Formulations

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Definitions I

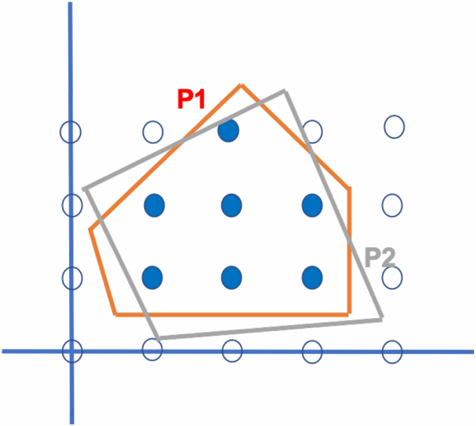
Polyhedron

A subset of $\mathbb{R}^n$ described by a finite set of linear constraints $\boldsymbol{P} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ is a polyhedron.

Formulation

A polyhedron $\boldsymbol{P} \subseteq \mathbb{R}^n$ is a formulation for a set $\boldsymbol{X} \subseteq \mathbb{Z}^n$ if and only if $\boldsymbol{X} = \boldsymbol{P} \cap \mathbb{Z}^n$.

Formulations - Example



Set $\boldsymbol{X}$ composed of BLUE points. P1 and P2 are two different formulations for $\boldsymbol{X}$.

Equivalent formulations- Knapsack

Consider the set of points

$$X = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(0,1,0,1),(0,0,1,1)\}$$

The three formulations below are formulations for X.

$$\begin{aligned}
P_1 &= \{x \in \mathbb{R}^4 : 0 \le x \le 1, && 83x_1 + 61x_2 + 49x_3 && +20x_4 \le 100 && \} \\
P_2 &= \{x \in \mathbb{R}^4 : 0 \le x \le 1, && 4x_1 + 3x_2 + 2x_3 && +x_4 \le 4 && \} \\
P_3 &= \{x \in \mathbb{R}^4 : 0 \le x \le 1, && 4x_1 + 3x_2 + 2x_3 && +x_4 \le 4 && \\
& && x_1 + x_2 + x_3 && \le 1 && \\
& && x_1 && +x_4 \le 1 && \}
\end{aligned}$$

Uncapacitated FLP

- $x_j = 1$ if a facility is placed at j; 0 otherwise.
- Let y_{ij} be the fraction of the demand of client i that is satisfied from a facility at j
- Let c_j be cost of placing a facility in j.
- Let h_{ij} be the cost of satisfying the demand of client i from a facility at j

$$\min \sum_{j \in N} c_j x_j + \sum_{i \in I} \sum_{j \in N} h_{ij} y_{ij} \tag{1}$$

$$\sum_{j \in N} y_{ij} = 1 \qquad \text{for } i \in I \tag{2}$$

$$y_{ij} - x_j \leq 0 \qquad \text{for } i \in I \text{ and } j \in N \tag{3}$$

$$x_j \in \{0, 1\}, y_{ij} \geq 0 \qquad \text{for } i \in I \text{ and } j \in N \tag{4}$$

Equivalent formulation - UFL

Constraints

$$y_{ij} - x_j \leq 0 \text{ for } i \in I \text{ and } j \in N,$$

express the condition: for each i, if $y_{ij} > 0$ then $x_j = 1$.

Stated a little differently: if any $y_{ij} > 0$, then $x_j = 1$, which can be written as

$$\sum_{i \in I} y_{ij} \leq m x_j \text{ for } i \in I \text{ and } j \in N.$$

Equivalent formulation - UFL

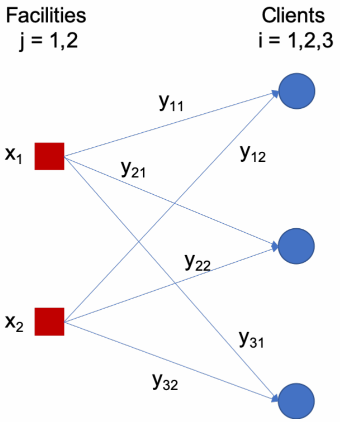
$$\min \sum_{j \in N} c_j x_j + \sum_{i \in I} \sum_{j \in N} h_{ij} y_{ij} \tag{5}$$

