Modelling with binary variables II

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Use of binary variables (I)

Packing constraint

$$\sum_j x_j \leq 1$$

At most one of a set of events is allowed to occur.

Cover constraint

$$\sum_j x_j \geq 1$$

At least one of a set of events is allowed to occur.

Partitioning constraint

$$\sum_j x_j = 1$$

Exactly one of a set of events is allowed to occur.

Use of binary variables (II)

- Neither or both events 1 and 2 must occur

$$x_2 - x_1 = 0, \text{ i.e., } x_2 = x_1,$$

- Event 2 can occur only if event 1 occurs

$$x_2 - x_1 \leq 0, \text{ i.e., } x_2 \leq x_1,$$

where

$$x_i = \begin{cases} 1 & \text{if event } i \text{ occurs} \\ 0 & \text{if event } i \text{ does not occur} \end{cases}$$

for $i = 1, 2$.

Use of binary variables (III)

- Consider an activity that can be operated at any level y from 0 to u, i.e., $0 \leq y \leq u$.
- The activity can be undertaken only if some event represented by the binary variable x occurs.

$$y - ux \leq 0, \text{ i.e., } y \leq ux,$$

where $x \in \{0, 1\}$ and $y \geq 0$.

$x = 0$ implies $y = 0$

$x = 1$ provides the original constraint $0 \leq y \leq u$.

The Facility Location Problem (FLP)

- We are given a set $N = \{1, \ldots, n\}$ of potential facility locations and a set of clients $I = \{1, \ldots, m\}$.
- A facility placed at j costs c_j for $j \in N$.
- Each client has a demand for a certain good.
- Cost of satisfying the demand of client i from a facility at j is h_{ij}.

The optimisation problem is to choose a subset of the locations at which to place facilities and then assign the clients to these facilities as to minimise the total cost.

Uncapacitated FLP

There is no restriction on the number of clients that a facility can serve.

Decision variables

$$x_j = \begin{cases} 1 & \text{if a facility is placed at } j \\ 0 & \text{otherwise} \end{cases}$$

$y_{ij} \in \mathbb{R}_+^{mn}$ is the fraction of the demand of client i that is satisfied from a facility at j.

Uncapacitated FLP

Constraints

- Each client's demand must be satisfied

$$\sum_{j \in N} y_{ij} = 1 \text{ for } i \in I.$$

- Client i cannot be served from j unless a facility is placed at j

$$y_{ij} - x_j \leq 0 \text{ for } i \in I \text{ and } j \in N.$$

Objective function

$$\min \sum_{j \in N} c_j x_j + \sum_{i \in I} \sum_{j \in N} h_{ij} y_{ij}$$

Capacitated FLP

It may be unrealistic to assume that a facility can serve any number of clients.

- Let u_j be the capacity of the facility located at j.
- Let b_i be the demand of the ith client.
- Let y_{ij} be the quantity of goods sent from facility j to client i
- Let h_{ij} be the shipping cost per unit

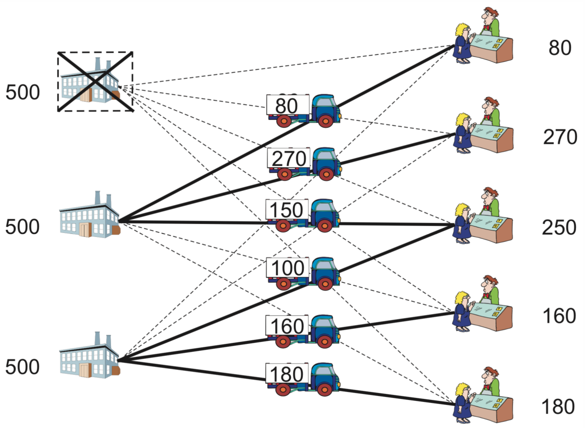
$$
\begin{aligned}
\min \sum_{j \in N} c_j x_j + \sum_{i \in I} \sum_{j \in N} h_{ij} y_{ij} & \\
\sum_{j \in N} y_{ij} = b_i & \qquad \text{for } i \in I \\
\sum_{i \in I} y_{ij} - u_j x_j \leq 0 & \qquad \text{for } j \in N \\
x_j \in \{0, 1\}, y_{ij} \geq 0 & \qquad \text{for } i \in I \text{ and } j \in N
\end{aligned}
$$

