# Deterministic Optimization

## Linear Optimization Modeling

## Network Flow Problems

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## Maximum Flow Problem

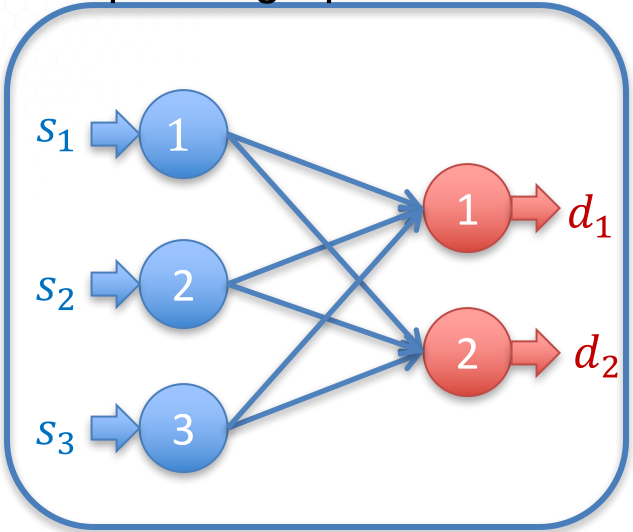
# Modeling using Linear Programs

## Learning Objectives

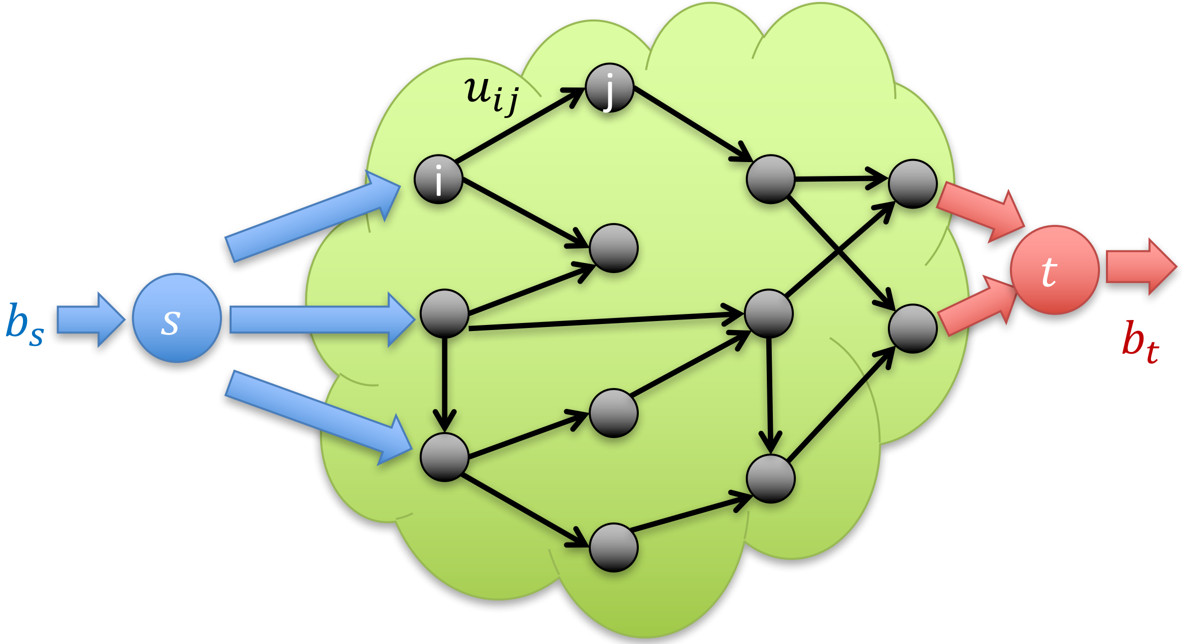
- Discover another (pair of) problem(s) related to the transportation problem, which is even more interesting, has a deep theory behind, and many applications.

# Maximum Flow Problem

Fixed supply and demand
Bipartite graph



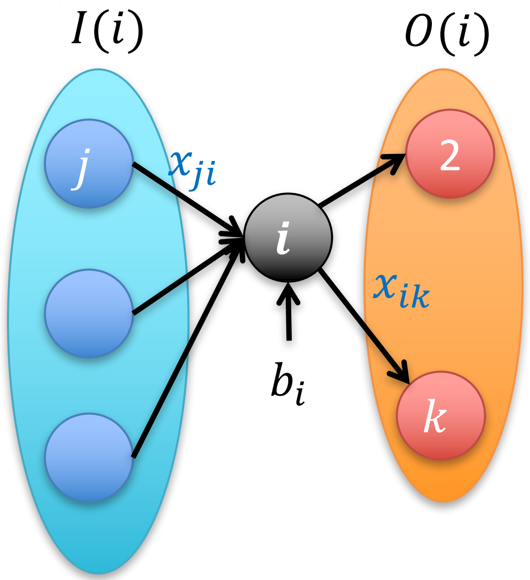

A general directed graph




A million-dollar question: **How much supply $b_s$ can be transported from source to target through the network with limited arc capacity?**

# Maximum Flow Problem: LP Model

Decision variables: $x_{ij}$ for $(i,j) \in \mathcal{A}$, where $\mathcal{A}$ is the set of arcs.




