# Deterministic Optimization

## Convexity

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### Convex Functions

# Convex Function

**Learning objectives:**

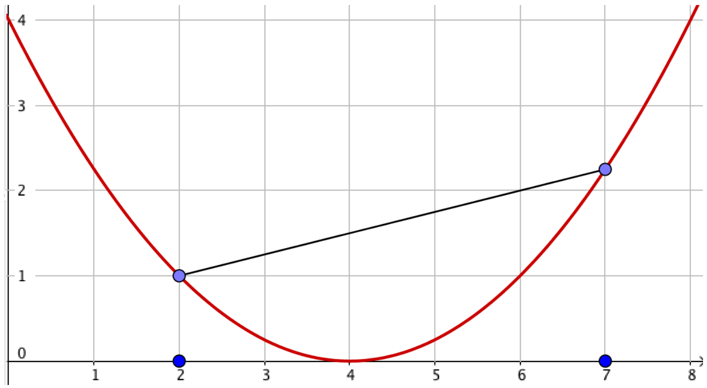
- Recall definition of convexity
- Recall properties of convex functions
- Recognize convex functions

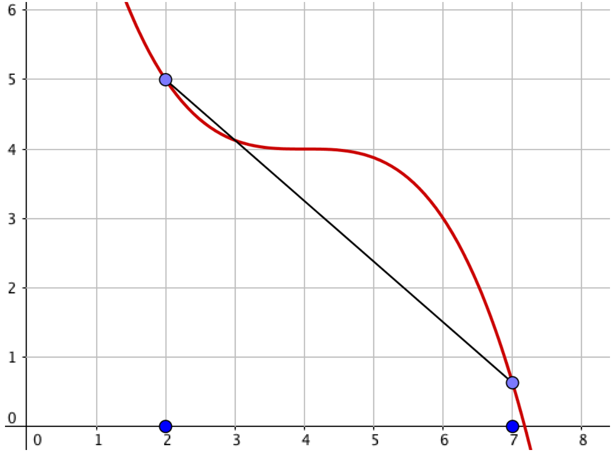
# Convex Function

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and } \lambda \in [0,1]$$

- "Function value at the average $\leq$ average of function values"



# Convex Function

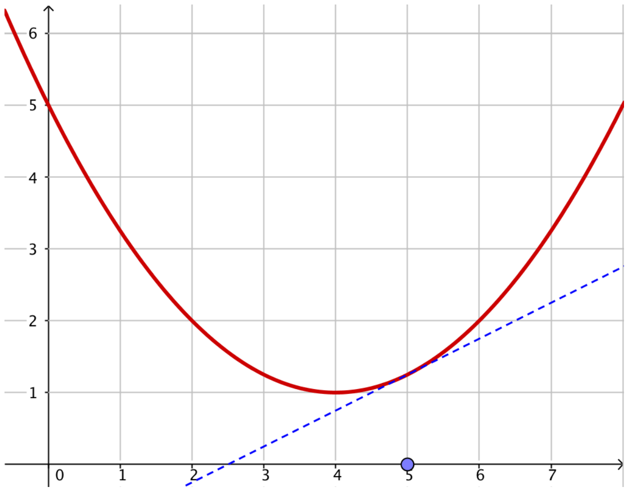
- A function $f$ is **concave** if $-f$ is convex, i.e. the inequality is reversed.
- A linear function is both convex and concave.
- Examples of convex functions:
    - $f(x) = x^2$
    - $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + b$
    - $f(\mathbf{x}) = \sum_{j=1}^{n} |x_j|$

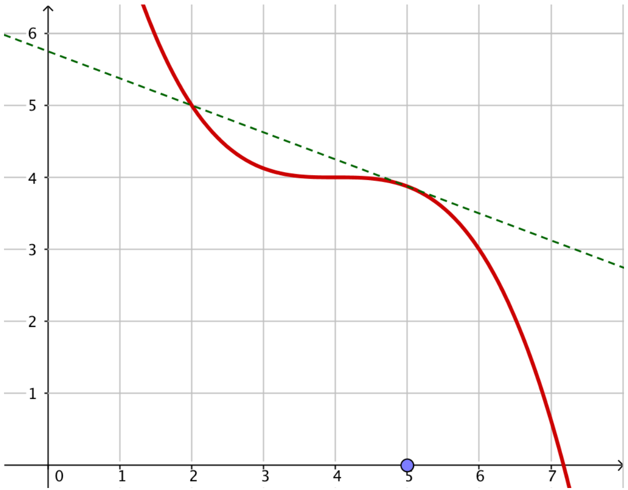
# First Order Condition

- A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \;\; \forall \, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

- That is, the first order Taylor's approximation is a global under-estimator.



