Drawing Math

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1 Introduction

- Numberphile video - I was bored and this looked fun - I wasn't paying attention in one of my lectures... - This paper is purely for funs... for now

2 Background

- Digital math - Euler's formula - I payed attention in some of my lectures

3 Definitions and questions

3.1 Definitions

Say you (yes you!) had a turtle living in D dimensional Euclidean space and in discrete time. At time step i, where $i \in \mathbb{Z}$ and $i > 0$, the turtle has position $p_i \in \mathbb{R}^D$. Then, lets define $\Delta p_{i+1} = p_{i+1} - p_i$; in other words, Δp_{i+1} is the change in position from time i to $i+1$.

Now say that the turtle's movement is determined by k seed parameter drawn from the same set. Then, for some state space $\mathcal{S}$, define $s_i^j \in \mathcal{S}$ to be some arbitrary state associated with timestamp i for the jth seed parameter where $j \in [k]$. Also, define $\boldsymbol{s}_i = (s_i^1, s_i^2, ..., s_i^k)$. Next we will define a set of functions $SU^j : \mathcal{S} \to \mathcal{S}$ (for State Updater) such that $s_{i+1}^j = SU^j(s_i^j, i)$. Note that for $j, a \in [m]$ where $j \neq a$, s_{i+1}^j is determined solely by s_i^j and i and not s_i^a.

Now that we have our machinery built up, lets define $Comb : \mathcal{S}^k \to \mathbb{R}^d$ such that

$$\Delta p_{i+1} = Comb\left(s_{i+1}^1, s_{i+1}^2, ..., s_{i+1}^k\right).$$

In other words, $Comb$ takes in the state of each seed and returns an update to the position of the turtle.

Finally, let us define

$$\Delta P_{a,b} = \sum_{i=a}^{b} \Delta p_i.$$

In other words, $\Delta P_{a,b}$ is the change in position from timestep a to b.

3.2 The problem

Say we are given, $Comb$, hhSU^j, p_0, and s_0^j for all $j \in [k]$. Informally, the question is whether the turtle draws a "closed" shape or not.
More formally, is there some period T such that

$$p_{i+\ell T} = p_i$$

for $i, \ell \in \mathbb{N}$. Then, note that if there exists a period T such that $\Delta P_{i,i+\ell T} = 0$ for all $i, \ell \in N$, $p_{i+\ell T} = p_i$ and the turtle forms a closed shape.

3.3 Specifying the task ahead of us

For our case, we consider $Comb, SU_i^j$ to all be memoryless (i.e. there output is uniquely determined by the current input). So, we can simplify the overall question. If, $\boldsymbol{s}_i = \boldsymbol{s}_{i+\ell T}$ for some $T \in \mathbb{N}$ and all $i \in \mathbb{N}$, then $\Delta p_i = \Delta p_{i+\ell T}$. So then, $\Delta P_{i,i+\ell T} = \Delta P_{i,i+\ell' T}$ for all $\ell, \ell' \in \mathbb{N}$. Thus, T is a period of the change in position. We can thus break down our problem into two parts:

1. Finding the period, T, of the state $\boldsymbol{s}$.

2. Checking whether $\Delta P_{i,i+T} = 0$.

3.4 Some more restrictions on our problem

We further restrict the problem by only considering $\mathcal{S} = \mathbb{N}^4$ where for $(n, d, b, \theta) \in \mathcal{S}$, n is the numerator of a rational in fraction form, d is the denominator, b is the base (i.e. base 10, base 12, etc.), and $\frac{2\theta}{b\pi}$ is an "angle" associated with the state.

Then, let $\phi^j : \mathbb{R} \to \mathbb{R}$ equal cos or sin.

Now, we will only consider

$$SU_i^j(n, b, d, \theta) = (n, b, d, \theta + \text{digit}(n, b, d, i) \mod b).$$

where digit(n, b, d, i) gives us the ith digit of the decimal expansion of $\frac{n}{d}$ in base b. For the sake of convenience, we will use the word "rational parameter" instead of "seed parameter" from here on out.

Moreover, we consider the case where

$$Comb((.,.,.,.,\theta^1),(.,.,.,.,\theta^2),...,(.,.,.,.,\theta^k)) = \left(\prod_{j=1}^{k} \phi^j\left(\frac{2\pi}{b^j}\cdot\theta^j\right)^{\text{incl}_1^j}, ..., \prod_{j=1}^{k} \phi^j\left(\frac{2\pi}{b^j}\cdot\theta^j\right)^{\text{incl}_D^j} \right)$$

where $\text{incl}_d^j \in \{0, 1\}$ for $d \in [D]$ indicates whether to include a given $x \in R$ determined by rational parameter j for position update in the dth dimension.

