Math 230a Problem Set 10

Lev Kruglyak

Due: November 20, 2024

Collaborators: *AJ LaMotta, Ignasi Vicente*

Problem L. et X be a smooth manifold. Suppose $\gamma : (a, b) \to X$ is a smooth motion. Assume $0 \in (a, b)$ and write $x = \gamma(0)$.

(a). Let G be a Lie group, let $\pi : P \to X$ be a principal G-bundle with connection. Suppose F is a smooth manifold equipped with a smooth left action of G on F. Let $\sigma : X \to F_P$ be a smooth section of the associated fiber bundle with fiber F. Explain how to use parallel transport to define a motion $\delta : (a, b) \to (F_P)_x$, where $(F_p)_x$ is the fiber of the associated bundle $F_P \to X$ at $x \in X$. Prove that δ is smooth.

Let $\tau_{\gamma(t)} : P_x \to P_{\gamma(t)}$ be the parallel transport map of the connection. Now let's choose some $p_t \in P_{\gamma(t)}$ such that $\sigma \circ \gamma(t) = [p_t, f_t] \in F_P$ for $f_t \in F$. We thus obtain a horizontal lift $\widetilde{\gamma}_t(s)$ in P with $\widetilde{\gamma}_t(0) = p_t$. The endpoint of this path lies in P_x and differs from a fixed point $p_0 \in P_x$ by some holonomy element $g \in G$. Let's call this element $g(t)$. We then define the motion as

$$\delta(t) = [p_0, g(t) \cdot f_t] \in (F_P)_x.$$

(b). Now suppose X has a linear connection. Use parallel transport to define a motion $\delta : (a, b) \to T_x X$. Prove that δ is smooth.

Let $\tau_{\gamma(t)} : T_x X \to T_{\gamma(t)} X$ be the parallel transport associated to the curve γ. Define the motion δ as

$$\delta(t) = \tau_{\gamma(t)}^{-1} \circ \dot{\gamma}(t),$$

i.e. the parallel transport of the velocity vector at $\gamma(t)$ back to the tangent space at the origin. This map can be seen to be smooth by expressing it as a solution to a system of linear ODEs. In particular, if γ is a geodesic motion, this induced flow on $T_x X$ will be the constant vector $\dot{\gamma}(0)$.

Problem 4. Let X be a Riemannian 2-manifold, and suppose x^1, x^2 is a local coordiante system. Compute the Gauss curvature in terms of the Riemann curvature tensor $R^i_{jk\ell}$.

Let's assume without loss of generality that we're working in an orthonormal frame. Recall that the Riemann curvature tensor can be given by

$$\Omega_i^j = \frac{1}{2} R^i_{jk\ell} \theta^k \wedge \theta^\ell.$$

The Gauss curvature is then given by $\Omega_{12} = K\theta^1 \wedge \theta^2$ so $K = R^1_{212}$ for some orthonormal frame.

Problem 6. Let $X \subset E$ be a submanifold of a Euclidean space. The dimensions of X and E are not fixed.

(a). Use the global parallelism of E to induce a parallelism – a covariant derivative – on X. So if $\xi \in T_x X$ is a tangent vector at some point $x \in X$ and η a vector field on X defined in a neighborhood of X, use the natural covariant derivative on E to define the covariant derivative $\nabla_\xi \eta$ on X.

Suppose $\xi \in T_x X$ is a tangent vector, and η a vector field on X in a neighborhood of $x \in X$. Let V be the vector space of translations of E so that $TE = V \times E$. Thus, ξ can be considered as an element of V, and η as a map $X \to V$. Define the covariant derivative as

$$\nabla_\xi \eta = \lim_{t \to 0} \frac{\eta \circ \gamma_\xi(t) - \eta}{t}$$

for some curve $\gamma_\xi : (-\delta, \delta) \to X$ with $\gamma_\xi(0) = x$ and $\gamma_\xi'(0) = \xi$.

(b). Prove that ∇ preserves the induced Riemannian metric on X.

To show this, we must prove that for any $\xi \in T_x X$ and $\eta_1, \eta_2 \in \Gamma(TX)$, we have

$$\xi\langle \eta_1, \eta_2 \rangle = \langle \nabla_\xi \eta_1, \eta_2 \rangle + \langle \eta_1, \nabla_\xi \eta_2 \rangle.$$

Expanding the definitions, we get

$$\begin{aligned}
\xi\langle \eta_1, \eta_2 \rangle = \lim_{t \to 0} \frac{\langle \eta_1 \circ \gamma_\xi(t), \eta_2 \circ \gamma_\xi(t) \rangle - \langle \eta_1, \eta_2 \rangle}{t} &= \lim_{t \to 0} \frac{\langle \eta_1 \circ \gamma_\xi(t) - \eta_1, \eta_2 \rangle + \langle \eta_1, \eta_2 \circ \gamma_\xi(t) - \eta_2 \rangle}{t} \\
&= \langle \nabla_\xi \eta_1, \eta_2 \rangle + \langle \eta_1, \nabla_\xi \eta_2 \rangle.
\end{aligned}$$

(c). Consider the example of a unit 2-sphere X in a 3-dimensional Euclidean space. Let C be the circle obtained by intersecting X with a plane whose distance from the nearest parallel tangent plane is $d < 1$. The holonomy of the parallel transport around C is rotation through some angle θ. Compute θ as a function of d. Make clear your orientations.