# Second Order Condition

- A twice differentiable univariate function $f : \mathbb{R} \to \mathbb{R}$ is convex if $f''(x) \geq 0$ for all $x \in \mathbb{R}$.
- That is, the slopes (or gradients) of $f$ are non-decreasing.
- A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its Hessian matrix $\nabla^2 f(\mathbf{x})$ is **positive semidefinite** (psd) for all $\mathbf{x} \in \mathbb{R}^n$.
- A $n \times n$ matrix $A$ is psd if $\mathbf{x}^\top A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
- Equivalently, $A$ is psd if all its eigenvalues are nonnegative.

# Examples

- $f(x) = e^x + x^2$. Then $f'(x) = e^x + 2x$ and $f''(x) = e^x + 2 \geq 0$ for all $x$. So $f$ is convex.

- $f(x) = x^3$. Then $f'(x) = 3x^2$ and $f''(x) = 6x$. For any $x < 0$, we have $f''(x) < 0$. Thus $f$ is not convex.

- $f(\mathbf{x}) = ||A\mathbf{x} - \mathbf{b}||^2 = (A\mathbf{x} - \mathbf{b})^\top(A\mathbf{x} - \mathbf{b})$.
  Then $\nabla f(\mathbf{x}) = 2A^\top(A\mathbf{x} - \mathbf{b})$ (check!), and $\nabla^2 f(\mathbf{x}) = 2A^\top A$ (check!).
  Now $\mathbf{y}^\top \nabla^2 f(\mathbf{x})\mathbf{y} = \mathbf{y}^\top A^\top A\mathbf{y} = ||A\mathbf{y}||^2 \geq 0$ for any $\mathbf{y}$.
  Thus $f$ is convex.

# Operations preserving convexity

- Nonnegative weighted sum of convex functions is convex,
  i.e. if $f_i$ is convex and $\alpha_i \geq 0$ for all $i = 1, \ldots, m$, then $g(\mathbf{x}) = \sum_{i=1}^{m} \alpha_i f_i(\mathbf{x})$ is convex.

- Maximum of convex functions is convex,
  i.e. if $f_i$ is convex for all $i = 1, \ldots, m$, then $g(\mathbf{x}) = \max_i \{f_i(\mathbf{x})\}$ is convex.

- Composition: Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and $g_i : \mathbb{R}^n \to \mathbb{R}$ be convex for all $i = 1, \ldots, m$. Then the composite function

$$h(\mathbf{x}) = f(g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_m(\mathbf{x}))$$

  is convex if either $f$ is nondecreasing or if each $g_i$ is a linear function.

# Checking convexity

- Apply the definition
- If twice differentiable, check second order condition
- Show that the function is obtained by applying convexity preserving operations to simple convex functions.

# Example

Is the function $f(\mathbf{x}) = e^{\sum_{i=1}^{m} |a_i^\top \mathbf{x} - b_i|}$ convex?

- Let $g_i(\mathbf{x}) = |a_i^\top \mathbf{x} - b_i|$. It is obtained by the composition of the convex function $|\cdot|$ and the linear function $a_i^\top \mathbf{x} - b_i$. So it is convex.
- The function $h(\mathbf{x}) = \sum_{i=1}^{m} g_i(\mathbf{x})$ is a sum of convex functions, hence convex.
- Finally, $f$ is obtained by taking composition of the function $m(a) = e^a$ with $h$, i.e. $f(\mathbf{x}) = m(h(\mathbf{x}))$. Since $m$ is nondecreasing and $h$ is convex, $f$ is convex.

# Summary

- We learned the definition of a convex function, some of its properties, and about operations that preserve convexity.
- In general checking convexity of a function is difficult.
- An often useful approach is to break the function down in to its simple constituents and checking convexity of the simple functions and the associated operations.