My Title

Lev Stambler
levstamb@umd.edu
University of Maryland, College Park,
NeverLocal Ltd.

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Abstract

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1 Introduction

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2 Core Idea: Datasets and Basic In-Distribution Testing

Ideally, we'd like to take the useful tools of Fourier Analysis (which assumes a product space) and generalize them to any distribution. Though, we do not have a formal method of reasoning about this, we do not think that it is quite possible.

Rather, we will attempt to think about analysis as over a *dataset:* i.e. we will think of our distribution as being defined by a finite-sized dataset, $\mathcal{D}$, and the probability vector will be defined as

$$p(x) = \begin{cases} \frac{1}{|\mathcal{D}|} & \text{if } \ x \in \mathcal{D} \\ 0 \text{ otherwise} \end{cases}.$$

More formally, we will be over a space $\mathcal{T}^n$ (think of $\mathcal{T}$ as either $\mathbb{R}$ or the space of tokens etc.) and $\mathcal{D} \subset \mathcal{T}^n$. Then, we will be focusing on functions $f \in \mathcal{T}^n \to \mathbb{R}$. f can either be a trained model, ideal labeling function on the dataset, or some other efficiently computable function.

Then, we want to reason about Fourier coefficients as

$$\hat{f}(S) = \mathbb{E}_{x \sim \mathcal{D}}[f(x)\chi_S^{-1}(x)].$$

Further, we will abuse notation to denote $\mathcal{D} : \mathcal{T}^n \to \{0, 1\}$ as the indicator function for the inclusion within the dataset. We will write $\hat{f}_{\text{OG}}(S)$ to denote the normal ("original") Fourier coefficient:

$$\hat{f}_{\text{OG}}(S) = \mathbb{E}_{x \sim \mathcal{T}^n}[f(x)\chi_S^{-1}(x)].$$

Importantly, notice that

$$\hat{f}(S) = \frac{|\mathcal{D}|}{|\mathcal{T}|^n}^{-1} \hat{f}_{\text{OG}}(\mathcal{D} \circ f) = \frac{|\mathcal{D}|}{|\mathcal{T}|^n}^{-1} \cdot \frac{1}{|\mathcal{T}|^n} \sum_{x \in \mathcal{D}} f(x)\chi_S(x).$$

where $\circ$ is the element wise composition. TODO: I think we need to use χ^{-1}?

We conveniently have our first lemma.

Lemma 2.1: For all $x \in \mathcal{D}$,

$$f(x) = \frac{|\mathcal{D}|}{|\mathcal{T}|^n} \sum_S \hat{f}(S)\chi_S(x)$$

Proof: First, note that $f(x) = \mathcal{D}(x) \cdot f(x)$ for $x \in \mathcal{D}$ and then $\widehat{\mathcal{D} \circ f}_{\text{OG}}(S) = \frac{|\mathcal{D}|}{|\mathcal{T}|^n} \cdot \hat{f}(S)$ and as such, using the standard Fourier identity

$$\mathcal{D}(x) \cdot f(x) = \sum_S \widehat{\mathcal{D} \circ f}(S)\chi_S(x) = \frac{|\mathcal{D}|}{|\mathcal{T}|^n} \sum_S \hat{f}(S).$$

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2.1 In Distribution Testing and Related Notions

Quite quickly, we run into a "design" decision: do we want our notion of analysis to include in-distribution testing (being able to distinguish whether an element is in x or not), or whether we want to *only* define things like influence over $\mathcal{D}$. We will do both, but start with the in-distribution notion because, as we will see, it is simpler and a warmup for the later. We will append our notation with $\mathrm{In}\mathcal{D}$ to denote the in-distribution case.

Definition 2.1 *(Distribution-Testing Coordinate Averaging)*: Let $E^i_{\mathrm{In}\mathcal{D}}$ for $i \in n$ be the i -th in distribution operator:

$$E^i_{\mathrm{In}\mathcal{D}}[f(x)] = \mathbb{E}_{a\in\mathcal{T}}[\mathcal{D}(x_i \mapsto a) \circ f(x^{x_i \mapsto a})].$$

Note that Definition 2.1 *zeros out* any coordinate setting which does not remain within the dataset. Intuitively, we can think about $E^i_{\mathrm{In}\mathcal{D}}$ as the generic coordinate averaging operator for a function which will test whether an input is in the dataset and output 0 if it is not.