Finally, for simplicity's sake, assume that $\theta = 0$ for all $(n, b, d, \theta) \in \boldsymbol{s}_0$, $n < d$, and $\frac{n}{d}$'s decimal expansion is periodic after some $N \geq 0$ decimal places and does not terminate in base b.

Also, lets set

$$\boldsymbol{b} = \operatorname*{lcm}_{(n,b,d,\theta)\in\boldsymbol{s}_0} b.$$

In other words, $\boldsymbol{b}$ can be thought of as a "common base" among all rational parameters.

3.4.1 Some intuition

While the restrictions may seem arbitrary, they aptly match our original problem statement. The original problem statement derives a spherical change in position based off of a rational number's digit at a particular timestep. The polar change in position also has a fixed radius. Translating from a polar to cartesian update then only requires products of sins and coss. See [Blu60] for more details.

Take the three dimensional case for instance. The turtle's update cartesian space is given by

$$\begin{aligned} x &= \cos(\alpha) \\ y &= \sin(\alpha)\cos(\beta) \\ z &= \sin(\alpha)\sin(\beta) \end{aligned}$$

where $\alpha = \frac{2\pi}{b^1}\cdot\theta^1$ and $\beta = \frac{2\pi}{b^2}\cdot\theta^2$. We can thus see that our definition of $Comb$ captures the three dimensional case.

4 Does it close?

In understanding whether a set of given rationals, bases, and updated functions draw a closed shape in D dimensional space, we first need to find the period of the update delta, Δp_i. We then know that the total update over a period will be repeated indefinitely. Consequently, we then seek to find the total change in position over a period. If the total change is 0, the shape will close as the Turtle will end up at its starting point after every period length. If the total update is nonzero, the Turtle will not draw a closed shape.

4.1 Finding period T

4.1.1 Finding the period of $\frac{n}{d}$

We will first aim to find period T of the state $\boldsymbol{s}$. For some $(n, b, d, \theta) \in \boldsymbol{s}_0$, by [ho], we have that the period of the decimal expansion of $\frac{n}{d}$ can be determined by finding the smallest $T^{j\prime}$ such that

$$b^{T^{j\prime}} \equiv 1 \mod d. \tag{1}$$

See, appendix A for more detail. More generally though, any nontrivial $T^{j\prime}$ satisfying equation 1 will be a period of $\frac{n}{d}$.

Next, let

$$T' = \underset{j \in [k]}{\mathrm{lcm}}\ T^{j\prime}.$$

Remark 4.1 (Complexity)**.** Interestingly, period finding of rational numbers is intimately tied to the discrete log problem and factoring. For more information, check out [ho]. This gives some intuition that this closure problem may not be in BPP (Bounded Error Polynomial Time), but may be in BQP (Bounded Error Quantum Polynomial Time) by [Sho97].

4.1.2 Digital sum

Next, we introduce the idea digital sums. For some number $N \in \mathbb{N}$, N can be represented in base b via

$$N = \sum_{i=0}^{m} d_i b^i \tag{2}$$

where $m = \lceil \log_b N \rceil$ and, $\forall i \in [m]$, $d_i \in \mathbb{Z}_b$. Then, we define function digSum : $\mathbb{N} \to \mathbb{Z}_b$ to give the digital sum such that

$$\mathrm{digSum}(N) = \sum_{i=0}^{m} d_i. \tag{3}$$

Moreover, define $\sigma^j \in \mathbb{Z}_b$ such that

$$\sigma^j = \sum_{i=i_0}^{i_0+T'} \mathrm{digit}(n, d, b, i). \tag{4}$$

In other words, T' is the digital sum over one period.

Remark 4.2 (Complexity)**.** For $d > 2$, prime, and coprime to b, we can find σ^j in polytime by multiplying $(b-1) \cdot \frac{d-1}{2} \mod b$ [KC81]. The authors are unsure as to the complexity of finding σ^j otherwise.

