# Deterministic Optimization

## Review of Mathematical Concepts

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## Linear Algebra

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# Linear Algebra

## Learning objectives:

- Recall basic concepts from linear algebra

# Vectors

- A vector $\mathbf{x} \in \mathbb{R}^n$ is an $n$-tuple of real numbers. It can be thought of as a point in $n$-dimensional space.

- Typically we will consider column vectors:
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \text{ Then } \mathbf{x}^\top = [x_1, x_2, \cdots, x_n]$$

- Recall standard vector operations $\mathbf{x} + \mathbf{y}$, $\mathbf{x} - \mathbf{y}$ and $\alpha\mathbf{x}$ (here $\alpha \in \mathbb{R}$)

- Typically, the magnitude of a vector will be in the $\ell_2$-norm, i.e. $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

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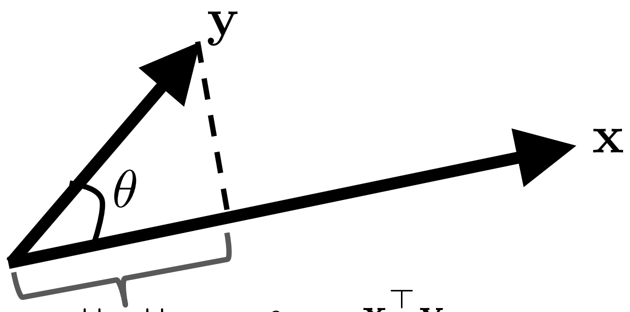
# Scalar Product

- The scalar (or inner or dot) product between two vectors (of the same dimension) is defined as $\mathbf{x}^\top \mathbf{y} = \sum_{j=1}^{n} x_j y_j$.

- Also, $\mathbf{x}^\top \mathbf{y} = ||\mathbf{x}|| \cdot ||\mathbf{y}|| \cos\theta$, where $\theta$ is the angle between the two vectors.

- The vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if $\mathbf{x}^\top \mathbf{y} = 0$, they make an acute angle if $\mathbf{x}^\top \mathbf{y} > 0$ and an obtuse angle if $\mathbf{x}^\top \mathbf{y} < 0$




$$||\mathbf{y}|| \cos\theta = \frac{\mathbf{x}^\top \mathbf{y}}{||\mathbf{x}||}$$

# Linear independence and basis

- A set of vectors $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^K \in \mathbb{R}^n$ are linearly independent if none of the vectors can be written as a linear combination of the others, i.e. the unique solution to the system of equations $\sum_{i=1}^K \alpha_i \mathbf{x}^i = \mathbf{0}$ is $\alpha_i = 0$ for all $i = 1, \dots, K$.

- Fact: There can be at most $n$ linearly independent vectors in $\mathbb{R}^n$.

- Any collection of $n$ linearly independent vectors in $\mathbb{R}^n$ defines a basis (or a coordinate system) of $\mathbb{R}^n$, i.e. any vector in $\mathbb{R}^n$ can written as a linear combination of the basis vectors.

- The unit vectors $\mathbf{e}^1 = [1, 0, \dots, 0]^\top, \mathbf{e}^2 = [0, 1, \dots, 0]^\top, \dots, \mathbf{e}^n = [0, 0, \dots, 1]^\top$ define the standard basis for $\mathbb{R}^n$.

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# Matrices

- A $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$ is a rectangular array of $mn$ number with $m$ rows and $n$ columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Alternatively, $A$ is a collection of $n$ column vectors $A_{\cdot 1}, \ldots, A_{\cdot n} \in \mathbb{R}^m$ or a collection of $m$ row vectors $A_{1\cdot}, \ldots, A_{m\cdot} \in \mathbb{R}^n$, i.e.

$$A = [A_{\cdot 1}, \ldots, A_{\cdot n}] \text{ or } A = \begin{bmatrix} A_{1\cdot}^\top \\ \vdots \\ A_{m\cdot}^\top \end{bmatrix}$$

# Matrix operations

- Let $A, B \in \mathbb{R}^{m\times n}$ and $C = A \pm B$, then $c_{ij} = a_{ij} \pm b_{ij}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
- Let $A \in \mathbb{R}^{m\times k}$, $B \in \mathbb{R}^{k\times n}$, and $C = AB$, then $c_{ij} = \sum_{\ell=1}^{k} a_{i\ell} b_{\ell j}$.
- Let $A \in \mathbb{R}^{n\times n}$ be a square matrix, then its inverse (if it exists) is a matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that $AA^{-1} = I$ where $I$ is the identity matrix (i.e. with zeros everywhere except the diagonal entries).
- Recall determinant of a matrix, i.e. $\det(A)$ or $|A|$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}, \qquad \text{and}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

# Matrix vector multiplication

Let $A \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{y} = A\mathbf{x}$. Then

- $\mathbf{y}$ is a linear combination of the columns of $A$, i.e. $\mathbf{y} = \sum_{j=1}^{n} A_{\cdot j} x_j$.
- The $i$-th row of $\mathbf{y}$ is the scalar product of the $i$-th row of A and $\mathbf{x}$, i.e.

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} A_{1\cdot}^\top \mathbf{x} \\ \vdots \\ A_{m\cdot}^\top \mathbf{x} \end{bmatrix}$$

# Rank

Let $A \in \mathbb{R}^{m \times n}$. Then

- The row rank of $A$ is the maximum number of linearly independent rows in $A$
- The column rank of $A$ is the maximum number of linearly independent columns in $A$
- Fact: row rank $=$ column rank $= \text{rank}(A)$
- $A$ is full rank if $\text{rank}(A) = \min\{m, n\}$

# Systems of Equations

Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Suppose that $A$ is full rank. Consider the system of $m$ equations $A\mathbf{x} = \mathbf{b}$ in the $n$ unknowns given by $\mathbf{x} \in \mathbb{R}^n$.

- If $m > n$, then the system has no solution.
- If $m = n$, then the system has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
- If $m < n$, then the system has an infinite number of solutions.

# Summary

- We reviewed some basic concepts from linear algebra
- Additional concepts will be introduced as needed