

Multiple Secret Leaders

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1 Preliminaries

1.1 Threshold FHE

[JRS17, BGG⁺18] defines threshold FHE encryption. For the sake of completeness, we will define it here.

Definition 1.1 (TFHE [JRS17]). Let $P = \{P_1, \dots, P_N\}$ be a set of N parties and \mathbb{S} be a class of access structures on P . A TFHE scheme for \mathbb{S} is a tuple of PPT algorithms

$$(\text{TFHE.Setup}, \text{TFHE.Encrypt}, \text{TFHE.Eval}, \text{TFHE.PartDec}, \text{TFHE.FinDec})$$

such that the following specifications are met

- $(pk, sk_1, \dots, sk_N) \leftarrow \text{TFHE.Setup}(1^\lambda, 1^s, \mathbb{A})$: Takes as input a security parameter λ , a depth bound on the circuit, and a structure $\mathbb{A} \in \mathbb{S}$. Outputs a public key pk and a secret key sk_i for each party P_i .
- $ct \leftarrow \text{TFHE.Encrypt}(pk, \mu)$: Takes as input a public key and a message $\mu \in \{0, 1\}$ and outputs a ciphertext ct .
- $\hat{ct} \leftarrow \text{TFHE.Eval}(C, ct_1, \dots, ct_k)$: Takes as input a circuit C of depth at most d and k ciphertexts ct_1, \dots, ct_k . Outputs a ciphertext $\hat{ct} = C(ct_1, \dots, ct_k)$.
- $p_i \leftarrow \text{TFHE.PartDec}(ct, sk_i)$: Takes as input a ciphertext ct and a secret key sk_i and outputs a partial decryption p_i .
- $\hat{\mu} \leftarrow \text{TFHE.FinDec}(B)$: Takes as input a set $B = \{p_i\}_{i \in S}$ for some $S \subseteq [N]$ and deterministically outputs a message $\hat{\mu} \in \{0, 1, \perp\}$.

Further, we remember the definitions of evaluation correctness, semantic security, and simulation security as outlined in, [BEHG20].

Definition 1.2 (Evaluation Correctness [JRS17]). We have that TFHE scheme is correct if for all λ , depth bounds d , access structure \mathbb{A} , circuit $C : \{0, 1\}^k \rightarrow \{0, 1\}$ of depth at most d , $S \in \mathbb{A}$, and $\mu_i \in \{0, 1\}$, we have the following. For $(pk, sk_1, \dots, sk_N) \leftarrow \text{TFHE.Setup}(1^\lambda, 1^d, \mathbb{A})$, $ct_i \leftarrow \text{TFHE.Encrypt}(pk, \mu_i)$ for $i \in [k]$, $\hat{ct} \leftarrow \text{TFHE.Eval}(pk, C, ct_1, \dots, ct_k)$,

$$\Pr [\text{TFHE.FinDec}(pk, \{\text{TFHE.PartDec}(pk, \hat{ct}, sk_i)\}_{i \in S}) = C(\mu_1, \dots, \mu_k)] = 1 - \text{negl}(\lambda).$$

Definition 1.3 (Semantic Security [JRS17]). We have that a TFHE scheme satisfies semantic security for for all λ , and depth bound d if the following holds. There is a stateful PPT algorithm $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ such that for any PPT adversary \mathcal{A} , the following experiment outputs 1 with negligible probability in λ :

1. On input 1^λ and depth 1^d , the adversary outputs $\mathbb{A} \in \mathbb{S}$
2. The challenger runs $(pk, sk_1, \dots, sk_N) \leftarrow \text{TFHE.Setup}(1^\lambda, 1^d, \mathbb{A})$ and provides pk to \mathcal{A} .
3. \mathcal{A} outputs a set $S \subseteq \{P_1, \dots, P_N\}$ such that $S \notin \mathbb{A}$.
4. The challenger provides $\{sk_i\}_{i \in S}$ and $\text{TFHE.Encrypt}(pk, \mu)$ to \mathcal{A} where $\mu \xleftarrow{\$} \{0, 1\}$.

