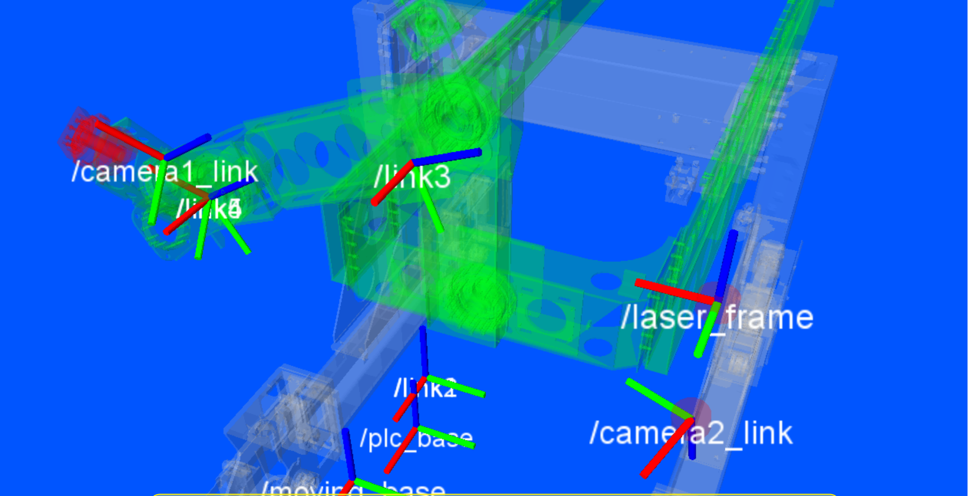


General Intelligent Mobile Systems I

CH09-320103

Prof. Dr. Kaustubh Pathak

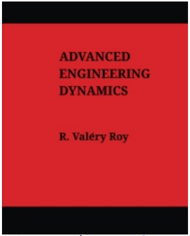
http://kpathak.user.jacobs-university.de/

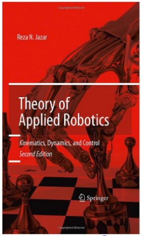
Jacobs University Bremen

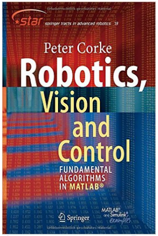
Contents

Textbooks

- Extensive slides are provided – they are the primary reference material.
- The notation and some of the material is from the following books:
  - *Advanced Engineering Dynamics*. [Roy(2015)]. Also see: `http://enggdynamics.blogspot.com/`.
  - *Theory of Applied Robotics: Kinematics, Dynamics, and Control*. [Jazar(2010)].
  - *Robotics, Vision and Control: Fundamental Algorithms in MATLAB*. [Corke(2011)].
- Reading these books is not strictly necessary. Just the slides, homeworks (solutions provided), and tutorial exercises are enough.







Teaching Assistants

- Alexandru Maiereanu, `a.maiereanu@jacobs-university.de`
- Sabin Bhandari, `sa.bhandari@jacobs-university.de`
- George Tudor, `g.tudor@jacobs-university.de`
- Ashish Khanal, `a.khanal@jacobs-university.de`

Grading

Home-work, Tutorial assignments, Programming projects	60 %	Almost every week	
Active Participation	5%	Lecture attendance is not mandatory.	Each person already gets 2.5%. You lose points for disturbing the class in any way. You gain points by answering questions in the class and asking thoughtful questions.
Mid-term exam	15%	Attendance mandatory	21st Oct., 09:00-11:00, Sports Center
Final exam	20%	Attendance mandatory	TBA

Tutorial Sessions

Fridays, 09:45-11:00

- The registrar has guaranteed that no scheduling conflict exists for these sessions.
- Almost each week, starting week-1, we will do practice-exercises. You will have to finish them during the tutorial and submit them at the end of each tutorial. Some will be conducted by the TAs and some by the instructor.
- **Mandatory Attendance**: Friday, 30th September, 09:45-11:00, in the tutorial: There will a Lab Safety Instructions presentation. Only students present will be allowed to attend the IMS Lab-1.

Rules of the Game

- Please do not be shy about asking relevant questions during lectures. You are also welcome to ask questions by email.
- Most information will be present in the slides.
- Some detailed derivations will be done on the board for ease of understanding. These will not be repeated in the slides. It is your responsibility to take notes from the board. There will be markers in the slides to signal that some material was done on the board.
- Attendance of lectures is not mandatory but, based on past experience, it will be very difficult to get a good grade without attending lectures.
- Please do not use laptops during the lectures unless asked to. Disturbing lectures in any way will be penalized by a reduction of the grade.

Rules of the Game

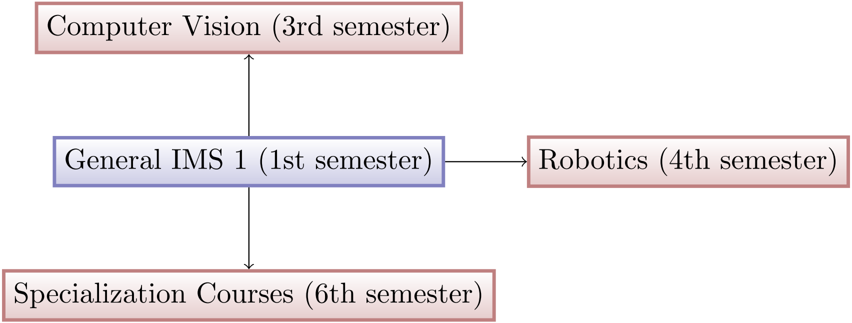
- A homework will be given almost every week, starting week 2. You will do it in groups of 2.
- There will be a separate "homeworks" section at the end of the slides.
- In case of suspected copying, both groups will get a zero.
- Since you get a week to submit a homework, it can be excused only if you have an official excuse from the registrar for that whole week.
- Homeworks will be submitted at http://jgrader.de/. You should upload scanned PDFs. If the size is bigger than allowed, scan at a lower resolution.
- A makeup exam will be provided only on submission of an official excuse from the registrar.

Rules of the Game

- Please copy your emails to all TAs. Put `[GIMS-1]` in the subject line.
- Any extension request should be made via jGrader. It may or may not be accepted.
- Class attendance is not mandatory but class discipline is.
- In case you need help please first ask the TAs during the tutorial. If anything is still unresolved, please write me an email to make a personal appointment.

Contents

Connections to Other Courses



Classical Mechanics

- Statics
- Kinematics
- Kinetics
- Dynamics

Definition 2.1 (Statics)

Statics analyzes forces and torques acting on systems of bodies in static equilibrium (either stationary or moving at constant velocity).
It is studied by civil and mechanical engineers. Stresses and strains inside the bodies are studied under "Solid Mechanics."

Figure 2.1: Bridge-design needs concepts from the fields of Statics and Solid Mechanics, among others. Image in public domain.

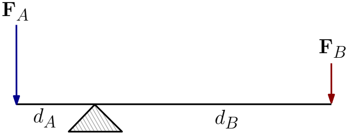


Figure 2.2: A static equilibrium of forces and moments.

Definition 2.2 (Kinematics)

Kinematics is the study of motions (velocities, accelerations) of points, bodies, and systems of bodies without consideration of their causes (forces, torques). It is the “geometry of motion.”

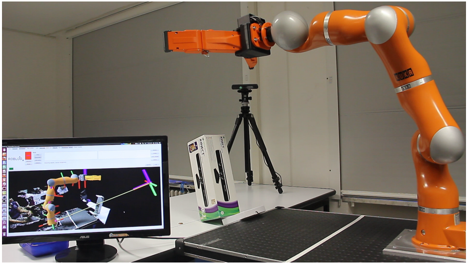


Figure 2.3: The KUKA Light-Weight-Arm.

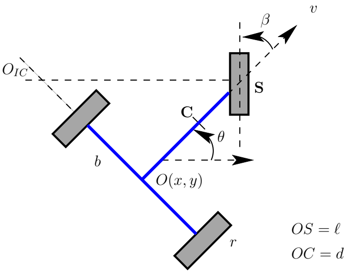


Figure 2.4: Kinematic model of a tricycle.

Definition 2.3 (Kinetics)

Kinetics combines kinematics with the mass-distribution in bodies, and thus studies concepts such as moments of inertia, linear momentum, angular momentum, and kinetic energy.

Figure 2.5: Motion of freely-floating bodies in space is made "interesting" due to the conservation of linear and angular momenta.[2]

[2] "Gravity Poster" by Source. Licensed under Fair use via Wikipedia - https://en.wikipedia.org/wiki/File:Gravity_Poster.jpg

Definition 2.4 (Dynamics)

Dynamics is the study of how forces and torques acting on bodies cause their relative motion, and how this motion depends on the inertial properties (masses and moments of inertia) of the bodies.

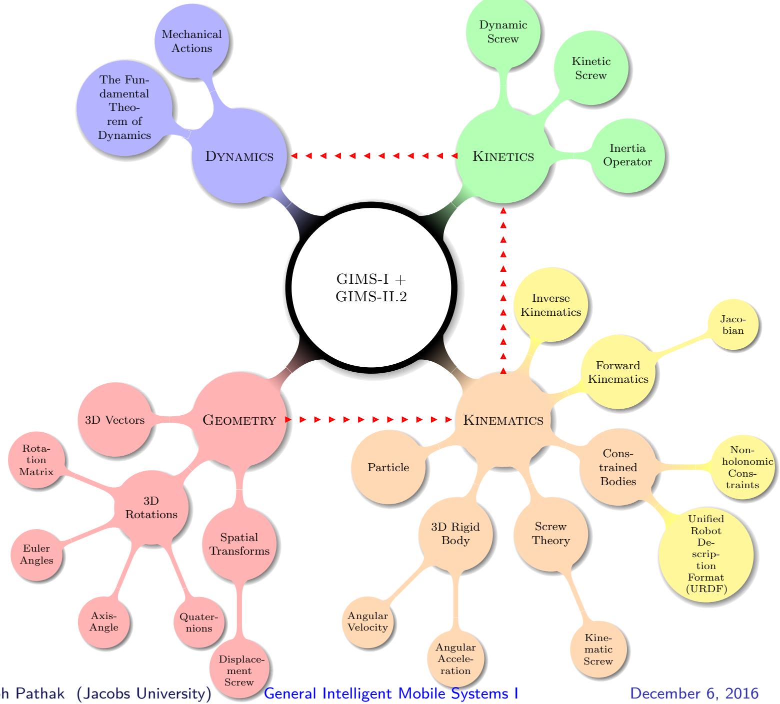
Figure 2.6: A dynamically balanced Segway RMP.

Figure 2.7: BigDog: A famous quadruped robot by Boston Dynamics. Image in public domain.

Dynamics in Action

Introducing Spot: https://www.youtube.com/watch?v=M8YjvHYbZ9w

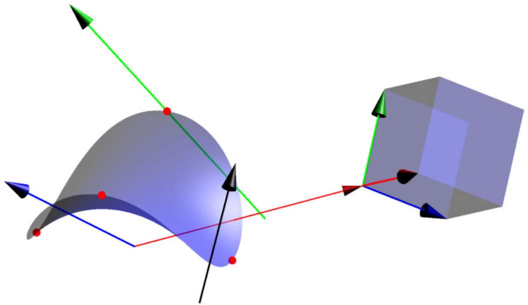
Launch external viewer.


GIMS-I + GIMS-II.2
Dynamics
Mechanical Actions
The Fundamental Theorem of Dynamics
Kinetics
Dynamic Screw
Kinetic Screw
Inertia Operator
Geometry
3D Vectors
3D Rotations
Rotation Matrix
Euler Angles
Axis-Angle
Quaternions
Spatial Transforms
Displacement Screw
Kinematics
Inverse Kinematics
Forward Kinematics
Jacobian
Constrained Bodies
Nonholonomic Constraints
Unified Robot Description Format (URDF)
Particle
3D Rigid Body
Angular Velocity
Angular Acceleration
Screw Theory
Kinematic Screw

Contents

Definition 3.1 (Vector)

In this course, a vector is just a 3D arrow. We emphasize this geometric interpretation rather than an algebraic one.



- Vectors are denoted as lower-case bold letters in type, e.g. $\mathbf{v}$, $\mathbf{w}$. On board, they are written as $\underset{\sim}{v}$, $\underset{\sim}{w}$ etc.
- The set of vectors is denoted as $\mathbb{V}$.

The Importance of a Geometric Definition

- A geometric definition of vectors allows us to proceed in a coordinate-system independent way.
- Any equation written in a vector-form is hence valid in all coordinate-systems!
- But what is a coordinate-system? For now, you can consider a coordinate-system as being comprised of some user-defined xyz axes.
- As an example, the force $\mathbf{f}$ on a current ($\mathbf{i}$) carrying conductor of length ℓ in the presence of a magnetic field $\mathbf{B}$ is given by:

$$\mathbf{f} = \ell\,\mathbf{i} \times \mathbf{B}. \tag{3.1}$$

We will review the vector cross-product operator $\times$ after a few slides.

Properties of a Vector

- A vector $\mathbf{v}$ has two properties which completely define it:
  1. The 3D direction of its arrow
  2. Its length, also called its magnitude, and denoted as $\|\mathbf{v}\|$.
- Two vectors $\mathbf{u}$ and $\mathbf{v}$ are equal, written, $\mathbf{u} = \mathbf{v}$, iff they have the same direction and $\|\mathbf{u}\| = \|\mathbf{v}\|$.
- Note: The starting points of vectors are hence immaterial for their equality.

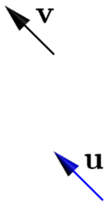


Figure 3.1: $\mathbf{u} = \mathbf{v}$.

Definition 3.2 (Unit Vector)

If $\|\mathbf{v}\| = 1$, we call it a unit-vector. We write a unit-vector as $\hat{\mathbf{v}}$.

Vector Field

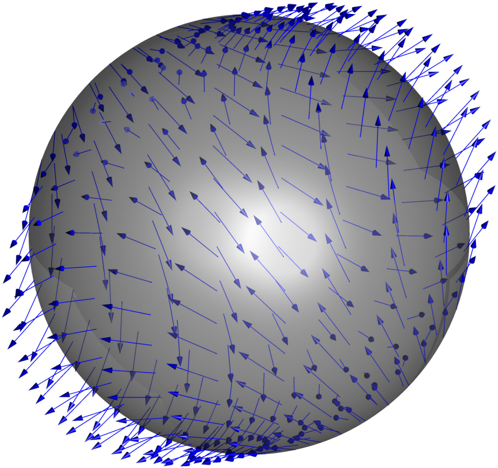


Figure 3.2: A vector-field maps each point (element) of a domain $\mathbb{D}$ to a vector. What is the domain here?

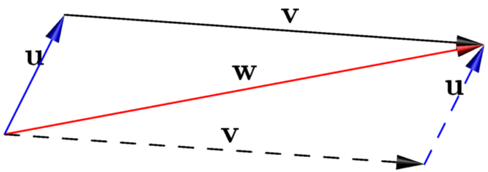
Here the domain is the surface of a unit-sphere.

Vector Field

Any other example of a vector-field?

Two Basic Operations on Vectors

1. **Scalar Multiplication**: If $\lambda \in \mathbb{R}$, $\mathbf{u} = \lambda \mathbf{v}$ is a vector such that:
   (i) $\|\mathbf{u}\| = |\lambda| \|\mathbf{v}\|$
   (ii) If $\lambda > 0$, $\mathbf{u}$ and $\mathbf{v}$ are in the same direction: they are parallel.
   (iii) If $\lambda < 0$, $\mathbf{u}$ and $\mathbf{v}$ are in opposite directions: they are anti-parallel.
2. **Vector Addition**: $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is defined by the parallelogram law.



- Clearly, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ lie in (hence, define) a 3D plane.
- Exception: If $\mathbf{u} = \lambda \mathbf{v}$, $\lambda \neq 0$, $\mathbf{u}$ and $\mathbf{v}$ are called collinear.
- From the figure, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$, i.e. vector addition is commutative.

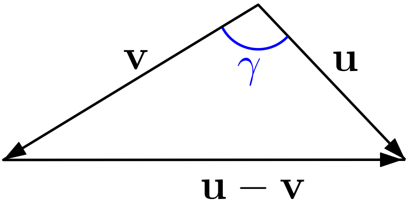
Vector Normalization

Any non-zero vector can be normalized to make it a unit-vector $\hat{\mathbf{v}} = \lambda \mathbf{v}$, where, $\lambda = 1/\|\mathbf{v}\|$.

Vector Subtraction

Using the above two properties, we can intuitively define:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} \tag{3.2}$$

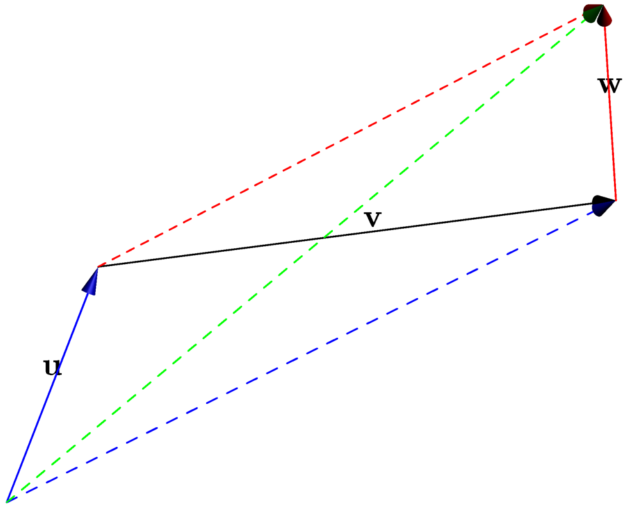


The Vector-Space of 3D Vectors

- The set of vectors $\mathbb{V}$ with the above 2 operations together satisfy the following axioms, all of which can be proven geometrically:
  1. Commutativity of vector addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
  2. Associativity of vector addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
  3. $\exists$ a zero-vector $\mathbf{0}$ such that $\forall\,\mathbf{v}$, we have $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
  4. $\forall\,\mathbf{v} \in \mathbb{V}$, its additive-inverse $-\mathbf{v} \in \mathbb{V}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
  5. Compatibility of scalar multiplication: For any $\alpha, \beta \in \mathbb{R}$, $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$.
  6. Multiplicative identity: $1\mathbf{v} = \mathbf{v}$.
  7. Distributivity of scalar multiplication over vector addition: $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.
  8. Distributivity of vector addition w.r.t. scalar addition: $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.
- For this reason, we can say that the set of vectors $\mathbb{V}$ forms a vector-space (in the jargon of Linear Algebra) over the set of real numbers. We name this vector-space $\mathbb{E}$.

Geometrically Showing Associativity of Vector Addition

$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$



Plane Spanned by Two Vectors

As you vary α and β, $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$ traces out a plane. This is called the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

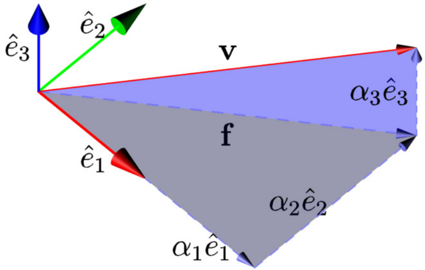
Linear Dependence

- Any three coplanar vectors ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) are called linearly-dependent (LD).
- Assuming $\mathbf{u}$ and $\mathbf{v}$ are not collinear, we can write $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$ where α, β are not both zero.

Linear Independence

- If a vector $\mathbf{w}$ cannot be written as $\alpha\mathbf{u} + \beta\mathbf{v}$ then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is called linearly-independent (LI).
- Are three mutually orthogonal (perpendicular) vectors LI?

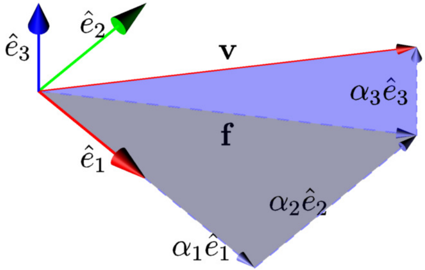
Basis



Given 3 LI (non-coplanar) vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and a fourth 3D vector $\mathbf{v}$, do the following construction:

- Move all the vectors until their tails touch together.
- From the tip (head) of $\mathbf{v}$ draw a line in direction $\mathbf{e}_3$.
- The point of intersection of this line with the plane spanned by $\mathbf{e}_1, \mathbf{e}_2$ defines the vector $\mathbf{f}$, as shown.

Basis

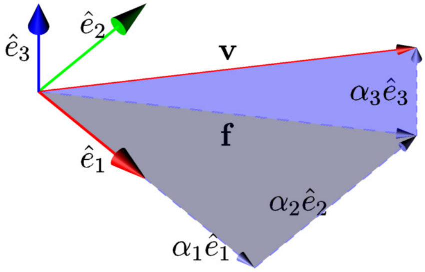


- Thus, given 3 LI vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, a fourth 3D vector $\mathbf{v}$ can always be written as

$$\begin{aligned} \mathbf{v} &= \mathbf{f} + \alpha_3\mathbf{e}_3, && \mathbf{f} \text{ lies in the plane spanned by } \mathbf{e}_1, \mathbf{e}_2 \\ &= \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3, && \text{for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}. \end{aligned} \tag{3.3}$$

- We say that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (any 3 LI vectors) form a basis of $\mathbb{E}$.

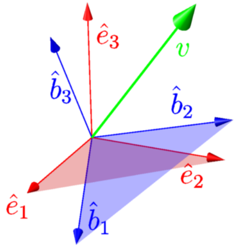
Orthogonal/Orthonormal Basis



$$\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3, \qquad \text{for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

- If the LI vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are mutually orthogonal they form an orthogonal basis.
- If the orthogonal basis consists of unit-vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$, they form an orthonormal basis.

Orthonormal Basis (Plural: Bases)



- In this course, we will predominantly use an orthonormal basis, with basis-vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. We call this $\mathscr{B}_E$ (basis "E").
- Similarly, a different orthonormal basis, with basis-vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3$ will be called $\mathscr{B}_B$ (basis "B"). Refer to the figure above.

What happened to the "high-school basis"?

- In high-school, we only had one basis $\mathscr{B}_{High-School}$. It was implicitly orthonormal, with unit-vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. We generally wrote a vector in terms of this basis as

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \tag{3.4}$$

- In this course, we will have many bases being used simultaneously, hence, a better notation is needed for conciseness. So, in our new notation,
    - The orthonormal vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are like $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ for $\mathscr{B}_E$.
    - The orthonormal vectors $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3$ are like $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ for $\mathscr{B}_B$.

What happened to the "high-school basis"?

- We can now write (3.4) compactly as follows in $\mathscr{B}_E$

$$\mathbf{v} = \alpha_1 \,\hat{\mathbf{e}}_1 + \alpha_2 \,\hat{\mathbf{e}}_2 + \alpha_3 \,\hat{\mathbf{e}}_3 \tag{3.5}$$

$$= \sum_{i=1}^{3} \alpha_i \,\hat{\mathbf{e}}_i \tag{3.6}$$

- Similarly, we can write (3.4) compactly as follows in $\mathscr{B}_B$

$$\mathbf{v} = \beta_1 \,\hat{\mathbf{b}}_1 + \beta_2 \,\hat{\mathbf{b}}_2 + \beta_3 \,\hat{\mathbf{b}}_3 \tag{3.7}$$

$$= \sum_{i=1}^{3} \beta_i \,\hat{\mathbf{b}}_i \tag{3.8}$$

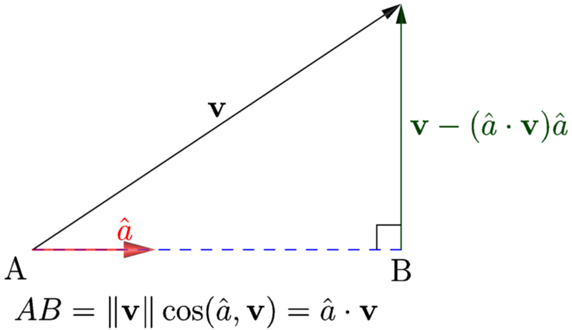
Scalar (Dot) Product of Vectors

- Let us define $\cos(\mathbf{u}, \mathbf{v})$ as the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$. Other trigonometric functions can be similarly defined.
- Since $\cos(x) = \cos(-x) = \cos(2\pi - x)$, it does not matter which way you measure this angle in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.
- Now the scalar (dot) product is defined as

$$\mathbf{u} \cdot \mathbf{v} \triangleq \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos(\mathbf{u}, \mathbf{v}) \tag{3.9}$$

- As $\cos \frac{\pi}{2} = 0$, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal $\mathbf{u} \cdot \mathbf{v} = 0$.
- Clearly, $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
- Also, from definition, the dot product is commutative.

Component/Projection of a Vector along a Unit-Vector



- As seen above, by simple trigonometry,

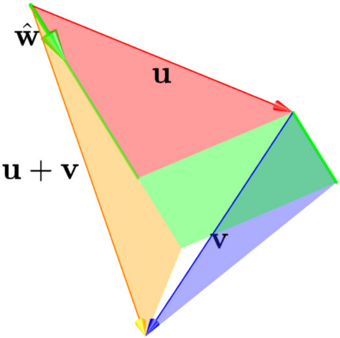
$$AB = \hat{\mathbf{a}} \cdot \mathbf{v} \tag{3.10}$$

  is the component of $\mathbf{v}$ in the direction of the unit-vector $\hat{\mathbf{a}}$.
- The remaining vector, shown in green above, is such that

$$\mathbf{v} - (\hat{\mathbf{a}} \cdot \mathbf{v})\, \hat{\mathbf{a}} \text{ is orthogonal to } \hat{\mathbf{a}} \tag{3.11}$$

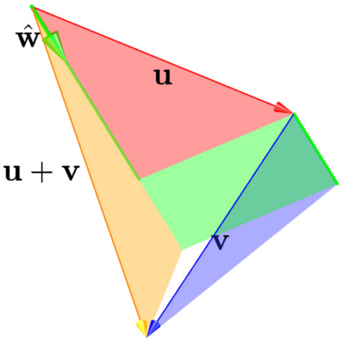
The Dot Product Distributes Over Vector Addition

Geometric Proof



We want to show $\mathbf{w}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{v}$. Since this is trivially satisfied if $\mathbf{w}=\mathbf{0}$, we assume $\|\mathbf{w}\| \neq 0$ and divide both sides by it. So, we can equivalently prove:

$$\hat{\mathbf{w}}\cdot(\mathbf{u}+\mathbf{v}) = \hat{\mathbf{w}}\cdot\mathbf{u}+\hat{\mathbf{w}}\cdot\mathbf{v} \tag{3.12}$$

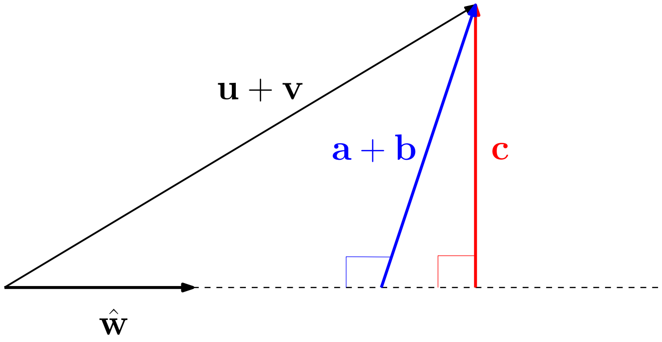


- $\mathbf{a} = \mathbf{u} - (\mathbf{u}\cdot\hat{\mathbf{w}})\,\hat{\mathbf{w}}$ is $\perp \hat{\mathbf{w}}$ from (3.11). Also, $\mathbf{b} = \mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{w}})\,\hat{\mathbf{w}}$ is $\perp \hat{\mathbf{w}}$.
- Hence, $\hat{\mathbf{w}}$ is the normal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$.
- Since $\mathbf{a}+\mathbf{b}$ lies in this plane, it is also $\perp \hat{\mathbf{w}}$, and,

$$\mathbf{a}+\mathbf{b} = \mathbf{u}+\mathbf{v} - (\mathbf{u}\cdot\hat{\mathbf{w}})\,\hat{\mathbf{w}} - (\mathbf{v}\cdot\hat{\mathbf{w}})\,\hat{\mathbf{w}} \quad = (\mathbf{u}+\mathbf{v}) - (\mathbf{u}\cdot\hat{\mathbf{w}} + \mathbf{v}\cdot\hat{\mathbf{w}})\,\hat{\mathbf{w}}$$

- Also, $\mathbf{c} = \mathbf{u}+\mathbf{v} - ((\mathbf{u}+\mathbf{v})\cdot\hat{\mathbf{w}})\,\hat{\mathbf{w}}$ is $\perp \hat{\mathbf{w}}$ from (3.11).

- $\mathbf{a} + \mathbf{b} = (\mathbf{u} + \mathbf{v}) - (\mathbf{u} \cdot \hat{\mathbf{w}} + \mathbf{v} \cdot \hat{\mathbf{w}})\,\hat{\mathbf{w}}$ which is $\perp \hat{\mathbf{w}}$ and clearly lies in the plane spanned by $(\mathbf{u} + \mathbf{v})$ and $\hat{\mathbf{w}}$.
- Furthermore, $\mathbf{c} = (\mathbf{u} + \mathbf{v}) - ((\mathbf{u} + \mathbf{v}) \cdot \hat{\mathbf{w}})\,\hat{\mathbf{w}}$ which is also $\perp \hat{\mathbf{w}}$, and also lies in the plane spanned by $(\mathbf{u} + \mathbf{v})$ and $\hat{\mathbf{w}}$.
- These statements cannot be both true unless $\mathbf{a} + \mathbf{b} = \mathbf{c}$ which implies (3.12).



Components of a Vector in a Basis

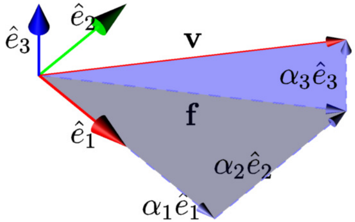


Figure 3.3: A vector resolved into its components on a basis.

We want to find the components $\alpha_1, \alpha_2, \alpha_3$.

- Using the result (3.10) and (3.12), it is now easy for us to do this.
- Assuming an orthonormal basis, we can write

$$\mathbf{v} = \alpha_1\hat{\mathbf{e}}_1 + \alpha_2\hat{\mathbf{e}}_2 + \alpha_3\hat{\mathbf{e}}_3, \text{ dot both sides with } \hat{\mathbf{e}}_1 \text{ and use (3.12)}$$

$$\mathbf{v}\cdot\hat{\mathbf{e}}_1 = \alpha_1\|\hat{\mathbf{e}}_1\|^2 + \alpha_2\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 + \alpha_3\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_1$$

$$\mathbf{v}\cdot\hat{\mathbf{e}}_1 = \alpha_1. \tag{3.13}$$

Proceeding in a similar way for α_2 and α_3, we finally have:

$$\mathbf{v} = \sum_{i=1}^{3} (\mathbf{v} \cdot \hat{\mathbf{e}}_i)\, \hat{\mathbf{e}}_i. \tag{3.14}$$

Recall the notation $\mathscr{B}_E$ (basis "E") to represent the orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. The dot-product allows us to map each vector (3D arrow) to a triplet of components in $\mathbb{R}^3$

$$\begin{bmatrix} \mathbf{v} \cdot \hat{\mathbf{e}}_1 \\ \mathbf{v} \cdot \hat{\mathbf{e}}_2 \\ \mathbf{v} \cdot \hat{\mathbf{e}}_3 \end{bmatrix}_{\mathscr{B}_E\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}} = \begin{bmatrix} {}^E v_1 \\ {}^E v_2 \\ {}^E v_3 \end{bmatrix} \triangleq {}^E\mathbf{v}. \tag{3.15}$$

The left-superscript notation ${}^E\mathbf{v}$ is from [Craig(2004)]. Some books start with this as the definition of a vector, but this is dependent on a basis – the physical geometric vector is independent of any basis.

Recall: Matrix Transpose

- Let $\mathbf{A}$ be an $m \times n$ matrix. Its transpose, denoted $\mathbf{A}^\mathsf{T}$ is an $n \times m$ matrix, whose kth column is the kth row of $\mathbf{A}$, or equivalently, whose kth row is the kth column of $\mathbf{A}$. In other words,

$$\mathbf{A}^\mathsf{T}[i,j] = \mathbf{A}[j,i], \qquad i = 1,\ldots,n. \quad j = 1,\ldots,m. \tag{3.16}$$

- As an example,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad \mathbf{A}^\mathsf{T} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}. \tag{3.17}$$

- If $\mathbf{A} = \mathbf{A}^\mathsf{T}$, it is called symmetric.
- If $\mathbf{A} = -\mathbf{A}^\mathsf{T}$, it is called skew-symmetric.
- Since, as shown in (3.15), ${}^{E}\mathbf{v}$ is a column-matrix, its transpose is a row-matrix

$$({}^{E}\mathbf{v})^\mathsf{T} = \begin{bmatrix} {}^{E}v_1 & {}^{E}v_2 & {}^{E}v_3 \end{bmatrix} \tag{3.18}$$

Dot-Product in terms of Vector Components

- As in the last slide, let us resolve vectors **u** and **v** in a basis $\mathscr{B}_E$.

$$\mathbf{u} = \sum_{i=1}^{3} {}^E u_i \, \hat{\mathbf{e}}_i, \qquad \mathbf{v} = \sum_{j=1}^{3} {}^E v_j \, \hat{\mathbf{e}}_j$$

- Then their dot-product is

$$\mathbf{u} \cdot \mathbf{v} = \left(\sum_{i=1}^{3} {}^E u_i \, \hat{\mathbf{e}}_i \right) \cdot \left(\sum_{j=1}^{3} {}^E v_j \, \hat{\mathbf{e}}_j \right)$$

- Since we showed in (3.12) that the dot-product distributes over vector addition, we can open the brackets:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} = & \left(\sum_{i=1}^{3} {}^E u_i \, \hat{\mathbf{e}}_i \right) \cdot ({}^E v_1 \, \hat{\mathbf{e}}_1) + \left(\sum_{i=1}^{3} {}^E u_i \, \hat{\mathbf{e}}_i \right) \cdot ({}^E v_2 \, \hat{\mathbf{e}}_2) \\ & + \left(\sum_{i=1}^{3} {}^E u_i \, \hat{\mathbf{e}}_i \right) \cdot ({}^E v_3 \, \hat{\mathbf{e}}_3) \end{aligned} \tag{3.19}$$

Dot-Product in terms of Vector Components

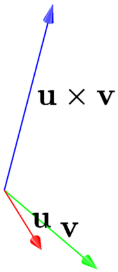
- We apply the distributivity of the dot product once more and use the fact that $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 1$ if $i = j$, and $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0$ if $i \neq j$. Thus,

$$\mathbf{u} \cdot \mathbf{v} = {}^{E}u_1 \, {}^{E}v_1 + {}^{E}u_2 \, {}^{E}v_2 + {}^{E}u_3 \, {}^{E}v_3. \tag{3.20a}$$

$$= ({}^{E}\mathbf{u})^{\mathsf{T}} \, {}^{E}\mathbf{v} \tag{3.20b}$$

$$= ({}^{E}\mathbf{v})^{\mathsf{T}} \, {}^{E}\mathbf{u}$$

Cross-Product of Vectors



- The cross-product of vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \times \mathbf{v}$ and is defined such that:
  1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.
  2. Its magnitude is given by $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\||\sin(\mathbf{u}, \mathbf{v})|$.
  3. The direction of $\mathbf{u} \times \mathbf{v}$ is found by the right-hand rule: If you curl the fingers of your right-hand from $\mathbf{u}$ to $\mathbf{v}$, your thumb will point in the direction of $\mathbf{u} \times \mathbf{v}$.

Uses of the Cross-Product

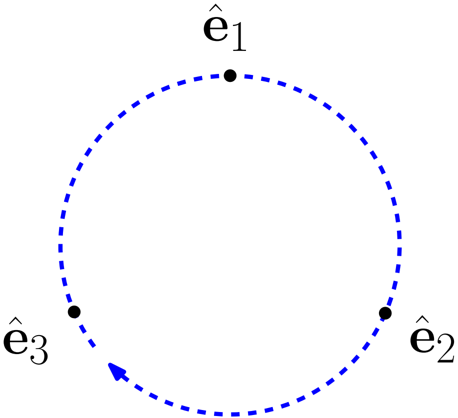
- We will learn later that the cross-product is used to define the relationship between:
  - The angular-velocity of a rigid-body and its velocity-field;
  - A force acting on a rigid-body and its moment about a point;
  - The angular-momentum of a rigid-body and its linear-momentum, etc.
- We already mentioned the following equation which plays a key role in the operation of electric motors:

$$\mathbf{f} = \ell\,\mathbf{i} \times \mathbf{B}.$$

- In two of the famous Maxwell's equations for electromagnetic waves, the cross-product makes an implicit appearance as the "curl" operator.

Properties of the Cross-Product

- If $\mathbf{v} = \lambda \mathbf{u}$, $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
- From the definition, one sees that $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$. Thus, the cross-product is anti-commutative.
- If $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$, the orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ is called right-handed (RH). We will assume henceforth that any basis $\mathscr{B}_E\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ is orthonormal and right-handed (ORH).
- Other names commonly used for right-handed orthonormal bases are $\{\hat{\imath}, \hat{\jmath}, \hat{\mathbf{k}}\}$ and $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$.



