# ISyE 6669 HW 1

Fall 2025

1. Consider the following maximization problem

$$\begin{aligned} \max \quad & x^2 + (y-1)^2 \\ \text{s.t.} \quad & x + 2y \leq 6 \\ & x - y \leq 0 \\ & x \geq 0,\ y \geq 0. \end{aligned}$$

Plot the feasible region of this problem with the feasible area shaded. Draw (in dashed lines) the contours of the objective function. Based on your drawing, find all the optimal solutions and the optimal objective value of this problem. There may be multiple optimal solutions. Find all optimal solutions.

**Solution:** To find the optimal solution, we need to graph contours of the objective against the feasible region. All points on a single contour take on the same objective values. In this case we need to graph the circles $x^2 + (y-1)^2 = c$ for various scalars $c$. We start off by graphing $x^2 + (y-1)^2 = c$, against the feasible region, for small values of $c$. We gradually increase $c$, until the contour $x^2 + (y-1)^2 = c$ no longer intersects the feasible region. The last contour intersecting the feasible region indicates the largest objective value of any point in the feasible region. Hence, it indicates the optimal solution of this problem. Figure 1 shows that last contour intersecting the feasible region is $x^2 + (y-1)^2 = 5$ and it intersects the feasible region at $(2, 2)$. Thus, the optimal solution is $(2, 2)$ with optimal objective value of 5.

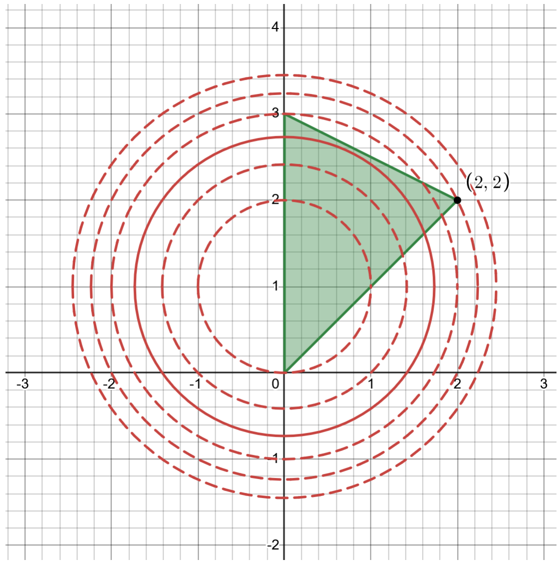

Figure 1: Feasible region and contours

2. Solve the following problem using basic calculus:

$$\max\{-10x + 5x^2 + 9x^3 + 8x^4 + 7x^5 : -1 \le x \le 1\}$$

What is the optimal solution and the optimal objective value? Are there any local maxima that are not global maxima?

**Solution:**

First derivative of the objective function $f(x) = -10x+5x^2 + 9x^3 + 8x^4 + 7x^5$ is:

$$f'(x) = -10 + 10x + 27x^2 + 32x^3 + 35x^4.$$

By solving the equation:

$$f'(x) = 0, \ -1 \le x \le 1,$$

we get solutions:

$$x_1^* = -0.8570,$$
$$x_2^* = 0.3764.$$

Now we need to analyze sign of the first derivative on intervals $(-1, -0.857)$, $(-0.857, 0.3764)$ and $(0.3764, 1)$ :

- We can select any point from interval $(-1, -0.857)$, for example $x = -0.9$ and calculate $f'(-0.9) = 2.5055 \ge 0$. From there, we know that for $x \in (-1, 0.8570)$ we have $f'(x) > 0$, and hence for $x \in (-1, 0.8570)$ function $f(x)$ is increasing.

- By selecting point $x = 0 \in (-0.857, 0.3764)$ and calculating $f'(0) = -10$, we conclude that for $x \in (-0.857, 0.3764)$ we have $f'(x) < 0$. Thus for $x \in (-0.857, 0.3764)$ function $f(x)$ is decreasing.
- By selecting point $x = 0.5 \in (0.3764, 1)$ and calculating $f'(0.5) = 7.9375$, we conclude that for $x \in (0.3764, 1)$ we have $f'(x) > 0$. Thus for $x \in (0.3764, 1)$ function $f(x)$ is increasing.