Flow conservation

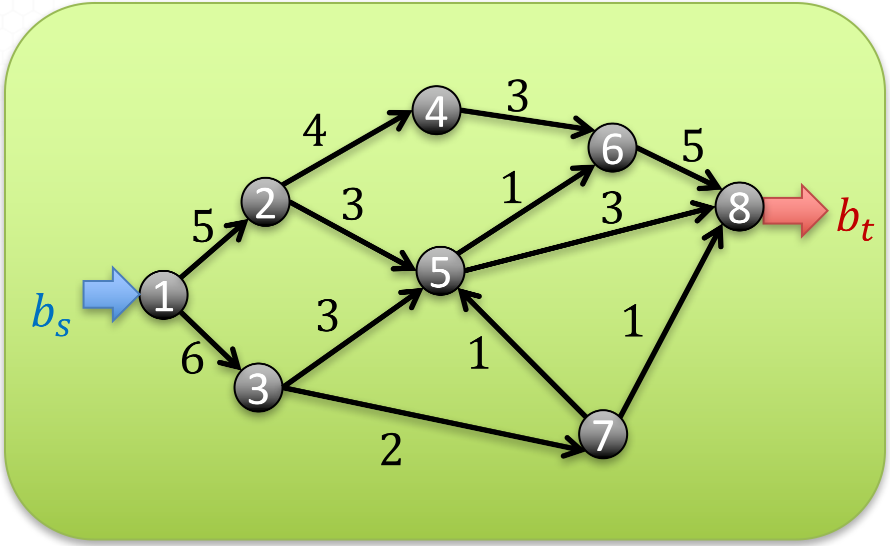
$$
\begin{aligned}
\max \quad & b_s \\
\text{s.t.} \quad & \sum_{k \in O(i)} x_{ik} - \sum_{j \in I(i)} x_{ji} = b_i \quad \forall i \\
& b_t = -b_s \\
& b_i = 0, \quad \forall i \neq s, t \\
& 0 \leq x_{ij} \leq u_{ij}, \quad \forall (i,j) \in \mathcal{A}.
\end{aligned}
$$