4.1.3 Finding a period of θ^j

For $(n^j, b^j, d^j, \theta_i^j) = s_i^j$, recall that $\theta_{i+1}^j = \theta_i^j + \text{digit}(n, b, d, i) \mod b$. So, after period T',

$$\begin{aligned} \theta_{i+T'} &= \left(\theta_i + \sum_{\ell=i}^{T'+i} \text{digit}(n, b, d, \ell)\right) \mod b \\ &= \left(\theta_i + \sigma^j\right) \mod b. \end{aligned}$$

So, after p periods of length T' where $p \cdot \sigma^j \equiv 0 \mod b$,

$$\theta_{i+pT'} \equiv \theta_i + 0 \equiv \theta_i.$$

For simplicity, lets define

$$T^j = pT'$$

where T^j is a period of the state for rational parameter j.

4.1.4 Finding the period of $\boldsymbol{s}$

We can first see that for $s^j \in \boldsymbol{s}$, s^j has period of T^j. So, $\boldsymbol{s}$ must have a period, T, of

$$\operatorname*{lcm}_{j \in [k]} T^j.$$

I.e. $\boldsymbol{s}_i = \boldsymbol{s}_{i+T}$ for all $i \in \mathbb{N}$.

4.2 Finding the change in position over a period

So now that we know the period of $\boldsymbol{s}$, we can ask if $\Delta P_{i,i+T} = 0$.

Note that

$$\Delta P_{i,i+T} = \Delta P_{q,q+T}$$

for all $i, q \in \mathbb{N}$ by definition of periodicity. So, we will drop the i and replace it with a 0. Then,

$$\begin{aligned} \Delta P_{0,T} &= \sum_{i=1}^{T} \Delta p_i \\ &= \sum_{i=1}^{T} Comb\left(s_i^1, s_i^2, ..., s_i^k\right) \\ &= \sum_{i=1}^{T} \left(\prod_{j=1}^{k} \phi^j \left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_1^j}, ..., \prod_{j=1}^{k} \phi^j \left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_D^j} \right) \\ &= \left(\sum_{i=1}^{T} \prod_{j=1}^{k} \phi^j \left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_1^j}, ..., \sum_{i=1}^{T} \prod_{j=1}^{k} \phi^j \left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_D^j} \right). \end{aligned}$$

We can thus see that $\Delta P_{0,T} = \mathbf{0} = (0, ..., 0)$ iff

$$\sum_{i=1}^{T} \prod_{j=1}^{k} \phi^j \left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_d^j} = 0 \tag{5}$$

for all $d \in D$.

Remark 4.3 (Complexity)**.** The algorithm we provide in equation (5) runs in time exponential in the size of the input assuming the Word RAM model. The period for the rational generated from rational parameter j, $1 \le T^{j'} \le d^j$. Then, the period over all rationals generated from parameters is at most

$$\operatorname*{lcm}_{j\in[k]} T^{j'} \le \prod_{j\in[k]} T^{j'} \le \left(\max_{j\in[k]} d^j\right)^k.$$

Then, $0 \le T \le T' \cdot \operatorname*{lcm}_{j\in[k]} b^j \le T' \left(\max_{j\in[k]} b^j\right)^k$. And because evaluating the product in (5) takes $O(k)$ time, we have that the time for (5) is at most

$$O\left(\left[\max_{j\in[k]} (b^j d^j)\right]^k\right).$$

Because (5) must be computed for each dimension, the algorithm runs in

$$O\left(D\left[\max_{j\in[k]} (b^j d^j)\right]^k\right)$$

time.

5 Conclusion

6 Open Questions

Acknowledgments

A Proving Property 1

Before getting to the main result, we need to first prove the following lemma

Lemma A.1. *For all $j \in [k]$ and $x, y \in \mathbb{N}$ where $y < T'$, we have that*

$$\theta^j_{xT'+y} = x \cdot \sigma^j + \sum_{q=0}^{y} \mathrm{digit}(n, b, d, q)$$

Proof. We can then see that for $(n, b, d, \theta^j_{xT'+y}) \in \boldsymbol{s}_{xT'+y}$,

$$\begin{aligned}
\theta^j_{xT'+y} &= \sum_{i=0}^{xT'+y} \mathrm{digit}(n, b, d, i) \\
&= \sum_{p=0}^{(x-1)T'} \sum_{q=0}^{T'-1} \mathrm{digit}(n, b, d, pT' + q) + \sum_{q=xT'}^{xT'+y} \mathrm{digit}(n, b, d, q) \\
&= x \cdot \sigma^j + \sum_{q=xT'}^{T'+y} \mathrm{digit}(n, b, d, q) \\
&= x \cdot \sigma^j + \sum_{q=0}^{y} \mathrm{digit}(n, b, d, q)
\end{aligned}$$

because $\mathrm{digit}(n, b, d, xT' + \ell) = \mathrm{digit}(n, b, d, \ell)$ for any $\ell \in \mathbb{N}$ by definition of periodicity. □

A.0.1 Closing along a single dimension

Let $\Delta P^d_{0,T}$ be the change of position along dimension d from timestep 0 to T. We are now ready to determine if we "close" along one dimension. I.e. does $\Delta P^d_{0,T} = 0$?

