Math 231a Problem Set 4

Lev Kruglyak

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Problem 1. The *cone* on a space X is the quotient space $CX = X \times I / X \times \{0\}$. The cone is a pointed space with basepoint $*$ given by the "cone point", i.e. the image of $X \times \{0\}$. Regard X as the subspace of CX of all points of the form $(x, 1)$.
Define the *suspension* of a space X to be $SX = CX/X$. Make SX a pointed space by declaring the image of $X \subset CX$ to be the basepoint in SX. The quotient map induces a map of pairs $f : (CX, X) \to (SX, *)$.

(a) Show that CX is contractible.

For any $a, b \in I$ with $a \leq b$, let $C_a^b X$ denote the image of $X \times [a, b]$ in CX. Thus $C_0^1 X = CX$, $C_0^0 X = *$, and $C_1^1 X = X$.

Let $p : CX \to CX$ send (x, t) to $(x, 3t)$ for $t \leq 1/3$ and to $(x, 1)$ if $t \geq 1/3$.

(b) Show that p defines a homotopy equivalence of pairs $(C_0^{2/3} X, C_{1/3}^{2/3} X) \to (CX, X)$.

(c) Show that the evident $e : (C_0^{2/3} X, C_{1/3}^{2/3} X) \to (SX, C_{1/3}^1 X/X)$ is an excision.

(d) Show that p defines a homotopy equivalence of pairs $(SX, C_{1/3}^1 X/X) \to (SX, *)$.

(e) Conclude from the commutativty of

$$\begin{array}{ccc} (C_0^{2/3} X, C_{1/3}^{2/3} X) & \xrightarrow{e} & (SX, C_{1/3}^1 X/X) \\ \downarrow & & \downarrow \\ (CX, X) & \xrightarrow{f} & (SX, *) \end{array}$$

that f induces an isomorphism in homology.

(f) Show that there is a natural isomorphism between augmented and reduced homology groups, $H_{n-1}(X) \to \widetilde{H_n}(SX)$, for any n.

(a) Let $H : (X \times I) \times I \to X \times I$ be the map sending $((x, s), t) \mapsto (x, st)$. Notice that $H((x, 0), t) = (x, 0)$ so we can pass to the quotient (since I is compact Hausdorff) to get a map $\widetilde{H} : CX \times I \to CX$. Then $\widetilde{H}(X, 1) = CX$ and $\widetilde{H}(X, 0) = *$, where $*$ is the cone point. So $\widetilde{H}$ is a homotopy between c_* and id_X and hence CX is contractible.

(b) Let $q : (CX, X) \to (C_0^{2/3} X, C_{1/3}^{2/3} X)$ be the map given by $q(x, s) = (x, s/3)$. Observe that it is well-defined with respect to the quotient. Then $q \circ p : CX \to CX$ is given by

$$(q \circ p)(x, s) = \begin{cases} (x, s) & 0 \leq s \leq \frac{1}{3} \\ (x, 1/3) & \frac{1}{3} < s \leq 1 \end{cases}.$$

Consider the the homotopy $H_{q\circ p} : C_0^{2/3}X \times I \to C_0^{2/3}X$ given by

$$H_{q\circ p}((x,s),t) = \begin{cases} (x,s) & 0 \le s \le \frac{1}{3}(1-t)+t \\ (x, \frac{1}{3}(1-t)+t) & \text{otherwise} \end{cases}$$

Clearly $H_{q\circ p}((x,s),0) = (q\circ p)(x,s)$ and $H_{q\circ p}((x,s),1) = (x,s)$. Conversely, note that $p\circ q : (C_0^{2/3}X, C_{1/3}^{2/3}X) \to (C_0^{2/3}X, C_{1/3}^{2/3}X)$ is just the identity, so we are done.

(c) Let $U_a^b X$ be the image of $X \times (a,b]$ in SX. Then $U_{2/3}^1 X \subset C_{2/3}^1 X/X$ is closed in SX, and $U_{1/3}^1 X \subset C_{1/3}^1 X/X$ is open in SX. Then we have $\overline{U_{2/3}^1} \subset C_{2/3}^1 X/X \subset U_{1/3}^1 X \subset \operatorname{int}(C_{1/3}^1 X/X)$. Then the domain of e can be written as $(SX - U_{2/3}^1, C_{1/3}^1 X/X - U_{2/3}^1)$ so we are done.

(d) Notice that the homotopy from (a) is well defined in the quotient by $X9$, and so passes to a homotopy equivalence $(SX, C_{1/3}^1 X/X) \to (SX, *)$.

(e) By the excision theorem and functorial properties of homology, we can see that the isomorphism

$$H_*(C_0^{2/3}X, C_{1/3}^{2/3}X) \cong H_*(SX, C_{1/3}^1 X/X)$$

gives us an isomorphism $H_*(CX, X) \cong H_*(SX, *)$.

