

Sparse Graph Label Randomization

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1 Preliminaries

1.1 Bounded Functional Encryption

We will use the notation of static, bounded functional encryption as presented in [GGLW22].

Security

We will slightly weaken the security notion such that the adversary does not choose which circuits it can learn the functional secret key for. Indeed, this is a weaker notion of functional encryption which fixes the adversary's output circuit. We will assume that we get circuit C_1, \dots, C_d .

For completeness, we have the original security definition of [GGLW22] below:

$$\left\{ \begin{array}{l} (1^n, 1^q) \leftarrow \mathcal{A}^{(1)} \\ (\text{MPK}, \text{MSK}) \leftarrow \text{Setup}(1^n, 1^q) \\ m \leftarrow \mathcal{A}^{\text{KeyGen}(\text{MSK}, \cdot)}(\text{CT}) \\ \text{CT} \leftarrow \text{Enc}(\text{MPK}, m) \end{array} \right\}_{\lambda \in \mathbb{N}} \approx^c \left\{ \begin{array}{l} (1^n, 1^q) \leftarrow \mathcal{A}(1^\lambda) \\ (\text{MPK}, \text{st}_0) \leftarrow \text{Sim}_0(1^\lambda, 1^n, q) \\ m \leftarrow \mathcal{A}^{S_1(\text{st}_0)}(\text{MPK}) \\ (\text{CT}, \text{st}_2) \leftarrow \text{Sim}_2(\text{st}_1, \Pi^m) \end{array} \right\}_{\lambda \in \mathbb{N}}$$

whenever the following admissibility constraints and properties are satisfied:

- $\text{Sim}_1, \text{Sim}_3$ are stateful in that after each invocation, they updated their states st_1, st_3 respectively which is carried over to the next invocation.
- Π^m contains a list of functions f_i queried by \mathcal{A} in the pre-challenge phase along with their output on the challenge message m . That is, if f_i is the i -th function queried by \mathcal{A} to oracle Sim_1 and q_{pre} be the number of queries \mathcal{A} makes before outputting m , then $\Pi^m = ((f_1, f_1(m)), \dots, (f_{q_{\text{pre}}}, f_{q_{\text{pre}}}(m)))$.
- \mathcal{A} makes at most q queries combined to key generation oracle in both games.
- Sim_3 for each queried function f_i , in the post challenge phase, makes a single query to its message oracle U_m on the same f_i itself.

Our modified security definition is as follows:

$$\left\{ \begin{array}{l} (1^n, 1^q) \leftarrow \mathcal{A}^{(1)} \\ (\text{MPK}, \text{MSK}) \leftarrow \text{Setup}(1^n, 1^q) \\ m \leftarrow \mathcal{A}(\text{MPK}, \text{SK}_{C_1}, \dots, \text{SK}_{C_d}) \\ \text{CT} \leftarrow \text{Enc}(\text{MPK}, m) \end{array} \right\}_{\lambda \in \mathbb{N}} \approx^c \left\{ \begin{array}{l} (1^n, 1^q) \leftarrow \mathcal{A}(1^\lambda) \\ (\text{MPK}, \text{st}_0) \leftarrow \text{Sim}_0(1^\lambda, 1^n, q) \\ m \leftarrow \mathcal{A}^{S_1(\text{st}_0)}(\text{MPK}, C_1, \dots, C_d) \\ (\text{CT}, \text{st}_2) \leftarrow \text{Sim}_2(\text{st}_1, \Pi^m) \end{array} \right\}_{\lambda \in \mathbb{N}} \quad (1)$$

where the admissibility constraints remain the same.

1.2 Non-malleable Bounded FE

Here, we introduce the notion of non-malleable bounded functional encryption.

