Graphs and Algorithms

Algorithmic Design

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Is There a Unique Way to Represent. . .

- dynamic systems
- information flows
- infectious disease spread
- knowledge relations
- dependency relations
- computer network
- document network (e.g., WWW)
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Yes, by using graphs

Basics

Graphs (Graph Theory)

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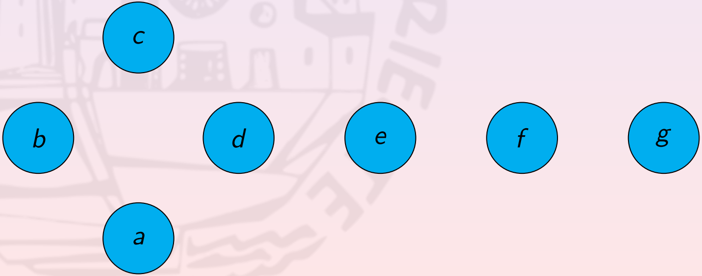

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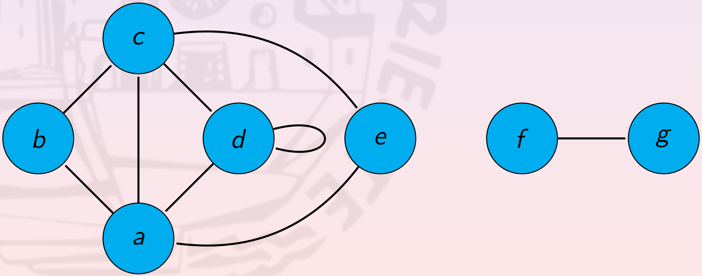

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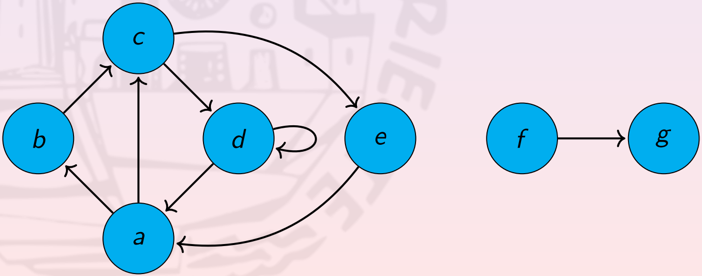
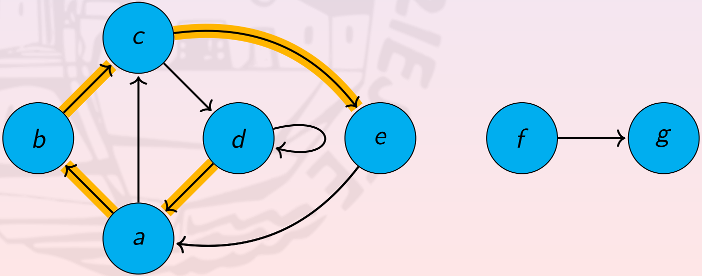
If the edges are (un)directed, the graph is (un)directed



Paths and Cycles

A path of length n between $a, b \in V$ is a sequence $e_1, \ldots, e_n$ s.t.

- e_1 involves a
- e_n involves b
- e_i and e_{i+1} involve a common node n_i



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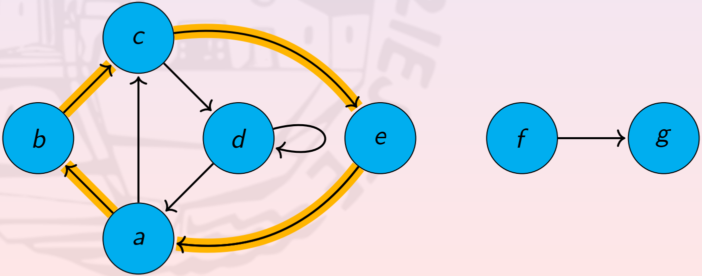
A cycle is a path whose initial and final node coincide.



Connected and Acyclic Graphs (Graph Theory)

A graph is connected if there is a path between every pairs of nodes

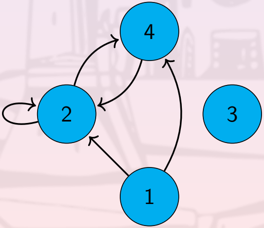
A connected component of a undirected graph G is a maximum connected sub-graph of G.

A graph is acyclic if it does not contain cycles

Directed Acyclic Graphs are also known as DAGs

Representing Graphs

Two main ways:



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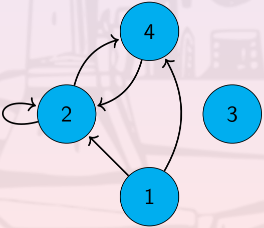

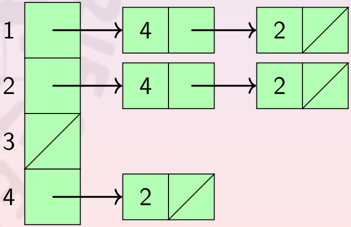

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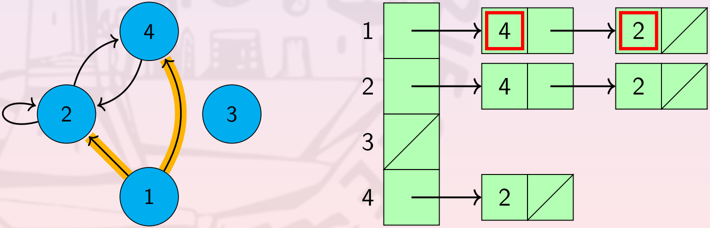

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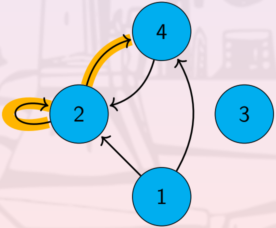

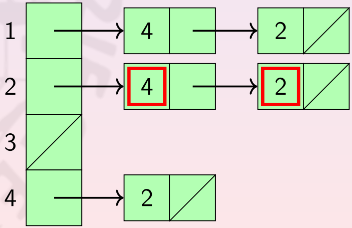

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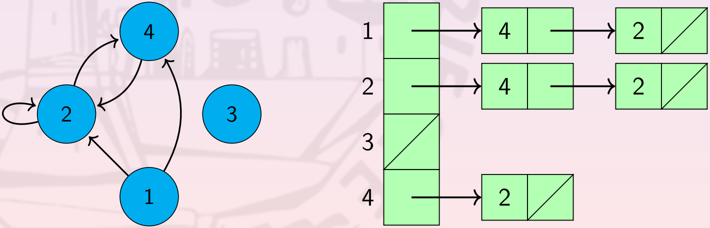

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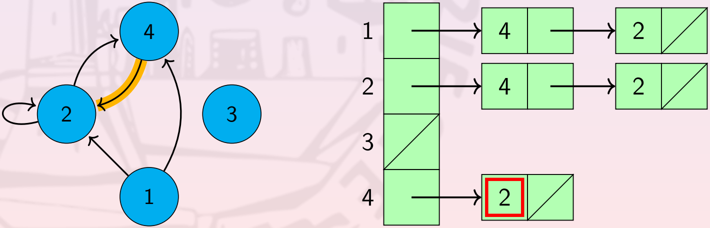

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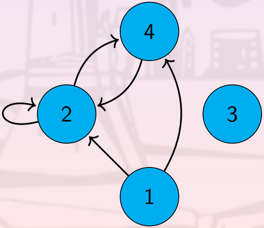

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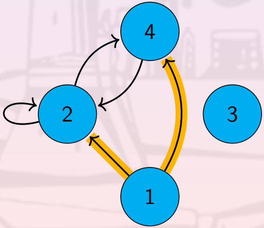

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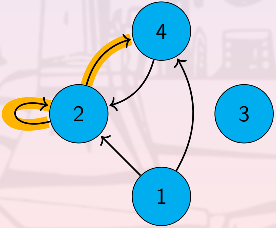

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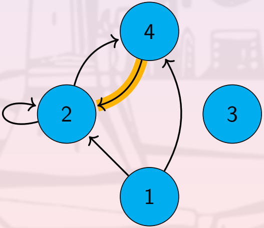

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Deapth-First-Search search “deeper” in the graph whenever possible

Breadth-First Search

Breadth-First-Search (BFS)

Visiting order is related to the distance from a source s: the lesser the distance of a node, the sooner it will be visited

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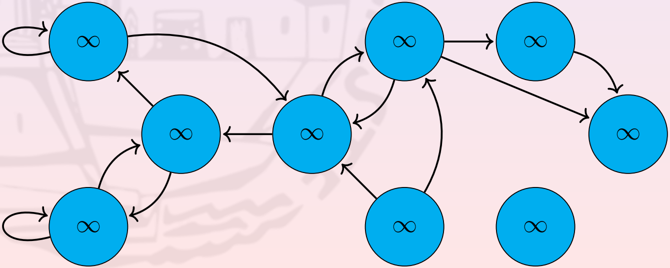
Because of this, BFS is used to compute source-node distances

It also produces the breadth-first tree i.e., the tree of shortest paths from s, and returns the shortest path from s to any reachable node

Breadth-First-Search (BFS): Coloring and Example



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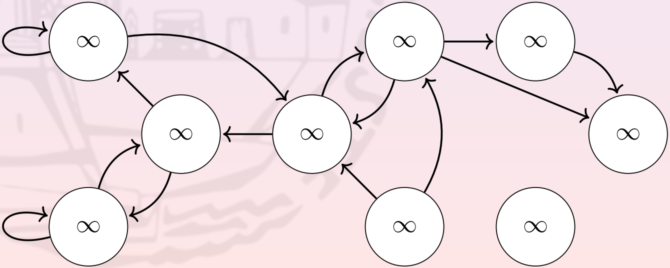