5. \mathcal{A} outputs a guess μ' . The experiment outputs 1 if $\mu' = \mu$.

Definition 1.4 (Simulation Security [JRS17]). We say that a TFHE scheme is simulation secure if for all λ , depth bound d , and access structure \mathbb{A} if there exists a stateful PPT simulator, \mathcal{S} , such that for any PPT adversary \mathcal{A} , we have that the experiments $\text{Expt}_{\mathcal{A}, \text{Real}}(1^\lambda, 1^d)$ and $\text{Expt}_{\mathcal{A}, \text{Sim}}(1^\lambda, 1^d)$ are statistically close as a function of λ . The experiments are defined as follows:

- $\text{Expt}_{\mathcal{A}, \text{Real}}(1^\lambda, 1^d)$:

1. On input the security parameter 1^λ and depth bound d , the adversary outputs $\mathbb{A} \in \mathbb{S}$.
2. Run $\text{TFHE.Setup}(1^\lambda, 1^d, \mathbb{A})$ to obtain (pk, sk_1, \dots, sk_N) . The adversary is given pk .
3. The adversary outputs a set $S \subseteq \{P_1, \dots, P_N\}$ such that $S \notin \mathbb{A}$ together with plaintext messages $\mu_1, \dots, \mu_k \in \{0, 1\}$. The adversary is handed over $\{sk_i\}_{i \in S}$
4. For each μ_i , the adversary is given $\text{TFHE.Encrypt}(pk, \mu_i) \rightarrow \text{ct}_i$.
5. The adversary issues a polynomial number of queries, $(S_i \subseteq \{P_1, \dots, P_N\}, C_i)$. for circuits $C_i : \{0, 1\}^k \rightarrow \{0, 1\}$. After each query the adversary receives for $l \in S_i$ the value

$$\text{TFHE.PartDec}(\text{TFHE.Eval}(C_i, \text{ct}_1, \dots, \text{ct}_k), sk_l) \rightarrow p_l$$

6. \mathcal{A} outputs out , the experiment's output.

- $\text{Expt}_{\mathcal{A}, \text{Sim}}(1^\lambda, 1^d)$:

1. On input the security parameter 1^λ and depth bound d , the adversary outputs $\mathbb{A} \in \mathbb{S}$.
2. Run $\text{TFHE.Setup}(1^\lambda, 1^d, \mathbb{A})$ to obtain (pk, sk_1, \dots, sk_N) . The adversary is given pk .
3. \mathcal{A} outputs a set $S^* \subseteq \{P_1, \dots, P_N\}$ such that $S \notin \mathbb{A}$ and plaintexts $\mu_1, \dots, \mu_k \in \{0, 1\}$. The simulator is given pk, \mathbb{A}, S^* as input and outputs $\{sk_i\}_{i \in S^*}$ and the state state . The adversary is given $\{sk_i\}_{i \in S^*}$
4. For each μ_i , the adversary is given $\text{TFHE.Encrypt}(pk, \mu_i) \rightarrow \text{ct}_i$.
5. \mathcal{A} issues a polynomial number of queries of the form $(S_i \subseteq \{P_1, \dots, P_N\}, C_i)$
6. for circuits $C_i : \{0, 1\}^k \rightarrow \{0, 1\}$. After each query, the simulator computes

$$\text{Sim}_{\text{TFHE}}(C_i, \{\text{ct}_l\}_{l=1}^k, C_i(\mu_1, \dots, \mu_k), \text{state}) \rightarrow \{p_l\}_{l \in S_i}$$

and sends $\{p_l\}_{l \in S_i}$ to the adversary.

7. \mathcal{A} outputs out , the experiment's output.