We define non-malleable security of bounded functional encryption in almost the exact notion of [Pas06] for public key encryption. First, let $NM(m_1, \dots, m_q, \mathcal{A})$ be a game as follows for $q = \text{poly}(\lambda)$:

1. $(\text{MPK}, \text{MSK}) \leftarrow \text{FE.Setup}(1^\lambda)$
2. $\text{CT}_1, \dots, \text{CT}_q \leftarrow \text{FE.Enc}(\text{MPK}, m_1), \dots, \text{FE.Enc}(\text{MPK}, m_q)$
3. $\text{CT}'_1, \dots, \text{CT}'_\ell \leftarrow \mathcal{A}(\text{MPK}, \text{CT}_1, \dots, \text{CT}_q, 1^{|m|})$
4. $m'_i \leftarrow \perp$ is $\text{CT}_i = \text{CT}'_j$ for any $i \in [q]$, $j \in [\ell]$ and $\text{FE.Dec}(\text{SK}_{\text{identity}}, c_i)$ otherwise.

Then, we say that a bounded functional encryption scheme is non-malleable if for all PPT \mathcal{A} and every PPT \mathcal{D} , there exists a negligible function negl such that for all $\{m\}_0, \{m\}_1 \in \{0, 1\}^{nq}$, we have

$$\left| \Pr[\mathcal{D}(NM(\{m\}_0, \mathcal{A})) = 1] - \Pr[\mathcal{D}(NM(\{m\}_1, \mathcal{A})) = 1] \right| \leq \text{negl}. \quad (2)$$

As outlined in [Pas06], we can equivalently define non-malleability in terms of a PPT recognizable relation R such that

$$\left| \Pr \left[NM(m_1, \dots, m_q, \mathcal{A}(z)) \in \bigcup_{m \in \{m\}} R(m) \right] - \Pr \left[c \leftarrow \text{Sim}_{NM}(1^n, z); m' = \text{FE.Dec}(\text{SK}_{\text{identity}}, c); m' \in \bigcup_{m \in \{m\}} R(m) \right] \right| \leq \text{negl}(\lambda). \quad (3)$$

Note that in the above definition, we do not give the adversary access to any SK_{C_i} . We simply require that the scheme is public key (many message) non-malleable.

2 Using Weak Extractible Obfuscation

2.1 Graph Randomized Traversal

Say that we have a sparse, potentially exponentially sized, graph $\mathcal{G} = (V, E)$ and $\forall v \in V, \deg(v) = d$. Moreover, if the graph is a DAG, for simplicity, assume that for all v ,

$$\deg^{-1}(v) = |\{u \in V \mid \exists j \in [d], \Gamma(u)_j = v\}| \leq d.$$

In words, there are at most d edges into a vertex. As a note, our construction just requires that $\deg^{-1}(\cdot) = O(1)$ but for the sake of simplicity we fix $\deg^{-1}(\cdot) \leq d$.

We also require that \mathcal{G} is equipped with a neighbor function, Γ , which can be computed in polynomial time. We define a randomized and keyed labelling function $\phi : \{0, 1\}^\lambda \times V \rightarrow \{0, 1\}^{\text{poly}(\lambda)}$ such that given, $\phi(K, v_0)$ for root v_0 , an adversary, \mathcal{A} , which does not know a path from v_0 to v ,

$$\Pr[\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, v, \phi(K, v_0)) = \phi(K, v)] \leq \epsilon \quad (4)$$

for function C_Γ where $C_\Gamma(\phi(K, u)) = \phi(K, \Gamma(u)_1), \dots, \phi(K, \Gamma(u)_d)$ if $\Gamma(u) \neq \emptyset$ and otherwise $\Gamma(u)$ returns a \perp string; and, \mathcal{O} represents an indistinguishable obfuscator. We fix $\epsilon \leq \text{negl}(\lambda)$.

2.2 Instantiation

We define

$$\phi(K, v) = F(K, v).$$

We can now define C_Γ :

Algorithm 1 The circuit for the neighbor function, C_Γ .

