

ISyE 6669 HW 3

1. Consider the following optimization problem:

$$\begin{array}{ll}\min & x \\ \text{s.t.} & xy \geq 1 \\ & x \geq 0, y \geq 0\end{array}$$

Does this problem have an optimal solution? Explain your answer.

Answer: No, this problem does not have an optimal solution.

To see this, consider any $\varepsilon > 0$. The point $(\varepsilon, 1/\varepsilon)$ is feasible since:

- $\varepsilon \geq 0$ and $1/\varepsilon \geq 0$ (non-negativity constraints satisfied)
- $\varepsilon \cdot (1/\varepsilon) = 1 \geq 1$ (constraint $xy \geq 1$ satisfied)

The objective value at this point is $f(\varepsilon) = \varepsilon$. As $\varepsilon \rightarrow 0$, we have $f(\varepsilon) \rightarrow 0$, which means the infimum of the objective function is 0.

However, the point $x = 0$ is infeasible because if $x = 0$, then the constraint $xy \geq 1$ cannot be satisfied for any $y \geq 0$ (since $0 \cdot y = 0 < 1$).

Since we can make the objective value arbitrarily close to 0 but never actually achieve it with a feasible solution, the minimum is not attained. Therefore, no optimal solution exists.

2. Consider the following optimization problem

$$\begin{array}{ll}\min & (x^2 - 2x + 1)(x^2 + 6x + 9) \\ \text{s.t.} & x \in \mathbb{R}.\end{array}$$

- Find all the global minimum solutions. Explain how you find them.
Hint: there may be multiple ones.
- Is there any local minimum solution that is not a global minimum solution?
- Is the objective function $f(x) = (x^2 - 2x + 1)(x^2 + 6x + 9)$ a convex function on \mathbb{R} ?

Answer:

(a) **Global minimum solutions:**

First, we can factorize the function as $f(x) = (x - 1)^2(x + 3)^2$.

Since both $(x - 1)^2$ and $(x + 3)^2$ are always greater than or equal to zero, the minimum value occurs only when both terms are zero simultaneously.

That happens at $x = 1$ and $x = -3$, where the function value is 0.

Therefore, the global minimum value is 0, achieved at $x = 1$ and $x = -3$.

(b) **Local minimum that is not global:**

Taking the first derivative of $f(x)$, we get:

$$f'(x) = 4(x - 1)(x + 1)(x + 3).$$

From the sign of $f'(x)$, we can see the function decreases before $x = -3$, reaches a minimum at $x = -3$, increases until $x = -1$, reaches a local maximum there, decreases again until $x = 1$, and finally increases for $x > 1$.

Thus, the local minima occur at $x = -3$ and $x = 1$, and since $f(-3) = f(1) = 0$, these are also the global minima.

Therefore, there is no local minimum that is not global.

(c) **Convexity:**

To check whether $f(x)$ is convex, we take the second derivative:

$$f''(x) = 12x^2 + 24x - 4.$$

Looking at the discriminant of this quadratic, we have:

$$\Delta = 24^2 - 4 \cdot 12 \cdot (-4) = 768 > 0,$$

which means $f''(x)$ has two distinct real roots.

Since the sign of $f''(x)$ changes across these roots, the second derivative is not always non-negative.

Therefore, $f(x)$ is not convex over \mathbb{R} .

3. Consider the following optimization problem

$$\begin{aligned} \min \quad & e^x + y^3 \\ \text{s.t.} \quad & x + y \leq 1 \\ & x + 2y \geq 6 \\ & 2x + y \geq 6. \end{aligned}$$

Does this problem have an optimal solution? Explain your answer.

Answer: No, this problem does not have an optimal solution.

First, we analyze the constraints and find that they are mutually inconsistent, resulting in an empty feasible set.

Therefore, the optimization problem is infeasible, and no optimal solution exists.

4. Consider the following problem

$$\begin{aligned} \min \quad & x^2 + f(x) \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

where the function $f(x)$ is defined as

$$f(x) = \begin{cases} x, & -1 < x < 1 \\ 2, & x \in \{-1, 1\} \\ +\infty, & x > 1 \text{ or } x < -1 \end{cases}.$$

- (a) Is the objective function a convex function defined on \mathbb{R} ? Explain your answer by checking the definition of convexity.
- (b) Find an optimal solution, or explain why there is no optimal solution.

Answer:

(a) **Convexity of the objective function:**

The objective function is $g(x) = x^2 + f(x)$ where $f(x)$ is defined as above.

First, consider the domain $-1 < x < 1$ where $f(x) = x$. In this region, $g(x) = x^2 + x$. Taking the second derivative, we get $g''(x) = 2$, which is non-negative. Therefore, $g(x)$ is convex on $(-1, 1)$.

For the other cases, we use Jensen's inequality to check convexity:

Case A: One point is at boundary -1 , the other is an interior point $y \in (-1, 1)$.

For any $\theta \in [0, 1]$, let $z = \theta(-1) + (1 - \theta)y$.

- If $\theta = 1$, then $z = -1$ and $f(z) = 2 \leq \theta \cdot 2 + (1 - \theta)f(y) = 2$, which trivially holds.
- If $\theta \in [0, 1)$, then $z \in (-1, 1)$ (convex combination of -1 and $y > -1$ exceeds -1).

