Drawing Math

Lev Stambler

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1 Introduction

- Numberphile video - I was bored and this looked fun - I wasn't paying attention in one of my lectures... - This paper is purely for funs... for now

2 Background

- Digital math - Euler's formula - I payed attention in some of my lectures

3 Definitions and questions

3.1 Definitions

Say you (yes you!) had a turtle living in D dimensional Euclidean space and in discrete time. At time step i, where $i \in \mathbb{Z}$ and $i > 0$, the turtle has position $p_i \in \mathbb{R}^D$. Then, lets define $\Delta p_{i+1} = p_{i+1} - p_i$; in other words, Δp_{i+1} is the change in position from time i to $i+1$.

Now say that the turtle's movement is determined by k seed parameters drawn from the same set. Then, for some state space $\mathcal{S}$, define $s_i^j \in \mathcal{S}$ to be some arbitrary state associated with timestamp i for the jth seed parameter where $j \in [k]$. Also, define $\boldsymbol{s}_i = (s_i^1, s_i^2, ..., s_i^k)$. Next we will define a set of functions $SU^j : \mathcal{S} \to \mathcal{S}$ (for State Updater) such that $s_{i+1}^j = SU^j(s_i^j, i)$. Note that for $j, a \in [m]$ where $j \neq a$, s_{i+1}^j is determined solely by s_i^j and i and not s_i^a.

Now that we have our machinery built up, lets define $Comb : \mathcal{S}^k \to \mathbb{R}^d$ such that

$$\Delta p_{i+1} = Comb\left(s_{i+1}^1, s_{i+1}^2, ..., s_{i+1}^k\right).$$

In other words, $Comb$ takes in the state of each seed and returns an update to the position of the turtle.

Finally, let us define

$$\Delta P_{a,b} = \sum_{i=a}^{b} \Delta p_i.$$

In other words, $\Delta P_{a,b}$ is the change in position from timestep a to b.

3.2 The problem

Say we are given, $Comb$, hhSU^j, p_0, and s_0^j for all $j \in [k]$. Informally, the question is whether the turtle draws a "closed" shape or not.
More formally, is there some period T such that

$$p_{i+\ell T} = p_i$$

for $i, \ell \in \mathbb{N}$. Then, note that if there exists a period T such that $\Delta P_{i,i+\ell T} = 0$ for all $i, \ell \in N$, $p_{i+\ell T} = p_i$ and the turtle forms a closed shape.

3.3 Specifying the task ahead of us

For our case, we consider $Comb, SU_i^j$ to all be memoryless (i.e. there output is uniquely determined by the current input). So, we can simplify the overall question. If, $\boldsymbol{s}_i = \boldsymbol{s}_{i+\ell T}$ for some $T \in \mathbb{N}$ and all $i \in \mathbb{N}$, then $\Delta p_i = \Delta p_{i+\ell T}$. So then, $\Delta P_{i,i+\ell T} = \Delta P_{i,i+\ell' T}$ for all $\ell, \ell' \in \mathbb{N}$. Thus, T is a period of the change in position. We can thus break down our problem into two parts:

1. Finding the period, T, of the state $\boldsymbol{s}$.
2. Checking whether $\Delta P_{i,i+T} = 0$.

3.4 Some more restrictions on our problem

We further restrict the problem by only considering $\mathcal{S} = N^5$ where for $(n, d, b, \theta) \in \mathcal{S}$, n is the numerator of a rational in fraction form, d is the denominator, b is the base (i.e. base 10, base 12, etc.), and $\frac{2\theta}{b\pi}$ is an "angle" associated with the state. Moreover, we will only consider

$$SU_i^j(n, b, d, \theta) = (n, b, d, \theta + \text{digit}(n, b, d, i) \mod b).$$

where $\text{digit}(n, b, d, i)$ gives us the ith digit of the decimal expansion of $\frac{n}{d}$ in base b.