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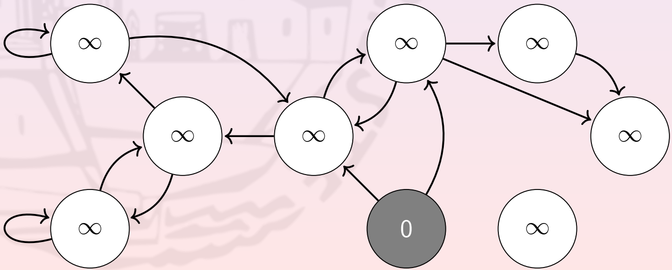

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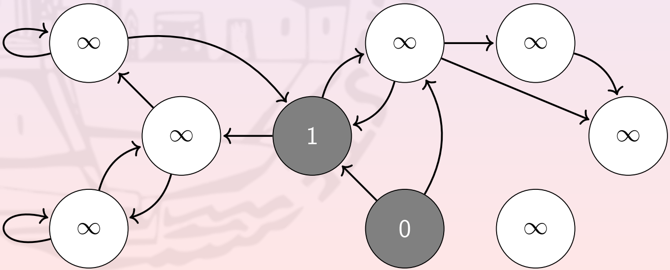

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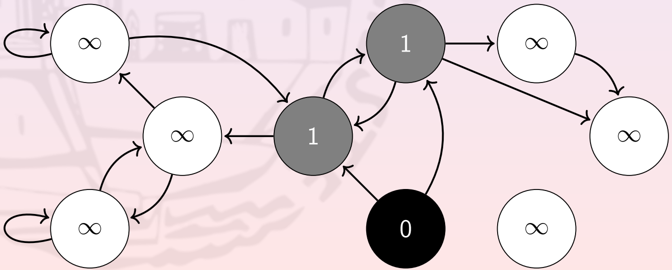

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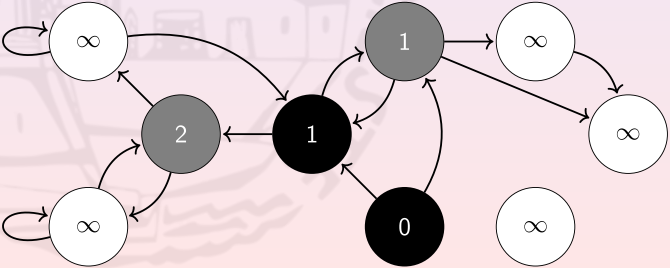

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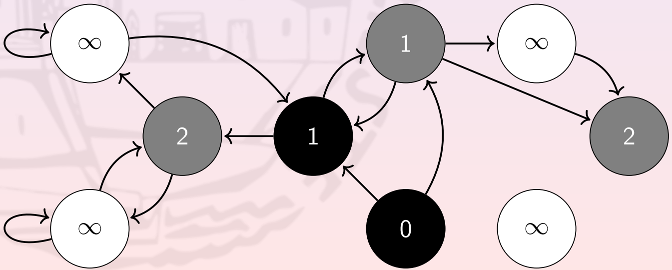

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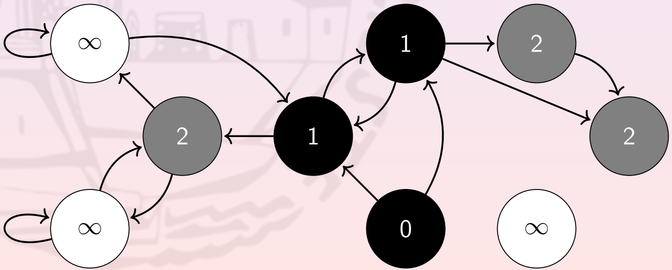

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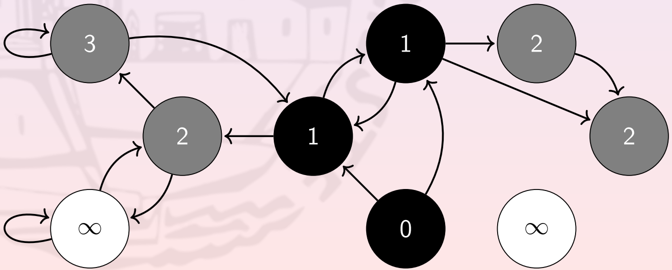

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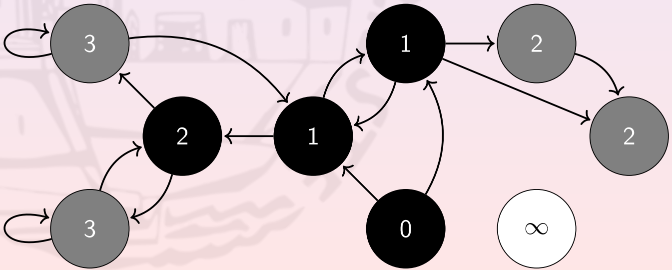

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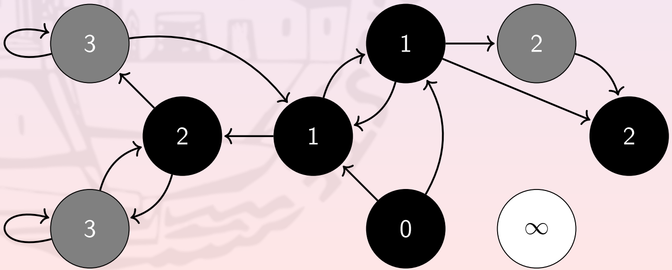

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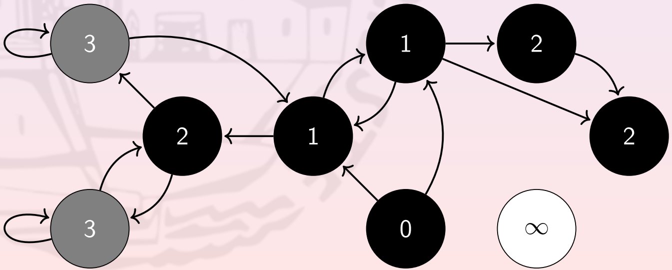

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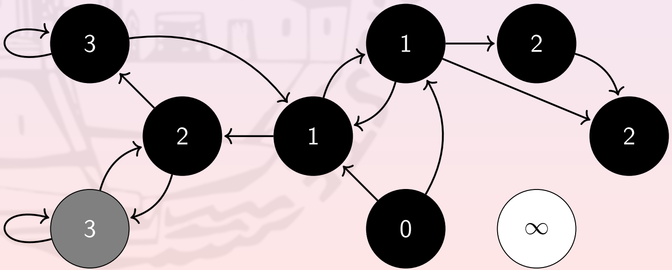

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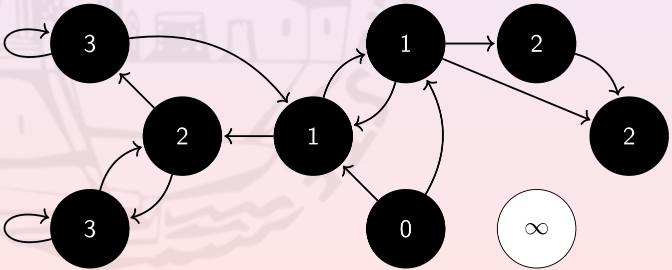

Breadth-First-Search (BFS): Pseudo-Code

```
def BFS_SET(v, color, d, pred):
    v.color ← color
    v.d ← d
    v.pred ← pred
enddef

def BFS_INIT(G,s):
    for v in G.V:
        BFS_SET(v, WHITE, ∞, NIL)
    endfor
    BFS_SET(s, GRAY, 0, s)

    return BUILD_QUEUE([s])
enddef
```

Breadth-First-Search (BFS): Pseudo-Code (Cont'd)

```
def BFS(G, s):
  Q ← BFS_INIT(G, s)
  while Q ≠ ∅:
    u ← DEQUEUE(Q)

    for v in G.Adj[u]:
      if v.color = WHITE:
        BFS_SET(v, GRAY, u.d+1, u)
        ENQUEUE(Q, v)
      endif
    endfor
    u.color ← BLACK
  endwhile

  return G.pred, G.d
enddef
```

Breadth-First-Search (BFS): Complexity

An iteration of the `while` extracts a u from Q and GREY colors it

The `for` loop costs $\Theta(|Adj[u]|)$ per `while` iteration

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`BFG` has asymptotic complexity $O(|V| + |E|)$

Breadth-First-Search (BFS): Code Properties

Lemma

Let $Q = [v_1, \ldots, v_n]$ be the queue during BFS. Then $v_i.d \leq v_{i+1}.d$ for all $i \in [1, n-1]$ and $v_n.d \leq v_1.d + 1$.

