Deterministic Optimization

Outcomes of Optimization

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Improving Search

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Learning objective:

- Identify the main ideas of the improving search for optimization

Improving Search

- Most optimization algorithms are based on the paradigm of improving search
    1. Start from a feasible solution
    2. Move to a new feasible solution with a better objective value; STOP if not possible
    3. Repeat step 2

- Typically we are only able to look in the “neighborhood” of the current solution in search of a better feasible solution, i.e., solutions that are within a small positive distance from the current solution

Moves

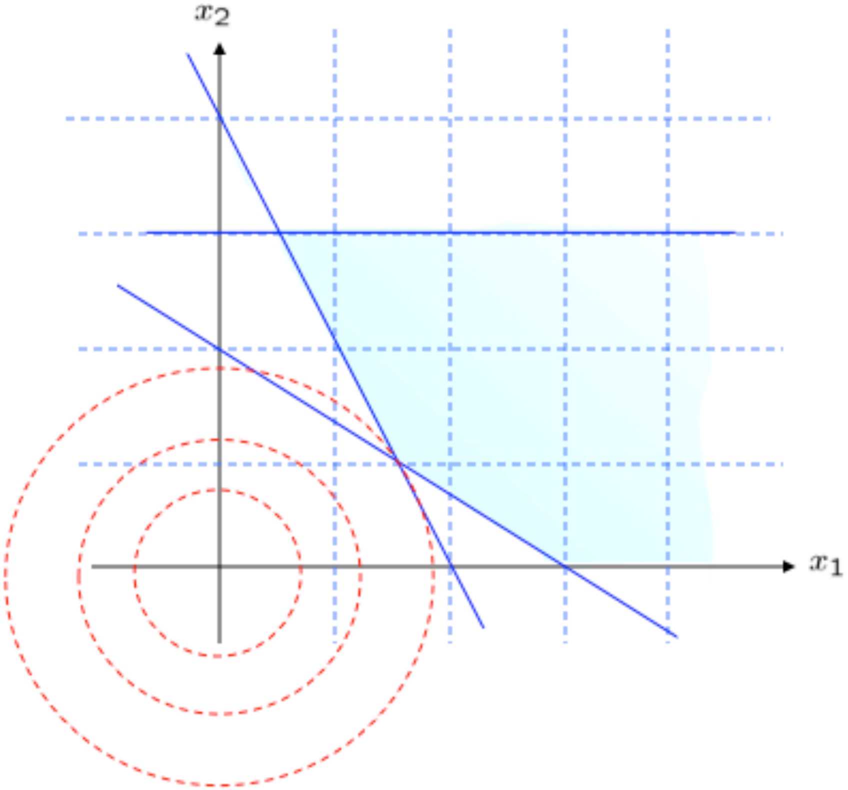
- The moves are executed by first finding a direction to move in and then taking a positive step in that direction

$$
\begin{aligned}
\mathbf{x}^i &= \text{Current point} \\
\Delta\mathbf{x} &= \text{Move direction} \\
\lambda &= \text{Step size}
\end{aligned}
$$

The new point: $\mathbf{x}^{i+1} = \mathbf{x}^i + \lambda \Delta \mathbf{x}$

Example

$$\begin{array}{ll} \text{min} & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