$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_i = \mathbf{0}, \qquad i = 1, 2, 3 \tag{3.21a}$$

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_k, \qquad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \tag{3.21b}$$

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = -\hat{\mathbf{e}}_k, \qquad (i, j, k) = (2, 1, 3), (3, 2, 1), (1, 3, 2) \tag{3.21c}$$

The Cross-Product Distributes over Vector-Addition

We want to show that:

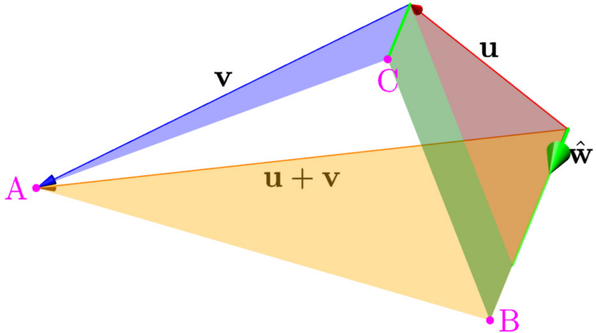
$$\mathbf{w} \times (\mathbf{u} + \mathbf{v}) = \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v} \tag{3.22}$$

Since this is trivially satisfied if $\mathbf{w} = \mathbf{0}$, we assume $\|\mathbf{w}\| \neq 0$ and divide both sides by it. So, we can equivalently prove:

$$\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v}) = \hat{\mathbf{w}} \times \mathbf{u} + \hat{\mathbf{w}} \times \mathbf{v} \tag{3.23}$$

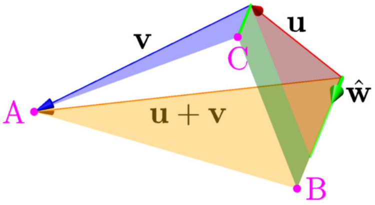
The Cross-Product Distributes over Vector-Addition

Geometric Proof

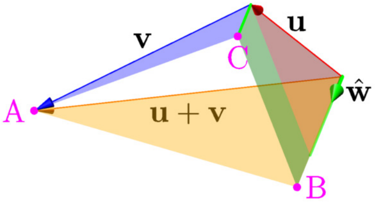


Using $\|\hat{\mathbf{w}}\| = 1$, we see from the figure that,

$$AB = \|\mathbf{u} + \mathbf{v}\| \sin(\hat{\mathbf{w}}, \mathbf{u} + \mathbf{v}) = \|\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v})\|$$
$$AC = \|\mathbf{v}\| \sin(\hat{\mathbf{w}}, \mathbf{v}) = \|\hat{\mathbf{w}} \times \mathbf{v}\|$$
$$CB = \|\mathbf{u}\| \sin(\hat{\mathbf{w}}, \mathbf{u}) = \|\hat{\mathbf{w}} \times \mathbf{u}\|$$

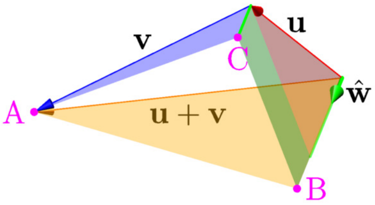


- Since their lengths match, we may be tempted to conclude that $\overrightarrow{AC} = \hat{\mathbf{w}} \times \mathbf{v}$, $\overrightarrow{CB} = \hat{\mathbf{w}} \times \mathbf{u}$, $\overrightarrow{AB} = \hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v})$ and as $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$, we have proved, $\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v}) = \hat{\mathbf{w}} \times \mathbf{u} + \hat{\mathbf{w}} \times \mathbf{v}$.
- But, direction-wise, $\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v})$ is not parallel to $\overrightarrow{AB}$ but is actually orthogonal to it and to the orange-plane, and points down.
- Similarly, $\hat{\mathbf{w}} \times \mathbf{u}$ is orthogonal to the red-plane and points to the left, and finally, $\hat{\mathbf{w}} \times \mathbf{v}$ is orthogonal to the blue-plane and points down and right. The vectors $\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v}), \hat{\mathbf{w}} \times \mathbf{u}, \hat{\mathbf{w}} \times \mathbf{v}$ are coplanar in the plane of $\triangle ABC$: shown in the next slide.



We want to first show that $\hat{\mathbf{w}} \times \mathbf{u}$, $\hat{\mathbf{w}} \times \mathbf{v}$, and $\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v})$ lie in the plane of $\triangle ABC$.

- Using (3.11), $\overrightarrow{BC} = \mathbf{u} - (\mathbf{u} \cdot \hat{\mathbf{w}})\,\hat{\mathbf{w}}$ is $\perp$ to $\hat{\mathbf{w}}$.
- Using (3.11), $\overrightarrow{CA} = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{w}})\,\hat{\mathbf{w}}$ is $\perp$ to $\hat{\mathbf{w}}$.
- Using (3.11), $\overrightarrow{BA} = (\mathbf{u} + \mathbf{v}) - ((\mathbf{u} + \mathbf{v}) \cdot \hat{\mathbf{w}})\,\hat{\mathbf{w}}$ is $\perp$ to $\hat{\mathbf{w}}$.
- Hence, $\triangle ABC$ lies in the plane $\perp$ to $\hat{\mathbf{w}}$.



- We showed that $\triangle ABC$ lies in the plane $\perp$ to $\hat{\mathbf{w}}$.
- By definition of the cross-product, $\hat{\mathbf{w}} \times \mathbf{u}$, $\hat{\mathbf{w}} \times \mathbf{v}$, and $\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v})$ all lie in a plane $\perp$ to $\hat{\mathbf{w}}$. Hence, they are coplanar with $\triangle ABC$.

The Cross-Product Distributes over Vector-Addition

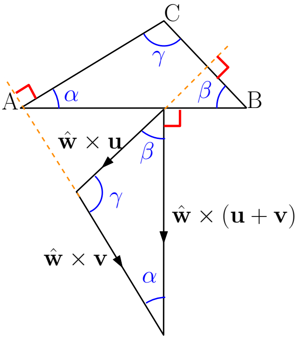


Figure 3.4: How the vectors look in the plane of $\triangle ABC$ which is $\perp$ to $\hat{\mathbf{w}}$.

- We use the property that in a plane if 2 segments a and b have an angle α, and if $c \perp a$ and $d \perp b$, then c and d have an angle α also.
- Finally, as shown above, using congruent triangles, the three vectors satisfy $\hat{\mathbf{w}} \times (\mathbf{u} + \mathbf{v}) = \hat{\mathbf{w}} \times \mathbf{u} + \hat{\mathbf{w}} \times \mathbf{v}$.

Other Cross-Product Results

- What is the general solution of $\mathbf{x} \times \mathbf{v} = \mathbf{w} \times \mathbf{v}$ for $\mathbf{x}$?
- The solution is

$$\mathbf{x} = \mathbf{w} + \lambda \mathbf{v}, \text{ where, } \lambda \text{ is a free parameter.} \tag{3.24}$$

The Cross-Product in terms of Vector Components

- Since we have shown in (3.22) that cross-product distributes over vector addition, we can use it to derive the cross-product in a more familiar expression. For this, we resolve all vectors in $\mathscr{B}_E$ as in (3.15)

$$\mathbf{u} \times \mathbf{v} = \left(\sum_{i=1}^{3} {}^{E}u_i \, \hat{\mathbf{e}}_i \right) \times \left(\sum_{j=1}^{3} {}^{E}v_j \, \hat{\mathbf{e}}_j \right) \tag{3.25}$$

- Using the distributivity property (3.22) twice, we can expand the product, and use (3.21)

$$\begin{aligned} \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_i &= \mathbf{0}, & i &= 1, 2, 3 \\ \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j &= \hat{\mathbf{e}}_k, & (i, j, k) &= (1, 2, 3), (2, 3, 1), (3, 1, 2) \\ \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j &= -\hat{\mathbf{e}}_k, & (i, j, k) &= (2, 1, 3), (3, 2, 1), (1, 3, 2) \end{aligned}$$

The Cross-Product in terms of Vector Components

This expansion gives

$$\begin{aligned}\mathbf{u} \times \mathbf{v} = &\left({}^{E}u_2\, {}^{E}v_3 - {}^{E}u_3\, {}^{E}v_2 \right) \hat{\mathbf{e}}_1 \\ &+ \left({}^{E}u_3\, {}^{E}v_1 - {}^{E}u_1\, {}^{E}v_3 \right) \hat{\mathbf{e}}_2 \\ &+ \left({}^{E}u_1\, {}^{E}v_2 - {}^{E}u_2\, {}^{E}v_1 \right) \hat{\mathbf{e}}_3 \end{aligned} \tag{3.26}$$

Since this looks messy, we usually use a mnemonic: we informally expand the following determinant symbolically.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ {}^{E}u_1 & {}^{E}u_2 & {}^{E}u_3 \\ {}^{E}v_1 & {}^{E}v_2 & {}^{E}v_3 \end{vmatrix} \tag{3.27}$$

Algebraically, it does not make any sense since the first row has vectors and the rest scalars. It can be easily verified that on doing the determinant expansion we get (3.26) again.

Triple Cross-Product

- The cross product is not associative, e.g.

$$\hat{\mathbf{e}}_1 \times (\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2) \neq (\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1) \times \hat{\mathbf{e}}_2$$

- The following expansion of the triple cross-product can be shown by resolving on an arbitrary basis:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w} \tag{3.28}$$

- The above result can be used to verify the Jacobi identity:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) = \mathbf{0}. \tag{3.29}$$

Triple Scalar Product

Definition 3.3 (Triple Scalar Product)

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) \triangleq (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \tag{3.30}$$

The Scalar Triple Product as a Determinant

To evaluate the scalar triple product (3.30) using components, we can combine the component-form of the cross-product (3.26) with the distributivity of the dot-product as follows:

$$
\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) \\
&= \Big(\sum_{j=1}^{3} {}^{E}w_j\, \hat{\mathbf{e}}_j\Big) \cdot \Big[\big({}^{E}u_2\, {}^{E}v_3 - {}^{E}u_3\, {}^{E}v_2\big)\, \hat{\mathbf{e}}_1 \\
&\quad + \big({}^{E}u_3\, {}^{E}v_1 - {}^{E}u_1\, {}^{E}v_3\big)\, \hat{\mathbf{e}}_2 + \big({}^{E}u_1\, {}^{E}v_2 - {}^{E}u_2\, {}^{E}v_1\big)\, \hat{\mathbf{e}}_3\Big] \\
&= {}^{E}w_1\big({}^{E}u_2\, {}^{E}v_3 - {}^{E}u_3\, {}^{E}v_2\big) \\
&\quad + {}^{E}w_2\big({}^{E}u_3\, {}^{E}v_1 - {}^{E}u_1\, {}^{E}v_3\big) + {}^{E}w_3\big({}^{E}u_1\, {}^{E}v_2 - {}^{E}u_2\, {}^{E}v_1\big)
\end{aligned}
\tag{3.31a}
$$

$$
= \begin{vmatrix} {}^{E}w_1 & {}^{E}w_2 & {}^{E}w_3 \\ {}^{E}u_1 & {}^{E}u_2 & {}^{E}u_3 \\ {}^{E}v_1 & {}^{E}v_2 & {}^{E}v_3 \end{vmatrix} = \begin{vmatrix} {}^{E}u_1 & {}^{E}u_2 & {}^{E}u_3 \\ {}^{E}v_1 & {}^{E}v_2 & {}^{E}v_3 \\ {}^{E}w_1 & {}^{E}w_2 & {}^{E}w_3 \end{vmatrix} \tag{3.31b}
$$

Properties of the Scalar Triple Product

- Using the properties of the determinant in (3.31a), it can be verified that the scalar triple product remains unchanged under a circular permutation of the vectors.

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{v}, \mathbf{w}, \mathbf{u}) = (\mathbf{w}, \mathbf{u}, \mathbf{v}). \tag{3.32}$$

- It changes sign under permutation of any two vectors, e.g.

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -(\mathbf{v}, \mathbf{u}, \mathbf{w}) \tag{3.33}$$

Linear Dependence Revisited

- The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$ is LD if $\mathbf{u}_2 = \lambda\mathbf{u}_1$ (collinear vectors).
- The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is LD if either of the following is true:
  - A subset $\{\mathbf{u}_i, \mathbf{u}_j\}$, $i \neq j$ is LD, or,
  - The vectors are coplanar, i.e., if, for some distinct i, j, k, the following is true: $\mathbf{u}_k = \lambda_i\mathbf{u}_i + \lambda_j\mathbf{u}_j$.
- We can write the above more succinctly. The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is LD iff $\exists\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, not all zero, s.t.:

$$\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 = \mathbf{0}. \tag{3.34}$$

- If the only solution of (3.34) is $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the vectors are LI.
- Since we're in 3D, any set of 4 or more vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \ldots, \mathbf{u}_n\}$ is always LD, i.e. $\exists\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots \in \mathbb{R}$, not all zero, s.t. $\sum_{i=1}^{n} \lambda_i\mathbf{u}_i = \mathbf{0}$, $n \geq 4$.

When is the Triple Scalar Product 0?

As $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$,
What does $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$ imply? It implies at least one of the following:

1. Any two vectors are collinear, or,
2. All three vectors are coplanar: think geometrically!

Linear Operators on Vectors

Definition 3.4 (Linear Operator)

An operator $\mathcal{L} : \mathbb{E} \to \mathbb{E}$ is linear if it satisfies:

$$\mathcal{L}(\lambda \mathbf{u}) = \lambda \mathcal{L}(\mathbf{u}) \qquad \text{Scaling Property} \tag{3.35a}$$

$$\mathcal{L}(\mathbf{u} + \mathbf{v}) = \mathcal{L}(\mathbf{u}) + \mathcal{L}(\mathbf{v}) \qquad \text{Superposition Property} \tag{3.35b}$$

- Example: Consider the operator $\mathcal{L}(\mathbf{x}) \triangleq \mathbf{v} \times \mathbf{x}$. Is this linear?
  - It satisfies the scaling property $\mathcal{L}(\lambda\mathbf{x}) = \mathbf{v} \times (\lambda\mathbf{x}) = \lambda\mathbf{v} \times \mathbf{x} = \lambda\mathcal{L}(\mathbf{x})$.
  - It satisfies the superposition property since it comes from the distributivity of the cross-product over vector-addition (3.22).
  - Hence, the cross-product operator is linear.

Properties of Linear Operators

- If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two linear operators, then their composition

$$(\mathcal{L}_1 \circ \mathcal{L}_2)(\mathbf{v}) \triangleq \mathcal{L}_1\big(\mathcal{L}_2(\mathbf{v})\big) \tag{3.36}$$

  is also a linear operator.
  - Example: $\mathcal{L}(\mathbf{x}) \triangleq \mathbf{v} \times (\mathbf{v} \times \mathbf{x})$ is a linear operator.
- For any linear operator $\mathcal{L}$, $\mathcal{L}(\mathbf{0}) = \mathbf{0}$.

Matrix of a Linear Operator on a Basis

How a linear operator modifies the unit-vectors of a basis $\mathscr{B}_E$ completely defines it. Let,

$$\mathcal{L}(\hat{\mathbf{e}}_1) = \lambda_{11}\hat{\mathbf{e}}_1 + \lambda_{21}\hat{\mathbf{e}}_2 + \lambda_{31}\hat{\mathbf{e}}_3 \tag{3.37}$$

$$\mathcal{L}(\hat{\mathbf{e}}_2) = \lambda_{12}\hat{\mathbf{e}}_1 + \lambda_{22}\hat{\mathbf{e}}_2 + \lambda_{32}\hat{\mathbf{e}}_3 \tag{3.38}$$

$$\mathcal{L}(\hat{\mathbf{e}}_3) = \lambda_{13}\hat{\mathbf{e}}_1 + \lambda_{23}\hat{\mathbf{e}}_2 + \lambda_{33}\hat{\mathbf{e}}_3 \tag{3.39}$$

Then, if $\mathbf{v} = {}^{E}v_1\hat{\mathbf{e}}_1 + {}^{E}v_2\hat{\mathbf{e}}_2 + {}^{E}v_3\hat{\mathbf{e}}_3$,

$$\mathbf{u} = \mathcal{L}(\mathbf{v}) = \mathcal{L}\left({}^{E}v_1\hat{\mathbf{e}}_1 + {}^{E}v_2\hat{\mathbf{e}}_2 + {}^{E}v_3\hat{\mathbf{e}}_3\right)$$

From scaling and superposition properties of the linear operator

$$= {}^{E}v_1\mathcal{L}(\hat{\mathbf{e}}_1) + {}^{E}v_2\mathcal{L}(\hat{\mathbf{e}}_2) + {}^{E}v_3\mathcal{L}(\hat{\mathbf{e}}_3) \tag{3.40}$$

$$\begin{aligned} &= \left({}^{E}v_1\lambda_{11} + {}^{E}v_2\lambda_{12} + {}^{E}v_3\lambda_{13}\right)\hat{\mathbf{e}}_1 \\ &\quad + \left({}^{E}v_1\lambda_{21} + {}^{E}v_2\lambda_{22} + {}^{E}v_3\lambda_{23}\right)\hat{\mathbf{e}}_2 \\ &\quad + \left({}^{E}v_1\lambda_{31} + {}^{E}v_2\lambda_{32} + {}^{E}v_3\lambda_{33}\right)\hat{\mathbf{e}}_3 \end{aligned} \tag{3.41}$$

From the last slide,

$$\mathbf{u} = \mathcal{L}(\mathbf{v}) = \mathcal{L}\left({}^{E}v_1\hat{\mathbf{e}}_1 + {}^{E}v_2\hat{\mathbf{e}}_2 + {}^{E}v_3\hat{\mathbf{e}}_3\right)$$

From scaling and superposition properties of the linear operator

$$\begin{aligned} &= {}^{E}v_1\mathcal{L}(\hat{\mathbf{e}}_1) + {}^{E}v_2\mathcal{L}(\hat{\mathbf{e}}_2) + {}^{E}v_3\mathcal{L}(\hat{\mathbf{e}}_3) \\ &= ({}^{E}v_1\lambda_{11} + {}^{E}v_2\lambda_{12} + {}^{E}v_3\lambda_{13})\,\hat{\mathbf{e}}_1 \\ &\quad + ({}^{E}v_1\lambda_{21} + {}^{E}v_2\lambda_{22} + {}^{E}v_3\lambda_{23})\,\hat{\mathbf{e}}_2 \\ &\quad + ({}^{E}v_1\lambda_{31} + {}^{E}v_2\lambda_{32} + {}^{E}v_3\lambda_{33})\,\hat{\mathbf{e}}_3 \end{aligned}$$

This shows that

$${}^{E}\mathbf{u} = \begin{bmatrix} {}^{E}v_1\lambda_{11} + {}^{E}v_2\lambda_{12} + {}^{E}v_3\lambda_{13} \\ {}^{E}v_1\lambda_{21} + {}^{E}v_2\lambda_{22} + {}^{E}v_3\lambda_{23} \\ {}^{E}v_1\lambda_{31} + {}^{E}v_2\lambda_{32} + {}^{E}v_3\lambda_{33} \end{bmatrix} \tag{3.42}$$

$$= \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} {}^{E}v_1 \\ {}^{E}v_2 \\ {}^{E}v_3 \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} {}^{E}\mathbf{v} \tag{3.43}$$

Matrix of a Linear Operator on a Basis

If we now define the matrix of $\mathcal{L}$ on $\mathscr{B}_E$ as follows:

$$^{E}[\mathcal{L}] \triangleq \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}, \qquad \text{note the columns!} \qquad (3.44)$$

$$^{E}[\mathcal{L}](i,j) = \mathcal{L}(\hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_i \qquad \text{a useful result} \qquad (3.45)$$

Then (3.43), can be written as:

$$\text{If } \mathbf{u} = \mathcal{L}(\mathbf{v}) \text{ then,} \qquad {}^{E}\mathbf{u} = {}^{E}[\mathcal{L}]\ {}^{E}\mathbf{v}. \qquad (3.46)$$

Example: Matrix of Cross-Product Operator on $\mathscr{B}_E$

We saw earlier that $\mathcal{L}(\mathbf{x}) \triangleq \mathbf{v} \times \mathbf{x}$ is a linear operator. Its matrix on $\mathscr{B}_E$ can be found as follows:

- Let $\mathbf{v} = {}^E v_1 \hat{\mathbf{e}}_1 + {}^E v_2 \hat{\mathbf{e}}_2 + {}^E v_3 \hat{\mathbf{e}}_3$.
- Then,

$$\mathcal{L}(\hat{\mathbf{e}}_1) = \mathbf{v} \times \hat{\mathbf{e}}_1 = -{}^E v_2 \hat{\mathbf{e}}_3 + {}^E v_3 \hat{\mathbf{e}}_2 \quad (3.47)$$

$$\mathcal{L}(\hat{\mathbf{e}}_2) = \mathbf{v} \times \hat{\mathbf{e}}_2 = {}^E v_1 \hat{\mathbf{e}}_3 - {}^E v_3 \hat{\mathbf{e}}_1 \quad (3.48)$$

$$\mathcal{L}(\hat{\mathbf{e}}_3) = \mathbf{v} \times \hat{\mathbf{e}}_3 = -{}^E v_1 \hat{\mathbf{e}}_2 + {}^E v_2 \hat{\mathbf{e}}_1 \quad (3.49)$$

- So, the desired matrix is (check column-wise)

$${}^E[\mathcal{L}] = \begin{bmatrix} 0 & -{}^E v_3 & {}^E v_2 \\ {}^E v_3 & 0 & -{}^E v_1 \\ -{}^E v_2 & {}^E v_1 & 0 \end{bmatrix} \quad (3.50)$$

$$\triangleq \left[{}^E \mathbf{v} \times \right], \quad \text{a skew-symmetric matrix.} \quad (3.51)$$

Composition is Matrix Multiplication

It is easily seen that the composition of two linear operators has the following matrix on $\mathscr{B}_E$

$$(\mathcal{A} \circ \mathcal{B})(\mathbf{v}) = \mathcal{A}(\mathcal{B}(\mathbf{v})) \tag{3.52a}$$

$$\Rightarrow {}^{E}[\mathcal{A} \circ \mathcal{B}]\ {}^{E}\mathbf{v} = {}^{E}[\mathcal{A}]\ {}^{E}[\mathcal{B}]\ {}^{E}\mathbf{v}. \tag{3.52b}$$

$$\Rightarrow {}^{E}[\mathcal{A} \circ \mathcal{B}] = {}^{E}[\mathcal{A}]\ {}^{E}[\mathcal{B}] \tag{3.52c}$$

Contents

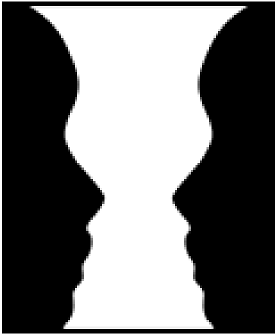
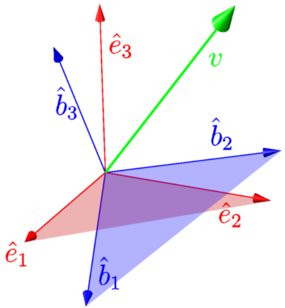


Figure 4.1: Two ways of looking at the same thing.[3]

[3] "Cup or faces paradox" by Bryan Derksen - Original image Image:Cup or faces paradox.jpg uploaded by User:Guam on 28 July 2005, SVG conversion by Bryan Derksen. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Cup_or_faces_paradox.svg

Two Ways of Looking At the Same Problem



1. We find a relation between the components of $\mathbf{v}$ on bases $\mathscr{B}_E\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ and $\mathscr{B}_B\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$.
   - We will see that ${}^{E}\mathbf{v} = {}^{E}_{B}\mathbf{R}\,{}^{B}\mathbf{v}$.
2. We define a linear-operator (recall (3.35)) of rotation such that $\mathcal{R}_{EB}(\hat{\mathbf{e}}_i) = \hat{\mathbf{b}}_i,\ i = 1, \ldots, 3$.

- Suppose we resolve a vector $\mathbf{v}$ in two bases $\mathscr{B}_E\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ and $\mathscr{B}_B\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$. What is the relation between the components of $\mathbf{v}$ on bases $\mathscr{B}_E$ and $\mathscr{B}_B$?
- Recall the left-superscript convention in (3.15).
- Since geometrically it is the same vector (arrow), we have

$$ {}^E v_1 \hat{\mathbf{e}}_1 + {}^E v_2 \hat{\mathbf{e}}_2 + {}^E v_3 \hat{\mathbf{e}}_3 = {}^B v_1 \hat{\mathbf{b}}_1 + {}^B v_2 \hat{\mathbf{b}}_2 + {}^B v_3 \hat{\mathbf{b}}_3 \tag{4.1} $$

- Dotting both sides by $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, we get:

$$ {}^E v_1 = {}^B v_1 (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{e}}_1) + {}^B v_2 (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_1) + {}^B v_3 (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{e}}_1) \tag{4.2} $$

$$ {}^E v_2 = {}^B v_1 (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{e}}_2) + {}^B v_2 (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_2) + {}^B v_3 (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{e}}_2) \tag{4.3} $$

$$ {}^E v_3 = {}^B v_1 (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{e}}_3) + {}^B v_2 (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_3) + {}^B v_3 (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{e}}_3) \tag{4.4} $$

- We can write this in matrix notation as

$$ \begin{bmatrix} {}^E v_1 \\ {}^E v_2 \\ {}^E v_3 \end{bmatrix} = \begin{bmatrix} (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{e}}_1) & (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_1) & (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{e}}_1) \\ (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{e}}_2) & (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_2) & (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{e}}_2) \\ (\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{e}}_3) & (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_3) & (\hat{\mathbf{b}}_3 \cdot \hat{\mathbf{e}}_3) \end{bmatrix} \begin{bmatrix} {}^B v_1 \\ {}^B v_2 \\ {}^B v_3 \end{bmatrix} \tag{4.5} $$

- On observing the columns of the 3×3 matrix:

$$
\begin{aligned}
{}^{E}\mathbf{v} &= \begin{bmatrix} {}^{E}\hat{\mathbf{b}}_1 & {}^{E}\hat{\mathbf{b}}_2 & {}^{E}\hat{\mathbf{b}}_3 \end{bmatrix} {}^{B}\mathbf{v} \qquad (4.6) \\
&= \begin{bmatrix} ({}^{B}\hat{\mathbf{e}}_1)^{\mathsf{T}} \\ ({}^{B}\hat{\mathbf{e}}_2)^{\mathsf{T}} \\ ({}^{B}\hat{\mathbf{e}}_3)^{\mathsf{T}} \end{bmatrix} {}^{B}\mathbf{v} \qquad (4.7)
\end{aligned}
$$

- This leads us to the definition of the matrix ${}^{E}_{B}\mathbf{R}$,

$$
{}^{E}\mathbf{v} = {}^{E}_{B}\mathbf{R}\, {}^{B}\mathbf{v} \qquad (4.8)
$$

- ${}^{E}_{B}\mathbf{R}$ is called the Direction-Cosines Matrix (DCM) of $\mathscr{B}_B$ with respect to $\mathscr{B}_E$ because its element at the ith row and jth column is

$$
{}^{E}_{B}\mathbf{R}[i, j] = \hat{\mathbf{b}}_j \cdot \hat{\mathbf{e}}_i = \cos(\hat{\mathbf{b}}_j, \hat{\mathbf{e}}_i), \qquad (4.9)
$$

A DCM is sometimes also called a Rotation Matrix.

The Determinant of a DCM is $+1$

- Using the row-vectors form (4.7) of the DCM, and by analogy of the determinant form of the scalar triple-product (3.31a), we get,

$$\det({}^{E}_{B}\mathbf{R}) = \begin{vmatrix} ({}^{B}\hat{\mathbf{e}}_1)^{\mathsf{T}} \\ ({}^{B}\hat{\mathbf{e}}_2)^{\mathsf{T}} \\ ({}^{B}\hat{\mathbf{e}}_3)^{\mathsf{T}} \end{vmatrix} \tag{4.10}$$

$$= ({}^{B}\hat{\mathbf{e}}_1 \times {}^{B}\hat{\mathbf{e}}_2) \cdot {}^{B}\hat{\mathbf{e}}_3 \tag{4.11}$$

$$= {}^{B}\hat{\mathbf{e}}_3 \cdot {}^{B}\hat{\mathbf{e}}_3 \tag{4.12}$$

$$= +1. \tag{4.13}$$

Hence, all DCMs are invertible!

- An orthogonal matrix with a determinant -1 represents a reflection rather than a rotation.
- The set of DCMs is also very commonly written as $SO(3)$ (The Special Orthogonal Group in 3D). It is special because the determinant is $+1$. "Groups" are structures studied in abstract algebra (beyond the scope of this course).
- Although, a DCM has 9 elements, these elements have the following 6 constraints:
  1. Each column is a unit-vector: 3 constraints.
  2. Each column is orthogonal to the other two: $\binom{3}{2} = 3$ constraints.
- Hence, a DCM has only $9 - 6 = 3$ real degrees of freedom.
- So, a 3D rotation between two bases can be parametrized by only 3 parameters, but unfortunately using such a minimal parameterization always leads to singularities at some points . We will see this when we study Euler angles.

The Transpose of a DCM is also its Inverse

- Since ${}^{E}\mathbf{v}$ and ${}^{B}\mathbf{v}$ represent the same physical vector $\mathbf{v}$, from (3.20),

$$\mathbf{v} \cdot \mathbf{v} = ({}^{B}\mathbf{v})^{\mathsf{T}}\, {}^{B}\mathbf{v} = ({}^{E}\mathbf{v})^{\mathsf{T}}\, {}^{E}\mathbf{v} \tag{4.14}$$

- But, from (4.8), and using the result $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$,

$$\begin{aligned} ({}^{E}\mathbf{v})^{\mathsf{T}}\, {}^{E}\mathbf{v} &= ({}^{E}_{B}\mathbf{R}\, {}^{B}\mathbf{v})^{\mathsf{T}}\, {}^{E}_{B}\mathbf{R}\, {}^{B}\mathbf{v} && (4.15) \\ &= ({}^{B}\mathbf{v})^{\mathsf{T}} ({}^{E}_{B}\mathbf{R})^{\mathsf{T}}\, {}^{E}_{B}\mathbf{R}\, {}^{B}\mathbf{v} && (4.16) \end{aligned}$$

- Combining (4.14) and (4.16), we get

$$\begin{aligned} ({}^{E}_{B}\mathbf{R})^{\mathsf{T}}\, {}^{E}_{B}\mathbf{R} &= \mathbf{I}_3 \qquad \text{The } 3 \times 3 \text{ Identity Matrix.} && (4.17) \\ \therefore\ ({}^{E}_{B}\mathbf{R})^{\mathsf{T}} &= ({}^{E}_{B}\mathbf{R})^{-1} = {}^{B}_{E}\mathbf{R}. && (4.18) \end{aligned}$$

- So finding the inverse of a DCM is easy and computationally inexpensive!

- Due to the property $({}^{E}_{B}\mathbf{R})^{\mathsf{T}}\,{}^{E}_{B}\mathbf{R} = \mathbf{I}_3$, DCMs belong to the set of orthogonal (or more percisely in this case, orthonormal) matrices.

An Example DCM

$$ {}^{A}_{B}\mathbf{R} = \frac{1}{25}\begin{bmatrix} 9 & 12 & -20 \\ 12 & 16 & 15 \\ 20 & -15 & 0 \end{bmatrix} \tag{4.19} $$

To verify that it is a DCM, you need to check two properties:

1. Its determinant is 1: (4.13)
2. Its transpose is its inverse: (4.17)

Composition of Rotations

- Suppose we have three bases $\mathscr{B}_E$, $\mathscr{B}_B$, $\mathscr{B}_A$. Then, from (4.8),

$$
\begin{aligned}
{}^{E}\mathbf{v} &= {}^{E}_{B}\mathbf{R}\ {}^{B}\mathbf{v} && \text{(4.20a)}\\
&= {}^{E}_{A}\mathbf{R}\ {}^{A}\mathbf{v} && \text{(4.20b)}\\
{}^{A}\mathbf{v} &= {}^{A}_{B}\mathbf{R}\ {}^{B}\mathbf{v} && \text{(4.20c)}
\end{aligned}
$$

- Substituting (4.20c) in (4.20b), we get

$$
{}^{E}\mathbf{v} = {}^{E}_{A}\mathbf{R}\ {}^{A}_{B}\mathbf{R}\ {}^{B}\mathbf{v}. \tag{4.20d}
$$

- Comparing this with (4.20a), we get the simple expression

$$
{}^{E}_{B}\mathbf{R} = {}^{E}_{A}\mathbf{R}\ {}^{A}_{B}\mathbf{R}. \tag{4.20e}
$$

- The order of multiplication is important!
- Multiplying two DCMs gives another one!

How are Matrices of a Linear Operator in Bases $\mathscr{B}_E$ and $\mathscr{B}_B$ related?

Let $\mathcal{L}(\mathbf{v}) = \mathbf{w}$ be a linear operator. Recall (3.44) and (3.46). Consider two bases $\mathscr{B}_E$ and $\mathscr{B}_B$.

$$
\begin{aligned}
{}^{E}\mathbf{w} &= {}^{E}_{B}\mathbf{R}\ {}^{B}\mathbf{w} \\
&= {}^{E}[\mathcal{L}]\ {}^{E}\mathbf{v} \qquad\qquad = {}^{E}[\mathcal{L}]\ {}^{E}_{B}\mathbf{R}\ {}^{B}\mathbf{v} && (4.21)\\
\therefore\ {}^{B}\mathbf{w} &= \underbrace{{}^{E}_{B}\mathbf{R}^{\mathsf{T}}\ {}^{E}[\mathcal{L}]\ {}^{E}_{B}\mathbf{R}}_{\equiv {}^{B}[\mathcal{L}]}\ {}^{B}\mathbf{v} && (4.22)\\
{}^{B}[\mathcal{L}] &= {}^{E}_{B}\mathbf{R}^{\mathsf{T}}\ {}^{E}[\mathcal{L}]\ {}^{E}_{B}\mathbf{R}. && (4.23)\\
{}^{E}[\mathcal{L}] &= {}^{E}_{B}\mathbf{R}\ {}^{B}[\mathcal{L}]\ {}^{E}_{B}\mathbf{R}^{\mathsf{T}}. && (4.24)
\end{aligned}
$$

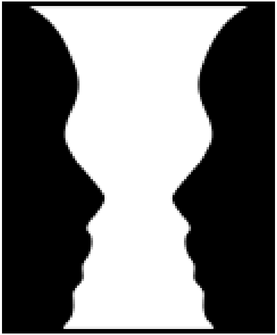


Figure 4.2: Two ways of looking at the same thing.[4]

[4] "Cup or faces paradox" by Bryan Derksen - Original image Image:Cup or faces paradox.jpg uploaded by User:Guam on 28 July 2005, SVG conversion by Bryan Derksen. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Cup_or_faces_paradox.svg

The Rotation Operator

Let Rotation Operator $\mathcal{R}_{EB}$ be the linear operator which maps (rotates) the basis $\mathscr{B}_E$ to $\mathscr{B}_B$ such that

$$\mathcal{R}_{EB}\left(\hat{\mathbf{e}}_i\right) = \hat{\mathbf{b}}_i, \qquad i = 1, \ldots, 3. \tag{4.25}$$

The Matrix of the Rotation Operator

Theorem 4.1

The matrix of the rotation-operator $\mathcal{R}_{EB}$ in both bases $\mathscr{B}_E$ to $\mathscr{B}_B$ is the direction-cosines matrix ${}^E_B\mathbf{R}$.

Proof.

First we show that ${}^E[\mathcal{R}_{EB}] = {}^E_B\mathbf{R}$.

$$ {}^E[\mathcal{R}_{EB}][i,j] \overset{(3.45)}{=} \mathcal{R}_{EB}(\hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_i \qquad = \hat{\mathbf{b}}_j \cdot \hat{\mathbf{e}}_i \qquad \overset{(4.9)}{=} {}^E_B\mathbf{R}[i,j] $$

Next we show that ${}^B[\mathcal{R}_{EB}] = {}^E_B\mathbf{R}$ too. From (4.23),

$$ \begin{aligned} {}^B[\mathcal{R}_{EB}] &= {}^E_B\mathbf{R}^\mathsf{T}\ {}^E[\mathcal{R}_{EB}]\ {}^E_B\mathbf{R} && (4.26) \\ &= {}^E_B\mathbf{R}^\mathsf{T}\ {}^E_B\mathbf{R}\ {}^E_B\mathbf{R} && (4.27) \\ &= {}^E_B\mathbf{R} && (4.28) \end{aligned} $$

Interlude

- What is ${}^{E}\hat{\mathbf{e}}_1$?
- What is ${}^{E}\hat{\mathbf{e}}_2$?
- What is ${}^{E}\hat{\mathbf{e}}_3$?
- What is ${}^{B}\hat{\mathbf{b}}_3$?
- What is ${}^{B}\hat{\mathbf{e}}_3$?

Hence, it should be true that

$$ {}^{E}\hat{\mathbf{b}}_i = {}^{E}_{B}\mathbf{R}\ {}^{E}\hat{\mathbf{e}}_i \tag{4.29} $$

$$ {}^{B}\hat{\mathbf{b}}_i = {}^{E}_{B}\mathbf{R}\ {}^{B}\hat{\mathbf{e}}_i \tag{4.30} $$

Composition of Rotation Operators

Theorem 4.2

Consider 3 bases $\mathscr{B}_E$, $\mathscr{B}_A$, and $\mathscr{B}_B$. Let us define the rotation-operators $\mathcal{R}_{EA}$ and $\mathcal{R}_{AB}$ as usual. Then, clearly (why?):

$$\mathcal{R}_{EB} = \mathcal{R}_{AB} \circ \mathcal{R}_{EA} \tag{4.31}$$

But, their matrix representations follow the rule

$$^{E}[\mathcal{R}_{EB}] = {}^{E}[\mathcal{R}_{EA}]\,{}^{A}[\mathcal{R}_{AB}] \tag{4.32}$$

Proof.

