Lecture 3: Introduction to Lie groups

Contents

As we said in (**1.34**), we organize geometric structures (after Klein) in terms of symmetry. In this lecture we introduce symmetry groups in differential geometry: Lie groups. A group is a structure on a set, and discrete groups play an important role in all of mathematics. In geometric situations one often combines the group structure with a geometric structure: topological groups, algebraic groups, group schemes, ...Here we combine with a smooth manifold structure on the underlying set to obtain the notion of a Lie group.

The smooth manifold that underlies a Lie group carries two global parallelisms, and we choose the *left* parallelism as primary. We begin the lecture with some general remarks on manifolds equipped with a global parallelism; see (**1.19**).

We then turn to the definition of a Lie group. The parallel (left invariant) vector fields are closed under Lie bracket and comprise the Lie algebra associated to a Lie group. We give several examples, including the "exceptional" Lie group G_2. (There is a classification of compact simply connected Lie groups that includes three infinite families and five exceptional groups; G_2 is one of the latter.)

The notion of a homomorphism of Lie groups (Definition 3.39) is evident, and by differentiation a homomorphism of Lie groups induces a homomorphism of their Lie algebras. Remark 3.42 on the converse sets up our discussion of distributions and the Frobenius theorem, which we take up in a few lectures. One-parameter groups are a special and important class of homomorphisms. They correspond to elements of the Lie algebra, and often are a useful way to work with such elements. This leads to the exponential map from the Lie algebra to the Lie group.

We conclude the lecture by working out one parameter groups and the exponential map for matrix groups.

In the next lecture we continue our introduction to Lie groups.

Parallelized manifolds

Let X be a smooth manifold equipped with a global parallelism (Definition 1.20). The structure (1.21) of the parallelism identifies each tangent space T_xX with a fixed vector space V. This provides an isomorphism

$$\mathfrak{X}(X) \xrightarrow{\cong} \mathrm{Map}(X, V). \tag{3.1}$$

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A vector field $\xi \in \mathfrak{X}(X)$ is *parallel* if it corresponds to a constant function under (3.1). Observe that a vector $\xi_x \in T_xX$ at any $x \in X$ extends to a unique parallel vector field $\xi \in \mathfrak{X}(X)$. Also, given ξ_x there is a distinguished motion germ with initial position x and initial velocity ξ_x defined by an integral curve of the parallel vector field ξ with initial position x. This is a *constant velocity motion* (germ).

Remark 3.2. On a smooth manifold equipped with a global parallelism, the Lie bracket of parallel vector fields is not necessarily parallel. However, for the special left and right parallelisms we define below on a Lie group, the Lie bracket of parallel vector fields is parallel.

The parallelism also induces an isomorphism

$$\Omega^\bullet(X) \xrightarrow{\cong} \mathrm{Map}(X, {\textstyle\bigwedge}^\bullet V^*), \tag{3.3}$$

and the parallel differential forms form a subalgebra

$$\Omega^\bullet_{\parallel}(X) \subset \Omega^\bullet(X). \tag{3.4}$$

As in Remark 3.2, the subspace of parallel differential forms is not necessarily closed under d.

Basic concepts of Lie groups

(3.5) *The definition.* A mathematical marriage imposes compatibility constraints, here smoothness of structure maps.

Definition 3.6. A *Lie group* is a set G endowed with the structures of a group and of a smooth manifold. The multiplication and inversion maps

$$\begin{aligned} m\colon G \times G &\longrightarrow G \\ i\colon G &\longrightarrow G \end{aligned} \tag{3.7}$$

are required to be smooth.

Observe that a Lie group has a distinguished point, the identity element $e \in G$. Therefore, it has a distinguished component, the *identity component* that contains e. Recall too Definition 2.33 of a Lie algebra. We will soon see that attached to a Lie group G is a Lie algebra $\mathrm{Lie}(G)$.

A Lie group can act as symmetries on a smooth manifold.

