数值分析与计算

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第五章：插值与逼近

多项式插值问题

Runge 现象

$$f(x)=\frac{1}{1+25x^2}, \quad [-1,1]$$

将区间 n 等分:

$$x_0 < x_1 < \cdots < x_n; \quad x_j = \frac{2j}{n} - 1$$

当 $|x| < 0.72$ 时收敛; 当 $|x| > 0.72$ 时误差很大.

Faber 定理

对区间 [a,b] 上的任意划分

$$
\begin{array}{llll}
x_0^0 & & & \\
x_0^1 & x_1^1 & & \\
x_0^2 & x_1^2 & x_2^2 & \\
 & \cdots & & \\
x_0^n & x_2^n & \cdots & x_n^n \\
 & \cdots & &
\end{array}
$$

总存在 $f(x) \in C[a,b]$ 使得按照上面三角矩阵的第 n 行为节点进行插值得到的插值函数 $p_n(x)$ 不能一致收敛到 $f(x)$.

即不管如何加密插值点，都不能保证收敛性.

$$f:(-\infty,\infty)\to C$$

以 2π 为周期. 将区间 $[0,2\pi)$ n 等分.

$$x_j=\frac{2\pi j}{n},\quad j=0,1,\cdots,n-1$$

逼近空间

$$T_n=span\{\phi_0(x),\phi_1(x),\cdots,\phi_{n-1}(x)\}$$

其中

$$\phi_k(x)=e^{ikx}=\cos(kx)+i\sin(kx)$$

三角函数插值问题: 求

$$s(x)=\sum_{k=0}^{n-1}c_k\phi_k(x)\in T_n$$

满足

$$s(x_j)=f(x_j),\quad j=0,1,\cdots,n-1$$

$$w = e^{i\frac{2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

其中 $i = \sqrt{-1}$ 为虚单位.

$$\phi_k(x_j) = \phi_k\left(\frac{2\pi j}{n}\right) = w^{jk}$$

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \end{pmatrix}$$

快速傅里叶变换

$w = e^{i\frac{2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$. 称矩阵:

$$F(w) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)(n-1)} \end{pmatrix}$$

为 n 阶傅里叶变换.

$$F(w) = \left(F_{j,k}\right) = \left(w^{jk}\right), \quad j,k \in [0,n)$$

普通的矩阵向量乘法 $y = F(w)x$, 复杂度为 $O(n^2)$.

当 $n = 2^p$ 时, 利用 w 的周期性: $w^{kn+j} = w^j$, 可以将其复杂度降为 $O(n \log n)$.

以 $n = 8$ 为例.

$$y_j = \sum_{k=0}^{7} x_k w^{jk}, \quad 0 \le j < 8 \tag{4}$$

将 j, k 表示为二进制形式:

$$\begin{aligned} j = (j_2, j_1, j_0) &= 4j_2 + 2j_1 + j_0 \\ k = (k_2, k_1, k_0) &= 4k_2 + 2k_1 + k_0 \end{aligned}$$

将公式 (4) 也转换为二进制形式:

$$\begin{aligned} y_j &= y(j_2, j_1, j_0) = \sum_{k=0}^{7} x_k w^{jk} \\ &= \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{(k_2,k_1,k_0)(j_2,j_1,j_0)} \end{aligned}$$

$$(k_2, k_1, k_0)(j_2, j_1, j_0) =$$
$$4k_2(j_2, j_1, j_0) + 2k_1(j_2, j_1, j_0) + k_0(j_2, j_1, j_0)$$

$$w^8 = 1,$$

$$w^{(k_2,k_1,k_0)(j_2,j_1,j_0)} = w^{k_2(j_0,0,0)} w^{k_1(j_1,j_0,0)} w^{k_0(j_2,j_1,j_0)}$$

