Deterministic Optimization

Linear Optimization Modeling

Network Flow Problems

Andy Sun
Assistant Professor
School of Industrial and Systems Engineering

Shortest Path Problem

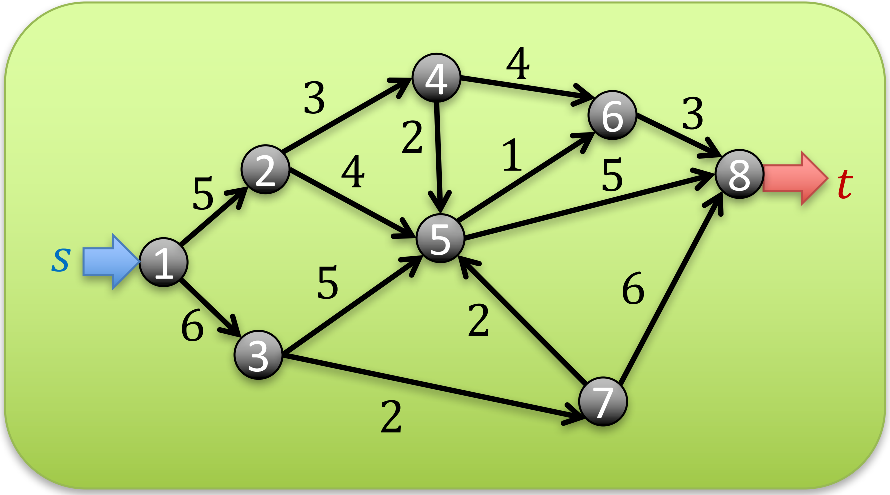
Modeling using Linear Programs

Learning Objectives

- Examine another important network flow problem called the Shorted Path Problem, probably one of the most solved optimization problems in our daily life.

Shorted Path Problem

A directed network, where the number on each edge is the "distance" for traversing that edge. We want to go from starting point s to the target point t.



A million-dollar question: **How to find a path with the shorted distance going from s to t?**

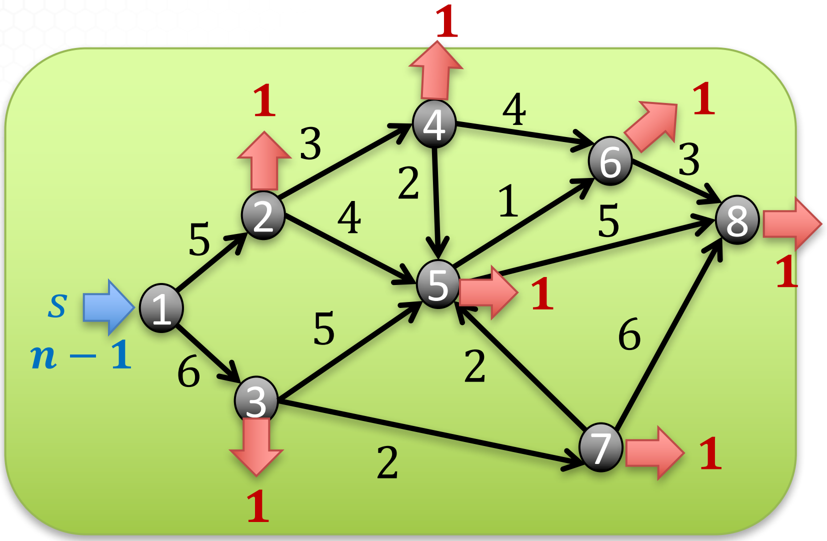
Shorted Path Problem Is Important:

Because:

1. Shorted path problems appear frequently in practice in a wide variety of applications, as we wish to send some material (internet data, cellphone calls, Amazon orders) between two points as quickly, as cheaply, or as reliably as possible.
2. They can be solved very efficiently.
3. They provide a bench mark for more complicated network flow problems.
4. They frequently arise as subproblems in network optimization.

Shorted Path Problem: LP model

Idea: Find a min cost way to ship 1 unit of flow from *s* to all other nodes.



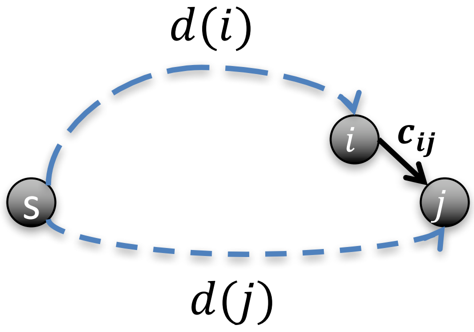
$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{k\in O(i)} x_{ik} - \sum_{j\in I(i)} x_{ji} = -1 \; \forall i \neq s \\
& \sum_{k\in O(s)} x_{sk} - \sum_{j\in I(s)} x_{js} = n-1 \\
& x_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{A}.
\end{aligned}
$$

What Characterizes a Shorted Path?

Optimality Condition for a Shorted Path:
Let $d(j)$ denote the length of some directed path from source node s to node j.
Then $d(j)$'s are the shortest path distances if and only if they satisfy the following shortest path optimality conditions:

$$\boldsymbol{d(j) \leq d(i) + c_{ij}}$$

for all $(i, j) \in A$.



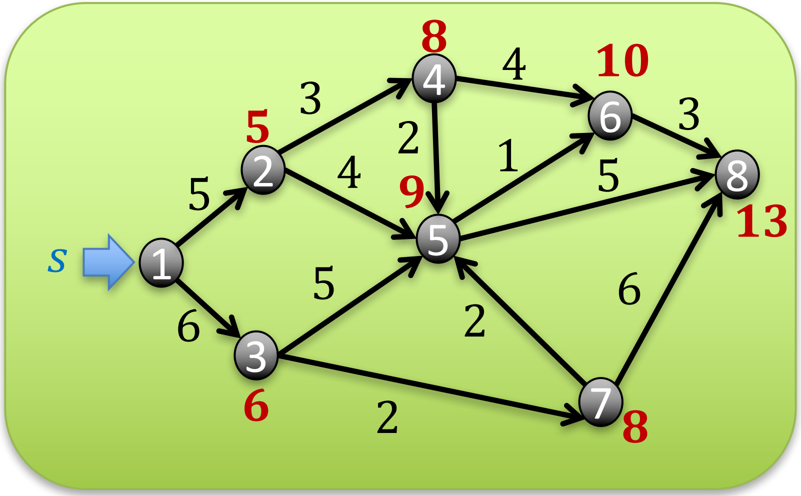
Dynamic Programming Principle (**Bellman's Equation**):

$$d(j) = \min_{\{i:(i,j)\in A\}} \{d(i) + c_{ij}\}$$

Very useful for designing algorithms

An Example

Let us compute the short paths starting from s:



Where do we solve shorted path?

Inside every GPS navigator, there is an efficient algorithm solving the shorted path problem in the background.

Can you estimate how many times the shorted path problem is solved in a day in this country?

Summary

- We have created a LP model for the famous shorted path problem.
- We have discovered a condition that characterizes an optimal shorted path solution.
- Shorted path problems are widely used in practice.