Sparse Graph Label Randomization

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1 Preliminaries

1.1 Punctured PRF

A punctured PRF is a simple type of constrained PRF ([BW13, BGI14, KPTZ13]) where a PRF is well defined on all inputs except for a specified, polynomial-sized set. We will adopt the notion specified in [SW14].

Definition 1.1 (Punctured PRF)**.** A puncturable family of PRF s F mapping is given by a tuple of algorithms $(\mathrm{Key}_F, \mathrm{Puncture}_F, \mathrm{Eval}_F)$. satisfying the following conditions:

- Functionality preserved under puncturing: For every PPT adversary $\mathcal{A}$, $S \subseteq \{0,1\}^n$ and every $x \in \{0,1\}^n$ where $x \notin S$, we have that

$$\mathbf{Pr}\left[\mathtt{Eval}_F(K,x) = \mathtt{Eval}_F(K_S,x) \mid K \leftarrow \mathtt{Key}_F(1^\lambda), \mathtt{K}_S = \mathtt{Puncture}_F(K,S)\right] = 1.$$

- Pseudorandom at punctured points: For every PPT adversary $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A}(1^\lambda)$ outputs a set S and state $\mathbf{st}$, consider an experiment where $K \leftarrow \mathtt{Key}_F(1^\lambda)$ and $K_S = \mathtt{Puncture}_F(K,S)$. Then, we have that

$$\left|\mathbf{Pr}\left[\mathcal{B}(\mathtt{st}, K_S, S, \mathtt{Eval}_F(K,S)) = 1\right] - \mathbf{Pr}\left[\mathcal{B}(\mathtt{st}, K_S, S, U_{m\cdot|S|})\right]\right| \leq \mathtt{negl}(\lambda).$$

1.2 Indistinguishable Obfuscation

We will use the definition of indistinguishable obfuscation as presented in [GGH+16].

Definition 1.2 (Indistinguishable obfuscation)**.** A uniform PPT machine $\mathcal{O}$ is an indistinguishable obfuscator for a class of circuits $\mathcal{C}$ if for every circuit $C \in \mathcal{C}$ we have that

$$\mathbf{Pr}[C'(x) = C(x) \mid C' \leftarrow \mathcal{O}(C)] \leq \mathtt{negl}(\lambda)$$

and for any PPT distinguisher $\mathcal{D}$ and two pairs of circuits C_0, C_1 such that $C_0(x) = C_1(x)$ for all x, then

$$\Bigg|\mathbf{Pr}\left[\mathcal{D}(\mathcal{O}(\lambda, C_0)) = 1\right] - \mathbf{Pr}\left[\mathcal{D}(\mathcal{O}(\lambda, C_1)) = 1\right]\Bigg|.$$

2 Using Weak Extractible Obfuscation

2.1 Graph Randomized Traversal

Say that we have a sparse, potentially exponentially sized, graph $\mathcal{G} = (V, E)$ and $\forall v \in V, \deg(v) = d$. Moreover, if the graph is a DAG, for simplicity, assume that for all v,

$$\deg^{-1}(v) = \big|\, \{\, u \in V \mid \exists j \in [d], \Gamma(u)_j = v \,\} \,\big| \leq d.$$

In words, there are at most d edges into a vertex. As a note, our construction just requires that $\deg^{-1}(\cdot) = O(1)$ but for the sake of simplicity we fix $\deg^{-1}(\cdot) \leq d$.

We also require that $\mathcal{G}$ is equipped with a neighbor function, Γ, which can be computed in polynomial time. We define a randomized and keyed labelling function $\phi : \{0,1\}^\lambda \times V \to \{0,1\}^{\mathrm{poly}(\lambda)}$ such that given, $\phi(K, v_0)$ for root v_0, an adversary, $\mathcal{A}$, which does not know a path from v_0 to v,

$$\mathbf{Pr}[\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, v, \phi(K, v_0)) = \phi(K, v)] \leq \epsilon \tag{1}$$

for function C_Γ where $C_\Gamma(\phi(K, u)) = \phi(K, \Gamma(u)_1), \ldots, \phi(K, \Gamma(u)_d)$ if $\Gamma(u) \neq \emptyset$ and otherwise $\Gamma(u)$ returns a $\perp$ string; and, $\mathcal{O}$ represents an indistinguishable obfuscator. We fix $\epsilon \leq \mathtt{negl}(\lambda)$.