Definition 3.8. Let G be a Lie group and let X be a smooth manifold. A (left/right) *action* of G on X is a smooth map

$$\begin{aligned} L\colon G \times X &\longrightarrow X \\ R\colon X \times G &\longrightarrow X \end{aligned} \tag{3.9}$$

that satisfies the associativity law required of a group action: for $g_1, g_2 \in G$ and $x \in X$, we have

$$\begin{aligned} L\big(g_1, L(g_2, x)\big) &= L\big(g_1 g_2, x\big) \\ R\big(R(x, g_1), g_2\big) &= R\big(x, g_1 g_2\big) \end{aligned} \tag{3.10}$$

for a left action L and a right action R, respectively.

As usual, we write $L(g, x) = g \cdot x = gx$ and $R(x, g) = x \cdot g = xg$.

(3.11) *Parallelisms on a Lie group.* A Lie group G carries canonical left and right actions on itself by left and right multiplication: for $g, h \in G$ and $x \in G$ define

$$\begin{aligned} L_g(x) &= gx \\ R_h(x) &= xh \end{aligned} \tag{3.12}$$

The maps $L_g, R_h \colon G \to G$ are diffeomorphisms. Left multiplication and right multiplication commute, so together define a left action of $G \times G$ on G: the pair $(g, h) \in G \times G$ maps $x \mapsto gxh^{-1}$. The differentials of left and right multiplication each define a global parallelism that identifies each tangent space with T_eG. The *left parallelism* is

$$\begin{aligned} L_g \colon T_eG &\longrightarrow T_gG \\ \xi &\longmapsto (L_g)_*\xi \end{aligned} \tag{3.13}$$

and the *right parallelism* is

$$\begin{aligned} R_h \colon T_eG &\longrightarrow T_hG \\ \xi &\longmapsto (R_h)_*\xi \end{aligned} \tag{3.14}$$

Convention 3.15. We use 'parallel' to mean the *left* parallelism of a Lie group.

Remark 3.16. A parallel vector field ξ is also called 'left-invariant': it is invariant under left translation (3.13). Namely, ξ is parallel iff $(L_g)_*\xi = \xi$ for all $g \in G$. In other words, ξ is L_g-related to itself for all $g \in G$. Similarly, a differential form α is parallel iff $L_g^*\alpha = \alpha$ for all $g \in G$.

Notation 3.17. The subspace of parallel vector fields is denoted $\mathfrak{g} \subset \mathfrak{X}(G)$.

Evaluation at the identity is an isomorphism of vector spaces

$$\mathrm{ev}_e \colon \mathfrak{g} \longrightarrow T_eG \tag{3.18}$$

(3.19) *The Lie algebra of a Lie group.* The space $\mathfrak{g}$ of parallel vector fields has a Lie algebra structure induced from the Lie bracket of vector fields.

Lemma 3.20. *If $\xi_1, \xi_2 \in \mathfrak{g}$ are parallel vector fields, then so too is their Lie bracket $[\xi_1, \xi_2]$.*

Proof. By Remark 3.16 each ξ_i is L_g-related to itself for all $g \in G$, hence by Proposition 2.66 this also holds for $[\xi_1, \xi_2]$. Therefore, $[\xi_1, \xi_2]$ is parallel. $\square$

We use the isomorphism (3.18) to transport the Lie algebra structure on $\mathfrak{g}$, defined by Lie bracket of vector fields, to the vector space T_eG. This is often convenient. For example, if $G = \mathrm{O}_n$ is the group of orthogonal $n \times n$ real matrices, then $T_e\mathrm{O}_n$ is the vector space of skew-symmetric $n \times n$ real matrices. We compute the induced Lie bracket on matrices below in Proposition 3.70.

Examples of Lie groups

Example 3.21 (discrete Lie groups)**.** A finite or countable group is a 0-dimensional Lie group when equipped with the discrete topology. (An uncountable discrete set is not a manifold: it violates second countability.) The associated Lie algebra is the zero vector space equipped with the zero bracket.

Example 3.22 (real vector spaces)**.** A finite dimensional real vector space V is a Lie group. The group law is vector addition and the manifold structure is as usual. A vector space is an *abelian* Lie group; the group law is commutative.

