Math 231b Problem Set 8

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Problem 1. Homology with local coefficients.

Let X denote a path-connected and semilocally simply-connected space, and let $\widetilde{X} \to X$ denote its universal cover.

a. Prove that $S_*(\widetilde{X}; R)$ is a complex of free $R[\pi_1(X)]$-modules, where $\pi_1(X)$ acts via deck transformations on $\widetilde{X}$.

Recall that $S_n(\widetilde{X}; R) = R\mathrm{Sin}_n(\widetilde{X})$ is a free R-module. Now recall that we also have an action of $\pi_1(X)$ on $\mathrm{Sin}_n(\widetilde{X})$, and the induced map $\mathrm{Sin}_n(\widetilde{X}) \to \mathrm{Sin}_n(X)$ has fibers exactly the orbits of this action. Similarly, $R[\pi_1(X)]$ acts on $S_n(\widetilde{X}; R)$, and the orbits of this action are the fibers of the map $S_n(\widetilde{X}; R) \to S_n(X; R)$. Notice that since Δ^n is simply connected, we can lift any $\sigma : \Delta^n \to X$ to some $\widetilde{\sigma} : \Delta^n \to \widetilde{X}$. Then $\widetilde{\sigma} \cdot R[\pi_1(X)]$ is exactly the fiber of $R\sigma$ so we have the direct sum decomposition:

$$S_n(\widetilde{X}; R) = \bigoplus_{\sigma \in \mathrm{Sin}_n(X)} \widetilde{\sigma} \cdot R[\pi_1(X)]$$

and thus it is a free $R[\pi_1(X)]$-module. The boundary maps are clearly seen to be $R[\pi_1(X)]$-module homomorphisms, since the inclusion of faces doesn't affect the choice of lifting. Thus $S_*(\widetilde{X}; R)$ is a complex of $R[\pi_1(X)]$-modules.

b. In the setting of (a), prove that a short exact seuqnce of $R[\pi_1(X)]$-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ gives rise to a long exact sequence:

$$\cdots \longrightarrow H_{n+1}(X; M_3) \longrightarrow H_n(X; M_1) \longrightarrow H_n(X; M_2) \longrightarrow H_n(X; M_3) \longrightarrow H_{n-1}(X; M_1) \longrightarrow \cdots$$

Recall that tensoring with free modules is exact. Thus, we have an SES of chain complexes of $R[\pi_1(X)]$-modules:

$$0 \longrightarrow S_*(\widetilde{X}; R) \otimes_{R[\pi_1(X)]} M_1 \longrightarrow S_*(\widetilde{X}; R) \otimes_{R[\pi_1(X)]} M_2 \longrightarrow S_*(\widetilde{X}; R) \otimes_{R[\pi_1(X)]} M_3 \longrightarrow 0$$

This leads to a LES in homology, and the homology of these chain complexes are exactly $H_*(X; M_*)$ by construction.

c. Prove that $H_*(K(G,1); M) \cong \mathrm{Tor}_*^{R[G]}(R, M)$ by noting that $S_*(\widetilde{K(G,1)}; R)$ is a resolution of R by free $R[G]$-modules. This is usually called the *group homology* of G with cofficients in M and is denoted $H_*(G; M)$.

Consider the canonical map $S_0(\widetilde{K(G,1)};R) \to R$ which sends any $r \cdot \sigma$ to r. The kernel of this map is generated by $r\cdot(a-b)$, where $a, b : \Delta^0 \to \widetilde{K(G,1)}$. This is exactly the image of the differential $S_1(\widetilde{K(G,1)};R) \to S_0(\widetilde{K(G,1)};R)$, so $S_*(\widetilde{K(G,1)};R)$ is naturally a resolution of R by free $R[G]$-modules. Then, by construction of Tor and homology with local coefficients, we have:

$$H_*(K(G,1);M) = H_*(S_*(\widetilde{K(G,1)};R) \otimes_{R[G]} M) = \mathrm{Tor}_*^{R[G]}(R, M).$$

Problem 2. Let $\mathbb{Z}(-1)$ denote the $\mathbb{Z}[C_2]$-module on which the generator of C_2 acts by -1. Compute $H_*(\mathbb{RP}^n;\mathbb{Z}(-1))$.

