Deterministic Optimization

Optimality Certificates

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Lagrangian Relaxation and Duality

Lagrangian Relaxation and Duality

Learning objectives:

- Examine the construction of a Lagrangian relaxation
- Recognize the dual problem

Relaxation

$$(P) : \min_x \{f(x) : x \in X\} \qquad (Q) : \min_x \{g(x) : x \in Y\}$$

Problem (Q) is a **relaxation** of (P) if

- $X \subseteq Y$
- $f(x) \geq g(x) \;\; \forall\, x \in X$

Lagrangian Relaxation

$$
(P): \begin{array}{lll} \min & f(x) & \\ \text{s.t.} & g_i(x) \le b_i & \forall\, i \in I \\ & h_j(x) = d_j & \forall\, j \in J \end{array}
$$

Let $\lambda_i \ge 0$ for all $i \in I$ and μ_j for $j \in J$ be given numbers, then the following problem is a relaxation of (P):

$$
(Q): \ \min\{f(x) + \sum_{i \in I} \lambda_i [g_i(x) - b_i] + \sum_{j \in J} \mu_j [h_j(x) - d_j]\}
$$

Lagrangian Relaxation

$$v_P = \min\{f(x) : \; g_i(x) \leq b_i \; i \in I, \; h_j(x) = d_j \; j \in J\}$$

For $\lambda_i \geq 0$, let

$$\mathcal{L}(\lambda, \mu) = \min\{f(x) + \sum_{i \in I} \lambda_i [g_i(x) - b_i] + \sum_{j \in J} \mu_j [h_j(x) - d_j]\}$$

Then $\mathcal{L}(\lambda, \mu) \leq v_p \; \forall \; \lambda \geq 0$

Lagrangian Dual and Weak Duality

$$(P): \; v_P = \min_x \{f(x): \; g_i(x) \leq b_i \; i \in I, \; h_j(x) = d_j \; j \in J\}$$

$$(D): \; v_D = \max_{\lambda,\mu} \{\mathcal{L}(\lambda, \mu) : \lambda \geq 0\}$$

where

$\mathcal{L}(\lambda, \mu) = \min_x \{f(x) + \sum_i \lambda_i [g_i(x) - b_i] + \sum_j \mu_j [h_j(x) - d_j]\}$

Weak duality: $v_D \leq v_P$

Example

$$
(P):\quad \begin{array}{ll} \min & 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}
$$

$$
(P):\quad \begin{array}{lll} \min & 2x_1 + 3x_2 & \\ \text{s.t.} & x_1 + x_2 = 1 & (\mu) \\ & -x_1 \leq 0 & (\lambda_1) \\ & -x_2 \leq 0 & (\lambda_2) \end{array}
$$

$$
\begin{aligned} \mathcal{L}(\lambda_1, \lambda_2, \mu) &= \min_{x_1, x_2} \{2x_1 + 3x_2 + \lambda_1[-x_1] + \lambda_2[-x_2] + \mu[x_1 + x_2 - 1]\} \\ &= \min_{x_1, x_2} \{(2 - \lambda_1 + \mu)x_1 + (3 - \lambda_2 + \mu)x_2\} - \mu \end{aligned}
$$

Example (contd.)

$$(D):\ v_D = \max_{\lambda_1\geq 0, \lambda_2 \geq 0, \mu} \left\{ \min_{x_1, x_2} \{(2-\lambda_1+\mu)x_1 + (3-\lambda_2+\mu)x_2\} - \mu \right\}$$

We must have $(2-\lambda_1+\mu) = 0$ and $(3-\lambda_2+\mu) = 0$, otherwise the inner problem is unbounded!

$$(D):\quad \begin{array}{rll} v_D = & \max & -\mu \\ & \text{s.t.} & (2-\lambda_1+\mu) = 0 \\ & & (3-\lambda_2+\mu) = 0 \\ & & \lambda_1 \geq 0,\ \lambda_2 \geq 0 \end{array}$$

$$(D):\quad \begin{array}{rll} v_D = & \max_\mu & -\mu \\ & \text{s.t.} & (2+\mu) \geq 0 \\ & & (3+\mu) \geq 0 \end{array}$$

Example (contd.)

$$(P): \quad \begin{array}{ll} \min & 2x_1 + 3x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

$$(D): \quad \begin{array}{ll} \max & y \\ \text{s.t.} & y \leq 2 \\ & y \leq 3 \end{array}$$

Summary

- Lagrangian relaxation is a specific way to construct a relaxation of an optimization problem
- The dual problem attempts to find the relaxation with the tightest bound
- Weak duality: dual optimal value ≤ original optimal value
- Some times we get strong duality: dual opt. val. = original opt. val.