From Theorem 4.1, $^{E}[\mathcal{R}_{EB}] = {}^{E}_{B}\mathbf{R}$, $^{E}[\mathcal{R}_{EA}] = {}^{E}_{A}\mathbf{R}$, $^{A}[\mathcal{R}_{AB}] = {}^{A}_{B}\mathbf{R}$. Substituting in (4.32), we need to prove that

$$^{E}_{B}\mathbf{R} = {}^{E}_{A}\mathbf{R}\;{}^{A}_{B}\mathbf{R}$$

But, we already proved this property of DCMs in (4.20e). □

Theorem 4.3 (A Property of the DCM)

Let $\mathbf{R}$ *be a DCM. Then,* $\det(\mathbf{R} - \mathbf{I}) = 0$*.*

Proof.

$$\begin{aligned}
\det(-\mathbf{R}) &= (-1)^3 \det(\mathbf{R}) && = -\det(\mathbf{R}) \\
&= -1. && \qquad (4.33) \\
\det(\mathbf{R}^{-1}) &= \det(\mathbf{R}^{\mathsf{T}}) && = \det(\mathbf{R}) \\
&= 1. && \qquad (4.34) \\
\det(\mathbf{R} - \mathbf{I}) &= \det((\mathbf{R} - \mathbf{I})^{\mathsf{T}}) && = \det(\mathbf{R}^{\mathsf{T}} - \mathbf{I}) \\
&= \det(\mathbf{R}^{-1} - \mathbf{I}) && = \det(-\mathbf{R}^{-1}(\mathbf{R} - \mathbf{I})) \\
&= \det(-\mathbf{R}^{-1}) \det(\mathbf{R} - \mathbf{I}) && = -\det(\mathbf{R} - \mathbf{I}) \qquad (4.35) \\
\Rightarrow \det(\mathbf{R} - \mathbf{I}) &= 0. && \qquad (4.36)
\end{aligned}$$

We have used the result $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$, which we prove by the Matlab® symbolic-toolbox for the 3×3 case we need, and by intimidation for the general $n \times n$ case. □

- Let ${}^{E}_{B}\mathbf{R}$ be a DCM.
- From (4.7),

$$
{}^{E}_{B}\mathbf{R} - \mathbf{I} = \begin{bmatrix} \left({}^{B}\hat{\mathbf{e}}_1 - {}^{B}\hat{\mathbf{b}}_1\right)^{\mathsf{T}} \\ \left({}^{B}\hat{\mathbf{e}}_2 - {}^{B}\hat{\mathbf{b}}_2\right)^{\mathsf{T}} \\ \left({}^{B}\hat{\mathbf{e}}_3 - {}^{B}\hat{\mathbf{b}}_3\right)^{\mathsf{T}} \end{bmatrix} \tag{4.37}
$$

$$
\begin{aligned}
\det({}^{E}_{B}\mathbf{R} - \mathbf{I}) &\overset{(3.31a)}{=} \left({}^{B}\hat{\mathbf{e}}_1 - {}^{B}\hat{\mathbf{b}}_1\right) \cdot \left[\left({}^{B}\hat{\mathbf{e}}_2 - {}^{B}\hat{\mathbf{b}}_2\right) \times \left({}^{B}\hat{\mathbf{e}}_3 - {}^{B}\hat{\mathbf{b}}_3\right)\right] \\
&= \left(\left({}^{B}\hat{\mathbf{e}}_1 - {}^{B}\hat{\mathbf{b}}_1\right), \left({}^{B}\hat{\mathbf{e}}_2 - {}^{B}\hat{\mathbf{b}}_2\right), \left({}^{B}\hat{\mathbf{e}}_3 - {}^{B}\hat{\mathbf{b}}_3\right)\right) \qquad (4.38) \\
&\overset{(4.36)}{=} 0 \qquad (4.39)
\end{aligned}
$$

- Recall ▸ When is the Triple Scalar Product 0?

- From the last two expressions, we can see that the vectors:

$$\left({}^B\hat{\mathbf{e}}_1 - {}^B\hat{\mathbf{b}}_1\right), \left({}^B\hat{\mathbf{e}}_2 - {}^B\hat{\mathbf{b}}_2\right), \left({}^B\hat{\mathbf{e}}_3 - {}^B\hat{\mathbf{b}}_3\right) \text{ are coplanar} \tag{4.40}$$

Left-superscripts can be dropped.

- Therefore, there exists a vector $\hat{\mathbf{a}}$ which is normal/orthogonal to this plane (i.e. all 3 vectors)! A degenerate case can be when all three are parallel. How does the conclusion about $\hat{\mathbf{a}}$ change?

$$\left(\hat{\mathbf{e}}_i - \hat{\mathbf{b}}_i\right) \cdot \hat{\mathbf{a}} = 0, \qquad i = 1, \ldots, 3 \tag{4.41}$$

$$\Rightarrow \left({}^B\hat{\mathbf{e}}_i - {}^B\hat{\mathbf{b}}_i\right)^{\mathsf{T}}\, {}^B\hat{\mathbf{a}} = 0, \qquad i = 1, \ldots, 3 \tag{4.42}$$

$$\Rightarrow \begin{bmatrix} \left({}^B\hat{\mathbf{e}}_1 - {}^B\hat{\mathbf{b}}_1\right)^{\mathsf{T}} \\ \left({}^B\hat{\mathbf{e}}_2 - {}^B\hat{\mathbf{b}}_2\right)^{\mathsf{T}} \\ \left({}^B\hat{\mathbf{e}}_3 - {}^B\hat{\mathbf{b}}_3\right)^{\mathsf{T}} \end{bmatrix} {}^B\hat{\mathbf{a}} = \mathbf{0} \tag{4.43}$$

$$\Rightarrow \left({}^E_B\mathbf{R} - \mathbf{I}\right)\, {}^B\hat{\mathbf{a}} = \mathbf{0} \tag{4.44}$$

$$\Rightarrow {}^E_B\mathbf{R}\, {}^B\hat{\mathbf{a}} = {}^B\hat{\mathbf{a}}. \tag{4.45}$$

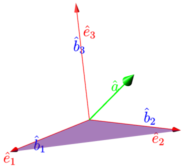
Implications

- As we know from (4.8) that ${}^{E}_{B}\mathbf{R}\,{}^{B}\hat{\mathbf{a}} = {}^{E}\hat{\mathbf{a}}$, (4.45) implies that

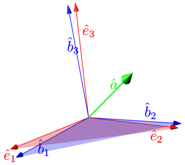
$$ {}^{E}\hat{\mathbf{a}} = {}^{B}\hat{\mathbf{a}} \tag{4.46} $$

- As an aside: It further implies that (you will learn in your Linear-Algebra class) that ${}^{B}\hat{\mathbf{a}}$ is an eigenvector of ${}^{E}_{B}\mathbf{R}$ with eigenvalue 1.
- Effectively, ${}^{B}[\mathcal{R}_{EB}]\,{}^{B}\hat{\mathbf{a}} = {}^{B}\hat{\mathbf{a}}$ shows that for the vector rotation operator $\mathcal{R}_{EB}(\hat{\mathbf{a}}) = \hat{\mathbf{a}}$. Hence, the rotation leaves $\hat{\mathbf{a}}$ unrotated – all other vectors rotate around it!
- Hence, $\hat{\mathbf{a}}$ is called the rotation-axis.
- We just proved that a rotation-axis exists for each 3D rotation.

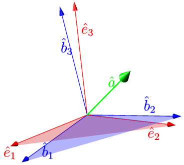
Rotating about $\hat{\mathbf{a}}$ by $\alpha = \pi/6$



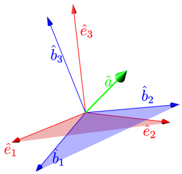
(a) $\alpha = 0$



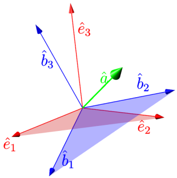
(b) $\alpha = 5^\circ$



(c) $\alpha = 10^\circ$



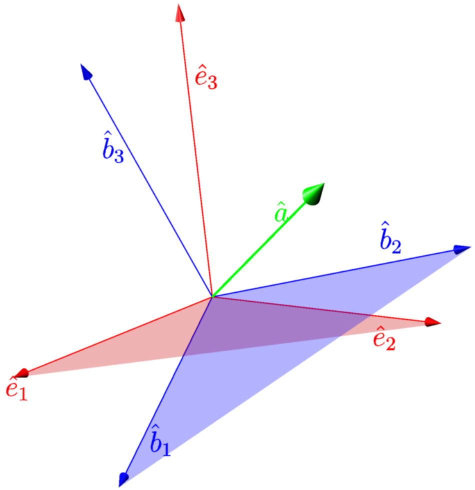
(d) $\alpha = 20^\circ$



(e) $\alpha = 30^\circ$

Rotation $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$

${}^{E}\hat{\mathbf{a}} = [0.27, 0.80, 0.53]^{\mathsf{T}}$ and the angle $\alpha = \pi/6$.



Rotation of a Vector in the Plane ⊥ to the Rotation Axis

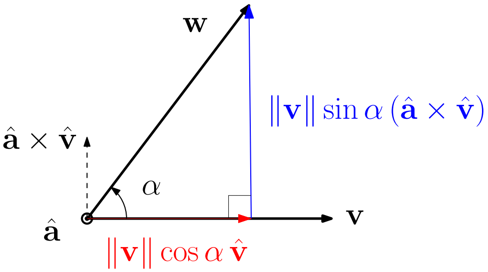


Figure 4.3: The rotation-axis $\hat{\mathbf{a}}$ is coming out of the plane. The operator $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$ rotates a vector $\mathbf{v}$ in the plane by an angle α to $\mathbf{w}$. Hence, $\|\mathbf{v}\| = \|\mathbf{w}\|$.

We define the unit-vector in the direction of $\mathbf{v}$ as $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$. From the figure it is clear that:

$$\begin{aligned} \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v}) = \mathbf{w} &= \|\mathbf{v}\| \cos\alpha \, \hat{\mathbf{v}} + \|\mathbf{v}\| \sin\alpha \, (\hat{\mathbf{a}} \times \hat{\mathbf{v}}) \\ &= \cos\alpha \, \mathbf{v} + \sin\alpha \, (\hat{\mathbf{a}} \times \mathbf{v}) \end{aligned} \tag{4.47}$$

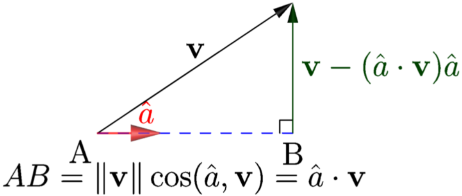
Now we go back to general 3D rotations: We consider what happens to a 3D vector $\mathbf{v}$ if the rotation-axis $\hat{\mathbf{a}}$ is not orthogonal to it. The central result here is the famous Rodrigues formula.

Theorem 4.4 (Rodrigues Formula)

A rotation *operator* $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$ *of angle* α *about any arbitrary axis unit-vector* $\hat{\mathbf{a}}$ *can be represented as*

$$\mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v}) = \mathbf{v} + \sin\alpha\, \hat{\mathbf{a}} \times \mathbf{v} + (1 - \cos\alpha)\, \hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \mathbf{v}) \tag{4.48}$$

Proof.



Recall that any vector $\mathbf{v}$ can be resolved as a sum of a vector collinear with $\hat{\mathbf{a}}$ and a vector orthogonal to $\hat{\mathbf{a}}$. Let this vector which is orthogonal to $\hat{\mathbf{a}}$ be:

$$\mathbf{v}_\perp \triangleq \mathbf{v} - \lambda\, \hat{\mathbf{a}}, \text{ where, } \lambda = \mathbf{v} \cdot \hat{\mathbf{a}}. \tag{4.49}$$

We have,

$$\mathbf{v} = \lambda\,\hat{\mathbf{a}} + \mathbf{v}_\perp\,, \qquad \mathbf{v}_\perp \cdot \hat{\mathbf{a}} = 0. \tag{4.50}$$

The rotation operator $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$ satisfies

$$\mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\hat{\mathbf{a}}) = \hat{\mathbf{a}} \tag{4.51}$$

$$\mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v}_\perp) \overset{(4.47)}{=} \cos\alpha\,\mathbf{v}_\perp + \sin\alpha\,\hat{\mathbf{a}} \times \mathbf{v}_\perp \tag{4.52}$$

$$\therefore \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v}) = \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\lambda\,\hat{\mathbf{a}} + \mathbf{v}_\perp) \tag{4.53}$$

$$= \lambda\,\mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\hat{\mathbf{a}}) + \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v}_\perp) \tag{4.54}$$

$$= \lambda\,\hat{\mathbf{a}} + \cos\alpha\,\mathbf{v}_\perp + \sin\alpha\,\hat{\mathbf{a}} \times \mathbf{v}_\perp \tag{4.55}$$

$$\overset{(4.49)}{=} (\mathbf{v}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}} + \cos\alpha\,(\mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}}) + \sin\alpha\,\hat{\mathbf{a}} \times (\mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}})$$

$$= \mathbf{v} + ((\mathbf{v}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}} - \mathbf{v}) - \cos\alpha((\mathbf{v}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}} - \mathbf{v}) + \sin\alpha\,\hat{\mathbf{a}} \times \mathbf{v} \tag{4.56}$$

$$= \mathbf{v} + (1 - \cos\alpha)((\mathbf{v}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}} - \mathbf{v}) + \sin\alpha\,\hat{\mathbf{a}} \times \mathbf{v} \tag{4.57}$$

$$= \mathbf{v} + \sin\alpha\,\hat{\mathbf{a}} \times \mathbf{v} + (1 - \cos\alpha)\,\hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \mathbf{v}) \tag{4.58}$$

This completes the proof. □

Let the linear operator $\mathcal{L}(\mathbf{x}) \triangleq \hat{\mathbf{a}} \times \mathbf{x}$. The linear-operator which is the composition of $\mathcal{L}$ with itself is

$$\mathcal{L}^2(\mathbf{x}) = \mathcal{L}(\mathcal{L}(\mathbf{x})) \qquad = \hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \mathbf{x}). \tag{4.59}$$

We would like to find ${}^E[\mathcal{L}^2]$: this is easy because we know ${}^E[\mathcal{L}]$ from (3.51). Hence, on using (3.52), we have:

$${}^E[\mathcal{L}^2] = \left({}^E[\mathcal{L}]\right)^2 \qquad = \left[{}^E\hat{\mathbf{a}}\times\right]^2 \tag{4.60}$$

Let ${}^E\hat{\mathbf{a}} = [a_1,\, a_2,\, a_3]^\mathsf{T}$, where, $a_1^2 + a_2^2 + a_3^2 = 1$. It can be verified using (3.51) that

$$\left[{}^E\hat{\mathbf{a}}\times\right]^2 = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}^2 = \begin{bmatrix} a_1^2 - 1 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 - 1 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 - 1 \end{bmatrix}, \tag{4.61}$$

a symmetric matrix.

The Matrix Representation of $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$

To get the matrix representation of the linear operator (4.48) on any arbitrary $\mathscr{B}_E$, we first see that it is a linear-combination of three linear-operators:

$$\mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v}) = \underbrace{\mathbf{v}}_{\mathcal{I}(\mathbf{v})} + \sin\alpha\, \underbrace{\hat{\mathbf{a}} \times \mathbf{v}}_{\mathcal{L}(\mathbf{v})} + (1-\cos\alpha)\, \underbrace{\hat{\mathbf{a}} \times (\hat{\mathbf{a}} \times \mathbf{v})}_{\mathcal{L}^2(\mathbf{v})}$$

Hence, to find the matrix of $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$ on $\mathscr{B}_E$, we just take the linear-combination of the matrices on $\mathscr{B}_E$ of these three linear-operators. Using (3.51) and (4.61), we get,

$$^{E}\left[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}\right] = \underbrace{\mathbf{I}_3}_{\text{symmetric}} + \underbrace{\sin\alpha \left[{}^{E}\hat{\mathbf{a}}\times\right]}_{\text{skew-symmetric}} + \underbrace{(1-\cos\alpha)\left[{}^{E}\hat{\mathbf{a}}\times\right]^2}_{\text{symmetric}}. \qquad (4.62)$$

To use it, we recall the interpretation (3.46), which we rewrite for $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$ as

$$\text{If } \mathbf{u} = \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v}) \text{ then, } {}^{E}\mathbf{u} = {}^{E}\left[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}\right]\, {}^{E}\mathbf{v}. \qquad (4.63)$$

Two Equivalent Interpretations

1. Rotate the vector $\mathbf{v}$ by the linear-operator and view it from the same basis $\mathscr{B}_E$:

$$^{E}\left[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}\right]\, {}^{E}\mathbf{v} = {}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{v})] \tag{4.64}$$

2. Keep the vector $\mathbf{v}$ static, but view it from two different bases $\mathscr{B}_E$ and $\mathscr{B}_B$ rotated w.r.t. each other:

$$^{E}_{B}\mathbf{R}\, {}^{B}\mathbf{v} = {}^{E}\mathbf{v}. \tag{4.65}$$

Trace of a Matrix

Trace of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the sum of its diagonal elements

$$\mathrm{trace}(\mathbf{A}) \triangleq \sum_{i=1}^{n} \mathbf{A}(i, i) \tag{4.66}$$

Finding the Rotation Angle from the DCM

We need to solve the reverse problem of the Rodrigues Theorem, we're given ${}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}]$ and we would like to recover the rotation axis ${}^{E}\hat{\mathbf{a}}$ and α. For getting α, we can look at the trace of the expression (4.62):

$$\begin{aligned} \mathrm{trace}({}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}]) &= 3 + (1-\cos\alpha)\left(a_1^2 + a_2^2 + a_3^2 - 3\right) \\ &= 3 + (1-\cos\alpha)(-2) \\ &= 1 + 2\cos\alpha. \end{aligned} \tag{4.67}$$

Thus, $\alpha \in [0, \pi]$ can be recovered using the arccosine function.
What happens if the actual $\alpha \in (-\pi, 0)$?

Finding the Rotation Axis from the DCM

Now that we have α, we want ${}^{E}\hat{\mathbf{a}}$, for which we can look at the skew-symmetric part of ${}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}]$.

$$\frac{1}{2}\left({}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}] - \left({}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}]\right)^{\mathsf{T}}\right) \overset{(4.62)}{=} \sin\alpha\,\left[{}^{E}\hat{\mathbf{a}}\times\right] \tag{4.68}$$

$$= \sin\alpha \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{4.69}$$

Since the LHS and α are known, this allows us to read-off ${}^{E}\hat{\mathbf{a}} = [a_1,\ a_2,\ a_3]^{\mathsf{T}}$ unless $\alpha = \pi$ or $\alpha = 0$.

Finding the Rotation Axis from the DCM

- When $\alpha = 0$, ${}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}]$ is $\mathbf{I}_3$ (easily detected) and the rotation-axis is not defined.
- When $\alpha = \pi$ (or close), we could use (4.40) to compute

$$ {}^{B}\mathbf{a} = ({}^{B}\hat{\mathbf{e}}_i - {}^{B}\hat{\mathbf{b}}_i) \times ({}^{B}\hat{\mathbf{e}}_j - {}^{B}\hat{\mathbf{b}}_j), \tag{4.70} $$

  where, we select the first combination $i \neq j$ which gives $\mathbf{a} \neq \mathbf{0}$.
  - We normalize $\mathbf{a}$ to get the rotation-axis $\hat{\mathbf{a}}$.
  - Remember from (4.46) that ${}^{B}\hat{\mathbf{a}} = {}^{E}\hat{\mathbf{a}}$ for the rotation-axis.
  - We need to do an additional check to see if π or $-\pi$ is the real rotation. Does it matter?
- For $\alpha = \pi$, the other option is to expand (4.62) and see what it gives us.

Euler Angles

Another way to paramterize $\mathcal{R}_{EB}$ using three parameters is as follows:

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \xrightarrow{\mathcal{R}_{\alpha,\hat{\mathbf{e}}_i}} (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3), \qquad i = 1 \text{ or } 2 \text{ or } 3 \tag{4.71a}$$

$$(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3) \xrightarrow{\mathcal{R}_{\beta,\hat{\mathbf{u}}_j}} (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3), \qquad j = 1 \text{ or } 2 \text{ or } 3 \tag{4.71b}$$

$$(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3) \xrightarrow{\mathcal{R}_{\gamma,\hat{\mathbf{v}}_k}} (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3), \qquad k = 1 \text{ or } 2 \text{ or } 3 \tag{4.71c}$$

- It may seem like there are $3 \times 3 \times 3 = 27$ possibilities. But, note that if $i = j$ or $j = k$ we have successive rotations about the same axis, which can be replaced by a single rotation. So, we need to exclude such cases.
- The total number of independent ways to define Euler Angles is 12.

$$27 - \underbrace{3}_{i=j=k} - \underbrace{3 \times 2}_{i=j\neq k} - \underbrace{3 \times 2}_{i\neq j=k} = 12. \tag{4.72}$$

Bryant (Cardan) Angles

Roll, Pitch, Yaw

- Take $i = 3,\ j = 2,\ k = 1$ in (4.71). This is also called the zyx convention.

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \xrightarrow{\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}} (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3 = \hat{\mathbf{e}}_3), \qquad \text{Yaw} \tag{4.73a}$$

$$(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3) \xrightarrow{\mathcal{R}_{\theta,\hat{\mathbf{u}}_2}} (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 = \hat{\mathbf{u}}_2, \hat{\mathbf{v}}_3), \qquad \text{Pitch} \tag{4.73b}$$

$$(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3) \xrightarrow{\mathcal{R}_{\phi,\hat{\mathbf{v}}_1}} (\hat{\mathbf{b}}_1 = \hat{\mathbf{v}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3), \qquad \text{Roll} \tag{4.73c}$$

- Using the composition of rotations from Theorem 4.2 and the matrix-representation of the axis-angle rotation operator (4.62), we have:

$$^{E}[\mathcal{R}_{EB}] = {}^{E}[\mathcal{R}_{EU}]\ {}^{U}[\mathcal{R}_{UV}]\ {}^{V}[\mathcal{R}_{VB}] \tag{4.74}$$

$$= {}^{E}\left[\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}\right]\ {}^{U}\left[\mathcal{R}_{\theta,\hat{\mathbf{u}}_2}\right]\ {}^{V}\left[\mathcal{R}_{\phi,\hat{\mathbf{v}}_1}\right] \tag{4.75}$$

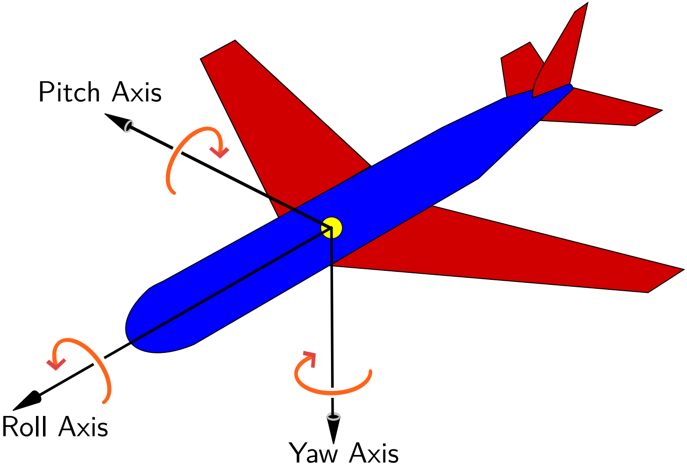


Figure 4.4: Yaw is also called "bearing". The overall rotation of the craft is also called "attitude" in Aeronautics. Image credits: Wikipedia [6]

[6] "Yaw Axis Corrected" by Yaw_Axis.svg: Auawisederivative work: Jrvz (talk) - Yaw_Axis.svg. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Yaw_Axis_Corrected.svg

Board Example

Showing that

$$ {}^{E}\left[\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}\right] = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.76} $$

Using $s_\theta = \sin\theta$, $c_\theta = \cos\theta$ etc,

$$
\begin{aligned}
{}^E[\mathcal{R}_{EB}] &= {}^E[\mathcal{R}_{EU}]\ {}^U[\mathcal{R}_{UV}]\ {}^V[\mathcal{R}_{VB}] && (4.77)\\
&= {}^E\left[\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}\right]\ {}^U\left[\mathcal{R}_{\theta,\hat{\mathbf{u}}_2}\right]\ {}^V\left[\mathcal{R}_{\phi,\hat{\mathbf{v}}_1}\right] \\
&= \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} && (4.78)\\
&= \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} && (4.79)
\end{aligned}
$$

Back-Computing Roll, Pitch, Yaw from a DCM

$$
{}^{E}[\mathcal{R}_{EB}] = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}
$$

- From the element $(3, 1)$ and the arcsine function, we get the pitch $\theta \in [-\pi/2, \pi/2]$.
- Using this θ, elements $(1, 1)$, $(2, 1)$ and the atan2 function, we get the yaw $\psi \in (-\pi, \pi]$.
- Using θ, elements $(3, 2)$, $(3, 3)$ and the atan2 function, we get the roll $\phi \in (-\pi, \pi]$.
- Hence, we see that at pitch $\theta = \pm\pi/2$, yaw and roll are indeterminate. This singularity is called the gimbal lock.
- This is a problem of this rotation representation. The DCM does not have this problem but it needs to store 9 elements rather than 3.

Roll, Pitch, Yaw can be looked at in another equivalent way...

Roll, Pitch, Yaw Revisited

- Instead of *zyx*, we now use *xyz* rotations, but about the axes of the original basis $\mathscr{B}_E$.

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \xrightarrow{\mathcal{R}_{\phi,\hat{\mathbf{e}}_1}} (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3), \quad \text{Roll} \tag{4.80a}$$

$$(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3) \xrightarrow{\mathcal{R}_{\theta,\hat{\mathbf{e}}_2}} (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3), \quad \text{Pitch} \tag{4.80b}$$

$$(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3) \xrightarrow{\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}} (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3), \quad \text{Yaw} \tag{4.80c}$$

- Compare with (4.73). How do we know it works?

- The above is a composition of three linear-operators in the following order:

$$\mathcal{R}_{EB} = \mathcal{R}_{\psi,\hat{\mathbf{e}}_3} \circ \mathcal{R}_{\theta,\hat{\mathbf{e}}_2} \circ \mathcal{R}_{\phi,\hat{\mathbf{e}}_1} \tag{4.81}$$

- Hence, we can use (3.52) for the matrix of the operator in $\mathscr{B}_E$.

$${}^{E}[\mathcal{R}_{EB}] = {}^{E}[\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}]\,{}^{E}[\mathcal{R}_{\theta,\hat{\mathbf{e}}_2}]\,{}^{E}[\mathcal{R}_{\phi,\hat{\mathbf{e}}_1}] \tag{4.82}$$

- But, earlier in (4.75), we found that:

$${}^{E}[\mathcal{R}_{EB}] = {}^{E}\left[\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}\right]\ {}^{U}\left[\mathcal{R}_{\theta,\hat{\mathbf{u}}_2}\right]\ {}^{V}\left[\mathcal{R}_{\phi,\hat{\mathbf{v}}_1}\right]$$

- Comparing the two, we see that the two are actually equal in terms of the final matrix expression!
- However, this space-fixed rotation **xyz** about $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ is much harder to visualize than the body-fixed rotation **zyx** about $\hat{\mathbf{e}}_3, \hat{\mathbf{u}}_2, \hat{\mathbf{v}}_1$.

Accelerometer

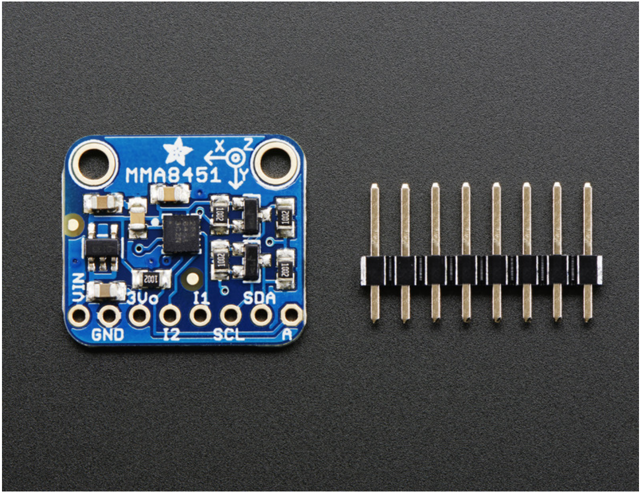


Figure 4.5: The Adafruit MMA8451 accelerometer mounted on a breakout board. Notice the marked XYZ axes.

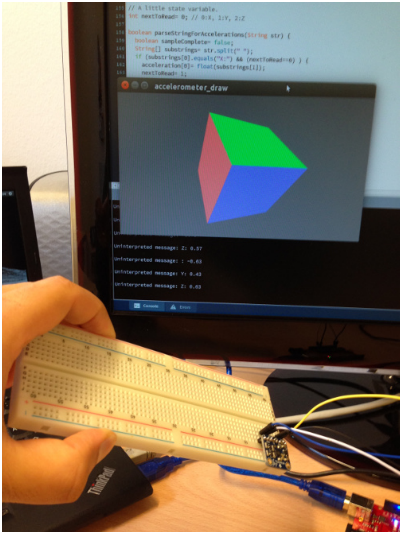


Figure 4.6: The Arduino+Processing experiment in the IMS-Lab involving the MMA8451 accelerometer. The RGB cube changes its orientation depending on the tilt of the accelerometer.

How does an Accelerometer work?

https://www.youtube.com/watch?v=i2U49usFo10

Launch external viewer.

What does an Accelerometer measure?

- An accelerometer measures the g-force: the force which is stopping the accelerometer from going into a free fall.
- Hence, if you just hold an accelerometer, it will register a force in a direction opposite to that of the gravity.
- In free-fall, an accelerometer will just return zeros, i.e. not register any force.
- The force (or, actually, the acceleration caused by it) is reported as multiples of $|\mathbf{g}| = 9.8\, m/s^2$.

What does an Accelerometer measure?

Example

- Let's say you have an accelerometer strapped to your chest and you are sitting in a car.
- If the car suddenly accelerates, you will be pinned to the seat.
- The accelerometer will hence report a forward acceleration as a multiple of $|\mathbf{g}|$.
- It will report this forward force because this is the force exerted by your chest to accelerate it along with the car.

What does an Accelerometer measure?

Example

- If you're sitting on a static platform, and you gently tilt the accelerometer, you can use its readings to determine the tilt. Let's see how...

- Let is have a space-fixed basis $\mathscr{B}_E$ in which $\hat{\mathbf{e}}_3 = -\hat{\mathbf{g}}$, where $\mathbf{g}$ is the acceleration due to gravity vector pointing down.
- Let us have a body-fixed basis $\mathscr{B}_B$ fixed to the accelerometer.
- Initially, let the unit-vectors of $\mathscr{B}_E$ and $\mathscr{B}_B$ be the same.
- Assume that you tilt the accelerometer smoothly from the initial orientation to another arbitrary orientation $\mathscr{B}_B$.
- What will the accelerometer show?
- It will register a force $\mathbf{f}$ opposite to the gravity direction which is keeping the accelerometer from going into a free-fall. Note the only non-zero component of ${}^E\mathbf{f}$ is in the direction $\hat{\mathbf{e}}_3$.
- The accelerometer will return three values c_x, c_y, c_z such that

$$
{}^B\mathbf{f} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}, \qquad \text{and, } {}^E\mathbf{f} \overset{(4.8)}{=} {}^E_B\mathbf{R}\, {}^B\mathbf{f}. \tag{4.83}
$$

- Since right now we're just interested in finding the tilt, let's work with normalized vectors, so:

$$ {}^{E}\hat{\mathbf{f}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad {}^{B}\hat{\mathbf{f}} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad = \frac{1}{\sqrt{c_x^2 + c_y^2 + c_z^2}} \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} \tag{4.84} $$

$$ {}^{B}\hat{\mathbf{f}} = {}^{E}_{B}\mathbf{R}^{\mathsf{T}}\, {}^{E}\hat{\mathbf{f}} \tag{4.85} $$

- Let's write ${}^{E}_{B}\mathbf{R}^{\mathsf{T}}$ terms of roll (ϕ), pitch (θ), and yaw (ψ) angles. From (4.79),

$$ {}^{B}\hat{\mathbf{f}} = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} -s_\theta \\ c_\theta s_\phi \\ c_\theta c_\phi \end{bmatrix} \tag{4.86} $$

This is independent of yaw (ψ), as expected.

- We can compute ${}^B\hat{\mathbf{f}}$ by just normalizing the output of the accelerometer.
- What we really want is to use (4.86) to compute roll $\phi \in (-\pi, \pi]$ and pitch $\theta \in [-\pi/2, \pi/2]$

$$ {}^B\hat{\mathbf{f}} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} -s_\theta \\ c_\theta s_\phi \\ c_\theta c_\phi \end{bmatrix} \tag{4.87} $$

- From the first component, we get

$$ \theta = \operatorname{asin}(-a_x) \quad \in [-\pi/2, \pi/2] \tag{4.88} $$

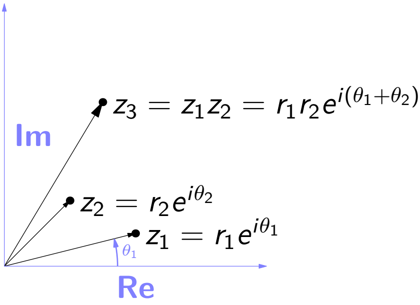
- From the remaining components, we get (if we're not in gimbal-lock):

$$ \sin\phi = \frac{a_y}{\cos\theta}, \quad \cos\phi = \frac{a_z}{\cos\theta} \tag{4.89} $$

$$ \Rightarrow \phi = \operatorname{atan2}\left(\frac{a_y}{\cos\theta}, \frac{a_z}{\cos\theta}\right) \quad \in (-\pi, \pi] \tag{4.90} $$

- Hence, we know the tilt of the accelerometer w.r.t. gravity, which we can now use as we wish, e.g. to visualize in Processing.

Recall Complex Numbers



- The imaginary unit $i^2 = -1$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.
- Cartesian and polar representation of a complex number.

$$z = x + i\,y \qquad = re^{i\theta} \tag{4.91}$$

- So, complex-numbers are like 2D vectors and complex-multiplication is like a rotation in 2D.

Quaternions

Quaternions are to rotation of 3D vectors what complex-numbers are to rotation of 2D vectors.

The set of quaternions is written $\mathbb{H}$. Its elements are written as $Q \in \mathbb{H}$.

$$Q = \underbrace{q_0}_{\text{Scalar Part}} + \underbrace{q_1\,\hat{\mathbf{e}}_1 + q_2\,\hat{\mathbf{e}}_2 + q_3\,\hat{\mathbf{e}}_3}_{\text{Vector Part}} \tag{4.92}$$

$$= q_0 + \mathbf{q} \tag{4.93}$$

$${}^{E}Q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{4.94}$$

In some software libraries, an object `Q` of type quaternion has the following fields (or methods with similar names):

$$
\begin{aligned}
\texttt{Q.w} &\equiv q_0 \\
\texttt{Q.x} &\equiv q_1 \\
\texttt{Q.y} &\equiv q_2 \\
\texttt{Q.z} &\equiv q_3
\end{aligned}
$$

The set of quaternions $\mathbb{H}$ is equivalent to $\mathbb{R}^4$ and hence forms a 4-dimensional vector-space with the usual two operations. Assuming $Q, Q' \in \mathbb{H}$,

1. **Scalar Multiplication**:

$$\lambda\, Q = (\lambda\, q_0) + (\lambda\, q_1)\,\hat{\mathbf{e}}_1 + (\lambda\, q_2)\,\hat{\mathbf{e}}_2 + (\lambda\, q_3)\,\hat{\mathbf{e}}_3 \tag{4.95}$$

2. **Addition**:

$$\begin{aligned} Q + Q' = (q_0 + q_0') \\ + (q_1 + q_1')\,\hat{\mathbf{e}}_1 \\ + (q_2 + q_2')\,\hat{\mathbf{e}}_2 \\ + (q_3 + q_3')\,\hat{\mathbf{e}}_3 \end{aligned} \tag{4.96}$$

Quaternion Algebra

Hamilton Rules

$\mathbb{H}$ is different from $\mathbb{R}^4$ due to the additional structure imposed by the following fundamental rules of quaternion algebra.

$$\hat{\mathbf{e}}_1^2 = \hat{\mathbf{e}}_2^2 = \hat{\mathbf{e}}_3^2 = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 = -1 \tag{4.97}$$

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 \tag{4.98}$$

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 \tag{4.99}$$

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 \tag{4.100}$$

Thus, the unit-vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ behave like imaginary-units when they multiply themselves. When they multiply each other, they behave like cross-products.