Based on the above we conclude that function $f(x)$, $-1 \leq x \leq 1$ has two local maxima at points $x = -0.857$ and $x = 1$. By evaluating function at those two values:

$$
\begin{aligned}
f(-0.857) &= 7.6568 \\
f(1) &= 19
\end{aligned}
$$

Finally we can conclude that optimal solution is $x = 1$ with optimal value equal to 19 (global maximum). Function also has a local maximum at point $x = 0.857$ with value of 7.6568.

We can verify all this by looking at the graph of the function:

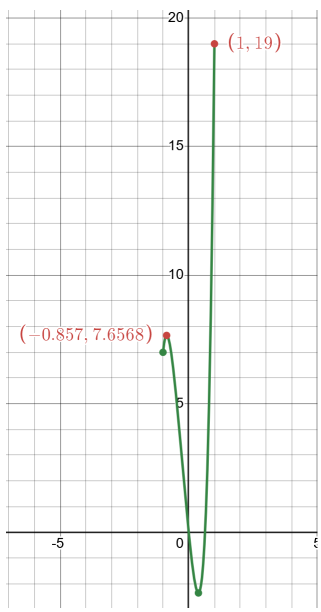


Figure 2: Graph of the function $f(x) = -10x + 5x^2 + 9x^3 + 8x^4 + 7x^5$

3. Consider the following optimization problem:

$$
\begin{aligned}
(P) \quad \max \quad & x(z^2 - y^2) \\
\text{s.t.} \quad & y + |z| \leq 1, \\
& x \in \{0, 1\}, \; y \geq 0.
\end{aligned}
$$

Answer the questions:

(a) Is (P) a linear program, a mixed integer nonlinear program, or a mixed integer quadratic program? Choose all descriptions that apply.

(b) Write a minimization problem that is equivalent to (P).

(c) Find all the optimal solutions.

**Solution:**

(a) (P) is a mixed integer nonlinear program.

(b) The following is equivalent to (P)

$$\begin{aligned} \min \quad & x(y^2 - z^2) \\ \text{s.t.} \quad & y + |z| \leq 1, \\ & x \in \{0, 1\}, \; y \geq 0. \end{aligned}$$

(c) First note that from constraints $y + |z| \leq 1$ and $y \geq 0$ we have $0 \leq y \leq 1 - |z|$ and thus both $y$ and $z$ are bounded i.e. finite values. Thus difference $y^2 - z^2$ is also a finite number. Since $x$ is independent of $y$ and $z$, and binary the two factors can be maximized independently. If $x = 0$ objective function becomes zero. If x = 1: problem becomes:

$$\begin{aligned} \max \quad & z^2 - y^2 \\ \text{s.t.} \quad & 0 \leq y + |z| \leq 1, \end{aligned}$$

Objective value $x^2 - y^2$ is largest for smallest value of $y$, hence $y = 0$. Now we have:

$$\begin{aligned} \max \quad & z^2 \\ \text{s.t.} \quad & |z| \leq 1. \end{aligned}$$

and hence $z = \pm 1$. Optimal objective value in this case is 1. Thus, optimal solutions are:

$$\{(1, 0, 1), (1, 0, -1)\}.$$

4. Recall the portfolio optimization problem solved in Module 2, Lesson 3. Use the provided code file (`portopt_cvxpy_python3_HW1.py`) and provided data file `monthly_prices_HW1.csv` to solve the exact same portfolio problem using this new data. Compare and contrast this new solution to the one obtained in the lesson.

**Solution:**
The output is shown below.

```
----------------------
MSFT: Exp ret = 0.024328, Risk = 0.062160
V: Exp ret = 0.019058, Risk = 0.040485
WMT: Exp ret = 0.030167, Risk = 0.056756
----------------------
Optimal portfolio
----------------------
x[MSFT] = 0.217135
x[V] = 0.581500
x[WMT] = 0.201364
----------------------
Exp ret = 0.022439
risk = 0.034429
----------------------
```

Any reasonable comparison should be awarded full points.