Capacitated FLP - Exercise

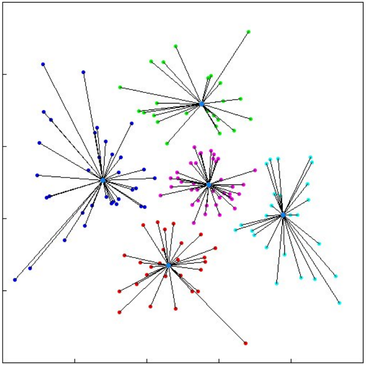
Consider a company with three potential sites for installing its facilities/warehouses and five demand points. Each site j has a yearly activation cost f_j, i.e., an annual leasing expense that is incurred for using it, independently of the volume it services. This volume is limited to a given maximum amount that may be handled yearly, M_j. Additionally, there is a transportation cost c_{ij} per unit serviced from facility j to the demand point i.

Customer i	1	2	3	4	5		
Annual demand d_i	80	270	250	160	180		
Facility j			c_{ij}			f_j	M_j
1	4	5	6	8	10	1000	500
2	6	4	3	5	8	1000	500
3	9	7	4	3	4	1000	500

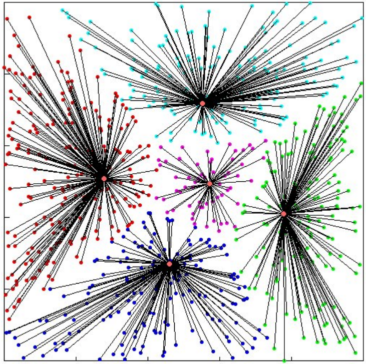
Capacitated FLP - Exercise



FLP - example



(a) 150 clients



(b) 750 clients

The Network Flow Problem (NFP)

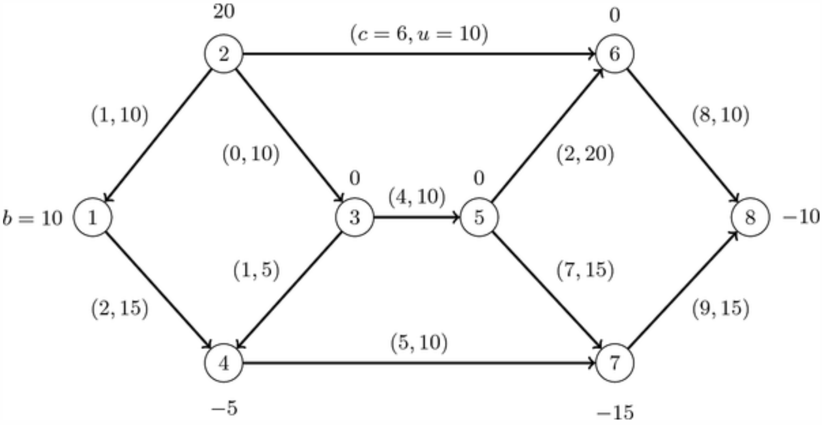
- A network is composed of a set on nodes $\boldsymbol{V}$ (e.g., facilities) and a set of arcs $\boldsymbol{\mathcal{A}}$.
- An arc $\boldsymbol{e = (i, j)}$ that points from node $\boldsymbol{i}$ to node $\boldsymbol{j}$ means that there is a direct shipping route (i.e., a flow) from node $\boldsymbol{i}$ to node $\boldsymbol{j}$.
- Associated with each node $\boldsymbol{i}$, there is a demand $\boldsymbol{b_i}$.