Theorem

Let $\delta(s, v)$ be the distance from s to v. After BFS:

- $v.d \neq \infty$ *iff* v *is reachable from* s
- *if* $v.d \neq \infty$, *then* $v.d = \delta(s, v)$
- *the shortest path from* s *to* v *ends with* $(v.pred, v)$

Depth-First Search

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DFS labels all the nodes with discovery time and finishing time

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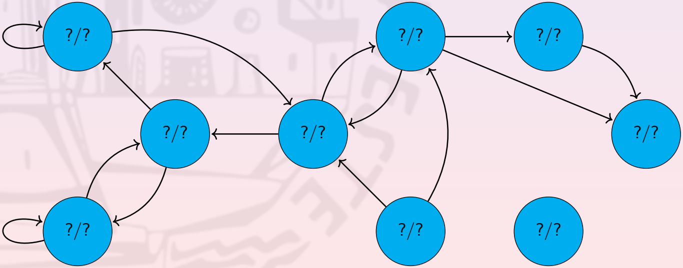

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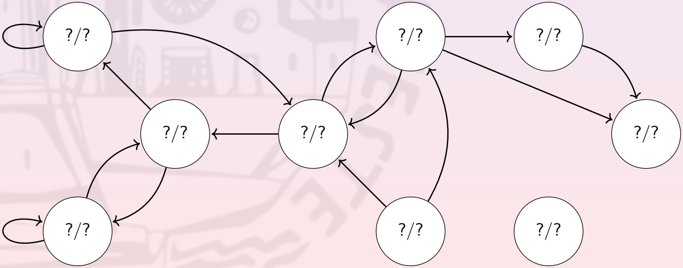

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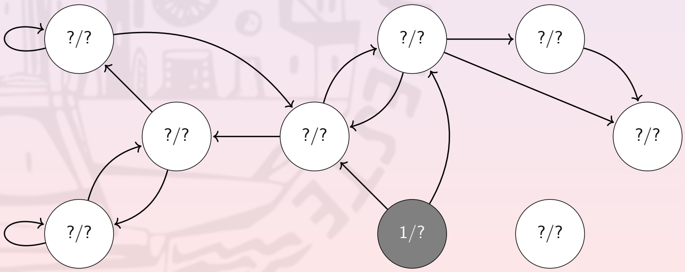

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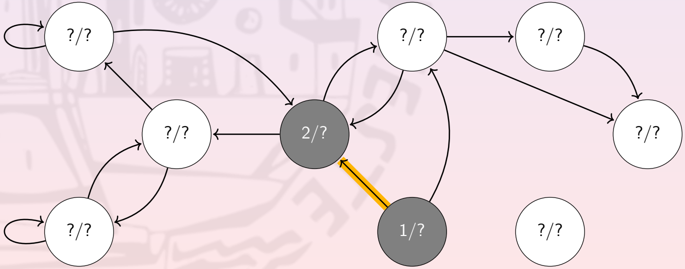

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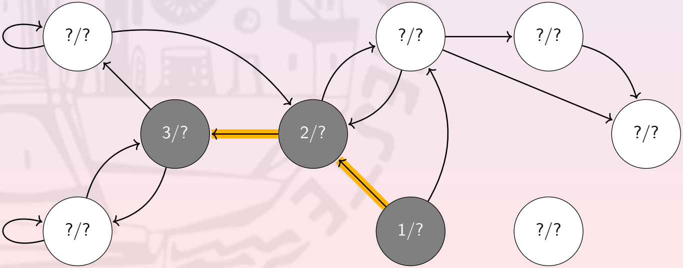

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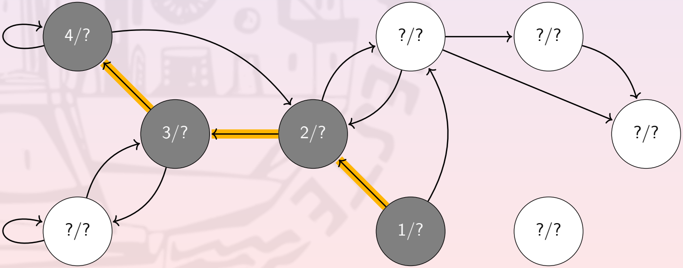

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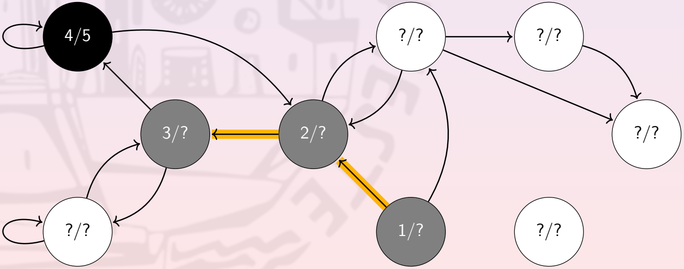

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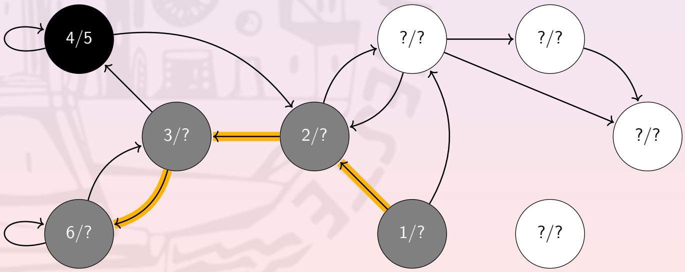

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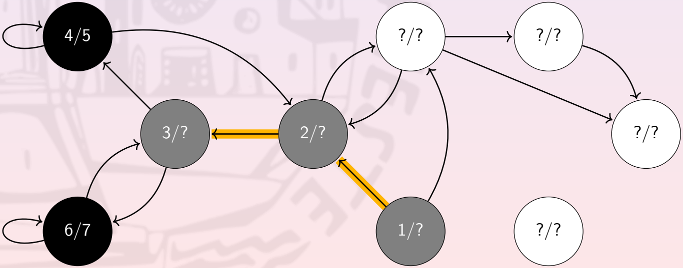

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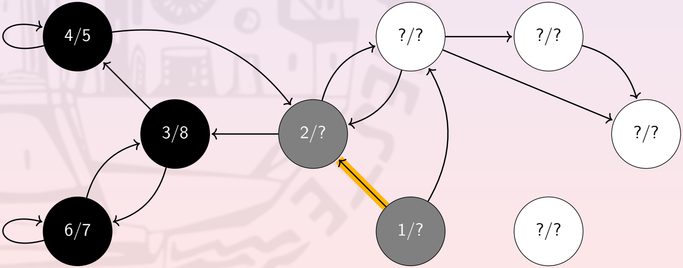

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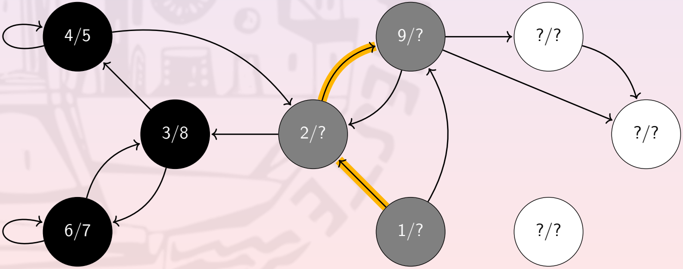

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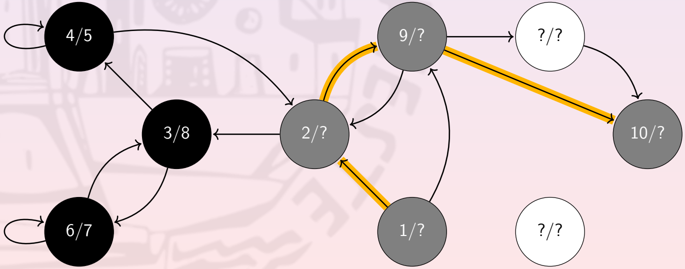

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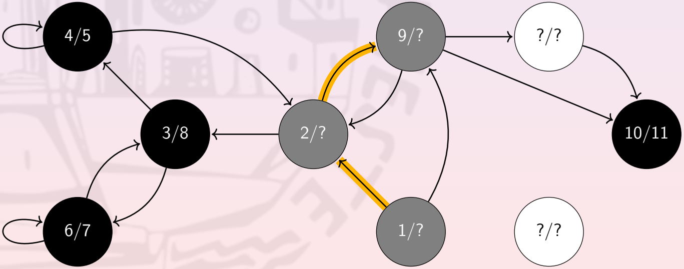

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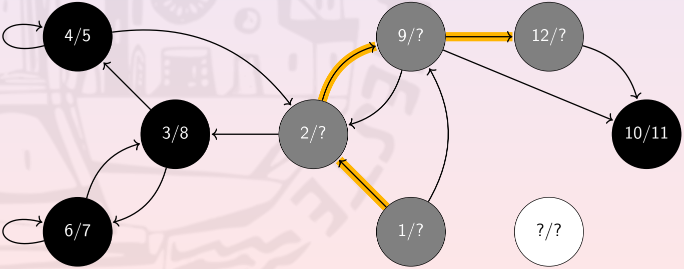

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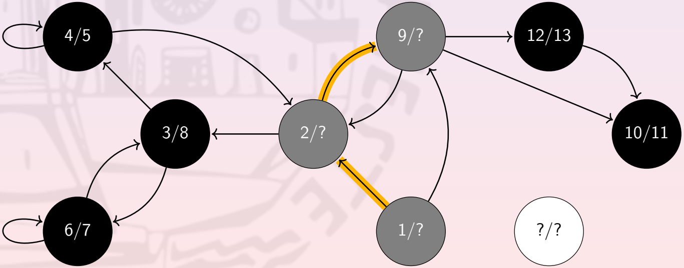

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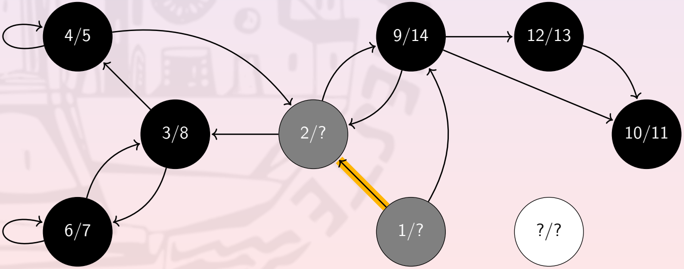

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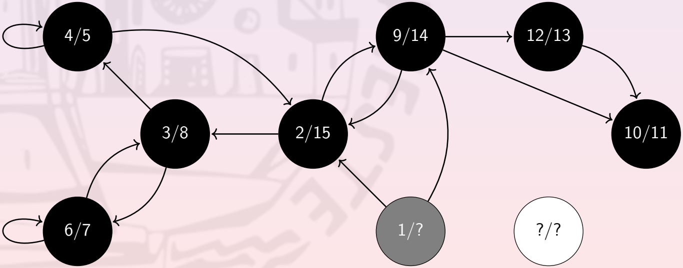

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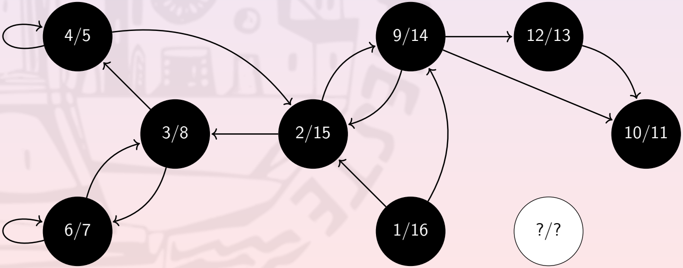