1.2 Non Interactive Zero-Knowledge

The following definition is taken almost verbatim from [CFG22]. A non-Interactive Zero-Knowledge (NZIK) proof system for relationship Rel is a tuple of PPT algorithms $(\text{NZIK.G}, \text{NZIK.P}, \text{NZIK.V})$ such that NZIK.G generates a common reference string, NZIK.crs , $\text{NZIK.P}(\text{NZIK.crs}, x, w)$ given $(x, w) \in Rel$, outputs a proof π , and $\text{NZIK.V}(\text{NZIK.crs}, x, \pi)$ outputs 1 if $(x, w) \in Rel$ and 0 otherwise. A NZIK is correct if for every NZIK.crs and all $(x, w) \in Rel$, we have that $\text{NZIK.V}(\text{NZIK.crs}, x, \text{NZIK.P}(\text{NZIK.crs}, x, w)) = 1$ holds with probability 1. We also require that our NZIKs satisfy the notions of weak simulation extractability [Sah99] and zero-knowledge [FLS90].

Weak simulation extractability guarantees the extractability of proofs produced by the adversary that are not equal to proofs previously observed. Thus, we make each proof “unique” by implicitly

adding a session ID to the statement. For more of a commentary, see section 2.7 of [CFG22]. We will not detail how to handle these session IDs.

We recall the notion of simulation security from [Sah99]:

Definition 1.5 (NZIK simulation security). We say that a proof system for relationship Rel is simulation secure if proof system $(NZIK.G, NZIK.P, NZIK.V)$ is a non-interactive proof system and Sim_1, Sim_2 are PPT algorithms such that for all PPT adversaries $\mathcal{A}_1, \mathcal{A}_2$ we have that $|\Pr[\text{Expt}_{\mathcal{A}, \text{Real}}(\lambda) = 1] - \Pr[\text{Expt}_{\mathcal{A}, \text{Sim}}(\lambda) = 1]|$ is negligible in λ . The experiments are defined as follows:

- $\text{Expt}_{\mathcal{A}, \text{Real}}$:
 1. $NZIK.crs \leftarrow NZIK.G(1^\lambda)$
 2. \mathcal{A}_1 outputs (x, w, state_1)
 3. $\pi \leftarrow NZIK.P(NZIK.crs, x, w)$
 4. return $\mathcal{A}_2(\pi, \text{state}_1)$
- $\text{Expt}_{\mathcal{A}, \text{Sim}}$:
 1. $NZIK.crs \leftarrow Sim_1(1^\lambda)$
 2. \mathcal{A}_1 outputs (x, w, state)
 3. $\pi \leftarrow Sim_2(NZIK.crs, x, w)$
 4. return $\mathcal{A}_2(\pi, \text{state})$

We also have that a NZIK has ideal functionality \mathcal{F}_{NZIK}^{Rel} which is defined as follows:

The \mathcal{F}_{NZIK}^{Rel} functionality for zero-knowledge:

Upon receiving $(\text{prove}, sid, x, w)$ from P_i , with sid being used for the first time, if $(x, w) \in Rel$, broadcast $(\text{proof}, sid, i, \pi)$, otherwise broadcast \perp .

1.3 Data Independent Priority Queue

In this work, we will use data independent queues as studied in [Tof11, MZ14, MDPB23]. Data independent data structures are unique as their control flow and memory access do not depend on input data ([MZ14]).

Definition 1.6 (Word RAM model [MZ14]). In the word RAM model, the RAM has a constant number of public and secret registers and can perform arbitrary operations on a constant number of registers in constant time.

Definition 1.7 (Data Independent Data Structure [MZ14]). In the word RAM model, a data independent data structure is a collection of algorithms where all the algorithms uses RAM such that the RAM can only set its control flow based on registers that are public.

Data independent queues are especially useful as they allow for efficient computation within MPC and FHE as control flow is not dependent on underlying ciphertexts data. We use a data independent queue as outlined in [MDPB23] which allows for

- **PQ.Insert**: Inserts a tag and value, (p, x) into PQ according to the tag's priority.

- **PQ.ExtractFront**: Removes and returns the (p, y) with highest tag priority.
- **PQ.Front**: Returns the (p, y) with highest tag priority without removing the element.

Moreover, we note that the order is stable. I.e. the first inserted among equal tagged elements has a higher priority.

