Math 132
Problem Set 1
Spring, 2023

This problem set is due on Thursday, Feb. 2nd. Please make your answers as complete and clear as possible. You are encouraged to discuss these problems with others in the class, but your writing should be your own.

1. (This is GP, Problem 4 of Chapter 1, §2). Suppose that $f : X \to Y$ is a diffeomorphism. Prove that at each $x \in X$, the derivative

$$df_x : T_x X \to T_{f(x)} Y$$

is an isomorphism of tangent spaces.

2. This is a thought problem you will return to with more tools in the next problem set. Write elements of $\mathbb{R}^{2n}$ as $n \times 2$ matrices, which you should think of as pairs of column vectors $[v_1, v_2]$. With this in mind, consider the set $V \subset \mathbb{R}^{2n}$ of orthonormal pairs $[v_1, v_2]$. By definition this means the pairs $[v_1, v_2]$ satisfying

$$\begin{aligned} v_1 \cdot v_2 &= 0 \\ v_1 \cdot v_1 &= 1 \\ v_2 \cdot v_2 &= 1. \end{aligned}$$

This turns out to be a smooth manifold. It's not so easy to show this directly, but by next week you will have a tool do to so. The manifold is called a Stiefel manifold. The thought problem is this: can you guess the dimension of this manifold? More generally there is a Stiefel manifold of orthonormal k-tuples

$$[v_1, \ldots, v_k]$$

of vectors $v_i \in \mathbb{R}^n$. It is naturally a subspace of $\mathbb{R}^{nk}$. Can you make a reasonable guess for the dimension of this manifold?

3. This problem involves the definition of smooth functions.

(a) Suppose that $f : M \to N$ is a function between smooth manifolds. Show that f is smooth if and only if for each smooth $g : N \to \mathbb{R}$ the composition $g \circ f$ is smooth.

(b) Suppose that M is a smooth manifold, $\{U_i\}$ is a covering of M by open subsets. Show that a function $f : M \to \mathbb{R}$ is smooth if and only if the restriction of f to each U_i is smooth. (If this seems a little complicated, just do it for a covering $M = U \cup V$ by two open sets.)

(c) The above result is not true if the condition that the U_i be open is dropped. Can you find a counterexample? (HINT: Think about Problem 3 of Problem Set 0.)

4. (GP, §2, Problem 11).

(a) Suppose that $f : X \to Y$ is a smooth map, and let $F : X \to X \times Y$ be $F(x) = (x, f(x))$. Show that

$$dF_x(v) = (v, df_x(v)).$$

(b) Prove that the tangent space to graph of f at the point $(x, f(x))$ is the graph of $df_x : T_x(X) \to T_{f(x)}(Y)$.

5. (GP, §2, Problem 12). A *curve* in a manifold X is a smooth map $t \to c(t)$ of an interval of $\mathbb{R}^1$ into X. The *velocity vector* of the curve c at time t_0 (denoted simply $dc/dt(t_0)$) is defined to be the vector $dc_{t_0}(1) \in T_{x_0(X)}$, where $x_0 = c(t_0)$, and $dc_{t_0} : \mathbb{R}^1 \to T_{x_0}X$. In case $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t_0), \ldots, c_k'(t_0)).$$

Prove that every vector in $T_x(X)$ is the velocity vector of some curve in X, and conversely.