2.2 Instantiation

We define

$$\phi(K, v) = F(K, v).$$

We can now define C_Γ:

Algorithm 1 The circuit for the neighbor function, C_Γ.

> **function** $C_\Gamma(X, v)$
>   **if** $f(X) \neq f(F(K, v))$ **then**
>   **return** $\perp$
>   **if** $\Gamma(v) = \emptyset$ **then**
>   **return** $\perp$
> $u_1, \ldots u_d = \Gamma(v)$
>   **return** $F(K, u_1), F(K, u_2), \ldots, F(K, u_d)$

We are going to show that eq. (1) holds by first showing that the non-existence of an extractor to find a path from v_0 to v implies that $\mathcal{A}$ necessarily does not know $\phi(K, c)$ for a $c \in C_V \subset V$ where the vertices in C_V border a graph cut which separates v_0 and v. Then, we inductively build up a series of games to show that $\mathcal{A}$ cannot learn *any* $\phi(K, v)$ for $v \in V_1$ where V_1 are the vertices on the right-hand side of the cut.

Lemma 2.1 (Base Case Game)**.** *Assuming that there is no extractor E such that $\mathbf{Pr}[E(\Gamma, v_0, v) = P] \geq \frac{1}{p(\lambda)}$ where $P \in \mathcal{P}$, then for any PPT $\mathcal{A}$, there exists some graph cut $C_E \subset E$ which separates v_0 and v and a set C_V such that*

$$\mathbf{Pr}[\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, v, \phi(K, v_0)) \in \phi(K, C_V)] < \epsilon. \tag{2}$$

We define $C_V \subset V$ to be

$$\{\, u \mid (w, u) \in C_E \text{ and } u \text{ on the side of } v \,\} \bigcup \{\, v \mid (w, u) \in C_E \text{ and } u \text{ on the side of } v \,\}.$$

In words, C_V are the vertices just adjacent to the cut and on the same side as v.

Proof. We will show that if $\mathcal{A}$ can break eq. (2), then we can construct an extractor, E, which finds a path from v_0 to v with non-negligible probability.

Assume that for every possible cut, $\mathcal{A}$ is able to produce a single label in this cut for a vertex w. Then, we note that there must be at least 1 path from v_0 to w and v as otherwise, w would not be in the cut. Moreover, we note that $\mathcal{A}$ must be able to produce a label for all vertices on at least one path from v_0 to w as otherwise, we can change the cut to include the edges between where $\mathcal{A}$ is able to produce a label and not able to produce a label. Using the same argument, we can show that $\mathcal{A}$ must be able to produce all labels on a path from w to v.

Note that $\mathcal{A}$ is not given the specific cut C_E but rather C_E is chosen based off of the adversary. So, we can build an extractor to do the following:

1. Create an iO obfuscated circuit with a random key, K', for C_Γ and create circuit $\mathcal{O}(C_\Gamma)$ as well as $\phi(K', v_0)$
2. Run $\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, v, \phi(K', v_0))$ to get all labels $\phi(K', v_0), \ldots \phi(K', v)$ for some path from v_0 to v.
3. Recreate the path from v_0 to v via checking which vertex matches to adjacent labels in the path: I.e. starting with $\ell = 0$, we can learn the $\ell + 1$ vertex via finding $j \in [d]$ such that $C_\Gamma(\phi(K', v_\ell), v_\ell)_j \in \{ \phi(K', v_0), \ldots, \phi(K', v) \}$ and then setting $v_{\ell+1} = \Gamma(v_\ell)_j$.

□

We can look at lemma 2.1 as a "base case" of sorts. We now inductively build up a series of games such that $\mathcal{A}$ cannot find any label in V_1 where V_1 are the vertices on side of the cut (as defined in lemma 2.1) which contain v.

Lemma 2.2 (Inductive Game Hypothesis). *Let $H \subset V$ be a "hard" set of vertices such that $\mathcal{A}$ cannot, with non-negligible probability, produce $\phi(K, h)$ where $h \in H$. Note that the base case has $H = C_V$. Assuming BLAH Then, for any $v \notin H$ and $w \in \Gamma(h)$ for all $h \in H$, we have that*

$$\mathbf{Pr}[\mathcal{A}(\mathcal{O}(C_\Gamma), v_0, w, \phi(K, v_0)) = \phi(K, w)] < \epsilon.$$

Proof. We are going to use a series of indistinguishable hybrids along with the circuit defined in 2 to show the above