1.4 Reservoir Sampling

Reservoir sampling is an online algorithm which allows for randomly selecting k elements from a stream of n elements while using $\tilde{O}(k)$ space. Algorithm R ([Vit85]) is a simple algorithm which relies on a priority queue with interface:

- **Reservoir.Init**(k) initialize the reservoir sampling data structure \mathcal{R} such that $\mathcal{R}.\text{PQ}$ is an empty priority queue.
- **Reservoir.Insert**($\mathcal{R}, \mu_i, e_i, \text{coin}_i$) where μ_i is i -th item, e_i is independently sampled randomness, and coin_i is a random coin with probability $1/m$ of equaling 1.
 - If $i \leq k$, insert the item into the queue along with e_i as its tag via $\mathcal{R}.\text{PQ.Insert}(e_i, \mu_i)$.
 - If $i > k$ and $\text{coin}_i = 1$, replace the smallest labeled item in the queue with the new item if the coin is 1.
 - If $i > k$ and $\text{coin}_i = 0$, do nothing.
 - Return \mathcal{R}
- $\mu_{a_1}, \mu_{a_2}, \dots, \mu_{a_k} \leftarrow \text{Reservoir.Output}(\mathcal{R})$ where a_1, \dots, a_k are a uniformly random ordered subset of $[n]$ if e_i, coin_i are independently sampled uniformly at random for all $i \in n$ where n is the number of **Reservoir.Insert** calls.
 - Call **PQ.ExtractFront** k times setting μ_{a_ℓ} to the ℓ -th call to **PQ.ExtractFront** where $\ell \in [k]$.

2 Ideal MSLE Functionality

We use a similar notion of ideal functionality for a multi-secret leader election from the ideal functionality of single secret leader election of [CFG22] except that we add a `register_elect` phase for each election.

The MSLE functionality $\mathcal{F}_{\text{MSLE}}$: Set $\mathcal{E} \leftarrow \emptyset$ to denote the set of finished elections. Fix some $k \in \mathbb{N}$ to denote the number of rounds. Upon receiving,

- `register_elect`(eid) from party P_i . If $eid \in \mathcal{E}$ send \perp to P_i and do nothing. If S_{eid} is not defined, set $S_{eid} \leftarrow \{i\}$. Otherwise, set $S_{eid} \leftarrow S_{eid} \cup \{i\}$ and store S_{eid} .
- `elect`(eid) from all honest participants.
If $|S_{eid}| \geq k$, randomly sample $W^{eid} \subseteq S_{eid}$ where $|W^{eid}| = k$. Then, assign a random ordering to W^{eid} to get ordered set E^{eid} . Next, send `(outcome, eid, ℓ)` to P_j if $E_\ell^{eid} = j$ and `(outcome, eid, \perp)` to P_i if $i \notin E^{eid}$. Store E^{eid} and set $\mathcal{E} \leftarrow \mathcal{E} \cup \{eid\}$.
- `reveal`(eid, ℓ) from P_i : If E^{eid} is not defined, send \perp to P_i and do nothing. Otherwise, retrieve E^{eid} . If $i = E_\ell^{eid}$, broadcast `(result, eid, ℓ, i)`. Otherwise, broadcast `(rejected, eid, ℓ, i)`.

Figure 1: Description of the Multi Secret Leader Election functionality

3 Semi Honest Reservoir Sampling Based Protocol

We outline a semi-honest protocol in [fig. 2](#) in the common reference string model for the MSLE functionality.

We define the setup, **setup**, for the protocol as follows:

- Set $k_{\text{TFHE}} \xleftarrow{\$} \mathbb{F}_q$.
- Publish CRS, $CRS = \text{Enc}_{\text{TFHE}}(k_{\text{TFHE}})$.

The MSLE Protocol, Protocol, π_i , for party P_i in the semi-honest setting:

Each party has as input a TFHE secret key share sk_i of secret key sk with associated public key pk . Initialize a set of finished elections \mathcal{E} .