Example 3.23 (circle group)**.** Let $\mathbb{T} \subset \mathbb{C}$ denote the set of complex numbers λ of unit norm: $\lambda\bar{\lambda} = 1$. Then $\mathbb{T}$ is a 1-dimensional Lie group; the group law is multiplication of complex numbers. Regarding $\mathbb{T} \subset \mathbb{C}$, it is natural to identify $\mathrm{Lie}(\mathbb{T}) \cong T_1\mathbb{T} \cong \sqrt{-1}\,\mathbb{R} \subset \mathbb{C}$.

Example 3.24 (general linear group)**.** Let V be a finite dimensional real vector space. Then $\mathrm{End}(V)$ is the vector space of linear maps $A\colon V \to V$, and $\mathrm{Aut}(V) \subset \mathrm{End}(V)$ is the open subset of invertible linear maps $P\colon V \to V$. The group $\mathrm{Aut}(V)$ is a Lie group whose group law is composition of automorphisms. Its dimension is $(\dim V)^2$. The tangent space at the identity is canonically $T_{\mathrm{id}}\,\mathrm{Aut}(V) = \mathrm{End}(V)$. Note that $\mathrm{End}(V)$ is an associative algebra under composition, a structure we exploit. Quite generally, it is pleasant to compute in Lie groups that are subgroups of associative algebras.

Observe that $\mathrm{Aut}(V)$ has two parallelisms. First, the affine parallelism induced from $\mathrm{End}(V)$ identifies each tangent space with $\mathrm{End}(V)$. Second, as a Lie group $\mathrm{Aut}(V)$ has the left parallelism (3.13). (Of course, it has a third parallelism—right parallelism—but we do not use that.) These two parallelisms do not agree. Namely, if we use the affine parallelism to identify $T_P\,\mathrm{Aut}(V) \cong \mathrm{End}(V)$ for $P \in \mathrm{Aut}(V)$, then the left parallelism is left composition by P:

$$\begin{aligned} (L_P)_*\colon T_{\mathrm{id}}\,\mathrm{Aut}(V) \cong \mathrm{End}(V) &\longrightarrow \mathrm{End}(V) \cong T_P(V) \\ A \;&\longmapsto\; P \circ A \end{aligned} \tag{3.25}$$

We compute the Lie bracket on $T_e\,\mathrm{Aut}(V) \cong \mathrm{End}(V)$ in **(3.68)** below.

If $V=\mathbb{R}^n$, then $\mathrm{End}(\mathbb{R}^n)=M_n\mathbb{R}$ is the algebra of $n\times n$ real matrices under matrix multiplication, and $\mathrm{Aut}(\mathbb{R}^n)=\mathrm{GL}_n\mathbb{R}$ is the subgroup of invertible matrices.

Definition 3.26. A Lie group G is a *matrix group* if it is a Lie subgroup $G\subset \mathrm{Aut}(V)$ of a general linear group.

As we spell out in the next lecture, this means that G has a Lie group structure and the inclusion map into the general linear group is a homomorphism of Lie groups (defined below).

Example 3.27 (matrix groups cut out by a bilinear form)**.** Let V be a finite dimensional real vector space and $B\colon V\times V\to\mathbb{R}$ a bilinear form. Define

$$\mathrm{Aut}_B(V)=\{P\in\mathrm{Aut}(V): B(P\xi_1,P\xi_2)=B(\xi_1,\xi_2)\text{ for all }\xi_1,\xi_2\in V\};\tag{3.28}$$

it is a Lie subgroup of $\mathrm{Aut}(V)$. The tangent space at $\mathrm{id}\in\mathrm{Aut}_B(V)$ is computed by differentiating the equation in (3.28) in the variable P:

$$\mathrm{End}_B(V)=\{A\in\mathrm{End}(V): B(A\xi_1,\xi_2)+B(\xi_1,A\xi_2)=0\text{ for all }\xi_1,\xi_2\in V\}.\tag{3.29}$$

