Math 230a Problem Set 2

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Due: September 18, 2024

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Problem 4. For each of the following Lie groups G, answer the questions: If G abelian? Is G compact? Is G connected? Is G simply connected? What is the Lie algebra of G?

A finite dimensional real vector space V.

This is an abelian, non-compact, connected, simply connected Lie group. Since it's abelian, it's Lie bracket vanishes and the Lie algebra can be canonically identified with the trivial Lie algebra V itself.

$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

This is an abelian, compact, connected, not simply connected Lie group. Since it is abelian, it's Lie bracket vanishes and the Lie algebra is the trivial Lie algebra $\mathbb{R}$.

SU_2.

This is a non-abelian, compact, connected, simply connected Lie group. This is evident if we view SU_2 as

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1 \right\},$$

which gives us the homeomorphism $\mathrm{SU}_2 \to S^3 \subset \mathbb{C}^2$ which sends a matrix to (α, β). We know the Lie algebra $\mathfrak{su}_2$ must be 3-dimensional, and indeed we have generators of SU_2 given by the matrices

$$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Equipped with the commutator relations $[X, Y] = 2Z$, $[Y, Z] = 2X$, and $[Z, X] = Y$, these give us generators for $\mathfrak{su}_2$ as a Lie algebra.

SL_2.

This is a non-abelian, non-compact, connected, and not simply connected Lie group. Its Lie algebra is the space of traceless 2×2 matrices, which is three dimensional, with basis E, F, H and commutator relations

$$[E, F] = H, \quad [H, F] = -2F, \quad \text{and} \quad [H, E] = 2E.$$

GL_n.

For $n = 1$, we know that $\mathrm{GL}_1 \cong \mathbb{R}^\times$, so it is abelian, non-compact, non-connected, and simply connected. More generally, when $n \geq 2$, GL_n is non-abelian, non-compact, non-connected, and not simply connected Lie group. In any case GL_n is not compact or connected because it's image under det is $\mathbb{R}^\times$, which is not connected or compact. It's not simply connected because there is a deformation retraction of GL_n^+ onto SO_n, and SO_n fits into the fiber bundle

$$\mathrm{SO}_{n-1} \longrightarrow \mathrm{SO}_n \longrightarrow S^{n-1}.$$

The long exact sequence on homotopy shows that $\pi_1(\mathrm{SO}_n) \cong \pi_1(\mathrm{SO}_3)$ when $n \geq 3$, but $\mathrm{SO}_3 \cong \mathbb{RP}^3$, which is not simply connected. In the case that $n = 2$, we know that $\mathrm{SO}_2 \simeq S^1$. In either case, SO_n is not simply connected and so neither is GL_n^+, which shows that GL_n isn't simply connected either.

We showed in class that the Lie algebra of GL_n is $\mathfrak{gl}_n$ – the space of $n \times n$ matrices with Lie bracket given by the commutator of matrices.

O_n.

For $n = 1$, we know that $O_1 \cong \mathbb{Z}_2$, so it is abelian, compact, non-connected, and simply connected. More generally, when $n \geq 2$, O_n is non-abelian, compact, non-connected, and not simply connected. The simple connectedness follows the same argument as for GL_n, only now we don't have to perform a deformation retract since $\mathrm{O}_n^+ = \mathrm{SO}_n$ directly.

Under the identification of $\mathfrak{gl}_n$ with $M_n(\mathbb{R})$, Lie algebra $\mathfrak{so}_n$ is the space of $n \times n$ skew-symmetric matrices with Lie bracket given by the commutator of matrices and determinant 1.

Problem 5. Let G be a Lie group.

(a). Let V be a finite dimensional real vector space. Define a real line $|\operatorname{Det} V|$ such that an ordered n-tuple $\xi_1, \dots, \xi_n \in V$ defines an element $|\xi_1 \wedge \cdots \wedge \xi_n| \in |\operatorname{Det} V|$ which transforms by the absolute value of the determinant of a change of basis matrix. Identify $|\operatorname{Det} V^*|$ as a certain space of functions $V^n \to \mathbb{R}$. Show that positive functions determine an orientation of $|\operatorname{Det} V^*|$. Interpret a positive function as a notion of volume for n-dimensional parallelepipeds in V. Does this induce a notion of volume for lower dimensional parallelipipeds? Identify positive elements as translationally invariant positive measures on V. Construct such a positive element from an inner product on V.