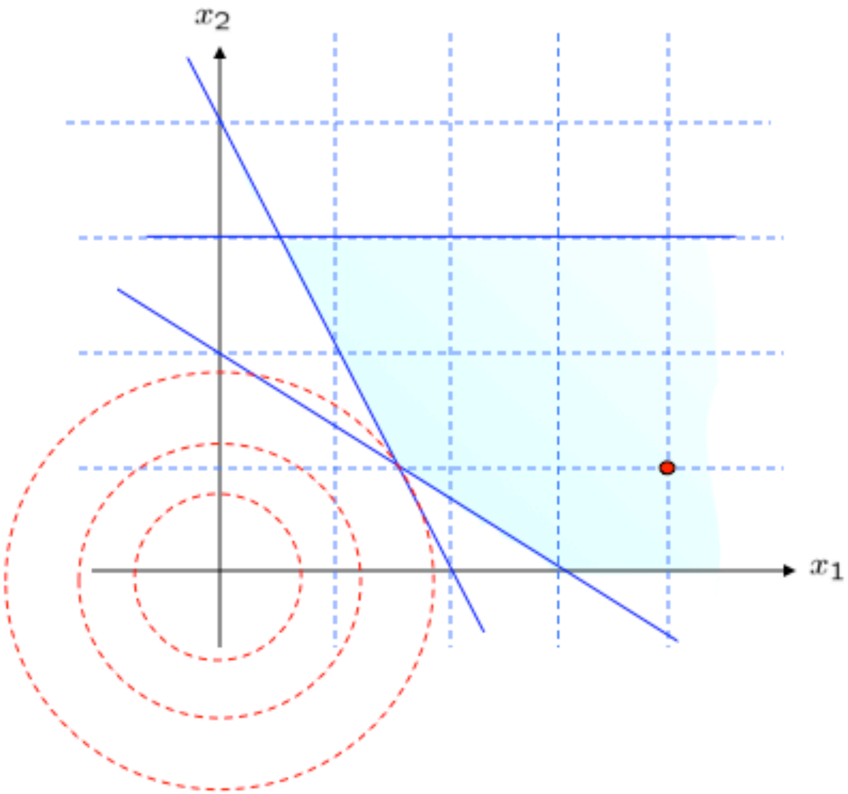


Example

$$\begin{array}{ll} \text{min} & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

Suppose

$$\mathbf{x}^i = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



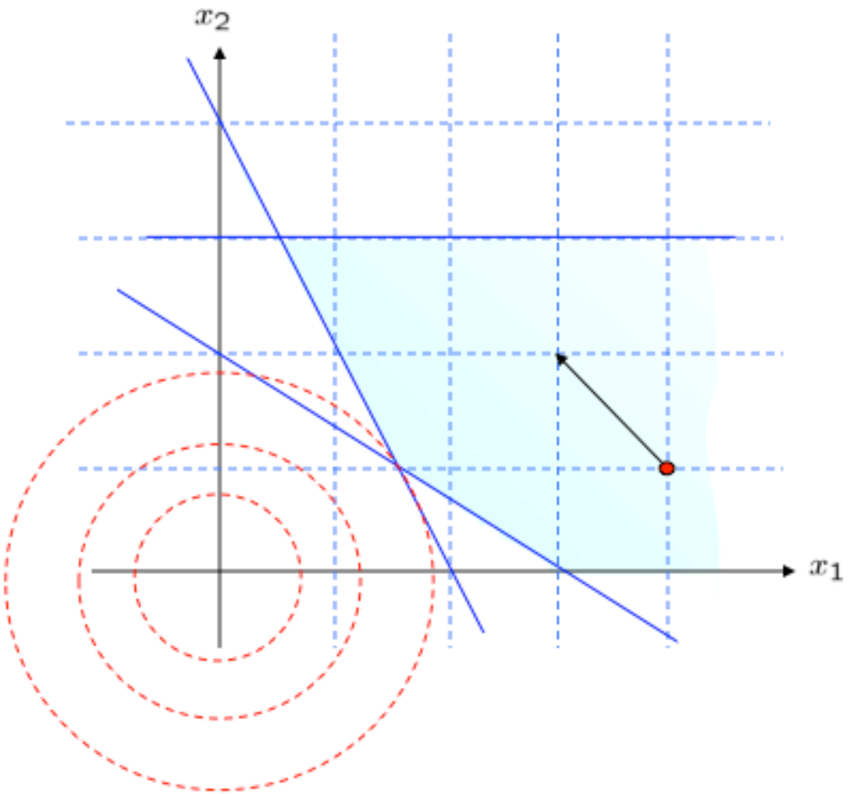
Example

$$\begin{array}{ll} \text{min} & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

Suppose

$$\mathbf{x}^i = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\Delta \mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Example

$$\begin{array}{ll} \text{min} & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