Figure 4.7: Quaternion plaque on Brougham (Broom) Bridge, Dublin. [7]

[7] "William Rowan Hamilton Plaque - geograph.org.uk - 347941" by JP. Licensed under CC BY-SA 2.0 via Commons - https://commons.wikimedia.org/wiki/File:William_Rowan_Hamilton_Plaque_-_geograph.org.uk_-_347941.jpg

Special Quaternions

- A pure (or purely imaginary) quaternion is one whose scalar part is zero, $Q = 0 + \mathbf{q}$. We denote their set by $\mathbb{H}_0 \subset \mathbb{H}$.
- If $Q \in \mathbb{H}$, then its conjugate (analogous to $\mathbb{C}$) is denoted by $\overline{Q}$, s.t.

$$\overline{Q} = q_0 - \mathbf{q}. \tag{4.101}$$

Quaternion Multiplication

Using Hamilton rules and distributivity, we get the definition of quaternion multiplication. Given $Q, P \in \mathbb{H}$,

$$\begin{aligned} PQ &= (p_0 + p_1\,\hat{\mathbf{e}}_1 + p_2\,\hat{\mathbf{e}}_2 + p_3\,\hat{\mathbf{e}}_3)(q_0 + q_1\,\hat{\mathbf{e}}_1 + q_2\,\hat{\mathbf{e}}_2 + q_3\,\hat{\mathbf{e}}_3) \\ &= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) \\ &\quad + (p_0 q_1 + q_0 p_1 + p_2 q_3 - p_3 q_2)\,\hat{\mathbf{e}}_1 \\ &\quad + (p_0 q_2 + q_0 p_2 + p_3 q_1 - p_1 q_3)\,\hat{\mathbf{e}}_2 \\ &\quad + (p_0 q_3 + q_0 p_3 + p_1 q_2 - p_2 q_1)\,\hat{\mathbf{e}}_3 \qquad (4.102) \\ &= (p_0 q_0 - \mathbf{p}\cdot\mathbf{q}) + p_0\,\mathbf{q} + q_0\,\mathbf{p} + \mathbf{p}\times\mathbf{q} \qquad (4.103) \end{aligned}$$

Multiplying a quaternion by its conjugate gives a scalar.

$$\therefore\ Q\overline{Q} = \overline{Q}Q = q_0^2 + \mathbf{q}\cdot\mathbf{q} \qquad = q_0^2 + \|\mathbf{q}\|^2 \qquad \triangleq \|Q\|^2 \qquad (4.104)$$

We can now define the inverse of a quaternion

$$Q^{-1} \triangleq \frac{\overline{Q}}{\|Q\|^2}, \qquad QQ^{-1} = Q^{-1}Q = 1. \qquad (4.105)$$

Quaternion Triple Product

Let $Q \in \mathbb{H}$ and $V \in \mathbb{H}_0$, i.e. $V = 0 + \mathbf{v}$. Then, if we form the triple-product,

$$
\begin{aligned}
P &= Q\,V\,\overline{Q} \\
&= (q_0 + \mathbf{q})(0 + \mathbf{v})(q_0 - \mathbf{q}) \\
&\overset{(4.103)}{=} (-\mathbf{q}\cdot\mathbf{v} + q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})\,(q_0 - \mathbf{q}) \\
&= -q_0\,\mathbf{q}\cdot\mathbf{v} + (q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})\cdot\mathbf{q} \\
&\quad + (\mathbf{q}\cdot\mathbf{v})\mathbf{q} + q_0(q_0\mathbf{v} + \mathbf{q}\times\mathbf{v}) - (q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})\times\mathbf{q} && (4.106) \\
&= (\mathbf{q}\cdot\mathbf{v})\mathbf{q} + q_0^2\mathbf{v} + q_0\mathbf{q}\times\mathbf{v} + q_0\mathbf{q}\times\mathbf{v} + \underbrace{\mathbf{q}\times(\mathbf{q}\times\mathbf{v})}_{=(\mathbf{q}\cdot\mathbf{v})\,\mathbf{q} - \|\mathbf{q}\|^2\,\mathbf{v}} && (4.107) \\
&= (q_0^2 - \|\mathbf{q}\|^2)\,\mathbf{v} + 2(\mathbf{q}\cdot\mathbf{v})\,\mathbf{q} + 2q_0\,\mathbf{q}\times\mathbf{v} && (4.108)
\end{aligned}
$$

So, $P \in \mathbb{H}_0$!

Unit Quaternion

Versor

- If $\|Q\| = 1$, we call Q a unit-quaternion.
- Then, using (4.105), we get that $Q^{-1} = \overline{Q}$.
- A unit-quaternion can always be parameterized as

$$Q = \cos\theta + \sin\theta\, \hat{\mathbf{u}} \tag{4.109}$$

$$\|Q\|^2 = \cos^2\theta + \sin^2\theta\,(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) = 1. \tag{4.110}$$

- A unit-quaternion can be considered to be a point on the surface of a 4D unit-hypersphere.

Theorem 4.5 (Rotation by a Unit-Quaternion)

Given a vector $\mathbf{v}$*, we first make it into a pure quaternion* $V = 0 + \mathbf{v} \in \mathbb{H}_0$*. Then, given a unit-quaternion* $Q = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\,\hat{\mathbf{u}}$*, the operator*

$$\mathcal{R}_Q : V \in \mathbb{H}_0 \longmapsto Q\,V\,\overline{Q} \in \mathbb{H}_0 \tag{4.111}$$

corresponds to a rotation $\mathcal{R}_{\alpha,\hat{\mathbf{u}}}(\mathbf{v})$ *of angle* α *about the unit-vector* $\hat{\mathbf{u}}$*. Furthermore,* Q *and* $-Q$ *represent the same rotation.*

Proof.

$$Q\,V\,\overline{Q} = (\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\,\hat{\mathbf{u}})(0+\mathbf{v})(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\,\hat{\mathbf{u}}) \tag{4.112}$$

$$\overset{(4.108)}{=} (\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2})\mathbf{v} + 2\sin^2\frac{\alpha}{2}(\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}} + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\hat{\mathbf{u}}\times\mathbf{v}$$

$$= \cos\alpha\,\mathbf{v} + (1-\cos\alpha)\ \underbrace{(\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}}}_{=\hat{\mathbf{u}}\times(\hat{\mathbf{u}}\times\mathbf{v})+\mathbf{v}}\ + \sin\alpha\,\hat{\mathbf{u}}\times\mathbf{v} \tag{4.113}$$

$$= \mathbf{v} + \sin\alpha\,\hat{\mathbf{u}}\times\mathbf{v} + (1-\cos\alpha)\,\hat{\mathbf{u}}\times(\hat{\mathbf{u}}\times\mathbf{v})$$

We have recovered the Rodrigues formula (4.48). □

Composition of Rotations

$$V \in \mathbb{H}_0 \stackrel{\mathcal{R}_{Q_1}}{\longmapsto} V_1 = Q_1\, V\, \overline{Q}_1 \stackrel{\mathcal{R}_{Q_2}}{\longmapsto} V_2 = Q_2\, V_1\, \overline{Q}_2 \tag{4.114}$$

$$V_2 = Q_2\, Q_1\, V\, \overline{Q}_1\, \overline{Q}_2 \tag{4.115}$$

$$= Q_2\, Q_1\, V\, \overline{Q_2 Q_1} \tag{4.116}$$

Hence, $\mathcal{R}_{Q_2} \circ \mathcal{R}_{Q_1}$ is equivalent to the unit-quaternion $Q = Q_2 Q_1$.

Advantages of Quaternions for Representing Rotations

- A unit-quaternion uses only 4-parmeters (with one constraint) to represent a 3D rotation.
- It has no singularities like the Euler angles, but $Q \equiv -Q$ for rotations.
- After several rotation compositions, numerical errors may cause the resultant quaternion to not be a strict unit-quaternion.
- But, fixing this is easy — just renormalize.
- What will you do if, you are using DCMs, and after several compositions you run into a similar problem, namely, the resultant matrix is no longer a DCM?
- Quaternions can be used to smoothly interpolate between two rotations using the SLERP (Spherical Linear Interpolation) algorithm. This is useful for animations.

Quaternion SLERP

Spherical Linear Interpolation

- From (4.109) and (4.116),

$$\begin{aligned}\mathcal{R}_Q \circ \mathcal{R}_Q &= Q^2 \\ &= (\cos\theta + \sin\theta\, \hat{\mathbf{u}})(\cos\theta + \sin\theta\, \hat{\mathbf{u}}) \\ &= \cos 2\theta + \sin 2\theta\, \hat{\mathbf{u}} \qquad (4.117) \\ \text{Similarly, } Q^n &= \cos n\theta + \sin n\theta\, \hat{\mathbf{u}} \qquad (4.118)\end{aligned}$$

- This is generalized for fractional powers $t \in [0, 1]$

$$Q^t = \cos t\theta + \sin t\theta\, \hat{\mathbf{u}} \qquad (4.119)$$

- Using (4.119), SLERP smoothly interpolates between two given rotations Q_0 and Q_1 as

$$\text{SLERP}(Q_0, Q_1, t \in [0, 1]) = Q_0(Q_0^{-1} Q_1)^t \qquad (4.120)$$

Contents

Definition 5.1 (Rigid Body)

A rigid-body $\mathcal{B}$ is defined as a connected set of points which remain at constant distance from each other.

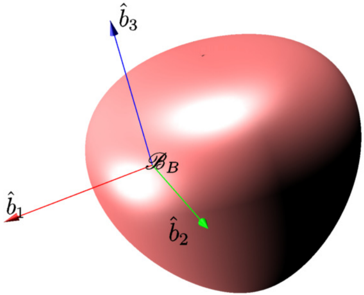


Figure 5.1: A rigid body. Typically, we consider a basis $\mathscr{B}_B$ fixed to the body. By fixing the basis, at this point, we have just fixed three orthogonal directions on the body, from the perspective of an observer on the body.

Rigid Extension of a Rigid Body

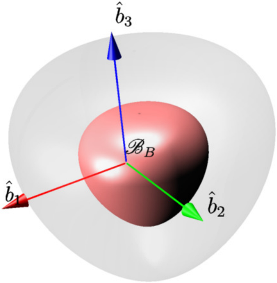


Figure 5.2: Sometimes in analysis, it is useful to consider points outside a physical rigid-body as if they were rigidly connected to it. These points are considered massless and do not affect any physical properties of the rigid body under consideration.

Referential

The frame of reference or just "frame"

Definition 5.2 (Referential)

A referential is synonymous with an observer (usually fixed to a rigid body) who observes relative-motions of other rigid-bodies from its perspective.

Example 5.3

An astronaut in a space-station, a sensor mounted on a robot arm.

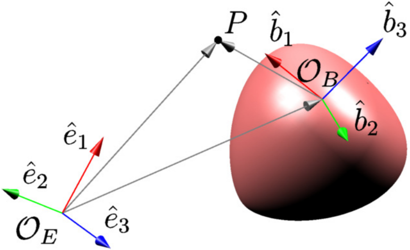
Referential= Basis + Origin

- Assume that a referential is attached to the rigid-body $\mathcal{E}$.
- This referential (or frame "E") will be denoted as as $\mathscr{F}_E$.
- $\mathscr{F}_E$ is equipped with a basis $\mathscr{B}_E(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ fixed to it.
- A point $\mathcal{O}_E$ attached to $\mathscr{F}_E$ is chosen as its origin.
- More than one coordinate-system (coming slides) can be defined within a referential.

Definition 5.4 (Position Vector)

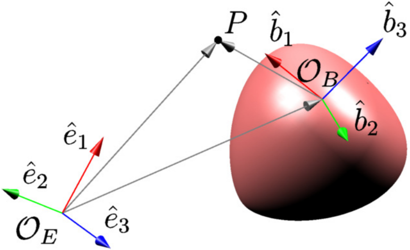
The position vector is a vector that represents the position of a point in space in relation to an arbitrary starting origin. Examples:

- $\mathbf{r}_{\mathcal{O}_E P} \equiv \overrightarrow{\mathcal{O}_E P}$ is the position-vector of P with respect to (w.r.t.) $\mathcal{O}_E$.
- $\mathbf{r}_{\mathcal{O}_B P} \equiv \overrightarrow{\mathcal{O}_B P}$ is the position-vector of P w.r.t. or from $\mathcal{O}_B$.
- $\mathbf{r}_{\mathcal{O}_E \mathcal{O}_B} \equiv \overrightarrow{\mathcal{O}_E \mathcal{O}_B}$ is the position-vector from $\mathcal{O}_E$ to $\mathcal{O}_B$.



Parameterizing one referential with respect to another

- $\mathscr{F}_B\{\mathcal{O}_B, \mathscr{B}_B\}$ can be completely parameterized with respect to $\mathscr{F}_E\{\mathcal{O}_E, \mathscr{B}_E\}$ if:
  1. We know their relative-rotation $\mathcal{R}_{EB}$ (parameterized as a DCM, as Euler angles, or as a unit-quaternion), and,
  2. Their relative translation vector $\mathbf{r}_{\mathcal{O}_E\mathcal{O}_B} \equiv \overrightarrow{\mathcal{O}_E\mathcal{O}_B}$. How can we parameterize it?



Coordinate Systems

Parametrizing Position-Vectors

Definition 5.5 (Coordinate System)

The position of a point P in referential $\mathscr{F}_E$ is defined by a coordinate-system, i.e., a set of 3 scalar-parameters (q_1, q_2, q_3) called coordinates of P. Given an origin $\mathcal{O}_E$, the position-vector $\mathbf{r}_{\mathcal{O}_E P}$ is a function of (q_1, q_2, q_3).

$$\mathbf{r}_{\mathcal{O}_E P} = \mathbf{r}(q_1, q_2, q_3). \tag{5.1}$$

Three important types of coordinate systems are the Cartesian, cylindrical, and spherical coordinate-systems.

Cartesian Coordinate System

We already did this in (3.15). The Cartesian coordinates of P in $\mathscr{F}_E$ are simply its three components of $\mathbf{r}_{\mathcal{O}_E P}$ on the basis $\mathscr{B}_E$. In other words, the Cartesian coordinates are:

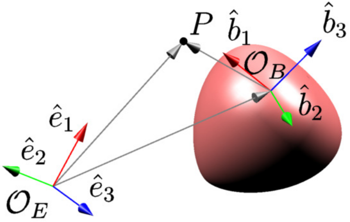
$$\begin{aligned} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} &= \begin{bmatrix} \mathbf{r}_{\mathcal{O}_E P} \cdot \hat{\mathbf{e}}_1 \\ \mathbf{r}_{\mathcal{O}_E P} \cdot \hat{\mathbf{e}}_2 \\ \mathbf{r}_{\mathcal{O}_E P} \cdot \hat{\mathbf{e}}_3 \end{bmatrix} \\ &= {}^E[\mathbf{r}_{\mathcal{O}_E P}] \\ &\triangleq {}^E\mathbf{r}_P \end{aligned} \tag{5.2}$$

Parameterizing one referential with respect to another

Translation Parameterization

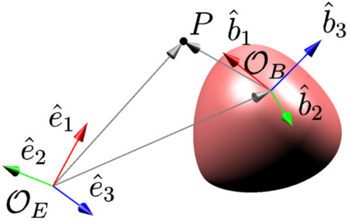
- $\mathscr{F}_B\{\mathcal{O}_B, \mathscr{B}_B\}$ can be completely parameterized with respect to $\mathscr{F}_E\{\mathcal{O}_E, \mathscr{B}_E\}$ if:
  1. We know their relative-rotation $\mathcal{R}_{EB}$ (parameterized as a DCM, as Euler angles, or as a unit-quaternion), and,
  2. Their relative translation vector $\mathbf{r}_{\mathcal{O}_E\mathcal{O}_B} \equiv \overrightarrow{\mathcal{O}_E\mathcal{O}_B}$. To parameterize it, we define

$$ {}^E_B\mathbf{t} \triangleq {}^E\mathbf{r}_{\mathcal{O}_B} \equiv {}^E[\mathbf{r}_{\mathcal{O}_E\mathcal{O}_B}] \tag{5.3} $$

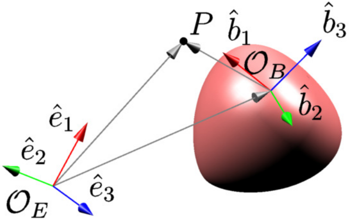


Coordinates transform from one frame to another

How are ${}^{E}\mathbf{r}_P$ and ${}^{B}\mathbf{r}_P$ related?



Coordinates transform from one frame to another



We start from a coordinate-free vector equation.

$$\overrightarrow{\mathcal{O}_E P} = \overrightarrow{\mathcal{O}_E \mathcal{O}_B} + \overrightarrow{\mathcal{O}_B P} \tag{5.4}$$

We now resolve this vector equation in $\mathscr{F}_E$ using (5.2) and (5.3).

$$
\begin{aligned}
{}^E\mathbf{r}_P &= {}^E_B\mathbf{t} + {}^E[\overrightarrow{\mathcal{O}_B P}] && (5.5)\\
&\overset{(4.8)}{=} {}^E_B\mathbf{t} + {}^E_B\mathbf{R}\,{}^B[\overrightarrow{\mathcal{O}_B P}] && (5.6)\\
&\overset{(5.2)}{=} {}^E_B\mathbf{t} + {}^E_B\mathbf{R}\,{}^B\mathbf{r}_P. && (5.7)
\end{aligned}
$$

3D Transforms

Homogeneous Transforms

We thus arrive at an extremely useful expression:

$$ {}^{E}\mathbf{r}_P = {}^{E}_{B}\mathbf{R}\, {}^{B}\mathbf{r}_P + {}^{E}_{B}\mathbf{t} \tag{5.8} $$

This can be written as a matrix-multiplication by the following trick:

$$ \underbrace{\begin{bmatrix} {}^{E}\mathbf{r}_P \\ 1 \end{bmatrix}}_{\triangleq\, {}^{E}\underline{\mathbf{r}}_P} = \underbrace{\begin{bmatrix} {}^{E}_{B}\mathbf{R} & {}^{E}_{B}\mathbf{t} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{\triangleq\, {}^{E}_{B}\mathbf{T}} \underbrace{\begin{bmatrix} {}^{B}\mathbf{r}_P \\ 1 \end{bmatrix}}_{\triangleq\, {}^{B}\underline{\mathbf{r}}_P} \tag{5.9} $$

The 4-vectors ${}^{E}\underline{\mathbf{r}}_P$ and ${}^{B}\underline{\mathbf{r}}_P$ are called homogeneous coordinates and the 4×4 matrix ${}^{E}_{B}\mathbf{T}$ is called a homogeneous transform.

Inverse Transform

- You can find ${}^{B}_{E}\mathbf{T} = {}^{E}_{B}\mathbf{T}^{-1}$ but this is not very computationally efficient.
- We should use the property that ${}^{B}_{E}\mathbf{R} = {}^{E}_{B}\mathbf{R}^{-1} = {}^{E}_{B}\mathbf{R}^{\mathsf{T}}$. Using this we can rearrange (5.8) and identify the inverse transform by comparison

$$
\begin{aligned}
{}^{B}\mathbf{r}_{P} &= {}^{E}_{B}\mathbf{R}^{\mathsf{T}}\, {}^{E}\mathbf{r}_{P} - {}^{E}_{B}\mathbf{R}^{\mathsf{T}}\, {}^{E}_{B}\mathbf{t} && (5.10)\\
\therefore\ {}^{B}_{E}\mathbf{t} &= -\,{}^{E}_{B}\mathbf{R}^{\mathsf{T}}\, {}^{E}_{B}\mathbf{t} && (5.11)
\end{aligned}
$$

- Substituting back, the inverse transform is

$$
{}^{B}_{E}\mathbf{T} = \begin{bmatrix} {}^{E}_{B}\mathbf{R}^{\mathsf{T}} & -\,{}^{E}_{B}\mathbf{R}^{\mathsf{T}}\, {}^{E}_{B}\mathbf{t} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{5.12}
$$

Composition of 3D Transforms

We follow the same strategy as in (4.20).

- Suppose we have three frames $\mathscr{F}_E$, $\mathscr{F}_B$, $\mathscr{F}_A$. Then, from (5.8),

$$ {}^{E}\underline{\mathbf{r}}_P = {}^{E}_{B}\mathbf{T}\ {}^{B}\underline{\mathbf{r}}_P \tag{5.13a}$$

$$ = {}^{E}_{A}\mathbf{T}\ {}^{A}\underline{\mathbf{r}}_P \tag{5.13b}$$

$$ {}^{A}\underline{\mathbf{r}}_P = {}^{A}_{B}\mathbf{T}\ {}^{B}\underline{\mathbf{r}}_P \tag{5.13c}$$

- Substituting (5.13c) in (5.13b), we get

$$ {}^{E}\underline{\mathbf{r}}_P = {}^{E}_{A}\mathbf{T}\ {}^{A}_{B}\mathbf{T}\ {}^{B}\underline{\mathbf{r}}_P. \tag{5.13d}$$

- Comparing this with (5.13a), we get the simple expression

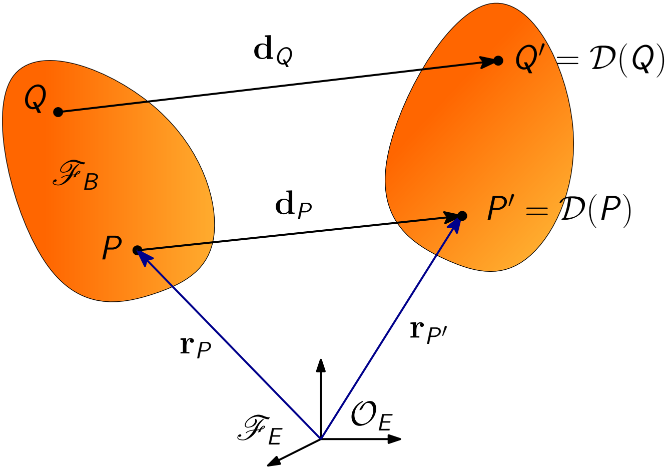
$$ {}^{E}_{B}\mathbf{T} = {}^{E}_{A}\mathbf{T}\ {}^{A}_{B}\mathbf{T}. \tag{5.13e}$$

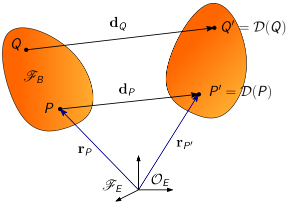
Up to now we have seen how the position-vectors (and Cartesian Coordinates) of an arbitrary point in space with respect to two referentials/frames are related.

Now we will learn how the points belonging to a rigid-body can move so as to "keep the body together" i.e. maintain the rigidity constraints.

Displacements

A displacement is a mapping which takes a configuration of a rigid-body $\mathcal{B}$ (with referential $\mathscr{F}_B$ attched) into a new configuration with respect to a referential $\mathscr{F}_E$.





Hence, each point $P \in \mathcal{B}$ is mapped by displacement $\mathcal{D}(P)$ into a point P'. Let $P, Q, R \in \mathcal{B}$, then the following constraints on their displacements are required to maintain rigidity:

$$\|\mathbf{r}_{\mathcal{O}_E P} - \mathbf{r}_{\mathcal{O}_E Q}\| = \|\mathbf{r}_{\mathcal{O}_E P'} - \mathbf{r}_{\mathcal{O}_E Q'}\|, \quad \text{or,} \quad \|\mathbf{r}_{QP}\| = \|\mathbf{r}_{Q'P'}\| \tag{5.14a}$$

i.e., distances are conserved, and,

$$\mathbf{r}_{PQ} \cdot \mathbf{r}_{PR} = \mathbf{r}_{P'Q'} \cdot \mathbf{r}_{P'R'} \tag{5.14b}$$

i.e., angles are conserved.

The field of displacements can be defined as the vector-field:

$$P \in \mathcal{B} \longmapsto \mathbf{d}_P = \mathbf{r}_{\mathcal{O}_E P'} - \mathbf{r}_{\mathcal{O}_E P} \tag{5.15}$$

The set of all possible displacements contains translations, rotations, and their compositions.

Translation of a Rigid Body

$\mathcal{T}_{\mathbf{t}}$, a translation by vector $\boldsymbol{t}$ is a kind of displacement transformation in which

$$\forall P \in \mathcal{B}, \;\; \mathbf{r}_{\mathcal{O}_E P'} = \mathbf{r}_{\mathcal{O}_E P} + \boldsymbol{t}. \tag{5.16}$$

Clearly (how?), the rigidity constraints (5.14) are satisfied.

Rotation of a Rigid Body

- A rotation is a kind of displacement transformation which leaves a point C of the (rigid extension of) the rigid-body $\mathcal{B}$ invariant, i.e. $C' = C$.
- The rotation $\mathcal{R}_C$, about the center of rotation $C \in \mathcal{B}$, of angle α, and axis $\hat{\mathbf{a}}$ is such that

$$\forall P \in \mathcal{B}, \quad \mathbf{r}_{CP'} = \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{r}_{CP}) \tag{5.17}$$

where, the rotation operator $\mathcal{R}_{\alpha,\hat{\mathbf{a}}}$ is defined by the Rodrigues formula (4.48).

Rotation

Rigidity Constraints

- We need to show that the transform (5.17) is a rigid displacement satisfying (5.14). Let $P, Q, R \in \mathcal{B}$ be undergoing the transform (5.17), then,

$$\mathbf{r}_{CP'} = \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CP}\right), \qquad \mathbf{r}_{CQ'} = \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CQ}\right) \tag{5.18}$$

- We subtract one from the other and use the property that $\mathcal{R}$ is a linear operator.

$$\mathbf{r}_{CP'} - \mathbf{r}_{CQ'} = \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CP}\right) - \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CQ}\right) \tag{5.19}$$

$$= \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CP} - \mathbf{r}_{CQ}\right) \tag{5.20}$$

$$\Rightarrow \mathbf{r}_{Q'P'} = \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{QP}\right). \tag{5.21}$$

- Since a rotation-operator does not change the length of a vector, $\|\mathbf{r}_{Q'P'}\| = \|\mathbf{r}_{QP}\|$. This proves (5.14a). Hence, a rotation-transform about a point is a valid displacement.

Rotation

Rigidity Constraints: Alternative, More Explicit Proof

- (5.21) can be resolved in any basis $\mathscr{B}_E$ using (4.62). For convenience, define a DCM $\mathbf{R} = {}^E[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}]$. Then,

$$
\begin{aligned}
{}^E\mathbf{r}_{Q'P'} &= \mathbf{R}\,{}^E\mathbf{r}_{QP} && (5.22)\\
\left({}^E\mathbf{r}_{Q'P'}\right)^\mathsf{T}\,{}^E\mathbf{r}_{Q'P'} &= \left(\mathbf{R}\,{}^E\mathbf{r}_{QP}\right)^\mathsf{T}\mathbf{R}\,{}^E\mathbf{r}_{QP} && (5.23)\\
\|\mathbf{r}_{Q'P'}\|^2 &= ({}^E\mathbf{r}_{QP})^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{R}\,{}^E\mathbf{r}_{QP} && (5.24)\\
&= ({}^E\mathbf{r}_{QP})^\mathsf{T}\,{}^E\mathbf{r}_{QP} = \|\mathbf{r}_{QP}\|^2. && (5.25)
\end{aligned}
$$

- This proves (5.14a). (5.14b) can be proven similarly. Hence, a rotation-transform about a point is a valid displacement.

General Displacement

The most general displacement is a composition of a rotation about some point $C \in \mathcal{B}$ or its rigid extension and translation

$$\mathcal{D} : P \in \mathcal{B} \longmapsto P'$$
$$P' = \mathcal{D}(P) = \mathcal{T}_{\mathbf{t}} \circ \mathcal{R}_C(P) \tag{5.26}$$

This can be written as

$$\mathbf{r}_{CP'} = \mathbf{t} + \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{r}_{CP}) \tag{5.27}$$

- Note that this is a vector equation and can be resolved in any basis.
- Is $\mathcal{D} : \mathbf{r}_{CP} \longmapsto \mathbf{r}_{CP'}$ a linear operator?
- No, because if $\mathbf{r}_{CP} = \mathbf{0}$, $\mathbf{r}_{CP'}$ need not be $\mathbf{0}$.
- What is $\mathbf{r}_{CC'}$?

$$\mathbf{r}_{CP'} = \mathbf{t} + \mathcal{R}_{\alpha,\hat{\mathbf{a}}}(\mathbf{r}_{CP})$$

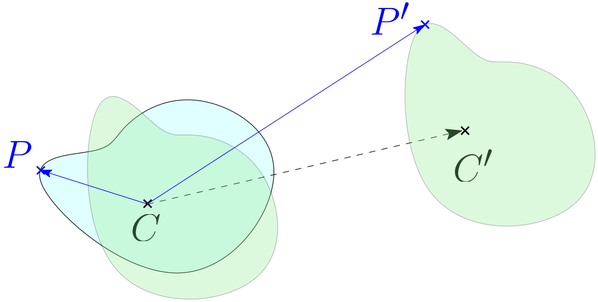


Figure 5.3: Rotation about C, followed by a translation.

- The center of rotation $C \in \mathcal{B}$ does not move during rotation.
- However, C moves to C' during the subsequent translation.
- The left-hand side is still w.r.t. the original point C.

General Displacement

Rigidity Constraints

To show that (5.27) is a rigid displacement satisfying (5.14), we consider $P, Q \in \mathcal{B}$.

$$\mathbf{r}_{CP'} = \mathbf{t} + \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CP}\right), \qquad \mathbf{r}_{CQ'} = \mathbf{t} + \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CQ}\right) \tag{5.28}$$

We subtract one from the other and use the property that $\mathcal{R}$ is a linear operator.

$$\begin{aligned}
\mathbf{r}_{CP'} - \mathbf{r}_{CQ'} &= \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CP}\right) - \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CQ}\right) \\
&= \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CP} - \mathbf{r}_{CQ}\right) \\
\Rightarrow \mathbf{r}_{Q'P'} &= \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{QP}\right).
\end{aligned}$$

We reach (5.21) again. From there, the rigidity proof is the same as before.

General Displacement

Resolved to a Frame

$$\mathbf{r}_{CP'} = \mathbf{t} + \mathcal{R}_{\alpha,\hat{\mathbf{a}}}\left(\mathbf{r}_{CP}\right)$$

can be resolved in a frame $\mathscr{F}_E$ as

$${}^{E}\mathbf{r}_{CP'} = {}^{E}\left[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}\right]{}^{E}\mathbf{r}_{CP} + {}^{E}\mathbf{t}$$

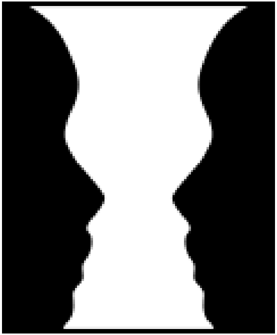


Figure 5.4: Two ways of looking at the same thing.[8]

[8] "Cup or faces paradox" by Bryan Derksen - Original image Image:Cup or faces paradox.jpg uploaded by User:Guam on 28 July 2005, SVG conversion by Bryan Derksen. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Cup_or_faces_paradox.svg

Two Different Semantics

- The first is the change of position-vector of the same physical point w.r.t. two different referentials: This was (5.8):

$$ {}^{E}\mathbf{r}_P = {}^{E}_{B}\mathbf{R}\, {}^{B}\mathbf{r}_P + {}^{E}_{B}\mathbf{t} $$

  - In other words, the referential moves, but the physical point remains the same.
- The second is (5.27), the displacement of a point $P \in \mathcal{B}$, resolved in a referential $\mathscr{F}_E$

$$ {}^{E}\mathbf{r}_{CP'} = {}^{E}\left[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}\right] {}^{E}\mathbf{r}_{CP} + {}^{E}\mathbf{t} \tag{5.29} $$

Here, $C \in \mathcal{B}$ (or its rigid extension) is the center of rotation.

  - In other words, the referential remains stationary, but the physical rigid-body moves.
  - **Special Case**: If C is initially coincident with $\mathcal{O}_E$, then (5.29) can also be written as

$$ {}^{E}\mathbf{r}_{P'} = {}^{E}[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}]\, {}^{E}\mathbf{r}_P + {}^{E}\mathbf{t} \tag{5.30} $$

Homogeneous Transforms Revisited

Eq. (5.30) can be rewritten using homogeneous transforms (5.9) as follows for $P \in \mathcal{B}$

$$ {}^{E}\underline{\mathbf{r}}_{P'} = \begin{bmatrix} {}^{E}\left[\mathcal{R}_{\alpha,\hat{\mathbf{a}}}\right] & {}^{E}\mathbf{t} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} {}^{E}\underline{\mathbf{r}}_{P} \triangleq {}^{E}\mathbf{T}\,{}^{E}\underline{\mathbf{r}}_{P} \tag{5.31} $$

Remember that this is only valid if $\mathcal{B}$ is undergoing a rotation about a $C \in \mathcal{B}$ with C coincident with $\mathcal{O}_E$, and a translation.
In other words, this is only valid if the initial rotation is about the origin of $\mathscr{F}_E$.

Composition of Homogeneous Transforms

Two successive motions (5.31) of body $\mathcal{B}$ relative to $\mathscr{F}_E$ can be composed as follows:

$$ {}^{E}\underline{\mathbf{r}}_{P'} = {}^{E}\mathbf{T}_1\, {}^{E}\underline{\mathbf{r}}_{P} \tag{5.32}$$

$$ {}^{E}\underline{\mathbf{r}}_{P''} = {}^{E}\mathbf{T}_2\, {}^{E}\underline{\mathbf{r}}_{P'} \tag{5.33}$$

Combining the two we get:

$$ {}^{E}\underline{\mathbf{r}}_{P''} = {}^{E}\mathbf{T}_2\, {}^{E}\mathbf{T}_1\, {}^{E}\underline{\mathbf{r}}_{P} \tag{5.34}$$

$$ = {}^{E}\mathbf{T}_{2\circ 1}\, {}^{E}\underline{\mathbf{r}}_{P} \tag{5.35}$$

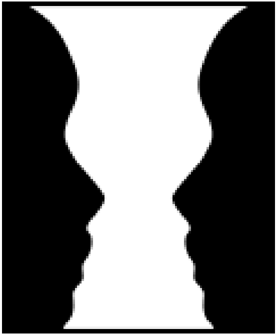


Figure 5.5: Two ways of looking at the same thing.[9]

[9] "Cup or faces paradox" by Bryan Derksen - Original image Image:Cup or faces paradox.jpg uploaded by User:Guam on 28 July 2005, SVG conversion by Bryan Derksen. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Cup_or_faces_paradox.svg

We now compare (5.13e) and (5.35)

- **Body-Fixed Transforms (BFT) or Relative Transforms**: In (5.13e) we are multiplying successive transforms left to right.

$$ {}^{E}_{B}\mathbf{T} = {}^{E}_{A}\mathbf{T}\ {}^{A}_{B}\mathbf{T} $$

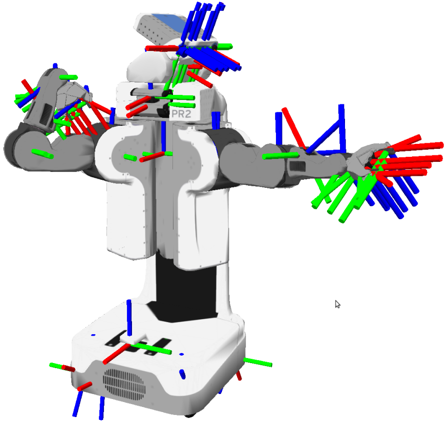
- **Space-Fixed Transforms (SFT) or Absolute Transforms**: In (5.35) we are multiplying successive transforms right to left.

$$ {}^{E}\mathbf{T}_{2\circ 1} = {}^{E}\mathbf{T}_{2}\,{}^{E}\mathbf{T}_{1} $$

- Given a series of successive homogeneous-transforms multiplying together, we are free to interpret the sequence of multiplications as we choose: either as BFT or SFTs.
- We usually choose the interpretation which is easier to understand intuitively.

Willow Garage PR2 Robot

(a) PR2. Source: Link



(b) Link frames. Source: Link

Frame Transform Trees in Robotics

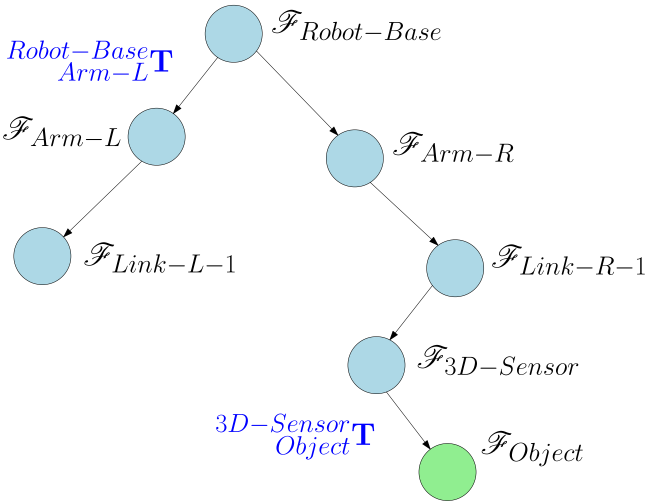


Figure 5.6: Nodes are frames, edges are transforms. Going down from root is equivalent to BFT.