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1: function  $C_\Gamma(X, v)$ 
2:   if  $f(X) \neq f(F(K, v))$  then
3:     return  $\perp$ 
4:   if  $\Gamma(v) = \emptyset$  then
5:     return  $\perp$ 
6:    $u_1, \dots, u_d = \Gamma(v)$ 
7:   return  $F(K, u_1), F(K, u_2), \dots, F(K, u_d)$ 

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We are going to show that [eq. \(4\)](#) holds by first showing that the non-existence of an extractor to find a path from v_0 to v implies that \mathcal{A} necessarily does not know $\phi(K, c)$ for a $c \in C_V \subset V$ where the vertices in C_V border a graph cut which separates v_0 and v . Then, we inductively build up a series of games to show that \mathcal{A} cannot learn *any* $\phi(K, v)$ for $v \in V_1$ where V_1 are the vertices on the right-hand side of the cut.

Lemma 2.1 (Base Case Game). *Assuming that there is no extractor E such that $\Pr[E(\Gamma, v_0, v) = P] \geq \frac{1}{p(\lambda)}$ where $P \in \mathcal{P}$, then for any PPT \mathcal{A} , there exists some graph cut $C_E \subset E$ which separates v_0 and v and a set C_V such that*

$$\Pr[\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, v, \phi(K, v_0)) \in \phi(K, C_V)] \leq \text{negl}(\lambda). \quad (5)$$

We define $C_V \subset V$ to be

$$\{u \mid (w, u) \in C_E \text{ and } u \text{ on the side of } v\} \cup \{v \mid (w, u) \in C_E \text{ and } u \text{ on the side of } v\}.$$

In words, C_V are the vertices just adjacent to the cut and on the same side as v .

Proof. We will show that if \mathcal{A} can break [eq. \(5\)](#), then we can construct an extractor, E , which finds a path from v_0 to v with non-negligible probability.

Assume that for every possible cut, \mathcal{A} is able to produce a single label in this cut for a vertex w . Then, we note that there must be at least 1 path from v_0 to w and v as otherwise, w would not be in the cut. Moreover, we note that \mathcal{A} must be able to produce a label for all vertices on at least one path from v_0 to w as otherwise, we can change the cut to include the edges between where \mathcal{A} is able to produce a label and not able to produce a label. Using the same argument, we can show that \mathcal{A} must be able to produce all labels on a path from w to v .

Note that \mathcal{A} is not given the specific cut C_E but rather C_E is chosen based off of the adversary. So, we can build an extractor to do the following:

1. Create an iO obfuscated circuit with a random key, K' , for C_Γ and create circuit $\mathcal{O}(C_\Gamma)$ as well as $\phi(K', v_0)$
2. Run $\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, v, \phi(K', v_0))$ to get all labels $\phi(K', v_0), \dots, \phi(K', v)$ for some path from v_0 to v .
3. Recreate the path from v_0 to v via checking which vertex matches to adjacent labels in the path: I.e. starting with $\ell = 0$, we can learn the $\ell + 1$ vertex via finding $j \in [d]$ such that $C_\Gamma(\phi(K', v_\ell), v_\ell)_j \in \{\phi(K', v_0), \dots, \phi(K', v)\}$ and then setting $v_{\ell+1} = \Gamma(v_\ell)_j$.

□

We can look at [lemma 2.1](#) as a “base case” of sorts. We now inductively build up a series of games such that \mathcal{A} cannot find any label in V_1 where V_1 are the vertices on side of the cut (as defined in [lemma 2.1](#)) which contain v .

Lemma 2.2 (Inductive Game Hypothesis). *Let $H \subset V$ be a “hard” set of vertices such that \mathcal{A} cannot, with non-negligible probability, produce $\phi(K, h)$ where $h \in H$. Note that the base case has $H = C_V$. Then, for any $v \notin H$ and $w \in \Gamma(h)$ for all $h \in H$, we have that*

$$\Pr[\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, w, \phi(K, v_0)) = \phi(K, w)] < \text{negl}(\lambda).$$

Proof. We are going to use a series of indistinguishable hybrids along with the circuit defined in [2](#) to show the above

