Deterministic Optimization

Review of Mathematical Concepts

Shabbir Ahmed
Anderson-Interface Chair and Professor
School of Industrial and Systems Engineering

Properties of functions

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Properties of Functions

Learning objective:

- Recall basic concepts for multivariate functions

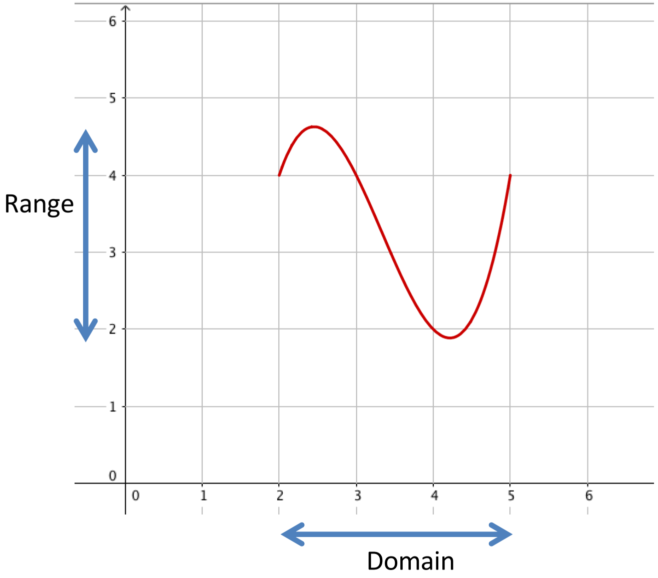
Functions

- A function $f : D \to R$ takes as an argument an element of its domain D and returns an element of its range R.

- We are interested in multivariate real valued functions whose arguments are vectors, i.e. domain $D \subseteq \mathbb{R}^n$, and whose range is a real number, i.e. $R \subseteq \mathbb{R}$.

- Examples:

  - o $f(\mathbf{x}) = \sum_{j=1}^{n} |x_j|$, here $D = \mathbb{R}^n$ and $R = \mathbb{R}_+$.
  - o $f(x) = \log x$, here $D = \mathbb{R}_{++}$ and $R = \mathbb{R}$.



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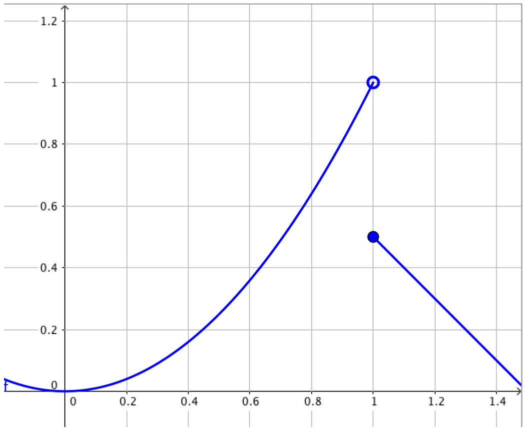
Continuity

- A function $f : D \to R$ is continuous at a point $\mathbf{x}^0 \in D$, if for any sequence $\{\mathbf{x}^i\}$ such that $\lim_{i\to\infty} \mathbf{x}^i = \mathbf{x}^0$, it holds $\lim_{i\to\infty} f(\mathbf{x}^i) = f(\mathbf{x}^0)$.

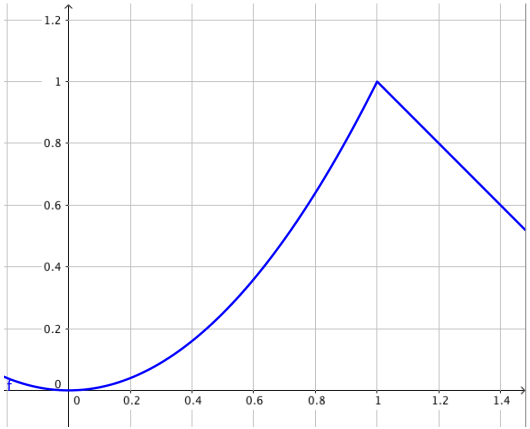
- A function is continuous if it is continuous at every point in its domain.

- Roughly, a function is continuous if it does not have any "jumps".

Not continuous



Continuous



Differentiability

- A univariate function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x_0 if the derivative
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$
exists.

- A multivariate function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{x}^0$ if all partial derivatives
$$\left.\frac{\partial f(\mathbf{x})}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}^0} = \lim_{h \to 0} \frac{f(\mathbf{x}^0 + h\mathbf{e}^j) - f(\mathbf{x}^0)}{h} \quad j = 1, \ldots, n$$
exists and are continuous.

Differentiability

- A function f is differentiable if it is differentiable at every point in its domain.

- The gradient of f at $\mathbf{x}^0$ is a vector of the n-partial derivatives:

$$\nabla f(\mathbf{x}^0) = \left[\left. \frac{\partial f(\mathbf{x})}{\partial x_1} \right|_{\mathbf{x}=\mathbf{x}^0}, \left. \frac{\partial f(\mathbf{x})}{\partial x_2} \right|_{\mathbf{x}=\mathbf{x}^0}, \ldots, \left. \frac{\partial f(\mathbf{x})}{\partial x_n} \right|_{\mathbf{x}=\mathbf{x}^0} \right]^\top$$

- Examples:
  - The function $|x|$ is not differentiable at 0.
  - The gradient of $f(x, y) = x^2 + y$ at $(1, 0)$ is $\nabla f(1, 0) = [2, 1]^\top$.