Total supply = total demand

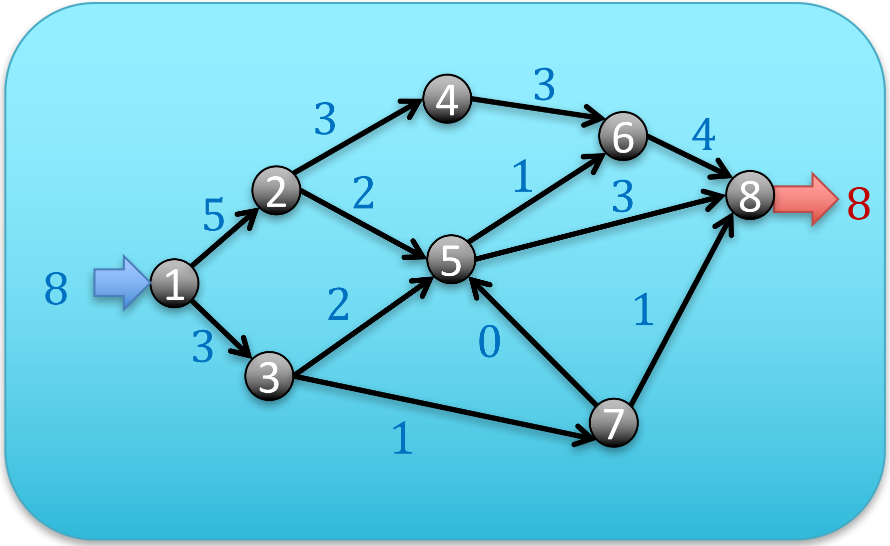
Arc capacity

# A Concrete Example

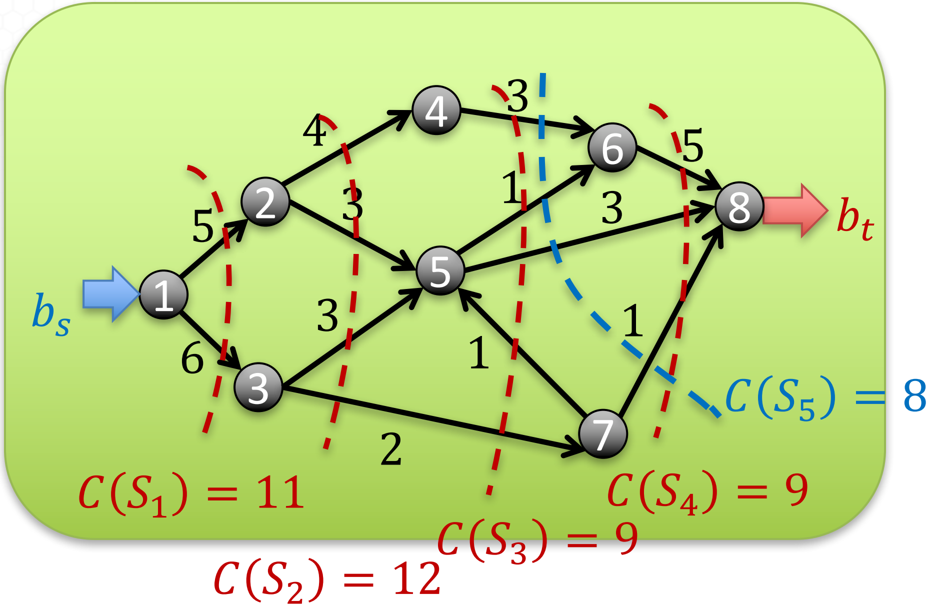
Capacity constrained network




Max flow solution



# Minimum Cut Problem




A s-t **cut** $S$ is a subset of nodes
Such that $s \in S$ and $t \notin S$

So a **cut** $S$ is a separation of
Source node from target node

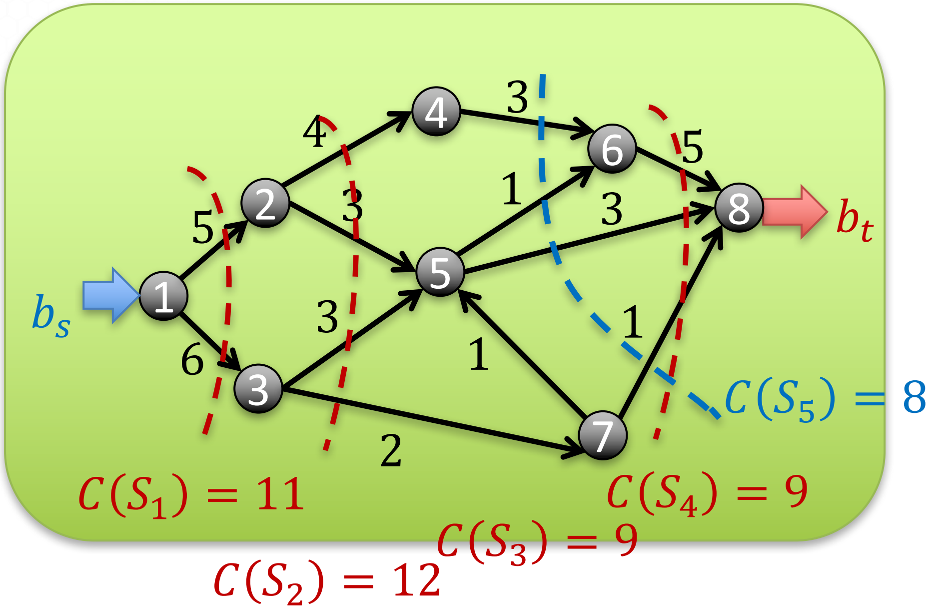
**Capacity of a cut $S$** is the total capacity of arcs that cross from $S$ to its complement
Denoted as $C(S) \coloneqq \sum_{(i,j)\in A, i\in S, j\notin S} c_{ij}$

A million-dollar question:
**Can you find a cut with minimum capacity?**

**Minimum cut** $= \boldsymbol{C(S_5) = 8}$

# Minimum cut = Max Flow

**Minimum cut = $C(S_5) = 8$ = Max Flow**




Is this a Coincidence?

Not at all! There is a deep theory behind it – LP duality.

Max-flow and min-cut are two LPs dual to each other.

Intuitively, it makes sense too.

# Summary

- We constructed a LP model for maximizing the amount of flow that can be pushed through a network.
- We discovered another related LP, Minimum cut problem, which is "dual" to the max flow problem.