- **Hyb₀**: In the first hybrid, the following game is played
 1. $K \leftarrow \{0, 1\}^{\lambda'}$ and $\phi(K, v_0) = (F(K, v_0), v)$ where K is some fixed secret drawn from a random distribution
 2. The challenger generates $\mathcal{O}(C_\Gamma)$ and gives the program to \mathcal{A}
 3. The challenger gives the adversary w^* in plaintext.
 4. \mathcal{A} outputs guess g and wins if $g = \phi(K, w^*)$
- **Hyb₁**: We replace C_Γ with $C_\Gamma^{w^*}$ as defined in [2](#). Fix the constant $z^* = f(F(K, w^*))$
- **Hyb_{2,1}**: We replace [algorithm 2](#) with [algorithm 3](#) where we set $Y^* = (1, y)$ such that $\Gamma(y)_1 = w^*$. So then, we have that $F(K, \Gamma(y)_1) = \perp$
- **Hyb_{2,j}** for $j \in 2, \dots, \deg^{-1}(w^*)$: We replace Y^* with $Y^* \cup (j, y)$ such that $\Gamma(y)_j = w^*$. Note after the last of these hybrids, we have that $F(K, w^*)$ is always set to \perp .

- Hyb_3 : Set $z^* = f(t)$ where t is chosen at random

Finally, we can note that if $\text{Hyb}_0 \stackrel{c}{\approx} \text{Hyb}_2$,

$$\Pr[\mathcal{A}(C_\Gamma, v_0, w, \phi(K, v_0)) = \phi(K, w)] \stackrel{c}{\approx} \Pr[\mathcal{A}(C_\Gamma^*, v_0, w, \phi(K, v_0)) = \phi(K, w)]$$

where z^* in C_Γ^* is the image on a OWF of a randomly chosen point. As we will show in [lemma 2.3](#), [lemma 2.4](#), and [lemma 2.5](#), an adversaries advantage between games in Hyb_0 and Hyb_3 is at most $\epsilon/2$. Thus, if \mathcal{A} can produce $\phi(K, v) = (\sigma_v, v)$ with advantage $\epsilon/2$ in Hyb_3 , then \mathcal{A} can find a pre-image for z^* under f with non-negligible probability and thus break the security of a one way function. We then have that the advantage of the adversary in Hyb_0 cannot be more than ϵ . \square

Lemma 2.3. *Hyb_0 and Hyb_1 are distinguishable with advantage at most $\epsilon/8$.*

Proof. Note that for all inputs (z, v) to C_Γ as defined in [algorithm 1](#) and [algorithm 2](#) are equivalent and thus indistinguishable by the definition of indistinguishable obfuscation. So, if $\epsilon \in \text{poly}(\lambda)$, then an adversary cannot distinguish the hybrids with probability more than $\epsilon/8$. \square

Lemma 2.4. *Each hybrid from Hyb_1 to $\text{Hyb}_{2,1}$ and $\text{Hyb}_{2,j-1}$ to $\text{Hyb}_{2,j}$ for $j \in 2, \dots, \deg^{-1}(w^*)$ is distinguishable with advantage at most $\epsilon/(8d)$. Thus, Hyb_1 and $\text{Hyb}_{2, \deg^{-1}(w^*)}$ are distinguishable with advantage at most $\epsilon/8$.*

Proof. This proof will follow very closely the simple case of weak extractible obfuscation as defined in (TODO: cite). The key idea is that if a hybrid is distinguishable with advantage more than $\epsilon/(8d)$, then \mathcal{A} can produce a label $\phi(K, h)$ for $h \in H$.

