Problem Statement and why we should care

Say we had an n dimensional lattice Λ where $\Lambda = \{Bz : z \in \mathbb{Z}\}$ for $B \in \mathbb{Z}^{n \times n}$ where $b_1, b_2, ..., b_n$ are column vectors of B. Moreover, we are only going to worry about orthogonal lattices, i.e. all the basis vectors are orthogonal.

Now say we have a radially monotone function $f : \mathbb{R}^n \to \mathbb{R}$. WLOG (we'll touch on the other case later), assume that $f(x) \geq f(x')$ for all $||x|| \leq ||x'||$ where $||.||$ denotes the L2 norm.

Now, we will provide an algorithm for computing upper and lower bounds on

$$\sum_{p \in V \bigcap \Lambda} f(p)$$

where $V \subseteq \mathbb{R}^n$ and we can find $\int_{x \in V} f(x)\, dx$.

Algorithm Outline

Say that we are looking for an approximation summing over convex, continuous region R containing the origin. We can parametrize R with a function $r : [0, 2\pi]^{n-1} \to \mathbb{R} \setminus \mathbb{R}^-$. In other words, as a function of $n-1$ angles $\theta_1, ...\theta_{n-1}$ where $r(\theta_1, ..., \theta_{n-1})$ gives the radius of the furthest point from the origin for the ray starting at the origin with angle $\theta_1, ..., \theta_{n-1}$. Then,

$$R = \{ \bigcup_{\theta_1, ..., \theta_{n-1}} [0, r(\theta_1, ..., \theta_{n-1})] \} .$$

Now, define

$$\tilde{bl} = \sqrt{\sum_{i \in [n]} ||b_i||^2}.$$

In other words, $\tilde{bl}$ is the length of the largest vector in a Voroni region.

Now, define the region R_l (l for larger!) by

$$\{ \bigcup_{\theta_1, ..., \theta_{n-1}} [0, r(\theta_1, ..., \theta_{n-1}) + \tilde{bl}] \} .$$

and R_s (s for smaller) by

$$\{ \bigcup_{\theta_1, ..., \theta_{n-1}} [\tilde{bl}, \max\Big(0, r(\theta_1, ..., \theta_{n-1}) - \tilde{bl}\Big)] \} .$$

R_l and R_s can be respectively thought of as an expanded out and contracted in version of R by $\tilde{bl}$ respectively. Moreover, for $S \subseteq [n]$, define P_S as a projector to the subspace spanned by $b_{S_1}, b_{S_2}, ..., b_{S_k}$ for $k = |S|$.

A general outline of the algorithm follows:

Algorithm 1 Estimate for a radially non-decreasing function

$ub = 0$
$lb = 0$
sets = Powerset($[n]$)
for $S \in$ sets **do**
  $ub \mathrel{+}= \frac{\int_{x \in P_S R_l} f(x)dx}{\mathrm{Vol}(P_S V)}$
  $lb \mathrel{+}= \frac{\int_{x \in P_S R_s} f(x)dx}{\mathrm{Vol}(P_S V)}$
end for

Proof

To show the correctness of the algorithm, we first need to prove a few lemmas

Lemma 0.1. *For some $x \in \Lambda$, $x = Bz$ for $z \in \mathbb{Z}^n$, and $S \subseteq [n]$,*

$$||x + \sum_{i \in S} \pm b_i||^2 = x^2 + \sum_{i \in [S]} (\pm 2 z_i + 1)||b_i||^2.$$

Proof.

$$\begin{aligned} ||x + \sum_{i \in S} \pm b_i||^2 &= ||x||^2 + \sum_{i \in S} \pm 2 \langle b_i, x \rangle + \sum_{i,j \in S} \Delta_i \Delta_j \langle b_i, b_j \rangle \\ &= x^2 + \sum_{i \in S} \pm 2 \langle b_i, x \rangle + \sum_{i \in S} ||b_i||^2 \end{aligned}$$

and

$$\begin{aligned} \langle b_i, x \rangle &= \langle b_i, z_i b_i \rangle + \sum_{j \in [n] \setminus \{i\}} \langle b_i, z_j b_j \rangle \\ &= z_i \langle b_i,\, b_i \rangle = z_i ||b_i||^2. \end{aligned}$$

□

Theorem 0.2 (Unique Furthest Point)**.** *For a Voroni region, V, in a lattice defined by Λ, there is a furthest $p \in \Lambda \cap V$ from the origin (i.e. has the largest L2 norm). Moreover, this point has a larger L2 norm than all other points in V.*

Proof. First we show that each V has only 1 furthest point. Note that each V can be defined by a point p_v such that

$$V \cap \Lambda = \{\, p_v + \sum_{i \in S} b_i : S \subseteq [n] \,\}.$$

Then, we will explicitly construct a unique furthest point in V by specifying an $S_l \subseteq [n]$ where $p_v \sum_{i \in S_l} b_i$ is the furthest point.