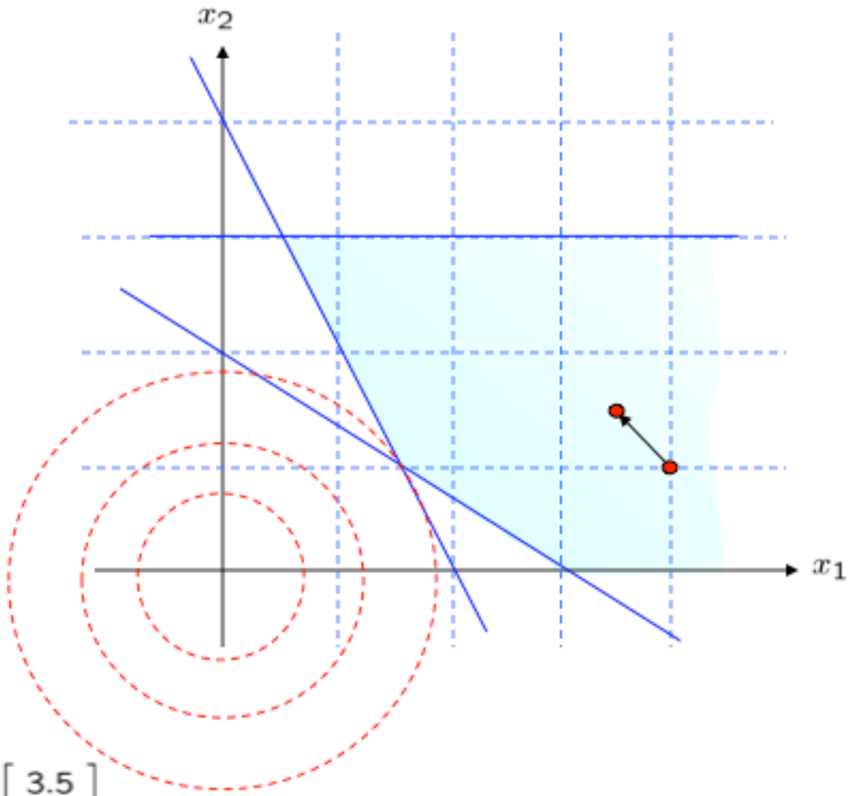
Suppose

$$\mathbf{x}^i = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\Delta\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda = 0.5$$

$$\mathbf{x}^{i+1} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 1.5 \end{bmatrix}$$



Example

$$\begin{array}{ll} \text{min} & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

