MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY
HOMEWORK 3
DUE: TUESDAY, FEBRUARY 21 AT 12:00AM (MIDNIGHT) ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Closure properties of cofibrations (10 points)

Prove that cofibrations are closed under the following operations:

(1) Products with a space Y: if $A \to X$ is a cofibration, then the induced map $A \times Y \to X \times Y$ is as well.
(2) Compositions: if $A \to B$ and $B \to X$ are cofibrations, then so is $A \to B \to X$.
(3) Countably transfinite compositions: if $X_i \to X_{i+1}$ are cofibrations for integers $i \geq 0$, then $X_0 \to \varinjlim_n X_n$ is a cofibration. [1]

2. Problem 2: Homotopy pushouts (15 points)

Given a diagram of spaces of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ B, & \end{tikzcd}$$

construct the *homotopy pushout*, which is universal for diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \arrow[dl, Leftarrow] \\ B \arrow[r] & Y. \end{tikzcd}$$

Furthermore:

- Explain how to modify your construction to apply to the setting of pointed spaces and express the homotopy cofiber as a special case of a homotopy pushout of pointed spaces.
- Give an example where the homotopy pushout (of unpointed spaces) is *not* a pushout in the homotopy category. (Hint: consider the case where $X = B = *$.)

3. Problem 3: Space of little cubes (15 points)

Let $\square^n = (0,1)^n$ denote the open unit cube of dimension n. A map $f : \square^n \hookrightarrow \square^n$ is said to be a *rectilinear embedding* if it is of the form

$$f(x_1, \ldots, x_n) = (a_1 x_1 + b_1, \ldots, a_n x_n + b_n)$$

for real numbers $a_i > 0$ and b_i. A map $\coprod_{i=1}^k \square^n \to \square^n$ is said to be a *rectilinear embedding* if it is an open embedding and its restriction to each $\square^n$ in the domain is a rectilinear embedding in the above sense.

We let $C_k(n)$ denote the space of rectilinear embeddings of k disjoint n-cubes $\coprod_{i=1}^k \square^n \to \square^n$, topologized as an open subspace of $(\mathbb{R}^{2n})^k$. As we discussed in class for $k = 2$, there is a continous map $C_k(n) \times (\Omega^n X)^k \to \Omega^n X$, which allows us to view $C_k(n)$ as a space parametrizing k-ary products on n-fold loop spaces.

[1] This is true for general transfinite compositions as well, with the same proof.

On the other hand, let $\mathrm{Conf}_k(\mathbb{R}^n)$ denote the space of k disjoint ordered points in $\mathbb{R}^n$, viewed as an open subspace of $(\mathbb{R}^n)^k$.

(1) Prove that there are homotopy equivalences $C_k(n) \simeq \mathrm{Conf}_k(\mathbb{R}^n)$.
(2) Prove that there are homotopy equivalences $\mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$. (Hint: one direction is given by the map $S^{n-1} \simeq \mathbb{R}^n - \{0\} \to \mathrm{Conf}_2(\mathbb{R}^n)$ which sends $x \in \mathbb{R}^n - \{0\}$ to the ordered pair of points $(0, x)$.)

Remark: The maps $C_k(n) \times (\Omega^n X)^k \to \Omega^n X$ fit together into what is called an *action of the* $\mathbb{E}_n$- or *little n-cubes operad* on an n-fold loop space $\Omega^n X$. In *The Geometry of Iterated Loop Spaces*, May proved that this action completely charactertizes connected n-fold loop spaces: any connected space Y with an action of the $\mathbb{E}_n$-operad is homotopy equivalent to $\Omega^n X$ for a space X called the *n-fold delooping* of Y.

4. Problem 4: Eckmann–Hilton argument (10 points)

An *H-space* is pointed space $(X, *)$ equipped with a multiplication map $m : X \times X \to X$ which is homotopy unital in the sense that the diagram

$$\begin{tikzcd}
X \cong * \times X \arrow[r] \arrow[dr, "id_X"'] & X \times X \arrow[d, "m"] & X \times * \cong X \arrow[l] \arrow[dl, "id_X"] \\
& X &
\end{tikzcd}$$

commutes up to homotopy. (We do *not* include the homotopies as part of the data of an H-space.)

Prove that if X is an H-space, then $\pi_1(X, *)$ is an abelian group. Moreover, prove that the map $\pi_1(X, *) \times \pi_1(X, *) \to \pi_1(X, *)$ induced by m is equal to the group operation. (Hint: make an argument in pure algebra using the fact that m induces a unital group homomorphism $\pi_1(X, *) \times \pi_1(X, *) \to \pi_1(X, *)$.)

Remark: Taking $X = \Omega^{n-1} Y$, this gives a different proof that $\pi_n(Y, *)$ is abelian for $n \geq 2$, as well as the fact that the group operation doesn't depend on which coordinate we use. It also proves that for an H-space X (for example, a topological group), the group operation in $\pi_n(X, *)$ is equal to the operation given by pointwise multiplication of two representatives $S^n \to X$ using the H-space product.