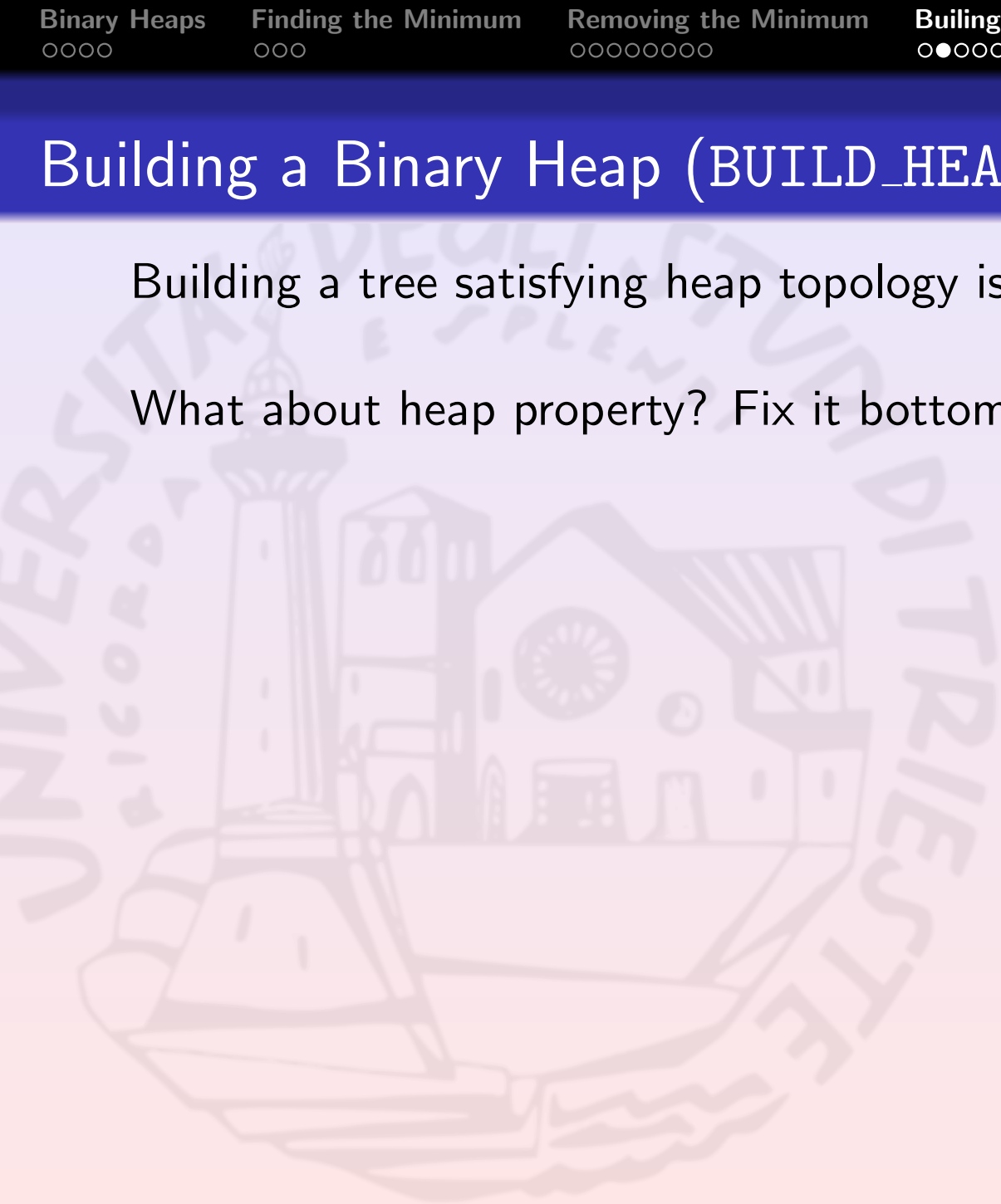
Building a tree satisfying heap topology is easy

What about heap property?

Building a Binary Heap (BUILD_HEAP)

Building a tree satisfying heap topology is easy

What about heap property? Fix it bottom-up by using HEAPIFY



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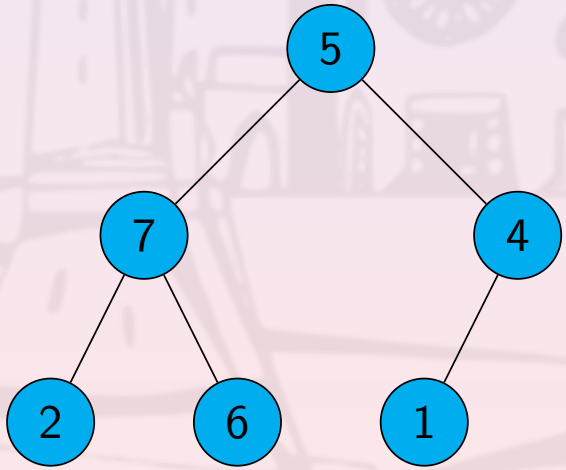

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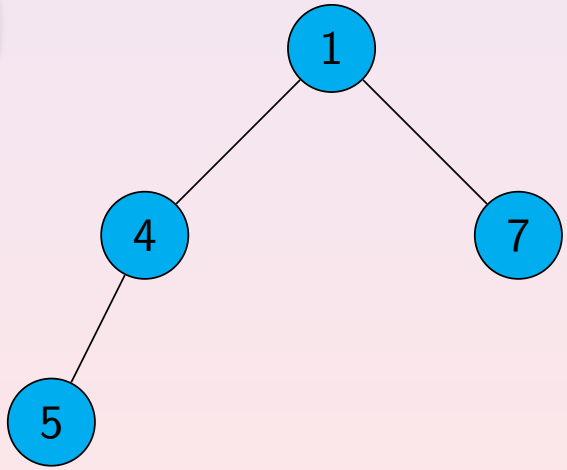

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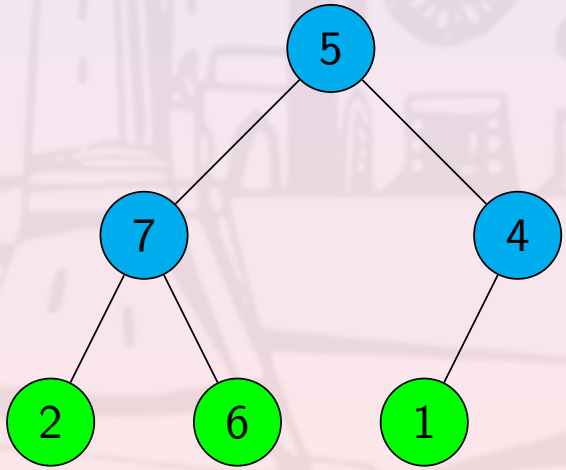

