Numerical Analysis — Lec 4 LH — 15.10.2020

General Interpolation problem from $\mathcal{X} := C^0([a,b])$ to the space $\mathcal{Y} := V := \text{span}\{v_i\}_{i=0}^n$ with fixed support points $\{x_i\}_{i=0}^n$

finite dimensional with dimension $(n+1)$

$$\mathbb{I} : C^0([a,b]) \longrightarrow V$$
$$u(x) \longrightarrow p(x) := \sum_{i=0}^n p^i v_i(x)$$

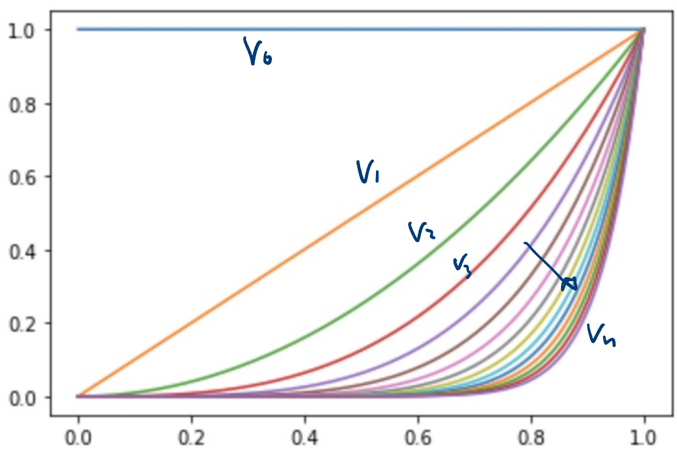
Such that p and u coincides in $\{x_i\}_{i=0}^n$

$$\sum_{j=0}^n p^j v_j(x_i) = u(x_i)$$
$$\equiv$$

$$\underline{\underline{V}}_{ij} := v_j(x_i) \qquad \underline{\underline{V}}\ \underline{p} = \underline{u}$$
$$\{\underline{p}\}^j = p^j$$
$$\{\underline{u}\}_i = u(x_i)$$

The basis functions $v_i \in C^0([a,b])$ (polynomials, cos/sin, sigmoids)

Example : $[a,b] = [0,1]$ $\qquad v_i = \text{pow}(x,i) = x^i$



fix $n+1 = 5$ $\quad \underline{p} := [1,0,0,2,3]$

$$p(x) = 1 + 2x^3 + 3x^4$$
$$= p^i v_i(x)$$

$$\underline{u} + \underline{v} = \underline{(u+v)}$$
$$(\underline{u}+\underline{v})_i := u(x_i) + v(x_i)$$

$$\mathbb{I}(u+v) = \mathbb{I}(u) + \mathbb{I}(v)$$

$$\mathbb{I}(u) = (\underline{\underline{V}}^{-1}\underline{u})^i v_i(x) \qquad \mathbb{I}(v) = (\underline{\underline{V}}^{-1}\underline{v})^i v_i(x)$$
$$\mathbb{I}(u+v) = (\underline{\underline{V}}^{-1}(\underline{u}+\underline{v}))^i v_i(x)$$

Condition number for interpolation problem

We use as norms in $C^0([a,b])$ $\|\cdot\|_\infty$, the same in V:

$$\|\Pi(u) - \Pi(\hat{u})\|_\infty = \|\Pi(u-\hat{u})\|_\infty \le \underbrace{\|\Pi\|_*}_{\simeq K_{abs}} \|u-\hat{u}\|_\infty$$

What is $\|\Pi\|_*$?

$$\|\Pi\|_* := \sup_{0\ne u\in C^0([a,b])} \frac{\|\Pi(u)\|_\infty}{\|u\|_\infty} = \sup_{0\ne u\in\mathbb{R}} \frac{\|\underbrace{\overbrace{V^{ij}u(x_j)}^{p^i}\, v_i(x)}_{p=\Pi u}\|_\infty}{\|u\|_\infty}$$

$$\le \sup_{0\ne u\in\mathbb{R}} \frac{\|V^{-1}\|_\infty \cancel{\|u\|_\infty} \max_i \|v_i\|_\infty}{\cancel{\|u\|_\infty}} \le \|V^{-1}\|_\infty \max_i \|v_i\|_\infty$$