Matlab hgtransform Hierarchy

$\mathbf{T}_c \, \mathbf{T}_b \, \mathbf{T}_a$

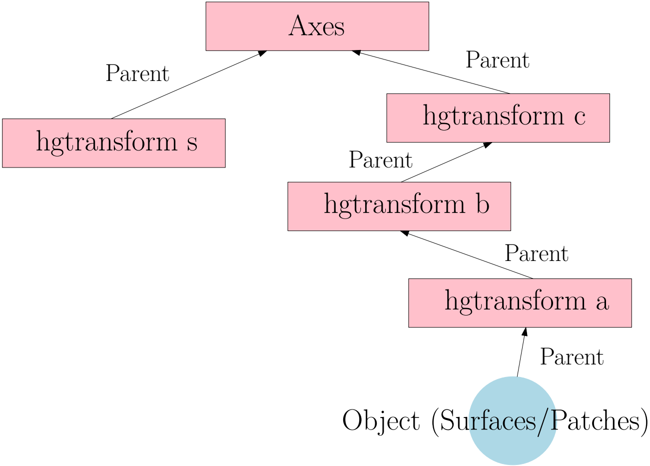


Figure 5.7: Nodes are transforms, objects, or axes. The parent edges implicitly represent frames.

Illustration Of Transforms in Matlab

illustrate_frames.m

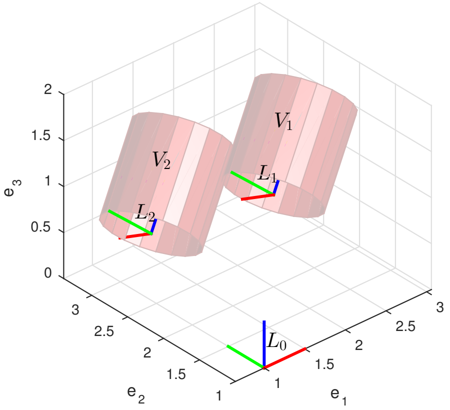


Figure 5.8: L_1 does rotation first (right to left), L_2 does translation first (right to left).

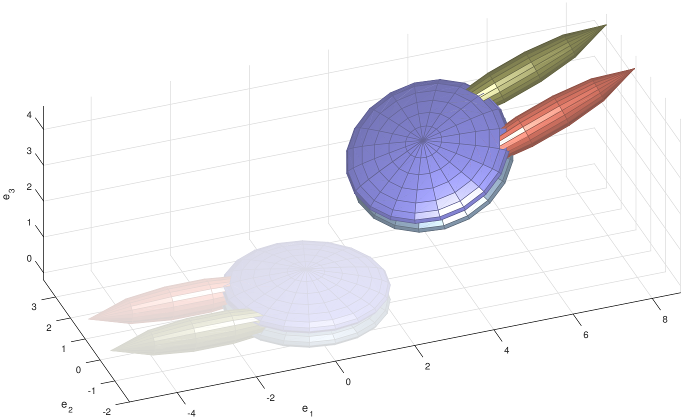


Figure 5.9: A ship undergoing a motion: Look at `show_object_hierarchy.m`

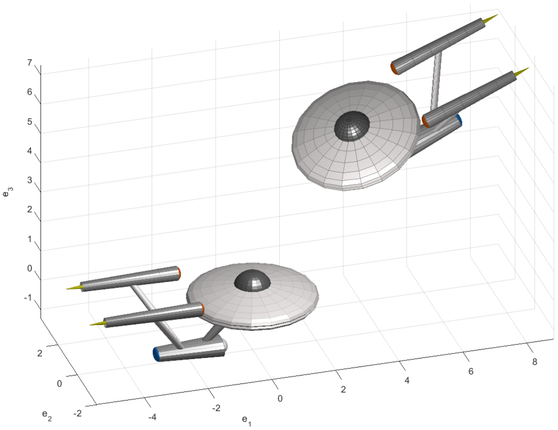


Figure 5.10: A much better space-ship model created by IMS student Miraj Sheth.

Robot Links and Joints

Example: SCARA (Selective Compliance Assembly Robot Arm)

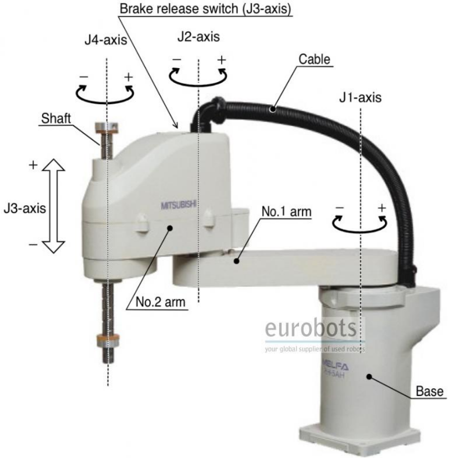


Figure 5.11: SCARA: Mitsubishi Melfa RH-10AH85. Image credits: Link

Robot Links and Joints

Example: SCARA (Selective Compliance Assembly Robot Arm)

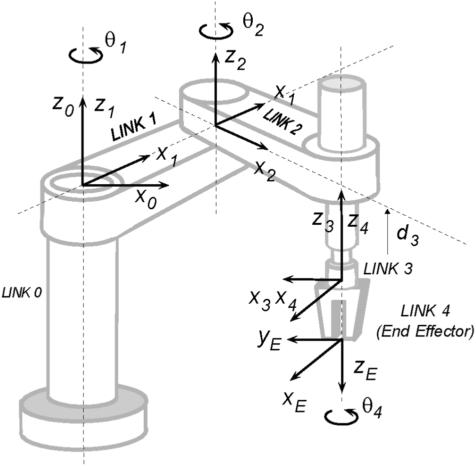


Figure 5.12: SCARA schematic. Image credits: Link. Three revolute and one prismatic joints are shown.

Specifying the Robot Geometry in URDF Format

Unified Robot Description Format (URDF)

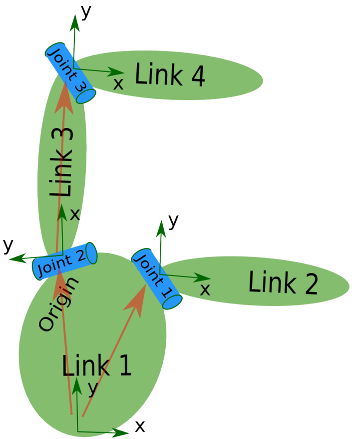


Figure 5.13: A robot consisting of links and joints. Image credits: ROS website.

Specifying the Robot Geometry/Kinematics in URDF Format

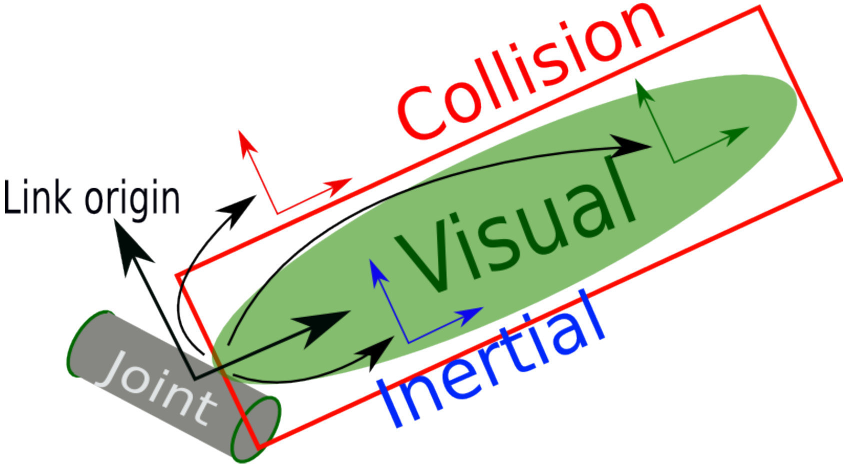


Figure 5.14: A robot link. Image credits: ROS website.

Specifying the Robot Geometry/Kinematics in URDF Format

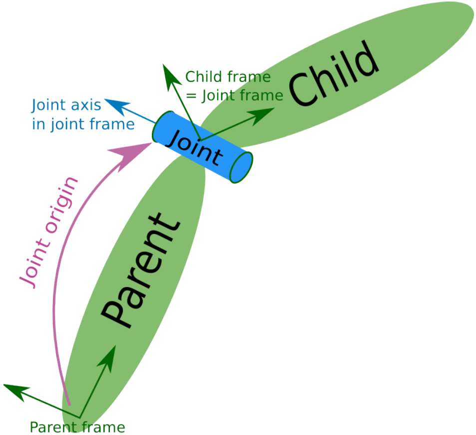


Figure 5.15: A joint. Note: The joint-frame is not exactly the same as the child-frame: They are the same when the joint-parameter (e.g. the angle for a revolute joint) is 0. Image credits: ROS website.

A Simple Example

Unified Robot Description Format (URDF)

```
<robot name="test_robot">
  <link name="link1" />
  <link name="link2" />
  <link name="link3" />
  <link name="link4" />

  <joint name="joint1" type="continuous">
    <parent link="link1"/>
    <child link="link2"/>
    <origin xyz="5 3 0" rpy="0 0 0" />
    <axis xyz="-0.9 0.15 0" />
  </joint>

  <joint name="joint2" type="continuous">
    <parent link="link1"/>
    <child link="link3"/>
    <origin xyz="-2 5 0" rpy="0 0 1.57" />
    <axis xyz="-0.707 0.707 0" />
  </joint>

  <joint name="joint3" type="continuous">
    <parent link="link3"/>
    <child link="link4"/>
    <origin xyz="5 0 0" rpy="0 0 -1.57" />
    <axis xyz="0.707 -0.707 0" />
  </joint>
</robot>
```

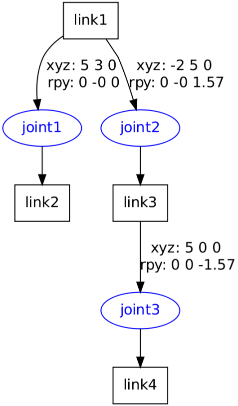


Figure 5.16: The graph.

Robot Links and Joints

Example: Extended SCARA Matlab Model

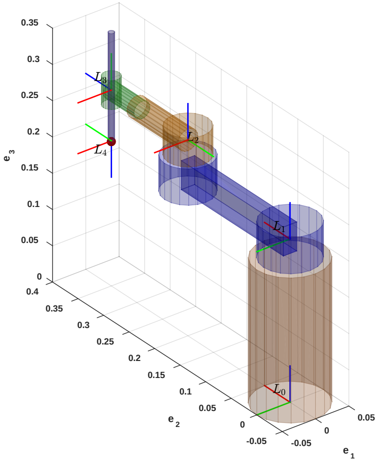
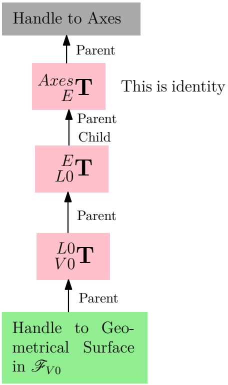


Figure 5.17: Code make_scara_robot.m. The model is based on this URDF file.

Step 0: Create Base-Frame & Link-0

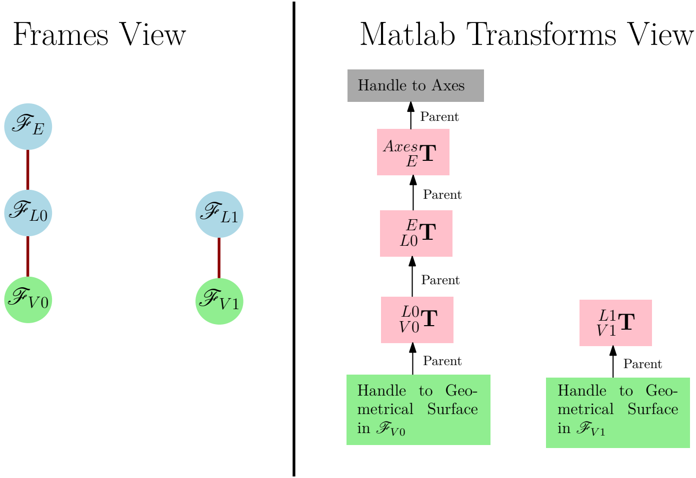
Frames View

Matlab Transforms View




```
handle_axes= axes('XLim', [-0.4,0.4], 'YLim', [-0.2,0.4], 'ZLim', [0,0.4]);
trf_E_axes= hgtransform('Parent', handle_axes);
trf_link0_E= make_transform([0, 0, 0], 0, 0, pi/2, trf_E_axes);
trf_viz0_link0= make_transform([0, 0, 0.1], 0, 0, 0, trf_link0_E);
handle_g0 = link_cylinder(radius0, length0, trf_viz0_link0);
```

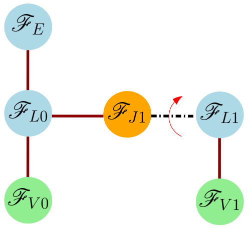
Step 1: Create Link-1



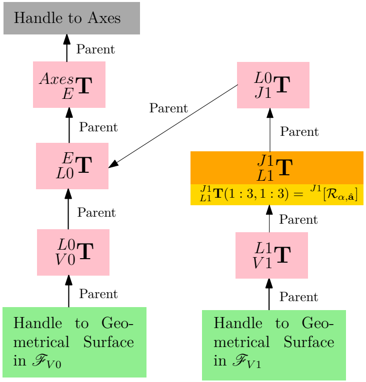

```
trf_viz1_link1= make_transform([0, 0, 0], 0, 0, 0); % Do not specify parent yet: It will be done in the joint
handle_g1 = link_cylinder(radius1, length1, trf_viz1_link1);
```

Step 2, Case I: Revolute Joint between Links 0 & 1

Frames View

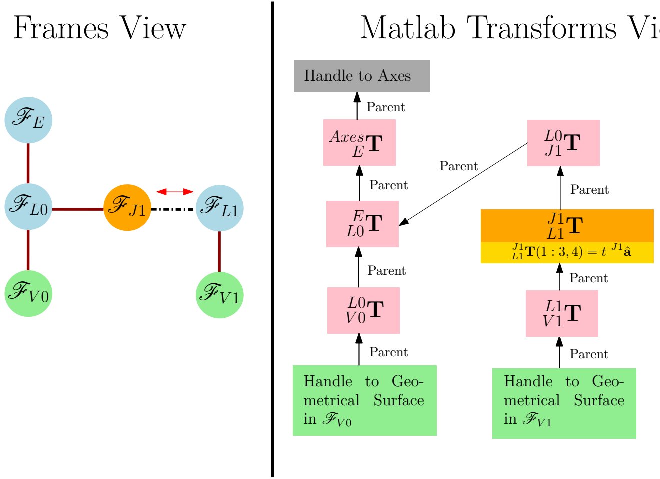


Matlab Transforms View




```
j1_rot_axis_j1= [0,0,1]'; % Joint rotation-axis resolved in joint-frame.
j1_rot_angle= 0; % Joint d.o.f. range [-pi/2, pi/2]
trf_joint1_link0= make_transform([0, 0, 0.225], 0, 0, 0, trf_link0_E); % From URDF
trf_link1_joint1= make_transform_revolute(j1_rot_axis_j1, j1_rot_angle, trf_joint1_link0); % Joint transform
make_child(trf_link1_joint1, trf_viz1_link1); % Make argument 2 the child of 1
```

Step 2, Case II: Prismatic Joint between Links 0 & 1




```
j1_translation_axis_j1= [1,0,0]'; % Joint translation direction resolved in joint-frame.
j1_translation= 1.5; % Joint d.o.f. Range [0, 2.8]
trf_joint1_link0= make_transform([0, 0, 0.225], 0, 0, 0, trf_link0_E); % From URDF
trf_link1_joint1= make_transform_prismatic(j1_translation_axis_j1, j1_translation, trf_joint1_link0);
make_child(trf_link1_joint1, trf_viz1_link1); % Make argument 2 the child of 1
```

- Recall that a referential (basis + origin) can have several different coordinate-systems attached to it.
- We looked at the Cartesian coordinate-system.
- Now we look at two other types of coordinate-systems: cylindrical and spherical.

Cylindrical Coordinate System

$$\mathbf{r}_{\mathcal{O}_E P} = r\,\hat{\mathbf{e}}_r + z\,\hat{\mathbf{e}}_z \tag{5.36}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} \tag{5.37}$$

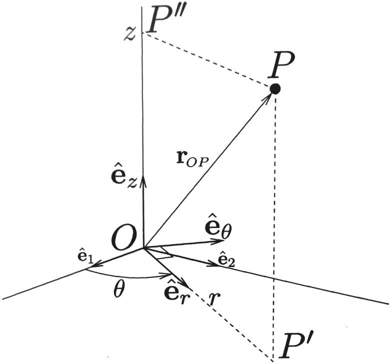


Figure 5.18: Cylindrical coordinates. Image credits: [Roy(2015)].

Spherical Coordinate System

$$\mathbf{r}_{\mathcal{O}_E P} = \rho\, \hat{\mathbf{e}}_\rho \tag{5.38}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} \tag{5.39}$$

- ϕ is called the colatitude angle.
- θ is called the azimuthal or longitude angle.
- The unit-vector $\hat{\mathbf{e}}_\phi$ is tangential to the arc of constant θ at P in the direction of increasing ϕ.
- The unit-vector $\hat{\mathbf{e}}_\theta$ is tangential to the arc of constant ϕ at P in the direction of increasing θ.

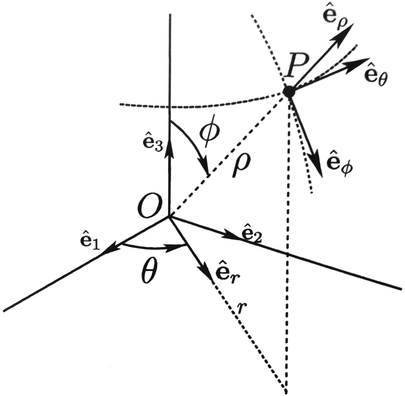


Figure 5.19: Spherical coordinates. Image credits: [Roy(2015)].

Contents

Assumptions in Classical Mechanics

- Recall that a referential $\mathscr{F}_E$ is a 3D space, usually associated with a rigid-extension of a physical rigid-body $\mathcal{E}$.
- The observer associated with $\mathscr{F}_E$ has a clock using which he/she can measure the time t at which an event took place. Hence, it can be decided if two events in $\mathscr{F}_E$ took place simultaneously.
- Similarly, the observer associated with another referential $\mathscr{F}_B$, which is in motion relative to $\mathscr{F}_E$, also has a clock.

Assumptions in Classical Mechanics

- In Classical Mechanics, we assume that the clocks in $\mathscr{F}_E$ and $\mathscr{F}_B$ are synchronized and indicate the same time t for the occurrence of an event observed from both the referentials: two events which occur simultaneously in $\mathscr{F}_E$ also occur simultaneously in $\mathscr{F}_B$.
- Another assumption made in Classical Mechanics is that observers in $\mathscr{F}_E$ and $\mathscr{F}_B$ agree on the lengths measured between any two points.
- These assumptions agree with our common sense, but they break down if the magnitude of the relative velocity of $\mathscr{F}_E$ and $\mathscr{F}_B$ is close to the speed of light: In this case, we have to use the Special Theory of Relativity.

Time-Derivative of a Scalar Function

If $u(t)$ is a scalar function of time (e.g. temperature) its time-derivative, written $\dot{u} \triangleq du/dt$ is the same in all referentials.
In other words, observers in any two referentials will observe the same value of the time-derivative for the scalar function using their clocks!

Dependence of the Time-Derivative of a Vector on the Referential

- In Kinematics, we are interested in $d\mathbf{f}/dt$ of a vector-function $\mathbf{f}(t)$ of time, e.g. the position-vector $\mathbf{r}_{OP}$ of a moving point P.
- For vectors, the time-derivative $d\mathbf{f}/dt$ is different in different referentials in relative motion w.r.t. each other. Example?
- So, for $d\mathbf{f}/dt$ relative to $\mathscr{F}_E$ we write:

$$\left(\frac{d\mathbf{f}}{dt}\right)_E \tag{6.1}$$

- This is a vector (3D arrow) – we have not yet resolved it in any basis.

Time-Derivative of a Vector in a Referential

- Consider a referential $\mathscr{F}_E$ with associated basis $\mathscr{B}_E$. In it, a vector $\mathbf{f}$ can be written as

$$\mathbf{f} = {}^E f_1 \hat{\mathbf{e}}_1 + {}^E f_2 \hat{\mathbf{e}}_2 + {}^E f_3 \hat{\mathbf{e}}_3 \tag{6.2}$$

- Since $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are fixed in $\mathscr{F}_E$, we have

$$\left(\frac{d\hat{\mathbf{e}}_i}{dt}\right)_E = \mathbf{0}, \qquad i = 1, 2, 3. \tag{6.3}$$

- Hence, we get,

$$\left(\frac{d\mathbf{f}}{dt}\right)_E = {}^E \dot{f}_1 \hat{\mathbf{e}}_1 + {}^E \dot{f}_2 \hat{\mathbf{e}}_2 + {}^E \dot{f}_3 \hat{\mathbf{e}}_3 \tag{6.4}$$

- This formulation is adequate for particle kinematics, but for rigid-body kinematics it will get cumbersome. Hence, for rigid-body kinematics, we will use another formulation.

Notation

- Note that (6.4) can be written as

$$\left(\frac{d\mathbf{f}}{dt}\right)_E = {}^E\dot{f}_1\hat{\mathbf{e}}_1 + {}^E\dot{f}_2\hat{\mathbf{e}}_2 + {}^E\dot{f}_3\hat{\mathbf{e}}_3$$

$$\Rightarrow {}^E\left(\frac{d\mathbf{f}}{dt}\right)_E = \begin{bmatrix} {}^E\dot{f}_1 \\ {}^E\dot{f}_2 \\ {}^E\dot{f}_3 \end{bmatrix} \tag{6.5}$$

- This example illustrates the difference between "relative to" and "resolved in".

Some Useful Results

The following results can be shown using (6.4) and the chain-rule of differentiation

$$\left(\frac{d(\mathbf{v}+\mathbf{w})}{dt}\right)_E = \left(\frac{d\mathbf{v}}{dt}\right)_E + \left(\frac{d\mathbf{w}}{dt}\right)_E \tag{6.6a}$$

$$\left(\frac{d(\lambda\mathbf{v})}{dt}\right)_E = \dot{\lambda}\,\mathbf{v} + \lambda\left(\frac{d\mathbf{v}}{dt}\right)_E \tag{6.6b}$$

$$\frac{d(\mathbf{v}\cdot\mathbf{w})}{dt} = \left(\frac{d\mathbf{v}}{dt}\right)_E \cdot \mathbf{w} + \mathbf{v}\cdot\left(\frac{d\mathbf{w}}{dt}\right)_E \tag{6.6c}$$

$$\left(\frac{d(\mathbf{v}\times\mathbf{w})}{dt}\right)_E = \left(\frac{d\mathbf{v}}{dt}\right)_E \times \mathbf{w} + \mathbf{v}\times\left(\frac{d\mathbf{w}}{dt}\right)_E \tag{6.6d}$$

Theorem 6.1

A vector $\mathbf{v}(t)$ *remains of constant magnitude iff* $\mathbf{v}$ *remains perpendicular to its derivative* $d\mathbf{v}/dt$ *relative to any referential* $\mathscr{F}_E$.

Proof.

As $\mathbf{v}(t)$ is of constant magnitude

$$\frac{d\|\mathbf{v}\|^2}{dt} = 0$$

$$\Rightarrow \frac{d(\mathbf{v}\cdot\mathbf{v})}{dt} \overset{(6.6c)}{=} \left(\frac{d\mathbf{v}}{dt}\right)_E \cdot \mathbf{v} + \mathbf{v}\cdot\left(\frac{d\mathbf{v}}{dt}\right)_E \quad = 2\,\mathbf{v}\cdot\left(\frac{d\mathbf{v}}{dt}\right)_E \quad = 0$$

Hence, $\mathbf{v}$ is orthogonal to $\left(\frac{d\mathbf{v}}{dt}\right)_E$ relative to any arbitrary referential $\mathscr{F}_E$. □

Velocity of a Particle Relative to a Referential

- Let a particle P be in motion relative to $\mathscr{F}_E$. Its position-vector (recall Def. 5.4) in $\mathscr{F}_E$ is defined by $\mathbf{r}_{\mathcal{O}_E P}$ relative to an origin point $\mathcal{O}_E$ fixed in $\mathscr{F}_E$.
- The velocity of P relative to $\mathscr{F}_E$ is defined as

$$\mathbf{v}_{P/E} \triangleq \left(\frac{d\mathbf{r}_{\mathcal{O}_E P}}{dt} \right)_E \tag{6.7}$$

- The speed of P relative to $\mathscr{F}_E$ is defined as

$$v_{P/E} \triangleq \left\| \mathbf{v}_{P/E} \right\| \tag{6.8}$$

Theorem 6.2

$\mathbf{v}_{P/E}$ *is independent of the choice of the origin* $\mathcal{O}_E$ *of* $\mathscr{F}_E$.

Proof.

Let O' be another point fixed in (attached to) $\mathscr{F}_E$, then,

$$\mathbf{r}_{O'P} = \mathbf{r}_{O'\mathcal{O}_E} + \mathbf{r}_{\mathcal{O}_E P} \tag{6.9}$$

$$\Rightarrow \left(\frac{d\mathbf{r}_{O'P}}{dt}\right)_E = \left(\frac{d\mathbf{r}_{O'\mathcal{O}_E}}{dt}\right)_E + \left(\frac{d\mathbf{r}_{\mathcal{O}_E P}}{dt}\right)_E$$

$$= \left(\frac{d\mathbf{r}_{\mathcal{O}_E P}}{dt}\right)_E \tag{6.10}$$

$$= \mathbf{v}_{P/E}$$

□

In general, if $\mathscr{F}_E$ and $\mathscr{F}_B$ are in relative motion, $\mathbf{v}_{P/E} \neq \mathbf{v}_{P/B}$.

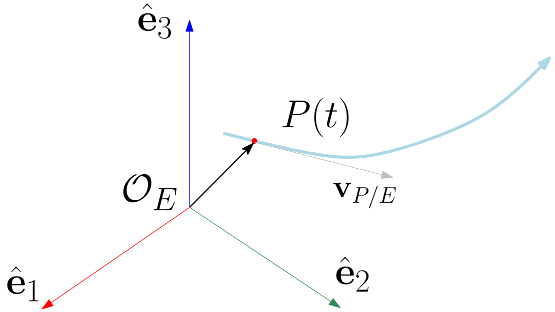
Acceleration of a Particle Relative to a Referential

The acceleration of P relative to $\mathscr{F}_E$ is defined as follows

$$\mathbf{a}_{P/E} \triangleq \left(\frac{d\mathbf{v}_{P/E}}{dt}\right)_E \tag{6.11}$$

The motion of P is called accelerated if $\mathbf{v}_{P/E} \cdot \mathbf{a}_{P/E} > 0$.
The motion of P is called decelerated if $\mathbf{v}_{P/E} \cdot \mathbf{a}_{P/E} < 0$. How do the vectors look?

Trajectory of a Particle in a Referential



The trajectory of a particle P in $\mathscr{F}_E$ is the locus described by $P(t)$ as time varies from $t = t_0$ to $t = t_1$. The velocity $\mathbf{v}_{P/E}$ is tangent to the trajectory and is oriented in the direction corresponding to increasing time.

Digression: Trajectory Scaling For the Space-Ship Example

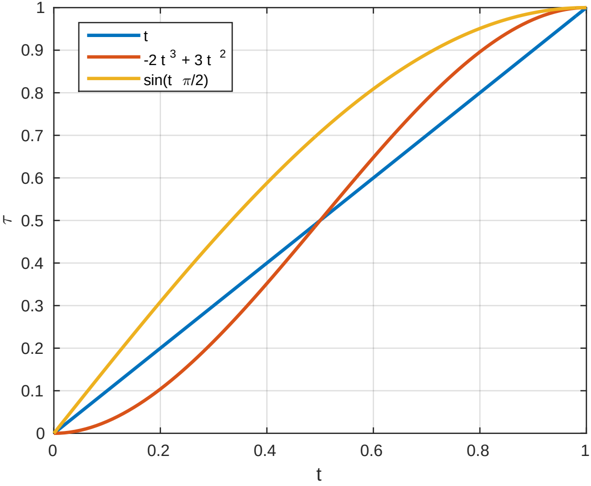
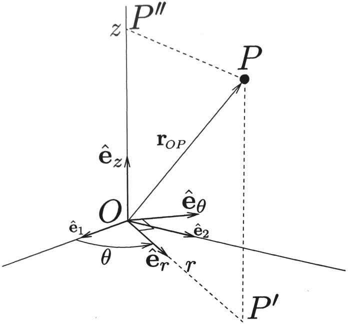


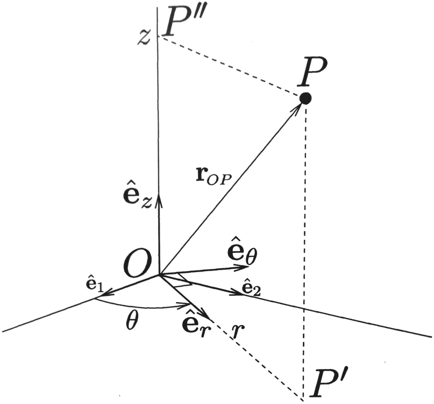
Figure 6.1: Scaling the time for the trajectory of the space-ship example in `animate_spaceship.m`

Velocity in Cylindrical Coordinate-System

Recall Fig. 5.18

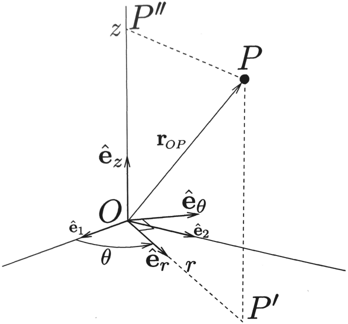


- In $\mathscr{F}_E$, now we have two bases attached, the usual Cartesian basis $\mathscr{B}_E\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ and the cylindrical basis $\mathscr{B}_{Cyl}\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_3\}$.
- In the basis $\mathscr{B}_{Cyl}\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z\}$, the position-vector of P is $\mathbf{r}_{\mathcal{O}_E P} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$.
- The unit-vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are functions of θ and hence time dependent.



$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \tag{6.12}$$

$$\hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2 \tag{6.13}$$



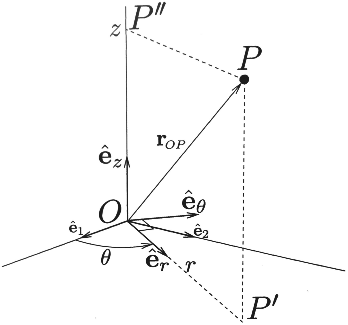
$$\left(\frac{d\hat{\mathbf{e}}_r}{dt}\right)_E = \left(\frac{d}{dt}\right)_E (\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2) \tag{6.14}$$

$$= \dot{\theta}\,(-\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2) \qquad = \dot{\theta}\,\hat{\mathbf{e}}_\theta. \tag{6.15}$$

$$\left(\frac{d\hat{\mathbf{e}}_\theta}{dt}\right)_E = \left(\frac{d}{dt}\right)_E (-\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2) \tag{6.16}$$

$$= \dot{\theta}\,(-\cos\theta\,\hat{\mathbf{e}}_1 - \sin\theta\,\hat{\mathbf{e}}_2) \qquad = -\dot{\theta}\,\hat{\mathbf{e}}_r. \tag{6.17}$$

Clearly, the derivatives are orthogonal to the vectors as they should be. Why?



To find the velocity in $\mathscr{B}_{Cyl}$, we differentiate $\mathbf{r}_{\mathcal{O}_E P} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$

$$\mathbf{v}_{P/E} = \dot{r}\,\hat{\mathbf{e}}_r + r\left(\frac{d\hat{\mathbf{e}}_r}{dt}\right)_E + \dot{z}\,\hat{\mathbf{e}}_z \tag{6.18}$$

$$\overset{(6.15)}{=} \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta + \dot{z}\,\hat{\mathbf{e}}_z. \tag{6.19}$$

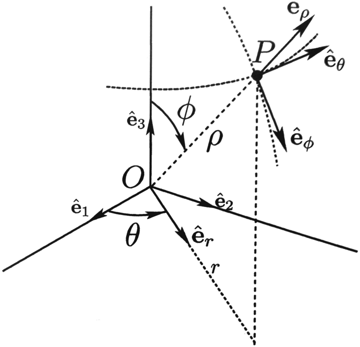
Acceleration in Cylindrical Coordinate-System

We differentiate (6.19) to get the acceleration:

$$
\begin{aligned}
\mathbf{a}_{P/E} &= \left(\frac{d\mathbf{v}_{P/E}}{dt}\right)_E \\
&= \left(\frac{d}{dt}\right)_E \left(\dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta + \dot{z}\,\hat{\mathbf{e}}_z\right) \\
&= \ddot{r}\,\hat{\mathbf{e}}_r + \dot{r}\,\left(\frac{d\hat{\mathbf{e}}_r}{dt}\right)_E + (r\ddot{\theta} + \dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta + r\dot{\theta}\,\left(\frac{d\hat{\mathbf{e}}_\theta}{dt}\right)_E + \ddot{z}\,\hat{\mathbf{e}}_z \qquad (6.20) \\
&\overset{(6.15),(6.17)}{=} (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{e}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\mathbf{e}}_\theta + \ddot{z}\,\hat{\mathbf{e}}_z.
\end{aligned}
$$

Spherical Coordinates

Recall Fig. 5.19



$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{e}}_z + \sin\phi\,\hat{\mathbf{e}}_r, \tag{6.21}$$

$$\hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_z + \cos\phi\,\hat{\mathbf{e}}_r \tag{6.22}$$

Spherical Coordinates

$$\left(\frac{d\hat{\mathbf{e}}_\rho}{dt}\right)_E \stackrel{(6.15)}{=} -\dot{\phi}\,\sin\phi\,\hat{\mathbf{e}}_z + \dot{\phi}\,\cos\phi\,\hat{\mathbf{e}}_r + \sin\phi\,\dot{\theta}\,\hat{\mathbf{e}}_\theta \tag{6.23}$$

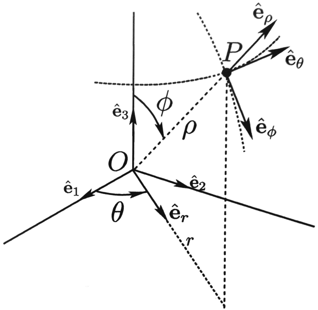
$$\stackrel{(6.22)}{=} \dot{\phi}\,\hat{\mathbf{e}}_\phi + \dot{\theta}\,\sin\phi\,\hat{\mathbf{e}}_\theta \tag{6.24}$$

$$\left(\frac{d\hat{\mathbf{e}}_\phi}{dt}\right)_E \stackrel{(6.22)}{=} -\dot{\phi}\cos\phi\,\hat{\mathbf{e}}_z - \dot{\phi}\,\sin\phi\,\hat{\mathbf{e}}_r + \cos\phi\,\dot{\theta}\,\hat{\mathbf{e}}_\theta \tag{6.25}$$

$$\stackrel{(6.21)}{=} -\dot{\phi}\,\hat{\mathbf{e}}_\rho + \dot{\theta}\,\cos\phi\,\hat{\mathbf{e}}_\theta \tag{6.26}$$

Are $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ orthogonal to their respective derivatives?

Velocity in Spherical Coordinates



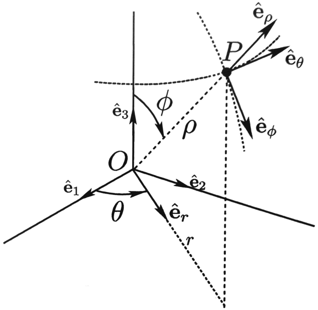
$$\mathbf{r}_{\mathcal{O}_E P} = \rho\,\hat{\mathbf{e}}_\rho \tag{6.27}$$

$$\mathbf{v}_{P/E} \overset{(6.24)}{=} \dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\left(\dot{\phi}\,\hat{\mathbf{e}}_\phi + \dot{\theta}\,\sin\phi\,\hat{\mathbf{e}}_\theta\right) \tag{6.28}$$

Acceleration in Spherical Coordinates

$$
\begin{aligned}
\mathbf{a}_{P/E} &= a_\rho\,\hat{\mathbf{e}}_\rho + a_\phi\,\hat{\mathbf{e}}_\phi + a_\theta\,\hat{\mathbf{e}}_\theta,\ \text{where,} && (6.29a)\\
a_\rho &= \ddot{\rho} - \rho\,\dot{\phi}^2 - \rho\dot{\theta}^2\sin^2\phi && (6.29b)\\
a_\phi &= 2\dot{\rho}\dot{\phi} + \rho\ddot{\phi} - \rho\dot{\theta}^2\cos\phi\,\sin\phi && (6.29c)\\
a_\theta &= 2\dot{\rho}\dot{\theta}\sin\phi + 2\rho\dot{\phi}\dot{\theta}\,\cos\phi + \rho\ddot{\theta}\sin\phi && (6.29d)
\end{aligned}
$$

Spherical Joint



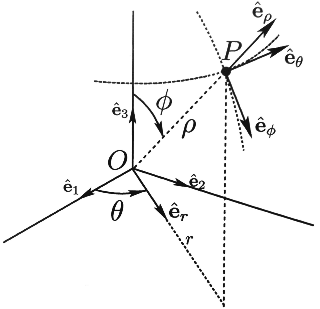
$$\hat{\mathbf{e}}_\rho = \cos\phi\,\hat{\mathbf{e}}_z + \sin\phi\,\hat{\mathbf{e}}_r, \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\,\hat{\mathbf{e}}_z + \cos\phi\,\hat{\mathbf{e}}_r$$

Consider 3 bases:

- The original $\mathscr{B}_E(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$.
- The cylindrical $\mathscr{B}_C(\hat{\mathbf{c}}_1 = \hat{\mathbf{e}}_r,\ \hat{\mathbf{c}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{c}}_3 = \hat{\mathbf{e}}_3)$.
- The spherical basis $\mathscr{B}_S(\hat{\mathbf{s}}_1 = \hat{\mathbf{e}}_\phi,\ \hat{\mathbf{s}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{s}}_3 = \hat{\mathbf{e}}_\rho)$.