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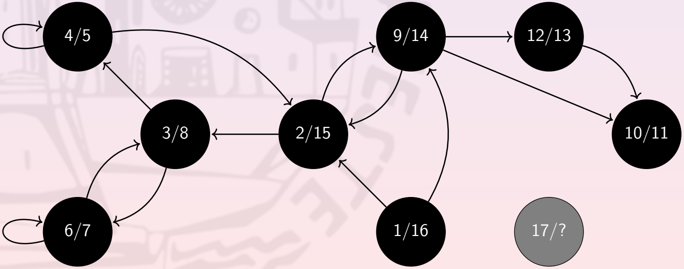

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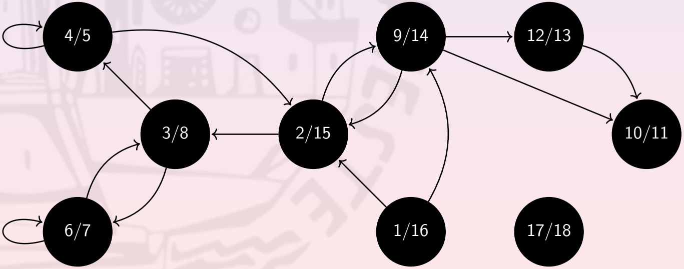

Depth-First-Search (DFS): Pseudo-Code

```
def DFS(G):
  for v in G.V:
    v.color ← WHITE
    v.pred ← NIL
  endfor

  time ← 0
  for v in G.V:
    if v.color = WHITE:
      time ← DFS_VISIT(G,v,time)
    endif
  endfor

  return G.pred, G.d, G.f
enddef
```

Depth-First-Search (DFS): Pseudo-Code (Cont'd 2)

```
def DFS_VISIT(G,v,time):
  # discovery
  time ← time + 1
  v.d ← time
  v.color ← GRAY

  # search for WHITE neighbors
  for u in G.Adj[v]:
    if u.color = WHITE:
      u.pred ← v
      time ← DFS_VISIT(G,v,time)
    endif
  endfor
```

Depth-First-Search (DFS): Pseudo-Code (Cont'd 3)

```
    # finalization
    time ← time + 1
    v.f ← time
    v.color = BLACK

    return time
enddef
```

Depth-First-Search (DFS): Complexity

Each `DFS_VISIT` call GRAY color the node parameter

The `for` loop costs $\Theta(|Adj[u]|)$ per `DFS_VISIT` call

Each iteration of the `for` calls `DFS_VISIT` on $v \in Adj[u]$ if only if it is WHITE

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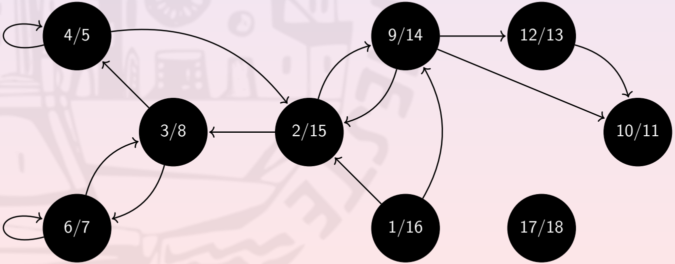
`DFS` has asymptotic complexity $\Theta(|V| + |E|)$

Depth-First-Search (DFS): Edge Classification



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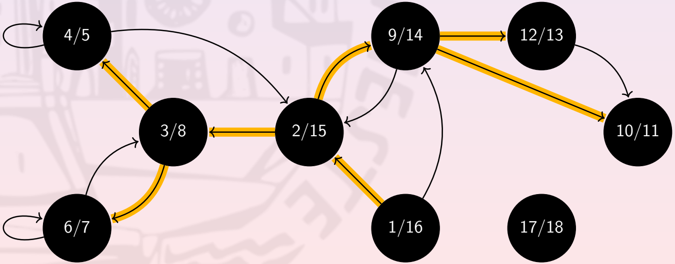
Tree Edges: belong to the depth first forest



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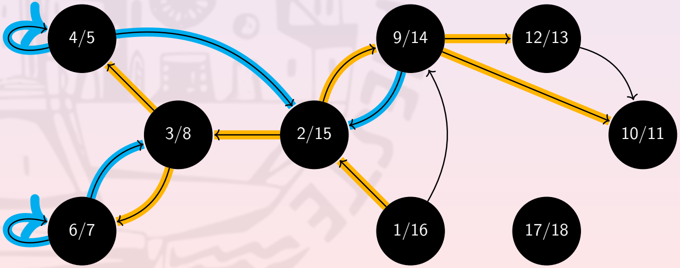

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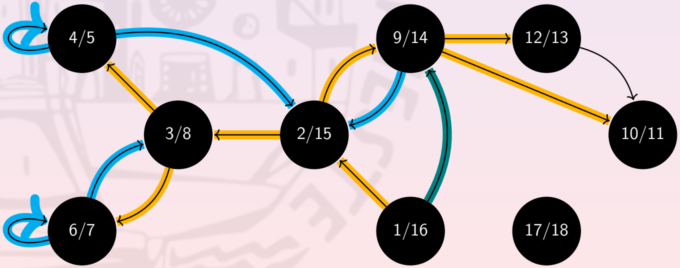

Depth-First-Search (DFS): Edge Classification

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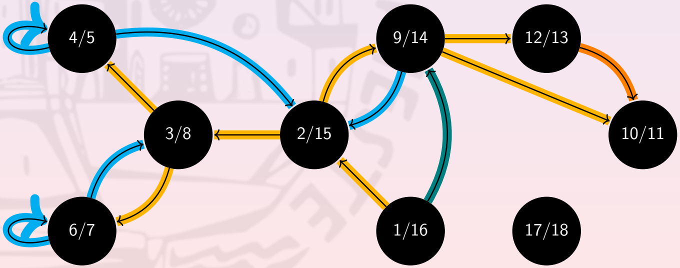
Cross Edges: the remaining edges



Depth-First-Search (DFS): Properties

Theorem (Parenthesis Theorem)

For any pair of nodes $v, u \in V$, either:

- $[u.d, u.f] \cap [v.d, v.f] = \emptyset$ *and neither* u *is a descendant of* v *nor* v *of* u
- $[u.d, u.f] \subsetneq [v.d, v.f]$ *and* u *is a descendant of* v
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Theorem (White-Path Theorem)

For any pair of nodes $v, u \in V$, u is a descendant of v iff at time $v.d - 1$ there exists a WHITE-only path from v to u.

Topological Sort

How to Prepare. . . Tiramisù

It is quite simple

We have to:

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We have to:

- beating the egg whites until still
- adding sugar
- preparing coffee
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Timing and sub-task ordering is fundamental

Installing Software

In a similar way, trying to install a software can be a “sorting” problem

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E.g., To install software A solve following dependencies

- Software A needs software B, C, D
- Software B depends on libraries E and F and on software G
- Software G depends on library E and on D
- ...

How to figure out what is needed and in which order?

Dependency Relations and Topological Sort

Dependency relations can be modeled by using directed graphs

Nodes represent sub-tasks, edges the needs, e.g., $(v, u) \in E$ iff $v=$"soaking cookies" needs $u=$"preparing coffee"

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On DAGs, there exists a topological order, $\preceq_T$, on nodes satisfying

$$\text{If } (u, v) \in E, \text{ then } u \preceq_T v$$

Computing Topological Sort and Complexity

```
def TOPOLOGICAL_SORT(G)
    call DFS(G)


enddef
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Asymptotic complexity $\Theta(|V| + |E|)$

Topological Sort: Correctness

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Let $[v_1, \ldots, v_n]$ be the output of TOPOLOGICAL_SORT(G). Then $v_i.f > v_{i+1}.f$ for all $i \in [1, n-1]$

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Theorem

If G is a DAG, then TOPOLOGICAL_SORT(G) produces G's topological sort

Strongly Connected Components

What Comes Before Egg or Hen?

When a graph is not a DAG, establishing a topological ordering is not possible

From the reachability point of view, all the nodes in a loop behave in the same way

If one of them is reachable from a node, so are all the others

If one of them can reach a node, so do all the others

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Discovering equivalent nodes is a useful task

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Are maximum sub-graphs (of a directed graph) s.t., for every pair of their nodes, there is a path from one to the other and vice versa

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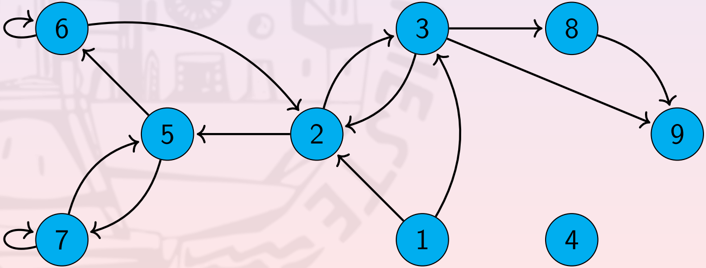

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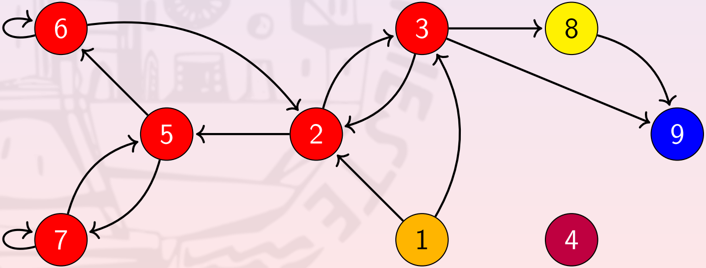

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Minimum Discovery Time from the Sub-Tree (lowlink): if it is smaller than the node discovery time, then a back edge must be reachable from it

One DFS_VISIT Call to Rule Them All

One DFS_VISIT call discovers all the nodes of the SCCs it "touches", i.e., no half visited SCC

DFS_VISITs do not interleave nodes of two distinct SCCs.

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So, if we have a way to:

- perform a DFS_VISIT call and update lowlinks when possible
- identify a SCC as soon as all its nodes have been visited
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then we have an algorithm to identify all the SCCs

Two Useful Intuitions

How are related the discovery time and the lowlink of the first node S_f in a SCC S to be visited?

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Which nodes in the visit belong to the just identified SCC if we do not consider the nodes of the already fully discovered SCCs?

The last ones!

We can use a stack to store finished nodes and use lowlink property to detect S_f

Tarjan's SCCs Algorithm: Pseudo-Code