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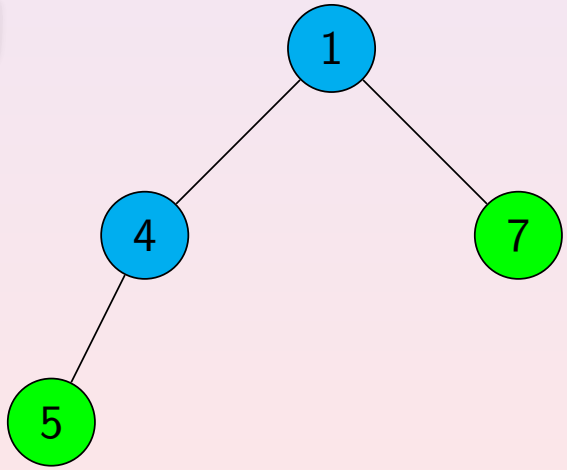

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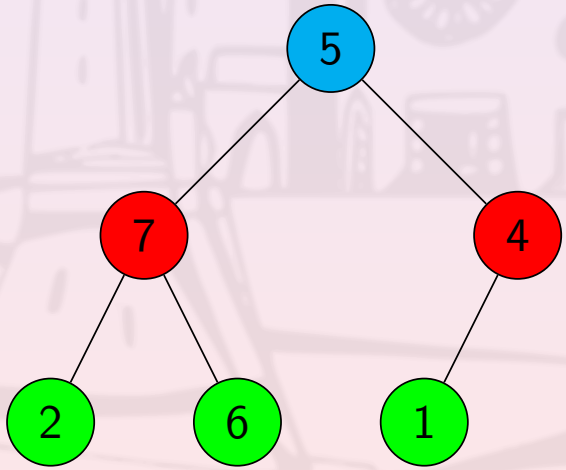
- fix the heaps rooted on the second-last level with children



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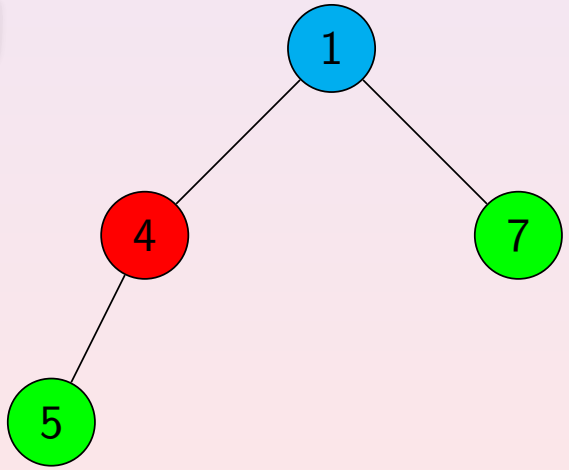

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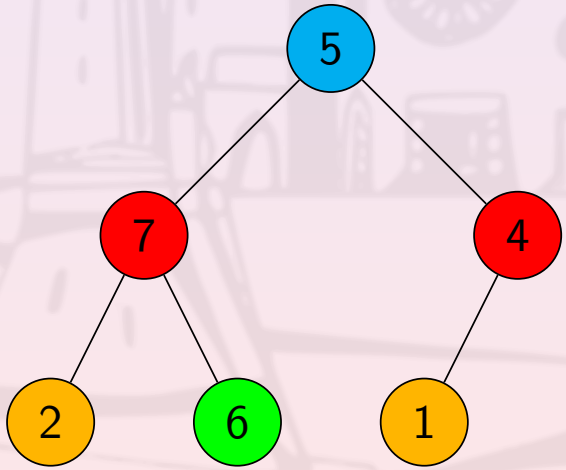

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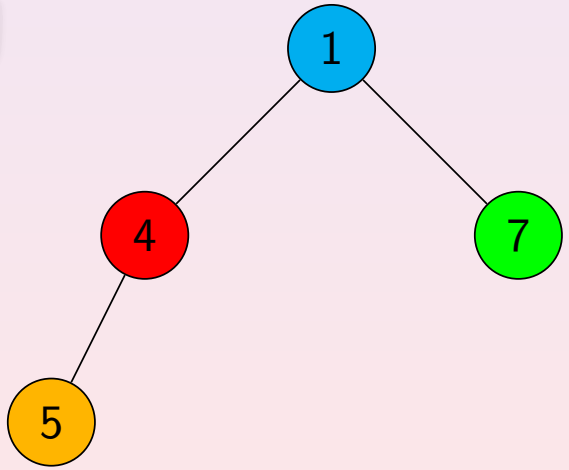

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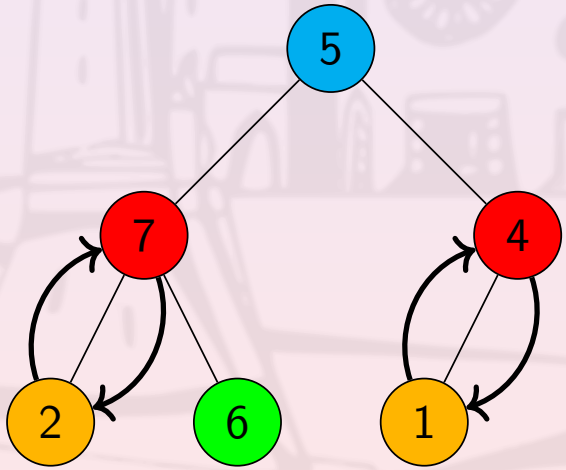

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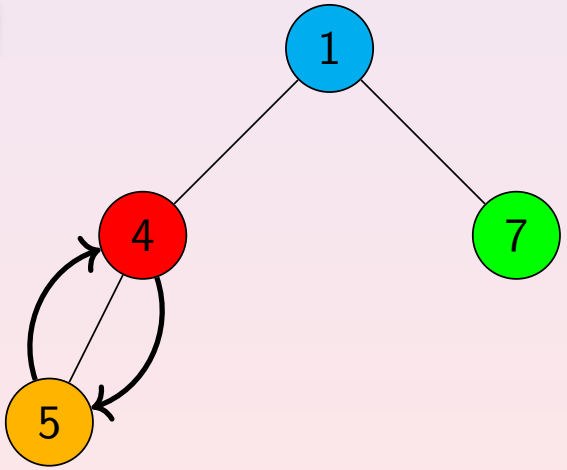

