Deterministic Optimization

Review of Mathematical Concepts

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Properties of Sets

Properties of Sets

Learning objectives:

- Recognize sets arising in optimization problems
- Recall some properties of sets

Sets in Optimization Problems

- Recall the generic optimization model

$$\min\{f(\mathbf{x}) : \ \mathbf{x} \in X\}$$

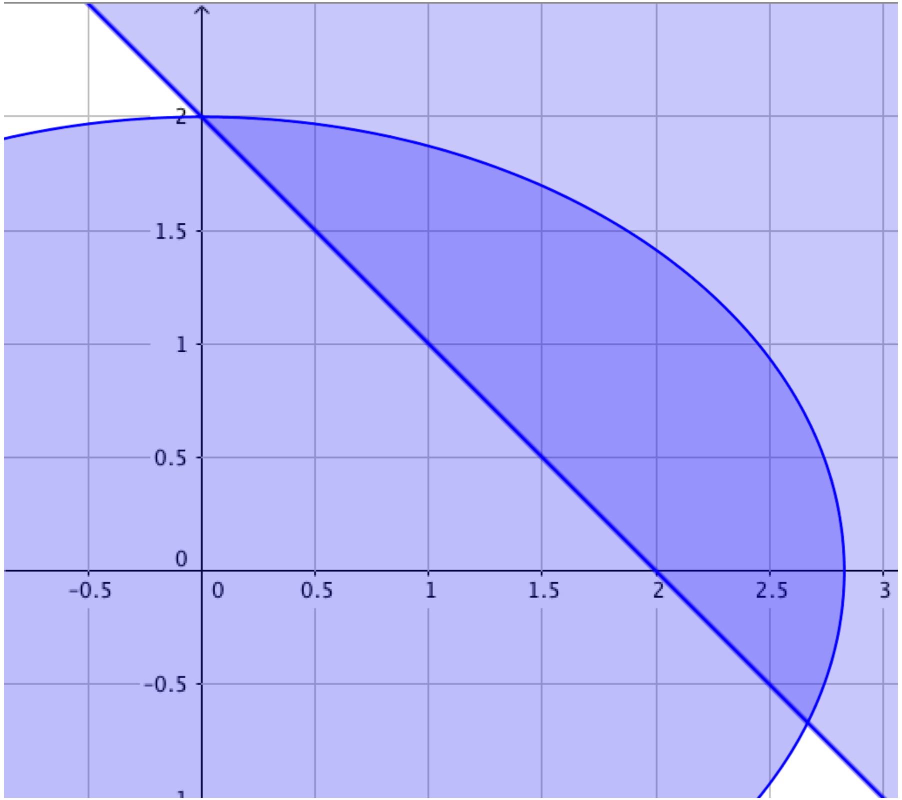
- Here $\mathbf{x}$ is the decision vectors whose values are constrained to be in the set X.
- Typically the constraint set is defined by domain restrictions on the variables and relationships between specified by inequalities or equalities, e.g.

$$X = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq b_i \ i = 1, \ldots, m\}$$

- Here X is the set of solutions to the specified inequalities.