(f) Recall that $H_*(SX, *) \cong \widetilde{H}_*(SX)$. Similarly, we have an exact sequence $H_n(CX) \to H_n(CX, X) \to H_{n-1}(X) \to H_{n-1}(CX)$. Since CX is contractible, this gives us an isomorphism $H_n(CX, X) \to H_{n-1}(X)$. Putting these isomorphisms together, we get

$$H_{n-1}(X) \cong \widetilde{H}_n(SX).$$

Problem 2.

(a) Verify the claim that the map $z \mapsto z^d$, sending the unit circle in the complex numbers to itself, has degree d.

(b) Regard S^{n-1} as the unit sphere in $\mathbb{R}^n$. Let L be a line through the origin in $\mathbb{R}^n$, and $L^\perp$ its orthogonal complement. Let ρ_L be the linear map given by -1 on L and $+1$ on $L^\perp$. What is $\deg(\rho_L\big|_{S^{n-1}})$?

(c) What is the degree of the "antipodal map", $\alpha : S^{n-1} \to S^{n-1}$ sending x to $-x$?

(d) The tangent space to a point x on the sphere S^{n-1} can be regarded as the subspace of $\mathbb{R}^n$ of vectors perpendicular to x. A "vector field" on S^{n-1} is thus a continuous function $v : S^{n-1} \to \mathbb{R}^n$ such that $v(x) \perp x$ for all $x \in S^{n-1}$. Show that if n is odd then every vector field vanishes at some point on the sphere. On the other hand, construct a nowhere vanishing vector field on S^{n-1} for any even n.

(a) A common result from the degree theory of the circle using the fundamental group shows that the degree of a map $f : I/\{0,1\} \to S^1$ is given by the complex integral

$$\text{wind}(f) = \frac{1}{2\pi i}\int_0^1 \frac{f'(t)}{f(t)}\,dt = \frac{1}{2\pi i}\int_0^1 \frac{2\pi i p \cdot e^{2\pi i p t}}{e^{2\pi i p t}}\,dt = p.$$

So the degree of such a map is p.

(b) We claim that $\deg \rho_L = -1$. Let $L = \mathbb{R}v_1$ with v_1 a unit vector, and let $v_2, \ldots, v_n$ be a basis for $L^\perp$ so that $v_1, v_2, \ldots, v_n$ be an orthonormal basis. Then the map ρ_L sends $\alpha_1 v_1 + \cdots + \alpha_n v_n$ to $(-\alpha_1)v_1 + \cdots + \alpha_n v_n$. Let's regard S^{n-1} as the unit sphere in $\mathbb{R}^n$. Let S_+^{n-1} be the upper hemisphere along L and let S_-^{n-1} be the lower hemisphere along L. Let $U_+ \subset S^{n-1}$ be some ϵ-expansion of S_+^{n-1} and U_- be the same but along the bottom hemisphere. Since ρ_L preserves $\mathcal{U} = \{U_+, U_-\}$, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(U_+ \cap U_-) & \longrightarrow & C_*(U_+) \oplus C_*(U_-) & \longrightarrow & C_*^{\mathcal{U}}(S^{n-1}) & \longrightarrow & 0 \\
 & & \big\downarrow \rho_L & & \big\downarrow \rho_L \oplus \rho_L & & \big\downarrow \rho_L & & \\
0 & \longrightarrow & C_*(U_+ \cap U_-) & \longrightarrow & C_*(U_+) \oplus C_*(U_-) & \longrightarrow & C_*^{\mathcal{U}}(S^{n-1}) & \longrightarrow & 0
\end{array}$$

By naturality of the connecting map ∂, this gives us a commutative diagram

$$\begin{array}{ccc}
H_{n-1}(S^{n-1}) & \xrightarrow{\partial} & H_{n-2}(S^{n-2}) \\
\big\downarrow \rho_{L*} & & \big\downarrow \rho_{L*} \\
H_{n-1}(S^{n-1}) & \xrightarrow{\partial} & H_{n-2}(S^{n-2})
\end{array}$$

where ∂ is an isomorphism. It then suffices to show that ρ_L has degree -1 for S^1, the rest will follow inductively. Note that in S^1, it follows that $\rho_L(\zeta) = 1/\zeta$ for a particular choice of L. (It doesn't really matter, since we can always rotate using a degree 1 rotation.) This has degree -1 by (a) so we are done.

(c) Letting $L_1, L_2, \ldots, L_n$ be orthogonal in $\mathbb{R}^n$, then $\alpha = \rho_{L_1} \circ \cdots \circ \rho_{L_n}$. By elementary properties of degrees, we get $\deg \alpha = \deg \rho_{L_1} \cdots \deg \rho_{L_n} = (-1)^n$.

