ISyE 6669 HW 2

Fall 2025

1. Expand the following summations:
   (For example, the answer to part (a) is $x_1 + x_2 + x_3$.)

(a)	$\sum_{i=1}^{3} x_i$	(d)	$\sum_{i=1}^{3} \sum_{j=2}^{4} (x_i + y_{ij})$
(b)	$\sum_{t=1}^{3} \frac{x^{2t}}{t!}$	(e)	$\sum_{k=-1}^{3} (2k+1) x_{k+1}$
(c)	$\sum_{i=1}^{3} \sum_{j=1}^{i} x_{ij}$	(f)	$\sum_{n=3}^{5} \sum_{m=n+1}^{n+3} x_n y_m$

   Note that by definition $t! = 1 \cdot 2 \cdots (t-1) \cdot t$ for integer $t \geq 1$.

   Solution:

   (a) $\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3$

   (b) $\sum_{t=1}^{3} \frac{x^{2t}}{t!} = \frac{x^{2\cdot 1}}{1!} + \frac{x^{2\cdot 2}}{2!} + \frac{x^{2\cdot 3}}{3!} = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

   (c) $$\sum_{i=1}^{3} \sum_{j=1}^{i} x_{ij} = \sum_{j=1}^{1} x_{1,j} + \sum_{j=1}^{2} x_{2,j} + \sum_{j=1}^{3} x_{3,j} =$$
   $$= x_{1,1} + x_{2,1} + x_{2,2} + x_{3,1} + x_{3,2} + x_{3,3}$$

   (d) $$\sum_{i=1}^{3} \sum_{j=2}^{4} (x_i + y_{ij}) = \sum_{i=1}^{3} ((x_i + y_{i,2}) + (x_i + y_{i,3}) + (x_i + y_{i,4})) =$$
   $$= \sum_{i=1}^{3} (3x_i + y_{i,2} + y_{i,3} + y_{i,4}) =$$
   $$= 3x_1 + y_{1,2} + y_{1,3} + y_{1,4} + 3x_2 + y_{2,2} + y_{2,3} + y_{2,4} + 3x_3 + y_{3,2} + y_{3,3} + y_{3,4}$$

   (e) $$\sum_{k=-1}^{3} (2k+1) x_{k+1} = (2 \cdot (-1) + 1) x_{-1+1} + (2 \cdot 0 + 1) x_{0+1} + (2 \cdot 1 + 1) x_{1+1} + (2 \cdot 2 + 1) x_{2+1} + (2 \cdot 3 + 1) x_{3+1} =$$
   $$= -x_0 + x_1 + 3x_2 + 5x_3 + 7x_4$$

   (f) $$\sum_{n=3}^{5} \sum_{m=n+1}^{n+3} x_n y_m = \sum_{m=4}^{6} x_3 y_m + \sum_{m=5}^{7} x_4 y_m + \sum_{m=6}^{8} x_5 y_m =$$
   $$= x_3 y_4 + x_3 y_5 + x_3 y_6 + x_4 y_5 + x_4 y_6 + x_4 y_7 + x_5 y_6 + x_5 y_7 + x_5 y_8$$

2. Consider the following two vectors: $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$, and a matrix $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & 0 & -1 \end{bmatrix}$.

   (a) Let n be the dimension of $\boldsymbol{x}$ and $\boldsymbol{y}$. What is the value of n?

   (b) Compute $3\boldsymbol{x} - 2\boldsymbol{y}$.

   (c) Compute the inner product $\boldsymbol{x}^\top \boldsymbol{y}$.

   (d) Compute $\boldsymbol{x}\boldsymbol{y}^\top$.

   (e) Compute the Euclidean norm $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. Also called the ℓ_2-norm.

   (f) Compute the ℓ_1-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = \sum_{i=1}^{n} |x_i - y_i|$.

   (g) Compute the ℓ_∞-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|$.

   (e) Compute $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}$.

   Solution:

(a) The dimension of $\boldsymbol{x}$ and of $\boldsymbol{y}$ are 3. Thus, $n = 3$.

(b) $3x - 2y = 3\begin{bmatrix}2\\1\\4\end{bmatrix} - 2\begin{bmatrix}2\\0\\5\end{bmatrix} = \begin{bmatrix}6\\3\\12\end{bmatrix} - \begin{bmatrix}4\\0\\10\end{bmatrix} = \begin{bmatrix}2\\3\\2\end{bmatrix}$.

(c) $\boldsymbol{x}^\top \boldsymbol{y} = \begin{bmatrix}2 & 1 & 4\end{bmatrix}\begin{bmatrix}2\\0\\5\end{bmatrix} = 2\cdot 2 + 1\cdot 0 + 4\cdot 5 = 24.$

(d) $\boldsymbol{x}\boldsymbol{y}^\top = \begin{bmatrix}2\\1\\4\end{bmatrix}\begin{bmatrix}2 & 0 & 5\end{bmatrix} = \begin{bmatrix}2\cdot 2 & 2\cdot 0 & 2\cdot 5\\ 1\cdot 2 & 1\cdot 0 & 1\cdot 5\\ 4\cdot 2 & 4\cdot 0 & 4\cdot 5\end{bmatrix} = \begin{bmatrix}4 & 0 & 10\\ 2 & 0 & 5\\ 8 & 0 & 20\end{bmatrix}$.