```
def TARJAN_SCC(G):
  for v in G.V:
    v.color ← WHITE
  endfor

  time ← 0
  S ← BUILD_STACK()
  for v in G.V:
    if v.color = WHITE:
      time ← TARJAN_SCC_VISIT(G,v,S,time)
    endif
  endfor
enddef
```

Tarjan's SCCs Algorithm: Pseudo-Code (Cont'd)

```
def TARJAN_SCC_VISIT(G,v,S,time):
  time ← time + 1
  v.d ← time
  v.color = GRAY

  v.lowlink ← time
  S.push(v)
  v.onStack ← True

  for u in G.Adj[v]:
    if u.color = WHITE:
      time ← TARJAN_SCC_VISIT(G,u,S,time)
```

Tarjan's SCCs Algorithm: Pseudo-Code (Cont'd 2)

```
        v.lowlink ← min(v.lowlink, u.lowlink)
      else if u.onStack:
        v.lowlink ← min(v.lowlink, u.d)
      endif
    endfor

    if v.lowlink = v.d:
      yield EXTRACT_SCC_FROM_STACK(S, v)
    endif

    return time
enddef
```

Tarjan's SCCs Algorithm: Pseudo-Code (Cont'd 3)

```
def EXTRACT_SCC_FROM_STACK(S, v):
  L ← EMPTY_LIST()

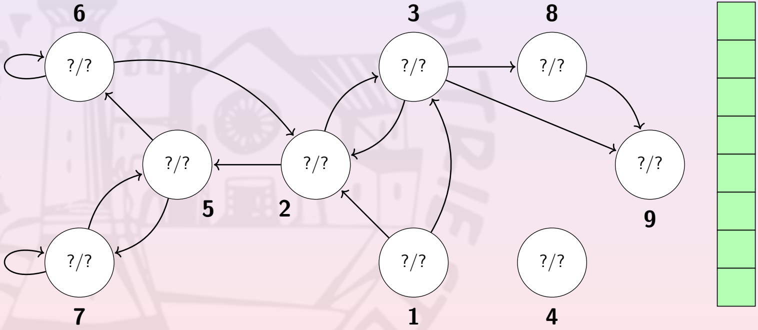
  repeat:
    w ← S.pop()
    w.onStack ← False