$$\sim \text{cond}_N(V^{-1}) \max_i \|v_i\|_\infty$$

Two things: $\{x_i\}_{i=1}^n$, v_i

$\Rightarrow \min \text{cond}(V^{-1}) \equiv \{v_i\}_{i=0}^n$ Lagrange basis functions:

$$l_i := \prod_{\substack{j\ne i\\ j=0}}^n \frac{(x-x_j)}{(x_i-x_j)}$$

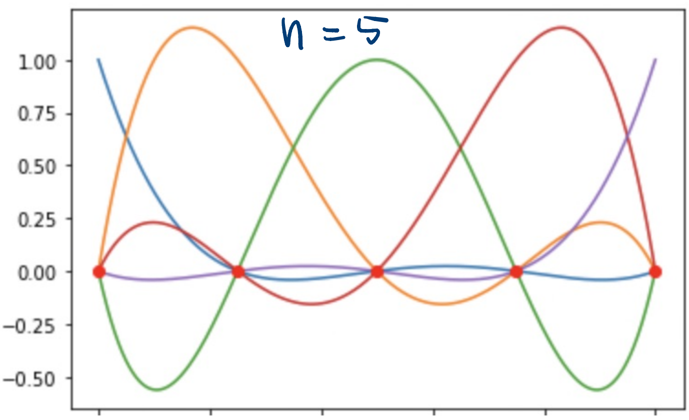
$$\rightsquigarrow V_{ij} := l_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\ne j\end{cases}$$

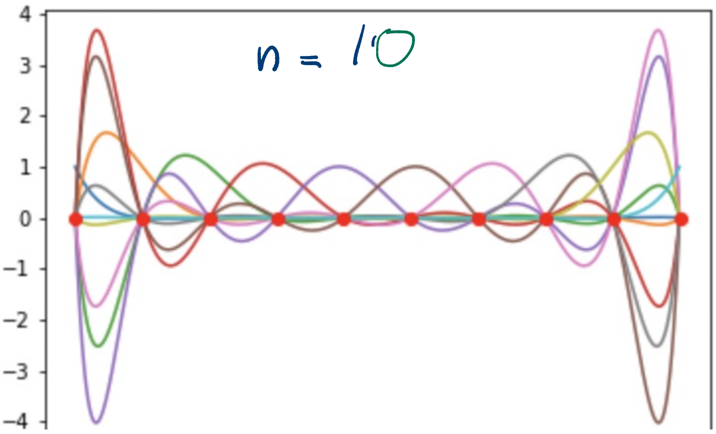
We should choose x_i s.t. $\max_i \|l_i\|_\infty$ is small

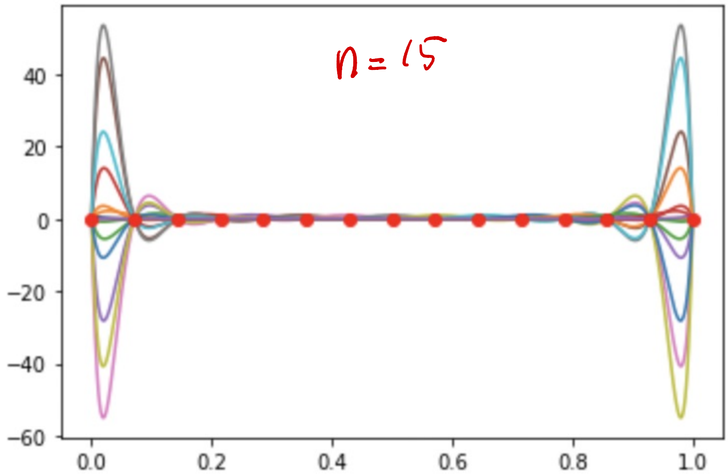
For power basis:

```
n = 1 : 1.0
n = 2 : 2.6180339887498953
n = 3 : 15.099657722502098
n = 4 : 98.86773850722759
n = 5 : 686.4349418185955
n = 6 : 4924.371056611224
n = 7 : 36061.16088021232
n = 8 : 267816.7009075794
n = 9 : 2009396.3800421846
n = 10 : 15193229.677753646
n = 11 : 115575244.54431371
n = 12 : 883478687.0721825
n = 13 : 6780588492.454725
n = 14 : 52214926084.1525
n = 15 : 403234616528.72504
```

