Deterministic Optimization

Convexity

Shabbir Ahmed
Anderson-Interface Chair and Professor
School of Industrial and Systems Engineering

Convex Optimization Problems

Convex Optimization Problems

Learning objectives:

- Recall definition of a convex optimization problem
- Recognize if a given optimization problem convex

Convex Optimization Problem

- An optimization problem (in minimization) form is a convex optimization problem, if the objective function is a convex function and constraint set is a convex set.

- The problem $\min_{\mathbf{x}}\{f(\mathbf{x}) : \ \mathbf{x} \in X\}$ is a convex optimization problem if f is a convex function and X is a convex set.

- Examples

  o $\min\{x^2 : -1 \leq x \leq 1\}$ is a convex opt. problem.

  o $\min\{-x^2 : -1 \leq x \leq 1\}$ is not a convex opt. problem.

  o $\min\{x^2 : -5 \leq x \leq 5, x \in \mathbb{Z}\}$ is not a convex opt. problem.

Recognizing a Convex Optimization Problem

The optimization problem

$$
\begin{array}{lll}
\min & f(\mathbf{x}) & \\
\text{s.t.} & g_i(\mathbf{x}) \leq b_i & i = 1, \ldots, m \\
 & h_j(\mathbf{x}) = d_j & j = 1, \ldots, \ell \\
 & \mathbf{x} \in \mathbb{R}^n &
\end{array}
$$

is convex if f is a convex functions, each g_i is a convex function and each h_j is a linear function.

Why?

- Note that we can write an equality constraint $h_j(\mathbf{x}) = d_j$ as two inequalities $h_j(\mathbf{x}) \leq d_j$ and $-h_j(\mathbf{x}) \leq -d_j$.
- Now define $X_{g_i} = \{\mathbf{x} : g_i(\mathbf{x}) \leq b_i\}$. Note that this is a level set of a convex function, and so is a convex set.
- Also define $X_{h_j} = \{\mathbf{x} : h_j(\mathbf{x}) \leq d_j\}$ and $X_{-h_j} = \{\mathbf{x} : -h_j(\mathbf{x}) \leq -d_j\}$. Since h_j is linear h_j is convex and $-h_j$ is convex. Thus both sets are convex.

Why? (contd)

- The constraint set of the problem can be written as

$$X = X_{g_1} \cap \dots X_{g_m} \cap X_{h_1} \cap \dots X_{h_\ell} \cap X_{-h_1} \cap \dots X_{-h_\ell}$$

- Since intersection of convex sets is convex, so X is a convex set.
- Also f is a convex function, hence the problem is a convex optimization problem.

Checking convexity

1. Check that all variables are continuous

2. Check that the objective function is convex

3. Check each equality constraint to see if it is linear

4. Write each constraint as an inequality constraint in $\leq$ form with a constant on the right-hand-side, and check the convexity of the function on the left-hand-side.

If it passes all the checks then you have a convex optimization problem. Otherwise, it may or may not be convex (the conditions are sufficient, not necessary).

Convexity of optimization problems

Functions	Variable domains	Problem Type	Convex?
All linear	Continuous variables	Linear Program (LP) or Linear Optimization problem	Yes
Some nonlinear	Continuous variables	Nonlinear Program (NLP) or Nonlinear Optimization Problem	Depends
Linear/nonlinear	Some discrete	Integer Program (IP) or Discrete Optimization Problem	Not in general

Summary

- A convex optimization problem (in min form) has a convex objective and convex set of solutions.
- To check if a given problem is convex, we can check convexity of each constraint separately. (This is a sufficient test, not necessary).
- Linear programs are convex optimization problems.