For each pair $\xi_1,\xi_2\in V$ the equation in (3.28) cuts out a closed set in $\mathrm{Aut}(V)$, and hence $\mathrm{Aut}_B(V)$ is an intersection of closed sets so is a closed subgroup of $\mathrm{Aut}(V)$. This alone is sufficient to prove that $\mathrm{Aut}_B(V)$ is a submanifold of $\mathrm{Aut}(V)$ and that the induced manifold structure is compatible with the multiplication. We explain the indicated theorem in the next lecture. If B is nondegenerate, then for any $A\in\mathrm{End}(V)$ define the *adjoint* operator $A^*\in\mathrm{End}(V)$ via the equation

$$B(\xi_1,A\xi_2)=B(A^*\xi_1,\xi_2),\qquad \xi_1,\xi_2\in V.\tag{3.30}$$

Then $P\in\mathrm{Aut}(V)$ is in $\mathrm{Aut}_B(V)$ iff $P^*P=\mathrm{id}$. One can prove that this equation defines a submanifold, and from there that $\mathrm{Aut}_B(V)$ is a Lie group.

For $V=\mathbb{R}^n$ this specializes to classical Lie subgroups of $\mathrm{GL}_n\mathbb{R}$. Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{R}^n$.

Example 3.31 (orthogonal groups)**.** Let $p,q\in\{0,1,\dots,n\}$ satisfy $p+q=n$. Define the bilinear form $B_{p,q}\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$B_{p,q}(e_k,e_\ell)=\begin{cases}+1, & k=\ell\le p;\\ -1, & k=\ell>p;\\ 0, & k\ne\ell.\end{cases}\tag{3.32}$$

Then B is symmetric and nondegenerate. The *orthogonal group* $\mathrm{Aut}_{B_{p,q}}(\mathbb{R}^n)$ is notated $\mathrm{O}_{p,q}$, and in the definite case we write $\mathrm{O}_{n,0}=\mathrm{O}_{0,n}=\mathrm{O}_n$.

An orthogonal matrix P satisfies $\det P=\pm1$. The orthogonal group is not connected; write $\mathrm{SO}_{p,q}\subset\mathrm{O}_{p,q}$ for the *identity component*, that is, the connected component that contains the identity element. In the definite case, write $\mathrm{SO}_n\subset\mathrm{O}_n$. This subgroup is called the *special orthogonal group*.

Your homework treats nondegenerate skew-symmetric forms.

Example 3.33 (Lie groups from complex vector spaces)**.** A finite dimensional complex vector space U has an underlying finite dimensional real vector space, so is a smooth manifold. Hence, so too is $\mathrm{Aut}(U) \subset \mathrm{End}(U)$, where now $\mathrm{End}(U)$ is the associative algebra of *complex linear* endomorphisms of U. For $U = \mathbb{C}^m$ we write $\mathrm{Aut}(U) = \mathrm{GL}_m\mathbb{C}$; it is the *complex general linear group* of (real) dimension $2m^2$. By contrast with the real case in Example 3.31, there is only a single isomorphism class of nondegenerate symmetric complex bilinear forms on $\mathbb{C}^m$; the corresponding group of symmetries is the *complex orthogonal group* $\mathrm{O}_m\mathbb{C}$. There is a new possibility: *hermitian forms*. Recall that a hermitian form on a complex vector space is a bilinear map[1]

$$h\colon \overline{U} \times U \longrightarrow \mathbb{C}, \tag{3.34}$$

which is equivalent to a bilinear map $U_{\mathbb{R}} \times U_{\mathbb{R}} \to \mathbb{C}$ of the underlying real vector spaces that is complex conjugate linear in the first variable and complex linear in the second variable. Symmetry or skew-symmetry is the equation

$$h(\bar{\xi}_1, \xi_2) = \pm\overline{h(\bar{\xi}_2, \xi_1)}. \tag{3.35}$$

A symmetric hermitian form has a signature, just as in Example 3.31, leading for $U = \mathbb{C}^m$ to the *unitary groups* $\mathrm{U}_{p,q}$, $p + q = m$. In the definite case we write $\mathrm{U}_{m,0} = \mathrm{U}_{0,m} = \mathrm{U}_n$

The next example is not defined by a bilinear form, but rather by a trilinear form.