    L.append(w)
  until v ≠ w

  return L
enddef
```

Tarjan's SCCs Algorithm: Example

Nodes are labeled by "Discovery Time" / "Lowlink"



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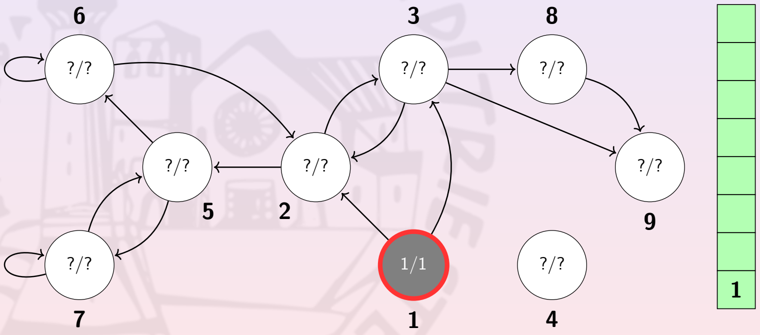

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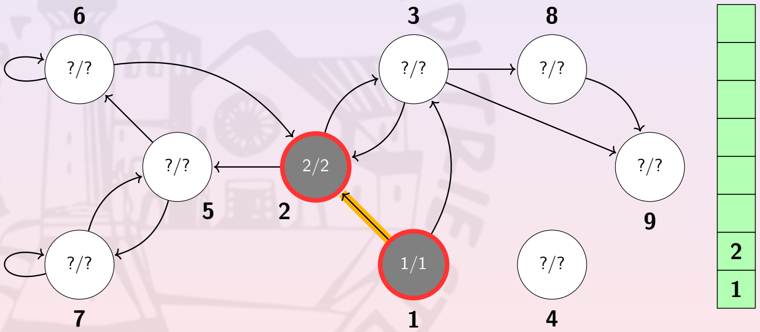

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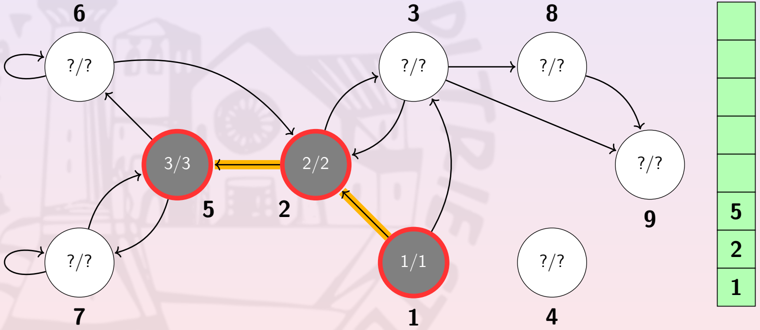

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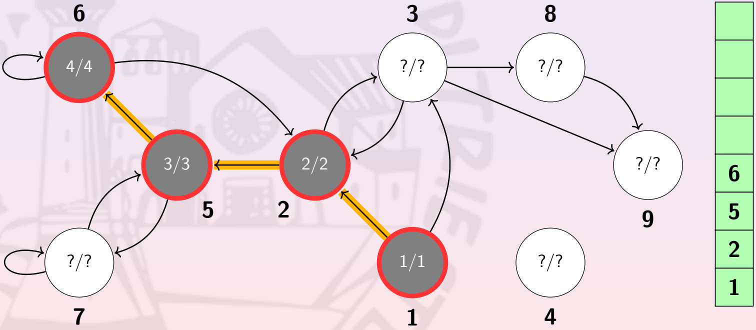

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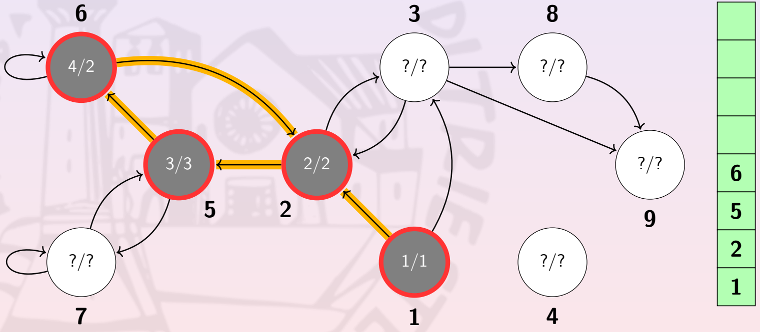

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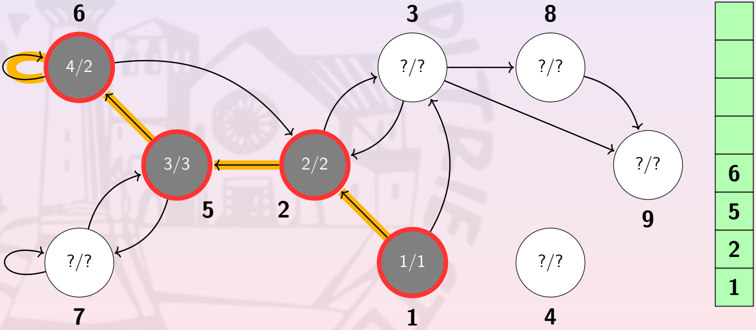

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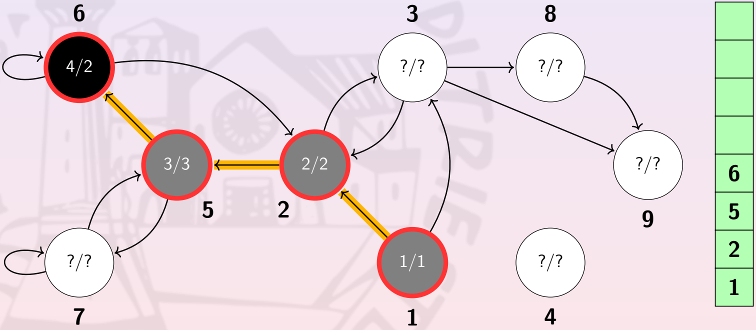

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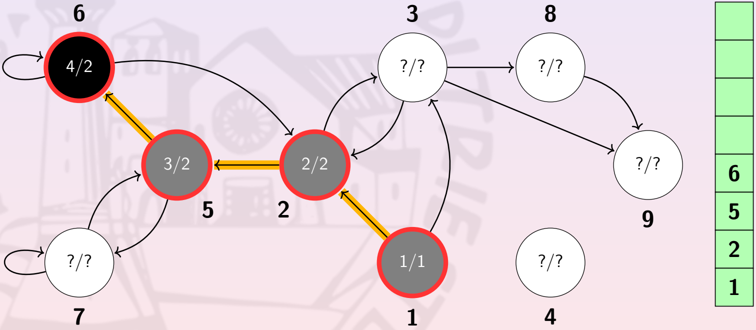

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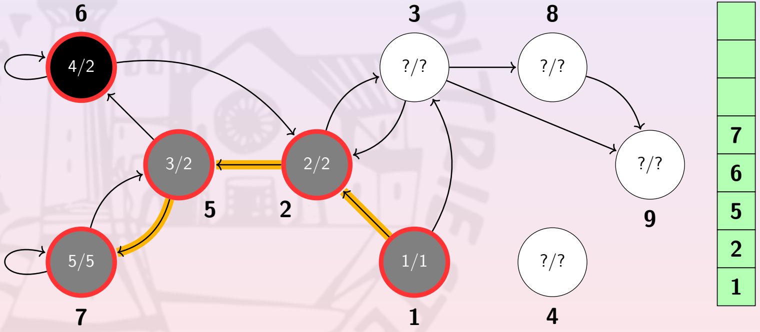

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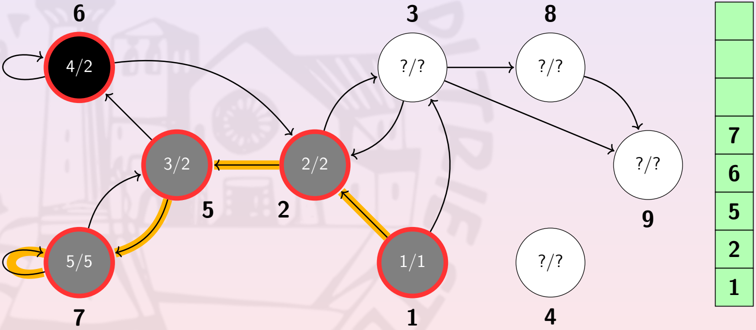

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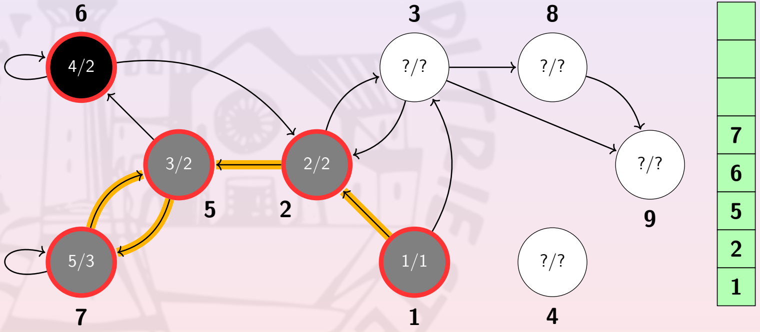

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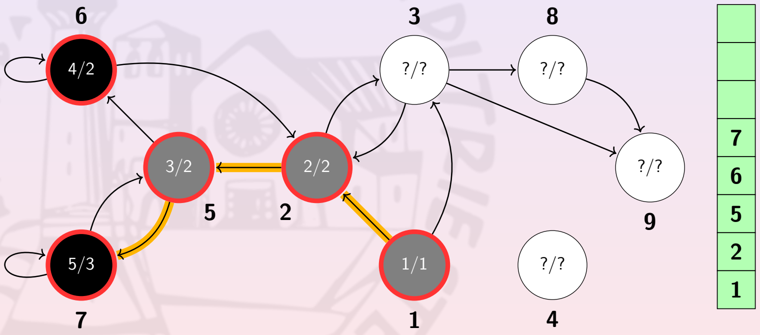

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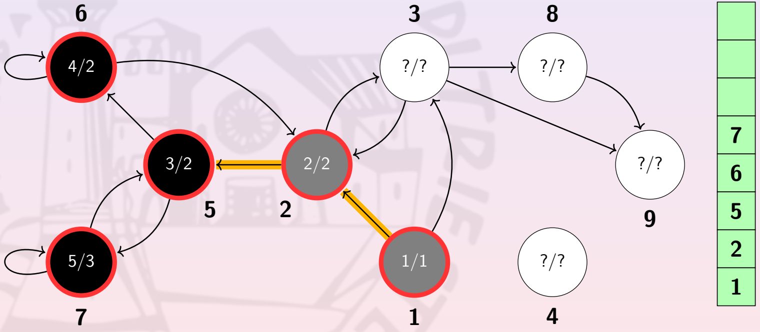

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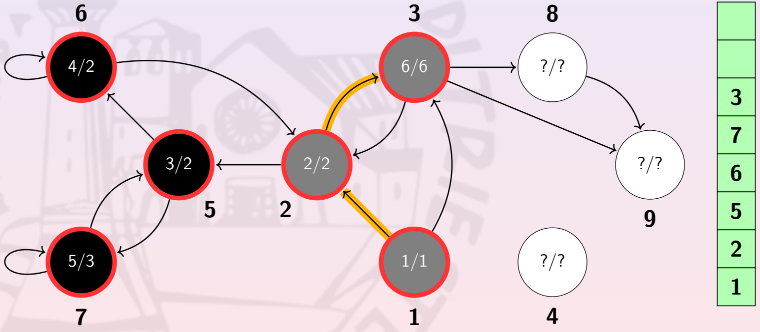

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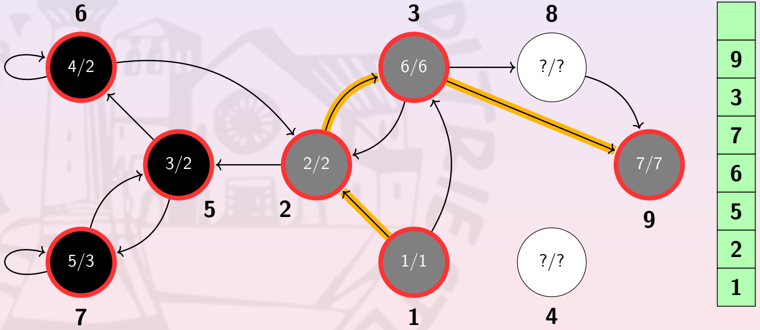

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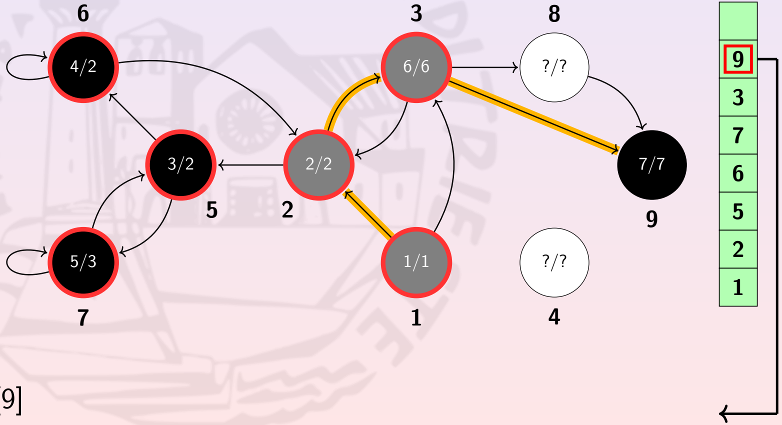

[9]

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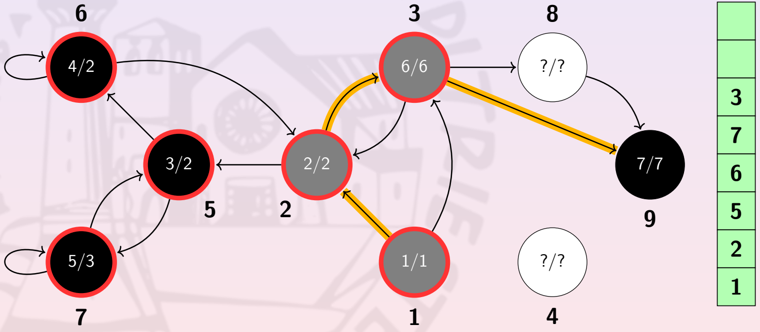

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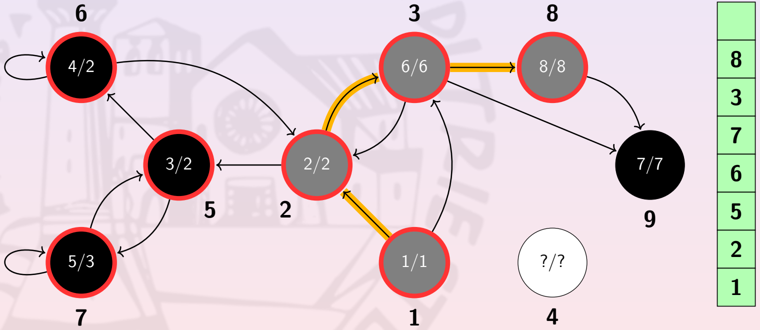

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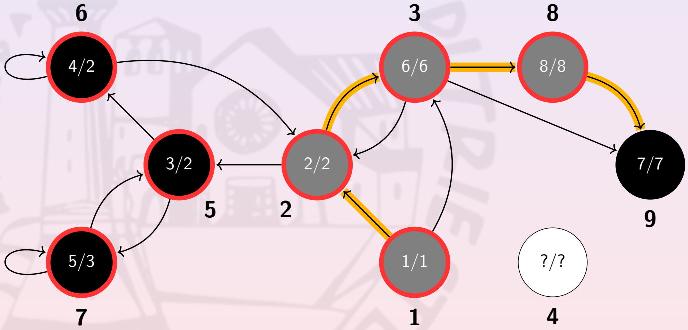

8
3
7
6
5
2
1

[9], [8]

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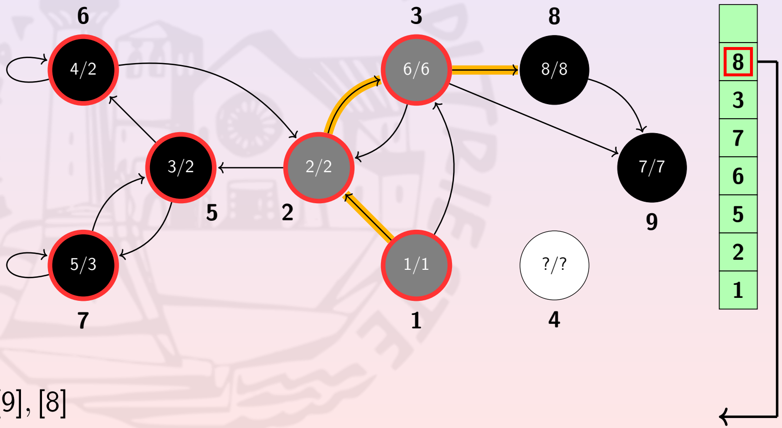

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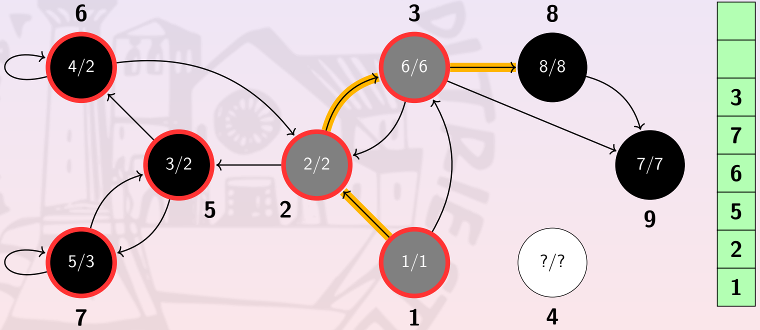

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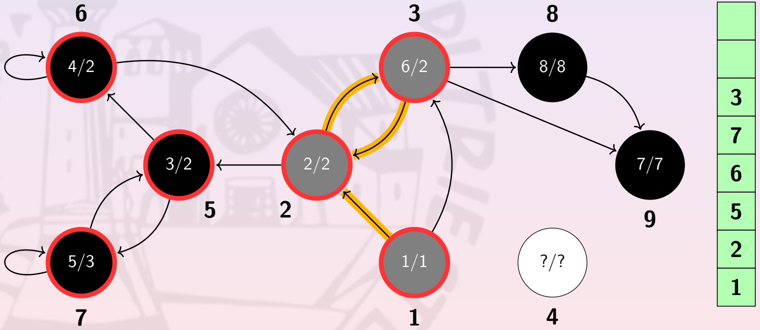

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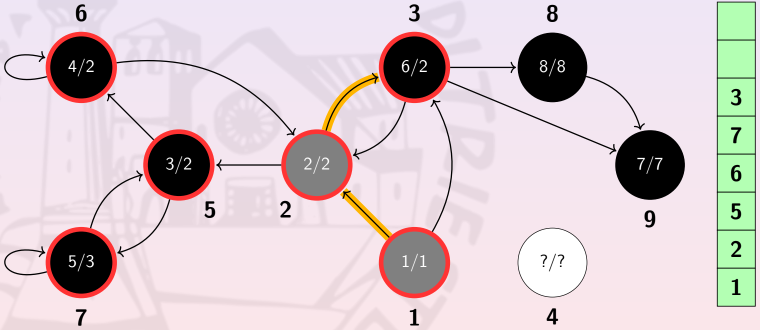

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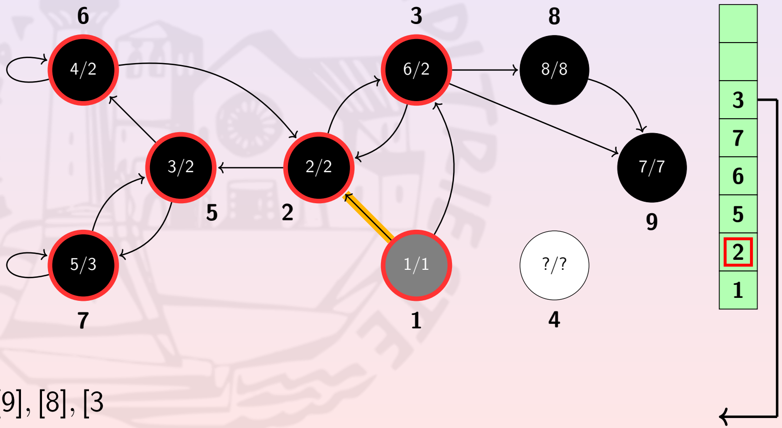

[9], [8], [3

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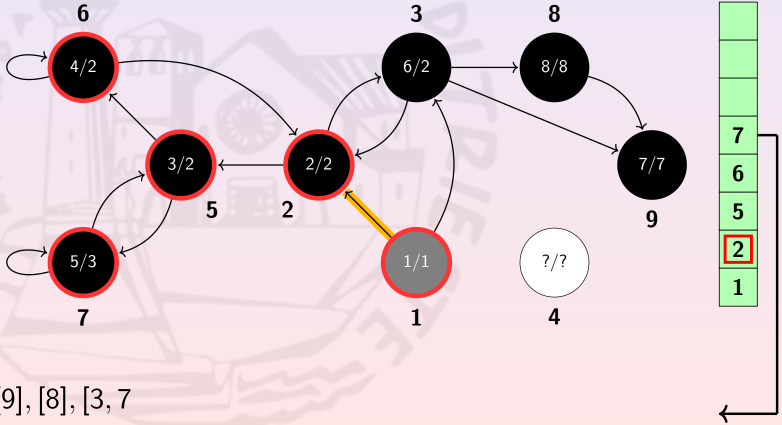

[9], [8], [3, 7