(d) Let n be odd, and suppose for the sake of contradiction that $v : S^{n-1} \to \mathbb{R}^n$ is some nonvanishing vector field on S^{n-1}. Since v is nonvanishing, consider the map $\widetilde{v} : S^{n-1} \to S^{n-1}$ given by $\widetilde{v}(\zeta) = v(\zeta)/\|v(\zeta)\|_1$. Note that this map still preserves the orthogonality condition $\widetilde{v}(x) \perp x$ for all $x \in S^{n-1}$. Now consider the homotopy $H : S^{n-1} \times I \to S^{n-1}$ given by $H(\zeta, t) = (\cos \pi t)\zeta + (\sin \pi t)\widetilde{v}(\zeta)$. This is well defined since ζ and $\widetilde{v}(\zeta)$ are orthogonal and both have norm 1. Then H gives a homotopy between the identity map and α since $H(\zeta, 0) = \zeta$ and $H(\zeta, 1) = -\zeta$. This is a contradiction, since by (c) the degree of α should be -1, while the degree of the identity is 1.

In the even case, we can explicitly construct a nonvanishing vector field. Consider the field

$$v(x_1, \ldots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1}).$$

Then we can check

$$v(x_1, \ldots, x_{2k}) \cdot (x_1, \ldots, x_{2k}) = x_2x_1 - x_1x_2 + \cdots + x_{2k}x_{2k-1} - x_{2k-1}x_{2k} = 0.$$

Problem 3. Let A denote an $n \times n$ matrix with positive entries. Prove that A admits an eigenvalue with positive entries and positive eigenvalue by following the steps below. Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\|x\|_1 = |x_1| + \cdots + |x_n|$ denote the L^1 norm.

(a) Prove that there is a continuous map $\varphi : \Delta^{n-1} \to \Delta^{n-1}$ given by $\varphi(x) = \frac{Ax}{\|Ax\|_1}$.

(b) Apply the Brouwer fixed point theorem to φ to prove that A admits an eigenvalue with positive entries and positive eigenvalue.

(a) Observe that every element $v \in \Delta^{n-1} \subset \mathbb{R}^n$ is nonzero, with each coordinate positive. Since every entry in A is positive, Av is nonzero with each coordinate positive. Thus $\varphi(x)$ is continuous since it is the quotient of a continuous function by a nonzero continuous function. We still need to establish that $\mathrm{Im}(\Delta^{n-1}) \subset \Delta^{n-1}$.

For any $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$ with $\sum_{i=1}^n \alpha_i = 1$, we have

$$\frac{Ax}{\|Ax\|_1} = \frac{\sum_{k=1}^n \alpha_k \sum_{i=1}^n a_{ki} e_i}{\| \sum_{k=1}^n \alpha_k \sum_{i=1}^n a_{ki} e_i \|} = \frac{\sum_{i=1}^n e_i \sum_{k=1}^n \alpha_k a_{ki}}{\| \sum_{i=1}^n e_i \sum_{k=1}^n \alpha_k a_{ki} \|} = \sum_{i=1}^n e_i \frac{\sum_{k=1}^n \alpha_k a_{ki}}{\sum_{i=1}^n |\sum_{k=1}^n \alpha_k a_{ki}|} \in \Delta^{n-1}.$$

(b) Since $\Delta^{n-1} \cong D^{n-1}$, by the Brouwer fixed point theorem, there exists a $v \in \Delta^{n-1}$ such that $\varphi(v) = v$. This means that $Av = \|Av\|_1 v$. Note that $\|Av\|_1$ is a positive eigenvalue and $v \in \Delta^{n-1}$ has all positive entries.

Problem 4. Let $\mathcal{A}$ be a cover of a space X. For any simplex in X, let $k(\sigma)$ be the smallest integer such that $\$^k\sigma$ is $\mathcal{A}$-small. Define a map $T : S_*(X) \to S_*^{\mathcal{A}}(X)$ by sending each simplex σ to $\$^{k(\sigma)}\sigma$. Show that this defines a homotopy inverse of the inclusion map.

This map isn't actually a chain map, because $k(d\sigma)$ need not equal $k(\sigma)$, thus we do not have the equality $dT(\sigma) = T(d\sigma)$. Consider for instance $\mathcal{A}$ consisting of two sets, one of which fully fits into $\mathrm{Im}(\sigma)$. Then $k(d\sigma) = 0$, since the boundary must be fully contained inside in the other set in $\mathcal{A}$. On the other hand, $k(\sigma)$ must be greater than zero since it intersects both sets in $\mathcal{A}$.