Define $A_d = \{\, j \mid j \in [k] \text{ and } \mathrm{incl}^j_d = 1 \,\}$, in other words, A_d is the set of rational parameters which are included in determining the position along the dth dimension. We can then see that

$$\begin{aligned}
\Delta P^d_{0,T} &= \sum_{i=1}^{T} \prod_{j=1}^{k} \phi^j \left(\frac{2\pi}{b^j} \cdot \theta^j \right)^{\mathrm{incl}^j_d} \\
&= \sum_{i=1}^{T} \prod_{j=1}^{k} \left(\tfrac{1}{2} \left(\exp\left(\frac{2\pi}{b^j} \theta^j_i I \right) + \exp\left(-\frac{2\pi}{b^j} \theta^j_i I \right) \right) \right)^{\mathrm{incl}^j_d} \\
&= 2^{-|A|} \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \prod_{j \in A_d} \left(\exp\left(\frac{2\pi}{b^j} \theta^j_{pT'+q} I \right) + \exp\left(-\frac{2\pi}{b^j} \theta^j_{pT'+q} I \right) \right)
\end{aligned}$$

by the Euler form of cos.

Next, observe that

$$\begin{aligned}
&\prod_{j \in A_d} \left(\exp\left(\frac{2\pi}{b^j} \theta^j_{pT'+q} I \right) + \exp\left(-\frac{2\pi}{b^j} \theta^j_{pT'+q} I \right) \right) \\
&= \exp(\frac{2\pi}{b^1}\theta^1_{pT'+q} + \frac{2\pi}{b^2}\theta^2_{pT'+q} + \ldots + \frac{2\pi}{b^d}\theta^d_{pT'+q}) + \exp(\frac{2\pi}{b^1}\theta^1_{pT'+q} - \frac{2\pi}{b^2}\theta^2_{pT'+q} + \ldots + \frac{2\pi}{b^d}\theta^d_{pT'+q}) + \ldots \\
&\quad + \exp(-\frac{2\pi}{b^1}\theta^1_{pT'+q} - \frac{2\pi}{b^2}\theta^2_{pT'+q} - \ldots - \frac{2\pi}{b^d}\theta^d_{pT'+q})
\end{aligned}$$

which then equals

$$\sum_{\beta \in \{0,1\}^{|A_d|}} \exp\left(\frac{2\pi}{\boldsymbol{b}} I \sum_{j \in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j} \theta^j_{pT'+q}\right) \tag{6}$$

where β can be though of as a bit string deciding whether the angle from seed $j \in A_d$ is added to or subtracted from the exponent.

Then, we have that

$$\begin{aligned}
\Delta P^d_{0,T} =& 2^{-|A|} \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \sum_{\beta \in \{0,1\}^{|A_d|}} \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right) \\
=& 2^{-|A|} \sum_{\beta \in \{0,1\}^{|A_d|}} \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right).
\end{aligned}$$

Then, lets fix some $\beta \in \{0,1\}^{|A_d|}$, define Q such that

$$Q = \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right). \tag{7}$$

We will simplify Q to show 2 distinct cases where $Q = 0$ for any choice of β.

Observe that

$$\begin{aligned}
\exp(\theta^j_{pT'+q} I) &= \exp\left(p \cdot \sigma^j + \frac{\boldsymbol{b}}{b^j} \sum_{\ell = pT'}^{pT'+q} \mathrm{digit}(n^j, b^j, d^j, \ell)\right) && \text{(by lemma A.1)} \\
&= \exp\left(p \cdot \sigma^j\right) \exp\left(\frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \mathrm{digit}(n, b, d, \ell)\right). && (8)
\end{aligned}$$

So then, by equation (8), we get that

$$\begin{aligned}
&\exp\left(\sum_{j \in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right) \\
=& \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \cdot p \cdot \sigma^j\right) \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \mathrm{digit}(n^j, b^j, d^j, \ell).\right). \qquad (9)
\end{aligned}$$