Problem 7. Let $S^3 \subset \mathbb{E}^4$ be a sphere of radius R in Euclidean 4-space.

(a). Introduce local coordinates. Compute the Christoffel symbols and Riemann curvature tensor in the coordinate system.

A common coordinate system is to use hyperspherical coordinates, i.e.

$$\begin{aligned}
x_0 &= R\cos\psi, \\
x_1 &= R\sin\psi\cos\theta, \\
x_2 &= R\sin\psi\sin\theta\cos\varphi, \\
x_3 &= R\sin\psi\sin\theta\sin\varphi.
\end{aligned}$$

In these coordinates, the metric is given by

$$ds^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right].$$

Using the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})$$

which is implemented in the `GREAT.m` package in Mathematica, we get the following Christoffel symbols:

$$\Gamma^\psi_{\mu\nu} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin(\psi)\cos(\psi) & 0 \\ 0 & 0 & \cos(\psi)\sin^2(\theta) \end{pmatrix}, \quad \Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & \cot(\psi) & 0 \\ \cot(\psi) & 0 & 0 \\ 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix},$$

$$\Gamma^\varphi_{\mu\nu} = -\begin{pmatrix} 0 & 0 & \cot(\psi) \\ 0 & 0 & \cot(\theta) \\ \cot(\psi) & \cot(\theta) & 0 \end{pmatrix},$$

Finally, using the formula

$$R^\mu_{\lambda\alpha\beta} = \partial_\alpha \Gamma^\mu_{\beta\lambda} - \partial_\beta \Gamma^\mu_{\alpha\lambda} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\lambda} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\lambda},$$

and the aforementioned Mathematica package, we get the Riemann tensors:

$$R^\psi_{\alpha\mu\nu} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sin^2(\psi) & 0 \\ -\sin^2(\psi) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \sin^2(\psi)\sin^2(\theta) \\ 0 & 0 & 0 \\ -\sin^2(\psi)\sin^2(\theta) & 0 & 0 \end{pmatrix}\right),$$

$$R^\theta_{\alpha\mu\nu} = \left(\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sin^2(\psi)\sin^2(\theta) \\ 0 & -\sin^2(\psi)\sin^2(\theta) & 0 \end{pmatrix}\right),$$

$$R^\varphi_{\alpha\mu\nu} = \left(\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin^2(\psi) \\ 0 & \sin^2(\psi) & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

Putting all of this together to get the scalar Riemann curvature $R = g^{\alpha\beta} g_{\mu\nu} R^\nu_{\mu\alpha\beta}$, we get $R = 6/R^2$.

(b). Write S^3 as a homogeneous manifold for a Lie group of isometries. Is there a homogeneous connection? Does it induce the Levi-Civita connection? If so, recover the curvature you computed in (a) from the curvature of the homogeneous connection.

We can write S^3 as the quotient $\mathrm{SO}_4 / \mathrm{SO}_3$. Recall that the Lie algebra $\mathfrak{so}_n$ consists of skew-symmetric $n \times n$ matrices. Thus, there is an SO_3-invariant complement $\mathfrak{p}$ to $\mathfrak{so}_4$ isomorphic to $\mathbb{R}^4$. There is a homogeneous connection $D_\mathfrak{p} \subset T\,\mathrm{SO}_4$ with curvature $\Omega \in \Omega^2(G; \mathfrak{so}_3)$ given by $\Omega_e(\xi, \eta) = -[\xi, \eta]_{\mathfrak{so}_3}$ for $\xi, \eta \in \mathfrak{p}$. Let's choose standard bases p_1, p_2, p_3 for $\mathfrak{p}$ and h_1, h_2, h_3 for $\mathfrak{so}_3$. The commutator relations are:

$$[p_1, p_2] = h_3, \quad [p_1, p_3] = -h_2, \quad [p_2, p_3] = h_1.$$

Recall that the Ricci tensor is given by:

$$\mathrm{Ricci}(p_i, p_j) = \sum_k \langle R(p_k, p_i)p_j, p_k \rangle = 2\delta_{ij}.$$

The scalar curvature R is the trace of the Ricci tensor, i.e. $R = \mathrm{Tr}(\mathrm{Ricci}) = 2 + 2 + 2 = 6$. This aligns with the calculation of scalar curvature from the previous problem assuming $R = 1$.

(c). Compute yet another way by constructing a local orthonormal frame.