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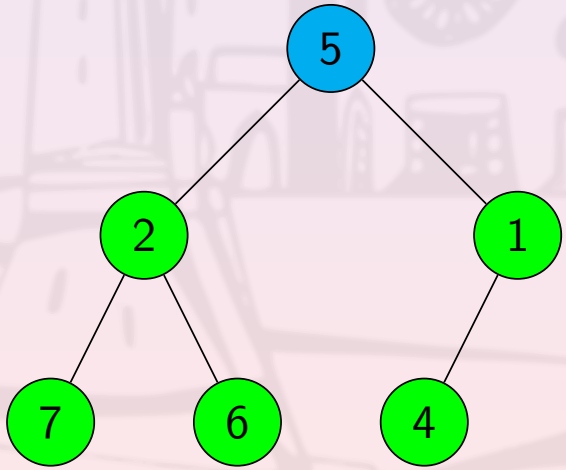

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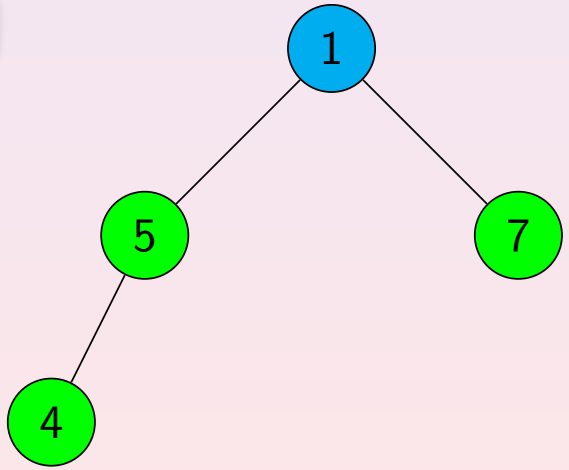

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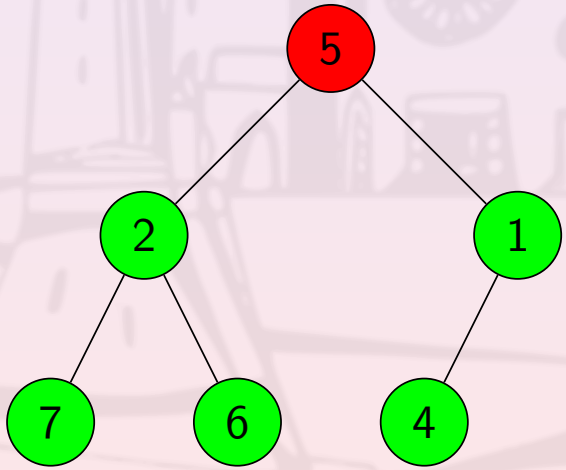

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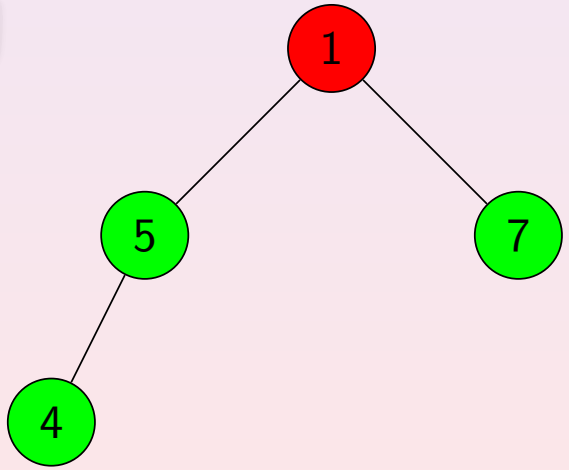

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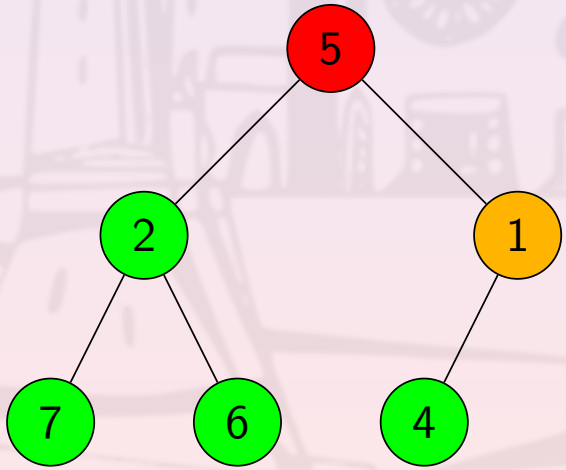

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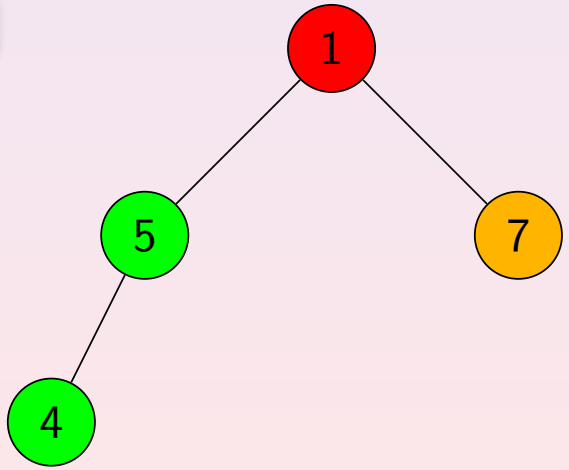

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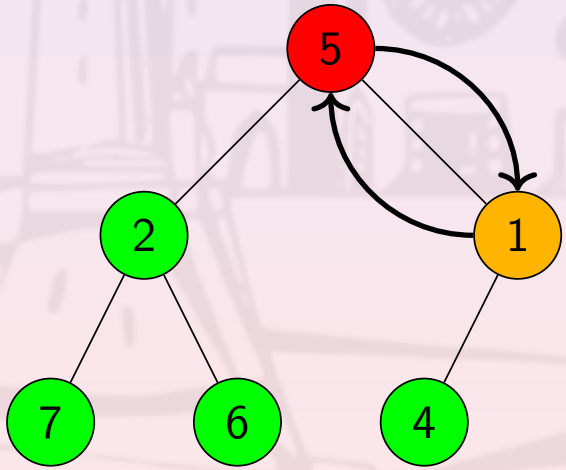

