Deterministic Optimization

Outcomes of Optimization

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Local and Global Optimal Solutions

Local and Global Optimal Solutions

Learning objectives:

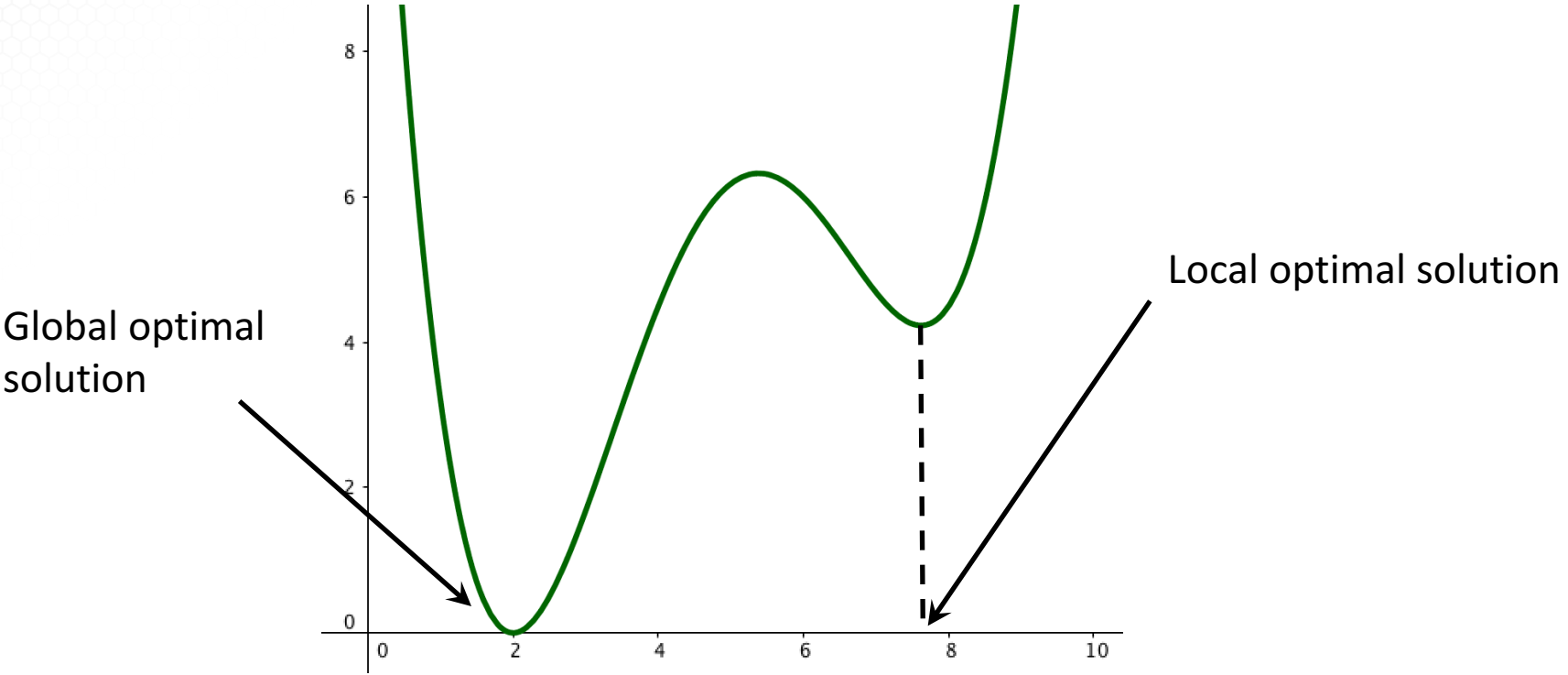
- Differentiate between local and global optimal solutions
- Recognize that local optimal solutions are also globally optimal for convex optimization problems

