Deterministic Optimization

Convexity

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Convex Sets

Convex Sets

Learning objectives:

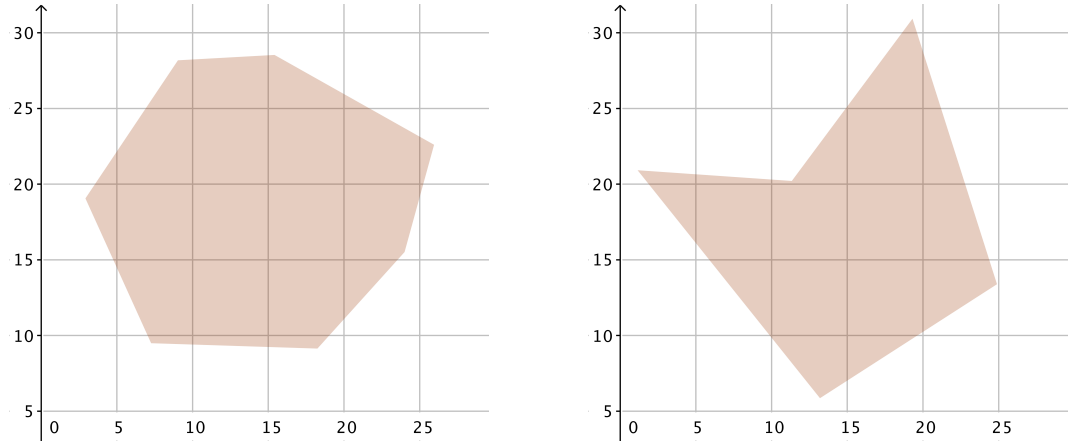
- Recall definition of a convex set
- Recall convexity preserving set operations
- Recognize connection between convex sets and functions

Convex Set

- A set $X \in \mathbb{R}^n$ is **convex** if

$$\forall\ \mathbf{x}, \mathbf{y} \in X \text{ and } \lambda \in [0, 1] \text{ it holds } (\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \in X$$

- A line segment connecting two points in the set lies entirely in the set.



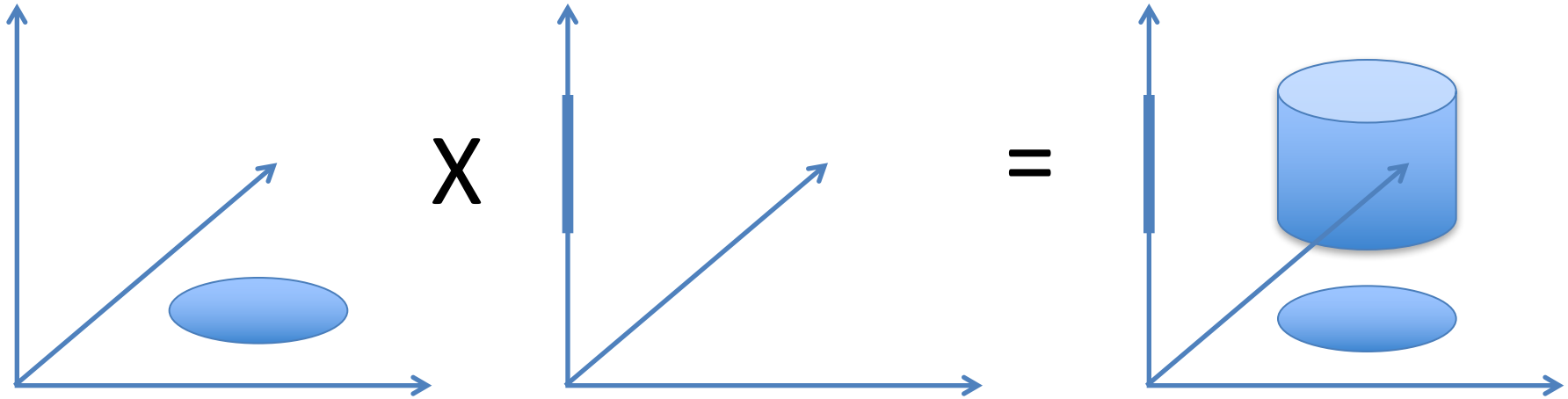
Convexity Preserving Set Operations

- Intersection: If $X_1, \ldots, X_m$ are convex sets, the $\cap_{i=1}^m X_i$ is a convex set.
- Union of two convex sets is not convex in general.
- Sum: Sum of two sets $X, Y \subseteq \mathbb{R}^n$ is defined as $X + Y = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in X, \mathbf{y} \in Y\}$. Sum of convex sets is convex.
- Product: Given two sets $X \subseteq \mathbb{R}^{n_1}$ and $Y \subseteq \mathbb{R}^{n_2}$, their (Cartesian) product $X \times Y$ is a set in $\mathbb{R}^{n_1+n_2}$ and is defined as

$$X \times Y = \{[\mathbf{x}^\top, \mathbf{y}^\top]^\top : \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

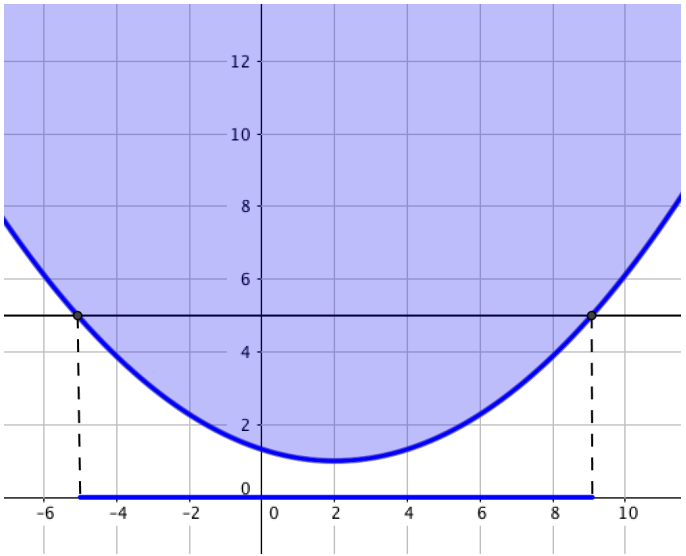
  Product of convex sets is convex.

Set Product



Connection to Convex Functions

- The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a set defined as
  $\text{epi} f = \{(y, \mathbf{x}) \in \mathbb{R}^{n+1} : \ y \geq f(\mathbf{x})\}$
- Given scalar α, the α-level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a set defined as:
  $X_\alpha = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq \alpha\}$
- The epigraph and the α-level set (for any $\alpha \in \mathbb{R}$) of a convex function are convex sets.



Summary

- A set is convex is the line segment connecting any two points in the set lies entirely in the set.
- Intersection, addition and product of convex sets is convex.
- Epigraph and level sets of convex functions are convex sets.