$$\sum_{j \in N} y_{ij} = 1 \qquad \text{for } i \in I \tag{6}$$

$$\sum_{i \in I} y_{ij} - m x_j \leq 0 \qquad \text{for } j \in N \tag{7}$$

$$x_j \in \{0, 1\}, y_{ij} \geq 0 \qquad \text{for } i \in I \text{ and } j \in N \tag{8}$$

UFL example - $m = 3, n = 2$



Comparing UFL formulations

Common constraints

$$y_{11} + y_{12} = 1$$
$$y_{21} + y_{22} = 1$$
$$y_{31} + y_{32} = 1$$

Constraints (3)

$$y_{11} - x_1 \leq 0$$
$$y_{21} - x_1 \leq 0$$
$$y_{31} - x_1 \leq 0$$
$$y_{12} - x_2 \leq 0$$
$$y_{22} - x_2 \leq 0$$
$$y_{32} - x_2 \leq 0$$

Constraints (7)

$$y_{11} + y_{21} + y_{31} - 3x_1 \leq 0$$
$$y_{12} + y_{22} + y_{32} - 3x_2 \leq 0$$

$\sum_i y_{ij} = m$ if and only if the entire demand of each client is fulfilled by the facility in j. In this case $\sum_i y_{ik} = 0$ for $k \neq j$.

Definitions II

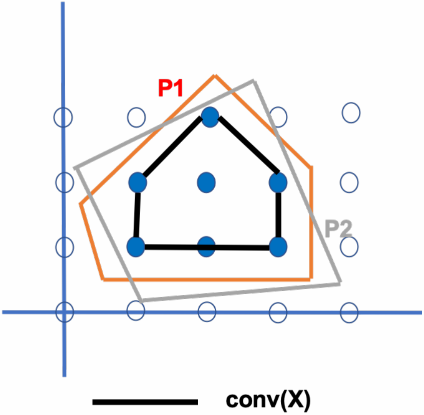
Convex hull

Given a set $\boldsymbol{X} \subseteq \mathbb{Z}^{\boldsymbol{n}}$, the convex hull of $\boldsymbol{X}$, denoted $\boldsymbol{conv(X)}$ is defined as

$$\boldsymbol{conv(X)} = \left\{ \boldsymbol{x} : \boldsymbol{x} = \sum_{\boldsymbol{i}=1}^{\boldsymbol{t}} \boldsymbol{\lambda_i x^i}, \sum_{\boldsymbol{i}=1}^{\boldsymbol{t}} \boldsymbol{\lambda_i} = \mathbf{1}, \boldsymbol{\lambda_i} \geq \mathbf{0} \right.$$

$$\left. \text{for } \boldsymbol{i}, \mathbf{1} \ldots, \boldsymbol{t} \text{ over all finite subsets } \{\boldsymbol{x^1}, \ldots, \boldsymbol{x^t}\} \text{ of } \boldsymbol{X} \right\}$$

Convex hull



Set $\boldsymbol{X}$ composed of BLUE points.

Propositions

Proposition 1

$\boldsymbol{conv(X)}$ is a polyhedron.

Proposition 2

The extreme points of $\boldsymbol{conv(X)}$ all lie in $\boldsymbol{X}$.

Hence,

$$\boldsymbol{X} \subseteq \boldsymbol{conv(X)} \subseteq \boldsymbol{P}, \text{ for all formulations } \boldsymbol{P}.$$

In other words, $\boldsymbol{conv(X)}$ is the smallest polyhedron containing $\boldsymbol{X}$.