- $\mathtt{Hyb}_0$: In the first hybrid, the following game is played
  1. $K \leftarrow \{0,1\}^{\lambda'}$ and $\phi(K, v_0) = (F(K, v_0), v)$ where K is some fixed secret drawn from a random distribution
  2. The challenger generates $\mathcal{O}(C_\Gamma)$ and gives the program to $\mathcal{A}$
  3. The challenger gives the adversary w^* in plaintext.
  4. $\mathcal{A}$ outputs guess g and wins if $g = \phi(K, w^*)$
- $\mathtt{Hyb}_1$: We replace C_Γ with C_Γ as defined in 2. Fix the constant $z^* = f(F(K, w^*))$
- $\mathtt{Hyb}_{2,1}$ We replace algorithm 2 with algorithm 3 where we set $Y^* = (1, y)$ such that $\Gamma(y)_1 = w^*$. So then, we have that have $F(K, \Gamma(y)_1) = \perp$. Moreover, we set the punctured set, S to $\emptyset$ (i.e. we do not puncture the PRF).
- $\mathtt{Hyb}_{2,j}$ for $j \in 2, \ldots, \deg^{-1}(w^*)$ We replace Y^* with $Y^* \cup (j, y)$ such that $\Gamma(y)_j = w^*$. Note after the last of these hybrids, we have that $F(K, w^*)$ is always set to $\perp$.

- $\texttt{Hyb}_3$: We puncture the PRF at w^* and set $S = \{ w^* \}$.
- $\texttt{Hyb}_4$: Set $z^* = f(t)$ where t is chosen at random

Finally, we can note that if $\texttt{Hyb}_0 \overset{c}{\approx} \texttt{Hyb}_2$,

$$\mathbf{Pr}[\mathcal{A}(C_\Gamma, v_0, w, \phi(K, v_0)) = \phi(K, w)] \overset{c}{\approx} \mathbf{Pr}[\mathcal{A}(C_\Gamma^*, v_0, w, \phi(K, v_0)) = \phi(K, w)]$$

where z^* in C_Γ^* is the image on a OWF of a randomly chosen point. As we will show in lemma 2.3, lemma 2.4, and lemma 2.6, an adversaries advantage between games in $\texttt{Hyb}_0$ and $\texttt{Hyb}_3$ is at most $\epsilon/2$. Thus, if $\mathcal{A}$ can produce $\phi(K, v) = (\sigma_v, v)$ with advantage $\epsilon/2$ in $\texttt{Hyb}_3$, then $\mathcal{A}$ can find a pre-image for $z*$ under f with non-negligible probability and thus break the security of a one way function. We then have that the advantage of the adversary in $\texttt{Hyb}_0$ cannot be more than ϵ. $\square$

Lemma 2.3. *$\texttt{Hyb}_0$ and $\texttt{Hyb}_1$ are distinguishable with advantage at most $\epsilon/10$.*

Proof. Note that for all inputs (z, v) to C_Γ as defined in algorithm 1 and algorithm 2 are equivalent and thus indistinguishable by the definition of indistinguishable obfuscation. So, if $\epsilon \in \text{poly}(\lambda)$, then an adversary cannot distinguish the hybrids with probability more than $\epsilon/8$. $\square$

Lemma 2.4. *Each hybrid from $\texttt{Hyb}_1$ to $\texttt{Hyb}_{2,1}$ and $\texttt{Hyb}_{2,j-1}$ to $\texttt{Hyb}_{2,j}$ for $j \in 2, \ldots, \deg^{-1}(w^*)$ is distinguishable with advantage at most $\epsilon/(10d)$. Thus, $\texttt{Hyb}_1$ and $\texttt{Hyb}_{2,\deg^{-1}(w^*)}$ are distinguishable with advantage at most $\epsilon/10$.*

Proof. This proof will follow very closely the simple case of weak extractible obfuscation as defined in (TODO: cite). The key idea is that if a hybrid is distinguishable with advantage more than $\epsilon/10d$, then $\mathcal{A}$ can produce a label $\phi(K, h)$ for $h \in H$.