Spherical Joint

The Rotation Sequence



$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \xrightarrow{\mathcal{R}_{\theta,\hat{\mathbf{e}}_3}} (\hat{\mathbf{c}}_1 = \hat{\mathbf{e}}_r,\ \hat{\mathbf{c}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{c}}_3 = \hat{\mathbf{e}}_3) \tag{6.30a}$$

$$(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3) \xrightarrow{\mathcal{R}_{\phi,\hat{\mathbf{c}}_2}} (\hat{\mathbf{s}}_1 = \hat{\mathbf{e}}_\phi,\ \hat{\mathbf{s}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{s}}_3 = \hat{\mathbf{e}}_\rho) \tag{6.30b}$$

Spherical Joint

The Rotation Sequence

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \xrightarrow{\mathcal{R}_{\theta,\hat{\mathbf{e}}_3}} (\hat{\mathbf{c}}_1 = \hat{\mathbf{e}}_r,\ \hat{\mathbf{c}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{c}}_3 = \hat{\mathbf{e}}_3)$$

$$(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3) \xrightarrow{\mathcal{R}_{\phi,\hat{\mathbf{c}}_2}} (\hat{\mathbf{s}}_1 = \hat{\mathbf{e}}_\phi,\ \hat{\mathbf{s}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{s}}_3 = \hat{\mathbf{e}}_\rho)$$

Using the composition of rotations from Theorem 4.2 and the matrix-representation of the axis-angle rotation operator (4.62), we have:

$$ {}^E_S\mathbf{R} = {}^E_C\mathbf{R}\ {}^C_S\mathbf{R} \tag{6.31a}$$

$$ = {}^E\left[\mathcal{R}_{\theta,\hat{\mathbf{e}}_3}\right]\ {}^C\left[\mathcal{R}_{\phi,\hat{\mathbf{c}}_2}\right] \tag{6.31b}$$

Note that this is the BFT interpretation, so we're multiplying from left to right.

Digression: BFT vs. SFT

`illustrate_rotations_spherical_joint.m`

- The DCM ${}^{E}_{S}\mathbf{R} = {}^{E}\left[\mathcal{R}_{\theta,\hat{\mathbf{e}}_3}\right] {}^{C}\left[\mathcal{R}_{\phi,\hat{\mathbf{c}}_2}\right]$ was computed using the BFT interpretation, hence multiplying from left to right.
- We can also compute it using the SFT interpretation, multiplying right to left. In this case: ${}^{E}_{S}\mathbf{R} = {}^{E}\left[\mathcal{R}_{\phi,\hat{\mathbf{e}}_\theta}\right] {}^{E}\left[\mathcal{R}_{\theta,\hat{\mathbf{e}}_3}\right]$ For this we need ${}^{E}\hat{\mathbf{e}}_\theta$. What is it?
- We get it from (6.13).

$$ {}^{E}\hat{\mathbf{e}}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{6.32} $$

- After algebraic simplification, (see `illustrate_rotations_spherical_joint.m`) it can be shown that the above two expressions are equal.

For Matlab and in URDF, it is more straightforward to use the BFT interpretation (6.31):

$$\begin{aligned} {}^{E}_{S}\mathbf{R} &= {}^{E}_{C}\mathbf{R}\ {}^{C}_{S}\mathbf{R} \\ &= {}^{E}\left[\mathcal{R}_{\theta,\hat{\mathbf{e}}_3}\right]\ {}^{C}\left[\mathcal{R}_{\phi,\hat{\mathbf{c}}_2}\right] \end{aligned}$$

Spherical Joint

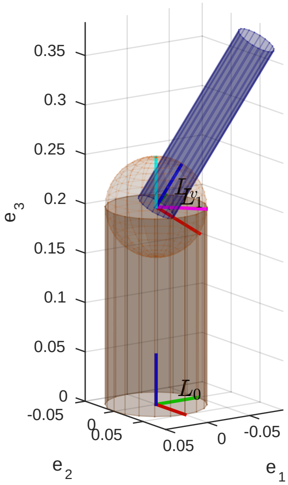


Figure 6.2: `animate_spherical_joint.m`

Contents

- We now study the kinematics of a rigid body $\mathcal{B}$ relative to a referential (or rigid body) $\mathcal{A}$.
- In particular, we want to find the velocity-field $P \in \mathcal{B} \longmapsto \mathbf{v}_{P/A}$.
- We will show that this field is entirely specified at any time by the knowledge of two quantities:
  1. The velocity of a particular point of $\mathcal{B}$
  2. A vector independent of position, called angular velocity, characterizing the change of orientation of $\mathcal{B}$ relative to $\mathcal{A}$.

- Consider a vector $\mathbf{u}$ attached to $\mathcal{B}$. Since $\mathbf{u} \in \mathcal{B}$, its components ${}^{B}\mathbf{u}$ are constant.
- But, the components ${}^{A}\mathbf{u}$ are time-varying, due to the motion of $\mathcal{B}$ relative to $\mathcal{A}$.
- We found one way to compute $\left(\frac{d\mathbf{u}}{dt}\right)_A$ in (6.4).
- However, this involved resolving $\mathbf{u}$ in $\mathscr{F}_A$, i.e. finding ${}^{A}\mathbf{u}$. This is not always convenient.
- We now present an easier way to find $\left(\frac{d\mathbf{u}}{dt}\right)_A$ for $\mathbf{u} \in \mathcal{B}$.

Quick Review of Linear Operators

Matrix of a linear operator in a basis: (3.45).

Angular Velocity as a Linear Operator

- Let us define a operator:

$$\mathcal{D}_A : \mathbf{u} \in \mathcal{B} \longmapsto \left(\frac{d\mathbf{u}}{dt}\right)_A \tag{7.1}$$

- This is a linear operator which can be shown by resolving the vectors involved in $\mathscr{F}_A$ and using (6.4) to show that scaling and superposition properties are satisfied by this operator.
- Consider two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{B}$. Since they are both fixed in $\mathcal{B}$, their dot-product $\mathbf{u} \cdot \mathbf{v}$ is constant.
- Hence, using (6.6c),

$$\frac{d(\mathbf{u} \cdot \mathbf{v})}{dt} = \mathcal{D}_A(\mathbf{u}) \cdot \mathbf{v} + \mathcal{D}_A(\mathbf{v}) \cdot \mathbf{u} \qquad = 0 \tag{7.2}$$

$$\therefore \mathcal{D}_A(\mathbf{u}) \cdot \mathbf{v} = -\mathcal{D}_A(\mathbf{v}) \cdot \mathbf{u} \tag{7.3}$$

Angular Velocity as a Linear Operator

- Let us now find the matrix of the linear operator in $\mathscr{F}_B$, i.e. ${}^{B}[\mathcal{D}_A]$.
- From (3.45), we have,

$$
\begin{aligned}
{}^{B}[\mathcal{D}_A](i,j) &= \mathcal{D}_A(\hat{\mathbf{b}}_j)\cdot\hat{\mathbf{b}}_i && (7.4)\\
&\overset{(7.3)}{=} -\mathcal{D}_A(\hat{\mathbf{b}}_i)\cdot\hat{\mathbf{b}}_j && (7.5)\\
&= -\,{}^{B}[\mathcal{D}_A](j,i) && (7.6)
\end{aligned}
$$

- Hence, ${}^{B}[\mathcal{D}_A]$ is a skew-symmetric matrix.
- Therefore, using (3.51), ${}^{B}[\mathcal{D}_A]$ can also be considered a cross-product matrix $\left[{}^{B}\boldsymbol{\omega}\times\right]$ of some vector $\boldsymbol{\omega} = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3$.

- A vector $\mathbf{u} \in \mathcal{B}$ can be expressed as $\mathbf{u} = u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2 + u_3\hat{\mathbf{b}}_3$, where the components u_i are constant.
- The above results can be summarized as

$$
\begin{aligned}
{}^{B}\left[\left(\frac{d\mathbf{u}}{dt}\right)_A\right] &= {}^{B}[\mathcal{D}_A]\,{}^{B}\mathbf{u} && (7.7)\\
&= \left[{}^{B}\boldsymbol{\omega}\times\right]\,{}^{B}\mathbf{u} && (7.8)\\
\Rightarrow \left(\frac{d\mathbf{u}}{dt}\right)_A &= (\omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3) \times (u_1\hat{\mathbf{b}}_1 + u_2\hat{\mathbf{b}}_2 + u_3\hat{\mathbf{b}}_3) && (7.9)\\
&= \boldsymbol{\omega} \times \mathbf{u} && (7.10)
\end{aligned}
$$

Theorem 7.1 (Angular Velocity)

There exists a unique vector denoted $\boldsymbol{\omega}_{B/A}$, called angular velocity of $\mathcal{B}$ relative to $\mathcal{A}$ such that for all vectors $\mathbf{u} \in \mathcal{B}$,

$$\left(\frac{d\mathbf{u}}{dt}\right)_A = \boldsymbol{\omega}_{B/A} \times \mathbf{u} \tag{7.11}$$

It is independent of the choice of basis for the rigid-body $\mathcal{B}$.

Proof.

Uniqueness is easily proven by contradiction. If $\boldsymbol{\omega}_{B/A}$ is not unique, let $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ be two such vectors satisfying (7.11). This gives,

$$\left(\frac{d\mathbf{u}}{dt}\right)_A = \boldsymbol{\omega}_1 \times \mathbf{u} = \boldsymbol{\omega}_2 \times \mathbf{u} \qquad \Rightarrow (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \times \mathbf{u} = \mathbf{0}. \tag{7.12}$$

Since (7.12) is true for all $\mathbf{u} \in \mathcal{B}$, this necessarily implies $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$.
The whole derivation can be repeated also using some other basis fixed to $\mathcal{B}$. Thus, $\boldsymbol{\omega}_{B/A}$ is independent of the choice of basis of $\mathcal{B}$. □

In general, $\boldsymbol{\omega}_{B/A}$ is not constant in either $\mathscr{F}_B$ or $\mathscr{F}_A$.

In (7.11), if we take $\mathbf{u}$ as $\mathbf{b}_1$, we get,

$$\left(\frac{d\hat{\mathbf{b}}_1}{dt}\right)_A = \boldsymbol{\omega}_{B/A} \times \hat{\mathbf{b}}_1 \tag{7.13a}$$

$$\Rightarrow \left(\frac{d\hat{\mathbf{b}}_1}{dt}\right)_A \cdot \hat{\mathbf{b}}_2 = (\boldsymbol{\omega}_{B/A} \times \hat{\mathbf{b}}_1) \cdot \hat{\mathbf{b}}_2 \tag{7.13b}$$

$$= (\hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2) \cdot \boldsymbol{\omega}_{B/A} \tag{7.13c}$$

$$= \hat{\mathbf{b}}_3 \cdot \boldsymbol{\omega}_{B/A} \tag{7.13d}$$

$$= {}^{B}\omega_3. \tag{7.13e}$$

Proceeding similarly, we get the remaining components of $\boldsymbol{\omega}_{B/A}$ in $\mathscr{F}_B$. These are summarized below:

$$\left(\frac{d\hat{\mathbf{b}}_2}{dt}\right)_A \cdot \hat{\mathbf{b}}_3 = {}^B\omega_1 \tag{7.14a}$$

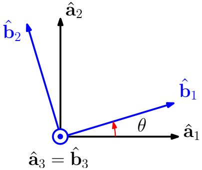
$$\left(\frac{d\hat{\mathbf{b}}_3}{dt}\right)_A \cdot \hat{\mathbf{b}}_1 = {}^B\omega_2 \tag{7.14b}$$

$$\left(\frac{d\hat{\mathbf{b}}_1}{dt}\right)_A \cdot \hat{\mathbf{b}}_2 = {}^B\omega_3 \tag{7.14c}$$

Unfortunately, these equations do not offer a practical means of finding $\boldsymbol{\omega}_{B/A}$ unless we resort to the methods of particle-kinematics and resolve $\hat{\mathbf{b}}_i$ in $\mathscr{B}_A$. However, for very simple cases, we can still use these equations.

Simple Relative Rotation

- For the simplest case, when the motion between $\mathscr{F}_A$ and $\mathscr{F}_B$ is a rotation about a common principal axis, (7.14) can still be used.
- Consider a rotation as shown.



$$(\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3) \xrightarrow{\mathcal{R}_{\theta,\hat{\mathbf{a}}_3}} (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3 = \hat{\mathbf{a}}_3)$$

$$\hat{\mathbf{b}}_1 = \cos\theta\,\hat{\mathbf{a}}_1 + \sin\theta\,\hat{\mathbf{a}}_2, \qquad \hat{\mathbf{b}}_2 = -\sin\theta\,\hat{\mathbf{a}}_1 + \cos\theta\,\hat{\mathbf{a}}_2 \tag{7.15a}$$

$$\left(\frac{d\hat{\mathbf{b}}_1}{dt}\right)_A = \dot{\theta}(-\sin\theta\,\hat{\mathbf{a}}_1 + \cos\theta\,\hat{\mathbf{a}}_2) \qquad = \dot{\theta}\,\hat{\mathbf{b}}_2 \tag{7.15b}$$

$$\left(\frac{d\hat{\mathbf{b}}_2}{dt}\right)_A = \dot{\theta}(-\cos\theta\,\hat{\mathbf{a}}_1 - \sin\theta\,\hat{\mathbf{a}}_2) \qquad = -\dot{\theta}\,\hat{\mathbf{b}}_1 \tag{7.15c}$$

Substituting (7.15) in (7.14),

$$
\begin{aligned}
{}^{B}\omega_1 &= -\dot{\theta}\,\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_3 && = 0 && (7.16a)\\
{}^{B}\omega_2 &= \mathbf{0} \cdot \hat{\mathbf{b}}_1 && = 0 && (7.16b)\\
{}^{B}\omega_3 &= \dot{\theta}\,\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{b}}_2 && = \dot{\theta} && (7.16c)
\end{aligned}
$$

This finally gives,

$$
\boldsymbol{\omega}_{B/A} = \dot{\theta}\,\hat{\mathbf{b}}_3 = \dot{\theta}\,\hat{\mathbf{a}}_3 \qquad (7.17)
$$

- We see a better approach for determining $\boldsymbol{\omega}_{B/A}$ next.
- Surprisingly, the simplest representation of $\boldsymbol{\omega}_{B/A}$ is neither resolved in $\mathscr{B}_B$ nor $\mathscr{B}_A$!

We now generalize Theorem 7.1 to the case where $\mathbf{u} \notin \mathcal{B}$. Hence, in general, $\mathbf{u}$ is varying in both $\mathcal{A}$ and $\mathcal{B}$ as they move relative to each other.

Generalization

For $\mathbf{u}$ moving relative to both $\mathcal{A}$ and $\mathcal{B}$, let:

$$\mathbf{u} = u_1\,\hat{\mathbf{b}}_1 + u_2\,\hat{\mathbf{b}}_2 + u_3\,\hat{\mathbf{b}}_3, \tag{7.18a}$$

where, the components u_i are not constant. Then,

$$\left(\frac{d\mathbf{u}}{dt}\right)_A = \dot{u}_1\,\hat{\mathbf{b}}_1 + \dot{u}_2\,\hat{\mathbf{b}}_2 + \dot{u}_3\,\hat{\mathbf{b}}_3$$

$$+ u_1\left(\frac{d\hat{\mathbf{b}}_1}{dt}\right)_A + u_2\left(\frac{d\hat{\mathbf{b}}_2}{dt}\right)_A + u_3\left(\frac{d\hat{\mathbf{b}}_3}{dt}\right)_A \tag{7.18b}$$

$$\overset{(6.4),(7.11)}{=} \left(\frac{d\mathbf{u}}{dt}\right)_B$$

$$+ u_1\,\boldsymbol{\omega}_{B/A}\times\hat{\mathbf{b}}_1 + u_2\,\boldsymbol{\omega}_{B/A}\times\hat{\mathbf{b}}_2 + u_3\,\boldsymbol{\omega}_{B/A}\times\hat{\mathbf{b}}_3 \tag{7.18c}$$

$$= \left(\frac{d\mathbf{u}}{dt}\right)_B + \boldsymbol{\omega}_{B/A}\times\left(u_1\,\hat{\mathbf{b}}_1 + u_2\,\hat{\mathbf{b}}_2 + u_3\,\hat{\mathbf{b}}_3\right) \tag{7.18d}$$

$$= \left(\frac{d\mathbf{u}}{dt}\right)_B + \boldsymbol{\omega}_{B/A}\times\mathbf{u} \tag{7.18e}$$

Theorem 7.2

The time-derivative of an arbitrary vector $\mathbf{u}$ *relative to referentials* $\mathscr{F}_A$ *and* $\mathscr{F}_B$ *are related by*

$$\left(\frac{d\mathbf{u}}{dt}\right)_A = \left(\frac{d\mathbf{u}}{dt}\right)_B + \boldsymbol{\omega}_{B/A} \times \mathbf{u} \tag{7.19}$$

Corollary 7.3

If we take $\mathscr{F}_A \equiv \mathscr{F}_B$ *in the above, we get,* $\forall\, \mathbf{u}$,

$$\begin{aligned} \left(\frac{d\mathbf{u}}{dt}\right)_A &= \left(\frac{d\mathbf{u}}{dt}\right)_A + \boldsymbol{\omega}_{A/A} \times \mathbf{u} \\ \mathbf{0} &= \boldsymbol{\omega}_{A/A} \times \mathbf{u} \\ \Rightarrow \boldsymbol{\omega}_{A/A} &= \mathbf{0}. \end{aligned} \tag{7.20}$$

Corollary 7.4

We can swap $\mathscr{F}_A$ and $\mathscr{F}_B$ in (7.19),

$$\left(\frac{d\mathbf{u}}{dt}\right)_A = \left(\frac{d\mathbf{u}}{dt}\right)_B + \boldsymbol{\omega}_{B/A} \times \mathbf{u}$$

$$\left(\frac{d\mathbf{u}}{dt}\right)_B = \left(\frac{d\mathbf{u}}{dt}\right)_A + \boldsymbol{\omega}_{A/B} \times \mathbf{u} \tag{7.21}$$

Adding the above two, we get, $\forall\, \mathbf{u}$,

$$(\boldsymbol{\omega}_{B/A} + \boldsymbol{\omega}_{A/B}) \times \mathbf{u} = \mathbf{0} \tag{7.22}$$

$$\Rightarrow \boldsymbol{\omega}_{B/A} = -\boldsymbol{\omega}_{A/B} \tag{7.23}$$

Corollary 7.5 (The Loop Equation)

Similarly, we can write (7.19) *for three referentials* $\mathscr{F}_A$, $\mathscr{F}_B$, $\mathscr{F}_C$.

$$\begin{aligned}\left(\frac{d\mathbf{u}}{dt}\right)_A &= \left(\frac{d\mathbf{u}}{dt}\right)_B + \boldsymbol{\omega}_{B/A} \times \mathbf{u}\\ \left(\frac{d\mathbf{u}}{dt}\right)_B &= \left(\frac{d\mathbf{u}}{dt}\right)_C + \boldsymbol{\omega}_{C/B} \times \mathbf{u}\\ \left(\frac{d\mathbf{u}}{dt}\right)_C &= \left(\frac{d\mathbf{u}}{dt}\right)_A + \boldsymbol{\omega}_{A/C} \times \mathbf{u}\end{aligned} \tag{7.24}$$

Adding the above, we get, $\forall\,\mathbf{u}$,

$$(\boldsymbol{\omega}_{B/A} + \boldsymbol{\omega}_{C/B} + \boldsymbol{\omega}_{A/C}) \times \mathbf{u} = \mathbf{0} \tag{7.25}$$

$$\Rightarrow \boldsymbol{\omega}_{B/A} + \boldsymbol{\omega}_{C/B} + \boldsymbol{\omega}_{A/C} = \mathbf{0}. \tag{7.26}$$

Angular Velocity Composition Rule

A Useful Result

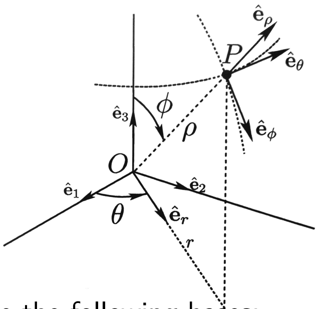
Combining (7.26) with (7.23), we get the useful result,

$$\boldsymbol{\omega}_{C/A} = \boldsymbol{\omega}_{C/B} + \boldsymbol{\omega}_{B/A} \tag{7.27}$$

This can be generalized by taking n intermediate frames

$$\boldsymbol{\omega}_{C/A} = \boldsymbol{\omega}_{C/B_n} + \boldsymbol{\omega}_{B_n/B_{n-1}} + \cdots + \boldsymbol{\omega}_{B_2/B_1} + \boldsymbol{\omega}_{B_1/A} \tag{7.28}$$

Example: Angular-Velocity of the Spherical Joint



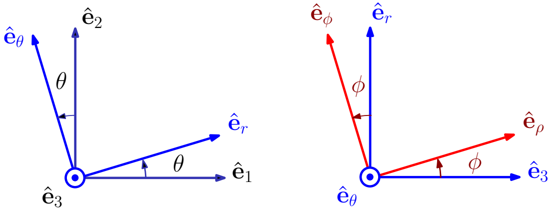
Recalling (6.30), we have the following bases:

- The original $\mathscr{B}_E(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$.
- The cylindrial $\mathscr{B}_C(\hat{\mathbf{c}}_1 = \hat{\mathbf{e}}_r,\ \hat{\mathbf{c}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{c}}_3 = \hat{\mathbf{e}}_3)$.
- The spherical basis $\mathscr{B}_S(\hat{\mathbf{s}}_1 = \hat{\mathbf{e}}_\phi,\ \hat{\mathbf{s}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{s}}_3 = \hat{\mathbf{e}}_\rho)$.

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \xrightarrow{\mathcal{R}_{\theta,\hat{\mathbf{e}}_3}} (\hat{\mathbf{c}}_1 = \hat{\mathbf{e}}_r,\ \hat{\mathbf{c}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{c}}_3 = \hat{\mathbf{e}}_3)$$

$$(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3) \xrightarrow{\mathcal{R}_{\phi,\hat{\mathbf{c}}_2}} (\hat{\mathbf{s}}_1 = \hat{\mathbf{e}}_\phi,\ \hat{\mathbf{s}}_2 = \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{s}}_3 = \hat{\mathbf{e}}_\rho)$$

Example: Angular-Velocity of the Spherical Joint



$$\boldsymbol{\omega}_{S/E} = \boldsymbol{\omega}_{S/C} + \boldsymbol{\omega}_{C/E} \tag{7.29a}$$

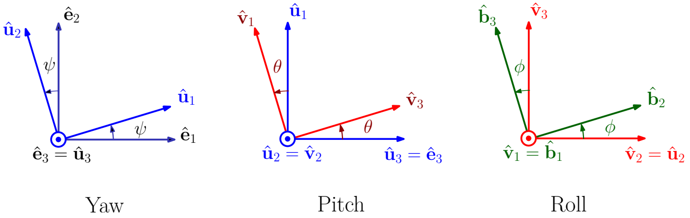
$$\boldsymbol{\omega}_{S/E} = \dot{\phi}\,\hat{\mathbf{e}}_\theta + \dot{\theta}\,\hat{\mathbf{e}}_3 \tag{7.29b}$$

From the figures above, we can also see at a glance that:

$$\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_\phi \tag{7.30a}$$

$$\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_\rho = \sin\phi\,\hat{\mathbf{e}}_\theta \tag{7.30b}$$

Angular Velocity $\omega_{B/E}$: Roll/Pitch/Yaw Frames



Recall (4.73):

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \xrightarrow{\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}} (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3 = \hat{\mathbf{e}}_3), \qquad \text{Yaw}$$

$$(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3) \xrightarrow{\mathcal{R}_{\theta,\hat{\mathbf{u}}_2}} (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2 = \hat{\mathbf{u}}_2, \hat{\mathbf{v}}_3), \qquad \text{Pitch}$$

$$(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3) \xrightarrow{\mathcal{R}_{\phi,\hat{\mathbf{v}}_1}} (\hat{\mathbf{b}}_1 = \hat{\mathbf{v}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3), \qquad \text{Roll}$$

$$\boldsymbol{\omega}_{B/E} = \boldsymbol{\omega}_{B/V} + \boldsymbol{\omega}_{V/U} + \boldsymbol{\omega}_{U/E} \tag{7.31a}$$

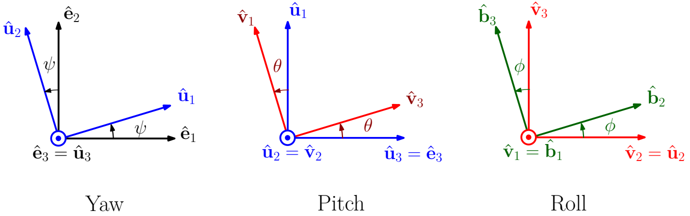
$$\boldsymbol{\omega}_{B/E} = \dot{\phi}\,\hat{\mathbf{v}}_1 + \dot{\theta}\,\hat{\mathbf{u}}_2 + \dot{\psi}\,\hat{\mathbf{e}}_3 \tag{7.31b}$$

Angular Velocity $\boldsymbol{\omega}_{B/E}$: Roll/Pitch/Yaw Frames

Let us define

$$^{B}\boldsymbol{\omega}_{B/E} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{7.32}$$

How can we find these components using (7.31)?



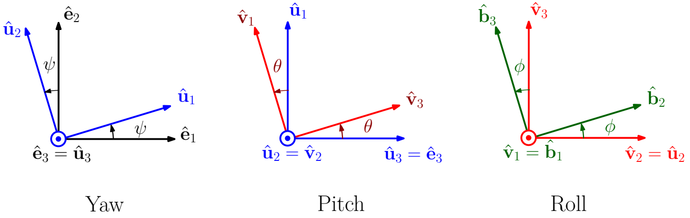
$$p = \boldsymbol{\omega}_{B/E} \cdot \hat{\mathbf{b}}_1 = (\dot{\phi}\,\hat{\mathbf{v}}_1 + \dot{\theta}\,\hat{\mathbf{u}}_2 + \dot{\psi}\,\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{b}}_1 \tag{7.33a}$$

We need the following dot products:

$$\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{b}}_1 = 1, \quad \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{b}}_1 = 0, \quad \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{b}}_1 = \cos(\theta + \frac{\pi}{2}) = -\sin\theta \tag{7.33b}$$

Substituting above, we get:

$$p = \dot{\phi} - \dot{\psi}\,\sin\theta \tag{7.33c}$$



$$q = \boldsymbol{\omega}_{B/E} \cdot \hat{\mathbf{b}}_2 \qquad = (\dot{\phi}\,\hat{\mathbf{v}}_1 + \dot{\theta}\,\hat{\mathbf{u}}_2 + \dot{\psi}\,\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{b}}_2 \tag{7.34a}$$

We need the following dot products:

$$\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{b}}_2 = 0, \tag{7.34b}$$

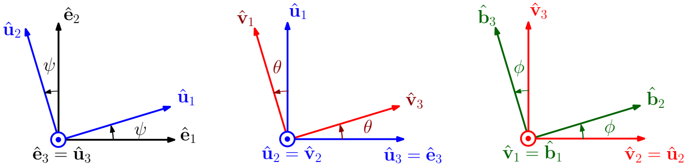
$$\hat{\mathbf{u}}_2 \cdot \hat{\mathbf{b}}_2 = \cos\phi \tag{7.34c}$$

$$\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{b}}_2 = (\cos\theta\,\hat{\mathbf{v}}_3 - \sin\theta\,\hat{\mathbf{v}}_1) \cdot (\cos\phi\,\hat{\mathbf{v}}_2 + \sin\phi\,\hat{\mathbf{v}}_3) \tag{7.34d}$$

$$= \sin\phi\,\cos\theta \tag{7.34e}$$

Substituting above, we get:

$$q = \dot{\theta}\,\cos\phi + \dot{\psi}\,\sin\phi\,\cos\theta \tag{7.34f}$$



Yaw Pitch Roll

$$r = \boldsymbol{\omega}_{B/E} \cdot \hat{\mathbf{b}}_3 = (\dot{\phi}\, \hat{\mathbf{v}}_1 + \dot{\theta}\, \hat{\mathbf{u}}_2 + \dot{\psi}\, \hat{\mathbf{e}}_3) \cdot \hat{\mathbf{b}}_3 \tag{7.35a}$$

After finding all required dot-products (homework), we get:

$$r = -\dot{\theta} \sin\phi + \dot{\psi} \cos\phi \cos\theta \tag{7.35b}$$

The Spinning Tube Trick

https://www.youtube.com/watch?v=wQTVcaA3PQw

Launch external viewer.

- The angular-velocity was derived in the class. The trick is explained after a few slides.

Velocity Field of a Rigid Body

Given: rigid bodies $\mathcal{A}$ and $\mathcal{B}$ are in relative motion

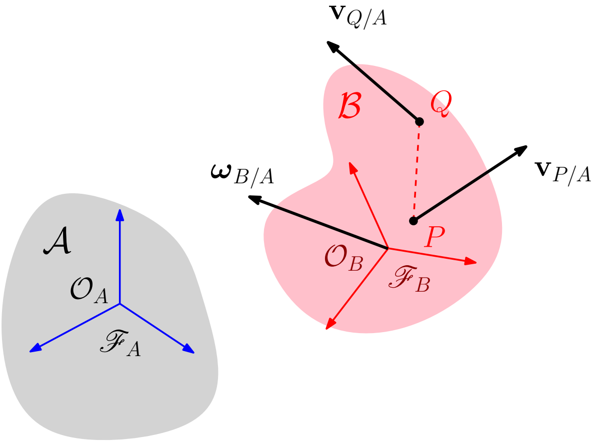
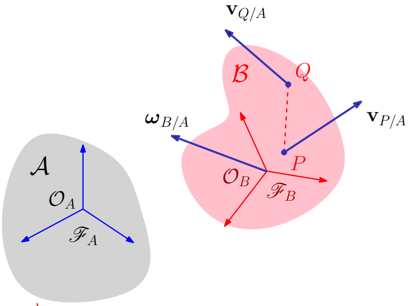


Figure 7.1: Given two arbitrary points $P \in \mathcal{B}$ and $Q \in \mathcal{B}$, what is the relationship between $\mathbf{v}_{P/A}$ and $\mathbf{v}_{Q/A}$?

Velocity Field of a Rigid Body



As the vector $\mathbf{r}_{PQ} \equiv \overrightarrow{PQ} \in \mathcal{B}$, we use Theorem 7.1 to get,

$$\left(\frac{d\mathbf{r}_{PQ}}{dt}\right)_A = \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{PQ} \tag{7.36}$$

$$\text{But, } \mathbf{r}_{PQ} = \mathbf{r}_{\mathcal{O}_A Q} - \mathbf{r}_{\mathcal{O}_A P} \tag{7.37}$$

$$\therefore \left(\frac{d\mathbf{r}_{\mathcal{O}_A Q}}{dt}\right)_A - \left(\frac{d\mathbf{r}_{\mathcal{O}_A P}}{dt}\right)_A = \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{PQ} \tag{7.38}$$

$$\Rightarrow \mathbf{v}_{Q/A} - \mathbf{v}_{P/A} = \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{PQ} \tag{7.39}$$

Theorem 7.6

The velocity field $P \in \mathcal{B} \longmapsto \mathbf{v}_{P/A}$ of any two points P and Q attached to $\mathcal{B}$ in motion relative to $\mathcal{A}$ satisfies the fundamental formula

$$\mathbf{v}_{Q\in B/A} = \mathbf{v}_{P\in B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{PQ} \tag{7.40}$$

Hence, at any instant, the velocity field is entirely specified from the knowledge of:

1. *The angular-velocity $\boldsymbol{\omega}_{B/A}$ and*
2. *The velocity of a particular point, say $P = P_*$, attached to $\mathcal{B}$.*

Screws

$$\mathbf{v}_{Q\in B/A} = \mathbf{v}_{P\in B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{PQ}$$

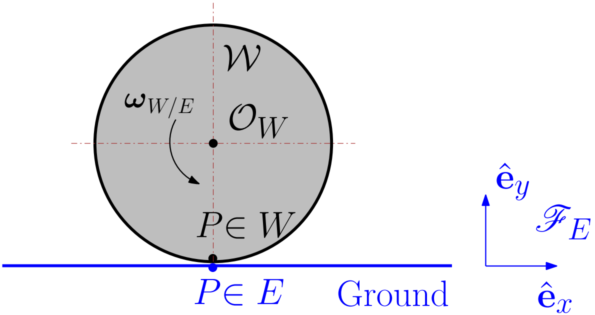
- This simple result is surprisingly useful!
- Vector-fields which satisfy (7.40) are called screws.
- The set

$$\{\mathcal{V}_{B/A}\} \triangleq \begin{Bmatrix} \boldsymbol{\omega}_{B/A} \\ \mathbf{v}_{P\in B/A} \end{Bmatrix} \tag{7.41}$$

is called the Kinematic Screw or twist.

No-Slip (Nonholonomic) Constraint

Simple 2D Example: A Wheel

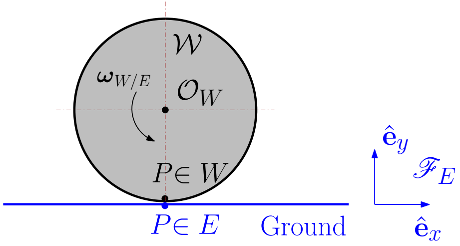


$$\text{No Slip} \Rightarrow \mathbf{v}_{P \in W/E} = \mathbf{v}_{P \in E/E} = \mathbf{0}$$

Figure 7.2: The radius of the wheel is R. Clearly, $\mathbf{v}_{P \in E/E} = \mathbf{0}$.

No-Slip (Nonholonomic) Constraint

Simple 2D Example: A Wheel



$$\text{No Slip} \;\Rightarrow \mathbf{v}_{P\in W/E} = \mathbf{v}_{P\in E/E} = \mathbf{0}$$

Using (7.40),

$$\mathbf{v}_{\mathcal{O}_W/E} = \mathbf{v}_{P\in W/E} + \boldsymbol{\omega}_{W/E} \times \mathbf{r}_{P\mathcal{O}_W} \tag{7.42}$$

$$= \mathbf{0} + \omega\,\hat{\mathbf{e}}_z \times R\,\hat{\mathbf{e}}_y \tag{7.43}$$

$$= -\omega\,R\,\hat{\mathbf{e}}_x \tag{7.44}$$

The Spinning Tube Trick

https://www.youtube.com/watch?v=7rAiZR_zasg

Launch external viewer.

- On board: Use the kinematic screw to derive the velocities of the points O and X and explain the trick.

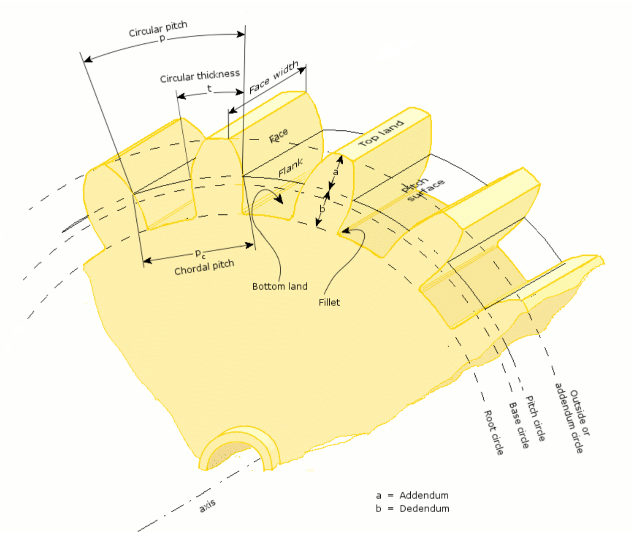


Figure 7.3: Gear terminology. For our kinematic analysis, we replace a gear by a disc of gear's pitch circle diameter.