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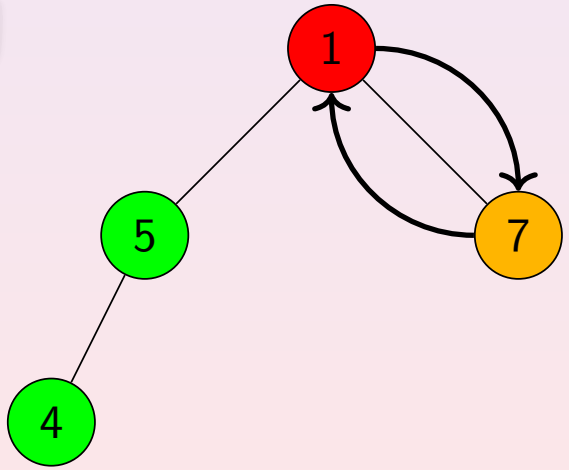

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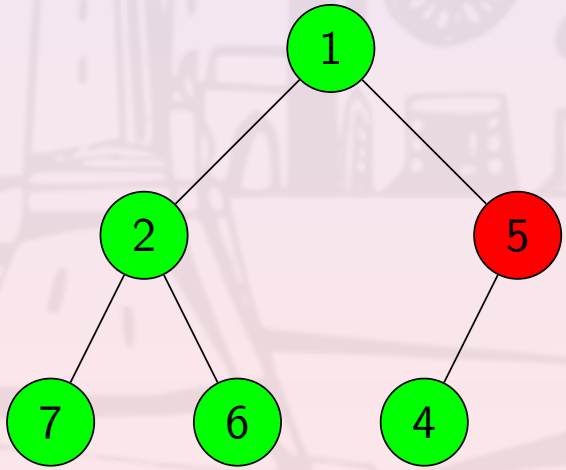

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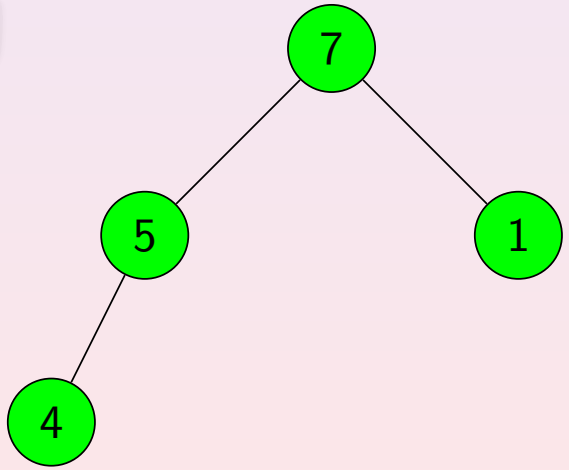

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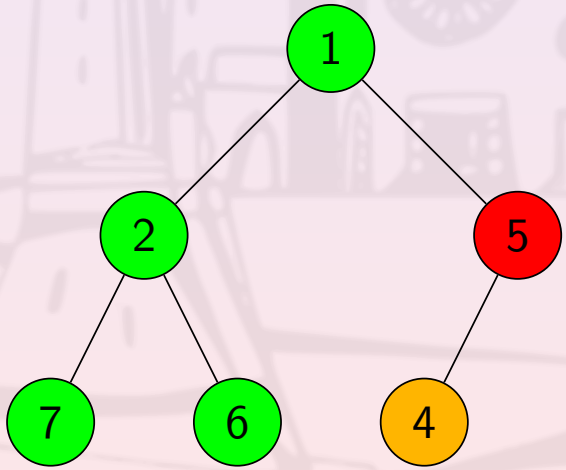

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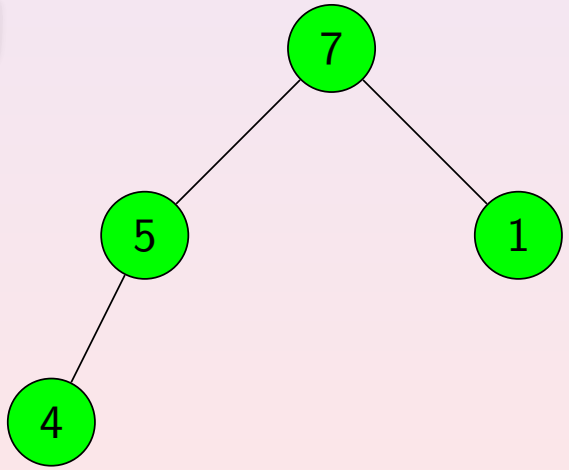

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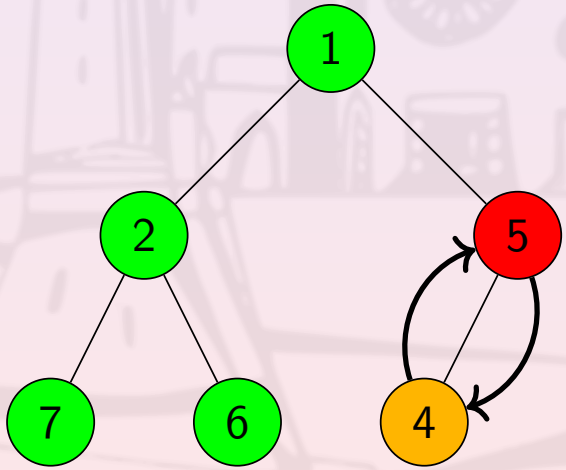

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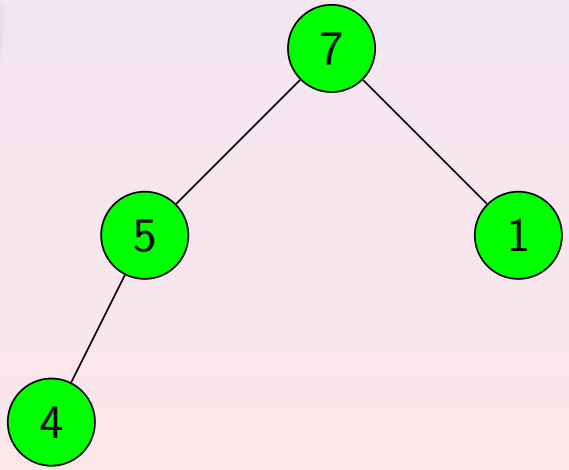