Global and Local Optimal Solutions

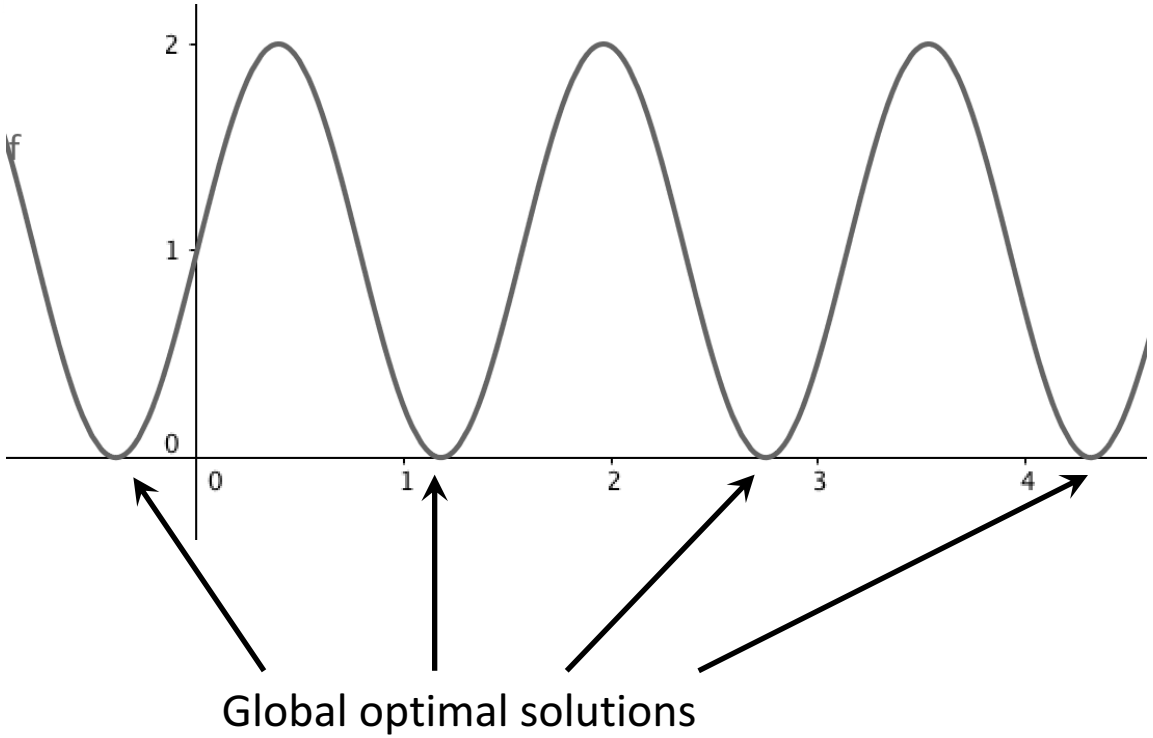
$$(P): \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

- A solution $\mathbf{x}^*$ is a **global** optimal solution of (P) if $\mathbf{x}^* \in X$ and $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$.
- Given a solution $\mathbf{x}^*$ and scalar $\epsilon > 0$, an ϵ-neighborhood of $\mathbf{x}^*$ is defined as $\mathbb{B}_\epsilon(\mathbf{x}^*) = \{\mathbf{x} : ||\mathbf{x} - \mathbf{x}^*|| \leq \epsilon\}$.
- A solution $\mathbf{x}^*$ is a **local** optimal solution of (P) if $\mathbf{x}^* \in X$ and if there is an $\epsilon > 0$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X \cap \mathbb{B}_\epsilon(\mathbf{x}^*)$.