Spur Gears

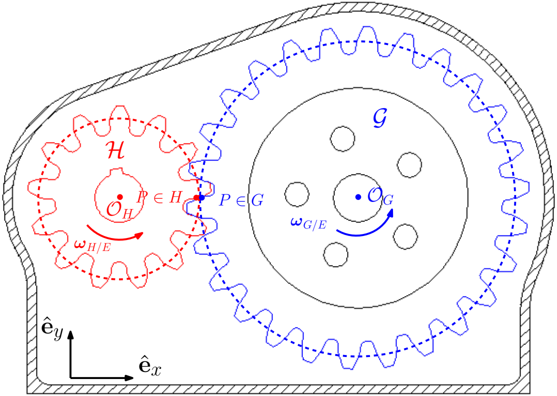
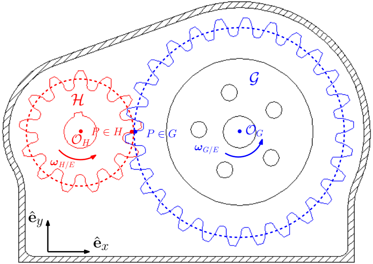


Figure 7.4: Spur gears. Let R_H and R_G be the pitch circle radii, and let N_H and N_G be the number of teeth on the respective frames.



$$\mathbf{v}_{P\in H/E} = \mathbf{v}_{P\in G/E} \qquad \text{No slip} \tag{7.45}$$

$$\underbrace{\mathbf{v}_{\mathcal{O}_H/E}}_{\mathbf{0}} + \boldsymbol{\omega}_{H/E} \times \mathbf{r}_{\mathcal{O}_H P} = \underbrace{\mathbf{v}_{\mathcal{O}_G/E}}_{\mathbf{0}} + \boldsymbol{\omega}_{G/E} \times \mathbf{r}_{\mathcal{O}_G P} \tag{7.46}$$

$$\omega_H \hat{\mathbf{e}}_z \times R_H \hat{\mathbf{e}}_x = \omega_G \hat{\mathbf{e}}_z \times (-R_G \hat{\mathbf{e}}_x) \tag{7.47}$$

$$\omega_H R_H \, \hat{\mathbf{e}}_y = -\omega_G R_G \, \hat{\mathbf{e}}_y \tag{7.48}$$

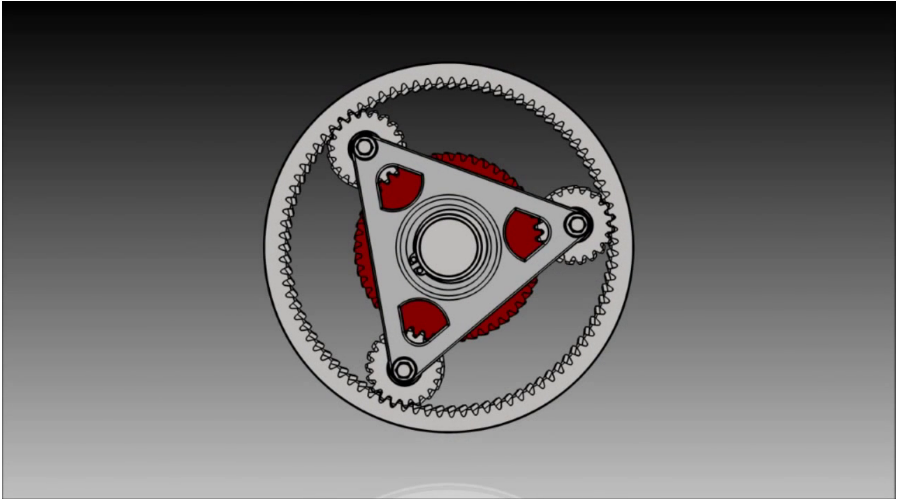
$$\frac{\omega_G}{\omega_H} = -\frac{R_H}{R_G} = -\frac{N_H}{N_G} \tag{7.49}$$

Con: The speed-reduction is not high, unless there is a big size difference.

Figure 7.5: Spiral Bevel gears. The output axle direction is orthogonal to the input axle direction.

Planetary/Epicyclic Gears

https://www.youtube.com/watch?v=JBB1sC7LCuQ



Launch external viewer.

Analysis of Planetary/Epicyclic Gears

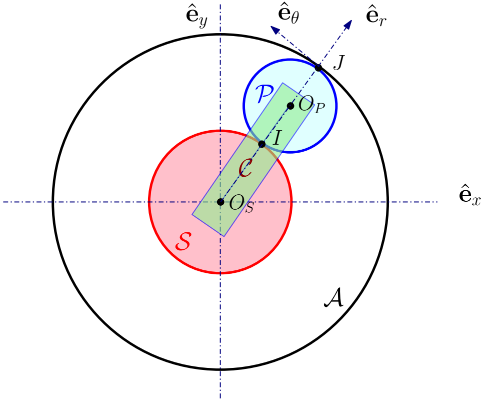
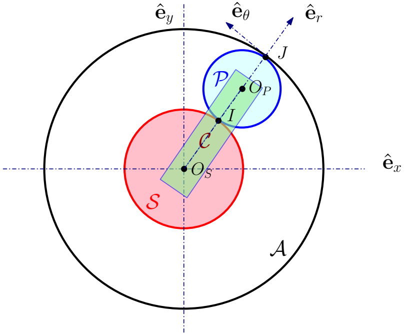
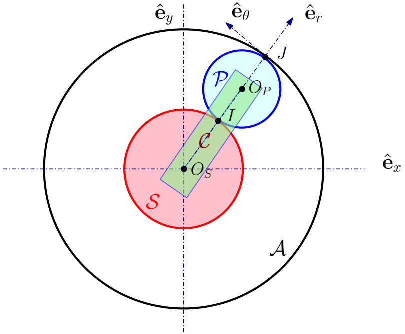


Figure 7.6: Analysis of Planetary Gear. $\mathcal{A}$ is the annular gear, $\mathcal{P}$ is a planet gear, $\mathcal{S}$ is the sun gear, and $\mathcal{C}$ is the carrier. $\mathcal{O}_S = \mathcal{O}_A$.

$$R_A = 2R_P + R_S, \qquad N_A = 2N_P + N_S \tag{7.50}$$



$$
\begin{aligned}
\mathbf{v}_{I\in P/E} &= \mathbf{v}_{I\in S/E} && \text{no slip} \qquad (7.51)\\
&= \mathbf{v}_{O_S/E} + \boldsymbol{\omega}_{S/E} \times \mathbf{r}_{O_S I} && (7.52)\\
&= \mathbf{v}_{O_S/E} + \omega_{S/E}\,\hat{\mathbf{e}}_z \times R_S\,\hat{\mathbf{e}}_r && (7.53)\\
&= \omega_{S/E}\,R_S\,\hat{\mathbf{e}}_\theta && (7.54)\\
\mathbf{v}_{J\in P/E} &= \mathbf{v}_{I\in P/E} + \boldsymbol{\omega}_{P/E} \times \mathbf{r}_{IJ} && (7.55)\\
&= \mathbf{v}_{I\in P/E} + \omega_{P/E}\,\hat{\mathbf{e}}_z \times (2\,R_P\,\hat{\mathbf{e}}_r) && (7.56)\\
&= (\omega_{S/E}\,R_S + 2\,R_P\,\omega_{P/E})\,\hat{\mathbf{e}}_\theta && (7.57)
\end{aligned}
$$



$$\mathbf{v}_{J\in A/E} = \mathbf{v}_{O_A/E} + \boldsymbol{\omega}_{A/E} \times \mathbf{r}_{O_A J} \tag{7.58}$$

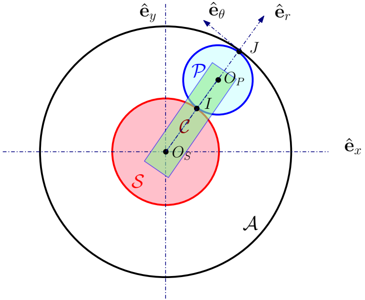
$$= \omega_{A/E}\, \hat{\mathbf{e}}_z \times R_A\, \hat{\mathbf{e}}_r \tag{7.59}$$

$$= \omega_{A/E}\, R_A\, \hat{\mathbf{e}}_\theta \tag{7.60}$$

But, due to no-slip, $\mathbf{v}_{J\in A/E} = \mathbf{v}_{J\in P/E}$. Hence, from the above and (7.57)

$$\omega_{S/E}\, R_S + 2\, R_P\, \omega_{P/E} = \omega_{A/E}\, R_A \tag{7.61}$$

But, we're interested in $\omega_{C/E}$ not $\omega_{P/E}$, so we seek to eliminate the latter.



$$\mathbf{v}_{O_P \in P/E} = \mathbf{v}_{I \in P/E} + \boldsymbol{\omega}_{P/E} \times \mathbf{r}_{IO_P} \tag{7.62}$$

$$= \mathbf{v}_{I \in P/E} + \omega_{P/E}\, \hat{\mathbf{e}}_z \times R_P\, \hat{\mathbf{e}}_r \tag{7.63}$$

$$\overset{(7.54)}{=} (\omega_{S/E}\, R_S + \omega_{P/E}\, R_P)\, \hat{\mathbf{e}}_\theta \tag{7.64}$$

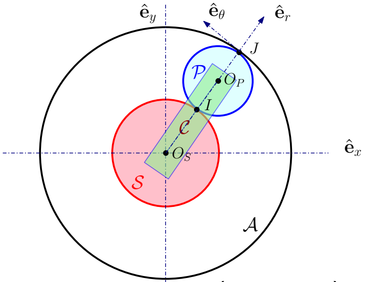
$$\mathbf{v}_{O_P \in C/E} = \mathbf{v}_{O_S \in C/E} + \boldsymbol{\omega}_{C/E} \times \mathbf{r}_{O_S O_P} \tag{7.65}$$

$$= \omega_{C/E}\, \hat{\mathbf{e}}_z \times (R_S + R_P)\, \hat{\mathbf{e}}_r \tag{7.66}$$

$$= \omega_{C/E}\, (R_S + R_P)\, \hat{\mathbf{e}}_\theta \tag{7.67}$$

$$\mathbf{v}_{O_P \in P/E} = \mathbf{v}_{O_P \in C/E}, \text{ hinged at } O_P \tag{7.68}$$

$$\Rightarrow \omega_{S/E}\, R_S + \omega_{P/E}\, R_P = \omega_{C/E}\, (R_S + R_P) \tag{7.69}$$



From (7.69), we get $\omega_{P/E}\, R_P = \omega_{C/E}\,(R_S + R_P) - \omega_{S/E}\, R_S$, which we substitute in (7.61) to eliminate $\omega_{P/E}$ and get

$$2\,\omega_{C/E}\,(R_S + R_P) - \omega_{S/E}\, R_S = \omega_{A/E}\, R_A \tag{7.70}$$

$$\stackrel{(7.50)}{=} \omega_{A/E}\,(R_S + 2\,R_P) \tag{7.71}$$

If we define $\eta \triangleq \frac{R_S}{R_P} \equiv \frac{N_S}{N_P}$, the above equation gives, finally,

$$2\,\omega_{C/E}\,(1+\eta) - \omega_{S/E}\,\eta = \omega_{A/E}\,(\eta + 2) \tag{7.72}$$

Case I: The Annular Gear is Kept Fixed

In (7.72), we set $\omega_{A/E} = 0$, to get:

$$\frac{\omega_{C/E}}{\omega_{S/E}} = \frac{\eta}{2(\eta + 1)} < 1 \tag{7.73}$$

This configuration gives a high speed reduction.

Case II: The Sun Gear is Kept Fixed

In (7.72), we set $\omega_{S/E} = 0$, to get:

$$\frac{\omega_{C/E}}{\omega_{A/E}} = \frac{\eta + 2}{2(\eta + 1)} \tag{7.74}$$

Case III: The Carrier is Kept Fixed

In (7.72), we set $\omega_{C/E} = 0$, to get:

$$\frac{\omega_{A/E}}{\omega_{S/E}} = \frac{-\eta}{\eta + 2} \tag{7.75}$$

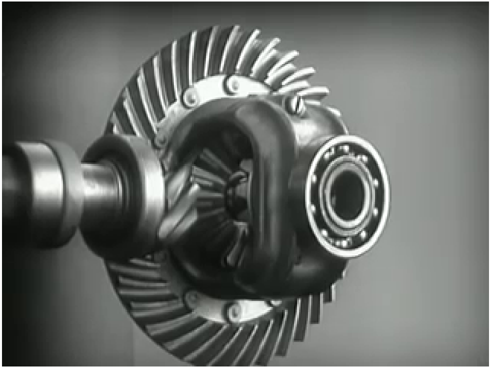
Note the minus sign!

- Planetary gear-set give a higher speed-reduction in a more compact size compared to pure spur-gears.
- Planetary gear-set is the key element which makes automatic transmission work (animation; the real thing). All speed-ratios of the various cases are obtained by using band-brakes to hold and release the carrier, the sun, or the annular gear as needed.

The automatic transmission of a Ford Escort 1992. Automatic transmission is more popular in north America. In other countries, the manual gear-shifting is more common, although automatic transmission is also available.

The Differential

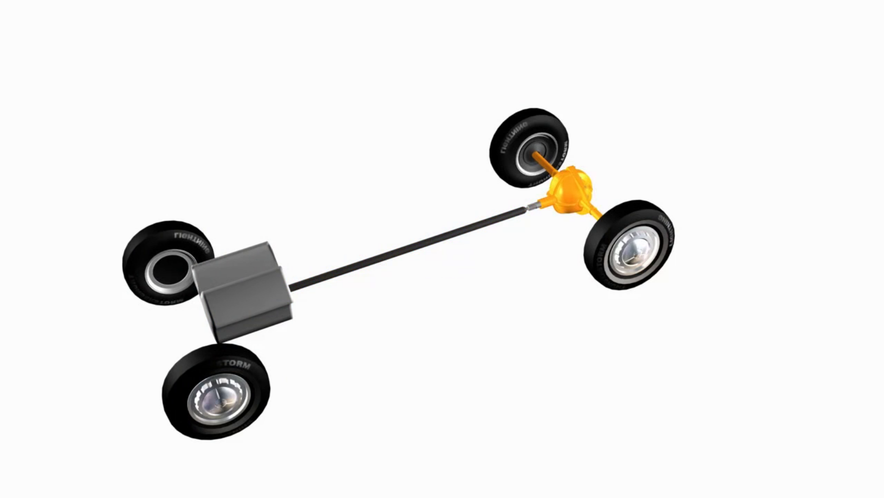
https://www.youtube.com/watch?v=yYAw79386WI



Launch external viewer.

The Differential

https://www.youtube.com/watch?v=SOgoejxzF8c



Launch external viewer.

Kinematic Analysis of the Differential

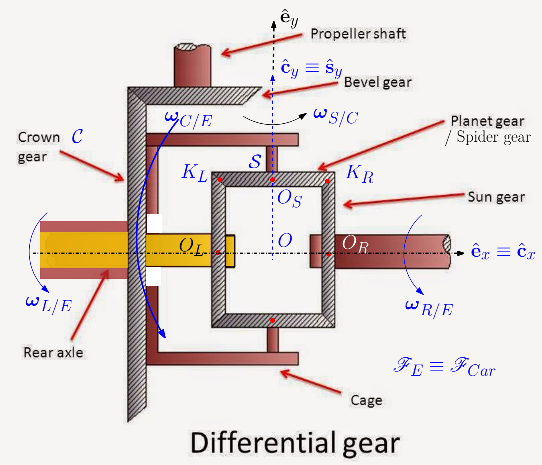
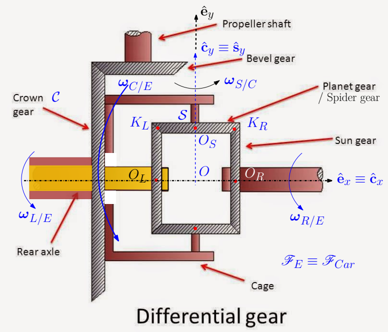
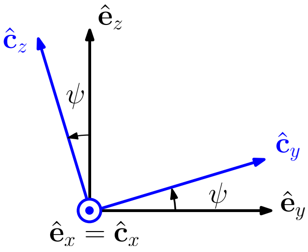
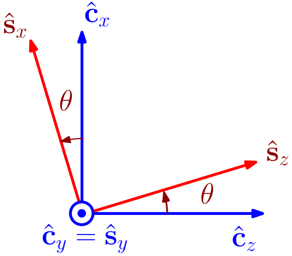


Figure 7.7: Note that the axles of the carrier (crown) and the left-wheel are coaxial but not connected. The radius of the spider-gear is R_S. The radii of the left and right sun gears is the same $= R$. Also, $OO_S = R$. Base image link.

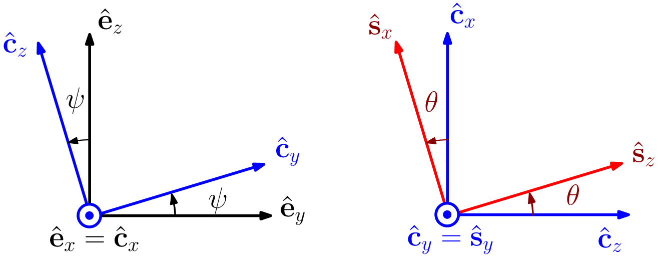




Crown-Rotation



Spider-Spin

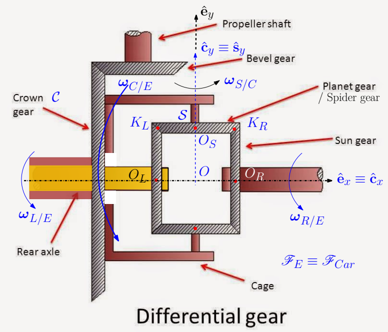


Crown-Rotation Spider-Spin

$$\boldsymbol{\omega}_{S/E} = \boldsymbol{\omega}_{S/C} + \boldsymbol{\omega}_{C/E} \tag{7.76}$$

$$\boldsymbol{\omega}_{S/E} = \dot{\theta}\, \hat{\mathbf{c}}_y + \dot{\psi}\, \hat{\mathbf{e}}_x \tag{7.77}$$

$$\boldsymbol{\omega}_{S/E} \triangleq \omega_{S/C}\, \hat{\mathbf{c}}_y + \omega_{C/E}\, \hat{\mathbf{e}}_x \tag{7.78}$$



$$\mathbf{v}_{O_S \in C/E} = \mathbf{v}_{O \in C/E} + \boldsymbol{\omega}_{C/E} \times \mathbf{r}_{OO_S} \tag{7.79}$$

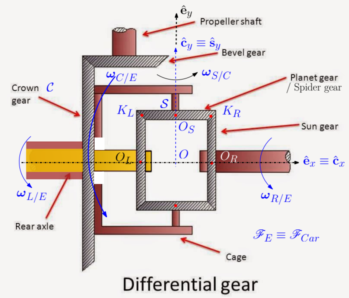
$$= \omega_{C/E}\, \hat{\mathbf{e}}_x \times (R\, \hat{\mathbf{c}}_y) \tag{7.80}$$

$$= \omega_{C/E}\, R\, \hat{\mathbf{c}}_z = \mathbf{v}_{O_S \in S/E} \tag{7.81}$$

$$\mathbf{v}_{K_R \in S/E} = \mathbf{v}_{O_S \in S/E} + \boldsymbol{\omega}_{S/E} \times \mathbf{r}_{O_S K_R} \tag{7.82}$$

$$= \omega_{C/E}\, R\, \hat{\mathbf{c}}_z + (\omega_{S/C} \hat{\mathbf{c}}_y + \omega_{C/E} \hat{\mathbf{e}}_x) \times R_S\, \hat{\mathbf{e}}_x \tag{7.83}$$

$$= (\omega_{C/E}\, R - \omega_{S/C}\, R_S)\, \hat{\mathbf{c}}_z \tag{7.84}$$



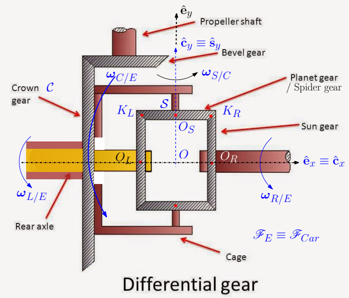
$$
\begin{aligned}
\mathbf{v}_{K_R \in S/E} &= \mathbf{v}_{O_S \in S/E} + \boldsymbol{\omega}_{S/E} \times \mathbf{r}_{O_S K_R} \\
&= \omega_{C/E}\, R\, \hat{\mathbf{c}}_z + (\omega_{S/C} \hat{\mathbf{c}}_y + \omega_{C/E} \hat{\mathbf{e}}_x) \times R_S\, \hat{\mathbf{e}}_x \\
&= (\omega_{C/E}\, R - \omega_{S/C}\, R_S)\, \hat{\mathbf{c}}_z
\end{aligned}
$$

$$
\mathbf{v}_{K_R \in R/E} = \mathbf{v}_{O_R \in R/E} + \boldsymbol{\omega}_{R/E} \times \mathbf{r}_{O_R K_R} \tag{7.85}
$$

$$
= \omega_{R/E}\, \hat{\mathbf{e}}_x \times (R\, \hat{\mathbf{c}}_y) = \omega_{R/E}\, R\, \hat{\mathbf{c}}_z \tag{7.86}
$$

Due to no-slip, the above two have to be equal, so,

$$
\omega_{C/E}\, R - \omega_{S/C}\, R_S = \omega_{R/E}\, R \tag{7.87}
$$



$$\mathbf{v}_{K_L \in S/E} = \mathbf{v}_{O_S \in S/E} + \boldsymbol{\omega}_{S/E} \times \mathbf{r}_{O_S K_L} \tag{7.88}$$

$$= \omega_{C/E}\, R\, \hat{\mathbf{c}}_z + (\omega_{S/C}\hat{\mathbf{c}}_y + \omega_{C/E}\hat{\mathbf{e}}_x) \times (-R_S\, \hat{\mathbf{e}}_x) \tag{7.89}$$

$$= (\omega_{C/E}\, R + \omega_{S/C}\, R_S)\, \hat{\mathbf{c}}_z \tag{7.90}$$

$$\mathbf{v}_{K_L \in L/E} = \mathbf{v}_{O_L \in L/E} + \boldsymbol{\omega}_{L/E} \times \mathbf{r}_{O_L K_L} \tag{7.91}$$

$$= \omega_{L/E}\, \hat{\mathbf{e}}_x \times (R\, \hat{\mathbf{c}}_y) = \omega_{L/E}\, R\, \hat{\mathbf{c}}_z \tag{7.92}$$

Due to no-slip, the above two have to be equal, so,

$$\omega_{C/E}\, R + \omega_{S/C}\, R_S = \omega_{L/E}\, R \tag{7.93}$$

We repeat (7.87) and (7.93) for convenience:

$$\omega_{C/E}\, R - \omega_{S/C}\, R_S = \omega_{R/E}\, R$$
$$\omega_{C/E}\, R + \omega_{S/C}\, R_S = \omega_{L/E}\, R$$

Adding them gets rid of $\omega_{S/C}$ and we get:

$$\omega_{C/E} = \frac{\omega_{L/E} + \omega_{R/E}}{2} \tag{7.94}$$

Thus, the carrier (crown) angular-speed is the average of the angular-speeds of the axles of the two wheels.

We repeat (7.87) and (7.93) for convenience:

$$\omega_{C/E}\,R - \omega_{S/C}\,R_S = \omega_{R/E}\,R$$
$$\omega_{C/E}\,R + \omega_{S/C}\,R_S = \omega_{L/E}\,R$$

Subtracting them gets rid of $\omega_{C/E}$ and we get:

$$\omega_{S/C} = (\omega_{L/E} - \omega_{R/E})\frac{R}{2R_S} \tag{7.95}$$

Thus, the spider spin angular-speed is proportional to the difference of the angular-speeds of the axles of the two wheels.

A General Screw

A screw, denoted $\{\mathcal{S}\}$ is a vector-field $P \in \mathcal{E} \longmapsto \mathbf{s}_P$ satisfying the equation

$$\mathbf{s}_Q = \mathbf{s}_P + \mathbf{u} \times \mathbf{r}_{PQ} \tag{7.96}$$

- The vector $\mathbf{u}$ is called the resultant of the screw $\{\mathcal{S}\}$. It is independent of position.
- The vector $\mathbf{s}_P$ is called moment of screw $\{\mathcal{S}\}$ about point P.
- We denote this screw as

$$\{\mathcal{S}\} = \begin{Bmatrix} \mathbf{u} \\ \mathbf{s}_P \end{Bmatrix} \tag{7.97}$$

- We say that the above screw is resolved at point P.

Example 1: The Kinematic Screw

We have already saw in (7.40) that the velocity field $P \in \mathcal{B} \longmapsto \mathbf{v}_{P\in B/A}$ of a rigid body $\mathcal{B}$ in motion relative to another rigid body $\mathcal{A}$ satisfies

$$\mathbf{v}_{Q\in B/A} = \mathbf{v}_{P\in B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{PQ}$$

- Hence it forms the kinematic screw of $\mathcal{B}$ relative to $\mathcal{A}$

$$\{\mathcal{V}_{B/A}\} \triangleq \begin{Bmatrix} \boldsymbol{\omega}_{B/A} \\ \mathbf{v}_{P\in B/A} \end{Bmatrix}$$

- The resultant of the screw $\{\mathcal{V}_{B/A}\}$ is the angular velocity $\boldsymbol{\omega}_{B/A}$.
- The moment of the screw $\{\mathcal{V}_{B/A}\}$ about point P is the velocity $\mathbf{v}_{P\in B/A}$.

Invariant of a General Screw

- If we dot both sides of (7.96), i.e. $\mathbf{s}_Q = \mathbf{s}_P + \mathbf{u} \times \mathbf{r}_{PQ}$ with the screw resultant $\mathbf{u}$, we get,

$$\mathbf{u} \cdot \mathbf{s}_Q = \mathbf{u} \cdot \mathbf{s}_P + \mathbf{u} \cdot (\mathbf{u} \times \mathbf{r}_{PQ}) \tag{7.98}$$

$$\Rightarrow \mathbf{u} \cdot \mathbf{s}_Q = \mathbf{u} \cdot \mathbf{s}_P \tag{7.99}$$

- Thus, the component of the moment of the screw about any point in the direction of the screw's resultant is constant, i.e. independent of the point about which the screw is resolved.

Example: Invariant of the Kinematic Screw

$$\mathbf{v}_{Q\in B/A}\cdot\boldsymbol{\omega}_{B/A}=\mathbf{v}_{P\in B/A}\cdot\boldsymbol{\omega}_{B/A} \tag{7.100}$$

Equiprojectivity of a General Screw

If we dot both sides of (7.96), i.e. $\mathbf{s}_Q = \mathbf{s}_P + \mathbf{u} \times \mathbf{r}_{PQ}$ with $\mathbf{r}_{PQ}$, we get,

$$\mathbf{r}_{PQ} \cdot \mathbf{s}_Q = \mathbf{r}_{PQ} \cdot \mathbf{s}_P + \mathbf{r}_{PQ} \cdot (\mathbf{u} \times \mathbf{r}_{PQ}) \tag{7.101}$$

$$\Rightarrow \mathbf{r}_{PQ} \cdot \mathbf{s}_Q = \mathbf{r}_{PQ} \cdot \mathbf{s}_P \tag{7.102}$$

Example: Equiprojectivity of the Kinematic Screw

$$\mathbf{v}_{Q\in B/A} \cdot \mathbf{r}_{PQ} = \mathbf{v}_{P\in B/A} \cdot \mathbf{r}_{PQ} \tag{7.103}$$

What is the geometric significance of this result?

The Axis of a General Screw

Definition 7.7

The axis Δ of a screw $\{\mathcal{S}\}$ with a non-zero resultant $\mathbf{u}$ is defined as the set of points Q about which the moment is collinear to $\mathbf{u}$.
This means that for $Q \in \Delta$, $\exists p \in \mathbb{R}$, such that

$$\mathbf{s}_Q = p\,\mathbf{u} = \mathbf{s}_P + \mathbf{u} \times \mathbf{r}_{PQ} \tag{7.104}$$

- To find its expression, we make use of the last problem (9.1) of HW-1. A necessary condition for the solution of (7.104) to exist is

$$(p\,\mathbf{u} - \mathbf{s}_P) \cdot \mathbf{u} = 0. \tag{7.105}$$

$$\Rightarrow p = \frac{\mathbf{u} \cdot \mathbf{s}_P}{\|\mathbf{u}\|^2} \tag{7.106}$$

- We see that the numerator is the screw-invariant (7.99). So, p is independent of the point P! It is a quantity intrinsic to the screw $\{\mathcal{S}\}$, called the pitch of the screw.
- So, the necessary condition is always satisfied by the same p.

Taking the cross-product with $\mathbf{u}$ on both sides of $p\,\mathbf{u} = \mathbf{s}_P + \mathbf{u} \times \mathbf{r}_{PQ}$,

$$-\mathbf{u} \times \mathbf{s}_P = \mathbf{u} \times (\mathbf{u} \times \mathbf{r}_{PQ}) \tag{7.107a}$$

$$= (\mathbf{u} \cdot \mathbf{r}_{PQ})\,\mathbf{u} - \|\mathbf{u}\|^2\,\mathbf{r}_{PQ} \tag{7.107b}$$

Let us choose a particular convenient $Q = Q_*$ such that, $\mathbf{u} \cdot \mathbf{r}_{PQ_*} = 0$. Eq. (7.107b) gives,

$$\mathbf{r}_{PQ_*} = \frac{\mathbf{u} \times \mathbf{s}_P}{\|\mathbf{u}\|^2}. \tag{7.107c}$$

The general solution of $p\,\mathbf{u} = \mathbf{s}_P + \mathbf{u} \times \mathbf{r}_{PQ}$ is hence

$$\mathbf{r}_{PQ} = \mathbf{r}_{PQ_*} + \lambda\,\mathbf{u} \tag{7.107d}$$

$$= \frac{\mathbf{u} \times \mathbf{s}_P}{\|\mathbf{u}\|^2} + \lambda\,\mathbf{u}. \tag{7.107e}$$

This is a 3D line passing through $\mathbf{r}_{PQ_*}$ in the direction of $\mathbf{u}$.

Theorem 7.8 (Axis of a Screw)

The axis Δ of a screw $\{\mathcal{S}\}$ with a non-zero resultant $\mathbf{u}$ is the 3D line defined by the equation

$$\mathbf{r}_{PQ} = \frac{\mathbf{u} \times \mathbf{s}_P}{\|\mathbf{u}\|^2} + \lambda\,\mathbf{u}. \tag{7.108}$$

The points Q of Δ satisy $\mathbf{s}_Q = p\,\mathbf{u}$, with,

$$p = \frac{\mathbf{u} \cdot \mathbf{s}_P}{\|\mathbf{u}\|^2}, \qquad \text{the pitch of } \{\mathcal{S}\}. \tag{7.109}$$

Example: The Axis of the Kinematic Screw

The axis of the Kinematic Screw is also called the Instantaneous Screw Axis.

Corollary 7.9

At any instant, whenever, $\boldsymbol{\omega}_{B/A} \neq \mathbf{0}$, the kinematic screw $\{\mathcal{V}_{B/A}\}$ has an axis $\Delta_{B/A}$, defined as the set of points whose velocity is collinear to $\boldsymbol{\omega}_{B/A}$,

$$\mathbf{v}_{Q\in\Delta/A} = p\,\boldsymbol{\omega}_{B/A} \tag{7.110}$$

$$p = \frac{\boldsymbol{\omega}_{B/A}\cdot\mathbf{v}_{P/A}}{\|\boldsymbol{\omega}_{B/A}\|^2}, \quad \textit{the pitch (an invariant of the screw)} \tag{7.111}$$

Thus, all points $Q \in \Delta$ of have the same velocity $\mathbf{v}_{Q\in B/A}$ at that instant.

The Instantaneous Kinematic Screw Axis

The equation of this axis (Theorem 7.8) is

$$\mathbf{r}_{PQ} = \underbrace{\frac{\boldsymbol{\omega}_{B/A} \times \mathbf{v}_{P\in B/A}}{\|\boldsymbol{\omega}_{B/A}\|^2}}_{\triangleq \mathbf{r}_{PQ*}} + \lambda\,\boldsymbol{\omega}_{B/A} \tag{7.112}$$

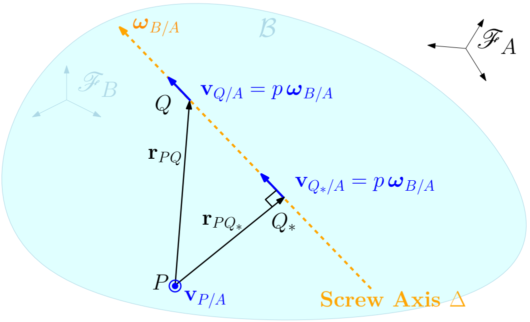


Figure 7.8: All points on Δ have the same velocity $p\,\boldsymbol{\omega}_{B/A}$, where p is the screw pitch (7.111).

The Instantaneous Kinematic Screw Axis

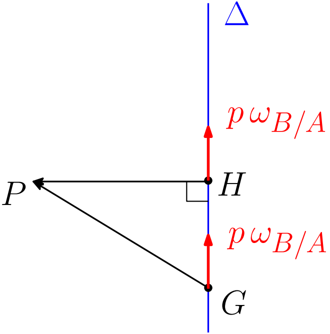
Let us resolve $\{\mathcal{V}_{B/A}\}$ about a point $G \in \Delta$. Then,

$$\mathbf{v}_{P\in B/A} = \mathbf{v}_{G\in B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{GP} \tag{7.113}$$

$$= p\,\boldsymbol{\omega}_{B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{GP} \tag{7.114}$$

$$= p\,\boldsymbol{\omega}_{B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{HP}, \tag{7.115}$$

where, H is the projection of P on Δ.



Canonical Velocity Decomposition

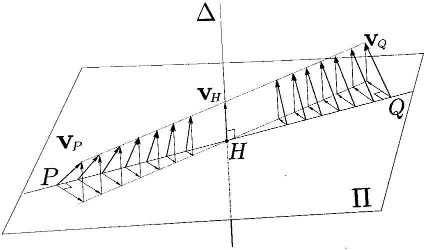


Figure 7.9: The instantaneous kinematic screw axis. [Roy(2015)]

$$\mathbf{v}_{P \in B/A} = p\,\boldsymbol{\omega}_{B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{HP}$$

The general infinitesimal motion of $\mathcal{B}$ relative to $\mathcal{A}$ at any instant can be decomposed as the sum of a translation along the instantaneous screw axis and a rotation about the instantaneous screw axis.

Does this remind you of a screw motion?

Special Case: Instantaneous Axis of Rotation

$$\mathbf{v}_{P\in B/A} = p\,\boldsymbol{\omega}_{B/A} + \boldsymbol{\omega}_{B/A} \times \mathbf{r}_{HP}$$

- If $\boldsymbol{\omega}_{B/A} \neq 0$, but the pitch $p = \frac{\boldsymbol{\omega}_{B/A}\cdot\mathbf{v}_{P/A}}{\|\boldsymbol{\omega}_{B/A}\|^2} = 0$, all points $Q \in \Delta$ have $\mathbf{v}_{Q/A} = \mathbf{0}$. Geometrically, when is $p = 0$?
- All other points in $\mathcal{B}$ appear, at that instant, to be rotating about the line $\Delta = (Q_*, \boldsymbol{\omega}_{B/A})$, where the point Q_* is given by (7.112).
- In this context, the point Q_* is usually renamed to I (for Instantaneous).
- Hence, in this case, $\Delta = (I, \boldsymbol{\omega}_{B/A})$ is called the instantaneous axis of rotation.
- Conversely, if we know that for a point $\mathbf{v}_{J\in B/A} = \mathbf{0}$, there exists an instantaneous axis of rotation, $\Delta = (J, \boldsymbol{\omega}_{B/A})$ as the pitch is 0.

Special Case of Special Case: Instantaneous Center of Rotation in 2D Motion

- In 2D motion, $\boldsymbol{\omega}_{B/A}$ is always orthogonal to the plane of motion.
- Hence, the pitch $p = \frac{\boldsymbol{\omega}_{B/A} \cdot \mathbf{v}_{P/A}}{\|\boldsymbol{\omega}_{B/A}\|^2} = 0$ always. This implies that an instantaneous axis of rotation, $\Delta = (I, \boldsymbol{\omega}_{B/A})$ exists, all points of which have velocity $\mathbf{0}$.
- The point of intersection of this axis with the plane of motion is called the instantaneous center of rotation I.
- Using (7.112) in 2D,

$$\mathbf{r}_{PI} = \frac{\boldsymbol{\omega}_{B/A} \times \mathbf{v}_{P \in B/A}}{\|\boldsymbol{\omega}_{B/A}\|^2} \tag{7.116}$$

- We see that the position vector $\mathbf{r}_{PI}$ is orthogonal to the velocity vector $\mathbf{v}_{P \in B/A}$.
- This is true for all points $P \in \mathcal{B}$. This property can be used to find the instantaneous center of rotation in 2D.

Instantaneous Center of Rotation in 2D Motion

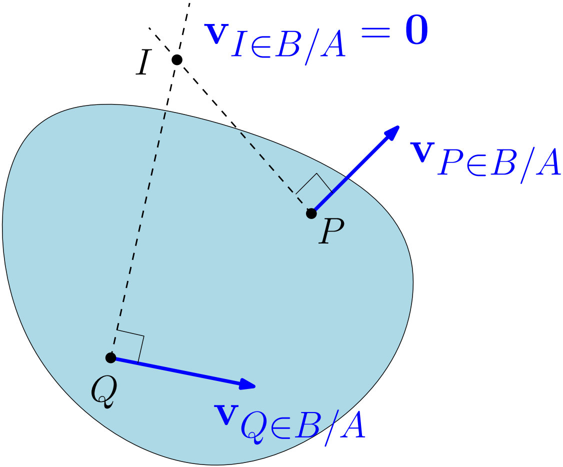


Figure 7.10: Finding the instantaneous center of rotation in 2D motion.

Contents

Ackermann Steering

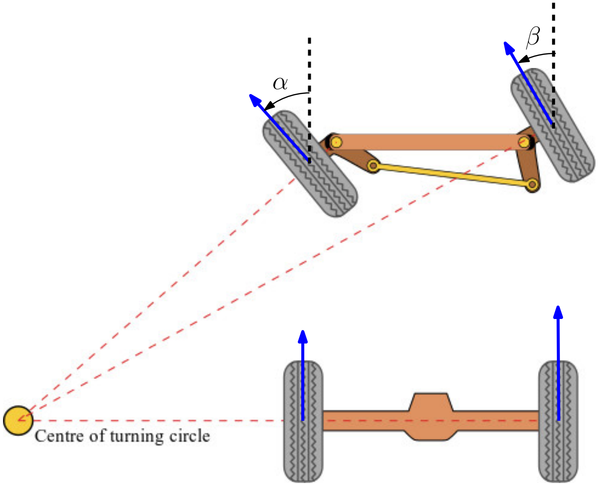


Figure 8.1: Note the instantaneous center of rotation.