傅里叶变换可以表示为

$$y(j_2, j_1, j_0) = \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{k_2(j_0,0,0)} w^{k_1(j_1,j_0,0)} w^{k_0(j_2,j_1,j_0)}$$

$$= \sum_{k_0=0}^{1} \left[\sum_{k_1=0}^{1} \left(\sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{k_2(j_0,0,0)} \right) w^{k_1(j_1,j_0,0)} \right] w^{k_0(j_2,j_1,j_0)}$$

令:

$$a(j_0,k_1,k_0)=\sum_{k_2=0}^{1}x(k_2,k_1,k_0)w^{k_2(j_0,0,0)}$$

则:

$$y(j_2,j_1,j_0)=$$

$$\sum_{k_0=0}^{1}\left[\sum_{k_1=0}^{1}\left(\sum_{k_2=0}^{1}x(k_2,k_1,k_0)w^{k_2(j_0,0,0)}\right)w^{k_1(j_1,j_0,0)}\right]w^{k_0(j_2,j_1,j_0)}$$

$$=\sum_{k_0=0}^{1}\left[\sum_{k_1=0}^{1}a(j_0,k_1,k_0)w^{k_1(j_1,j_0,0)}\right]w^{k_0(j_2,j_1,j_0)}$$

令

$$b(j_1,j_0,k_0)=\sum_{k_1=0}^{1}a(j_0,k_1,k_0)w^{k_1(j_1,j_0,0)}$$

则:

$$\begin{aligned}y(j_2,j_1,j_0) &= \sum_{k_0=0}^{1}\left[\sum_{k_1=0}^{1}a(j_0,k_1,k_0)w^{k_1(j_1,j_0,0)}\right]w^{k_0(j_2,j_1,j_0)}\\ &= \sum_{k_0=0}^{1}b(j_1,j_0,k_0)w^{k_0(j_2,j_1,j_0)}\end{aligned}$$

a 向量的计算格式为:

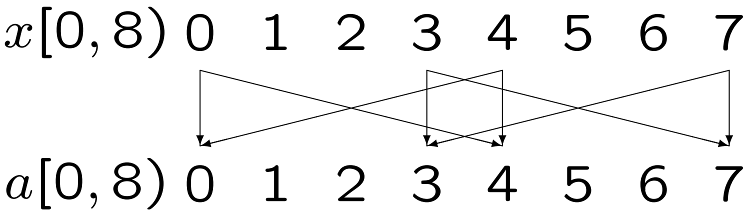
$$a(j_0, k_1, k_0) = \sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{k_2(j_0,0,0)}$$

$$= x(0, k_1, k_0) + x(1, k_1, k_0) w^{(j_0,0,0)}$$

$$a(0, k_1, k_0) = x(0, k_1, k_0) + x(1, k_1, k_0)$$

$$a(1, k_1, k_0) = x(0, k_1, k_0) - x(1, k_1, k_0)$$

可以用图示为:

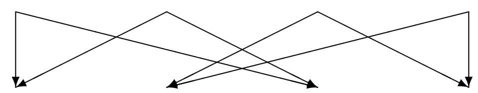


b 向量的计算格式为

$$
\begin{aligned}
b(j_1, j_0, k_0) &= \sum_{k_1=0}^{1} a(j_0, k_1, k_0) w^{k_1(j_1, j_0, 0)} \\
&= a(j_0, 0, k_0) + a(j_0, 1, k_0) w^{(j_1, j_0, 0)} \\
b(0, j_0, k_0) &= a(j_0, 0, k_0) + a(j_0, 1, k_0) w^{2j_0} \\
b(1, j_0, k_0) &= a(j_0, 0, k_0) - a(j_0, 1, k_0) w^{2j_0}
\end{aligned}
$$

可以用图示为:

$a[0,8)$ 0 1 2 3 4 5 6 7

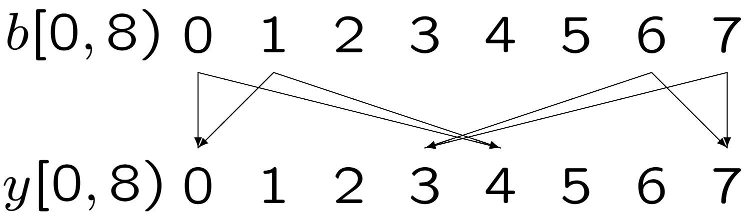


$b[0,8)$ 0 1 2 3 4 5 6 7

y 向量的计算格式为

$$
\begin{aligned}
y(j_2, j_1, j_0) &= \sum_{k_0=0}^{1} b(j_1, j_0, k_0) w^{k_0(j_2,j_1,j_0)} \\
&= b(j_1, j_0, 0) + b(j_1, j_0, 1) w^{(j_2,j_1,j_0)} \\
y(0, j_1, j_0) &= b(j_1, j_0, 0) + b(j_1, j_0, 1) w^{2j_1+j_0} \\
y(1, j_1, j_0) &= b(j_1, j_0, 0) - b(j_1, j_0, 1) w^{2j_1+j_0}
\end{aligned}
$$

可以用图示为:



二进制倒序变换

长度为 $n = 2^p$ 的数组，其二进制倒序是指将下标的二进制表示中的 0,1 串倒置.

长度为 8的数组

$$(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

的二进制倒序是

$$(x_0, x_4, x_2, x_6, x_1, x_5, x_3, x_7)$$

$$(0.0,\ 1.0,\ 2.0,\ 3.0,\ 4.0,\ 5.0,\ 6.0,\ 7.0)$$

的二进制倒序为:

$$(0.0,\ 4.0,\ 2.0,\ 6.0,\ 1.0,\ 5.0,\ 3.0,\ 7.0)$$

$n = 8$ 的二进制倒序变换

数组分量	下标	下标倒置	数组分量
x_0	000	000	x_0
x_1	001	100	x_4
x_2	010	010	x_2
x_3	011	110	x_6
x_4	100	001	x_1
x_5	101	101	x_5
x_6	110	011	x_3
x_7	111	111	x_7

B: n 维空间中的二进制倒置变换.

二进制模式串倒置后再倒置就变回自身. 所以

$$B^2 = I$$

当 $n = 8$ 时 B 相当于下面的排列:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 2 & 6 & 1 & 5 & 3 & 7 \end{pmatrix} = (0)(1,4)(2)(3,6)(5)(7)$$

B 是线性变换

$$\begin{aligned} B &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_0^T \\ e_4^T \\ e_2^T \\ e_6^T \\ e_1^T \\ e_5^T \\ e_3^T \\ e_7^T \end{pmatrix} \\ &= (e_0, e_4, e_2, e_6, e_1, e_5, e_3, e_7) \\ &= B^T = B^{-1} \end{aligned}$$

$n = 16$ 的二进制倒序变换

数组分量	下标	下标倒置	数组分量
x_0	0000	0000	x_0
x_1	0001	1000	x_8
x_2	0010	0100	x_4
x_3	0011	1100	x_{12}
x_4	0100	0010	x_2
x_5	0101	1010	x_{10}
x_6	0110	0110	x_6
x_7	0111	1110	x_{14}
x_8	1000	0001	x_1
x_9	1001	1001	x_9
x_{10}	1010	0101	x_5
x_{11}	1011	1101	x_{13}
x_{12}	1100	0011	x_3
x_{13}	1101	1011	x_{11}
x_{14}	1110	0111	x_7
x_{15}	1111	1111	x_{15}

$$(0)(1,8)(2,4)(3,12)(5,10)(6)(7,14)(9)(11,13)(15)$$

令 $k \in [0, 2^p)$, k 的 p 位二进制表示为

$$k = (b_{p-1} \cdots b_1 b_0)_2 = \sum_{j=0}^{p-1} b_j 2^j$$

记 k 的二进制倒置数为

$$\overleftarrow{k} = (b_0 b_1 \cdots b_{p-1})_2 = \sum_{j=0}^{p-1} b_j 2^{p-j-1}$$