We construct S_l with the following algorithm: for all $i \in [n]$, if and only if $||p_v + b_i|| > ||p_v||$ add i to S_l.

Then, AFSOC, that for some set $S \neq S_l$, $||p_v + \sum_{i \in S} b_i|| \geq ||p_v + \sum_{i \in S_l} b_i||$.
So

$$|| \sum_{i \in S} p_v + b_i||^2 = ||p_v||^2 + \sum_{i \in S}(1+2z_i)||b_i||^2 \geq || \sum_{i \in S_l} p_v + b_i||^2 = ||p_v||^2 + \sum_{i \in S_l}(1+2z_i)||b_i||^2$$

So,

$$\sum_{i \in S}(1+2z_i)||b_i||^2 \geq \sum_{i \in S_l}(1+2z_i)||b_i||^2.$$

But, this would mean that $i \notin S_l, (1+2z_i)||b_i||^2$ is positive for $i \in S$ or $(1+2z_i)||b_i||^2$ is negative for $i \in S_l$. But, both cases are not possible given our construction of S_l.

□

Theorem 0.3 (Furthest Point is Injective). *For Voroni regions V_1, V_2 and $V_1 \neq V_2$ and their respective furthest points p_1, p_2, we have that $p_1 \neq p_2$.*

Proof. AFSOC $V_1 \neq V_2$ but $p_1 = p_2$. Then by 0.2, there exists a unique $\Delta_1, \Delta_2 \in \{-1, 1\}^n$ such that for all $i \in [n]$, $||p_1|| > ||p_1 + \Delta_{1,i} b_i||$ and $||p_2|| > ||p_2 + \Delta_{2,i} b_i||$. Note that for each i and fixed p, Δ_i is unique for that p. So $\Delta_{1,i} = \Delta_{2,i}$ and $V_1 = V_2$. Thus, we reach our desired contradiction. □

Theorem 0.4 (Not a Furthest Point). *For an orthogonal lattice Λ, the only lattice points which are not uniquely associated to be the furthest point of a Voroni region are those which lie along some projection of a subset of the basis. More formally, p is not a furthest point if and only if $p \in T$ where*

$$T = \{Bz : z \in \mathbb{Z} \text{ and } \exists\, i \in [n] \text{ such that } z_i = 0\}.$$

Proof. First we will show that if p is not a unique furthest point then it is an element of T via the contrapositive. Assume that $x \in \Lambda \setminus T$. Then, let $z \in \mathbb{Z}^n$ such that $x = Bz$. We now want to show that x is a unique furthest point.

We will do this by explicitly constructing a Voroni region where x is the furthest point. We will show that for $\Delta \in \{-1, 1\}^n$ such that $\forall S \subseteq [n]$ except for the empty set,

$$||x + \sum_{i \in S} \Delta_i b_i|| < ||x||.$$

In other words, Δ gives us an "update vector" which explicitly constructs a Voroni region where x is the unique furthest point. We now give a way to construct such a Δ. For $i \in [n]$, if $x_i > 0$ then $||x - b_i|| < ||x||$, if $x_i < 0$, then $||x + b_i|| < ||x||$. Then, we fix Δ such that $\forall i \in [n]$, $||x + \Delta_i b_i|| < ||x||$. Next, we note that by lemma 0.1,

$$||x + \sum_{i \in S} \Delta_i b_i||^2 = x^2 + \sum_{i \in [S]} (2\Delta_i z_i + 1)||b_i||^2.$$

So, we can choose Δ such that $||x + \sum_{i \in S} \Delta_i b_i|| = \sqrt{x^2 - \sum_{i \in [S]}(2z_i - 1)||b_i||^2} < ||x||$.

Now to prove the backwards direction. Let $p \in T$ and $p = Bz$. Let $i \in [n]$ such that $z_i = 0$. Then note that any Voroni region containing p also contains either $p + b_i$ or $p - b_i$. So,

$$\begin{aligned}
||p \pm b_i||^2 &= ||p||^2 \pm 2\langle p, b_i \rangle + ||b_i||^2 \\
&= ||p||^2 \pm 0||b_i||^2 + ||b_i||^2 \\
&> ||p||.
\end{aligned}$$

We can thus see that p cannot be a unique furthest point.

$\square$

We are now ready to prove the correctness of the algorithm.

Some applications

Some Numerical Results