We can now define influence:

Definition 2.2 *(i-th Coordinate Distribution-Testing Influence Operator)*:

$$\mathrm{Inf}^i_{\mathrm{In}\mathcal{D}} f = \mathbb{E}_{x\in\mathcal{D}}\left[\left(f(x) - E^i_{\mathrm{In}\mathcal{D}}(x)\right)^2\right].$$

Proposition 2.1: We now prove that basic identities still hold:

$$\mathrm{Inf}^i_{\mathrm{In}\mathcal{D}} = \left\langle f, f - E^i_{\mathrm{In}\mathcal{D}}\right\rangle_{\mathcal{D}},$$

$$E^{i^i_{\mathrm{In}\mathcal{D}}}_{\mathrm{In}\mathcal{D}} = \frac{|\mathcal{D}|}{|\mathcal{T}|^n} \sum_{S, S_i = 0} \hat{f}(S)\chi_S(x)$$

$$\mathrm{Inf}^i_{\mathrm{In}\mathcal{D}} = \sum_{S, S_i \neq 0} \hat{f}(S)^2.$$

Proof: We start with the second equality. TODO: check constants Note that,

$$\begin{aligned} E^{i^i_{\mathrm{In}\mathcal{D}}}_{\mathrm{In}\mathcal{D}}(x) &= \mathbb{E}_{a\in\mathcal{T}}[\mathcal{D}(x_i \mapsto a) \circ f(x^{x_i \mapsto a})] \\ &= \frac{|\mathcal{D}|}{|\mathcal{T}|^n} \sum_{S, S_i = 0} \hat{f}(S)\chi_S(x). \end{aligned}$$

The first equality holds as we note that,

$$\begin{aligned} \mathrm{Inf}^i_{\mathrm{In}\mathcal{D}} f &= \left\langle f(x) - E^i_{\mathrm{In}\mathcal{D}}(x), f(x) - E^i_{\mathrm{In}\mathcal{D}}(x)\right\rangle \\ &= \mathbb{E}[f(x)^2] - 2E_x\left[f(x)E^i_{\mathrm{In}\mathcal{D}}(x)\right] + \mathbb{E}_{\mathcal{D}}\left[E^i_{\mathrm{In}\mathcal{D}}(x)^2\right]. \end{aligned}$$

TODO: norm constant Then, note that by the second equality and Parseval's theorem, $E_x\left[f(x)E^i_{\mathrm{In}\mathcal{D}}(x)\right] = \sum_{S, S_i = 0} \hat{f}(S)^2$ and $\mathbb{E}_{\mathcal{D}}\left[E^i_{\mathrm{In}\mathcal{D}}(x)^2\right] = \sum_{S, S_i = 0} \hat{f}(S)^2$ as desired.

Finally, we note that as $\mathrm{Inf}^i_{\mathrm{In}\mathcal{D}} f = \mathbb{E}_{\mathcal{D}}[f(x)^2] - \mathbb{E}_{\mathcal{D}}\left[E^i_{\mathrm{In}\mathcal{D}}(x)^2\right]$, we can see that

$$\mathbb{E}_{\mathcal{D}}[f(x)^2] - E_x\left[f(x)E^i_{\mathrm{In}\mathcal{D}}(x)\right] = \sum_S \hat{f}(S)^2 - \sum_{S, S_i = 0} \hat{f}(S)^2 = \sum_{S, S_i \neq 0} \hat{f}(S)^2.$$

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Just as in TODO: cite , we can define total influence and get some convenient corollaries:

Definition 2.3 *(Total Influence, $\boldsymbol{I}_{\mathrm{In}\mathcal{D}}$)*:

$$\boldsymbol{I}_{\mathrm{In}\mathcal{D}}[f] = \sum_{i\in[n]} \mathrm{Inf}^i_{\mathrm{In}\mathcal{D}}[f].$$

We immediately get:

Proposition 2.2:

$$\boldsymbol{I}_{\mathrm{In}\mathcal{D}}[f] = \sum_{S} \#S \ \hat{f}(S)^2$$

which, for finite groups, we get

$$\boldsymbol{I}_{\mathrm{In}\mathcal{D}}[f] = \sum_{S} k \cdot W^k[f].$$

as in page 213 of TODO: a where $\#S = |\mathrm{supp}\ (S)| = \mathrm{supp}\ (S) = \{i : S_i \neq 0\}$.