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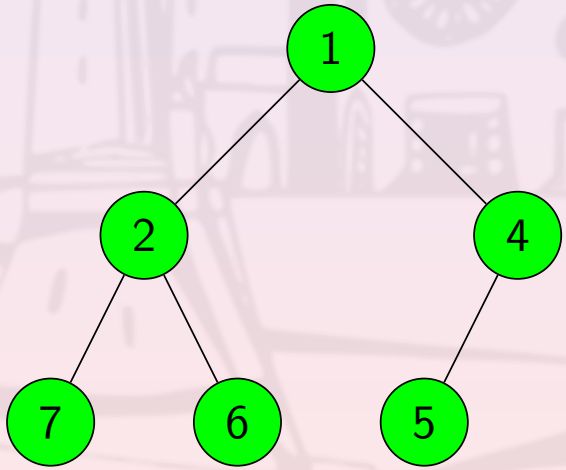

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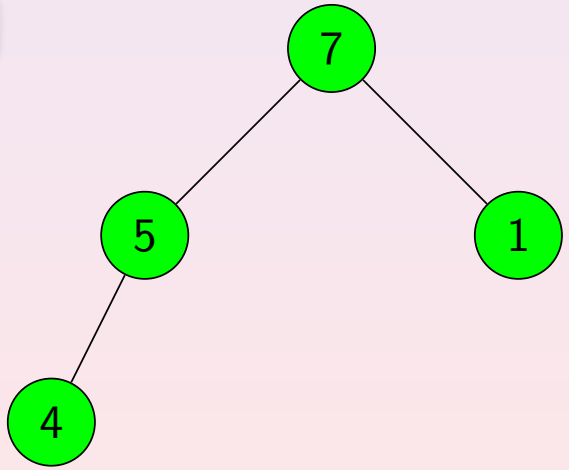

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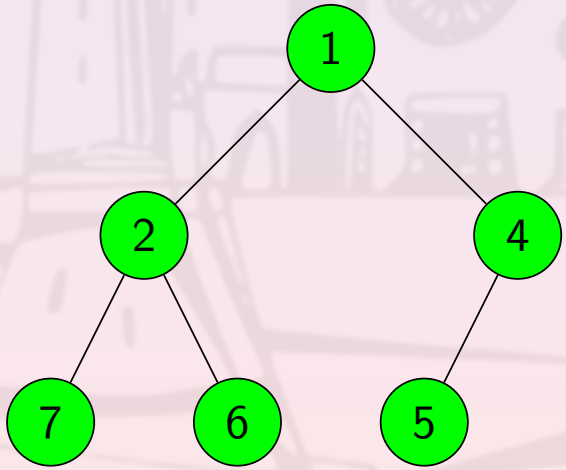

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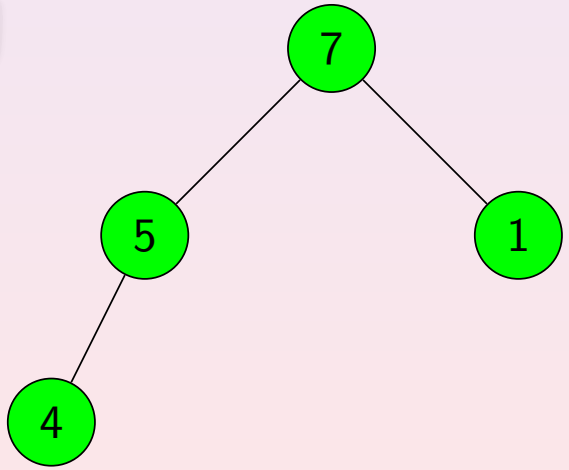


Max-heap

Complexity of BUILD_HEAP

HEAPIFY costs $O(h)$ (i.e., $\leq c * h$) on a tree having height h

If the considered tree has n nodes:

- its height is $\lfloor \log_2 n \rfloor$
- it contains at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ at height h

Complexity of BUILD_HEAP

The costs $T_{\mathtt{bh}}(n)$ of executing BUILD_HEAP on a n-sized tree is:

$$T_{\mathtt{bh}}(n) \leq \sum_{h=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil * (c * h) \leq \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h} * (c * h)$$

$$\leq c * n * \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{h}{2^h} \leq c * n * \sum_{h=0}^{\infty} \frac{h}{2^h}$$

$$\leq c * n * \frac{1/2}{(1-1/2)^2} = 2 * c * n \in O(n)$$

BUILD_HEAP: Pseudo-Code

```
def BUILD_HEAP_AUX(H, node):
    if IS_VALID_NODE(H, node):
        BUILD_HEAP_AUX(H, LEFT(node))
        BUILD_HEAP_AUX(H, RIGHT(node))

        HEAPIFY(H, node)
    endif
enddef

def BUILD_HEAP(A):
    H ← BUILD_HEAP_TREE(A)
    BUILD_HEAP_AUX(H, GET_ROOT(H))

    return H
enddef
```

BUILD_HEAP: Array-Based Pseudo-Code

The array-based representation helps in avoiding recursion

Finding the nodes of the i-th level is easy ...

... they are represented by elements in positions $\left[2^i, 2^{i+1} - 1\right]$

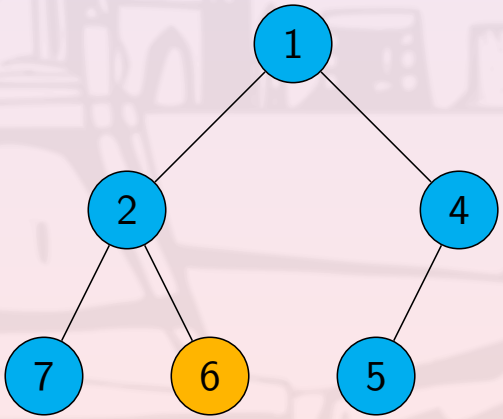
```
def BUILD_HEAP(A):
  A.size = |A|

  for i ← PARENT(A.size) downto 1:
    HEAPIFY(A, i)
  endfor

  return A
enddef
```