- (**register_elect**, eid) called by party P_i
 - Sample $s_i, r_i \xleftarrow{\$} \mathbb{F}_q$, $c_i \leftarrow \text{comm}(s_i, r_i)$, $\text{ct}_i \leftarrow \text{Enc}_{\text{TFHE}}(c_i)$
 - Broadcast (**register_elect_add**, eid, ct_i) to all parties and run (**register_elect_add**, eid, ct_i)
- (**register_elect_add**, eid, ct_j) from party P_j for $j \in [n]$
 - If b_{eid} is not stored, set $b_{eid} = 0$
 - $b_{eid} \leftarrow b_{eid} + 1$
 - If \mathcal{R}_{eid} is not stored, $\mathcal{R}_{eid} \leftarrow \text{Reservoir.Init}(k)$
 - Set $\text{Enc}_{\text{TFHE}}(e_i), \text{Enc}_{\text{TFHE}}(\text{coin}_i) \leftarrow \text{TFHE.Eval}(C_{\text{PRF}}, \text{CRS}, \text{ct}_j)$ where C is the PRF evaluation circuit and CRS is the key to the PRF.
 - Call $\mathcal{R}_{eid} \leftarrow \text{TFHE.Eval}(C, \mathcal{R}_{eid}, \text{ct}_j, \text{Enc}_{\text{TFHE}}(e_i), \text{Enc}_{\text{TFHE}}(\text{coin}_i))$ where C is the reservoir sampling circuit for **Reservoir.Insert**.
 - Store \mathcal{R}_{eid} .
- (**elect**, eid) called by party P_i when $b_{eid} = n$
 - For $\ell \in [k]$, set $\text{ct}_\ell \leftarrow \text{TFHE.Eval}(C_{\text{Reservoir.Output}}, \mathcal{R}_{eid})$
 - Set $p_i \leftarrow \text{TFHE.PartDec}(\text{ct}_1, \dots, \text{ct}_k, sk_i)$
 - Broadcast (**elect_done**, eid, p_i) and call (**elect_done**, eid, p_i)
- (**elect_done**, eid, p_j) from party P_j
 - Retrieve \mathcal{P}^{eid} if it is stored, otherwise set $\mathcal{P}^{eid} \leftarrow \emptyset$.
 - Set $\mathcal{P}^{eid} \leftarrow \mathcal{P}^{eid} \cup \{p_j\}$
 - If $|\mathcal{P}^{eid}| = n$, set $c_{a_1}, \dots, c_{a_\ell} \leftarrow \text{TFHE.FinDec}(\mathcal{P}^{eid})$
 - Store \mathcal{P}^{eid} .
- (**reveal**, eid, ℓ) from party P_i if $c_{a_\ell} = c_i$.
 - Broadcast (**result**, eid, ℓ, i)

Figure 2: Description of the MSLE protocol

3.1 Semi Honest Simulation Security

We will now show that the above protocol is semi-honest simulation secure.

Theorem 3.1 (Semi-honest simulation security). *Assuming the existence of a PRF, a TFHE scheme with semantic security ([definition 1.3](#)) and simulation security ([definition 1.4](#)), then the protocol outline in [fig. 2](#) is semi-honest simulation secure.*

Proof. We will show that the simulator outlined in [fig. 3](#) is a simulator for the view of corrupt parties $C \subset [n]$ and $|C| < t$ by showing that the simulator's view and the protocol's view are indistinguishable for all calls via the following three lemmas.

Lemma 3.2. *The views of `register_elect`, `register_elect_add` in the protocol and simulator are indistinguishable.*

Proof. The simulator firsts simulates the view of the protocol for all calls to `register_elect` from P_i , $i \in C$. Note that the simulator uses the same inputs, s_i, r_i as in the protocol and thus the view is identical. Moreover, the view of $(\text{register_elect_add}, \text{eid}, \text{ct}'_i)$ is identical to the real protocol as $\text{ct}'_i = \text{ct}_i$. For $j \notin C$, note that $c'_j \stackrel{C}{\equiv} c_j$ by the hiding property of commitments and thus $\text{ct}'_j \stackrel{C}{\equiv} \text{ct}_j$. Then, assuming that prior calls to `register_elect` by P_j are simulated by the simulator, the view of $(\text{register_elect_add}, \text{eid}, \text{ct}'_j)$ is indistinguishable to the real protocol as `register_elect_add` is completely determined by prior calls to `register_elect_add` and ct'_j .