Example 3.36 (An exceptional Lie group)**.** Let $V = \mathbb{R}^7$ with its standard inner product and standard (orthonormal) basis $e_1, \dots, e_7$. Denote the dual basis of V^* by $e^1, \dots, e^7$, and use the notation $e^{ijk} = e^i \wedge e^j \wedge e^k \in \bigwedge^3 V^*$. Introduce the 3-form $\omega \in \bigwedge^3 V^*$ defined as

$$\omega = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{3.37}$$

The exceptional (compact) Lie group G_2 is

$$\mathrm{G}_2 = \{P \in \mathrm{SO}_7 : P^*\omega = \omega\}. \tag{3.38}$$

It has dimension 14.

Homomorphisms of Lie groups

The next definition is in keeping with the compatible marriage in Definition 3.6.

Definition 3.39. Let G', G be Lie groups. A function $\psi\colon G' \to G$ is a (*Lie group*) *homomorphism* if it is a smooth map of manifolds and a homomorphism of groups.

[1] $\overline{U}$ is the complex conjugate vector space to U: the same real vector space with the complex conjugate action of the complex scalars $\mathbb{C}$.

Lemma 3.40. *A Lie group homomorphism $\psi\colon G' \to G$ induces a Lie algebra homomorphism $\dot\psi\colon \mathfrak{g}' \to \mathfrak{g}$ on parallel vector fields.*

Proof. Let $\xi' \in \mathfrak{g}'$ be a parallel vector field. Then for $g' \in G'$ we have

$$\begin{aligned} \psi_*(\xi'_{g'}) &= \psi_*(L_{g'})_*\xi'_{e'} \\ &= (\psi \circ L_{g'})_*\xi'_{e'} \\ &= (L_{\psi(g')} \circ \psi)_*\xi'_{e'} \\ &= (L_{\psi(g')})_*\xi_e \end{aligned} \tag{3.41}$$

where $\xi_e = \psi_*\xi'_{e'}$. It follows that ξ' pushes forward to a vector field ξ on the image of ψ, and that ξ is parallel on that image. Extend ξ to a parallel vector field on G, i.e., to an element of $\mathfrak{g}$. Proposition 2.66 implies that the map $\dot\psi\colon \mathfrak{g}' \to \mathfrak{g}$ is a Lie algebra homomorphism. Finally, observe that under (3.18) the map $\dot\psi$ is identified with the differential of ψ at the identity $e' \in G'$. □

Remark 3.42. Lemma 3.40 suggests the following question: Given a Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathfrak{g}$, does there exist a Lie group homomorphism $\psi\colon G' \to G$ such that $\dot\psi = \beta$? We address this question by seeking the graph of ψ, which is a submanifold $\Gamma \subset G' \times G$. The equation $\dot\psi = \beta$ is encoded by a *distribution* on $G' \times G$, that is, a subbundle of the tangent bundle such that the restriction of this subbundle to Γ is the tangent bundle to Γ. Specifically, identify

$$T_{(g',g)}(G' \times G) \cong \mathfrak{g}' \oplus \mathfrak{g} \tag{3.43}$$

and then the graph of $\dot\psi$ determines a subspace of (3.43). The local and global existence and uniqueness of Γ—called an *integral manifold* of the distribution—is the subject of the Frobenius theorem, which we take up in Lecture 5.

(3.44) *One-parameter groups.* Specialize to $G' = \mathbb{R}$. Then the distribution described in Remark 3.42 is 1-dimensional, and the graph we seek is an integral curve of a vector field.

Definition 3.45. A *one-parameter group* in a Lie group G is a homomorphism $\gamma\colon \mathbb{R} \to G$. We say γ is *generated* by $\dot\gamma(0) = \xi \in \mathfrak{g}$.

The following is an affirmative answer to the question in Remark 3.42 for $G' = \mathbb{R}$.

Theorem 3.46. *Let $\xi \in \mathfrak{g}$ be a parallel vector field on a Lie group G. Then ξ is complete. Furthermore, the integral curve $\gamma_\xi\colon \mathbb{R} \to G$ of ξ with $\gamma_\xi(0) = e$ is a one-parameter group with generator ξ.*

A one-parameter group is a constant velocity motion: its velocity is parallel.

Remark 3.47. Note that Definition 3.45 and Theorem 3.46 combine to a bijection between the Lie algebra $\mathfrak{g}$ and one-parameter groups in G. Often these motions in G provide a geometric way to work with an element of the Lie algebra of a Lie group, an idea we use often. In particular, a one-parameter group and its translates provide distinguished motions that represent a tangent vector on a Lie group.

