ISyE 6669 HW 3

1. Consider the following optimization problem:

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & xy \geq 1 \\ & x \geq 0, y \geq 0 \end{aligned}$$

Does this problem have an optimal solution? Explain your answer.

Solution: Problem is feasible (for example point $(1,1)$ is feasible point for the problem) and bounded (0 is lower bound of the problem), but it does not have a solution. To show this, lets assume opposite: that problem has a solution and lets denote that solution as (x^*, y^*). Then $(x^*, y^*) \in X$ where:

$$X = \{(x, y) \in \mathbb{R}^2 : xy \geq 1,\ x \geq 0,\ y \geq 0\}$$

and for all $(x, y) \in X$:

$$x^* \leq x.$$

Let's consider point $(x = x^*/2, y = 2y^*)$. Since $(x^*, y^*) \in X$, we have $(x^*/2, 2y^*) \in X$ and $x^*/2 < x^*$. So now we have:

$$x^* \leq x^*/2 < x^*$$

which is contradiction. Thus problem does not have a solution.

2. Consider the following optimization problem

$$\begin{aligned} \min \quad & (x^2 - 2x + 1)(x^2 + 6x + 9) \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned}$$

(a) Find all the global minimum solutions. Explain how you find them. Hint: there may be multiple ones.

(b) Is there any local minimum solution that is not a global minimum solution?

(c) Is the objective function $f(x) = (x^2 - 2x + 1)(x^2 + 6x + 9)$ a convex function on $\mathbb{R}$?

Solution:

(a) Since, for all $x \in \mathbb{R}$ we have:

$$\begin{aligned} f(x) &= (x^2 - 2x + 1)(x^2 + 6x + 9) = \\ &= (x-1)^2(x+3)^2 \geq 0, \end{aligned}$$

and $f(1) = f(-3) = 0$, global minimum is reached at points $x = 1$ and $x = -3$, and global optimal value is equal to 0.

(b) Since $f'(x) = 4(x-1)(x+3)(x+1)$, we know that function $f(x)$ is monotonically decreasing for $x \in (-\infty, -3) \cup (-1, 1)$ and monotonically increasing for $x \in (-3, -1) \cup (1, +\infty)$. Hence function has one local maximum $x = -1$, but all of its local minimums are global.

(c) Function is not convex. For example if we consider points $a = -3$ and $b = 1$ and $\lambda = 1/2$, then:

$$\begin{aligned} c &= \lambda a + (1-\lambda)b = \\ &= \frac{1}{2}(-3) + \left(1 - \frac{1}{2}\right)(1) = -1. \end{aligned}$$

However:

$$\begin{aligned} 16 = f(-1) = f(c) &= f(\lambda a + (1-\lambda)b) \leq \\ &\leq \lambda f(a) + (1-\lambda) f(b) = \\ &= \frac{1}{2} f(-3) + \left(1 - \frac{1}{2}\right) f(1) = \\ &= 0 \end{aligned}$$

is obviously not true.

All of the above can be seen by graphing the function:

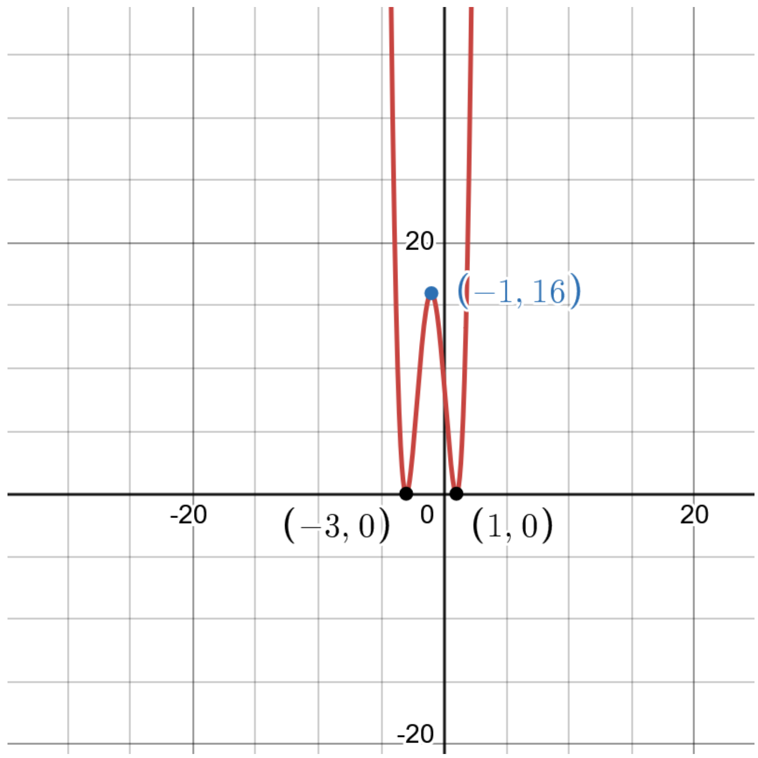

Figure 1: Graph of a function $f(x) = (x^2 - 2x + 1)(x^2 + 6x + 9)$ for $x \in \mathbb{R}$

3. Consider the following optimization problem

$$\begin{aligned} \min \quad & e^x + y^3 \\ \text{s.t.} \quad & x + y \leq 1 \\ & x + 2y \geq 6 \\ & 2x + y \geq 6. \end{aligned}$$

Does this problem have an optimal solution? Explain your answer.

Solution: Problem does not have an optimal solution because it is infeasible. By adding the last two constraints we get:

$$3x + 3y \geq 12 \quad \Leftrightarrow \quad x + y \geq 4$$

which contradicts the first constraint $x + y \leq 1$. Hence, feasible set is empty.

4. Consider the following problem

$$\begin{aligned} \min \quad & x^2 + f(x) \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

where the function $f(x)$ is defined as

$$f(x) = \begin{cases} x, & -1 < x < 1 \\ 2, & x \in \{-1, 1\} \\ +\infty, & x > 1 \text{ or } x < -1 \end{cases}.$$

(a) Is the objective function a convex function defined on $\mathbb{R}$? Explain your answer by checking the definition of convexity.

(b) Find an optimal solution, or explain why there is no optimal solution.

Solution:

(a) To prove that function $f(x)$ is convex we will used the definition and consider several different possibilities:

i. If $a, b \in (-1, 1)$ then, $f(a) = a,\ f(b) = b$, and for any $\lambda \in (0, 1)$ we have $\lambda a + (1 - \lambda)b \in (-1, 1)$. Hence:

$$f(\lambda a + (1 - \lambda)b) = \lambda a + (1 - \lambda)b = \lambda f(a) + (1 - \lambda)f(b)$$

ii. If $a = -1$ and $b \in (-1, 1)$ then $f(a) = 2,\ f(b) = b$, and for any $\lambda \in (0, 1)$ we have $\lambda a + (1-\lambda)b = -\lambda + (1-\lambda)b \in (-1, 1)$. Hence:

$$f(-\lambda+(1-\lambda)b) = -\lambda+(1-\lambda)b \leq 2\lambda+(1-\lambda)b = \lambda f(-1)+(1-\lambda)f(b)$$

iii. If $a \in (-1, 1)$ and $b = 1$, then $f(a) = 1,\ f(b) = 2$, and for any $\lambda \in (0, 1)$ we have $\lambda a + (1 - \lambda)b = \lambda a + 1 - \lambda \in (-1, 1)$. Hence:

$$f(\lambda a+1-\lambda) = \lambda a+1-\lambda \leq \lambda a+2(1-\lambda) = \lambda f(a)+(1-\lambda)f(b)$$

iv. If at least one of the points a, b belongs to $(-\infty, -1) \cup (1, +\infty)$ then $\lambda f(x) + (1 - \lambda)f(b) = +\infty$. From the fact that for any $y \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ we have $y \leq +\infty$ we get:

$$f(\lambda a + (1 - \lambda)b) \leq +\infty = \lambda f(x) + (1 - \lambda)f(b)$$

We've shown that any two points $a, b \in \mathbb{R}$ and for any $\lambda \in (0, 1)$ inequality:

$$f(\lambda a + (a - \lambda)b) \leq \lambda f(a) + (a - \lambda)f(b)$$

holds. Thus function $f(x),\ x \in \mathbb{R}$ is convex, and as stated before this proves that objective function $h(x) = g(x) + f(x),\ x \in \mathbb{R}$ is also convex.