Illustration

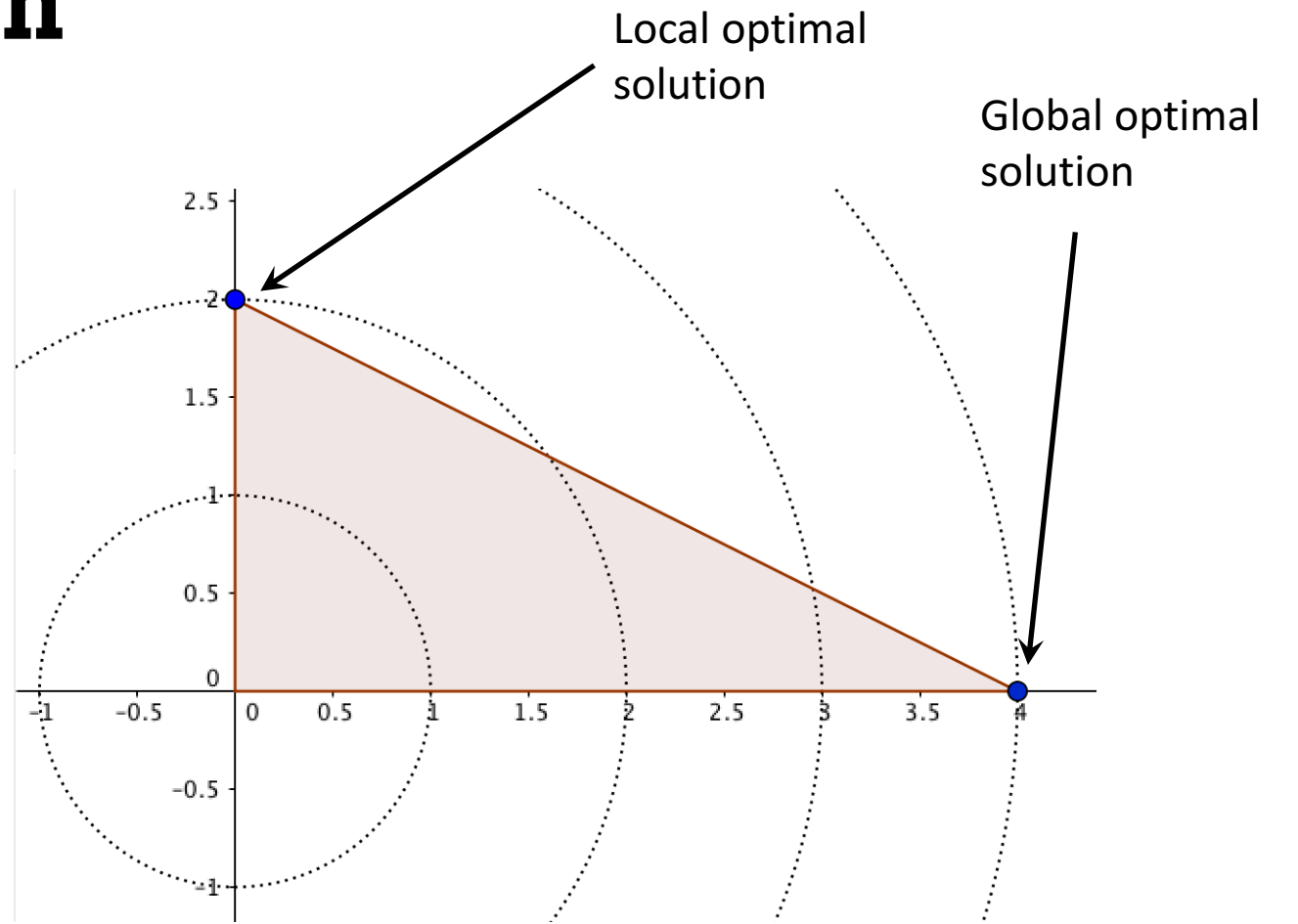


Illustration



Illustration

$$\begin{aligned} \min \quad & -x^2 - y^2 \\ \text{s.t.} \quad & x + 2y \leq 4 \\ & x \geq 0,\ y \geq 0 \end{aligned}$$



Convex Optimization Problems

$$(P): \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

If (P) is a convex optimization problem,
i.e. f is a convex function and X is a convex set,
then any local optimal solution (P) is also a
global optimal solution.

Proof (in one dimension)

Remarks

- Every global optimal solution is a local optimal solution, but not vice versa
- The objective function value at different local optimal solutions may be different
- The objective function value at all global optimal solutions must be the same
- Typical optimization algorithms are designed to find local optimal solutions (at best)
- If the problem is convex, then we are sure that any local optimal solution we find is also a global optimal solution

Summary

- A solution is locally optimal if it is optimal within a local neighborhood of the solution
- A solution is globally optimal if is optimal with respect to all feasible solutions
- For convex optimization problems, any local optimal solution is also a global optimal solution