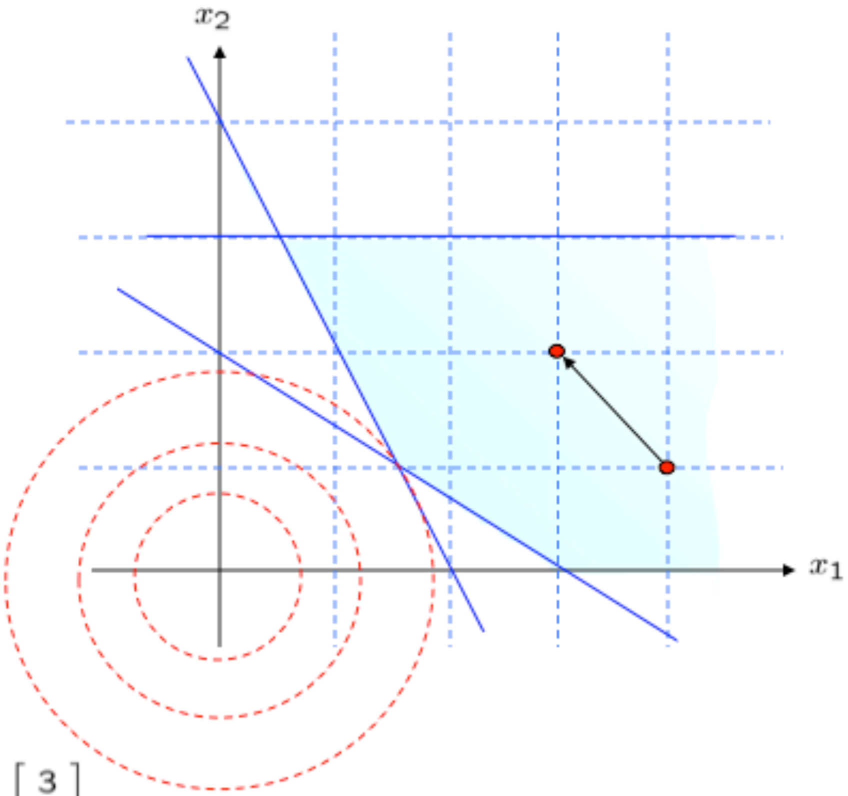
Suppose

$$\mathbf{x}^i = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\Delta \mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda = 1.0$$

$$\mathbf{x}^{i+1} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 1.0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Feasible and Improving Directions

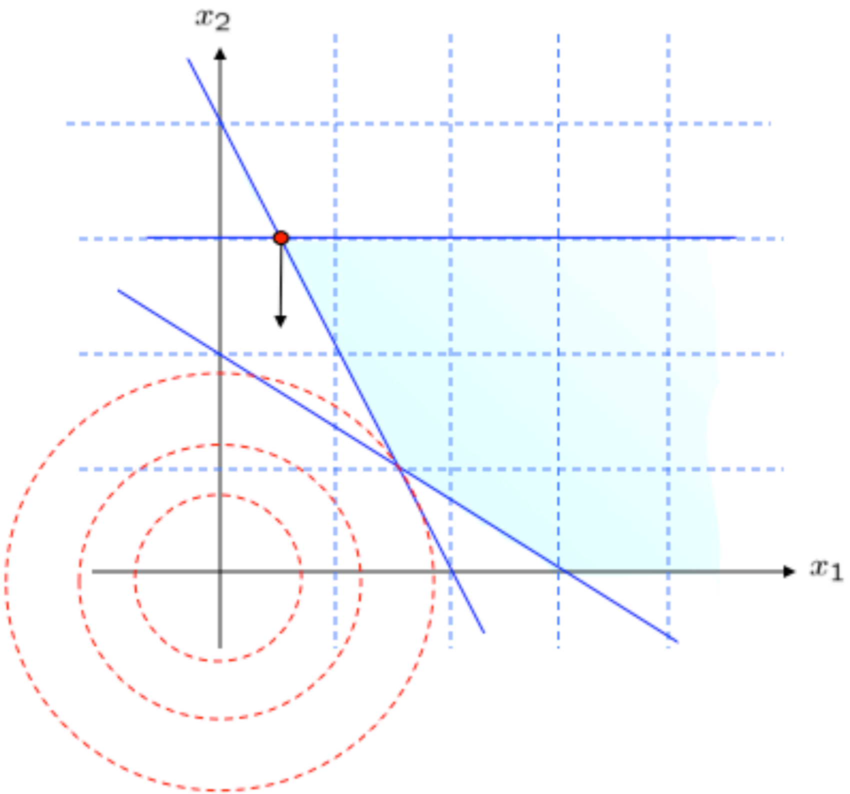
- The move direction and step size should ensure that the new point is feasible has an improved objective value

- A move direction $\Delta\mathbf{x}$ is feasible at the current solution $\mathbf{x}^i$ if the next point $\mathbf{x}^{i+1} = \mathbf{x}^i + \lambda\Delta\mathbf{x}$ is feasible for a sufficiently small step size $\lambda > 0$.

- A move direction $\Delta\mathbf{x}$ is improving at the current solution $\mathbf{x}^i$ if the next point $\mathbf{x}^{i+1} = \mathbf{x}^i + \lambda\Delta\mathbf{x}$ has a better objective function value for all step sizes $\lambda > 0$ sufficiently small.