Moreover, we consider the case where

$$Comb((.,.,.,.,\theta^1), (.,.,.,.,\theta^2), ..., (.,.,.,.,\theta^k)) = \left(\prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_1^j}, ..., \prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_D^j}\right)$$

where $\text{incl}_d^j \in \{0, 1\}$ for $d \in [D]$ indicates whether to include a given $x \in R$ determined by seed j for position update in the dth dimension.

Finally, for simplicity's sake, assume that $\theta = 0$ for all $(n, b, d, \theta) \in \boldsymbol{s}_0$, $n < d$, and $\frac{n}{d}$'s decimal expansion is periodic after some $N \geq 0$ decimal places and does not terminate in base b.

Also, lets set

$$\boldsymbol{b} = \operatorname*{lcm}_{(n,b,d,\theta) \in \boldsymbol{s}_0} b.$$

In other words, $\boldsymbol{b}$ can be thought of as a "common base" among all seeds.

3.4.1 Some intuition

While the restrictions may seem arbitrary, they aptly match our original problem statement. Take the three dimensional case with two seeds. The two seeds can be thought of as the fraction which generates the updates to two angles, θ, α. θ and α can then uniquely determine a direction in 3D space in which the turtle is pointing. So, at each time step $i + 1$, θ_{i+1} equals to θ_i, but rotated by some fraction of 2π determined by the ith digit of the decimal expansion of one fraction and its base.

4 Results

4.1 Finding period T

4.1.1 Finding the period of $\frac{n}{d}$

We will first aim to find period T of the state $\boldsymbol{s}$. For some $(n, b, d, \theta) \in \boldsymbol{s}_0$, by [ho], we have that the period of the decimal expansion of $\frac{n}{d}$ can be determined by finding the smallest $T^{j'}$ such that

$$b^{T^{j'}} \equiv 1 \mod d. \tag{1}$$

See, appendix A for more detail. More generally though, any nontrivial $T^{j'}$ satisfying equation 1 will be a period of $\frac{n}{d}$.

Next, let

$$T' = \operatorname*{lcm}_{j \in [k]} T^{j'}.$$

Remark 4.1 (Polytime)**.** TODO: remark about discrete log and similarity... (we think this is hard, but in QPP)...

4.1.2 Digital sum

Next, we introduce the idea digital sums. For some number $N \in \mathbb{N}$, N can be represented in base b via

$$N = \sum_{i=0}^{m} d_i b^i \tag{2}$$

where $m = \lceil \log_b N \rceil$ and, $\forall i \in [m]$, $d_i \in \mathbb{Z}_b$. Then, we define function $\mathrm{digSum} : \mathbb{N} \to \mathbb{Z}_b$ to give the digital sum such that

$$\mathrm{digSum}(N) = \sum_{i=0}^{m} d_i. \tag{3}$$

Moreover, define $\sigma^j \in \mathbb{Z}_b$ such that

$$\sigma^j = \sum_{i=i_0}^{i_0+T'} \mathrm{digit}(n, d, b, i). \tag{4}$$

In other words, T' is the digital sum over one period.

Remark 4.2 (Finding the digit sum of a rational)**.** aaaa

4.1.3 Finding a period of θ^j

For $(n^j, b^j, d^j, \theta_i^j) = s_i^j$, recall that $\theta_{i+1}^j = \theta_i^j + \mathrm{digit}(n, b, d, i) \mod b$. So, after period T',

$$\begin{aligned}
\theta_{i+T'} &= \left(\theta_i + \sum_{\ell=i}^{T'+i} \mathrm{digit}(n, b, d, \ell)\right) \mod b \\
&= \left(\theta_i + \sigma^j\right) \mod b \\
&= \theta_i \mod b + \sigma^j \mod b.
\end{aligned}$$

So, after p periods of length T' where $p \cdot \sigma^j \equiv 0 \mod b$,

$$\theta_{i+pT'} = \theta_i.$$

For simplicity, lets define

$$T^j = pT'$$

where T^j is a period of the state for seed j.

4.1.4 Finding the period of $\boldsymbol{s}$

We can first see that for $s^j \in \boldsymbol{s}$, s^j has period of T^j. So, $\boldsymbol{s}$ must have a period, T, of

$$\operatorname*{lcm}_{j \in [k]} T^j.$$

I.e. $\boldsymbol{s}_i = \boldsymbol{s}_{i+T}$ for all $i \in \mathbb{N}$.