Let's consider S^3 with radius R as the space of unit quaternions SU_2. Let e_1, e_2, e_3 be the vector fields corresponding to left multiplication by iR, jR, kR respectively, with dual coframe $\theta^1, \theta^2, \theta^3$. This gives a local orthonormal frame on S^3 with Lie bracket relations

$$[e_i, e_j] = \frac{2\epsilon_{ijk} e_k}{R} \quad \text{where } \epsilon_{123} = 1 \text{ alternates in indices.}$$

The structure equations give us connection 1-forms ω_{ij} satisfying $de_i + \omega_{ij}e_j = 0$. However since e_i are left-invariant under SU_2, the differentials are given by

$$de_i = -\frac{1}{2}[e_j, e_k]\,\theta^j \wedge \theta^k \quad \text{where} \quad \theta^i(e_j) = \delta^i_j.$$

Now $d\theta^k = \epsilon_{k\ell m}\,\theta^\ell \wedge \theta^m$. This means that the curvature has the form

$$\begin{aligned}\Omega_{ij} &= d\omega_{ij} + \omega_{ik} \wedge \omega_{kj} \\ &= -\frac{1}{R}\epsilon_{ijk}\left(\epsilon_{k\ell m}\,\theta^\ell \wedge \theta^m\right) + \frac{1}{R^2}\epsilon_{ik\ell}\epsilon_{kjm}\,\theta^\ell \wedge \theta^m \\ &= \frac{1}{R^2}\theta^i \wedge \theta^j.\end{aligned}$$

Recall now that the Riemann curvature tensor is given by

$$\Omega_{ij} = \frac{1}{2}R^i_{jk\ell}\,\theta^k \wedge \theta^\ell \quad \Longrightarrow \quad \begin{aligned} R^i_{jk\ell} &= \frac{1}{R^2}\left(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}\right) \\ R_{ij} &= \sum_k R^i_{kjk} = \frac{1}{R^2}\delta_{ij}\sum_k(\delta_{kk} - \delta_{ij}) = \frac{2}{R^2}\delta_{ij}.\end{aligned}$$

Taking the trace of R_{ij} gives us a scalar curvature of $6/R^2$, which agrees with the previous problem.

Problem 9. Let X be a smooth manifold and $E \subset TX$ a distribution of rank k. Let $\mathcal{B}_E(X) \subset \mathcal{B}(X)$ be the subbundle of frames for which the first k vectors form a basis for E. Prove that $\mathcal{B}_E(X) \to X$ admits a torsion-free connection if and only if E is involutive.

Note that $\mathcal{B}_E(X)$ is a principle $(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$-subbundle of $\mathcal{B}(X)$. It's easier to work with the covariant derivative associated to the connection, so we will do this throughout the problem. Let's first suppose that ∇ is a torsion-free affine connection on TX which preserves E. This means that for any $\xi, \eta \in \Gamma(E)$, we have $\nabla_\xi \eta \in \Gamma(E)$, and the torsion-free condition implies that $\nabla_\xi\eta - \nabla_\eta\xi = [\xi, \eta] \in \Gamma(E)$. This shows that the distribution is involutive.

Conversely, suppose E is involutive. By the Frobenius theorem, we can find local coordinates such that we have commuting vector fields $\partial/\partial x^1, \ldots, \partial/\partial x^k$ spanning E. We can extend these local coordinates by some vector fields $\partial/\partial x^{k+1}, \ldots, \partial/\partial x^n$ complement E to span TX. We then define the covariant derivative by

$$\nabla_\xi \eta = \sum_{0 \le i,j \le n} \xi^i \frac{\partial \eta^j}{\partial x^i}\frac{\partial}{\partial x^j}.$$

This is clearly torsion free, a covariant derivative, and preserves $\Gamma(E)$.

Problem 10. On a smooth manifold with linear connection, is it possible for a geodesic to intersect itself? Give an example or counter-proof.

There are many examples of such manifolds. For a simple 2-dimensional example, consider the "plus sign" as a closed subset of $\mathbb{A}^2$. If we identify adjacent edges as shown in Figure 1, we get a surface with boundary homeomorphic to a sphere with three disks cut out. Since the attaching maps are affine, the flat connection on $\mathbb{A}^2$ descends to this quotient surface. Clearly, the path pictured in Figure 1 is a geodesic motion on this surface – and an immersion of S^1 to a submanifold homeomorphic to $S^1 \vee S^1$.

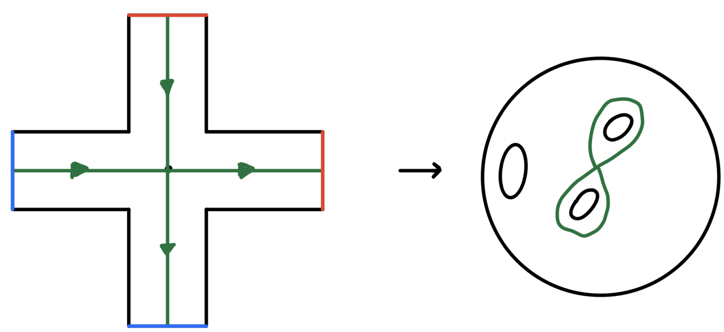

Figure 1: A surface with self intersecting geodesics.