Deterministic Optimization

Linear Optimization Modeling

Network Flow Problems

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Optimal Transportation Problem

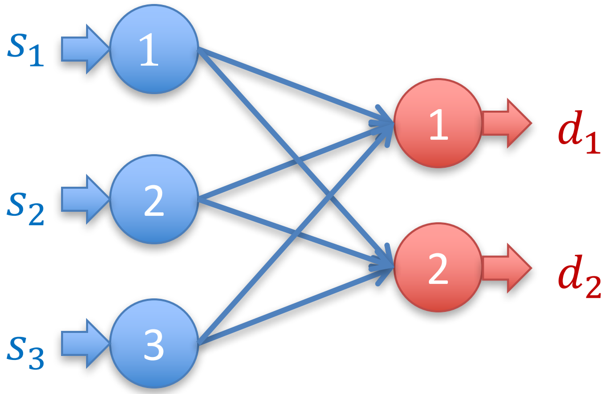
Modeling using Linear Programs

Learning Objectives

- Discover an interesting example of LP called the **transportation problem**, and discuss when two LP models are equivalent.
- Use CVX and read solver output.

Transportation Problem

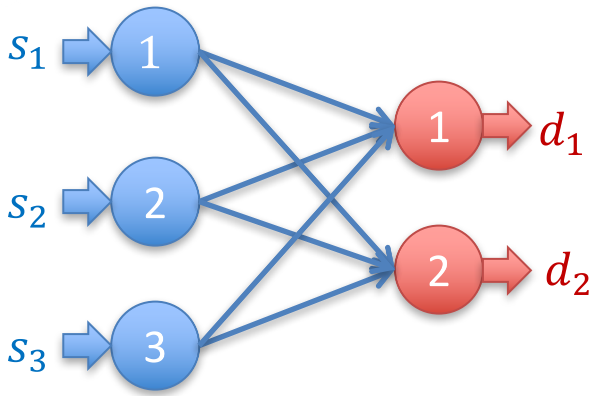
There are m suppliers, n customers. Supplier i can supply up to s_i units of supply, and customer j has d_j units of demand. It costs c_{ij} to transport a unit of product from supplier i to customer j. We want to find a transportation schedule to satisfy all the demand within minimum transportation cost.



Transportation Problem

Decision variables: x_{ij} for $i = 1, \ldots, m, j = 1, \ldots, n$

x_{ij}: the amount of product transported from supply i to warehouse j.



(M1)

$$\min \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Demand is exactly satisfied.

$$\text{s.t.} \quad \sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1, \ldots, n$$

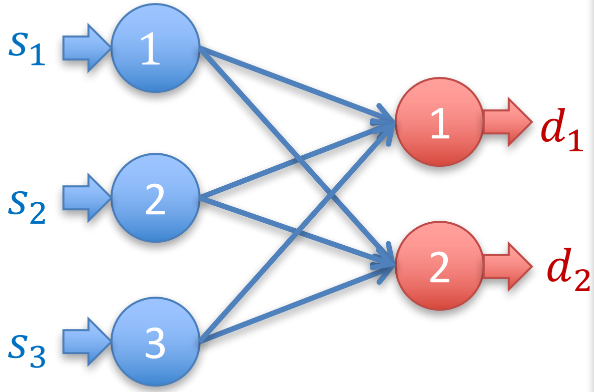
$$\sum_{j=1}^{n} x_{ij} \le s_i, \quad i = 1, \ldots, m$$

$$x_{ij} \ge 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

Another Formulation

Decision variables: x_{ij} for $i = 1, \ldots, m, j = 1, \ldots, n$

x_{ij}: the amount of product transported from supply i to warehouse j.



$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} && \text{(M2)} \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} \geq d_j, && j = 1, \ldots, n \\
& \sum_{j=1}^{n} x_{ij} \leq s_i, && i = 1, \ldots, m \\
& x_{ij} \geq 0, \quad i = 1, \ldots, m, && j = 1, \ldots, n.
\end{aligned}
$$

Demand is satisfied or exceeded.

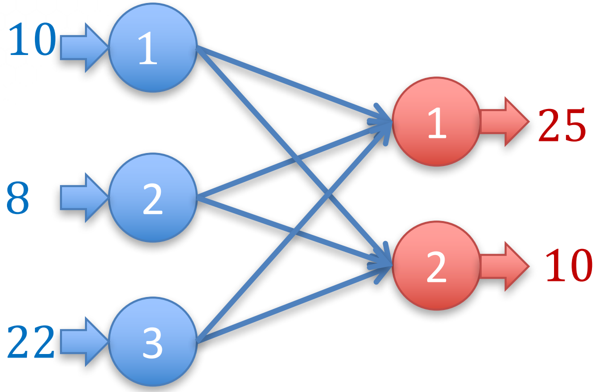
Are Two Formulations Equivalent?

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} = d_j, \quad \forall j \\ & \sum_{j=1}^{n} x_{ij} \leq s_i, \quad \forall i \\ & x_{ij} \geq 0, \quad \forall i, j. \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} \geq d_j, \quad \forall j \\ & \sum_{j=1}^{n} x_{ij} \leq s_i, \quad \forall i \\ & x_{ij} \geq 0, \quad \forall i, j. \end{aligned}$$

If $c_{ij} \geq 0$ for all i, j, then we can claim that the $\geq$ inequality in the second model will be satisfied as $=$ at optimal solution, thus, the two formulations are equivalent.

A Concrete Example



CVX code for (M2):

```
m = 3;
n = 2;
c = [8, 2;
     4, 10;
     6, 7];
s = [10; 8; 22];
d = [25; 10];
cvx_begin
variable x(3,2) nonnegative;
minimize sum(sum(c.*x));
subject to
sum(x,1) >= d';
sum(x,2) <= s;
cvx_end
```

Results

How to read the output log of Gurobi?

- Preprocessing
- Algorithm iterations
- Status
- Optimal solution

```
Calling Gurobi 6.00: 11 variables, 5 equality constraints
------------------------------------------------------------
Gurobi optimizer, licensed to CVX for CVX
Optimize a model with 5 rows, 11 columns and 17 nonzeros
Coefficient statistics:
  Matrix range    [1e+00, 1e+00]
  Objective range [2e+00, 1e+01]
  Bounds range    [0e+00, 0e+00]
  RHS range       [8e+00, 3e+01]
Presolve removed 0 rows and 5 columns
Presolve time: 0.00s
Presolved: 5 rows, 6 columns, 12 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    0.0000000e+00   3.500000e+01   0.000000e+00      0s
       3    1.5400000e+02   0.000000e+00   0.000000e+00      0s

Solved in 3 iterations and 0.01 seconds
Optimal objective  1.540000000e+02
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +154

>> x

x =

     0    10
     8     0
    17     0
```

Summary

- We have learned the transportation problem modeled as linear programs.
- Optimization problems need analysis: Different problems may be equivalent.
- CVX + Gurobi is a powerful tool for solving LPs.