Decreasing a Key

Decreasing a Node's Key w.r.t. $\preceq$



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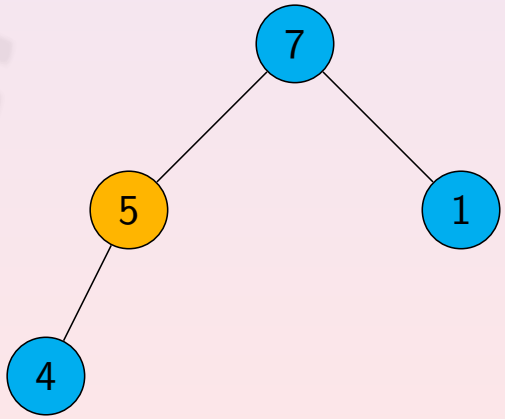
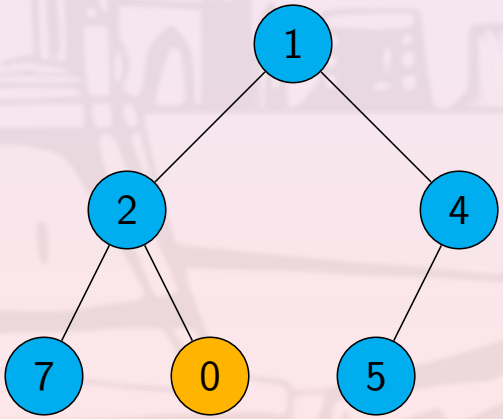


Max-heap

Decreasing a Node's Key w.r.t. $\preceq$

Preserves the heap property on the sub-tree rooted on the node



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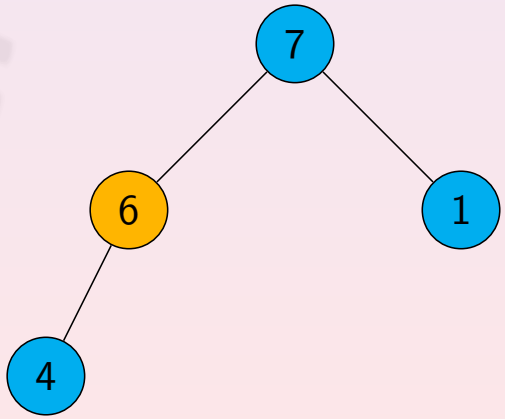
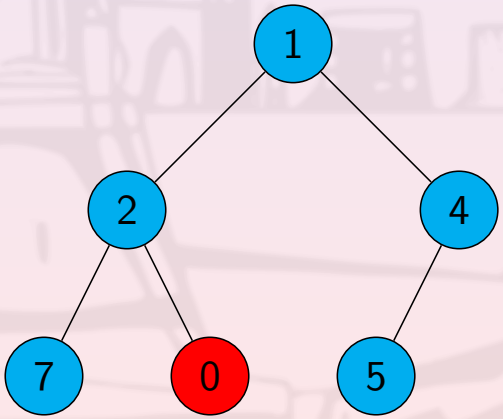


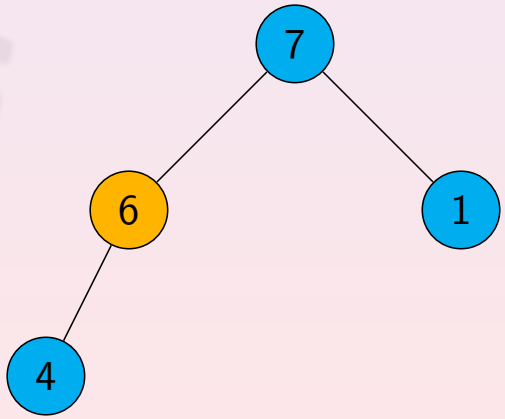
Max-heap

Decreasing a Node's Key w.r.t. $\preceq$

Preserves the heap property on the sub-tree rooted on the node, but it may broke the property w.r.t. its parent



Min-heap



Max-heap

Decreasing a Node's Key w.r.t. $\preceq$

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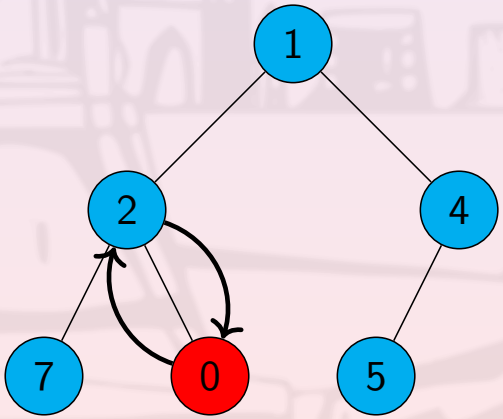
Swapping the keys of the node and its parent solves the problem on the subtree rooted on the parent



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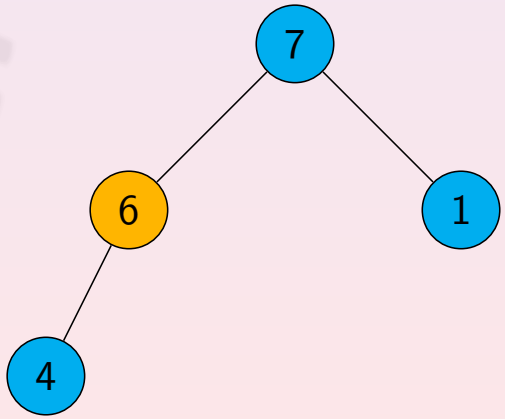

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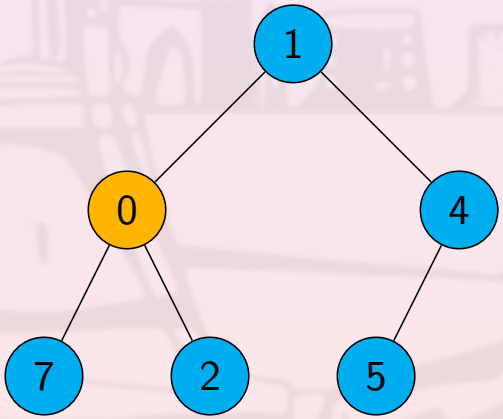

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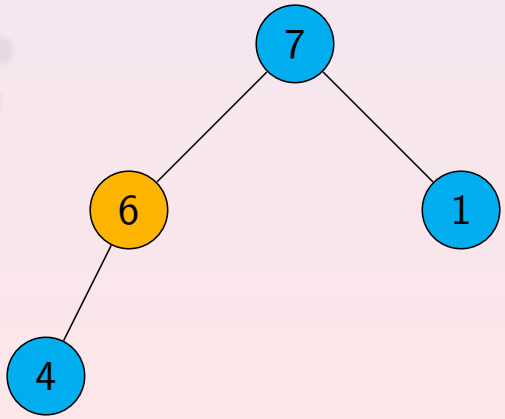

