1. Expand the following summations:
(For example, the answer to part (a) is $x_1 + x_2 + x_3$.)

(a)	$\sum_{i=1}^{3} x_i$	(d)	$\sum_{i=1}^{3}\sum_{j=2}^{4}(x_i + y_{ij})$
(b)	$\sum_{t=1}^{3} \frac{x^{2t}}{t!}$	(e)	$\sum_{k=-1}^{3}(2k+1)x_{k+1}$
(c)	$\sum_{i=1}^{3}\sum_{j=1}^{i} x_{ij}$	(f)	$\sum_{n=3}^{5}\sum_{m=n+1}^{n+3} x_n y_m$

Note that by definition $t! = 1 \cdot 2 \cdots (t-1) \cdot t$ for integer $t \geq 1$.

(a) $x_1 + x_2 + x_3$

(b) $x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

(c) $x_{11} + x_{21} + x_{22} + x_{31} + x_{32} + x_{33}$

(d) $(x_1 + y_{12}) + (x_1 + y_{13}) + (x_1 + y_{14})$
$+ (x_2 + y_{22}) + (x_2 + y_{23}) + (x_2 + y_{24})$
$+ (x_3 + y_{32}) + (x_3 + y_{33}) + (x_3 + y_{34})$

(e) $-x_0 + x_1 + 3x_2 + 5x_3 + 7x_4$

(f) $\sum_{m=4}^{6} x_3 y_m + \sum_{m=5}^{7} x_4 y_m + \sum_{m=6}^{8} x_5 y_m$

$= x_3 y_4 + x_3 y_5 + x_3 y_6 + x_4 y_5 + x_4 y_6 + x_4 y_7 + x_5 y_6 + x_5 y_7 + x_5 y_8$

2. Consider the following two vectors: $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$, and a matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & 0 & -1 \end{bmatrix}.$$

(a) Let n be the dimension of $\boldsymbol{x}$ and $\boldsymbol{y}$. What is the value of n?

(b) Compute $3\boldsymbol{x} - 2\boldsymbol{y}$.

(c) Compute the inner product $\boldsymbol{x}^\top \boldsymbol{y}$.

(d) Compute $\boldsymbol{x}\boldsymbol{y}^\top$.

(e) Compute the Euclidean norm $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. Also called the ℓ_2-norm.

(f) Compute the ℓ_1-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = \sum_{i=1}^n |x_i - y_i|$.

(g) Compute the ℓ_∞-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|$.

(e) Compute $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}$.

(a) $n = 3$

(b) $3\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

(c) $(2\ 1\ 4)\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = 2 \cdot 2 + 1 \cdot 0 + 4 \cdot 5 = 24$

(d) $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}(2\ 0\ 5) = \begin{pmatrix} 4 & 0 & 10 \\ 2 & 0 & 5 \\ 8 & 0 & 20 \end{pmatrix}$

(e) $\|x - y\|^2 = \sqrt{(2-2)^2 + (1-0)^2 + (4-5)^2} = \sqrt{2}$

(f) $\|x - y\| = |0| + |1| + |1| = 2$

(g) $(|0|^p + |1|^p + |1|^p)^{\frac{1}{p}} \xrightarrow[n \to \infty]{} \max(|0|, |1|, |1|) = 1$

(h) $(2\ 1\ 4)\begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$

$= (2\ 1\ 4)\begin{pmatrix} 12 \\ -9 \\ 1 \end{pmatrix}$

$= 2 \cdot 12 + 1 \cdot (-9) + 4 \cdot 1$

$= 19$

3. State whether each of the following sets is convex or not. Explain your reasoning.

   (a) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1^2 + 5|x_2| \leq 10\}$.

   (b) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1^2 - 5|x_2| \leq 10\}$.

   (c) $X = \{(x_1, x_2) \mid \frac{x_1}{(x_2-2)} \leq 2, \quad x_2 \geq 1\}$.

   (d) $X = \{(x_1, x_2) \mid \frac{x_1}{(x_2-2)} \leq 2, \quad x_2 \geq 2\}$.

Solution to Problem 3

(a) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1^2 + 5|x_2| \leq 10\}$

Not convex. Reason: The term $|x_2|$ is not a convex function. For example, $(0, 2)$ and $(0, -2)$ belong to the set, but their midpoint $(0, 0)$ satisfies $2(0)^2 + 5|0| = 0 \leq 10$ and belongs to the set, but in general convex combinations may not belong to the set.