Example

$$\begin{array}{ll} \text{min} & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$



Suppose

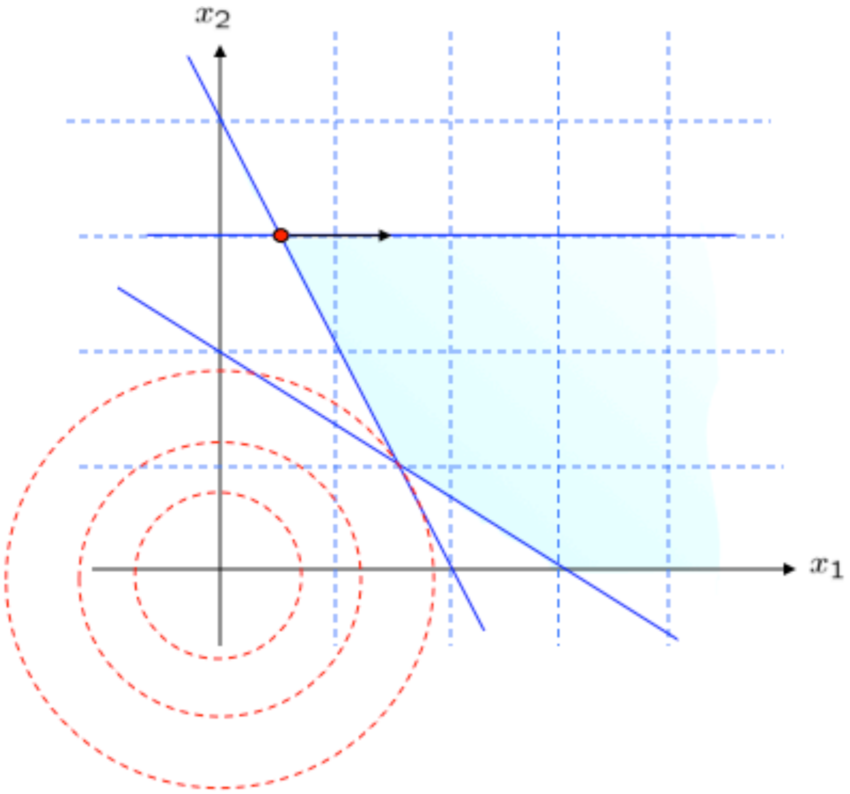
$$\mathbf{x}^i = \begin{bmatrix} 0.5 \\ 3 \end{bmatrix}$$

$$\Delta\mathbf{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Improving but not feasible

Example

$$\begin{array}{ll} \min & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$



Suppose

$$\mathbf{x}^i = \begin{bmatrix} 0.5 \\ 3 \end{bmatrix}$$

$$\Delta\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Feasible but not improving

Example

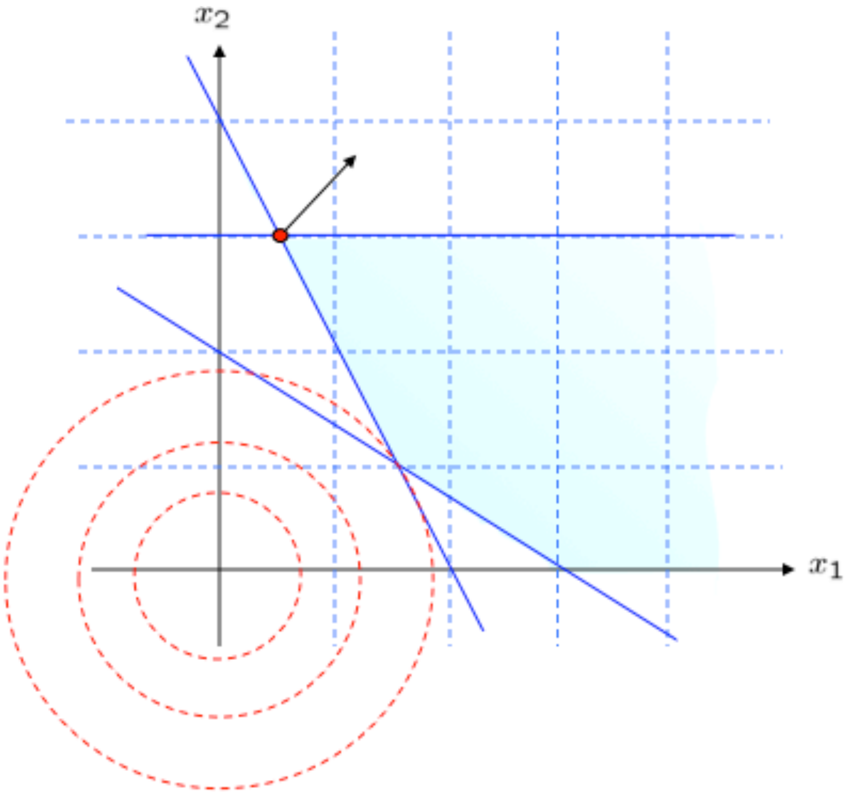
$$\begin{array}{ll} \text{min} & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