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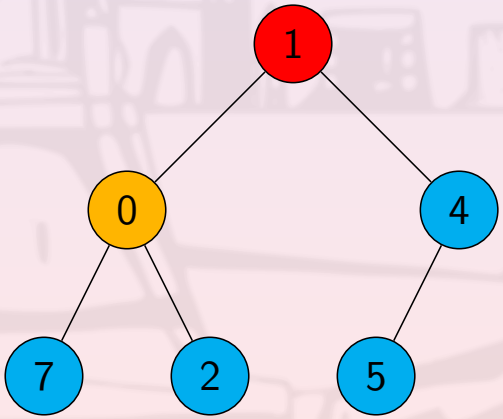

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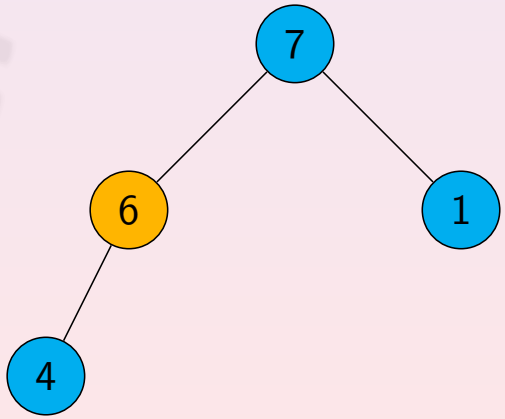

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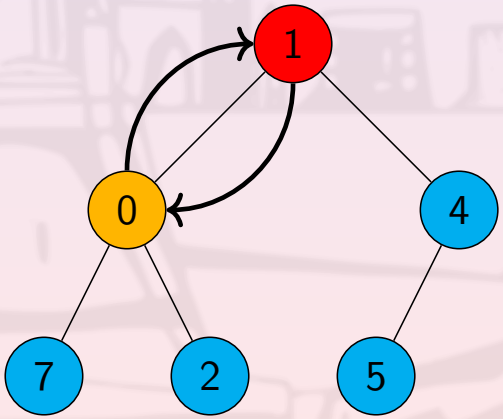
Repeat the process until the heap property is restored



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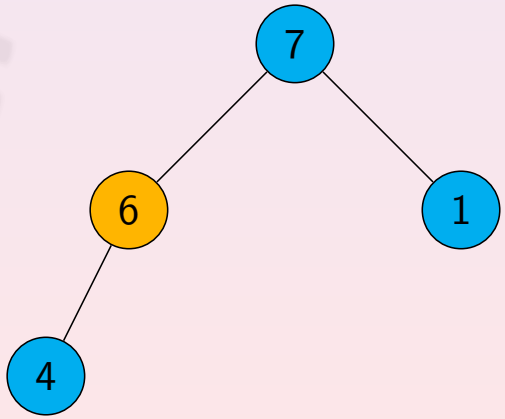

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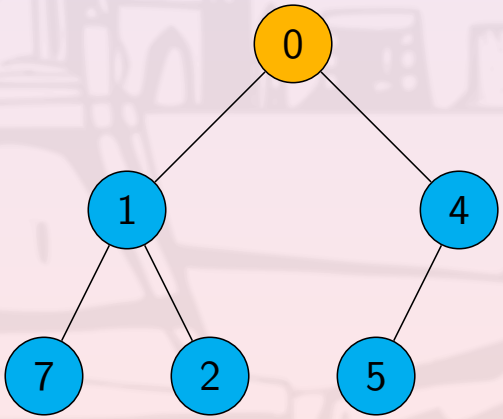
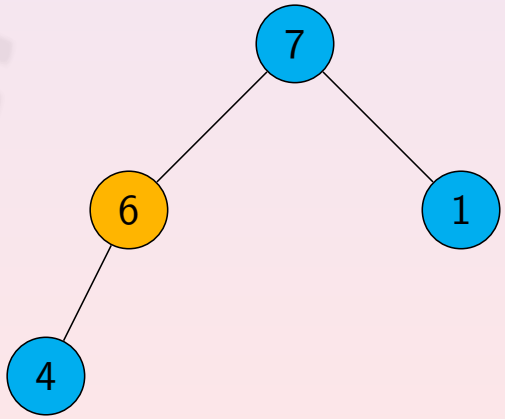


Min-heap



Max-heap

Decreasing a Key w.r.t. $\preceq$: Complexity

Each iteration either:

- ends the computation in time $\Theta(1)$ or
- pushes the problem one step closer to the root in time $\Theta(1)$

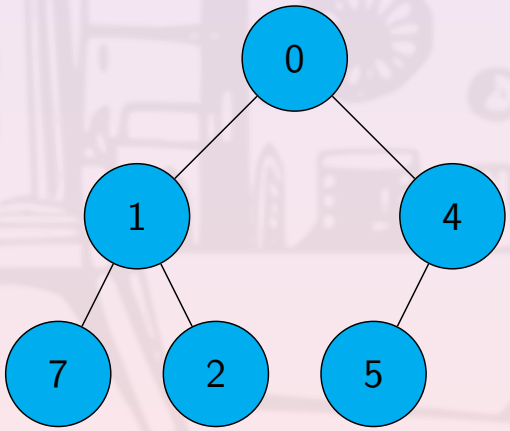
Since the heap height is $\lfloor \log_2 n \rfloor$, the complexity is $O(\log n)$

Decreasing a Key w.r.t. $\preceq$: Pseudo-Code

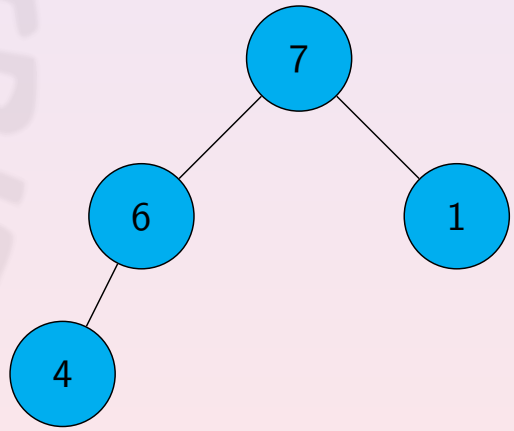
```
def DECREASE_KEY(H, i, value):
    if H[i] ⪯ value:
        error(value+" is not smaller than H["+i+"]")
    endif

    H[i] ← value

    while not (IS_ROOT(i) or H[PARENT(i)] ⪯ H[i]):
        swap(H,i,PARENT(i))

        i ← PARENT(i)
    endwhile
enddef
```