(b) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1^2 - 5|x_2| \leq 10\}$

Not convex. Reason: The term $-5|x_2|$ makes the boundary of the set non-convex. For example, $(0, 2)$ and $(0, -2)$ belong to the set, but their midpoint $(0, 0)$ satisfies $2(0)^2 - 5|0| = 0 \leq 10$ and belongs to the set, but in general convex combinations may not belong to the set.

(c) $X = \{(x_1, x_2) \mid \frac{x_1}{(x_2-2)} \leq 2, \quad x_2 \geq 1\}$

Not convex. Reason: The term $\frac{x_1}{(x_2-2)}$ is not a convex function. The denominator $(x_2 - 2)$ creates a singularity at $x_2 = 2$, and even though $x_2 \geq 1$, the function is not convex.

(d) $X = \{(x_1, x_2) \mid \frac{x_1}{(x_2-2)} \leq 2, \quad x_2 \geq 2\}$

Convex. Reason: The condition $x_2 \geq 2$ ensures that the denominator $(x_2 - 2) \geq 0$, so $\frac{x_1}{(x_2-2)} \leq 2$ becomes $x_1 \leq 2(x_2 - 2)$, which is a linear constraint. Linear constraints and linear inequalities form convex sets.

4. State whether the following problems are convex programs or not. Explain your reasoning.

   (a) $\min\{x_1^3 + x_2^2 \ : \ x_1 \leq 2,\ x_2 \leq 3\}$.

   (b) $\max\{2x_1 + 3x_2 + 4x_3 + 5x_4 \ : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}$.

   (c) $\min\{\sum_{i=1}^{n} 2^i (x_i)^{2i} \ : \ \sum_{i=1}^{n} x_i \geq 10\}$.

Solution to Problem 4

(a) $\min\{x_1^3 + x_2^2 \ : \ x_1 \leq 2, \ x_2 \leq 3\}$

Not a convex program. Reason: The objective function x_1^3 is not a convex function. Cubic functions are not convex.

(b) $\max\{2x_1 + 3x_2 + 4x_3 + 5x_4 \ : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}$

Convex program. Reason: The objective function is linear (convex), and the constraint $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$ defines a convex set (sphere). Although it's a maximization problem, maximizing a linear objective function can be treated as a convex program.

(c) $\min\{\sum_{i=1}^n 2^i (x_i)^{2i} \ : \ \sum_{i=1}^n x_i \geq 10\}$

Not a convex program. Reason: The terms $2^i(x_i)^{2i}$ in the objective function are not convex functions. Even though they involve even powers, the coefficients 2^i grow large, making them generally non-convex.

5. A quantity y is known to depend upon another quantity x. A set of n data pairs $\{y_i, x_i\}_{i=1}^n$ has been collected.

   (a) Formulate an optimization model for fitting the "best" straight line $y = a + bx$ to the data set, where best is with respect to the sum of absolute deviations. What kind of an optimization model is it?

   (b) Re-formulate the optimization model in part (a) where best is with respect to the maximum absolute deviation. What kind of an optimization model is it?

   (c) Formulate an optimization model for fitting the "best" quadratic curve $y = a + bx + cx^2$ to the data set, where best is with respect to the maximum absolute deviations. What kind of an optimization model is it ?

Solution to Problem 5

(a) Optimization model with respect to sum of absolute deviations:
$\min_{a,b} \sum_{i=1}^{n} |y_i - (a + bx_i)|$
This is a **Linear Programming (LP) problem**. The absolute values can be expressed using linear constraints, making it a linear programming problem.

(b) Optimization model with respect to maximum absolute deviation:
$\min_{a,b} \max_{i=1,\ldots,n} |y_i - (a + bx_i)|$
This is a **Linear Programming (LP) problem**. The maximum value can be expressed using linear constraints, making it a linear programming problem.

(c) Optimization model for quadratic curve with respect to maximum absolute deviation:
$\min_{a,b,c} \max_{i=1,\ldots,n} |y_i - (a + bx_i + cx_i^2)|$
This is a **Linear Programming (LP) problem**. The quadratic term cx_i^2 is treated as a constant since x_i are known data points, and the maximum value can be expressed using linear constraints, making it a linear programming problem.