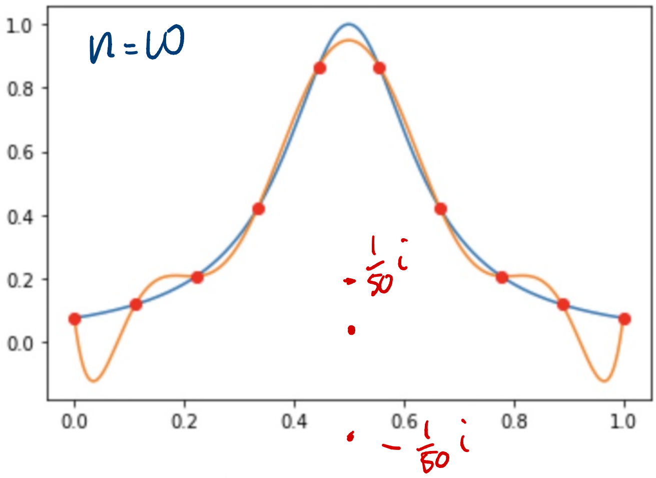
cond(V) for power basis







Is interpolation a good strategy for approximating a function?

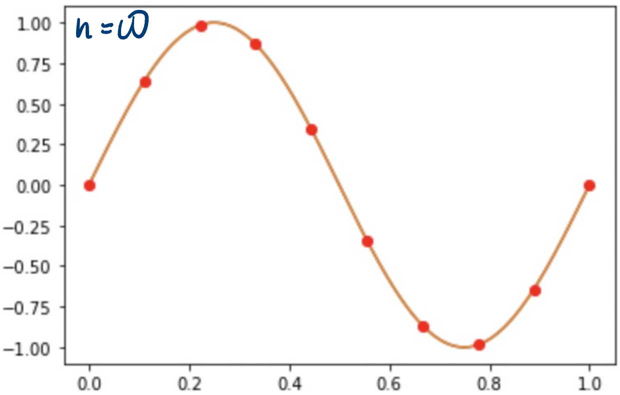


orange line:

$\sum_{i=0}^{n} u(x_i)\, l_i(x) = p(x)$ with Lagrange basis

blue line:

$$f(x) = \frac{1}{1+50(x-\frac{1}{2})^2}$$



$f(x) = \sin(2\pi x)$

Theo $f \in C^{n+1}([a,b])$ if $x_i \in (a,b)$, $\forall x \in (a,b)\ \exists \xi \in (a,b)$

$$(f - \Pi^n f)(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\, \omega(x)$$

where $\omega(x)$ is the characteristic polynomial of $\{x_i\}_{i=0}^{n}$

$$\omega(x) = \prod_{i=0}^{n} (x - x_i)$$

First consequence: $\|\Pi f - f\|_\infty \le \frac{\|\omega\|_\infty}{(n+1)!} \|f^{(n+1)}\|_\infty$

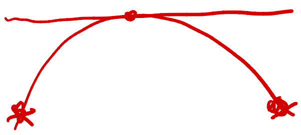
proof: $\forall x$, define $G(t)$ s.t.

$$G(t) = \big(f(t) - p(t)\big)\,\omega(x) - \big(f(x) - p(x)\big)\,\omega(t)$$

where $p(t) = \sum_{i=0}^{n} f(x_i)\, e_i(t) = (\Pi f)(t)$

$G(t)$ has $n+2$ zeros : $\{x_i\}_{i=0}^{n} \cup \{x\}$

Rolle's theorem : $\exists\, \xi \in (a,b)$ s.t. $\dfrac{d^{n+1} G(\xi)}{dt^{n+1}} = 0$



f with two zeros $\Rightarrow$ $\exists\, \xi$ s.t. $f'(\xi) = 0$

$$\frac{d^{n+1}}{dt^{n+1}}\big(\omega(t)\big)$$

$$\frac{d^{n+1}}{dt^{n+1}} G(\xi) = f^{(n+1)}(\xi)\cdot\omega(x) - \big(f(x) - p(x)\big)(n+1)! = 0$$