Key result

Because of these two results, we can replace

$$\boldsymbol{IP} = \{\max \boldsymbol{cx} : \boldsymbol{x} \in \boldsymbol{X}\}$$

by the equivalent linear program

$$\boldsymbol{LP} = \{\max \boldsymbol{cx} : \boldsymbol{x} \in \boldsymbol{conv(X)}\}\,.$$

Hence, to solve $\boldsymbol{IP}$ you just need to solve $\boldsymbol{LP}$. However,

- in most cases there is such an enormous (exponential) number of inequalities needed to describe $\boldsymbol{conv(X)}$,
- it may not be simple to characterise $\boldsymbol{conv(X)}$.

Comparing formulations

Given two formulations P_1 and P_2 for X, when can we say that one is better than the other?

Better formulation

Given a set $X \subseteq \mathbb{Z}^n$ and two formulations P_1 and P_2 for X, P_1 is a better formulation than P_2 if $P_1 \subset P_2$.

Ideal formulation

Since $X \subseteq conv(X) \subseteq P,$ for all formulations P, $conv(X)$ is the ideal formulation for X.

Equivalent formulations - Knapsack

We consider again

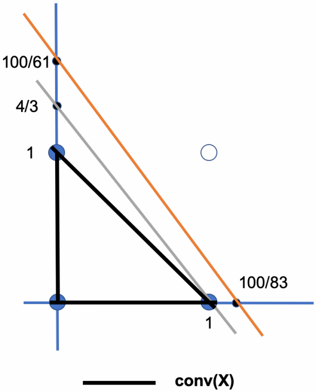
$$X = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(0,1,0,1),(0,0,1,1)\}$$

and its following formulations

$$\begin{aligned}
P_1 = \{x \in \mathbb{R}^4 : 0 \le x \le 1, \quad & 83x_1 + 61x_2 + 49x_3 && +20x_4 \le 100 && \} \\
P_2 = \{x \in \mathbb{R}^4 : 0 \le x \le 1, \quad & 4x_1 + 3x_2 + 2x_3 && +x_4 \le 4 && \} \\
P_3 = \{x \in \mathbb{R}^4 : 0 \le x \le 1, \quad & 4x_1 + 3x_2 + 2x_3 && +x_4 \le 4 && \\
& x_1 + x_2 + x_3 && \le 1 && \\
& x_1 && +x_4 \le 1 && \}
\end{aligned}$$

It can be seen that $P_3 \subset P_2 \subset P_1$. In addition $P_3 = conv(X)$ and thus P_3 is an ideal formulation.

Equivalent formulations - Knapsack



X = {(0,0), (1,0), (0,1)}

P1 = {x ∈ R^2_+: 83 x1 + 61 x2 ≤ 100}

P2 = {x ∈ R^2_+ : 4 x1 + 3 x2 ≤ 4}

P3 = {x ∈ R^2_+ : x1 + x2 ≤ 1}

Set $\boldsymbol{X}$ composed of BLUE points. We see that $\boldsymbol{P_3} \subset \boldsymbol{P_2} \subset \boldsymbol{P_1}$.

Equivalent formulations - UFL

Let P_1 the formulation with the single constraint (7) for each $j \in N$

$$\sum_{i \in I} y_{ij} \leq m x_j$$

and P_2 the formulation with m constraints (3) for each $j \in N$

$$y_{ij} \leq x_j \text{ for } i \in M.$$

We show that $P_2 \subset P_1$.

Equivalent formulations - UFL

a) Let (x, y) be a point that satisfies $y_{ij} \leq x_j$ for $i \in M$ and $j \in N$. Then $\sum_{i=1}^{m} y_{ij} \leq \sum_{i=1}^{m} x_j = mx_j$. Hence $P_2 \subseteq P_1$.

b) We show that there exists points that belong to P_1 but not to P_2. Let $m = kn$, with $k \geq 2$ and integer. Then a point in which each depot serves k clients such that

$$y_{ij} = \begin{cases} 1 & \text{for } i = k(j-1)+1, \ldots, k(j-1)+k, j = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

and $x_j = k/m$ for $j = 1 \ldots, n$ belongs to P_1 ($\sum_{i=1}^{m} y_{ij} = \sum_{i=1}^{k} 1 = k = m * k/m$) but not in P_2 (because $1 \not\leq k/m = 1/n$)

Hence $P_2 \subset P_1$.