Proof. Let $\gamma\colon(-\epsilon,\epsilon)\to G$ be a local integral curve of ξ with $\gamma(0)=e$. Suppose $t_1,t_2\in(-\epsilon,\epsilon)$ satisfy $t_1+t_2\in(-\epsilon,\epsilon)$. Then we claim

$$\gamma(t_1+t_2)=\gamma(t_1)\gamma(t_2). \tag{3.48}$$

To verify this, fix t_1 and let γ_L,γ_R denote the left and right hand sides of (3.48) as functions of $t=t_2$. Then

$$\dot\gamma_L(t)=\dot\gamma(t_1+t)=\xi_{\gamma(t_1+t)}=\xi_{\gamma_L(t)} \tag{3.49}$$

and

$$\dot\gamma_R(t)=\gamma(t_1)\dot\gamma(t)=(L_{\gamma(t_1)})_*\xi_{\gamma(t)}=\xi_{\gamma(t_1)\gamma(t)}=\xi_{\gamma_R(t)}. \tag{3.50}$$

Furthermore, the initial positions are

$$\gamma_L(0)=\gamma(t_1)=\gamma_R(0). \tag{3.51}$$

Now (3.48) follows from the uniqueness statement in the fundamental theorem of ordinary differential equations.

Next, we prove that $\gamma\colon(-\epsilon,\epsilon)\to G$ extends to an integral curve γ_ξ of ξ with domain $\mathbb{R}$. For any $t\in\mathbb{R}$ choose $N\in\mathbb{Z}^{>0}$ such that $|t/N|<\epsilon$, and define

$$\gamma_\xi(t)=\gamma\left(\frac{t}{N}\right)^N. \tag{3.52}$$

If also $M\in\mathbb{Z}^{>0}$ such that $|t/M|<\epsilon$, then using (3.48) we have

$$\gamma\left(\frac{t}{M}\right)^M=\left[\gamma\left(\frac{t}{NM}\right)^N\right]^M=\left[\gamma\left(\frac{t}{NM}\right)^M\right]^N=\gamma\left(\frac{t}{N}\right)^N. \tag{3.53}$$

This proves that (3.52) is well-defined. It follows from (3.48) that γ_ξ is a homomorphism.

Finally, for any $g\in G$ the map

$$\begin{aligned}\mathbb{R}&\longrightarrow G\\ t&\longmapsto g\,\gamma_\xi(t)\end{aligned} \tag{3.54}$$

is an integral curve of ξ with initial position g. This proves that ξ is a complete vector field. $\square$

Remark 3.55. Notice that (3.54) implies that the flow generated by a parallel (*left*-invariant) vector field is *right* multiplication by the integral curve through $e\in G$. Quite generally, signs and left vs. right are tricky in differential geometry, so it is worth paying attention.

(3.56) *The exponential map.* The integral curves in Theorem 3.46 fit together into a map from the Lie algebra $\mathfrak{g}$ of a Lie group G to the Lie group. Namely, define the *exponential map*

$$\begin{aligned} \exp\colon \mathfrak{g} &\longrightarrow G \\ \xi &\longmapsto \gamma_\xi(1) \end{aligned} \tag{3.57}$$

Proposition 3.58. *The exponential map* (3.57) *is smooth.*

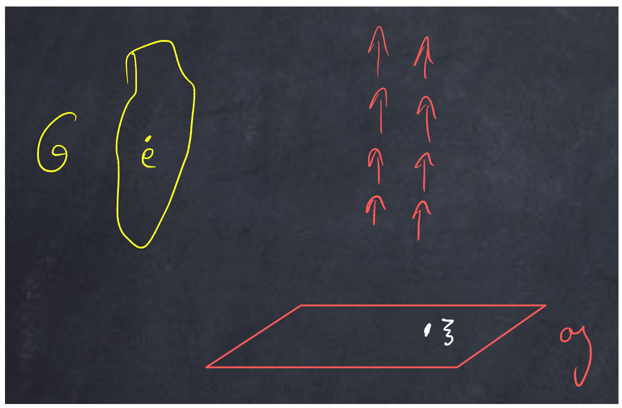

FIGURE 9. The universal parallel vector field on a Lie group

Proof. On $\mathfrak{g} \times G$ define the vector field $\eta_{\xi,g} = 0 \oplus \xi$; see Figure 9. It generates the flow

