# Deterministic Optimization

## Outcomes of Optimization

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### Existence of Optimal Solutions

# Existence of optimal solutions

**Learning objective:**

- Identify a sufficient condition for the existence of optimal solutions

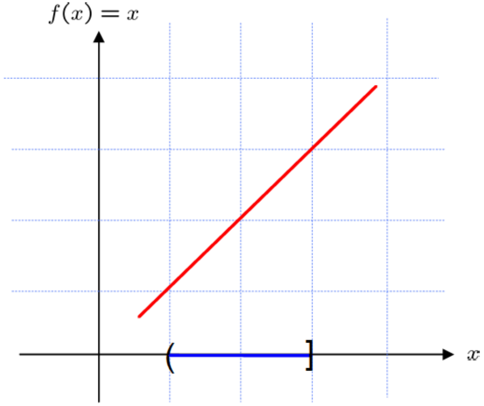
# Possible outcomes of optimization

$$(P): \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

1. Infeasible: $X = \emptyset$
2. Unbounded: $f(\mathbf{x}^i) \to -\infty$ for some $\{\mathbf{x}^i\} \in X$
3. Optimal solution exists: There is $\mathbf{x}^* \in X$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$
4. None of the above

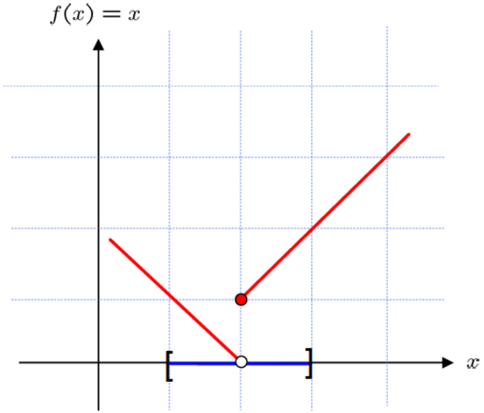
# No optimal solution

No Optimal
solution




$$\begin{array}{ll} \min & x \\ \text{s.t.} & 1 < x \leq 3 \end{array}$$

No Optimal
solution




$$f(x) = \begin{cases} 2 - x & \text{if } x < 2 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & 1 \leq x \leq 3 \end{array}$$

# Weierstrass's Theorem

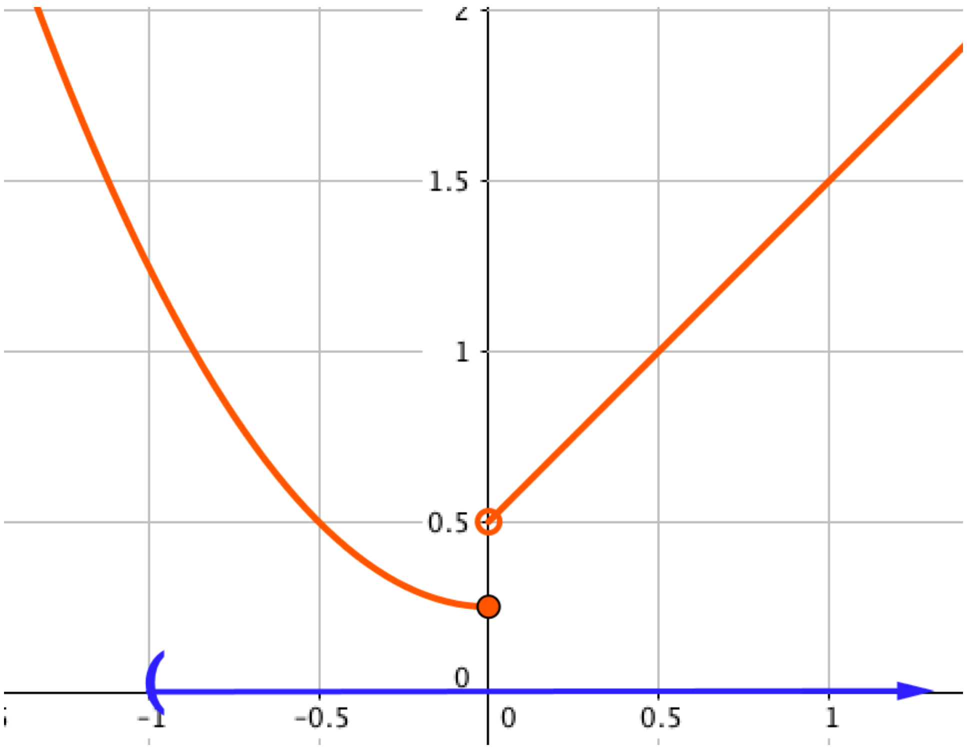
$$(P): \qquad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array}$$

If $f$ is a continuous function and $X$ is a nonempty, closed and bounded set, then $(P)$ has an optimal solution, i.e. there exists $x^* \in X$ such that $f(x^*) \leq f(x)$ for all $x \in X$

# Remarks

- $X$ is not empty, thus $(P)$ cannot be infeasible
- $X$ is bounded, this $(P)$ cannot be unbounded
- Then continuity and closedness guarantees an optimal solution exists

# Sufficient but not necessary



# Summary

- If the objective function is continuous, and the feasible region is nonempty, closed and bounded, then the problem must have an optimal solution
- When modeling use continuous objective functions and closed sets