Deterministic Optimization

Optimality Certificates

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Optimality Certificates and Relaxations

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Learning objectives:

- Recognize optimality certificates
- Examine relaxations of optimization problems

What is a “certificate”?

- Recall that optimization algorithms typically search for an optimal solution (e.g. by moving from one solution to another)
- An important question is how to know when an optimal solution or a “near-optimal” solution has been found and the search can stop
- An **certificate** or **a stopping condition** is an <u>easily checkable</u> condition such that if the current solution satisfies this condition then it is <u>guaranteed</u> to be optimal or near optimal
- Then the algorithm can check the condition every time it finds a new solution and stop when it is satisfied

Example 1

- Consider the following optimization problem

$$\min_{x,y} \{e^{(x^2-4x+y^3+4)^2}\}$$

- Suppose we have found a solution x=1.0, y=0.2. Is it optimal?
- The objective value of this solution is around 1.002
- But the least possible value the objective function can take is 1
- We do not know if this solution is optimal or not
- But we know that it is off by at most 0.2% from being optimal

Example 1

- Consider the same problem

$$\min_{x,y} \left\{ e^{(x^2-4x+y^3+4)^2} \right\}$$

- Is the solution x=2, y=0 optimal?
- Note that the objective value of this solution is 1.
- So then this solution (x=2,y=0) must be optimal!

Lower Bound

- In the previous example, we knew (a priori) that the objective value of any solution to the problem cannot be lower than 1.0.
- Thus we could compare the objective of any given solution to this lower bound
- If the solution has an objective close to this lower bound then we know we found a (near)-optimal solution
- Thus the lower bound of 1.0 is an easily checkable certificate

Optimality Gap

- Suppose we have a feasible solution x' to an optimization problem with an objective value of $f(x')$
- Suppose the optimal objective value of the problem is v^*
- Then the (absolute) optimality gap of the solution x' is :

$$gap(x') = f(x') - v^*$$

- If $|v^*| > 0$, then the relative optimality gap is:

$$rgap(x') = (f(x')-v^*)/|v^*|$$

- Note that gap and rgap are always nonnegative

Optimality Gap (contd.)

- But we do not know v^*
- Suppose we know a lower bound $L \leq v^*$
- Then the following holds:

$$L \leq v^* \leq f(x')$$

- Thus:

$$0 \leq gap(x') = f(x') - v^* \leq f(x') - L$$

- If $L > 0$ then:

$$rgap(x') = [f(x')-v^*]/v^* \leq [f(x')-L]/L$$

- Thus a lower bound allows us to get an upper bound on the gap

Example 2

- Consider the following example:

$$\begin{aligned} \min \quad & 2x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & -x_1 + x_2 \geq 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

- Consider the solution x1 = 0.5, x2 = 0.5
- Is it feasible? Easy to check
- Is it optimal? Not so easy to check
- Objective value of this solution is 3
- Need to find a lower bound to get an optimality gap

Example 2 (contd.)

$$\begin{array}{ll} \min & 2x_1 + 4x_2 \\ \text{s.t.} & x_1 + x_2 \geq 1 \\ & -x_1 + x_2 \geq 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

- Clearly 0 is a lower bound
- Is there a better one?
- Look at the first constraint, this implies that 2 is a lower bound, so the rel. optimality gap is at most 50%
- In fact, by considering the first and second constraints, we see that 3 is a lower bound, so the solution (x1=0.5, x2=0.5) is optimal

Relaxation

$$(P) : \min_x \{f(x) : x \in X\} \qquad (Q) : \min_x \{g(x) : x \in Y\}$$

Problem (Q) is a **relaxation** of (P) if

- $X \subseteq Y$
- $f(x) \geq g(x) \ \ \forall\, x \in X$

Relaxation and Lower Bound

- The relaxation of an optimization problem should be easier to solve
- Optimal value of the relaxation provides a lower bound on the original problem
- If the relaxation is infeasible then clearly the original problem is also infeasible
- Suppose only the constraints are relaxed, then if a solution to the relaxation is feasible to the original problem then it must be an optimal solution to the original problem

Examples

Summary

- A lower bound on the optimal value provides a way to certify the quality of a given solution
- The optimal value of relaxation of an optimization problem provides a lower bound