Let $\mathcal{R}_{\text{eid}, \text{Sim}_C}$ denote the reservoir sampling data-structure in the simulator. We then note that, after α , for $\alpha \in [n]$, calls to `register_elect_add`, the simulator's $\mathcal{R}_{\text{eid}, \text{Sim}_C} \stackrel{C}{\equiv} \mathcal{R}_{\text{eid}}$ of the protocol as \mathcal{R} is completely determined by the calls to `register_elect_add`. \square

Lemma 3.3. *The views of `elect`, `elect_done` in the protocol and simulator are indistinguishable.*

Proof. When `elect` is called by P_i for $i \in C$, we have that for $\text{ct}'_\ell = \text{TFHE.Eval}(C_{\text{Reservoir.Output}}, \mathcal{R}_{\text{eid}, \text{Sim}_C})$, $\text{ct}'_\ell \stackrel{C}{\equiv} \text{ct}_\ell$ for all $\ell \in [k]$. This is because $\mathcal{R}_{\text{eid}, \text{Sim}_C}$ is indistinguishable from that of the real protocol. We also have that, by the ideal functionality of `elect`, $\{a'_\ell\}_{\ell \in [k]}$ is a random subset of $[n]$ with random order with size k . Then, note that $e_i, \text{coin}_i = \text{PRF}(k_{\text{TFHE}}, c_i)$ in the protocol is indistinguishable from a randomly sampled e_i, coin_i by the semantic security of PRFs. Thus, by the correctness of `Reservoir.Output`(\mathcal{R}), $\{a_\ell\}_{\ell \in [k]}$ is a randomly chosen ordered subset of $[n]$. So then, $\{a'_\ell\}_{\ell \in [k]} \stackrel{C}{\equiv} \{a_\ell\}_{\ell \in [k]}$. Remembering that $c'_\beta \stackrel{C}{\equiv} c_\beta$ for all $\beta \in [n]$, we have that $c'_{a_1}, \dots, c'_{a_k} \stackrel{C}{\equiv} c_{a_1}, \dots, c_{a_k}$. By the simulation security of TFHE, we have that the decryption shares, $p'_1, \dots, p'_n = \text{Sim}_{\text{TFHE}}(C_{\text{Ident}}, (\{c'_\ell\}, c'_{a_1}, \dots, c'_{a_k}), [n], \text{st})$, for all $i \in [n]$ are simulated by the simulator such that $p'_i \stackrel{C}{\equiv} p_i$ in the real protocol. And thus, all calls to `elect_done` are indistinguishable from the real protocol as `elect_done` is completely determined by $\{p_i\}_{i \in [n]}$. \square

Lemma 3.4 (`reveal` is simulation secure). *Proof.* Finally, note that `reveal` in the simulator is identical to that of the real protocol as in the semi-honest setting, only parties which have won an election will call `reveal`. \square

We thus have that the view of the simulator is identical to that of the real protocol by the above three lemmas. \square

Simulator for Threshold MSLE Sim_C where $C \subseteq [n]$ and $|C| < t$:

Initialize a set of finished elections \mathcal{E} and set a random tape for the simulator.