Finally, we introduce one more definition which will capture the closeness of two functions, f, g over $\mathcal{D}$.

Definition 2.4 *(In-Distribution Closeness, $\epsilon_{\mathcal{D}}$-closeness)*: We say that a function g is $\epsilon_{\mathcal{D}}$ close to f if:

$$\mathop{\mathbb{E}}_{x \in \mathcal{D}}\left[(f(x) - g(x))^2\right] \leq \epsilon$$

or equivalently,

$$\frac{|\mathcal{D}|}{|\mathcal{T}|^n}^{-1} \mathop{\mathbb{E}}_{x \in \mathcal{T}^{\otimes n}}\left[(\mathcal{D} \circ f(x) - \mathcal{D} \circ g(x))^2\right] \leq \epsilon.$$

2.2 Immediate Consequences

Learning from Random Examples

One-Way Property Testing for Computable via Decision Tree

2.3 In Distribution Testing Definitions

Surprisingly, if want to characterize the complexity (or sensitivity or one of many Fourier properties) of our function f over distribution $\mathcal{D}$ and in-distribution testing, we can more or less use standard analysis.

In more detail, we will model our modified function $g : |\mathcal{T}|^n \to \mathbb{R}^{+^m} \cup \{0\}$ as

$$g(x) = \begin{cases} f(x) & \text{if} \ \ x \in \mathcal{D} \\ 0 & \text{otherwise} \end{cases}$$

And, taking the standard inner-product (TODO: we are over sphere/ real numbers, first take the inner-product then the other thing!)

Note that g is required to differentiate between in and out-of distribution.

Influence and Related Operators

We would like coordinate wise influence to be defined in the standard way, but adopted to our setting:

$$\mathrm{Inf}^i_{\mathcal{D}}[g] = \mathop{\mathbb{E}}_{x \in \mathcal{D}}\left[\left(g - E^i_{\mathcal{D}} g\right)^2\right]$$

where $E^i_{\mathcal{D}}$ is the i-th coordinate expectation operator:

$$\left[E^i_{\mathcal{D}} g\right](x) = \mathop{\mathbb{E}}_{x_i \in \mathcal{T}}\left[g(x^{i \mapsto x_i})\right].$$

In words, the i-th expectation operator "averages out" the i-th coordinate over the dataset while the Influence measures the difference.

TODO: define distribution coeffs! TODO: This prob dist thing doesn't work??? Just have a subset thingy for now!!! Then, we have the following proposition:

Proposition 2.3:
$$E^i_{\mathcal{D}} g = \sum_{s : s_i \neq 0} \widehat{p_{\text{scale}} \cdot} f(s) \hat{f}_{\mathcal{D}}$$

Lemma 2.2: We can re-write the influence as an inner-product
$$\mathrm{Inf}^i_{\mathcal{D}}[f] = \langle p \cdot (g - E^i_{\mathcal{D}} g), g \rangle = \langle p \cdot (g - E^i_{\mathcal{D}} g), g - E^i_{\mathcal{D}} g \rangle$$

Proof: The second equality follows directly from the definition. The first is because
$$\langle p \cdot (g - E^i_{\mathcal{D}} g), g - E^i_{\mathcal{D}} g \rangle = \mathbb{E}_x [g^2] - 2\mathbb{E}_x [g \cdot E^i_{\mathcal{D}} f] + \mathbb{E}_x \left[\left(E^i_{\mathcal{D}} f \right)^2 \right]$$

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3 Conclusion and Future Directions

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Acknowledgments

AI usage: the author would like to acknowledge the use of language models, Gemini and Claude, in generating the SVG…

Bibliography

A Appendix

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