有下面关系:

$$\begin{array}{ccc|ccc} k & = & (xyz01\cdots1)_2 & \overleftarrow{k} & = & (1\cdots10zyx)_2 \\ \hline k+1 & = & (xyz10\cdots0)_2 & \overleftarrow{k+1} & = & (0\cdots01zyx)_2 \end{array}$$

由 k 到 $k+1$ 可以由计算机硬件作加法实现.

由 $\overleftarrow{k}$ 到 $\overleftarrow{k+1}$ 的变换需要程序自己实现.

```
void biot(T* x, int p)          //二进制倒序变换
{
    if(p < 2)  return;
    int const n = (1 << p);
    int k = 1;
    int j = n/2;                    //j 为 k 的倒置
    while(k < n - 1) {
        if(k < j)  std::swap(x[k], x[j]);
        //求 k+1 的倒置
        p = n/2;
        while(j >= p) { j = j-p;   p = p/2; }
        j = j + p;                  //j 为 k+1 的倒置
        k = k + 1;
    }
}
```

$BF(w)$ 的计算格式

正序傅里叶变换:

$$y(j_2, j_1, j_0) = \sum_{k=0}^{7} x_k w^{jk}$$

$$= \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{(k_2,k_1,k_0)(j_2,j_1,j_0)}$$

倒序傅里叶变换:

$$y(j_0, j_1, j_2) = \sum_{k=0}^{7} x_k w^{jk}$$

$$= \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{(k_2,k_1,k_0)(j_2,j_1,j_0)}$$

$$y(j_0, j_1, j_2) = \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{k_2(j_0,0,0)} w^{k_1(j_1,j_0,0)} w^{k_0(j_2,j_1,j_0)}$$

$$= \sum_{k_0=0}^{1} \left[\sum_{k_1=0}^{1} \left(\sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{k_2(j_0,0,0)} \right) w^{k_1(j_1,j_0,0)} \right] w^{k_0(j_2,j_1,j_0)}$$

令:

$$a(j_0,k_1,k_0)=\sum_{k_2=0}^{1}x(k_2,k_1,k_0)w^{k_2(j_0,0,0)}$$

$$\begin{aligned}
&y(j_0,j_1,j_2)= \\
&\sum_{k_0=0}^{1}\left[\sum_{k_1=0}^{1}\left(\sum_{k_2=0}^{1}x(k_2,k_1,k_0)w^{k_2(j_0,0,0)}\right)w^{k_1(j_1,j_0,0)}\right]w^{k_0(j_2,j_1,j_0)} \\
&\qquad=\sum_{k_0=0}^{1}\left[\sum_{k_1=0}^{1}a(j_0,k_1,k_0)w^{k_1(j_1,j_0,0)}\right]w^{k_0(j_2,j_1,j_0)}
\end{aligned}$$

令:

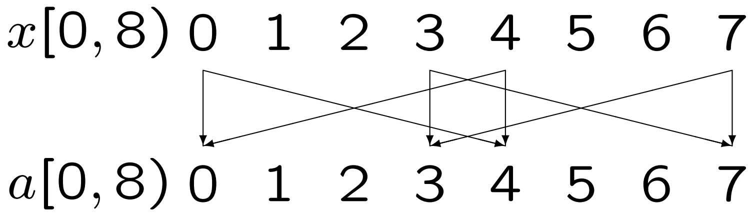
$$b(j_0,j_1,k_0)=\sum_{k_1=0}^{1}a(j_0,k_1,k_0)w^{k_1(j_1,j_0,0)}$$