Recall that the homology with local coefficients was defined in terms of the universal cover, as:

$$H_*(\mathbb{RP}^n;\mathbb{Z}(-1)) = H_*(S_*(S^n) \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}(-1)).$$

We can replace $S_*(-)$ here by cellular chains $C_*(-)$ without affecting the isomorphism class of the homology, but we need a choice of CW structure on S^n. Consider the cellular decomposition of S^n with two cells attached in each dimension in a way that respects the antipodal map. More explicitly, we start with two 0-cells e^0_+, e^0_- on antipodal points of the sphere. Next we add two 1-cells e^1_+ and e^1_- as arcs on a great circle, with $de^1_+ = e^0_+ - e^0_-$ and $de^1_- = e^0_- - e^0_+$. We keep building up S^n, alternating the signs in each dimension so the antipodality is preserved. Notice then that the action of the only nontrivial deck automorphism of the covering $S^n \to \mathbb{RP}^n$ simply transposes these two cells. Thus, as a chain complex of $\mathbb{Z}[C_2]$-modules, $C_*(S^n)$ looks like:

$$0 \longrightarrow \mathbb{Z}[C_2] \xrightarrow{\times(1+(-1)^n\tau)} \mathbb{Z}[C_2] \longrightarrow \cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{\times(1+\tau)} \mathbb{Z}[C_2] \xrightarrow{\times(1-\tau)} \mathbb{Z}[C_2] \longrightarrow 0$$

where τ is the generator of C_2. When we tensor this with $\mathbb{Z}(-1)$ over $\mathbb{Z}[C_2]$, we get the chain complex:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times(1-(-1)^n)} \mathbb{Z} \longrightarrow \cdots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow 0$$

From this, we get our desired homology:

$$H_k(\mathbb{RP}^n;\mathbb{Z}(-1)) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k < n, k \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3. Using the fibrations $U(n-1) \to U(n) \to U(n)/U(n-1) \cong S^{2n-1}$, prove by induction on n that $H^*(U(n);\mathbb{Z}) \cong \mathbb{Z}[x_1, x_3, \dots, x_{2n-1}]/(x_1^2, x_3^2, \dots, x_{2n-1}^2)$.

Let's begin with the base case of $U(2)$. Recall that $U(1) \simeq S^1$, so we have a fibration $S^1 \to U(2) \to S^3$. Using the cohomological Serre spectral sequence, we have the E_2-page:

$$\begin{array}{cc} E_2^{3,0} = \mathbb{Z}x_3 & \mathbb{Z}x_1x_3 \\ 0 & 0 \\ 0 & 0 \\ \mathbb{Z} & E_2^{0,1} = \mathbb{Z}x_1 \end{array}$$

Here note that if we let x_1 and x_3 be generators of $E_2^{0,1}$ and $E_2^{3,0}$ respectively, we get x_1x_3 as a generator of $E_2^{3,1}$. Notice that $x_1^2 = 0$ and $x_3^2 = 0$ follow by a simple degree check. Since there are no non-trivial differential maps, the spectral sequence collapses at E_2, so we get the ring structure $H^*(U(2);\mathbb{Z}) = \mathbb{Z}[x_1, x_3]/(x_1^2, x_3^2)$. Now suppose by induction that we had the desired ring structure on $H^*(U(n-1);\mathbb{Z})$. The fibration $U(n-1) \to U(n) \to S^{2n-1}$ gives us the E_2 page of a spectral sequence:

$$
\begin{array}{c|ccccccc}
s & & & & & & & \\
 & \mathbb{Z}x_{2n-1} & \mathbb{Z}x_1x_{2n-1} & 0 & \mathbb{Z}x_3x_{2n-1} & \mathbb{Z}x_1x_3x_{2n-1} & \mathbb{Z}x_5x_{2n-1} & \mathbb{Z}x_1x_5x_{2n-1} \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & \mathbb{Z}x_1 & 0 & \mathbb{Z}x_3 & \mathbb{Z}x_1x_3 & \mathbb{Z}x_5 & \mathbb{Z}x_1x_5
\end{array}
$$

Here the bottom row has the given multiplicative structure since it's simply the cohomology of $U(n-1)$. The top row must have the same multiplicative structure, but here we list a degree $2n-1$ generator in the $E_2^{2n-1,0}$ term, which generates the rest of the multiplicative structure. Again, there are no nontrivial differentials so we get $H^*(U(n)) = H^*(U(n-1))[x_{2n-1}]/(x_{2n-1}^2)$ as desired.