First, assume towards contradiction that there exists an adversary \mathcal{A} that can distinguish two consecutive hybrids with polynomial advantage $\epsilon' > \epsilon/8d$. Following the proof sketch in (TODO: cite), say that the input size to C_Γ is n . Also, let C_0 be the circuit from the first hybrid and C_1 the one from the second. Let C_i^{Mid} be a circuit such that $C_i^{\text{Mid}}(X) = C_0(X)$ if $X_i = 0$ and $C_i^{\text{Mid}}(X) = C_1(X)$ if $X_i = 1$. Note that C_0 and C_1 differs on at most 1 input (which is the appended vertex y to Y^*); call this input α . Then, $C_i^{\text{Mid}} = C_0$ if $\alpha_i = 0$ and $C_i^{\text{Mid}} = C_1$ if $\alpha_i = 1$. So, if we build an adversary \mathcal{B} to tell if $C_i^{\text{Mid}} = C_0$ or C_1 with probability γ , we have that \mathcal{B} can tell if α_i is 0 or 1 with probability γ . Thus, \mathcal{B} can reconstruct α with probability at least γ^n . Note that this implies that \mathcal{B} can learn $\phi(K, y)$ where $y \in H$ by construction and thus gives our desired contradiction. So now, we just need to build \mathcal{B} to tell if $C_i^{\text{Mid}} = C_0$ or C_1 with probability $\gamma^n \geq \frac{\epsilon}{8d}$.

Then, \mathcal{A} can distinguish between C^M via the following:

1. Run I iterations of the following experiment to estimate advantage ϵ'_b for $b \in \{0, 1\}$
 - (a) Sample a random obfuscation of C_b via re-obfuscating the existing C_b
 - (b) Sample a random obfuscation of C_i^{Mid} via re-obfuscating C_i^{Mid}
 - (c) Have \mathcal{A} distinguish between C_b and C^{Mid}
 - (d) Output 1 if successful.

Note that we can estimate ϵ'_b as the number of successful runs, which we will denote $\sum_{j \in [I]} S_{i,j}$, divided by I .

2. If $\epsilon'_1 > \epsilon'_0$, then $C^{\text{Mid}} = C_0$, otherwise, $C^{\text{Mid}} = C_1$.

WLOG, say that $C^{\text{Mid}} = C_0$, then

$$\begin{aligned}\Pr[\epsilon'_1 > \epsilon'_0] &= \Pr\left[\sum_j S_{1,j} > \sum_j S_{0,j}\right] \\ &\geq \Pr\left[\sum_j S_{1,j} > \frac{I\epsilon'}{2}\right] \cdot \Pr\left[\sum_j S_{0,j} < \frac{I\epsilon'}{2}\right].\end{aligned}$$

We then have that

$$\Pr\left[\sum_j S_{1,j} > I\epsilon' \cdot \frac{1}{2}\right] \geq 1 - \exp\left(-\frac{I\epsilon'}{2^2 \cdot 3}\right) = 1 - \exp\left(-\frac{I\epsilon'}{96}\right). \quad (\text{by the Chernoff bound})$$

And, if iO distinguishing advantage is at most α and $\delta = \frac{\epsilon'}{2\alpha} - 1$

$$\begin{aligned}\Pr\left[\sum_j S_{0,j} < \frac{I\epsilon'}{2}\right] &= 1 - \Pr\left[\sum_j S_{0,j} \geq (1 + \delta)I\alpha\right] \geq 1 - \exp\left(-I\alpha \left(\frac{\epsilon'}{2\alpha} - 1\right)^2 \cdot \frac{1}{3}\right) \\ &\quad (\text{by the Chernoff bound}) \\ &\geq 1 - \exp\left(-\frac{I\epsilon'^2}{12\alpha}\right) \geq 1 - \exp\left(-\frac{I\epsilon'}{12}\right) \quad (\text{as } \epsilon' > \alpha)\end{aligned}$$

So we finally have that

$$\Pr[\epsilon'_1 > \epsilon'_0] \geq 1 - \exp\left(-\frac{I\epsilon'}{12}\right) - \exp\left(-\frac{I\epsilon'}{96}\right) \geq 1 - 2\exp\left(-\frac{I\epsilon'}{96}\right). \quad (6)$$