  $\boldsymbol{b_i > 0}$ supply node
  $\boldsymbol{b_i < 0}$ demand node
  $\boldsymbol{b_i = 0}$ transit node

  We assume the net demand is zero, i.e., $\sum_{i \in V} \boldsymbol{b_i = 0}$.
- Each arc $\boldsymbol{(i, j)}$ has

  A flow capacity $\boldsymbol{u_{ij}}$
  A unit flow cost $\boldsymbol{c_{ij}}$

NFP - Example


20
0
$(c = 6, u = 10)$
2
6
$(1, 10)$
$(8, 10)$
$(0, 10)$
$(2, 20)$
0
0
$(4, 10)$
$b = 10$
1
3
5
8
-10
$(1, 5)$
$(7, 15)$
$(2, 15)$
$(9, 15)$
$(5, 10)$
4
7
-5
-15

The Network Flow Problem (NFP)

If $\boldsymbol{y_{ij}}$ is the flow on arc $(\boldsymbol{i}, \boldsymbol{j})$, the NFP is formulated as

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij} y_{ij} \tag{1}$$

$$y_{ij} \leq u_{ij} \qquad \text{for } (i,j) \in \mathcal{A} \tag{2}$$

$$\sum_{j\in V} y_{ij} - \sum_{j\in V} y_{ji} = b_i \qquad \text{for } i \in V \tag{3}$$

$$y \in \mathbb{R}_+^{|\mathcal{A}|} \tag{4}$$

Constraints (2) are the capacity constraints.

Constraints (3) are the flow conservation constraints.

The Fixed-Charged NFP

A fixed cost h_{ij} is imposed if there is a positive flow on arc (i, j).
A binary variable x_{ij} indicates whether arc (i, j) is used.

$$\min \sum_{(i,j)\in\mathcal{A}} (h_{ij}x_{ij} + c_{ij}y_{ij}) \tag{5}$$

$$y_{ij} - u_{ij}x_{ij} \le 0 \qquad \text{for } (i,j) \in \mathcal{A} \tag{6}$$

$$\sum_{j\in V} y_{ij} - \sum_{j\in V} y_{ji} = b_i \qquad \text{for } i \in V \tag{7}$$

$$x \in \{0,1\}^{|\mathcal{A}|}, y \in \mathbb{R}_+^{|\mathcal{A}|} \tag{8}$$

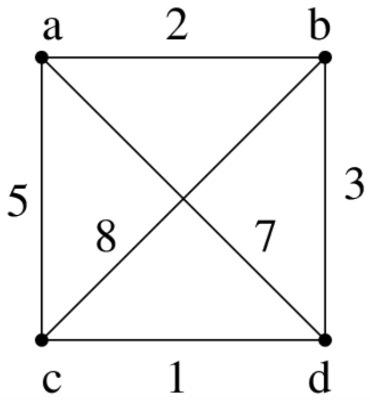
The Travelling Salesman Problem

- We are given a set on nodes $V = \{1, \ldots, n\}$ (e.g., cities) and a set of arcs $\mathcal{A}$.
- Arcs represent ordered pairs of cities between which direct travel is possible.
- For $(i, j) \in \mathcal{A}$, c_{ij} is the direct travel time from city i to city j.
- The TSP aims at finding a tour, starting at city 1, that
  a) visits each other city exactly once and then returns to city 1
  b) takes the least total travel time

The Travelling Salesman Problem (TSP)

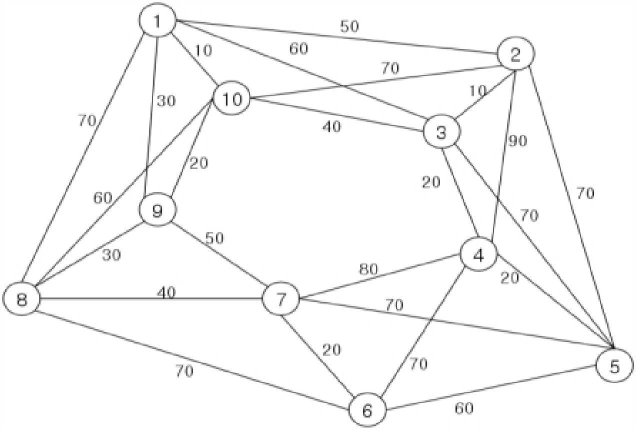
A tour that visits all nodes exactly once is called Hamiltonian tour. The TSP identifies the Hamiltonian tour of minimum cost.