We then use (9) to show that Q equals

$$\sum_{p=0}^{\frac{T}{T'}-1} \left[\exp\left(pI \sum_{j \in A_d} -1^{\beta(j)} \sigma^j\right) \left(\sum_{q=0}^{T'-1} \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \mathrm{digit}(n^j, b^j, d^j, \ell)\right)\right)\right]. \tag{10}$$

Case 1

Define

$$C_\beta = \sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \operatorname{digit}(n,b,d,\ell) \right).$$

Moreover, note that

$$\exp\left(pI \sum_{j\in A_d} -1^{\beta(j)} \sigma^j \right) = \prod_{j\in A_d} \exp\left(-1^{\beta(j)} pI \cdot \sigma^j \right)$$

and that

$$\exp\left(-1^{\beta(j)} pI \cdot \sigma^j \right) = \exp(0) = 1$$

when $p = \frac{T}{T'}$. So, we can see that

$$\prod_{j\in A_d} \exp\left(-1^{\beta(j)} pI \cdot \sigma^j \right) = 1$$

when $p = \frac{T}{T'}$.

Because σ^j is a constant, we can conclude that

$$\exp\left(I \sum_{j\in A_d} -1^{\beta(j)} \sigma^j \right)$$

is a $\frac{T}{T'}^{th}$ root of unity iff

$$\sum_{j\in A_d} -1^{\beta(j)} \sigma^j \neq 0$$

So, for $\sum_{j\in A_d} -1^{\beta(j)} \sigma^j \neq 0$, we have that

$$\begin{aligned}
\sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I \right) &= C_\beta \sum_{p=0}^{\frac{T}{T'}-1} \exp\left(pI \sum_{j\in A_d} -1^{\beta(j)} \sigma^j \right) \\
&= C_\beta \sum_{p=0}^{\frac{T}{T'}-1} \exp\left(W^p_{\frac{T}{T'}} \right) \\
&= 0.
\end{aligned}$$

where $W^p_{\frac{T}{T'}}$ is the $\frac{T}{T'}^{th}$ root of unity.

If $\sum_{j\in A_d} -1^{\beta(j)} \sigma^j = 0$, then

$$\begin{aligned}
\sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I \right) &= C_\beta \sum_{p=0}^{\frac{T}{T'}-1} \exp(0) \\
&= C_\beta.
\end{aligned}$$

So, we have now shown that (7) is always 0 or 1. In particular, we also have our first closure result. If, $\forall \beta \in \{0,1\}^{|A_d|}$, $\sum_{j \in A_d} -1^{\beta(j)} \neq 0$,

$$\begin{aligned}\sum_{i=1}^{T} \prod_{j=1}^{k} \phi^j \left(\frac{2\pi}{b^j} \cdot \theta^j \right)^{\mathrm{incl}_d^j} &= 2^{-|A|} \sum_{\beta \in \{0,1\}^{|A_d|}} \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp \left(\sum_{j \in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I \right) \\ &= 0\end{aligned}$$

for all $d \in D$. So then, $\Delta P_{0,T} = 0$ whenever no degree 1 multinomials with coefficients of -1 or 1 and $|A_d|$ variables has a root of $(\sigma^1, \sigma^2, ..., \sigma^{|A_d|})$. In other words, if variables, $x_1 = \sigma^1, x_2 = \sigma^2, ...$, there does not exist a multinomial of form

$$\pm x_1 \pm x_2 \pm ... \pm x_{|A_d|} = 0.$$

Said differently, let multinomial $m : \mathbb{Z}_{\boldsymbol{b}} \to \mathbb{Z}_{\boldsymbol{b}}$ be such that

$$m(x_1, x_2, ..., x_{|A_d|}) = \prod_{\beta \in \{0,1\}^{|A_d|}} \left(\sum_{j \in A_d} -1^{\beta_j} AAA \right).$$

Then, if $m(\sigma^1, \sigma^2, ..., \sigma^{|A_d|}) \neq 0$, we know that the shape must close.

There does exist

In the case that $\exists \beta \in \{0,1\}^{|A_d|}$ such that $\sum_{j \in A_d} -1^{\beta(j)} = 0$, it is still possible for

$$\sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp \left(\sum_{j \in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I \right) = 0$$

if $C_\beta = 0$. So,

$$\sum_{q=0}^{T'-1} \exp \left(\sum_{j \in A_d} -1^{\beta(j)} \sum_{\ell=0}^{q} \mathrm{digit}(n, b, d, \ell) \right) = 0$$

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