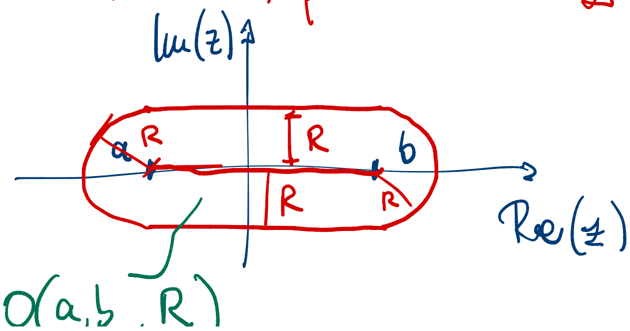
$$\Rightarrow f(x) - (\Pi f)(x) = \frac{f^{n+1}(\xi)}{(n+1)!}\,\omega(x) \qquad \text{QED}$$

How to apply this?

Theo 2 : If f is analytically extendible in a oval $O(a,b,R)$ with $R>0$

Then $$\| f^{(n+1)} \|_\infty \leq \frac{(n+1)!}{R^{n+1}} \| \tilde{f} \|_{L^\infty(O(a,b,R))}$$

Take x, and replace with $z \in \mathbb{C}$

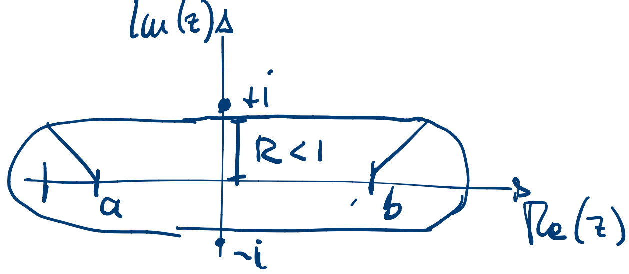


$f \in C^\infty$ in $O(a,b,R)$

Runge function: $f = \dfrac{1}{1+x^2}$

extension to $\mathbb{C}$

$$f(z) = \frac{1}{1+z^2}$$



f is analytical in $O(-\infty, +\infty, 1-\varepsilon)$ $\quad \forall \varepsilon > 0$

What are the consequences?

$$\| f^{(n+1)} \| \leq \frac{(n+1)!}{\underline{\underline{R^{n+1}}}} \| f \|_\infty \qquad R<1$$

$$\| p - f \|_\infty \leq \| f^{(n+1)} \|_\infty \frac{\| \omega \|_\infty}{(n+1)!} \leq \frac{\cancel{(n+1)!}}{\cancel{(n+1)!}} \frac{\| \omega \|_\infty}{R^{n+1}} \| f \|_\infty$$

$$\| \omega \|_\infty = \left\| \prod_{i=0}^{n} (x - x_i) \right\| \leq \underline{\underline{|b-a|^{(n+1)}}} \qquad x \in (a,b)$$

$$\| p - f \|_\infty \leq \left(\frac{|b-a|}{R} \right)^{n+1} \| f \|_\infty$$

$\Rightarrow$ Runge: $\dfrac{1}{1+x^2}$ in $\underline{(-1, 1)}$ everything $\underline{\text{ok}}$

Can we do better?

Best Approximation. Definition:

$\mathcal{X}$ Banach, reflexive, strictly convex

V subspace of $\mathcal{X}$

We call p the best approximation in V of f in $\mathcal{X}$ when:

$$\|f-p\| \leq \|f-q\| \quad \forall q \in V$$

$$\|f-p\| = E(f) = \inf_{q\in V} \|f-q\|$$

Theo: $\exists ! p$

How does polynomial interpolation compare to best approximation?