- **(register_elect, eid)** for party P_j
 - If $j \in C$ Follow the protocol's **register_elect** using P_j 's $s_j, r_j, c_j, \text{ct}_j$ from the real protocol and call **(register_elect_add, eid, ct_j)**. Set $c'_j \leftarrow c_j$ and store c'_j .
 - If $j \notin C$, $s'_j, r'_j \xleftarrow{\$} \mathbb{F}_q$, $c'_j \leftarrow \text{comm}(s'_j, r'_j)$, $\text{ct}'_j \leftarrow \text{Enc}_{\text{TFHE}}(c'_j)$ Store c'_j and follow the protocol by calling **(register_elect_add, eid, ct'_j)**.
- **(elect, eid, out)** from party P_j where **out** is $\mathcal{F}_{\text{MSLE}}.\text{elect}$ output $(\text{outcome}, \text{eid}, q)_i$ for all $i \in [n]$ where $q \in \{1, \dots, k, \perp\}$.
 - For $\ell \in [k]$, set $\text{ct}'_\ell \leftarrow \text{TFHE.Eval}(C_{\text{Reservoir.Output}}, \mathcal{R}_{\text{eid}})$
 - Let $a'_1, \dots, a'_k \in [n]$ such that if $q_i = \ell$ then $a'_\ell = i$.
 - Call $p'_1, \dots, p'_n \leftarrow \text{Sim}_{\text{TFHE}}(C_{\text{Ident}}, (\{\text{ct}'_\ell\}, c'_{a'_1}, \dots, c'_{a'_k}), [n], \text{st})$
 - Call **(elect_done, eid, p_j)** from the protocol.
- **(reveal, eid, ℓ)** for P_j and $\mathcal{F}_{\text{MSLE}}.\text{reveal}$ output **(result, eid, ℓ, j)** or **(reject, eid, ℓ, j)**.
 - Broadcast **(result, eid, ℓ, j)**

Figure 3: Description of the MSLE protocol

4 Malicious Adversary Secure Protocol

Before describing the protocol, we need to define two relationships which will be used in conjunction with two different NZIKs. A tuple $((pk, ct), \mu) \in Rel_{\text{Encr}}$ if $ct = \text{Enc}_{\text{TFHE}}(\mu)$ under TFHE public key pk . A tuple $((p_i, \text{Inps}, \text{Enc}_{\text{TFHE}}(s_{\text{TFHE}})), sk_i) \in Rel_{\text{Prot}}$ if

The MSLE Protocol π_{MSLE} : Initialize $b = 0$. Fix some $k \in \mathbb{N}$ to denote the number of rounds. Initialize an empty set of tickets, R , an empty lookup of sets of parties in each election, S , an empty lookup of reservoir sampling data structures \mathcal{R} , and a set of finished elections \mathcal{E} . Initialize an empty lookup of election results, E .

- **(register_elect, eid)** called by party P_i Every honest party does the following:
 - Sample $s_i, r_i \xleftarrow{\$} \mathbb{F}_q, c_i \leftarrow \text{comm}(s_i, r_i), ct_i \leftarrow \text{Enc}_{\text{TFHE}}(c_i)$. Store c_i and ct_i in a lookup table, C .
 - Generate a proof π_i^{Encr} to show $(c_i, ct_i) \in Rel_{\text{Encr}}$ where $(w, x) \in Rel_{\text{Encr}}$ if $x = \text{Enc}_{\text{TFHE}}(w)$.
 - Broadcast **(register_elect_add, eid, ct_i, π_i^{Encr})** and run **(register_elect_add, eid, ct_i, π_i^{Encr})**.
- **(register_elect_add, eid, ct_j, π_j^{Encr})** called by party P_j .
 - If $eid \in \mathcal{E}$ or $\text{NZIK.V}(\text{NZIK.crs}, ct_j, \pi_j^{\text{Encr}}) \neq 1$, return and do nothing.
 - If b_{eid} is not stored, $b_{eid} \leftarrow 0$
 - $b_{eid} \leftarrow b_{eid} + 1$
 - If \mathcal{R}_{eid} is not stored, $\mathcal{R}_{eid} \leftarrow \text{Reservoir.Init}(k)$
 - $\text{Enc}_{\text{TFHE}}(e_i), \text{Enc}_{\text{TFHE}}(\text{coin}_i) \leftarrow \text{TFHE.Eval}(C_{\text{PRF}}, CRS, ct_j)$
 - If Inps is not stored, $\text{Inps} \leftarrow []$.
 - $\text{Inps}[j] \leftarrow \text{Enc}_{\text{TFHE}}(c_j)$
 - Call $\mathcal{R}_{eid} \leftarrow \text{TFHE.Eval}(C, \mathcal{R}_{eid}, ct_j, \text{Enc}_{\text{TFHE}}(e_i), \text{Enc}_{\text{TFHE}}(\text{coin}_i))$ where C is the reservoir sampling circuit for **Reservoir.Insert**.
- **elect(eid)**. Each party, P_i does the following:
 - If $eid \in \mathcal{E}$, then broadcast \perp and do nothing.
 - The encrypted streaming sampler will output a list of k messages via calling **Reservoir^{eid}.Output()** k times: $\text{Enc}_{\text{TFHE}}(c_{a_1}), \text{Enc}_{\text{TFHE}}(c_{a_2}), \dots, \text{Enc}_{\text{TFHE}}(c_{a_k})$.
 - TODO: NEED TO CHECK ELECTION SET SIZE
 - Set $n_{\perp} \leftarrow 0$
 - Generate decryption share p_i and set $\text{Shares}_{eid} \leftarrow \{p_i\}$
 - Send **(prove, sid, (p_i, Inps, $\text{Enc}_{\text{TFHE}}(s_{\text{TFHE}})), sk_i)$** to $\mathcal{F}_{\text{NZIK}}^{\text{Rel}}$.
 - Upon receiving \perp , set $n_{\perp} \leftarrow n_{\perp} + 1$.