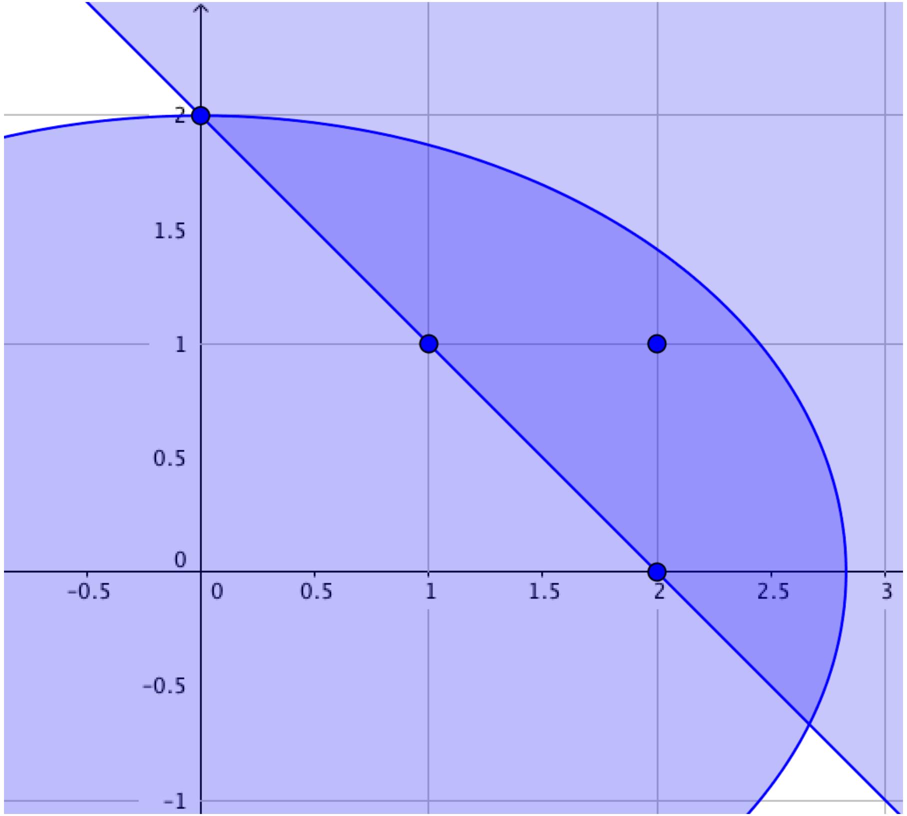
Example

$$\begin{array}{ll} \min & x + y^2 \\ \text{s.t.} & x + y \geq 2 \\ & x^2 + 2y^2 \leq 8 \end{array}$$



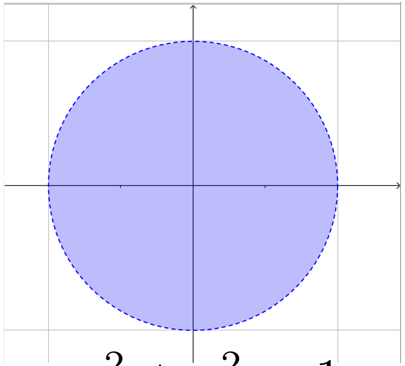
Example

$$\begin{array}{ll} \min & x + y^2 \\ \text{s.t.} & x + y \geq 2 \\ & x^2 + 2y^2 \leq 8 \\ & x, y \in \mathbb{Z} \end{array}$$

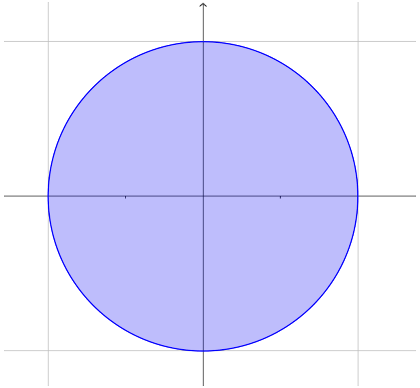


Closed Set

- A set is closed if it includes its boundary points.
- Formally, a set X is closed if for any convergent sequence in X, its limit point also belongs to X, i.e. if $\{\mathbf{x}^i\} \in X$ and $\lim_{i\to\infty} \mathbf{x}^i = \mathbf{x}^0$ then $\mathbf{x}^0 \in X$.
- Examples:
    - o $X = \mathbb{R}^2$ is closed
    - o $X = \{x : 0 < x \leq 1\}$ is not closed
- Intersection of closed sets is closed.
- Typically, if none of inequalities are strict, then the set is closed.



$x^2 + y^2 < 1$



$x^2 + y^2 \leq 1$

Bounded Set

- A set is bounded if it can be enclosed in a large enough (hyper)-sphere or a box.
- Formally, the set X is bounded if there exists $M \geq 0$ such that $||\mathbf{x}|| \leq M$ for all $\mathbf{x} \in X$.
- A set that is both bounded and closed is called compact.

Examples

- $X = \mathbb{R}^2$ is closed but not bounded
- $X = \{(x, y) : x^2 + y^2 < 1\}$ is bounded but not closed
- $X = \{(x, y) : x + y \geq 1\}$ is closed but not bounded
- $X = \{(x, y) : x^2 + y^2 \leq 1\}$ is closed and bounded (compact)

Summary

- The decision variables of an optimization problem are constrained to be in a set.
- Such sets are defined by domain restrictions and inequalities/equalities
- Remember the definitions of closed, bounded and compact sets.