$$\|f - \mathcal{I}^n f\|_\infty = \|f - p + \underset{=}{p} - \mathcal{I}^n f\|_\infty =$$

$$\|f - p + \mathcal{I}^n (p - f)\|_\infty$$

$$\leq \|f-p\|_\infty + \|\mathcal{I}^n\|_* \, \|p-f\|_\infty$$

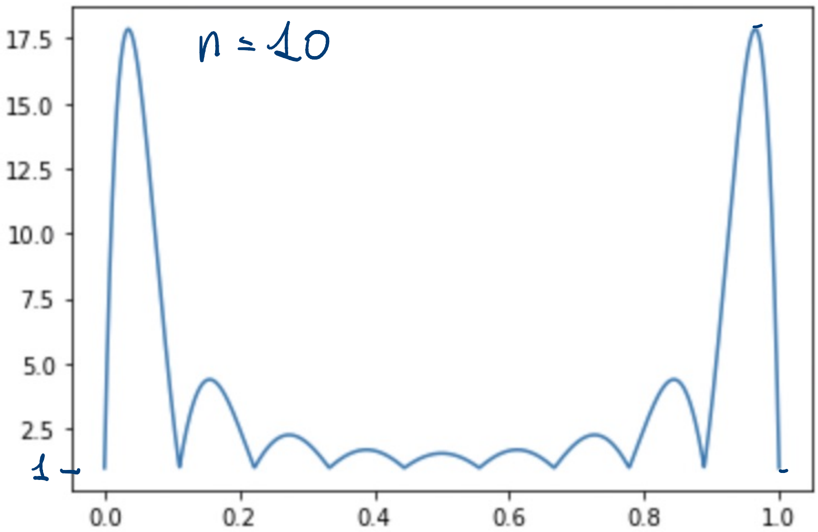
$$\leq \left(1 + \|\mathcal{I}^n\|_*\right) \|p-f\|_\infty$$

For Lagrange interpolation:

$$\|\mathcal{I}^n_*\| := \sup_{u \in C^0([a,b])} \frac{\|\sum_{i=0}^n u(x_i)\, \ell_i(x)\|_\infty}{\|u\|_\infty} \leq$$

$$= \|\sum_{i=0}^n |\ell_i(x)|\,\|_\infty \quad : \Lambda(x) := \sum_{i=1}^n |\ell_i(x)|$$

$$\| \mathcal{I}_*^n \| = \| \Lambda \|_\infty$$



$\|\Lambda\|_\infty$ for equispaced points

$$\|\Lambda\|_\infty \leq \frac{2^{n+1}}{e\, n \log n}$$

A collection of points: $\{x_i\}_{i=0}^n$ is a list of lists of points with increasing size.

Erdos

$\forall$ collection of points $\{x_i\}_{i=0}^n$ $\exists$ $c > 0$ s.t.

$$\|\Lambda\|_\infty \geq \frac{2}{\pi} \log(n+1) - c$$

Faber

$\forall$ collection of points $\{x_i\}_{i=0}^n$ $\exists$ f s.t.

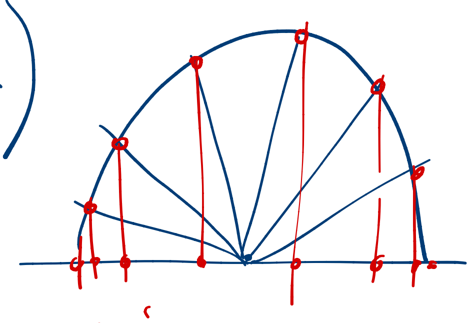
$$\lim_{n \to \infty} \|\mathcal{I}^n f - f\|_\infty \to \infty$$

The best is $\alpha := \{x_i\}_{i=0}^n$ s.t. $\|\Lambda^\alpha\|_\infty \leq \|\Lambda^\beta\|_\infty$ $\forall$ $\beta = \{\tilde{x}_i\}_{i=0}^n$

$\alpha := \{x_i\}_{i=0}^n$ are called Chebyshev points: between $[-1, 1]$

$$x_i = \cos\left(\frac{(2i+1)\pi}{2n+2}\right)$$

$$\|\Lambda^\alpha\|_\infty \leq \frac{2}{\pi} \log(n+1) + 1$$



Best case scenario

$$\frac{2}{\pi} \log(n+1) - c \leq \|\Lambda^\alpha\|_\infty \leq \frac{2}{\pi} \log(n+1) + 1$$