ISyE 6669 HW 3

1. Consider the following optimization problem:

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & xy \geq 1 \\ & x \geq 0, y \geq 0 \end{aligned}$$

   Does this problem have an optimal solution? Explain your answer.

2. Consider the following optimization problem

$$\begin{aligned} \min \quad & (x^2 - 2x + 1)(x^2 + 6x + 9) \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned}$$

   (a) Find all the global minimum solutions. Explain how you find them. Hint: there may be multiple ones.

   (b) Is there any local minimum solution that is not a global minimum solution?

   (c) Is the objective function $f(x) = (x^2 - 2x + 1)(x^2 + 6x + 9)$ a convex function on $\mathbb{R}$?

3. Consider the following optimization problem

$$\begin{aligned} \min \quad & e^x + y^3 \\ \text{s.t.} \quad & x + y \leq 1 \\ & x + 2y \geq 6 \\ & 2x + y \geq 6. \end{aligned}$$

   Does this problem have an optimal solution? Explain your answer.

4. Consider the following problem

$$\begin{aligned} \min \quad & x^2 + f(x) \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

   where the function $f(x)$ is defined as

$$f(x) = \begin{cases} x, & -1 < x < 1 \\ 2, & x \in \{-1, 1\} \\ +\infty, & x > 1 \text{ or } x < -1 \end{cases} .$$

(a) Is the objective function a convex function defined on $\mathbb{R}$? Explain your answer by checking the definition of convexity.

(b) Find an optimal solution, or explain why there is no optimal solution.

5. For each of the statements below, state whether it is true or false. Justify your answer.

(a) If I solve an optimization problem, then remove a constraint and solve it again, the solution must change.

(b) Consider the following optimization problem

$$\begin{aligned} (P) \quad \max \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{x}) \geq b_i, \ \ \forall i \in I. \end{aligned}$$

Suppose the optimal objective value of (P) is v_P. Then, the Lagrangian dual of (P) is given by

$$(D) \quad \min\{\mathcal{L}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{1}$$

where $\mathcal{L}(\boldsymbol{\lambda}) = \max_{\boldsymbol{x}}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i (g_i(x) - b_i)\}$. Furthermore, suppose the optimal objective value of (D) is v_D, then $v_P \leq v_D$.

(c) The following set is convex:

$$\{x \in \mathbb{R}^{10} \mid \|x\|_2 = 1\}$$

(d) Suppose $f : \mathbb{R} \to \mathbb{R}$ and suppose for any real number p, the set:

$$S_p := \{x \in \mathbb{R} \mid f(x) \leq p\},$$

is convex. Then f is a convex function.