4.2 Does it close? $\Delta P_{i,i+T} = 0$?

So now that we know the period of $\boldsymbol{s}$, we can ask if $\Delta P_{i,i+T} = 0$.

Note that

$$\Delta P_{i,i+T} = \Delta P_{q,q+T}$$

for all $i, q \in \mathbb{N}$ by definition of periodicity. So, we will drop the i and replace it with a 0. Then,

$$\begin{aligned}
\Delta P_{0,T} &= \sum_{i=1}^{T} \Delta p_i \\
&= \sum_{i=1}^{T} Comb\left(s_i^1, s_i^2, ..., s_i^k\right) \\
&= \sum_{i=1}^{T} \left(\prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_1^j}, ..., \prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_D^j} \right) \\
&= \left(\sum_{i=1}^{T} \prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_1^j}, ..., \sum_{i=1}^{T} \prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_D^j} \right).
\end{aligned}$$

We can thus see that $\Delta P_{0,T} = \mathbf{0} = (0, ..., 0)$ iff

$$\sum_{i=1}^{T} \prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\text{incl}_d^j} = 0 \tag{5}$$

for all $d \in D$.

Before getting to the main result, we need to first prove the following lemma

Lemma 4.3. *For all $j \in [k]$ and $x, y \in \mathbb{N}$ where $y < T'$, we have that*

$$\theta^j_{xT'+y} = x \cdot \sigma^j + \sum_{q=0}^{y} \text{digit}(n, b, d, q)$$

Proof. We can then see that for $(n, b, d, \theta^j_{xT'+y}) \in \boldsymbol{s}_{xT'+y}$,

$$\begin{aligned}
\theta^j_{xT'+y} &= \sum_{i=0}^{xT'+y} \text{digit}(n, b, d, i) \\
&= \sum_{p=0}^{(x-1)T'} \sum_{q=0}^{T'-1} \text{digit}(n, b, d, pT' + q) + \sum_{q=xT'}^{xT'+y} \text{digit}(n, b, d, q) \\
&= x \cdot \sigma^j + \sum_{q=xT'}^{T'+y} \text{digit}(n, b, d, q) \\
&= x \cdot \sigma^j + \sum_{q=0}^{y} \text{digit}(n, b, d, q)
\end{aligned}$$

because $\text{digit}(n, b, d, xT' + \ell) = \text{digit}(n, b, d, \ell)$ for any $\ell \in \mathbb{N}$ by definition of periodicity. $\square$

4.2.1 Closing along a single dimension

Let $\Delta P^d_{0,T}$ be the change of position along dimension d from timestep 0 to T. We are now ready to determine if we "close" along one dimension. I.e. does $\Delta P^d_{0,T} = 0$?

Define $A_d = \{ j \mid j \in [k] \text{ and } \mathrm{incl}^j_d = 1 \}$, in other words, A_d is the set of seeds which are included in determining the position along the dth dimension. We can then see that

$$
\begin{aligned}
\Delta P^d_{0,T} &= \sum_{i=1}^{T} \prod_{j=1}^{k} \cos\left(\frac{2\pi}{b^j} \cdot \theta^j\right)^{\mathrm{incl}^j_d} \\
&= \sum_{i=1}^{T} \prod_{j=1}^{k} \left(\tfrac{1}{2}\left(\exp\left(\frac{2\pi}{b^j}\theta^j_i I\right) + \exp\left(-\frac{2\pi}{b^j}\theta^j_i I\right)\right)\right)^{\mathrm{incl}^j_d} \\
&= 2^{-|A|} \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \prod_{j\in A_d} \left(\exp\left(\frac{2\pi}{b^j}\theta^j_{pT'+q} I\right) + \exp\left(-\frac{2\pi}{b^j}\theta^j_{pT'+q} I\right)\right)
\end{aligned}
$$

by the Euler form of cos.