- Upon receiving $(\text{proof}, \text{sid}, j, (p_j, \text{Inps}', \text{Enc}_{\text{TFHE}}(s_{\text{TFHE}})'))$ from $\mathcal{F}_{\text{NIZK}}^{\text{Rel}}$, check that $\text{Inps} = \text{Inps}'$ and that $\text{Enc}_{\text{TFHE}}(s_{\text{TFHE}}) = \text{Enc}_{\text{TFHE}}(s_{\text{TFHE}})'$. If the equalities hold, set $\text{Shares}_{\text{eid}} \leftarrow \text{Shares}_{\text{eid}} \cup \{p_j\}$.
 - If $|\text{Shares}_{\text{eid}}| \geq t$, use the decryption shares to get $\{c_{a_1}, \dots, c_{a_k}\} = \text{TFHE.FinDec}(\text{Shares})$. And add $E_{\text{eid}} = \{c_{a_1}, \dots, c_{a_k}\}$ to E .
 - If $n_{\perp} \geq (|\text{Inps}| - t) + 1$, then abort and output \perp .
 - TODO: When to abort? At time out, at next election? Etc.
- **reveal** $(\text{eid}, \ell, \text{Enc}_{\text{TFHE}}(c'_i))$ from P_i
 - P_i submits a proof that they know the opening to c_{a_ℓ} . If this proof verifies, send out $(\text{result}, \text{eid}, \ell, i)$ to all parties. Otherwise, send out $(\text{rejected}, \text{eid}, \ell, i)$ to all parties.

4.1 Static Malicious Adversary Security

Consider parties P_i to be corrupted where $i \in C$ and $C \subseteq [n]$ such that. $|C| < t$.

Lemma 4.1 (**register_elect** is simulation secure). *Idea: if \perp , just do same If: $j \notin C$, just do same If: $j \in C$, then use the NIZK to extract out the commitment from the FHE ciphertext or bot Then: S follows protocol*

Lemma 4.2 (**elect** is simulation secure). *Idea: if \perp , just do same If: $j \notin C$, just do same Output list from streaming sampler using TFHE sim, set to correct sequence (irrespective of advers) Ask advers to generate decryption shares for each corrupted party and broadcast proof For non-corrupted parties, use TFHE sim to generate decryption shares, simulate proofs For corrupted parties, ask advers to generate decryption shares and proofs If proofs check, use them, if not discard the decryption shares*

Lemma 4.3 (**reveal** is simulation secure). *If an honest party sends the req, we simulate a commitment opening according to the stored commitment (that we know) If corrupted party sends req, pretending to know opening to not their commitment, we say \perp (this is likely as we have the hiding property) If corrupted party sends req correctly, we use the de-commitment that they send*

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