Suppose

$$\mathbf{x}^i = \begin{bmatrix} 0.5 \\ 3 \end{bmatrix}$$

$$\Delta\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Not feasible not improving



Example

$$\begin{array}{ll} \min & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \geq 4 \\ & x_1 + 1.5x_2 \geq 3 \\ & x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{array}$$

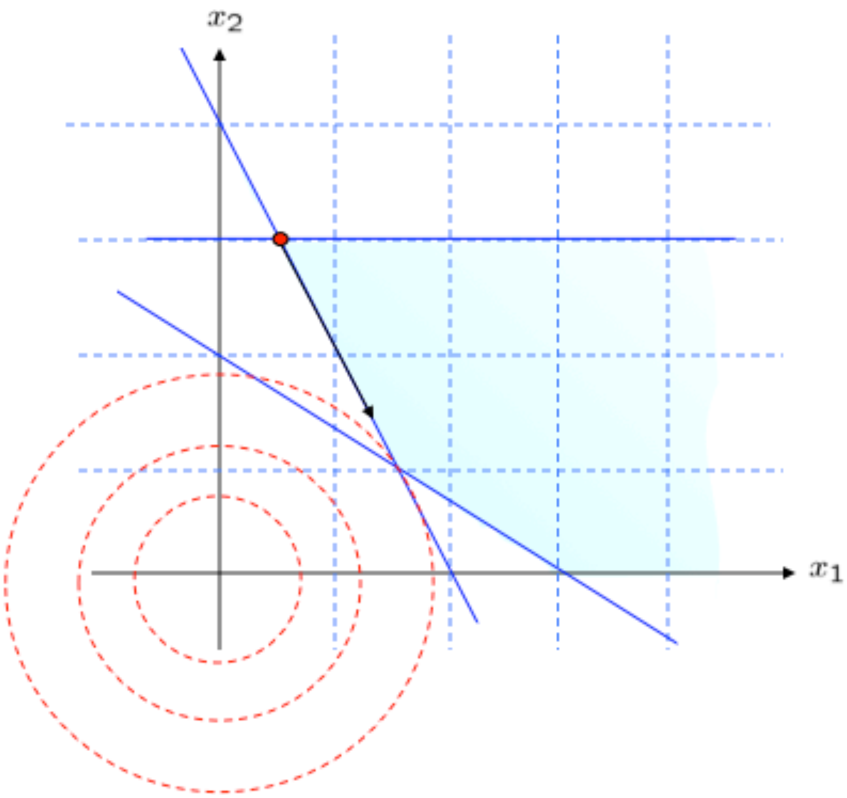
Suppose

$$\mathbf{x}^i = \begin{bmatrix} 0.5 \\ 3 \end{bmatrix}$$

$$\Delta \mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Feasible and improving if

$$\lambda < 11/5$$



Improving Search Method

Step 0. Find a starting feasible solution $\mathbf{x}^0$. If none exists, STOP the problem is infeasible. Initialize $i \leftarrow 0$.

Step 1. Find an improving feasible direction $\Delta\mathbf{x}^{i+1}$ at $\mathbf{x}^i$. If none exists STOP, $\mathbf{x}^i$ is a local optimal solution.