Next, observe that

$$
\begin{aligned}
&\prod_{j\in A_d} \left(\exp\left(\frac{2\pi}{b^j}\theta^j_{pT'+q} I\right) + \exp\left(-\frac{2\pi}{b^j}\theta^j_{pT'+q} I\right)\right) \\
&= \exp\left(\frac{2\pi}{b^1}\theta^1_{pT'+q} + \frac{2\pi}{b^2}\theta^2_{pT'+q} + \ldots + \frac{2\pi}{b^d}\theta^d_{pT'+q}\right) + \exp\left(\frac{2\pi}{b^1}\theta^1_{pT'+q} - \frac{2\pi}{b^2}\theta^2_{pT'+q} + \ldots + \frac{2\pi}{b^d}\theta^d_{pT'+q}\right) + \ldots \\
&\quad + \exp\left(-\frac{2\pi}{b^1}\theta^1_{pT'+q} - \frac{2\pi}{b^2}\theta^2_{pT'+q} - \ldots - \frac{2\pi}{b^d}\theta^d_{pT'+q}\right)
\end{aligned}
$$

which then equals

$$
\sum_{\beta\in\{0,1\}^{|A_d|}} \exp\left(\frac{2\pi}{\boldsymbol{b}} I \sum_{j\in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j}\theta^j_{pT'+q}\right) \tag{6}
$$

where β can be though of as a bit string deciding whether the angle from seed $j \in A_d$ is added to or subtracted from the exponent.

Then, we have that

$$
\begin{aligned}
\Delta P^d_{0,T} =& 2^{-|A|} \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \sum_{\beta\in\{0,1\}^{|A_d|}} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right) \\
=& 2^{-|A|} \sum_{\beta\in\{0,1\}^{|A_d|}} \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right).
\end{aligned}
$$

Then, lets fix some $\beta \in \{0,1\}^{|A_d|}$, define Q such that

$$
Q = \sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right). \tag{7}
$$

We will simplify Q to show 2 distinct cases where $Q = 0$ for any choice of β.

Observe that

$$\exp(\theta^j_{pT'+q} I) = \exp\left(p \cdot \sigma^j + \frac{\boldsymbol{b}}{b^j} \sum_{\ell=pT'}^{pT'+q} \mathrm{digit}(n^j, b^j, d^j, \ell)\right) \qquad \text{(by lemma 4.3)}$$
$$= \exp\left(p \cdot \sigma^j\right) \exp\left(\frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \mathrm{digit}(n, b, d, \ell)\right). \tag{8}$$

So then, by equation (8), we get that

$$\exp\left(\sum_{j \in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right)$$
$$= \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \cdot p \cdot \sigma^j\right) \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \mathrm{digit}(n^j, b^j, d^j, \ell).\right). \tag{9}$$

We then use (9) to show that Q equals

$$\sum_{p=0}^{\frac{T}{T'}-1} \left[\exp\left(pI \sum_{j \in A_d} -1^{\beta(j)} \sigma^j\right) \left(\sum_{q=0}^{T'-1} \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \mathrm{digit}(n^j, b^j, d^j, \ell)\right)\right)\right]. \tag{10}$$

Case 1

Define

$$C_\beta = \sum_{q=0}^{T'-1} \exp\left(\sum_{j \in A_d} -1^{\beta(j)} \frac{\boldsymbol{b}}{b^j} \sum_{\ell=0}^{q} \mathrm{digit}(n, b, d, \ell)\right).$$

Moreover, note that

$$\exp\left(pI \sum_{j \in A_d} -1^{\beta(j)} \sigma^j\right) = \prod_{j \in A_d} \exp\left(-1^{\beta(j)} pI \cdot \sigma^j\right)$$

and that

$$\exp\left(-1^{\beta(j)} pI \cdot \sigma^j\right) = \exp(0) = 1$$

when $p = \frac{T}{T'}$. So, we can see that

$$\prod_{j \in A_d} \exp\left(-1^{\beta(j)} pI \cdot \sigma^j\right) = 1$$

when $p = \frac{T}{T'}$.