(b) Objective function $h(x) = x^2 + f(x),\ x \in \mathbb{R}$ is a convex function because it is a sum of two convex functions, $g(x) = x^2,\ x \in \mathbb{R}$ and $f(x),\ x \in \mathbb{R}$.

(c) We are looking for the minimum of the objective function:

$$h(x) = g(x) + f(x) = \begin{cases} x^2 + x, & x \in (-1, 1) \\ 3, & x \in \{-1, 1\} \\ +\infty, & x \in (-\infty, -1) \cup (1, +\infty) \end{cases}$$

Since:

$$x^2 + x \leq 3 \leq +\infty, \quad x \in (-1, 1)$$

we can look for the solution by considering only $x \in (-1, 1)$. For $x \in (-1, 1)$ the objective is the quadratic function $x^2 + x$ whose minimum is equal to $-1/4$ and it is attained for $x = -1/2$.

5. For each of the statements below, state whether it is true or false. Justify your answer.

(a) If I solve an optimization problem, then remove a constraint and solve it again, the solution must change.

Solution: False.

Consider problem $\min\{(x-1)^2 : 0 \leq x \leq 4\}$. We know that its solution is $x = 1$. Now let's remove constraint $x \geq 0$. Problem now becomes $\min\{(x-1)^2 : x \leq 4\}$, and solution is still the same, $x = 1$.

(b) Consider the following optimization problem

$$\begin{aligned} (P) \quad & \max \quad f(\boldsymbol{x}) \\ & \text{s.t.} \quad g_i(\boldsymbol{x}) \geq b_i, \ \ \forall i \in I. \end{aligned}$$

Suppose the optimal objective value of (P) is v_P. Then, the Lagrangian dual of (P) is given by

$$(D) \quad \min\{\mathcal{L}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{1}$$

where $\mathcal{L}(\boldsymbol{\lambda}) = \max_{\boldsymbol{x}}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i(g_i(x) - b_i)\}$. Furthermore, suppose the optimal objective value of (D) is v_D, then $v_P \leq v_D$.

Solution: True.

Let $S = \{x : g_i(x) \geq b_i, \forall i \in I\}$. Let $x^* \in S$ be the optimal solution of (P). Thus, $v_P = f(x^*)$. Also, let $\lambda^* \geq 0$ be the optimal solution of (D) and so $v_D = \mathcal{L}(\lambda^*)$. Now, for any $\lambda \geq 0$,

$$\begin{aligned} \mathcal{L}(\lambda) &= \max_{\boldsymbol{x}}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i(g_i(x) - b_i)\} && (2) \\ &\geq \max_{\boldsymbol{x} \in S}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i(g_i(x) - b_i)\} && (3) \\ &\geq \max_{\boldsymbol{x} \in S}\{f(\boldsymbol{x})\} && (4) \\ &= v_P && (5) \end{aligned}$$

where Eq.(3) follows because the feasible space is being restricted and Eq.(4) follows because $\lambda \geq 0$ and $x \in S$ imply that $\lambda_i(g_i(x) - b_i) \geq 0$ for all $i \in I$. Since this holds for all $\lambda \geq 0$, it also holds for λ^*. Therefore, $v_D = \mathcal{L}(\lambda^*) \geq v_P$.

(c) The following set is convex:

$$\{x \in \mathbb{R}^{10} \,|\, \|x\|_2 = 1\}$$

Solution: False.

Consider points $a = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and $b = (-1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. Both points belong to the set $\{x \in \mathbb{R}^{10} : \|x\|_2 = 1\}$. However, for $\lambda = 1/2$, point:

$$c = \lambda a + (1 - \lambda)b = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

does not belong to the same set.

(d) Suppose $f : \mathbb{R} \to \mathbb{R}$ and suppose for any real number p, the set:

$$S_p := \{x \in \mathbb{R} \,|\, f(x) \leq p\},$$

is convex. Then f is a convex function.

Solution: False.
Consider the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$. This function is not convex. However, for every $p \in \mathbb{R}$, set:

$$\begin{aligned} S_p &= \{x \in \mathbb{R} : f(x) \leq p\} = \\ &= \{x \in \mathbb{R} : f(x) \leq p\} = \\ &= (-\infty, \sqrt[3]{p}] \end{aligned}$$

is convex.