Let's first define the real line $\operatorname{Det} V = \bigwedge^n V$. Any linearly independent tuple $\{v_i\}$ gives an identification of this space with $\mathbb{R}$, since $v_1 \wedge \cdots \wedge v_n$ spans $\bigwedge^n V$. We can then quotient by the antipodal relation to get a half-line, and gluing together two copies of these half lines by zero give us $|\operatorname{Det} V|$, i.e.

$$|\operatorname{Det} V| = \operatorname{Det}^+ V \cup \operatorname{Det}^- V / 0_+ \sim 0_-.$$

The orientation on this line is given by $\operatorname{Det}^+ V$ – this is an "arbitrary" part of our construction. Clearly, any tuple of elements $\{v_i\}$ can be sent to a non-negative element $|v_1 \wedge \cdots \wedge v_n|$ in the positive half line.

Now if we want to work with $|\operatorname{Det} V^*|$, recall that $\bigwedge^n V^*$ can be canonically identified with the space of alternating bilinear forms $\mathrm{Alt}_n(V) \subset \mathrm{Fun}(V^n, \mathbb{R})$. We can then identify $\operatorname{Det}^+ V^*$ with the image of $\mathrm{Alt}_n(V)$ after post-composition with the absolute value map, and $\operatorname{Det}^- V^*$ after post-composition with the negative absolute value map. This gives us another way to write $|\operatorname{Det} V^*|$, namely:

$$|\operatorname{Det} V^*| = \left\{ \omega \in (V^n)^* : \omega(b \cdot g) = \frac{\epsilon(b)}{|\det(g)|} \quad \text{for all} \quad b \in V^n, g \in \mathrm{End}(V) \right\}.$$

The volume of a parallelipiped spanned by $v_1, \ldots, v_n$ can then be measured by a positive element of $\omega \in |\operatorname{Det} V^*|$ as $\omega(v_1, \ldots, v_n)$. For degenerate parallelipipeds, it's clear that the volume under ω is 0, so the measure induced on lower dimensional parallelipipeds is trivial. Any parallelipiped in V can then be translated back to the origin in a concistent manner, and it's volume taken. Extending in a standard measure theoretic way, we obtain a translationally invariant measure over V corresponding to a positive element $\omega \in |\operatorname{Det} V^*|$.

(b). Apply to the tangent bundle of a smooth manifold. Define the notion of a smooth positive measure on a smooth manifold. Do they always exist?

On a smooth manifold M, we can construct a line bundle $|\operatorname{Det} TM^*|$. A positive section of this line bundle is a smooth positive measure (or in most texts, a density.) They always exist because the determinant line bundle is oriented, and all oriented line bundles are trivial.

(c). The real line $|\operatorname{Det} \mathfrak{g}^*|$ consists of left-invariant measures on G. Define an action of G on this line. Compute the action in case G is compact. Compute it for $G = \mathrm{GL}_n$ and $G = \mathrm{SL}_n$.

(d). A *Haar measure* on G is a bi-invariant positive smooth measure on G. Prove that a Haar measure exists if G is compact. Normalize it so the total volume of G is 1.

(e). Write a formula for the Haar measure on the circle group $\mathbb{T} \subset \mathbb{C}$; the formula should be in terms of $\lambda \in \mathbb{T}$. What about on the multiplicative group $\mathbb{R}^\times$. What about on the additive group $\mathbb{R}$? What about on the orthogonal group O_2?

Problem 6. Suppose G is a connected compact Lie group.

(a). Let $\Omega^\bullet_{\mathrm{linv}}(G) \subset \Omega^\bullet(G)$ denote the vector subspace of left-invariant differential forms. Show that $\Omega^\bullet_{\mathrm{linv}}(G)$ is in fact a sub-differential graded algebra, i.e. it is closed under multiplication and the differential d.

It's clear that the wedge product of two left-invariant differential forms is left-invariant since g acts as

$$L_g^*(\omega \wedge \eta) = (L_g^*\omega) \wedge (L_g^*\eta).$$

Similarly, a left-invariant form is still left-invariant after an exterior derivative since

$$L_g^*(d\omega) = d(L_g^*\omega).$$

(b). Construct an isomorphism

$$\textstyle\bigwedge^\bullet \mathfrak{g}^* \longrightarrow \Omega^\bullet_{\mathrm{linv}}(G).$$

Transfer the differential on $\Omega^\bullet_{\mathrm{linv}}(G)$ to $\bigwedge^\bullet \mathfrak{g}^*$ and write a formula for it. In this way you obtain a differential graded complex defined directly from the Lie algebra $\mathfrak{g}$. Observe that this definition of *any* Lie algebra.