Therefore, $f(z) = g(z)$ and $f(y) = g(y)$, and since $f(-1) = 2 \geq g(-1) = 0$, we have:

$$f(z) = g(z) \tag{1}$$

$$\leq \theta g(-1) + (1 - \theta)g(y) \quad (\text{convexity of } g) \tag{2}$$

$$= (1 - \theta)g(y) \tag{3}$$

$$\leq \theta \cdot 2 + (1 - \theta)g(y) \tag{4}$$

$$= \theta f(-1) + (1 - \theta)f(y). \tag{5}$$

This holds for any $y \in (-1, 1)$ and any $\theta \in [0, 1]$.

Case B: One point is at boundary 1, the other is an interior point $y \in (-1, 1)$.

Using the same argument as above with $f(1) = 2 \geq g(1) = 0$, the inequality holds similarly.

Case C: Both points are at boundaries $(-1$ and $1)$.

For any $\theta \in [0, 1]$, let $z = \theta(-1) + (1 - \theta)1 \in [-1, 1]$.

- If $z = \pm 1$, then $f(z) = 2 = \theta \cdot 2 + (1 - \theta) \cdot 2$ with equality.
- If $z \in (-1, 1)$, then $f(z) = g(z) \leq \theta g(-1) + (1 - \theta)g(1) = 0 \leq \theta \cdot 2 + (1 - \theta) \cdot 2$, so it holds.

Case D: Either point is outside the domain.

The right-hand side becomes $+\infty$, so the inequality holds trivially.

Therefore, the objective function $g(x) = x^2 + f(x)$ is convex on \mathbb{R} .

(b) **Optimal solution:**

To find the optimal solution, we analyze the objective function $g(x) = x^2 + f(x)$ in different regions:

- For $-1 < x < 1$: $g(x) = x^2 + x$. Taking the derivative, we get $g'(x) = 2x + 1$. Setting $g'(x) = 0$, we find $x = -\frac{1}{2}$ as the critical point. The value at this point is $g(-\frac{1}{2}) = (-\frac{1}{2})^2 + (-\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$.
- At the endpoints $x = \pm 1$: $g(\pm 1) = 1 + 2 = 3$.
- For $x > 1$ or $x < -1$: $g(x) = x^2 + \infty = +\infty$.

Therefore, the global minimum value is $-\frac{1}{4}$ achieved at $x = -\frac{1}{2}$.

5. For each of the statements below, state whether it is true or false. Justify your answer.

- (a) If I solve an optimization problem, then remove a constraint and solve it again, the solution must change.

Answer: False.

Counterexample: Consider the problem $\min x^2$ subject to $x \geq 0$ and $x \geq -1$. The optimal solution is $x = 0$. If we remove the constraint $x \geq -1$, the problem becomes $\min x^2$ subject to $x \geq 0$, which still has the same optimal solution $x = 0$. The solution does not change because the removed constraint was not binding at the optimal solution.

- (b) Consider the following optimization problem

$$(P) \quad \begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \geq b_i, \quad \forall i \in I. \end{aligned}$$

Suppose the optimal objective value of (P) is v_P . Then, the Lagrangian dual of (P) is given by

$$(D) \quad \min\{\mathcal{L}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \geq \mathbf{0}\}, \quad (6)$$

where $\mathcal{L}(\boldsymbol{\lambda}) = \max_{\mathbf{x}}\{f(\mathbf{x}) + \sum_{i \in I} \lambda_i(g_i(\mathbf{x}) - b_i)\}$. Furthermore, suppose the optimal objective value of (D) is v_D , then $v_P \leq v_D$.

Answer: False.

The statement is incorrect. In Lagrangian duality, we have the weak duality property: $v_P \leq v_D$ for maximization problems. However, the statement claims $v_P \leq v_D$ which is actually correct for maximization problems. But there's an error in the dual formulation. For a maximization problem with constraints $g_i(\mathbf{x}) \geq b_i$, the correct dual should be:

$$(D) \quad \min\{\mathcal{L}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \geq \mathbf{0}\},$$

where $\mathcal{L}(\boldsymbol{\lambda}) = \max_{\mathbf{x}}\{f(\mathbf{x}) - \sum_{i \in I} \lambda_i(g_i(\mathbf{x}) - b_i)\}$ (note the minus sign, not plus).

The given formulation with the plus sign is incorrect for this type of constraint.

- (c) The following set is convex:

$$\{x \in \mathbb{R}^{10} \mid \|x\|_2 = 1\}$$

Answer: False.

The set $\{x \in \mathbb{R}^{10} \mid \|x\|_2 = 1\}$ is the unit sphere in \mathbb{R}^{10} , which is not convex.

To see this, consider two points on the sphere: $x_1 = (1, 0, 0, \dots, 0)$ and $x_2 = (-1, 0, 0, \dots, 0)$. Both points satisfy $\|x_1\|_2 = \|x_2\|_2 = 1$. However, their convex combination $\frac{1}{2}x_1 + \frac{1}{2}x_2 = (0, 0, 0, \dots, 0)$ has $\|\frac{1}{2}x_1 + \frac{1}{2}x_2\|_2 = 0 \neq 1$, so it does not belong to the set. Therefore, the set is not convex.

- (d) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ and suppose for any real number p , the set:

$$S_p := \{x \in \mathbb{R} \mid f(x) \leq p\},$$

is convex. Then f is a convex function.

Answer: True.

This statement is true by definition. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex if and only if all its sublevel sets $S_p = \{x \in \mathbb{R} \mid f(x) \leq p\}$ are convex for every real number p .

This is a fundamental characterization of convex functions: a function is convex if and only if all its sublevel sets are convex sets. Therefore, if we are given that all sublevel sets S_p are convex, then by definition, f must be a convex function.