Inserting a Value

Inserting a New Value



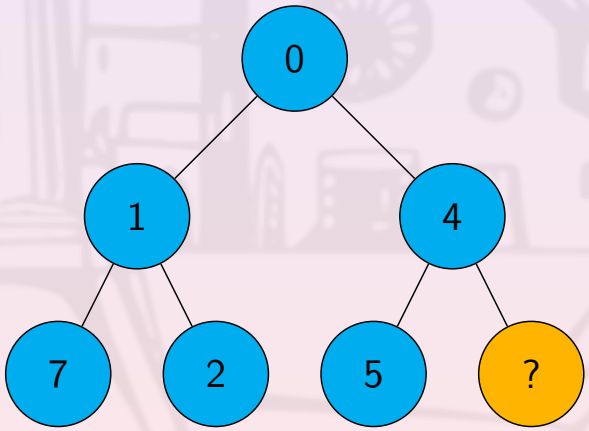
Min-heap



Max-heap

Inserting a New Value

- add a new node N preserving the heap topology



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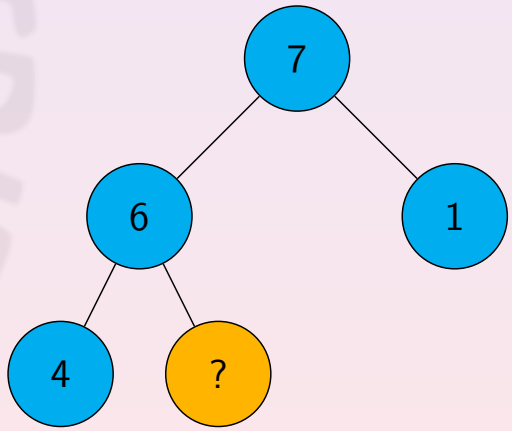
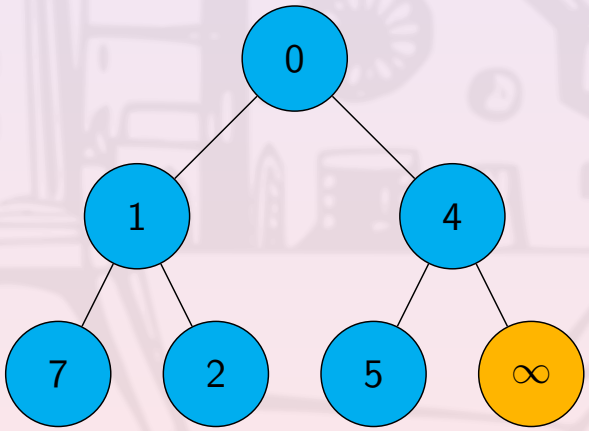


Max-heap

Inserting a New Value

- add a new node N preserving the heap topology
- set the key of N to the maximum value w.r.t. $\preceq$, e.g. ∞ for $\leq$



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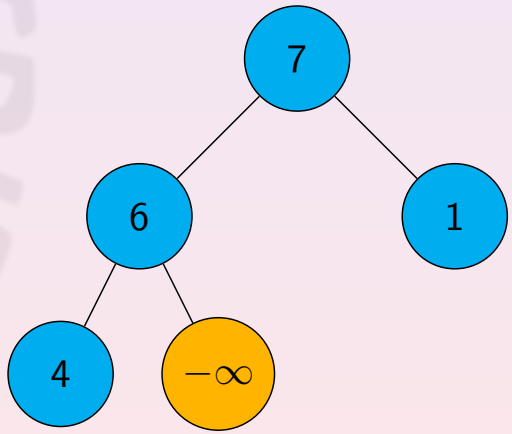
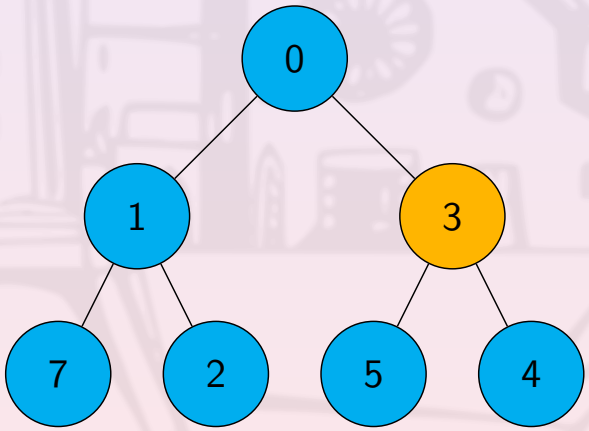


Max-heap

Inserting a New Value

- add a new node N preserving the heap topology
- set the key of N to the maximum value w.r.t. $\preceq$, e.g. ∞ for $\leq$
- decrease the key of N to the desired value



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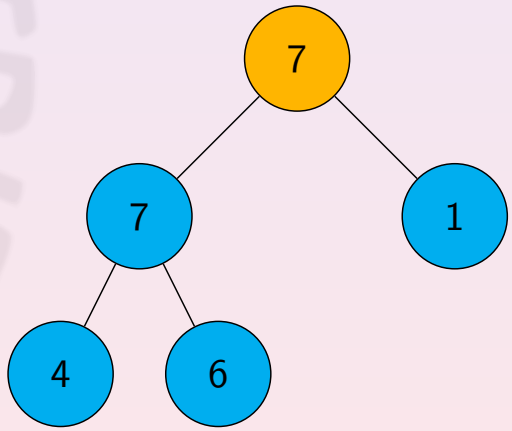

Max-heap

Inserting a New Value: Array-Based Pseudo-Code

```
def INSERT_VALUE(H, value):
  H.size ← H.size+1
  H[H.size] ← ∞_≼

  DECREASE_KEY(H, H.size, value)
enddef
```

Has the same complexity of DECREASE_KEY: $O(\log n)$

Summarizing complexity

DS	**Building**	**Extracting**	**Inserting**	**Decreasing**
Binary Heap	$\Theta(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
Fibonacci Heap (Amortized)	$\Theta(n)$	$O(\log n)$	$\Theta(1)$	$\Theta(1)$