TSP – seems easy

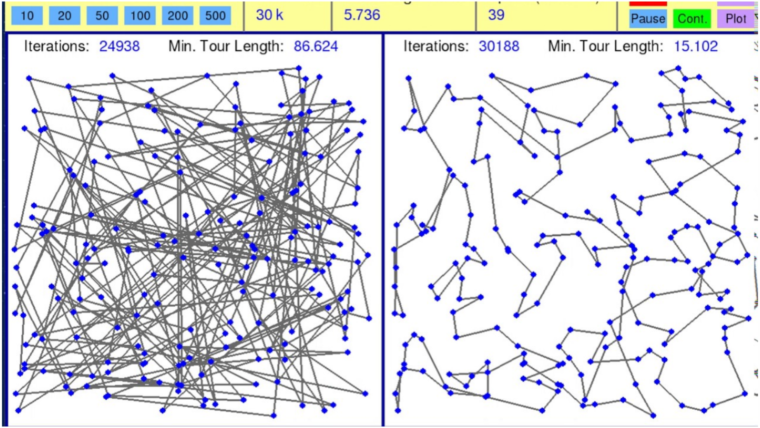


Three tours: A–B–D–C–A: 11; A-D-B-C-A: 23; A-D-C-B-A: 18.

TSP – maybe not too easy



TSP – it's difficult!!



TSP - Formulation

- Decision Variables

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ immediately follows } i \text{ on the tour} \\ 0 & \text{otherwise} \end{cases}$$

  Hence $x \in \{0, 1\}^{|\mathcal{A}|}$

- Objective function

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}$$

TSP - Constraint formulation

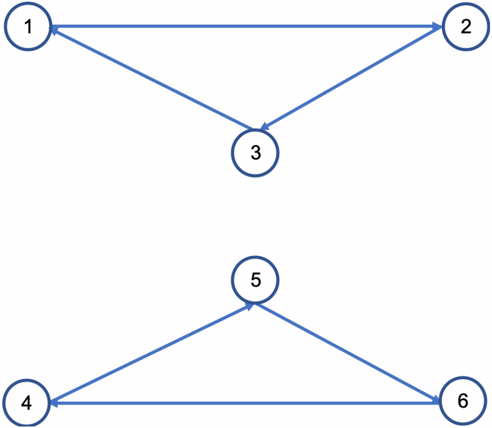
Each city is entered and left exactly once

$$\sum_{i:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } j \in V \tag{9}$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } i \in V \tag{10}$$

However, constraints (9) and (10) are not sufficient to define tours since they are also satisfied by subtours.

TSP - Subtours



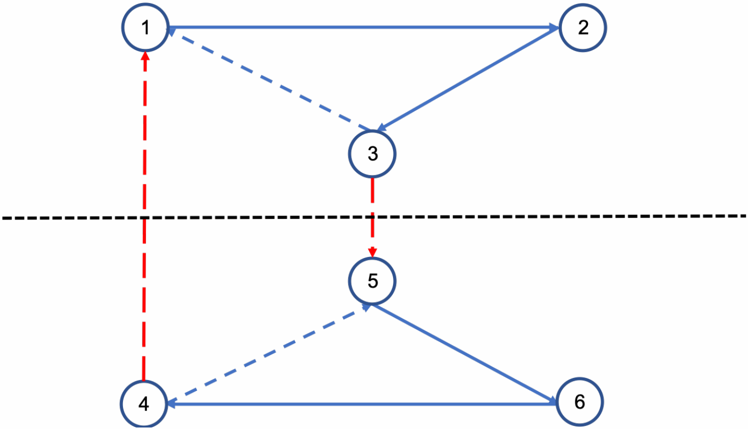
TSP - Subtour elimination (i)

In any tour there must be an arc that goes from $\{1, 2, 3\}$ to $\{4, 5, 6\}$ and an arc that goes from $\{4, 5, 6\}$ to $\{1, 2, 3\}$. In general, for any $U \subset V$ with $2 \leq |U| \leq |V| - 2$, constraints

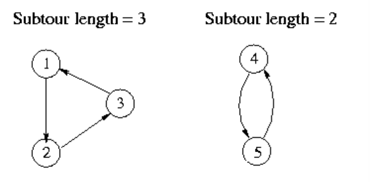
$$\sum_{\{(i,j)\in\mathcal{A}:i\in U,j\in V\setminus U\}} x_{ij} \geq 1 \qquad (11)$$

are satisfied by all tours, but every subtour violates at least one of them.