Why a good formulation?

Q: Why is it so important to look for “good” or “ideal” formulations?

A: Because most of the times it may not be trivial to solve an integer programming problem (IP) whereas it is always “easy” to solve a linear programming problem (LP).

How to solve an IP?

Enumeration. All feasible solutions are identified and the best one is picked up. It may not be practically viable. For instance, to solve the TSP in a complete graph with n nodes there are $(n-1)!$ feasible tours. Hence,

n	$n!$
10	3.6×10^{6}
100	9.33×10^{157}
1000	4.02×10^{2567}

Better ideas are needed.

How to solve an IP?

Solve the corresponding convex hull. If the convex hull of an IP problem is known, we just need to solve a LP on it.

- In some circumstances, it is easy to identify the convex hull of an IP problem, such as for the Network Flow Problem (or Minimum Cost Flow Problem) and its special cases (shortest path, maximum flow, transportation and assignment problems).
- In most circumstances to find the convex hull is as difficult as to solve the original problem

Even if the convex hull is not know, why not to solve an IP by disregarding variables' integrality?

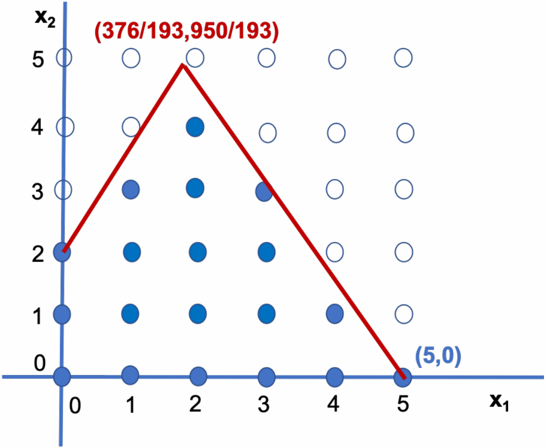
How to solve an IP?

Disregarding variables' integrality constraints. Consider the following problem:

$$
\begin{aligned}
\max Z = & 1.00x_1 + 0.64x_2 \\
& 50x_1 + 31x_2 \leq 250 \\
& 3x_1 - 2x_2 \geq -4 \\
& x_1, x_2 \geq 0 \text{ and integer.}
\end{aligned}
$$

- The optimal integer solution is $(5, 0)$
- The optimal solution without considering variables' integrality constraints is $(376/193, 950/193) = (1.948, 4.922)$

How to solve an IP?



How to solve an IP?

Disregarding variables' integrality constraints. Why not to round up and/or down the linear solution?

- The upper integer part $(\lceil \mathbf{1.948} \rceil, \lceil \mathbf{4.922} \rceil) = (\mathbf{2}, \mathbf{5})$ is NOT FEASIBLE (the first constraint is violated).
- The lower integer part $(\lfloor \mathbf{1.948} \rfloor, \lfloor \mathbf{4.922} \rfloor) = (\mathbf{1}, \mathbf{4})$ is NOT FEASIBLE (the second constraint is violated).
- The mixed choice $(\lfloor \mathbf{1.948} \rfloor, \lceil \mathbf{4.922} \rceil) = (\mathbf{1}, \mathbf{5})$ is NOT FEASIBLE (the second constraint is violated).
- The mixed choice $(\lceil \mathbf{1.948} \rceil, \lfloor \mathbf{4.922} \rfloor) = (\mathbf{2}, \mathbf{4})$ is feasible but NOT OPTIMAL: $\boldsymbol{Z}(\mathbf{2}, \mathbf{4}) = \mathbf{4.56}$, whereas $\boldsymbol{Z}(\mathbf{5}, \mathbf{0}) = \mathbf{5}$.

In addition, no rounding gives the values $(\mathbf{5}, \mathbf{0})$.

In conclusion, the linear solution appears to be useless to find the integer solution.