(e) $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{(2-2)^2 + (1-0)^2 + (4-5)^2} = \sqrt{2}.$

(f) $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = |2-2| + |1-0| + |4-5| = 2$

(g) $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max\{|2-2|, |1-0|, |4-5|\} = 1$

(e) $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix}2 & 1 & 4\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\ -2 & 1 & -1\\ 3 & 0 & -1\end{bmatrix}\begin{bmatrix}2\\0\\5\end{bmatrix} =$

$$= \begin{bmatrix}2\cdot 1 + 1\cdot(-2) + 4\cdot 3 & 2\cdot(-1) + 1\cdot 1 + 4\cdot 0 & 2\cdot 2 + 1\cdot(-1) + 4\cdot(-1)\end{bmatrix}\begin{bmatrix}2\\0\\5\end{bmatrix} =$$

$$= \begin{bmatrix}12 & -1 & -1\end{bmatrix}\begin{bmatrix}2\\0\\5\end{bmatrix} = 12\cdot 2 + (-1)\cdot 0 + (-1)\cdot 5 = 19.$$

3. State whether each of the following sets is convex or not. Explain your reasoning.

(a) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1^2 + 5|x_2| \leq 10\}$.

(b) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1^2 - 5|x_2| \leq 10\}$.

(c) $X = \{(x_1, x_2) \mid \frac{x_1}{(x_2-2)} \leq 2, \quad x_2 \geq 1\}$.

(d) $X = \{(x_1, x_2) \mid \frac{x_1}{(x_2-2)} \leq 2, \quad x_2 \geq 2\}$.

Solution:

(a) Set X is convex: Function $f(x_1, x_2) = 2X_1^2 + 5|x_2|$ is a convex function because it is nonnegative weighted sum of two convex functions (quadratic function and absolute value function). Set X is $\alpha-$level set of a convex function $f(x_1, x_2)$ and hence X is convex.

(b) Set X is not convex: For example points $a = (3, -2)$ and $b = (3, 2)$ are in the set X, but for $\lambda = \frac{1}{2}$ we get point:

$$\lambda a + (1-\lambda)b = \frac{1}{2}(3, -2) + \frac{1}{2}(3, 2) = (3, 0)$$

which does not belong to the set X.

(c) Set X is not convex: Notice that points for which $x_2 = 2$ do not belong to the set, because first inequality is not defined for those points. Then for $x_2 > 2$ we have $x_2 - 2 > 0$ and hence for points such that $x_2 > 2$ we have $\frac{x_1}{x_2-2} \leq 2 \Leftrightarrow x_1 \leq 2(x_2 - 2)$. On the other hand, for $x_2 < 2$ we have $x_2 - 2 < 0$ and hence for points such that $x_2 < 2$ we have $\frac{x_1}{x_2-2} \leq 2 \Leftrightarrow x_1 \geq 2(x_2 - 2)$ With that in mind we can rewrite set as:

$$X = \left\{(x_1, x_2) \;\middle|\; \frac{x_1}{(x_2-2)} \leq 2, \quad x_2 \geq 2\right\} =$$
$$= \{(x_1, x_2) : x_1 \geq 2(x_2 - 2), \ x_2 < 2\} \cup \{(x_1, x_2) : x_1 \leq 2(x_2 - 2), \ x_2 > 2\}$$

Set $\{(x_1, x_2) : x_1 \geq 2(x_2 - 2), \ x_2 < 2\}$ is defined using linear constraints and thus convex. To be more specific:

$$\{(x_1, x_2) : x_1 \geq 2(x_2 - 2), \ x_2 < 2\} =$$
$$\{(x_1, x_2) : x_1 \geq 2(x_2 - 2)\} \cap \{(x_1, x_2) : x_2 < 2\}$$

Sets $\alpha-$level sets $\{(x_1, x_2) : x_1 \geq 2(x_2 - 2)\}$ and $\{(x_1, x_2) : x_2 < 2\}$ are convex because they are $\alpha-$level sets of linear functions (and linear functions are convex). Set $\{(x_1, x_2) : x_1 \geq 2(x_2 - 2),\ x_2 < 2\}$ is intercept of convex sets and thus convex.

Similarly set $\{(x_1, x_2) : x_1 \leq 2(x_2 - 2),\ x_2 > 2\}$ is convex.

However union of two convex sets does not have to be convex and in this case it is not. For example points $a = (2, 1)$ and $b = (2, 3)$ belong to set X, but for $\lambda = \frac{1}{2}$ we get point:

$$\lambda a + (1 - \lambda)b = \frac{1}{2}(2, 1) + \frac{1}{2}(2, 3) = (2, 2)$$

which does not belong to set X.

