Math 132
Problem Set 11
Spring, 2023

This problem set is due on Tuesday, May 2nd. Please make your answers as complete and clear as possible. You are allowed to discuss these problems with others in the class, but your writing should be your own.

You need only do **one** of Problems 2 and 4 (there is some overlap between them.)

1. This problem is more or less worked out in your book, at the end of §4 of Chapter 4. See if you can do it on your own, looking at the book if you need a hint.

(a) In calculus courses, line integrals in $\mathbb{R}^3$ are usually written in the form

$$\oint_C \vec{F} \cdot d\vec{r}.$$

Given a vector function $\vec{F}$ show how to construct a 1-form θ with the property that for every oriented curve C,

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C \theta.$$

(b) In multivariable calculus surface integrals are written in the form

$$\iint_S \vec{F} \cdot \mathbf{n}\, dS,$$

and computed when S is the set of points $(u, v, \mathbf{X}(u, v))$, $(u, v) \in D$ using

$$\mathbf{n}\, dS = \mathbf{X}_u \times \mathbf{X}_v\, du\, dv$$

and

$$\iint_S \vec{F} \cdot \mathbf{n}\, dS, = \iint_D \vec{F} \cdot \mathbf{X}_u \times \mathbf{X}_v\, du\, dv$$

Reconcile this with our formulation of integration on manifolds. More specifically, show how to associate a 2-form $\omega \in \Omega^2\mathbb{R}^3$ to the vector function $\vec{F} = (f_1, f_2, f_3)$ with the property that for every surface S,

$$\iint_S \vec{F} \cdot \mathbf{n}\, dS = \int_S \omega.$$

(c) In multivariable calculus courses, Stokes Theorem is usually stated in the form

$$\iint_S \mathbf{curl}\, \vec{F} \cdot \mathbf{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}.$$

Reconcile this with our formulation of Stokes Theorem.

(d) When the surface S is the boundary of a 3-dimensional region $D \subset \mathbb{R}^3$, the *Divergence theorem* states

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \mathbf{n}\, dS.$$

Show that this is also a special case of our formulation of Stokes theorem.

2. Here are some familiar facts about *conservative vector fields* stated in the language of 1-forms.

(a) (This Problem 11 from §4 of Chapter 4 of Guillemin-Pollack.) Suppose that ω is a 1-form on a connected manifold X, having the property that $\oint_\gamma \omega = 0$ for all colsed curves γ. Show that if $p, q \in X$ are two points. Choose smooth map $c : [0,1] \to X$ is a smooth path with $c(0) = p$, and $c(1) = q$. Show that

$$\int_0^1 c^*\omega$$

is independent of the choice of c. In this case one can write

$$\int_p^q \omega = \int_0^1 c^*\omega$$

(Hint: There is a hint in the book.)

(b) (This Problem 12 from §4 of Chapter 4 of Guillemin-Pollack.) Prove that any 1-form ω on X with the property that $\oint_\gamma \omega = 0$ for all closed curves γ is *exact* in the sense that there is a smooth function f with $\omega = df$. A curve γ is *closed* $\gamma(0) = \gamma(1)$. (Hint: There is a hint in the book.)

3. There are many variations on bordism homology. An important one is *oriented bordism.* One defines an *oriented k-manfifold* over a space X to be a pair (M, f) consisting of an oriented k-manifold M and a continuous map $f : M \to X$. The notion of *cobordism* is a little more subtle in this case. A *cobordism* of (M_0, f_0) and (M_1, f_1) consists of an oriented $(k+1)$-manifold N, a continuous map $h : N \to X$, and an oriented diffeomorphism $M_1 \amalg (-M_0) \approx \partial N$ having the property that the for $i = 0, 1$, the composition

$$M_i \to N \xrightarrow{h} X$$

is f_i. As a "reminder" when M is an oriented manifold, symbol $(-M)$ refers to M with the opposite orientation. The *oriented bordism homology group* $MSO_k(X)$ is defined to be the set of equivalence classes of oriented closed k-manifolds over X modulo the equivalence relation of oriented cobordism. The set $MSO_k(X)$ is a commutative monoid under disjoint union.

(a) Suppose that $f : M \to X$ is an oriented k-manifold over X. Show that

$$(M, f) + (-M, f) = 0 \in MSO_k(X).$$

(Thus $MSO_k(X)$ is an abelian group.)

(b) Now suppose that X is a smooth manifold and $\omega \in \Omega^k(X)$ is a k-form which is *closed* in the sense that $d\omega = 0$. The proof that $MO_k(X)$ can be computed by smooth maps and smooth cobordism applies to $MSO_k(X)$. Show that sending $f : M \to X$ to $\int_M f^*\omega$ gives a well-defined homomorphism

$$MSO_k(X) \to \mathbb{R}.$$

(c) A k-form ω is *exact* if there is a $(k-1)$-form η with $d\eta = \omega$. Show that if ω is exact then the above homomorphism is zero. Thus there is a map

$$H^k_{\mathrm{DR}}(X) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\} \to \hom(MSO_k(X), \mathbb{R}).$$

4. There is a tried and true method for showing that a 1-form ω on a connected manifold X is exact. You choose a point $x_0 \in X$ and for $x \in X$ try to define $f(x)$ by

$$f(x) = \int_0^1 \gamma^*\omega$$

where $\gamma : [0,1] \to X$ is any smooth path with $\gamma(0) = x_0$ and $\gamma(1) = x$. The trick is to show that $f(x)$ is well-defined, in the sense that it is independent of the choice of γ. By the above exercise, this holds if $MSO_1(X) = 0$.

(a) Show that if $f(x)$ is well-defined then $df = \omega$.

(b) Show that if $n > 1$ then every closed 1-form on S^n is exact. (You may wish to review our computation of $MO_1(S^n)$ (Chapter VIII, Proposition 3.10 of the lecture notes).)

(c) Let $T^n = S^1 \times \cdots \times S^1$ (n factors). Show that if $n > 1$ the degree of any smooth map $S^n \to T^n$ is zero.

5. (GP, ch 4, Problem 6). Prove that for the $(n-1)$ sphere of radius r in $\mathbb{R}^n$, the Gaussian curvature is everywhere $1/r^{n-1}$. (Hint: There is a hint in the book).