Firstly, recall that there is a natural "extension by left-translation" injective map $\mathfrak{g}^* \to \mathfrak{X}^*(G)$ where $\mathfrak{X}^*(G) = \Gamma(T^*G)$ is the space of covector fields. More precisely, given some $\omega \in \mathfrak{g}^*$, there is a corresponding covector field L_ω on G is defined by:

$$L_\omega(\xi) = \omega \circ L^*_{g^{-1}}(\xi) \quad \text{for} \quad \xi \in T_gG.$$

Such a covector field is exactly a differential 1-form, so we've exhibited a map $\mathfrak{g}^* \to \Omega^1(G)$. The form L_ω is left-invariant because for any $h \in G$, we have

$$(L_h^* L_\omega)(\xi) = \omega \circ dL_{(hg)^{-1}} \circ dL_h(\xi) = \omega \circ dL_{g^{-1}}(\xi) = L_\omega(\xi) \quad \text{for all} \quad \xi \in T_g G.$$

This means that we actually have a linear map $\mathfrak{g}^* \to \Omega^1_{\text{linv}}(G)$. Since the inverse is given by $\omega = (L_\omega)_e$, we have an isomorphism. There is also an isomorphism $\mathbb{R} \to \Omega^0_{\text{linv}}(G)$ which sends a real number to its constant function on G. This is an isomorphism since the only left-invariant functions are the constant functions. This pair of isomorphisms uniquely extends to a graded algebra isomorphism $\bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet_{\text{linv}}(G)$.

To express the differential d as a coboundary map in $\bigwedge^\bullet \mathfrak{g}^*$, first note that $df = 0$ for any 0-form $f \in \Omega^0_{\text{linv}}(G)$ since left-invariant 0-forms are constant. To derive an expression for 1-forms, let $\omega \in \mathfrak{g}^*$ be a covector. Given vector fields $\xi_1, \xi_2 \in \mathfrak{X}(G)$, a corollary of Cartan's formula tells us that:

$$dL_\omega(\xi_1, \xi_2) = \xi_1(L_\omega(\xi_2)) - \xi_2(L_\omega(\xi_1)) - L_\omega([\xi_1, \xi_2]) = -L_\omega([\xi_1, \xi_2]).$$

Here the terms $\xi_1(L_\omega(\xi_2))$ and $\xi_2(L_\omega(\xi_1))$ vanish since L_ω is left-invariant. Let $\{\xi_i\}$ be a basis for $\mathfrak{g}$ and define structure coefficients $c^k_{i,j}$ by $[\xi_i, \xi_j] = c^k_{i,j}\xi_k$. Let $\{\theta^i\}$ be the corresponding dual basis for $\mathfrak{g}^*$. Note that:

$$d\theta^k(\xi_i, \xi_j) = -\widetilde{\theta^k}([\xi_i, \xi_j]) = -\widetilde{\theta^k}(c^q_{i,j}\xi_q) = -c^q_{i,j}\widetilde{\theta^k}(\xi_q) = -c^k_{i,j},$$

where the last equality follows since $\theta^k(\xi_q) = \delta^k_q$. Writing this this form in terms of $\bigwedge^\bullet \mathfrak{g}^*$, we get the expression

$$d\theta^k = -\frac{1}{2} c^k_{i,j} \theta^i \wedge \theta^j$$

where the 1/2 factor accounts for repeat terms. Along with the observation that $df = 0$ for any 0-form f, using the Leibniz rule this coboundary operator extends over the entire graded algebra so that the isomorphism $\bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet_{\text{linv}}(G)$ is an isomorphism of

(c). Prove that the inclusion in (a) induces an isomorphism on cohomology. A map of cochain complexes with this property is called a *quasi-isomorphism*.