(d) Set X is convex: As in the previous part, notice that points for which $x_2 = 2$ do not belong to the set. Then from $x_2 > 2$ we have $x_2 - 2 > 0$ and hence for points $(x_1, x_2) \in X$ we have $\frac{x_1}{x_2 - 2} \leq 2 \Leftrightarrow x_1 \leq 2(x_2 - 2)$ With that in mind we can rewrite set as:

$$\begin{aligned} X = &\left\{(x_1, x_2) \;\middle|\; \frac{x_1}{(x_2 - 2)} \leq 2,\quad x_2 \geq 2\right\} = \\ &= \{(x_1, x_2) : x_1 \leq 2(x_2 - 2),\ x_2 > 2\} = \\ &= \{(x_1, x_2) : x_1 \leq 2(x_2 - 2)\} \cap \{(x_1, x_2) : -x_2 < -2\} \end{aligned}$$

Set X is defined using linear constraints and thus it is convex. In particular, sets $\alpha-$level sets $\{(x_1, x_2) : x_1 \leq 2(x_2 - 2)\}$ and $\{(x_1, x_2) : -x_2 < -2\}$ are convex because they are $\alpha-$level sets of linear functions (and linear functions are convex). Set X is intercept of convex sets and thus convex.

4. State whether the following problems are convex programs or not. Explain your reasoning.

(a) $\min\{x_1^3 + x_2^2 \ :\ x_1 \leq 2,\ x_2 \leq 3\}$.

(b) $\max\{2x_1 + 3x_2 + 4x_3 + 5x_4 \ :\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}$.

(c) $\min\{\sum_{i=1}^n 2^i (x_i)^{2i} \ :\ \sum_{i=1}^n x_i \geq 10\}$.

Solution:

(a) This is not a convex program: Problem is minimization. The feasible region is a convex set because it is defined using linear constraints. However the objective function $f(x_1, x_2) = x_1^3 + x_2^2$ is not a convex function over $\{(x_1, x_2) : x_1 \leq 2, x_2 \leq 3\}$. To show this let's calculate its Hessian

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 & 0 \\ 0 & 2 \end{bmatrix}$$

.

Eigenvalues of $\nabla^2 f(x_1, x_2)$ are $6x_1$ and 2. Since for some points from feasible set (for example $(x_1 = -1, x_2 = 1)$) one of the eigenvalues will be negative, Hessian is not always psd, and thus function $f(x_1, x_2)$ is not convex.

(b) This is a convex program: The feasible region $\{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}$ is a ball, and thus a convex set. Problem is maximization and objective function is concave, hence problem is convex. Alternatively, problem is equal to $-\min\{-2x_1 - 3x_2 - 4x_3 - 5x_4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}$ which is minimization of a convex function over a convex set and thus convex problem.

(c) This is a convex program: The feasible region $\{(x_1, ..., x_n) : \sum_{i=1}^n x_i \geq 10\}$. is a half space (i.e. defined by linear inequality) and thus a convex set. The objective function $f(x_1, ..., x_n) = \sum_{i=1}^n 2^i (x_i)^{2i}$ is sum of nonnegative weighted (i.e. all coefficients 2^i are nonnegative numbers) convex functions (functions $f_i(x) = x_i^(2i)$ are convex for all $x_i \in \mathbb{R}^n$). Hence objective function is convex. Since we have minimization of a convex function over convex set, problem is convex.

5. A quantity y is known to depend upon another quantity x. A set of n data pairs $\{y_i, x_i\}_{i=1}^n$ has been collected.

(a) Formulate an optimization model for fitting the "best" straight line $y = a + bx$ to the data set, where best is with respect to the sum of absolute deviations. What kind of an optimization model is it?

(b) Re-formulate the optimization model in part (a) where best is with respect to the maximum absolute deviation. What kind of an optimization model is it?

(c) Formulate an optimization model for fitting the "best" quadratic curve $y = a + bx + cx^2$ to the data set, where best is with respect to the maximum absolute deviations. What kind of an optimization model is it ?

Solution:

(a) The optimization problem can be directly formulated as

$$\min_{\forall a,b \in \mathbb{R}} \quad \sum_{i=1}^{n} \mid y_i - a - bx_i \mid$$

which is an unconstrained convex nonlinear program.

(b) The optimization problem can be directly formulated as

$$\min_{\forall a,b \in \mathbb{R}} \max_{i=1,\ldots,n} \mid y_i - a - bx_i \mid$$

which is an unconstrained convex nonlinear program.

(c) The optimization problem can be directly formulated as

$$\min_{\forall a,b,c \in \mathbb{R}} \quad \max_{i=1,\ldots,n} \mid y_i - a - bx_i - cx_i^2 \mid$$

which is an unconstrained convex nonlinear program.