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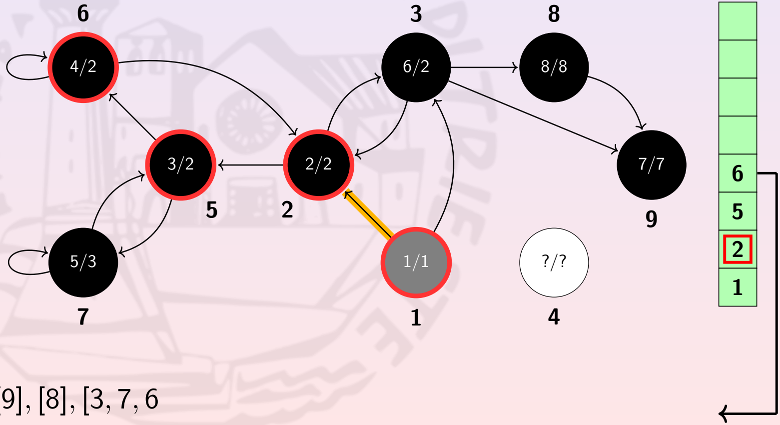

[9], [8], [3, 7, 6

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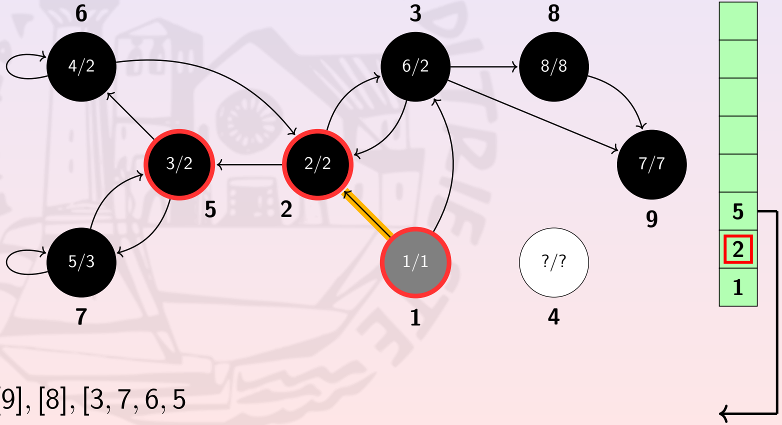

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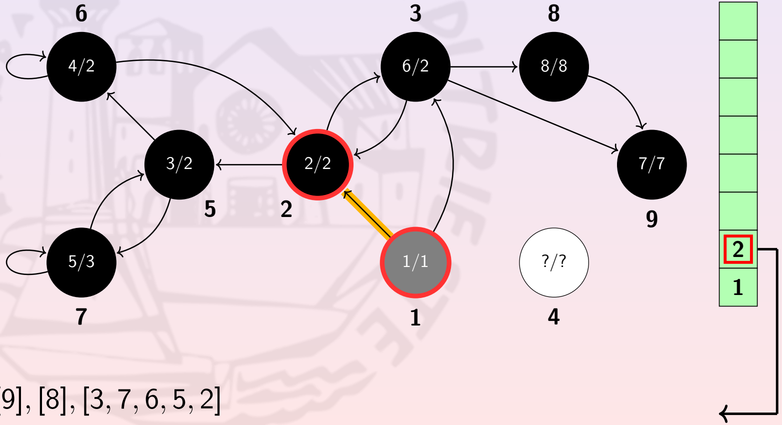

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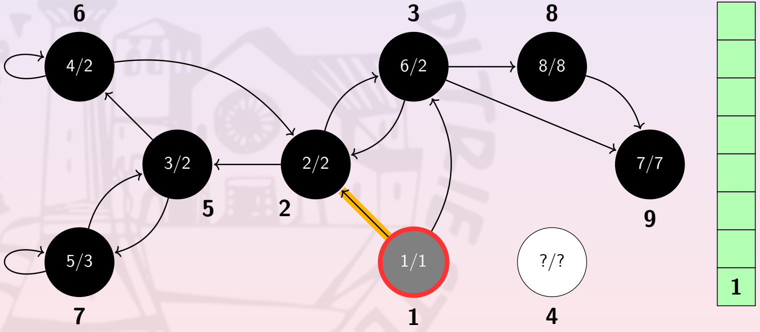

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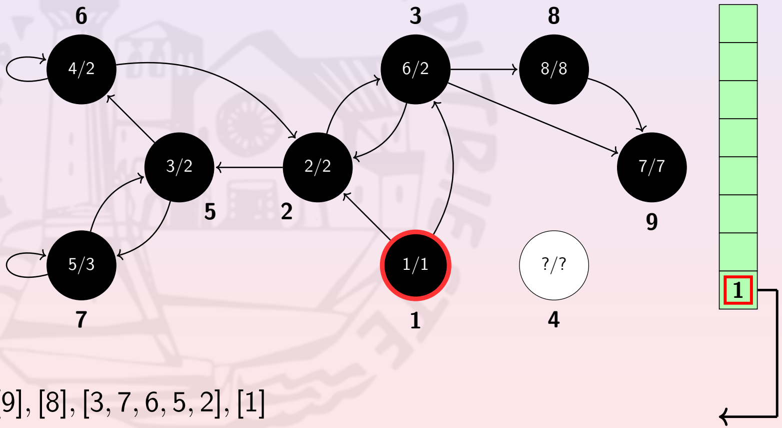

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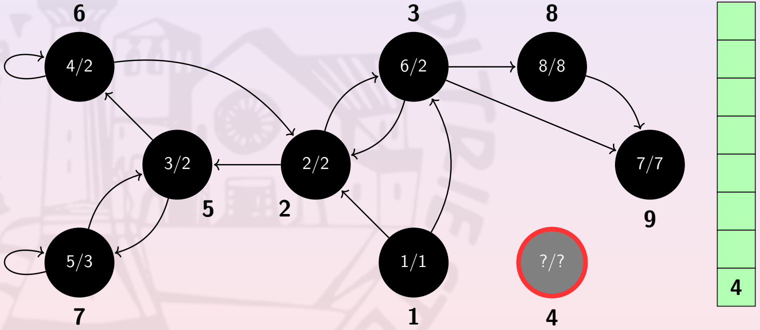

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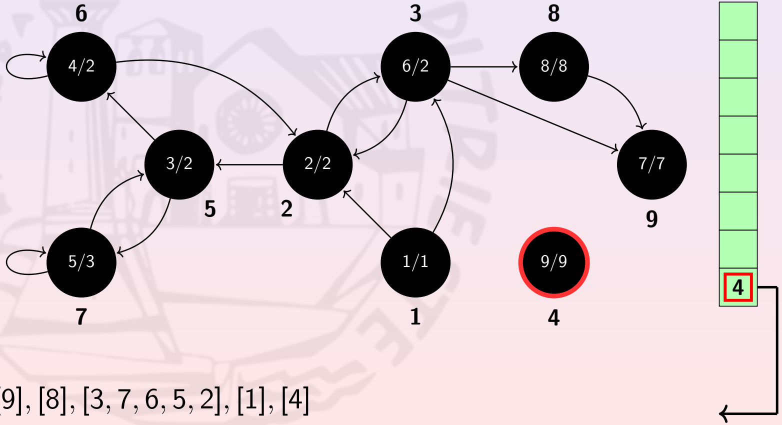

[9], [8], [3, 7, 6, 5, 2], [1], [4]

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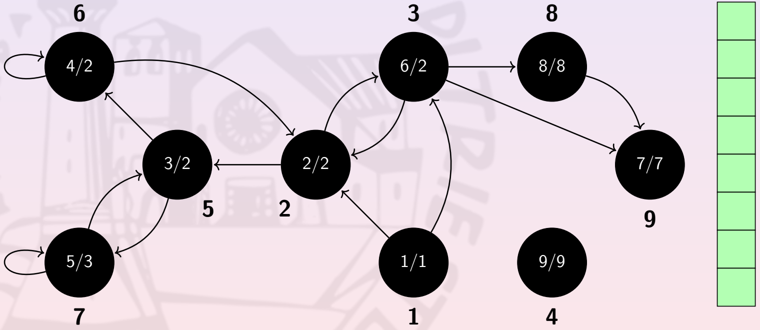

[9], [8], [3, 7, 6, 5, 2], [1], [4]

Tarjan's SCCs Algorithm: Complexity

The algorithm performs a DFS-like visit + stack handling

One single node is pushed in S during each `TARJAN_SCC_VISIT` call

`TARJAN_SCC_VISIT` is called on WHITE nodes and sets them to GRAY

So, the # of node inserted into S at some point is $|V|$

All `EXTRACT_SCC_FROM_STACK` calls cumulatively cost $\Theta(|V|)$

Tarjan's algorithm costs $\Theta(|V| + |E|)$

Transitive Closure

Transitive Closure: Definition and Naïve Solution

For each pair of nodes v and w, we would like to know whether there is a path from v to w

How to solve this problem?

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Naïve solution: evaluate BFS from all the graph nodes

Complexity? $|V| * O(|V| + |E|) = O(|V|^2 + |V| * |E|)$

Any other ideas?

Have a Look at the Matrix Formulation of the Problem

	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	0	0	1	0
4	0	$\boxed{1}$	0	1

- w has distance 1 from v iff $A[v][w] \neq 0$

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- w has distance 1 from v iff $A[v][w] \neq 0$
- w has distance 2 from v iff there exists z s.t. $A[v][z] \neq 0$ and $A[z][w] \neq 0$

Have a Look at the Matrix Formulation of the Problem

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 \\
4 & 0 & \boxed{1} & 0 & 1
\end{array}
\quad \times \quad
\begin{array}{c|cccc}
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- w has distance 1 from v iff $A[v][w] \neq 0$
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Have a Look at the Matrix Formulation of the Problem

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\hline
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- w has distance $\leq n$ from v iff $A^n[v][w] > 0$

