Deterministic Optimization

Outcomes of Optimization

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Possible outcomes of optimization

Outcomes of optimization

Learning objective:

- Discover outcomes of an optimization problem

The generic optimization problem

$$(P): \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

- $\mathbf{x}$ is the decision vector
- X is the set of feasible solutions
- f is the objective function

Feasible solutions, Infeasible problem

$$(P): \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

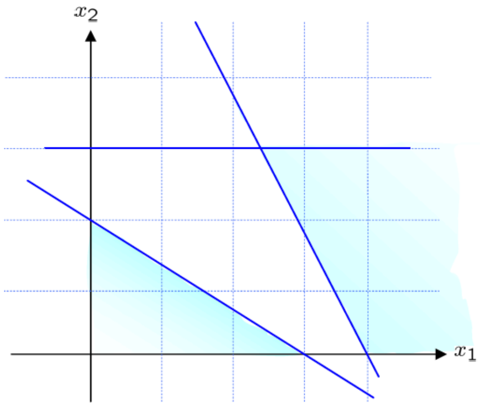
- Any $\mathbf{x} \in X$ is a feasible solution of (P)
- Feasible solution = A solution that satisfies all the constraints
- If $X = \emptyset$ then no feasible solutions exist, and the problem (P) is said to be infeasible.
- The problem $\min\{3x + 2y : x + y \leq 1, x \geq 2, y \geq 2\}$ is infeasible

Unbounded Problem

- The optimization problem (P) is unbounded, if there are feasible solutions with arbitrarily small objective values.
- Formally, (P) is unbounded if there exists a sequence of feasible solutions $\{\mathbf{x}^i\} \in X$ such that $\lim_{i \to \infty} f(\mathbf{x}^i) = -\infty$.
- An unbounded problem must be feasible.
- If X is a bounded set then (P) cannot be unbounded

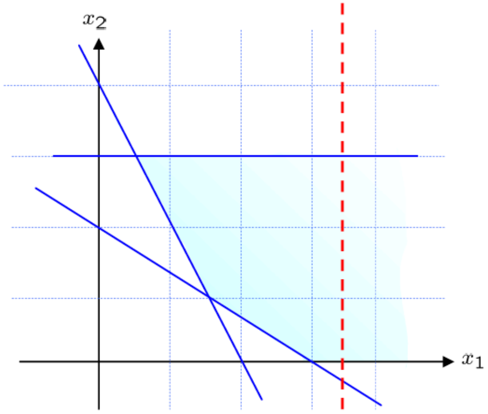
Examples

Infeasible



$$\begin{aligned} \text{min} \quad & x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 8 \\ & x_1 + 1.5x_2 \leq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Unbounded



$$\begin{aligned} \text{max} \quad & x_1 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Optimal solution

$$(P): \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

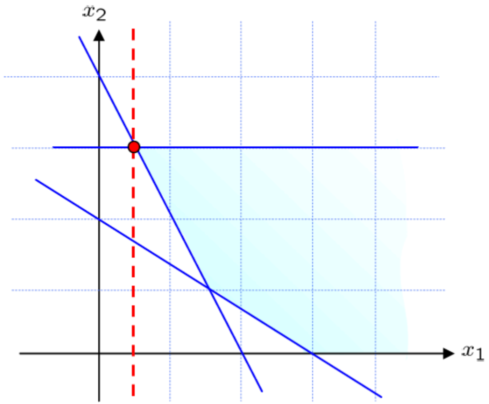
- A feasible solution $\mathbf{x}^*$ is an optimal solution of (P) if

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \; \forall \; \mathbf{x} \in X$$

- The objective value corresponding to an optimal solution (if it exists) is called the optimal objective value of (P)

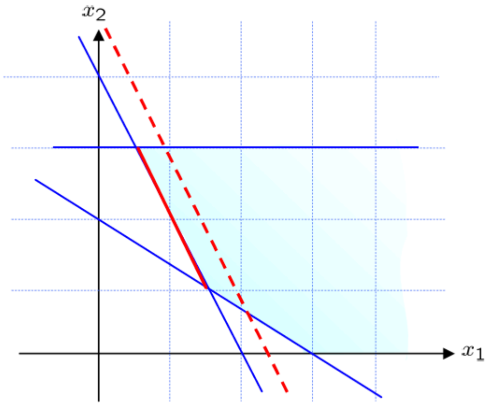
Examples

Unique Optimal solution



$$
\begin{aligned}
\text{min} \quad & x_1 \\
\text{s.t.} \quad & 2x_1 + x_2 \geq 4 \\
& x_1 + 1.5x_2 \geq 3 \\
& x_2 \leq 3 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

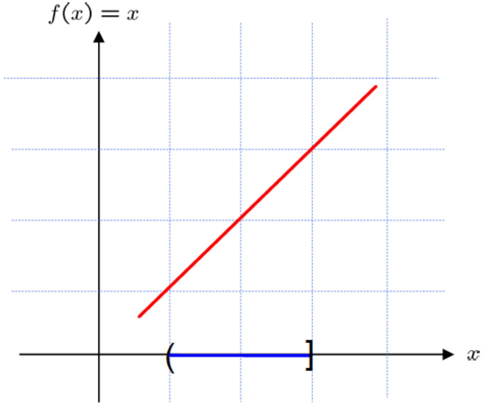
Multiple Optimal solutions



$$
\begin{aligned}
\text{min} \quad & 2x_1 + x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \geq 4 \\
& x_1 + 1.5x_2 \geq 3 \\
& x_2 \leq 3 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

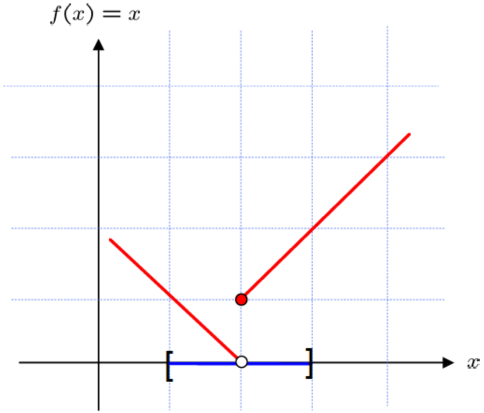
No Optimal Solution

No Optimal
solution



$$\begin{array}{ll} \min & x \\ \text{s.t.} & 1 < x \leq 3 \end{array}$$

No Optimal
solution



$$f(x) = \begin{cases} 2 - x & \text{if } x < 2 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & 1 \leq x \leq 3 \end{array}$$

Possible Outcomes Of Optimization

$$(P): \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

1. Infeasible: $X = \emptyset$
2. Unbounded: $f(\mathbf{x}^i) \to -\infty$ for some $\{\mathbf{x}^i\} \in X$
3. Optimal solution exists: There is $\mathbf{x}^* \in X$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$
4. None of the above

Summary

- An optimization problem can have 4 possible outcomes
- When modeling and solving a problem we should be aware of the possible outcomes