Step 2. Find the largest step size λ_{i+1} such that the point $\mathbf{x}^i + \lambda_{i+1}\Delta\mathbf{x}^{i+1}$ is feasible and has a better objective value than $\mathbf{x}^i$. If $\lambda_{i+1} = +\infty$ STOP the problem is unbounded.

Step 3. Update $\mathbf{x}^{i+1} \leftarrow \mathbf{x}^i + \lambda_{i+1}\Delta\mathbf{x}^{i+1}$ and $i \leftarrow i + 1$. Return to Step 1.

Example

$$\begin{array}{ll} \min & (x_1-2)^2+(x_2-2)^2 \\ \text{s.t.} & 2x_1+x_2 \le 4 \\ & x_1,\ x_2 \ge 0 \end{array}$$

Consider the point $\mathbf{x}^0 = (0,4)^\top$ and the improving feasible direction $\Delta\mathbf{x}^1 = (1,-2)^\top$

$$\mathbf{x}^1 = (0+\lambda, 4-2\lambda)^\top$$

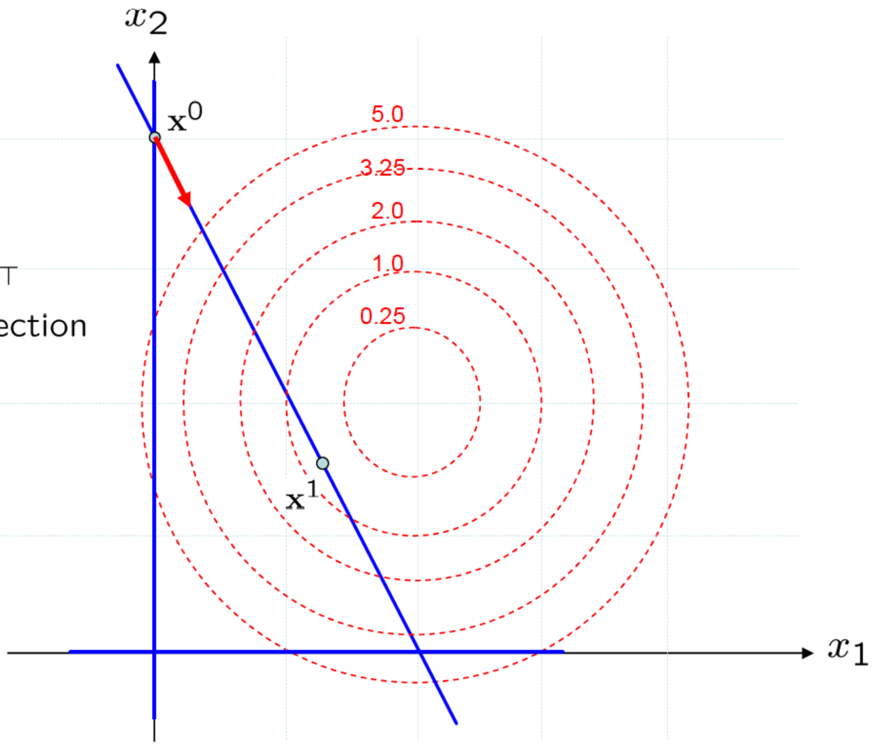
$$\Rightarrow \lambda \le 2$$

$$f(\mathbf{x}^1) = (\lambda-2)^2+(2-2\lambda)^2$$

$$\frac{df(\mathbf{x}^1)}{d\lambda} = 10\lambda - 12 = 0$$

$$\Rightarrow \lambda = 1.2$$

$$\mathbf{x}^1 = (1.2, 1.6)^\top$$



Remarks

- Methods for finding improving and feasible directions and corresponding step size depend on the problem structure
- The improving search method is designed to find a local solution at best
- So, for convex problems, it can provide global optimal solutions (if properly designed)

Summary

- Improving search is an algorithmic idea based on moving from one solution to a better solution
- For convex problems, it can (in principle) be designed to provide a global optimal solution