Hessian

The Hessian of a (twice) differentiable function f at a point $\mathbf{x}^0$ is an $n \times n$ matrix of second-order partial derivatives, i.e.

$$\nabla^2 f(\mathbf{x}^0) = \begin{bmatrix} \left.\frac{\partial^2 f(\mathbf{x})}{\partial^2 x_1}\right|_{\mathbf{x}=\mathbf{x}^0} & \left.\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2}\right|_{\mathbf{x}=\mathbf{x}^0} & \cdots & \left.\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n}\right|_{\mathbf{x}=\mathbf{x}^0} \\ \left.\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2}\right|_{\mathbf{x}=\mathbf{x}^0} & \left.\frac{\partial^2 f(\mathbf{x})}{\partial^2 x_2}\right|_{\mathbf{x}=\mathbf{x}^0} & \cdots & \left.\frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n}\right|_{\mathbf{x}=\mathbf{x}^0} \\ \vdots & \vdots & \ddots & \vdots \\ \left.\frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n}\right|_{\mathbf{x}=\mathbf{x}^0} & \left.\frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n}\right|_{\mathbf{x}=\mathbf{x}^0} & \cdots & \left.\frac{\partial^2 f(\mathbf{x})}{\partial^2 x_n}\right|_{\mathbf{x}=\mathbf{x}^0} \end{bmatrix}$$

Taylor's Approximation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $\mathbf{x}^0 \in \mathbb{R}^n$.

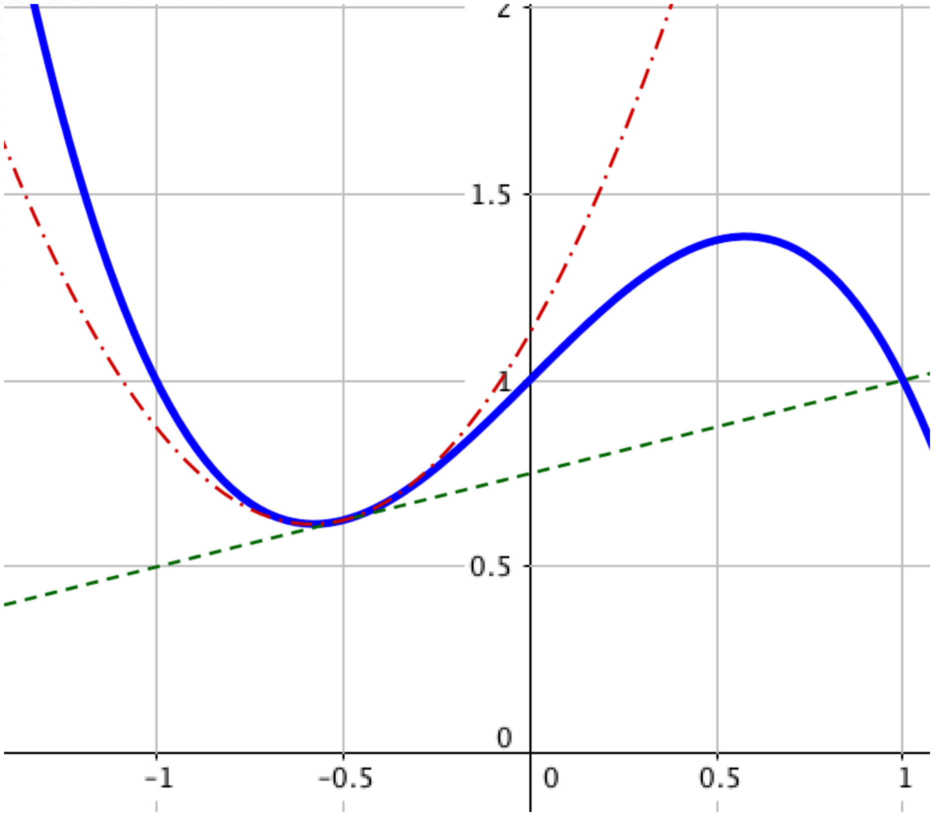
- First order Taylor's approximation of f at $\mathbf{x}^0$:

$$f(\mathbf{x}) \approx f(\mathbf{x}^0) + \nabla f(\mathbf{x}^0)^\top (\mathbf{x} - \mathbf{x}^0)$$

- Second order Taylor's approximation of f at $\mathbf{x}^0$:

$$f(\mathbf{x}) \approx f(\mathbf{x}^0) + \nabla f(\mathbf{x}^0)^\top (\mathbf{x} - \mathbf{x}^0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^0)^\top \nabla^2 f(\mathbf{x}^0)(\mathbf{x} - \mathbf{x}^0)$$

Example



$$f(x) = 1 + x - x^3$$

$$f'(x) = 1 - 3x^2$$

$$f''(x) = -6x$$

First order Taylor's approximation at $x = -0.5$ is $0.25x + 0.75$

Second order Taylor's approximation at $x = -0.5$ is $1.5x^2 + 1.75x + 1.125$

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Summary

- We reviewed some basic concepts about functions
- Make sure to verify the example