$$\begin{aligned}
y(j_0,j_1,j_2) &= \sum_{k_0=0}^{1}\left[\sum_{k_1=0}^{1}a(j_0,k_1,k_0)w^{k_1(j_1,j_0,0)}\right]w^{k_0(j_2,j_1,j_0)} \\
&= \sum_{k_0=0}^{1}b(j_0,j_1,k_0)w^{k_0(j_2,j_1,j_0)}
\end{aligned}$$

a 向量的计算格式为:

$$a(j_0, k_1, k_0) = \sum_{k_2=0}^{1} x(k_2, k_1, k_0) w^{k_2(j_0,0,0)}$$
$$= x(0, k_1, k_0) + x(1, k_1, k_0) w^{(j_0,0,0)}$$
$$a(0, k_1, k_0) = x(0, k_1, k_0) + x(1, k_1, k_0)$$
$$a(1, k_1, k_0) = x(0, k_1, k_0) - x(1, k_1, k_0)$$

可以用图示为:



b 向量的计算格式为

$$
\begin{aligned}
b(j_0, j_1, k_0) &= \sum_{k_1=0}^{1} a(j_0, k_1, k_0) w^{k_1(j_1, j_0, 0)} \\
&= a(j_0, 0, k_0) + a(j_0, 1, k_0) w^{(j_1, j_0, 0)} \\
b(j_0, 0, k_0) &= a(j_0, 0, k_0) + a(j_0, 1, k_0) w^{2 j_0} \\
b(j_0, 1, k_0) &= a(j_0, 0, k_0) - a(j_0, 1, k_0) w^{2 j_0}
\end{aligned}
$$

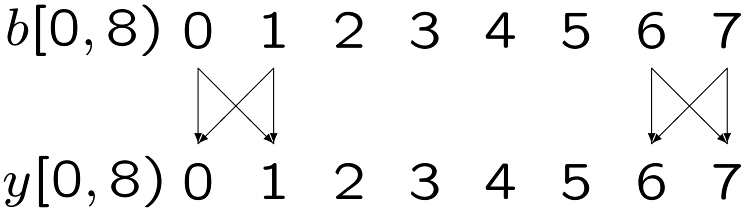
可以用图示为:

$$
\begin{array}{lcccccccc}
a[0,8) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
b[0,8) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

y 向量的计算格式为

$$
\begin{aligned}
y(j_0, j_1, j_2) &= \sum_{k_0=0}^{1} b(j_0, j_1, k_0) w^{k_0(j_2, j_1, j_0)} \\
&= b(j_0, j_1, 0) + b(j_0, j_1, 1) w^{(j_2, j_1, j_0)} \\
y(j_0, j_1, 0) &= b(j_0, j_1, 0) + b(j_0, j_1, 1) w^{2j_1 + j_0} \\
y(j_0, j_1, 1) &= b(j_0, j_1, 0) - b(j_0, j_1, 1) w^{2j_1 + j_0}
\end{aligned}
$$

可以用图示为:



傅里叶变换的逆变换

$$F(w)F(\overline{w}) = nI, \qquad F^{-1}(w) = \frac{1}{n}F(\overline{w})$$

多项式乘法与长整数乘法

$$\begin{aligned} f(x) &= f_0 + f_1 x + f_2 x^2 + \cdots + f_{n-1} x^{n-1} \\ g(x) &= g_0 + g_1 x + g_2 x^2 + \cdots + g_{n-1} x^{n-1} \\ h(x) &= f(x) g(x) \\ &= h_0 + h_1 x + h_2 x^2 + \cdots + h_{n-1} x^{n-1} \end{aligned}$$

在 n 个不同点上的值唯一确定一个多项式

$$x_0, x_1, \cdots, x_{n-1}$$

由系数计算多项式的值

$$\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

取 $x_j = w^j, \quad w = e^{i\frac{2\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$

$$\begin{pmatrix} f(w^0) \\ f(w^1) \\ f(w^2) \\ \vdots \\ f(w^{n-1}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{pmatrix}$$