First, assume towards contradiction that there exists an adversary $\mathcal{A}$ that can distinguish two consecutive hybrids with polynomial advantage $\epsilon' > \epsilon/10d$. Following the proof sketch in (TODO: cite), say that the input size to C_Γ is n. Also, let C_0 be the circuit from the first hybrid and C_1 the one from the second. Let C_i^{Mid} be a circuit such that $C_i^{\text{Mid}}(X) = C_0(X)$ if $X_i = 0$ and $C_i^{\text{Mid}}(X) = C_1(X)$ if $X_i = 1$. Note that C_0 and C_1 differs on at most 1 input (which is the appended vertex y to Y^*); call this input α. Then, $C_i^{\text{Mid}} = C_0$ if $\alpha_i = 0$ and $C_i^{\text{Mid}} = C_1$ if $\alpha_i = 1$. So, if we build an adversary $\mathcal{B}$ to tell if $C_i^{\text{Mid}} = C_0$ or C_1 with probability γ, we have that $\mathcal{B}$ can tell if α_i is 0 or 1 with probability γ. Thus, $\mathcal{B}$ can reconstruct α with probability at least γ^n. Note that this implies that $\mathcal{B}$ can learn $\phi(K, y)$ where $y \in H$ by construction and thus gives our desired contradiction. So now, we just need to build $\mathcal{B}$ to tell if $C_i^{\text{Mid}} = C_0$ or C_1 with probability $\gamma^n \geq \frac{\epsilon}{10d}$.

Then, $\mathcal{A}$ can distinguish between C^M via the following:

1. Run $I = \left\lceil \frac{\ln 2 \cdot 96\left(\ln n - \ln\left(1 - \frac{\epsilon}{10d}\right)\right)}{\epsilon'} \right\rceil$ iterations of the following experiment to estimate advantage ϵ'_b for $b \in \{ 0, 1 \}$
   (a) Sample a random obfuscation of C_b via re-obfuscating the existing C_b
   (b) Sample a random obfuscation of C_i^{Mid} via re-obfuscating C_i^{Mid}
   (c) Have $\mathcal{A}$ distinguish between C_b and C^{Mid}
   (d) Output 1 if successful.

   Note that we can estimate ϵ'_b as the number of successful runs, which we will denote $\sum_{j \in [I]} S_{i,j}$, divided by I.

2. If $\epsilon'_1 > \epsilon'_0$, then $C^{\mathrm{Mid}} = C_0$, otherwise, $C^{\mathrm{Mid}} = C_1$.

WLOG, say that $C^{\mathrm{Mid}} = C_0$, then

$$\begin{aligned}\gamma = Pr[\epsilon'_1 > \epsilon'_0] &= \mathbf{Pr}[\sum_j S_{1,j} > \sum_j S_{0,j}] \\ &\geq \mathbf{Pr}\left[\sum_j S_{1,j} > \frac{I\epsilon'}{2}\right] \cdot \mathbf{Pr}\left[\sum_j S_{0,j} < \frac{I\epsilon'}{2}\right].\end{aligned}$$

We then have that

$$\mathbf{Pr}\left[\sum_j S_{1,j} > I\epsilon' \cdot \tfrac{1}{2}\right] \geq 1 - \exp\left(-\frac{I\epsilon'}{2^2 \cdot 3}\right) = 1 - \exp\left(-\frac{I\epsilon'}{96}\right). \qquad \text{(by the Chernoff bound)}$$

And, if iO distinguishing advantage is at most α and $\delta = \frac{\epsilon'}{2\alpha} - 1$

$$\begin{aligned}\mathbf{Pr}\left[\sum_j S_{0,j} < \frac{I\epsilon'}{2}\right] = 1 - Pr\left[\sum_j S_{0,j} \geq (1+\delta)I\alpha\right] &\geq 1 - \exp\left(-I\alpha\left(\frac{\epsilon'}{2\alpha} - 1\right)^2 \cdot \frac{1}{3}\right) \\ &\qquad \text{(by the Chernoff bound)} \\ \geq 1 - \exp\left(-\frac{I\epsilon'^2}{12\alpha}\right) \geq 1 - \exp\left(-\frac{I\epsilon'}{12}\right).. &\qquad \text{(as } \epsilon' > \alpha\text{)}\end{aligned}$$

So we finally have that

$$\mathbf{Pr}[\epsilon'_1 > \epsilon'_0] \geq 1 - \exp\left(-\frac{I\epsilon'}{12}\right) - \exp\left(-\frac{I\epsilon'}{96}\right) \geq 1 - 2\exp\left(-\frac{I\epsilon'}{96}\right). \tag{3}$$