$$\varphi_t(\xi, g) = \bigl(\xi, g\exp(t\xi)\bigr). \tag{3.59}$$

The fundamental theorem of ODE asserts that φ is a smooth function jointly in t, ξ, g. Hence so too is $\exp\colon \xi \mapsto \pi\bigl(\varphi_1(\xi, e)\bigr)$, where $\pi\colon \mathfrak{g} \times G \to G$ is projection. $\square$

Remark 3.60. The device of passing to the universal example—in this case of a parallel vector field—by introducing a parameter space—in this case $\mathfrak{g}$—is a powerful maneuver in many situations.

Exponentials and Lie brackets for the general linear group

(3.61) *The exponential.* As in Example 3.24, let V be a finite dimensional real vector space. Suppose $A \in \operatorname{End} V$. Define its exponential by a power series:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{3.62}$$

A finite truncation is defined algebraically. To define the infinite sum, introduce[2] a norm on V and prove that the sequence of partial sums is Cauchy. By easy manipulations with the partial sums, and then by passage to the limit, you will prove that for $t, t_1, t_2 \in \mathbb{R}$,

$$e^{(t_1+t_2)A} = e^{t_1A} \circ e^{t_2A} \tag{3.63}$$

$$\frac{d}{dt}e^{tA} = A \circ e^{tA} = e^{tA} \circ A. \tag{3.64}$$

The first equation implies that e^A is invertible with inverse e^{-A}. For endomorphisms $A, B \in \operatorname{End}(V)$, if A and B commute then $e^{A+B} = e^A \circ e^B$; if they do not commute, then in general $e^{A+B} \neq e^A \circ e^B$.

In the sequel we drop the '$\circ$' symbol.

(3.65) *The exponential map.* Recall from (3.25) that the parallel vector field ξ_A on the general linear group $\operatorname{Aut}(V)$ with value $A \in \operatorname{End}(V) \cong T_{\mathrm{id}} \operatorname{Aut}(V)$ has value $PA \in \operatorname{End}(V)$ at $P \in \operatorname{Aut}(V)$. By (3.64) the integral curve of ξ_A with initial value $P_0 \in \operatorname{Aut}(A)$ is

$$P_t = P_0e^{tA}. \tag{3.66}$$

In particular, the integral curve with initial value id is $t \mapsto e^{tA}$. This implies the following.

Proposition 3.67. *For the general linear group* $\operatorname{Aut}(V)$, *the exponential map* (3.57) *is the exponential map* (3.62).

(3.68) *The Lie bracket.* Let $A, B \in \operatorname{End}(V)$ generate parallel vector fields ξ_A, ξ_B, and let φ be the flow generated by ξ_A. Then $\mathcal{L}_{\xi_A}\xi_B$ at id is

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} (\varphi_{-t})_*\xi_B(\mathrm{id}) &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} e^{tA}e^{sB}e^{-tA} \\ &= \frac{d}{dt}\bigg|_{t=0} e^{tA}Be^{-tA} \\ &= AB - BA. \end{aligned} \tag{3.69}$$

In the first line we use that the flow generated by ξ_A is right multiplication by e^{tA}; see (3.66). This computation implies the following.

Proposition 3.70. *For the general linear group* $\operatorname{Aut}(V)$, *the Lie bracket on the Lie algebra* $\operatorname{End}(V)$ *is the commutator* $[A, B] = AB - BA$.

Observe that we use (3.18) to identify the Lie algebra with $\operatorname{End}(V)$, and we use the associative algebra structure of $\operatorname{End}(V)$ to define the commutator.

Remark 3.71. Any associative algebra determines a Lie algebra whose Lie bracket is the commutator.

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[2]All norms are equivalent, so any works for this argument. You can find details of the arguments in this paragraph in the notes on Multivariable Analysis, and you should work through this paragraph carefully if you have never seen it before. This is an exhortation that applies universally in these notes.