Setting $I > \frac{\ln 2 \cdot 96 (\ln n - \ln(1 - \frac{\epsilon}{8d}))}{\epsilon'}$, we have that

$$\begin{aligned}(\Pr[\epsilon'_1 > \epsilon'_0])^n &\geq \left(1 - 2\exp\left(-\frac{I\epsilon'}{96}\right)\right)^n \\ &\geq 1 - 2n \cdot \exp\left(-\frac{I\epsilon'}{96}\right) = 1 - 2n \cdot \exp\left(-\left(\ln n + \ln\left(1 - \frac{\epsilon}{8d}\right)\right) \cdot 2\right) \\ &= 1 - \left(1 - \frac{\epsilon}{8d}\right) = \frac{\epsilon}{8d}\end{aligned}$$

as desired. □

Lemma 2.5. *AAA*

Algorithm 2 Circuit for the neighbor function, C_Γ with punctured PRF key $K(\{w^*\})$ and constant w^*, z^*

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1: function  $C_\Gamma(X, v)$ 
2:   if  $v \neq w$  and  $f(X) \neq f(F(K, v))$  then
3:     return  $\perp$ 
4:   if  $v = w$  and  $f(X) \neq z^*$  then
5:     return  $\perp$ 
6:   if  $\Gamma(v) = \emptyset$  then
7:     return  $\perp$ 
8:    $u_1, \dots, u_d = \Gamma(v)$ 
9:   return  $F(K, u_1), F(K, u_2), \dots, F(K, u_d)$ 
```

Algorithm 3 Circuit for the neighbor function, C_Γ with punctured PRF key $K(\{w^*\})$ and constant w^*, Y^*, J^*, z^*

```

1: function  $C_\Gamma(X, v)$ 
2:   if  $v \neq w$  and  $f(X) \neq f(F(K, v))$  then
3:     return  $\perp$ 
4:   if  $v = w$  and  $f(X) \neq z^*$  then
5:     return  $\perp$ 
6:   if  $\Gamma(v) = \emptyset$  then
7:     return  $\perp$ 
8:    $u_1, \dots, u_d = \Gamma(v)$ 
9:   if  $\exists j \in [d], (j^*, v^*) \in Y^*$  then
10:    Set  $F(K, u_{j^*}) = \perp$ 
11:  return  $F(K, u_1), F(K, u_2), \dots, F(K, u_d)$ 

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Lemma 2.6. *The game in $\text{Hyb}_1(1a)$ is indistinguishable from Hyb_0 .*

Proof. As the functionality of C_Γ in Hyb_0 equals that of $\text{Hyb}_1(1a)$, we have indistinguishable simply from the definition of indistinguishable obfuscation. \square

Lemma 2.7. *The game in $\text{Hyb}_1(1b)$ is indistinguishable from $\text{Hyb}_1(1a)$.*

Proof. Here we argue that if the game in $\text{Hyb}_1(1b)$ is distinguishable from $\text{Hyb}_1(1a)$, then we can construct an adversary, \mathcal{B} , which can break the security of the PRF at the punctured point. \square

Lemma 2.8. *The game in $\text{Hyb}_1(2a)$ is indistinguishable from Hyb_0 and, by the inductive hypothesis, all previous hybrids.*

Proof. Again, we have that the circuit for C_Γ is the same in Hyb_0 and $\text{Hyb}_1(2a)$. Thus, by the definition of indistinguishable obfuscation, these games are indistinguishable. \square

Lemma 2.9. *The game in $\text{Hyb}_1(2b)$ is indistinguishable from $\text{Hyb}_1(2a)$ and, by the inductive hypothesis, all previous hybrids.*

Proof. TODO: PRF security + extractor part \square

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Abstract

References

- [GGLW22] Rachit Garg, Rishab Goyal, George Lu, and Brent Waters. Dynamic collusion bounded functional encryption from identity-based encryption. In *Annual International Conference on the Theory and Applications of Cryptographic Techniques*, pages 736–763. Springer, 2022. [1.1](#), [1.1](#)
- [Pas06] Rafael Pass. Lecture 16: Non-malleability and public key encryption, October 2006. [1.2](#), [1.2](#)