Setting $I \geq \frac{\ln 2 \cdot 96\left(\ln n - \ln\left(1 - \frac{\epsilon}{10d}\right)\right)}{\epsilon'} \in \mathrm{poly}(n, 1/\epsilon, 1/\epsilon')$, we have that

$$\begin{aligned}\gamma^n &\geq \left(1 - 2\exp\left(-\frac{I\epsilon'}{96}\right)\right)^n \\ &\geq 1 - 2n \cdot \exp\left(-\frac{I\epsilon'}{96}\right) = 1 - 2n \cdot \exp\left(-\left(\ln n + \ln\left(1 - \frac{\epsilon}{10d}\right)\right) \cdot 2\right) \\ &= 1 - \left(1 - \frac{\epsilon}{10d}\right) = \frac{\epsilon}{10d}\end{aligned}$$

as desired. □

Lemma 2.5. *The game in* $\mathtt{Hyb}_{2,\deg^{-1}(w^*)}$ *is indistinguishable from* $\mathtt{Hyb}_3$.

Proof. The indistinguishably follows directly from the definition of indistinguishable obfuscation. □

Lemma 2.6. *The game in* $\mathtt{Hyb}_3$ *is indistinguishable from* $\mathtt{Hyb}_4$.

Proof. We now show that if the advantage of $\mathcal{A}$ is greater than $\epsilon/8$, then we can create a reduction, $\mathcal{B}$, which can break the selective security of the PRF at the punctured point. $\mathcal{B}$ first chooses a message w^* and submits this to the constrained PRF challenger and gets back the punctured PRF key $K(\{w^*\})$ and challenge a. $\mathcal{B}$ then runs the experiment in $\mathtt{Hyb}_{2,\deg^{-1}(w^*)}$ except that $z^* = f(a)$. If a is the output of the PRF, then we are in $\mathtt{Hyb}_{2,\deg^{-1}(w^*)}$, if a is the output of a random function, then we are in $\mathtt{Hyb}_3$. □

Algorithm 2 Circuit for the neighbor function, C_Γ with PRF key K and constant w^*, z^*

function $C_\Gamma(X, v)$
    if $v \neq w$ and $f(X) \neq f(F(K, v))$ **then**
        return $\perp$
    if $v = w$ and $f(X) \neq z^*$ **then**
        return $\perp$
    if $\Gamma(v) = \emptyset$ **then**
        return $\perp$
    $u_1, \ldots u_d = \Gamma(v)$
    return $F(K, u_1), F(K, u_2), \ldots, F(K, u_d)$

Algorithm 3 Circuit for the neighbor function, C_Γ with punctured PRF key $K(S)$ and constant w^*, Y^*, J^*, z^*

function $C_\Gamma(X, v)$
    if $v \neq w$ and $f(X) \neq f(F(K, v))$ **then**
        return $\perp$
    if $v = w$ and $f(X) \neq z^*$ **then**
        return $\perp$
    if $\Gamma(v) = \emptyset$ **then**
        return $\perp$
    $u_1, \ldots u_d = \Gamma(v)$
    if $\exists j \in [d], (j^*, v^*) \in Y^*$ **then**
        Set $F(K, u_{j^*}) = \perp$
    return $F(K, u_1), F(K, u_2), \ldots, F(K, u_d)$

Lemma 2.7. *The game in* $\texttt{Hyb}_1(1a)$ *is indistinguishable from* $\texttt{Hyb}_0$.

Proof. As the functionality of C_Γ in $\texttt{Hyb}_0$ equals that of $\texttt{Hyb}_1(1a)$, we have indistinguishable simply from the definition of indistinguishable obfuscation. □

Lemma 2.8. *The game in* $\texttt{Hyb}_1(1b)$ *is indistinguishable from* $\texttt{Hyb}_1(1a)$.

Proof. Here we argue that if the game in $\texttt{Hyb}_1(1b)$ is distinguishable from $\texttt{Hyb}_1(1a)$, then we can construct an adversary, $\mathcal{B}$, which can break the security of the PRF at the punctured point. □

Lemma 2.9. *The game in* $\texttt{Hyb}_1(2a)$ *is indistinguishable from* $\texttt{Hyb}_0$ *and, by the inductive hypothesis, all previous hybrids.*

Proof. Again, we have that the circuit for C_Γ is the same in $\texttt{Hyb}_0$ and $\texttt{Hyb}_1(2a)$. Thus, by the definition of indistinguishable obfuscation, these games are indistinguishable. □

Lemma 2.10. *The game in* $\texttt{Hyb}_1(2b)$ *is indistinguishable from* $\texttt{Hyb}_1(2a)$ *and, by the inductive hypothesis, all previous hybrids.*

Proof. TODO: PRF security + extractor part □

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Abstract

References

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