Problem 4. Let $f : S^2 \to S^2$ be a map of degree 2. Compute the homology of its homotopy fiber.

Since S^2 is simply-connected, we can make a lot of useful reductions in the Serre fibration. First note that by the note from class, the fiber sequence $F \to S^2 \to S^2$ can be used in the Serre fibration, so we get a spectral sequence with E^2-page given by

$$E^2_{s,t} = H_s(S^2; H_t(F)).$$

Note also that the fiber sequence gives us a SES of the form:

$$0 \longrightarrow \pi_1(S^2) \xrightarrow{2\times} \pi_1(S^2) \longrightarrow \pi_1(F) \longrightarrow 0$$

This implies that $\pi_1(F) \cong \mathbb{Z}/2\mathbb{Z}$, and by the Hurewicz isomorphism this also tells us that $H_1(F) \cong \mathbb{Z}/2\mathbb{Z}$. Next, looking at the spectral sequence, we get:

$$
\begin{tikzcd}
H_0(F) \arrow[ddr, "d_0^2"] & H_1(F) \arrow[ddr, "d_1^2"] & H_2(F) \arrow[ddr, "d_2^2"] & H_3(F) \arrow[ddr, "d_3^2"] & \cdots \\
0 & 0 & 0 & 0 & \\
H_0(F) & H_1(F) & H_2(F) & H_3(F) & \cdots
\end{tikzcd}
$$

Since each chain only has one potentitally non-trivial differential, the E^3-page is in fact the E^∞-page, so we get the E^∞-page:

$$
\begin{array}{ccccc}
\ker d_0^2 & \ker d_1^2 & \ker d_2^2 & \ker d_3^2 & \cdots \\
0 & 0 & 0 & 0 & \\
H_0(F) & \operatorname{coker}\, d_0^2 & \operatorname{coker}\, d_1^2 & \operatorname{coker}\, d_2^2 & \cdots \\
\hline
\mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 \quad \cdots
\end{array}
$$

here the bottom row corresponds to the homology of S^2, the total space. Firstly, note that $H_0(F) \cong \mathbb{Z}$, this is expected. Next, we remember from the Hurewicz argument that $H_1(F) \cong \mathbb{Z}/2\mathbb{Z}$, so since coker $d_0^2 = 0$, it follows that the map $H_0(F) \to H_1(F)$ is the reduction mod 2. This means that $\ker d_0^2 = 2\mathbb{Z}$ and so coker $d_1^2 = 0$. Next, we know that $\ker d_i^2 = 0$ for all $i \geq 1$, and coker $d_i^2 = 0$ for all $i \geq 2$. Combining this with the fact that coker $d_1^2 = 0$, this means that $d_k^2 : H_k(F) \to H_{k+1}(F)$ is an isomorphism for $k \geq 1$. So to conclude,

$$H_k(F) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \end{cases}$$

Problem 5. Induced homology isomorphisms.

Here we will prove some criteria for maps to induce homology isomorphisms.

a. Show that if $p : E \to B$ is a fibration and each fiber has the homology of a point, then p induces an isomorphism in homology.

Consider the Serre spectral sequence, which has $E^2_{s,t} = H_s(B; H_t(F_b))$. Since the fibers have the homology of a point, the only non-trivial groups here are $E^2_{s,0} = H_s(B)$. Note that there are no non-trivial differential maps, so the spectral sequence collapses here at the E^2 page. It thus follows that the edge homomorphisms $H_n(E) \to H_n(B)$, which are exactly the induced map p_*, are isomorphisms.

b. Show that any weak equivalence $f : X \to Y$ induces a homology isomorphism.

Notice that for any $y \in Y$, the homotopy fiber sequence $F_y(f) \to X \to Y$ extends to a long exact sequence of homotopy groups, however since $\pi_k(X) \to \pi_k(Y)$ is an isomorphism, it follows that $F_y(f)$ has trivial homotopy groups, so it is weakly contractible. This implies that it has the homology of a point by the Hurewicz homomorphism, so we can apply the previous problem to see that f induces a homology isomorphism.