Differential Drive Mobile Robots

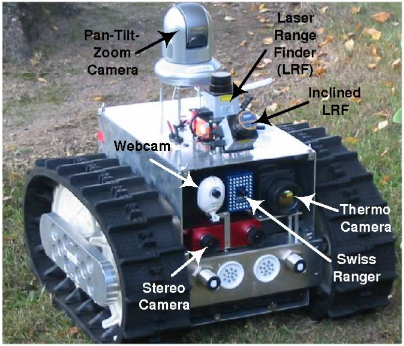
Also called the tank-track or unicycle model

(a) Front view with the blue SICK Laser Range Finder (LRF).

(b) Back view showing the passive castor wheel employed to avoid toppling over.

Figure 8.2: The Pioneer 3-DX: A popular research platform for 2D mapping. From: Youtube video.

Other Differential Drive Platforms



(a) Jacobs tank-track mobile robot Rugbot.

(b) A motorized wheelchair. Source link.

Figure 8.3: All such vehicles can "turn on a dime." Each actuated wheel has its own motor.

Kinematic Differential Drive Robot

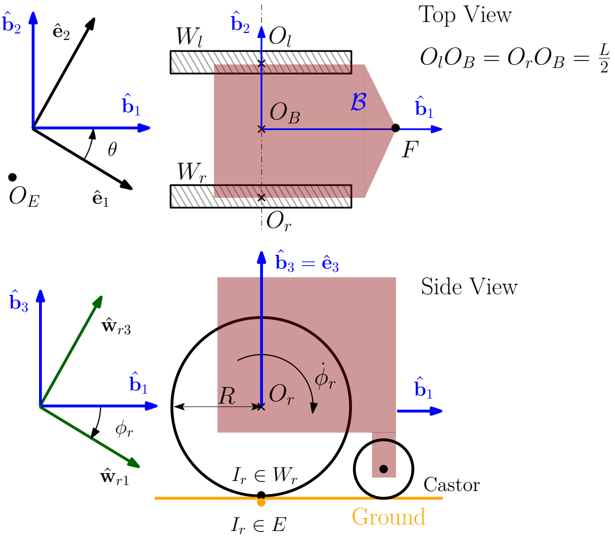
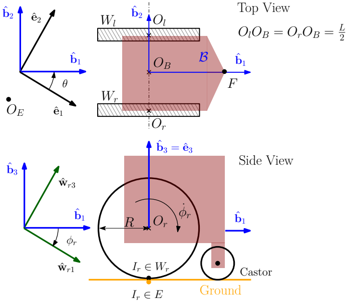


Figure 8.4: Various frames to analyze the kinematics.



- The various angular-velocities can be written immediately:

$$\boldsymbol{\omega}_{B/E} = \dot{\theta}\,\hat{\mathbf{e}}_3 \tag{8.1a}$$

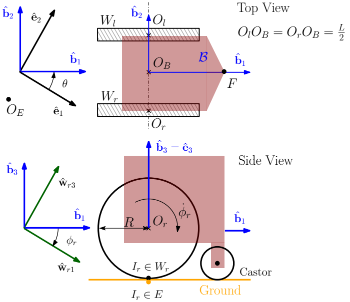
$$\boldsymbol{\omega}_{W_r/B} = \dot{\phi}_r\,\hat{\mathbf{b}}_2 \tag{8.1b}$$

$$\boldsymbol{\omega}_{W_\ell/B} = \dot{\phi}_\ell\,\hat{\mathbf{b}}_2 \tag{8.1c}$$

- Using the angular-velocities composition rule (7.27),

$$\boldsymbol{\omega}_{W_r/E} = \boldsymbol{\omega}_{W_r/B} + \boldsymbol{\omega}_{B/E} \qquad = \dot{\phi}_r\,\hat{\mathbf{b}}_2 + \dot{\theta}\,\hat{\mathbf{e}}_3 \tag{8.1d}$$

$$\boldsymbol{\omega}_{W_\ell/E} = \boldsymbol{\omega}_{W_\ell/B} + \boldsymbol{\omega}_{B/E} \qquad = \dot{\phi}_\ell\,\hat{\mathbf{b}}_2 + \dot{\theta}\,\hat{\mathbf{e}}_3 \tag{8.1e}$$



The equal velocity constraints are:

$$\mathbf{v}_{I_r \in W_r/E} = \mathbf{v}_{I_r \in E/E} = \mathbf{0} \tag{8.2a}$$

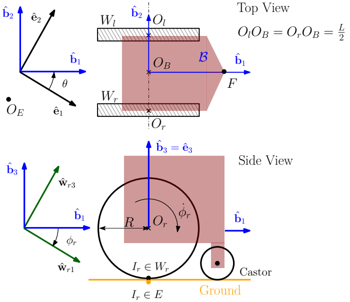
$$\mathbf{v}_{I_\ell \in W_r/E} = \mathbf{v}_{I_\ell \in E/E} = \mathbf{0} \tag{8.2b}$$

$$\mathbf{v}_{O_r \in B/E} = \mathbf{v}_{O_r \in W_r/E} \tag{8.2c}$$

$$= \mathbf{v}_{I_r \in W_r/E} + \boldsymbol{\omega}_{W_r/E} \times \mathbf{r}_{I_r O_r} \tag{8.2d}$$

$$= (\dot{\phi}_r\, \hat{\mathbf{b}}_2 + \dot{\theta}\, \hat{\mathbf{e}}_3) \times (R\hat{\mathbf{e}}_3) = \dot{\phi}_r\, R\, \hat{\mathbf{b}}_1 \tag{8.2e}$$

$$\text{Similarly, } \mathbf{v}_{O_\ell \in B/E} = \mathbf{v}_{O_\ell \in W_\ell/E} = \dot{\phi}_\ell\, R\, \hat{\mathbf{b}}_1 \tag{8.2f}$$



For the right-wheel, we can also write:

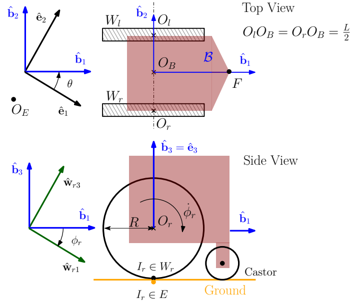
$$\mathbf{v}_{O_r \in B/E} = \dot{\phi}_r\, R\, \hat{\mathbf{b}}_1$$

$$= \mathbf{v}_{O_\ell \in B/E} + \boldsymbol{\omega}_{B/E} \times \mathbf{r}_{O_\ell O_r} \tag{8.3a}$$

$$= \dot{\phi}_\ell\, R\, \hat{\mathbf{b}}_1 + \dot{\theta}\, \hat{\mathbf{e}}_3 \times (-L\, \hat{\mathbf{b}}_2) \tag{8.3b}$$

$$\Rightarrow \dot{\phi}_r\, R\, \hat{\mathbf{b}}_1 = \dot{\phi}_\ell\, R\, \hat{\mathbf{b}}_1 + \dot{\theta}\, L\, \hat{\mathbf{b}}_1 \tag{8.3c}$$

$$\Rightarrow \dot{\theta} = \frac{R}{L}(\dot{\phi}_r - \dot{\phi}_\ell) \tag{8.3d}$$



We can find the speed of O_B in two different ways:

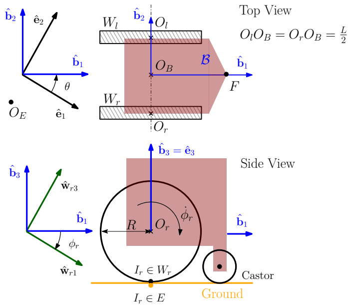
$$\mathbf{v}_{O_B \in B/E} = \mathbf{v}_{O_r \in B/E} + \boldsymbol{\omega}_{B/E} \times \mathbf{r}_{O_r O_B} \quad = \dot{\phi}_r \, R \, \hat{\mathbf{b}}_1 + \dot{\theta} \, \hat{\mathbf{e}}_3 \times \frac{L}{2} \hat{\mathbf{b}}_2 \tag{8.4a}$$

$$= (\dot{\phi}_r \, R - \dot{\theta} \, \frac{L}{2}) \, \hat{\mathbf{b}}_1 \tag{8.4b}$$

$$\mathbf{v}_{O_B \in B/E} = \mathbf{v}_{O_\ell \in B/E} + \boldsymbol{\omega}_{B/E} \times \mathbf{r}_{O_\ell O_B} \quad = \dot{\phi}_\ell \, R \, \hat{\mathbf{b}}_1 + \dot{\theta} \, \hat{\mathbf{e}}_3 \times \frac{-L}{2} \hat{\mathbf{b}}_2$$

$$= (\dot{\phi}_\ell \, R + \dot{\theta} \, \frac{L}{2}) \, \hat{\mathbf{b}}_1 \tag{8.4c}$$

We can now add (8.4b) and (8.4c) to eliminate $\dot{\theta}$.

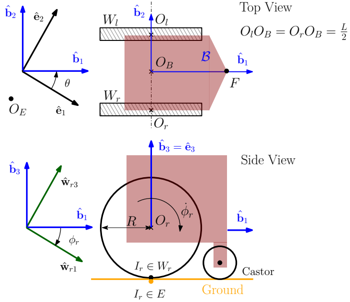


$$\mathbf{v}_{O_B \in B/E} = R\,\frac{\dot{\phi}_\ell + \dot{\phi}_r}{2}\,\hat{\mathbf{b}}_1 \qquad = R\,\frac{\dot{\phi}_\ell + \dot{\phi}_r}{2}\left(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2\right) \tag{8.5a}$$

Let the position vector of O_B in $\mathscr{F}_E$ be $\mathbf{r}_{O_E O_B} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$, then,

$$\mathbf{v}_{O_B \in B/E} = \left(\frac{d\mathbf{r}_{O_E O_B}}{dt}\right)_E \qquad = \dot{x}\,\hat{\mathbf{e}}_1 + \dot{y}\,\hat{\mathbf{e}}_2 \tag{8.5b}$$

We can equate (8.5a) and (8.5b).



This gives finally the kinematic equations:

$$\dot{x} = R\,\frac{\dot{\phi}_\ell + \dot{\phi}_r}{2}\,\cos\theta \tag{8.6a}$$

$$\dot{y} = R\,\frac{\dot{\phi}_\ell + \dot{\phi}_r}{2}\,\sin\theta \tag{8.6b}$$

$$\dot{\theta} = \frac{R}{L}(\dot{\phi}_r - \dot{\phi}_\ell) \tag{8.6c}$$

What is the use of such an equation?

The State-Space Representation For Designing Control-Systems

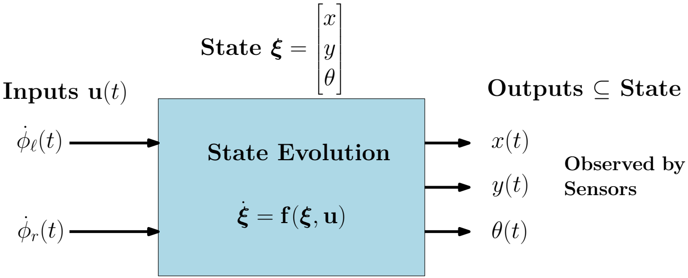
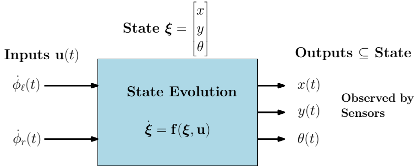


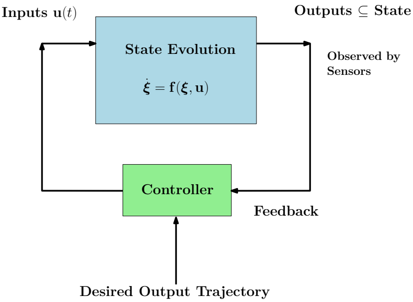
Figure 8.5: Question: Which inputs do I need to apply, if I want the outputs to follow a desired time-trajectory?

Open Loop Design



- Plan your inputs to achieve the desired output using the system evolution equation.
- Apply the planned inputs to the system and hope for the best.
- Without any feedback, the system may not exactly follow the desired output, due to unmodeled dynamics (e.g. motor-speed cannot instantaneously change to the commanded value), external disturbance, etc.
- When we use feedback to modify the inputs based on the observed output and sensor feedback, we call the control-system closed loop.

Closed Loop Design



- When we use feedback to modify the inputs based on the observed output (sensor feedback), we call the control-system closed loop.
- Feedback makes the closed-loop system much more robust to unmodeled dynamics and external disturbance than an open-loop system.
- We will study controller design in GIMS-2 next semester.

Some software libraries define:

$$v \triangleq R\,\frac{\dot{\phi}_\ell + \dot{\phi}_r}{2} \qquad \text{Heading Speed} \tag{8.7a}$$

$$\omega \triangleq \frac{R}{L}(\dot{\phi}_r - \dot{\phi}_\ell) \qquad \text{Turning Speed} \tag{8.7b}$$

These can be directly commanded as inputs by the software interface. In terms of these, the kinematic model is:

$$\dot{x} = v(t)\cos\theta \tag{8.7c}$$

$$\dot{y} = v(t)\sin\theta \tag{8.7d}$$

$$\dot{\theta} = \omega(t) \tag{8.7e}$$

These are the nonlinear Ordinary Differential Equations (ODE) of motion.

Alternate State-Space Representation

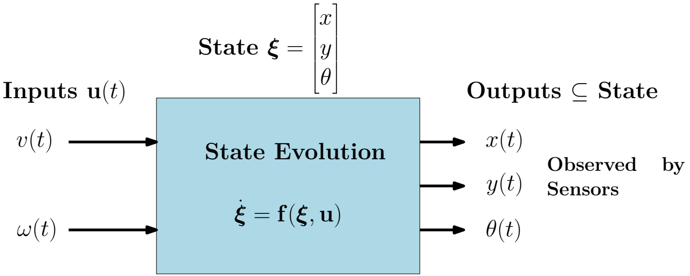
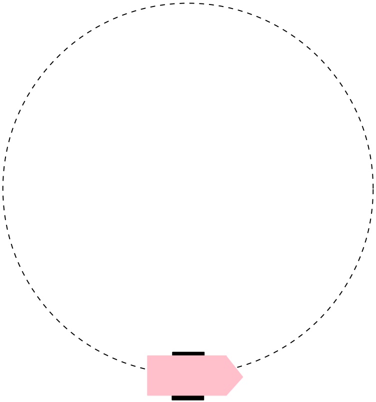


Figure 8.6: Question: Which inputs do I need to apply, if I want the outputs to follow a desired time-trajectory?

Open Loop Example

- Suppose we want the differential-drive robot to revolve in a circle of radius R, with one revolution taking T seconds. Which inputs $v(t)$, $\omega(t)$ can achieve this?



Open Loop Example

- First we explore what happens if we apply constant inputs $v(t) = v$, $\omega(t) = \omega$. Taking initial-state $x(0) = y(0) = \theta(0) = 0$.
- Eq. (8.7e) gives $\theta = \omega\, t$.
- Eqs. (8.7c), (8.7d) give:

$$x = v \int_0^t \cos(\omega\, t)\, dt \qquad = \frac{v}{\omega} \sin(\omega\, t) \tag{8.8}$$

$$y = v \int_0^t \sin(\omega\, t)\, dt \qquad = \frac{v}{\omega} (1 - \cos(\omega\, t)) \tag{8.9}$$

$$x^2 + \left(y - \frac{v}{\omega}\right)^2 = \frac{v^2}{\omega^2} \tag{8.10}$$

- This is a circle with radius v/ω and period of revolution $2\pi/\omega$. Hence, we can choose inputs as follows:

$$\omega = \frac{2\pi}{T}, \qquad v = \frac{2\pi\, R}{T}. \tag{8.11}$$

Ackermann Steering: Analysis Using a Tricycle Model

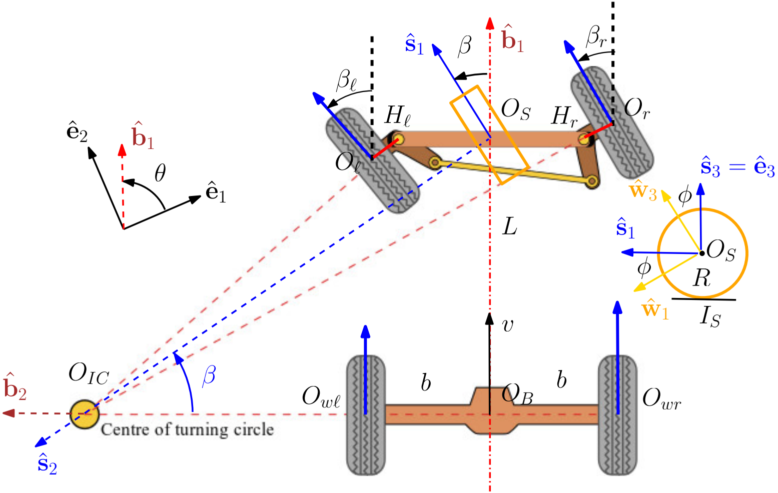


Figure 8.7: Note the instantaneous center of rotation.

We showed in the class that:

$$\dot{\theta} = \frac{v}{L} \tan\beta \tag{8.12}$$

Contents

Home-work 1 I

Due date: 16.9.2016, 23:59:59 on jGrader. Submission in groups of 2 or individually

1. Show (3.28) by resolving all the vectors in a common arbitrary basis $\mathscr{B}_E$. Show all steps. 15%
2. Show all the gory details of the steps between (3.25) and (3.26), where you have to use the distributivity property of the cross-product over vector-addition (3.22) twice. 15%
3. Let us say you have three LI vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$, not necessarily orthonormal – but since they are LI, they form a basis in $\mathbb{E}$. Given an arbitrary vector $\mathbf{v}$, how will you resolve it in this non-orthonormal basis? In other words, how will you find $\alpha_1, \alpha_2, \alpha_3$, s.t.

$$\mathbf{v} = \alpha_1\,\mathbf{f}_1 + \alpha_2\,\mathbf{f}_2 + \alpha_3\,\mathbf{f}_3$$

   Work directly with the given vectors and their dot-products – do not resolve them in any orthonormal basis. Hint: $\alpha_i \neq \mathbf{v} \cdot \mathbf{f}_i$ since the $\mathbf{f}_i$ are not orthonormal. 15%

Home-work 1 II

Due date: 16.9.2016, 23:59:59 on jGrader. Submission in groups of 2 or individually

4. Given two vectors **u** and **v** such that

$$
{}^{E}\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad {}^{E}\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.
$$

   1. What are the lengths of **u** and **v**? 5%
   2. What is the angle between them? 5%
   3. Write **u** as a sum of two vectors, one collinear to **v** and the other orthogonal to **v**. 7%
   4. Find a unit-vector orthogonal to the plane of **u** and **v**. 5%
   5. Find a vector in the plane of **u** and **v** which is orthogonal to **v**. 4%
   6. Find a vector in the plane of **u** and **v** which is orthogonal to **u**. 4%

   Hint: For the last two sub-parts you could make use of the triple cross-product.

Home-work 1 III

Due date: 16.9.2016, 23:59:59 on jGrader. Submission in groups of 2 or individually

5. Given non-zero vectors **a** and **b**:
   1. What is the necessary condition between **a** and **b** for an exact solution **z** of the following equation to exist? 5%

$$\mathbf{a} \times \mathbf{z} = \mathbf{b} \tag{9.1}$$

   2. Find the most general **z** satisfying (9.1) without resolving the vectors in any basis. Hints:
      - Refer to (3.24) for inspiration.
      - To find a particular $\mathbf{z} = \mathbf{z}_*$ satisfying (9.1), you can take cross-product of both sides of the equation by **a**, use (3.28), and further choose $\mathbf{z}_*$ to be orthogonal to **a**.
      - Now using the particular solution, find the general solution.

      20%

Home-work 2 I

Due date: 30.9. in class. Groups of 2.

1. You are given the following matrix:

$$ {}^{A}_{B}\mathbf{R} = \frac{1}{25} \begin{bmatrix} 9 & 12 & -20 \\ 12 & 16 & 15 \\ 20 & -15 & 0 \end{bmatrix} \tag{9.2} $$

   1. Verify that it is a valid DCM/rotation-matrix. 10%
   2. What are ${}^{A}\hat{\mathbf{b}}_1$, ${}^{B}\hat{\mathbf{b}}_2$, and ${}^{B}\hat{\mathbf{a}}_3$? 15%
   3. What is the angle between $\hat{\mathbf{a}}_3$ and $\hat{\mathbf{b}}_3$? 5%
   4. Given the vectors $\mathbf{x}$ and $\mathbf{y}$:

$$ \mathbf{x} = 2\,\hat{\mathbf{a}}_1 + 3\,\hat{\mathbf{b}}_3 + \hat{\mathbf{a}}_2 \tag{9.3} $$

$$ \mathbf{y} = \hat{\mathbf{b}}_1 + 2\,\hat{\mathbf{b}}_3 + \hat{\mathbf{a}}_1 \tag{9.4} $$

   find $\|\mathbf{x}\|$ and the angle between $\mathbf{x}$ and $\mathbf{y}$. 30%

Home-work 2 II

Due date: 30.9. in class. Groups of 2.

5. If ${}^{A}\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, what is ${}^{B}\mathbf{v}$? 15%

2. Given an $n \times n$ square matrix $\mathbf{A}$, show that,
   1. $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}})$ is symmetric. 5%
   2. $\mathbf{C} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathsf{T}})$ is skew-symmetric. 5%

   Clearly, $\mathbf{A} = \mathbf{B} + \mathbf{C}$.

3. Given ${}^{A}_{B}\mathbf{R}$, ${}^{B}_{C}\mathbf{R}$, ${}^{C}_{E}\mathbf{R}$, and ${}^{B}_{D}\mathbf{R}$, compute ${}^{D}_{E}\mathbf{R}$ in terms of the given matrices. 15%

Home-work 3 I

Due date: 7.10., 23:59:00, on jGrader

1. Do tasks 7,8,9 in the Matlab tutorial slides.

Home-work 4 I

Due date: 17.10., 23:59:00. Groups of 2. Submission on jGrader as zip file.

1. Find out in which principal intervals do the results of the following inverse trigonometric functions lie:
   $\operatorname{acos} x, \operatorname{asin} y, \operatorname{atan}(y/x), \operatorname{atan2}(y,x)$. You can find this by, e.g. typing "doc acos" on the Matlab command prompt. 4%
2. For each of the following, compute manually (using the principal intervals from above): $\operatorname{acos}(x), \operatorname{asin}(y), \operatorname{atan}(y/x), \operatorname{atan2}(y,x)$.
   1. $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$
   2. $x = -\frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$
   3. $x = -\frac{1}{2}$, $y = -\frac{\sqrt{3}}{2}$
   4. $x = \frac{1}{2}$, $y = -\frac{\sqrt{3}}{2}$

   You will have to plot the points on the XY plane. Explain clearly how you arrived at your answer. You can verify your results using Matlab. This will not be graded, so there is no need to submit the code. 4%

Home-work 4 II

Due date: 17.10., 23:59:00. Groups of 2. Submission on jGrader as zip file.

3. An airplane has the following attitude: yaw 130°, pitch 20°, roll 45°.
   1. Compute its DCM. 15%
   2. Now from this DCM compute the equivalent rotation-axis and rotation-angle. 10%
   3. Now compute the equivalent unit-quaternion. 10%

   You can use Matlab to do the computations: In this case, include the code.

4. Instead of the **zyx** (yaw, pitch, roll) Euler angles sequence, we can use the **zxz** sequence in the following order:
   - Precession $\mathcal{R}_{\psi,\hat{\mathbf{e}}_3}$
   - Nutation $\mathcal{R}_{\theta,\hat{\mathbf{u}}_1}$
   - Spin $\mathcal{R}_{\phi,\hat{\mathbf{v}}_3}$

   Sometimes, these are even called the Euler angles.

Home-work 4 III

Due date: 17.10., 23:59:00. Groups of 2. Submission on jGrader as zip file.

1. Find the equivalent DCM corresponding to these rotations. You can verify your answer from the first equation on page 26 of [Roy(2015)] (in IRC). 15%
2. Given a DCM how will you find the precession, nutation, and spin angles? 15%
3. In the last part, where will you encounter the singularity equivalent to the gimbal lock? 5%

5. For any two quaternions, show that 10%

$$\overline{Q}_1\,\overline{Q}_2 = \overline{Q_2\,Q_1} \tag{9.5}$$

6. How can you convert a quaternion $Q = q_0 + \mathbf{q}$ to an equivalent rotation-matrix directly? Hint: Use trigonometric identities for half angles in (4.62). 12%

Tutorial on Transforms I

1. What is the rotation-matrix and quaternion corresponding to no rotation?
2. Consider an omnidirectional robot moving on flat ground. Let us define $\mathscr{F}_A$ as a frame at its initial pose: i.e. position and oritientation considered together. Let $\hat{\mathbf{a}}_3$ point opposite to the gravity direction. It now makes motions in the following order:
    - A turning (yaw) rotation $\mathcal{R}_{\psi,\hat{\mathbf{a}}_3}$. We define $\mathscr{F}_B$ as the frame at its pose after this rotation;
    - A translation of ${}^B_C\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$, where $\mathscr{F}_C$ is the frame at its pose after this translation.

    Find the resultant transform ${}^A_C\mathbf{T}$. Hint: Define two transforms and compose them using (5.13e).
3. The same mobile robot now makes the motions in the following order, starting at $\mathscr{F}_A$:

Tutorial on Transforms II

- A translation of ${}^{A}_{B}\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$, where $\mathscr{F}_B$ is the frame at its pose after this translation.
- A rotation $\mathcal{R}_{\psi,\hat{\mathbf{b}}_3}$. Of couse, in this case $\hat{\mathbf{b}}_3 = \hat{\mathbf{a}}_3$. We define $\mathscr{F}_C$ is the frame at its pose after this rotation;

Find the resultant transform ${}^{A}_{C}\mathbf{T}$. Hint: Use (5.13e).

4. Consider two 3D sensors viewing a scene from two different vantage points. The first one measures the 3D coordinates of points which are in its field of view in a frame $\mathscr{F}_A$ attached to it; the second one measures them in a frame $\mathscr{F}_B$ attached to it. They both have a bright red LED L in their fields of view. According to sensor $\mathcal{A}$,

$$ {}^{A}\mathbf{r}_L = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} $$

Tutorial on Transforms III

Sensor $\mathcal{B}$ is located w.r.t. sensor $\mathcal{A}$ such that:

$$ {}^{A}_{B}\mathbf{t} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} $$

and the rotation ${}^{A}_{B}\mathbf{R}$ is composed of a yaw $\psi = 10^\circ$, a pitch $\theta = 15^\circ$ and a roll $\phi = 20^\circ$. What will be the position-vector of L reported by sensor $\mathcal{B}$? You can use Matlab to do numerical computations. You do not need to attach the code.

Home-work 6: Based on the Space-Ship Tutorial

Due date: 31.10.2016, 23:59. Groups of 1 or 2.

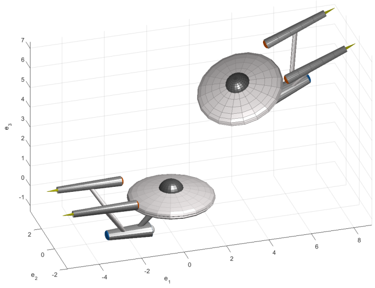
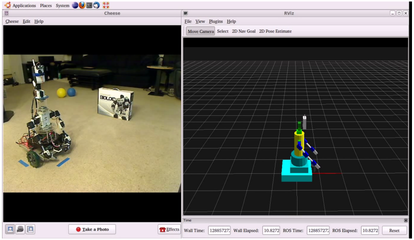


Figure 9.1: A much better space-ship model created by IMS student Miraj Sheth.

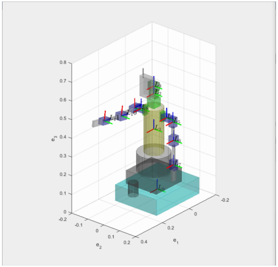
- Create a space-ship model (aim for the model shown above) starting from the Matlab code for a basic space-ship given in the tutorial.

Home-work 7: Short Project

Due date: 8.11.2016, 23:59. Groups of 1 or 2.



(a) Pi robot video from this web-page.



(b) Pi-Robot by Ivan Morales.

Figure 9.2: Create a model of the Pi-Robot using the material provided on the course's web-page. It also contains a description of the contents of the provided zip file. Upload your Matlab-file(s) as a zip-file on jGrader.

Home-work 8: Short Project I

Due date: 18th Nov., 23:59. Groups of 1 or 2.

1. On the course web-page, you are given a zipped folder containing the skeleton of code to create the space-ship animation shown in the class.
    - Fill-in the code necessary to run the main script `animate_spaceship.m` from the Matlab command-line: It will not run till you have filled-in the required code at all of the following places marked with `% Fill-in`

```
animate_spaceship.m: line 35
animate_spaceship.m: line 40
DCM2AA.m: line 5
Quat2AA.m: line 5
Quat2DCM.m: line 14
QuatConj.m: line 5
QuatMult.m: line 6
QuatPower.m: line 8
SLERP.m: line 8
```

Home-work 8: Short Project II

Due date: 18th Nov., 23:59. Groups of 1 or 2.

- After making it work, zip your folder and submit to jGrader.

Home-work 9 I

Due date: 25th Nov., 23:59. Groups of 1 or 2.

1. Watch the spinning-tube trick again, but this time on a glass table and viewed from underneath: https://www.youtube.com/watch?v=E9WUaBGH7_I. If from above you saw an X, from beneath you now see an O. Why's that? Explain using a sketch and by doing a derivation as done in the class.
2. Show all the intermediate steps from (7.35a) to (7.35b).

Home-work 10

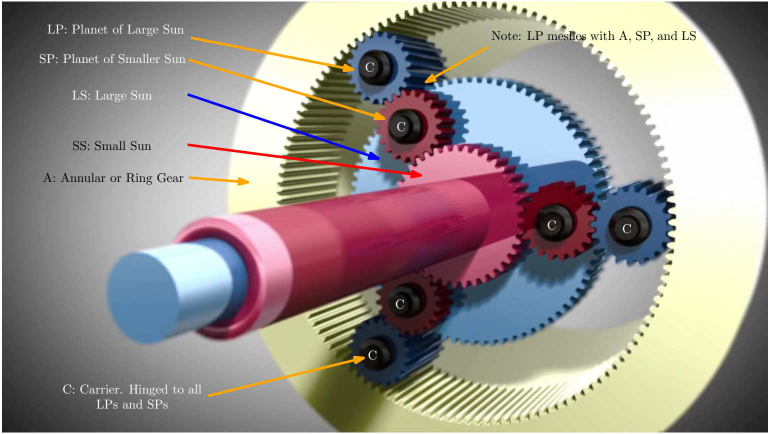


Figure 9.3: The double planetary gear-set of an automatic transmission. Watch this video for an animation. Note the initials assigned to various rigid bodies.

Home-work 10

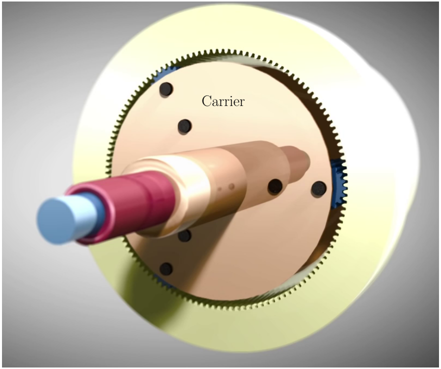


Figure 9.4: This figure shows how the carrier is actually hinged to the axles of all planetary gears.

Home-work 10

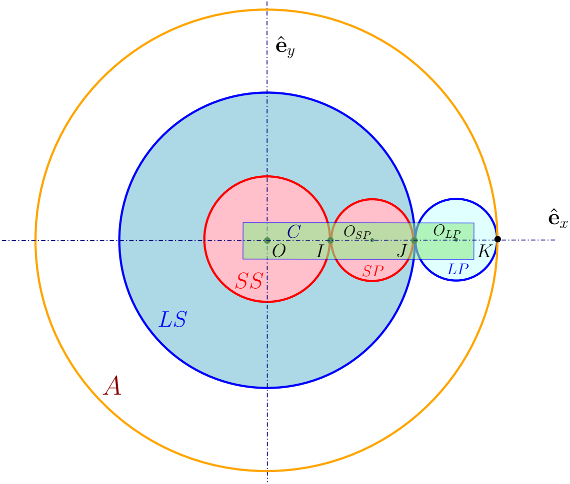
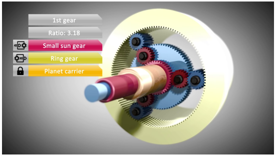
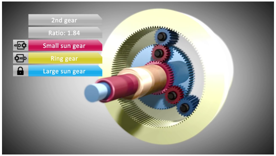


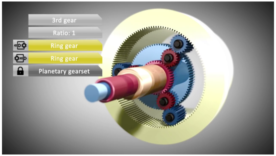
Figure 9.5: A simplified schematic for analysis of the double planetary gear-set.



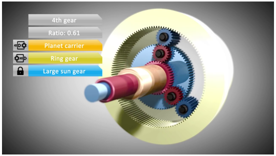
(a)



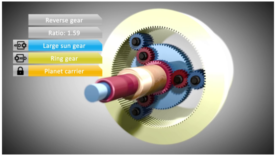
(b)



(c)



(d)



(e)

Figure 9.6: The various "gears" of the automatic transmission. In each case, the input shaft, the output shaft (to the wheels), and the locked shaft (not moving) are shown. For example, in the first gear, input is $\omega_{SS/E}$, output is $\omega_{A/E}$, $\omega_{C/E} = 0$, and the speed reduction ratio value (input/output) is 3.18.

Home-work 10: Due date: 2.12., 23:59 I

Individual or in Groups of 2

1. Look at the double planetary gear-set used in the automatic transmission in figures 9.3, 9.4, and 9.5, and watch this video. Just like we analyzed the normal planetary gear in Sec. 11, we want to now analyze the double planetary gear-set and compute the speed-reduction ratios (input/output) for the 5 cases mentioned in the video using the kinematic screw formula.
2. We have the following equal-velocity constraints:
   - $J \in SP$, $J \in LP$, $J \in LS$
   - $K \in A$, $K \in LP$
   - $I \in SP$, $I \in SS$
   - $O_{SP} \in C$, $O_{SP} \in SP$
   - $O_{LP} \in C$, $O_{LP} \in LP$

   The teeth of all gears have the same size. The number of teeth are:
   - $N_{SP} = N_{LP} = N$

Home-work 10: Due date: 2.12., 23:59 II

Individual or in Groups of 2

- N_{SS}
- We define $\eta = \frac{N_{SS}}{N}$. Note that the pitch radii of the various gears are in the same proportion as their respective number of teeth. The number of teeth of gears which are not given can be found from the given information and the Fig. 9.5.

Use the kinematic screw formula to derive the following two relationships:

$$\omega_{C/E}(\eta + 1) = \omega_{SS/E}\frac{\eta}{2} + \omega_{LS/E}(\frac{\eta}{2} + 1) \quad (9.6)$$

$$\frac{1}{2}\omega_{LS/E}(\frac{\eta}{2} + 1) + \frac{1}{2}\omega_{A/E}(\frac{\eta}{2} + 2) = \omega_{C/E}(\frac{\eta}{2} + \frac{3}{2}) \quad (9.7)$$

3. Using the given speed-reduction ratio value 3.18 for the first-gear (Fig. 6(a)), estimate the η which was used in the video.

Home-work 10: Due date: 2.12., 23:59 III

Individual or in Groups of 2

4. Using this η, check that the speed-reduction ratio values for the other gears in Fig. 9.6 agree (up to first decimal place) with your derivations. This part can be done also if you did not do the derivations.

Home-work 11 (Bonus): Due date: 12.12., 23:59 I

Individual or in Groups of 2

1. Refer to Fig. 8.7. For simplification assume that O_ℓ and O_r are coincident with H_ℓ and H_r respectively. If the vehicle is turning along a curve of radius ρ, show that:

$$\tan\beta = \frac{L}{\rho} \tag{9.8}$$

$$\tan\beta_\ell = \frac{L}{\rho - b} \tag{9.9}$$

$$\tan\beta_r = \frac{L}{\rho + b} \tag{9.10}$$

(9.11)

Contents

References I

Peter Corke.
Robotics, Vision and Control: Fundamental Algorithms in MATLAB.
Springer, 1st edition, 2011.
ISBN 978-3642201431.

John J. Craig.
Introduction to Robotics: Mechanics and Control.
Prentice Hall, 3rd edition, 2004.
ISBN 978-0201543612.

Reza N. Jazar.
Theory of Applied Robotics: Kinematics, Dynamics, and Control.
Springer, 2nd edition, 2010.
ISBN 978-1441917492.

References II

R. Valéry Roy.
Advanced Engineering Dynamics.
Hyperbolic Press, 1st edition, 2015.
ISBN 978-0990696957.