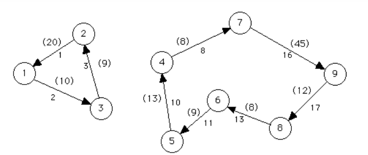
TSP - Subtour elimination



TSP - Subtours

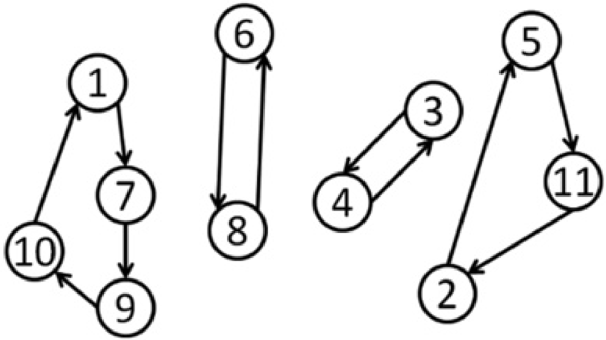


(c) 5 nodes

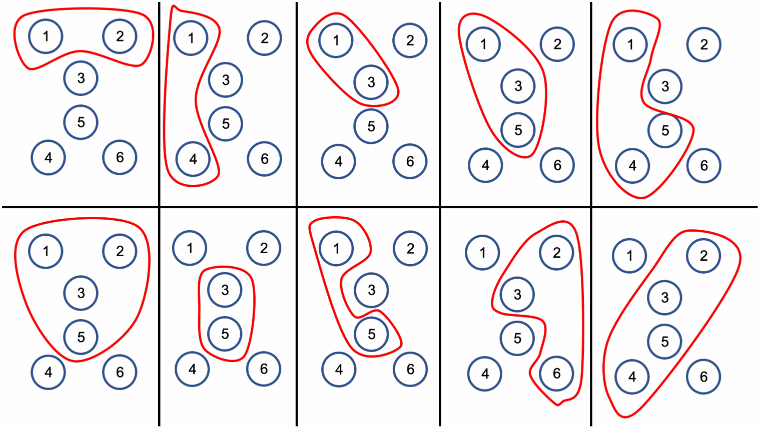


(d) 9 nodes

TSP - Subtours



TSP – Too many ways to choose U



TSP - Subtour elimination (ii)

An alternative way to eliminate subtours is to introduce constraints

$$\sum_{\{(i,j)\in\mathcal{A}:i\in U,j\in U\}} x_{ij} \leq |U| - 1 \;\forall U \subset V : 2 \leq |U| \leq |V| - 2 \tag{12}$$

But again we need a constraint for each $U \subset V$ such that $2 \leq |U| \leq |V| - 2$.

In both (11) and (12) the number of constraints is nearly $2^{|V|}$!!!

$$\frac{1}{2}\left[\binom{|V|}{2} + \binom{|V|}{3} + \cdots + \binom{|V|}{|V|-2}\right]$$

TSP - Formulation

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

$$\sum_{i:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } j \in V$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } i \in V$$

$$\sum_{\{(i,j)\in\mathcal{A}: i\in U, j\in V\setminus U\}} x_{ij} \geq 1 \qquad \forall U \subset V : 2 \leq |U| \leq |V| - 2$$

OR

$$\sum_{\{(i,j)\in\mathcal{A}: i\in U, j\in U\}} x_{ij} \leq |U| - 1 \qquad \forall U \subset V : 2 \leq |U| \leq |V| - 2$$

$$x \in \{0,1\}^{|\mathcal{A}|}$$