Because σ^j is a constant, we can conclude that

$$\exp\left(I \sum_{j \in A_d} -1^{\beta(j)} \sigma^j\right)$$

is a $\frac{T}{T'}^{th}$ root of unity iff

$$\sum_{j\in A_d} -1^{\beta(j)}\sigma^j \neq 0$$

So, for $\sum_{j\in A_d} -1^{\beta(j)}\sigma^j \neq 0$, we have that

$$\begin{aligned}
\sum_{p=0}^{\frac{T}{T'}-1}\sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)}\theta^j_{pT'+q} I\right) &= C_\beta \sum_{p=0}^{\frac{T}{T'}-1} \exp\left(pI\sum_{j\in A_d} -1^{\beta(j)}\sigma^j\right)\\
&= C_\beta \sum_{p=0}^{\frac{T}{T'}-1} \exp\left(W^p_{\frac{T}{T'}}\right)\\
&= 0.
\end{aligned}$$

where $W^p_{\frac{T}{T'}}$ is the $\frac{T}{T'}^{th}$ root of unity.

If $\sum_{j\in A_d} -1^{\beta(j)}\sigma^j = 0$, then

$$\begin{aligned}
\sum_{p=0}^{\frac{T}{T'}-1}\sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)}\theta^j_{pT'+q} I\right) &= C_\beta \sum_{p=0}^{\frac{T}{T'}-1} \exp(0)\\
&= C_\beta.
\end{aligned}$$

So, we have now shown that (7) is always 0 or 1. In particular, we also have our first closure result. If, $\forall \beta \in \{0,1\}^{|A_d|}$, $\sum_{j\in A_d} -1^{\beta(j)} \neq 0$,

$$\begin{aligned}
\sum_{i=1}^{T}\prod_{j=1}^{k}\cos\left(\frac{2\pi}{b^j}\cdot\theta^j\right)^{\mathrm{incl}^j_d} &= 2^{-|A|}\sum_{\beta\in\{0,1\}^{|A_d|}}\sum_{p=0}^{\frac{T}{T'}-1}\sum_{q=0}^{T'-1}\exp\left(\sum_{j\in A_d} -1^{\beta(j)}\theta^j_{pT'+q} I\right)\\
&= 0
\end{aligned}$$

for all $d \in D$. So then, $\Delta P_{0,T} = 0$ whenever no degree 1 multinomials with coefficients of -1 or 1 and $|A_d|$ variables has a root of $(\sigma^1, \sigma^2, ..., \sigma^{|A_d|})$. In other words, if variables, $x_1 = \sigma^1, x_2 = \sigma^2, ...$, there does not exist a multinomial of form

$$\pm x_1 \pm x_2 \pm ... \pm x_{|A_d|} = 0.$$

Said differently, let multinomial $m : \mathbb{Z}_{\boldsymbol{b}} \to \mathbb{Z}_{\boldsymbol{b}}$ be such that

$$m(x_1, x_2, ..., x_{|A_d|}) = \prod_{\beta\in\{0,1\}^{|A_d|}}\left(\sum_{j\in A_d} -1^{\beta_j} AAA\right).$$

Then, if $m(\sigma^1, \sigma^2, ..., \sigma^{|A_d|}) \neq 0$, we know that the shape must close.

There does exist

In the case that $\exists\beta \in \{0,1\}^{|A_d|}$ such that $\sum_{j\in A_d} -1^{\beta(j)} = 0$, it is still possible for

$$\sum_{p=0}^{\frac{T}{T'}-1} \sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \theta^j_{pT'+q} I\right) = 0$$

if $C_\beta = 0$. So,

$$\sum_{q=0}^{T'-1} \exp\left(\sum_{j\in A_d} -1^{\beta(j)} \sum_{\ell=0}^{q} \mathrm{digit}(n,b,d,\ell)\right) = 0$$

5 Conclusion

6 Open Questions

Acknowledgments

A Finding the period of $\frac{n}{d}$

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References

[ho] Glen O (https://math.stackexchange.com/users/67842/glen o). Length of period of decimal expansion of a fraction. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/2611737 (version: 2018-01-19). 4.1.1