To show that inclusion (we'll call it ι) is a quasi-isomorphism, we'll prove that $\Omega^\bullet_{\text{linv}}(G)$ is a deformation retract of $\Omega^\bullet(G)$. To do this, we'll have to construct two operators, or cochain maps:

$$A : \Omega^\bullet(G) \to \Omega^\bullet_{\text{linv}}(G) \quad \text{and} \quad H : \Omega^\bullet(G) \to \Omega^{\bullet-1}(G).$$

Here, A is a cochain map satisfying $A \circ \iota = \text{id}$ and $A \circ (\iota \circ A) = A$, and H is a linear map satisfying $dH + Hd = A - 1$. Put together, these maps would prove that ι induces an isomorphism on cohomology.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{k-1}_{\text{linv}}(G) \arrow[r, "d"] \arrow[d, shift left, "\iota"] & \Omega^{k}_{\text{linv}}(G) \arrow[r, "d"] \arrow[d, shift left, "\iota"] & \Omega^{k+1}_{\text{linv}}(G) \arrow[r] \arrow[d, shift left, "\iota"] & \cdots \\
\cdots \arrow[r] & \Omega^{k-1}(G) \arrow[r, shift left, "d"] \arrow[u, shift left, "A"] & \Omega^{k}(G) \arrow[r, shift left, "d"] \arrow[l, dashed, shift left, "H"] \arrow[u, shift left, "A"] & \Omega^{k+1}(G) \arrow[r] \arrow[l, dashed, shift left, "H"] \arrow[u, shift left, "A"] & \cdots
\end{tikzcd}$$

Using the assumption that G is compact, let μ be a left-invariant Haar measure on G. Since G is compact, we scale μ by a factor of $1/\mu(G)$ so that $\mu(G) = 1$. First, let's use this measure to construct A. A succinct form for A is:

$$A = \int_G L_h^* \, d\mu(h) \quad \Longrightarrow \quad A(\omega)_g(\xi_1, \ldots, \xi_k) = \int_G (L_h^*\omega)_g(\xi_1, \ldots, \xi_k) \, d\mu(h)$$

for all $g \in G$, $\omega \in \Omega^k(G)$, and $\xi_1, \ldots, \xi_k \in T_g G$. Clearly, if ω is already left-invariant, then $A(\omega) = \omega$ since $L^*h\omega = \omega$. For any $g' \in G$, we can act on A to get:

$$L_{g'}^* A = \int_G L_{hg'}^* \, d\mu(h) = A(\omega)$$

since the transformation $h \mapsto hg'$ is a bijection and left multiplication preserves the measure μ. This shows that $A \circ \iota$ is the identity on $\Omega^\bullet_{\text{linv}}(G)$ as well as $A \circ (\iota \circ A) = A$. A is a cochain map because differentials commute with integration, i.e. we have

$$A \circ d = \int_G L_h^* \circ d \, d\mu(h) = \int_G d \circ L_h^* \, d\mu(h) = d \circ A.$$

This proves A is a retract the cochain complexes. To show that A is a deformation retract, we must construct the cochain homotopy operator $H : \Omega^\bullet(G) \to \Omega^{\bullet-1}(G)$ which satisfies $dH + Hd = A - 1$. For any vector $\xi \in \mathfrak{g}$, define the operator

$$H_\xi = \int_0^1 L^*_{\overline{\exp}(t\xi)} \iota_{R_\xi} \, d\mu(h).$$

where R_ξ is the right-invariant vector field generated by ξ, and $\overline{\exp}$ is the right exponential map. Computing $dH_\xi + H_\xi d$, we get:

$$\begin{aligned}
dH_\xi + H_\xi d &= d \int_0^1 L^*_{\overline{\exp}(t\xi)} \iota_{R_\xi} \, d\mu(t) + \int_0^1 L^*_{\overline{\exp}(t\xi)} \iota_{R_\xi} d \, d\mu(t) \\
&= \int_0^1 L^*_{\overline{\exp}(t\xi)} (d\iota_{R_\xi} + \iota_{R_\xi} d) \, d\mu(t) \\
&= \int_0^1 L^*_{\overline{\exp}(t\xi)} \mathcal{L}_{R_\xi} \, d\mu(t) \\
&= \int_0^1 L^*_{\overline{\exp}(t\xi)} \frac{d}{ds}\bigg|_{s=0} L^*_{\overline{\exp}(s\xi)} \, d\mu(t) \\
&= \int_0^1 \frac{d}{ds}\bigg|_{s=0} L^*_{\overline{\exp}((s+t)\xi)} \, d\mu(t) \\
&= \int_0^1 \frac{d}{ds}\bigg|_{s=t} L^*_{\overline{\exp}(s\xi)} \, d\mu(t) \\
&= L^*_{\overline{\exp}(\xi)} - 1.
\end{aligned}$$

Now, let $\{U_\alpha\}$ be a locally finite open cover of G such that there are diffeomorphisms $\log_\alpha : U_\alpha \to V_\alpha \subset \mathfrak{g}$ with $\overline{\exp}(\log_\alpha(g)) = g$ for all $g \in U_\alpha$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to this open cover.