Transitive Closure By Matrix Multiplication

Every acyclic path has at most length $|V|$

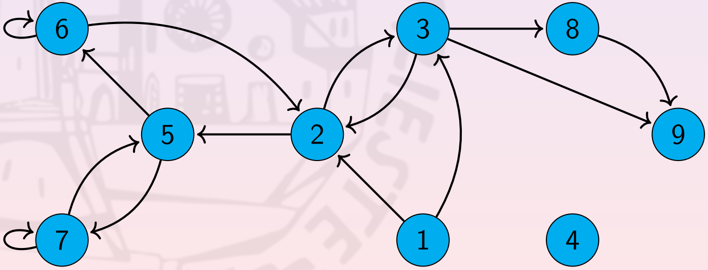
Wen can solve the problem by using Strassen's algorithm:
$(|V|-1) * \Theta(|V|^{\log_2 7})$

Which is worst than using BFS!!!

SCCs Are Unnecessarily Cumbersome

A.f.a. reachability concerns, all the nodes in a SCC are the same

So, collapse the nodes in SCCs and build the SCCs graph $\bar{G}$



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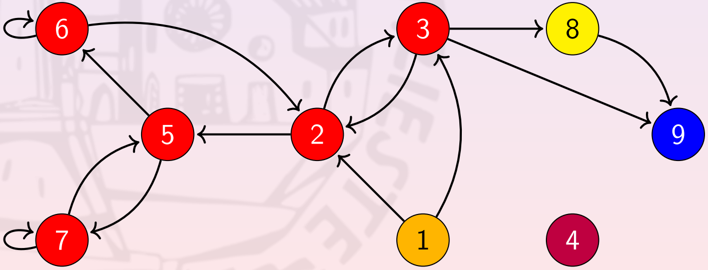

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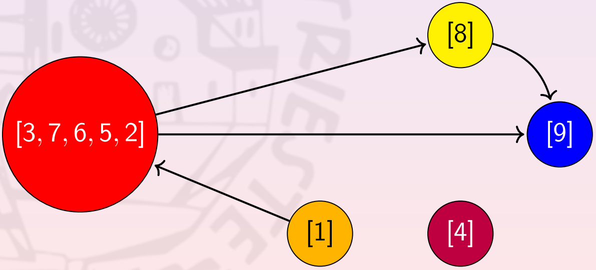

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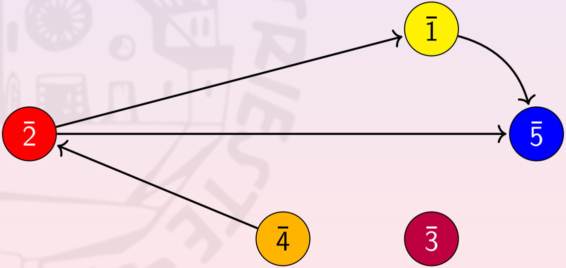

And Then What?

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Topological Sort!!!

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Why?

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Topological Sort!!!

Why? Have a look at the adjacency matrix of $\bar{G}$

	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{1}$	1	0	0	0	1
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$\Longrightarrow$

	$\bar{3}$	$\bar{4}$	$\bar{2}$	$\bar{1}$	$\bar{5}$
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$\bar{5}$	0	0	0	0	1

The new adjacency matrix is upper-triangular

And Then What?

Topological Sort!!! Tarjan's algorithm already did it!

Why? Have a look at the adjacency matrix of $\bar{G}$

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Why Upper Triangular Matrixes Are Nice?

$$\bar{G} = \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)$$

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Why Upper Triangular Matrixes Are Nice?

$$\bar{G} = \left(\begin{array}{c|c} \color{red}{\mathbf{A}} & \color{gray}{\mathbf{C}} \\ \hline 0 & \color{blue}{\mathbf{B}} \end{array}\right)$$

Why Upper Triangular Matrixes Are Nice?

$$\bar{G} = \left(\begin{array}{c|c} \mathbf{A} & \mathsf{C} \\ \hline 0 & \mathbf{B} \end{array}\right)$$

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$$\bar{G}^* = \left(\begin{array}{c|c} A^* & A^* \times C \times B^* \\ \hline 0 & B^* \end{array}\right)$$

The Complete Algorithm

- Tarjan's SCCs Algorithm on the input G: $\Theta(|V|+|E|)$
- Build the SCCs Graph $\bar{G}$ of G: $\Theta(|E|)$
- Topological sort of $\bar{G}$: $\Theta(|V|+|E|)$
- Compute $\bar{G}^*$ in time:

$$T(n) = 2 * T(n/2) + 2 * \Theta(n^{\log_2 7})$$

- Extend the transitive closure of $\bar{G}$ to G: $O(|V|^2)$

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The overall asymptotic complexity is $\Theta(|E| + |V|^{\log_2 7})$