Consider the operator:

$$H = \sum_\alpha \int_{U_\alpha} \psi_\alpha(h) \cdot H_{\log_\alpha(h)} \, d\mu(h).$$

Using our previous expression for $dH_\xi + H_\xi d$, we get:

$$\begin{aligned}
dH + Hd &= \sum_\alpha \int_{U_\alpha} \psi_\alpha(h) \cdot (dH_{\log_\alpha(h)} + H_{\log_\alpha(h)} d) \, d\mu(h) \\
&= \sum_\alpha \int_{U_\alpha} \psi_\alpha(h) \cdot (L_h^* - 1) \, d\mu(h) \\
&= \int_G L_h^* \, d\mu(h) - 1 \\
&= A - 1.
\end{aligned}$$

This completes the proof.

(d). Use the inverse map $g \mapsto g^{-1}$ to show that the differential of a *bi-invariant* differential form vanishes. Show that the de Rham cohomology of G is isomorphic to the algebra of bi-invariant forms.

I was not able to come up with a proof that the differential of a bi-invariant differential form vanishes.

Now, we could repeat the proof of (c) but with all parity flipped (left instead of right). This would show us that $\Omega^\bullet_{\text{binv}} \subset \Omega^\bullet_{\text{linv}} \subset \Omega^\bullet$ is a quasi-isomorphism, and since the differential of any bi-invariant differential form is zero, it follows that

$$\Omega^\bullet_{\text{binv}}(G) \cong \mathrm{H}^\bullet_{\text{dR}}(G),$$

since all image groups in the cohomology quotient are trivial.

(e). Use these ideas to compute $\mathrm{H}^\bullet_{\text{dR}}(\mathrm{SU}_2)$.

Recall that SU_2 is the Lie group of 2×2 unitary complex matrices with determinant 1, i.e.

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

There is a basis of $\mathfrak{su}_2$ by elements X, Y, Z satisfying the commutator relations

$$[X, Y] = 2Z, \quad [Y, Z] = 2X, \quad \text{and} \quad [Z, X] = 2Y.$$

This means that our defining relations for the differential d on $\bigwedge^\bullet \mathfrak{g}^*$ are

$$dZ = -2X \wedge Y, \quad dX = -2Y \wedge Z, \quad \text{and} \quad dY = -2Z \wedge X.$$

Letting $d^k : \bigwedge^k \mathfrak{g}^* \to \bigwedge^{k+1} \mathfrak{g}^*$ be the differential map and applying the Leibniz rule to the above relations, we can now compute the kernels and images of the differential map and obtain the cohomology:

$$\begin{aligned}
\operatorname{im}(d^{-1}) &= 0 & \ker(d^0) &= \mathbb{R} & &\Longrightarrow & \mathrm{H}^0_{\text{dR}}(\mathrm{SU}_2) &= \mathbb{R}, \\
\operatorname{im}(d^0) &= 0 & \ker(d^1) &= 0 & &\Longrightarrow & \mathrm{H}^1_{\text{dR}}(\mathrm{SU}_2) &= 0, \\
\operatorname{im}(d^1) &= \textstyle\bigwedge^2 \mathfrak{g}^* & \ker(d^2) &= \textstyle\bigwedge^2 \mathfrak{g}^* & &\Longrightarrow & \mathrm{H}^2_{\text{dR}}(\mathrm{SU}_2) &= 0, \\
\operatorname{im}(d^2) &= 0 & \ker(d^3) &= \textstyle\bigwedge^3 \mathfrak{g}^* & &\Longrightarrow & \mathrm{H}^3_{\text{dR}}(\mathrm{SU}_2) &= \mathbb{R}.
\end{aligned}$$

If we want to include multiplicative structure, we see that $\mathrm{H}^\bullet_{\text{dR}}(\mathrm{SU}_2) \cong \mathbb{R}[x]/(x^2)$ with $|x| = 3$. This makes sense, since $\mathrm{SU}_2 \simeq S^3$, and there are exactly the homology groups of the 3-sphere.