Math 132 Problem Set 6

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Due: March 24, 2023

Problem 1. Give an example of a 1-manifold M having the property that ∂M consists of 3 points. Show that there are infinitely many such manifolds, no two of which are diffeomorphic. Why doesn't this contradict our classification result.

The 3-manifold $[0,1]\sqcup[0,1)$ has exactly 3 points on its boundary. This isn't a compact manifold, so it doesn't contradict the classification result. Furthermore, to get an infinite amount of non-diffeomorphic manifolds of this form, we can simply disjoint union an arbitrary number of copies of S^1.

Problem 2. Suppose that X is a compact n-manifold, and we have smooth functions $\phi_i : X \to \mathbb{R}^n$, $\lambda_i : X \to [0,1]$ for $i = 1, .., k$ with the following properties:

(i) For all i, the image ϕ_i contains the open ball $B_2 \subset \mathbb{R}^n$ of radius 2.

(ii) Set $V_i = \phi_i^{-1}(B_2)$. The map ϕ_i restricts to a diffeomorphism $V_i \to B_2$.

(iii) The open subsets $U_i = \phi_i^{-1} B_1$ cover X, in which $B_1 \subset \mathbb{R}^n$ is the open ball of radius 1.

(iv) For $x \in U_i$, $\lambda_i(x) = 1$.

(v) For $x \in X \setminus V_i$, $\lambda_i(x) = 0$.

One can construct such data using some coordinate charts and a bump function. Using these, define a map $g : X \to (\mathbb{R}^n \times \mathbb{R}^1)^k = \mathbb{R}^{k(n+1)}$ by $g(x) = ((\phi_1, \lambda_1), \ldots, (\phi_k, \lambda_k))$. Show that the map g is an embedding. Since X is compact, this amounts to showing that it is an immersion and one-to-one.

Let's show that g is an immersion. This means that for any $x \in X$, the map $dg_x : T_xX \to T_{g(x)}(\mathbb{R}^n \times \mathbb{R})^k$ is injective. We can do this by showing that dg_x has a trivial kernel, so assume $dg_x(v) = 0$ for some $v \in T_xX$. Then since by (iii), we have some open set $U_i = \phi_i^{-1}(B_1)$ containing x, it follows that $(d\phi_i)_x(v) = 0$. However (ii) implies that $(d\phi_i)_x$ is an isomorphism, so $v = 0$. To show that g is injective, suppose we had $x, y \in X$ with $g(x) = g(y)$. Then using picking some U_i containing x, by (iv) we have $\lambda_i(x) = \lambda_i(y) = 1$. But since ϕ_i is a diffeomorphism on V_i by (ii), we get $x = y$. This completes the proof.

Problem 3. Recall the tubular neighborhood theorem: If $X \subset \mathbb{R}^n$ is a smooth manifold there is an open neighborhood $X \subset U \subset \mathbb{R}^n$ of X, and a smooth map $r : U \to X$ having the property that for all $x \in X$, $r(x) = x$.

Using the tubular neighborhood theorem, prove the following results.

a. Suppose that $Z \subset X$ is a compact submanifold of a compact manifold X (both without boundary), there is an open neighborhood $Z \subset U \subset X$ and a smooth retraction $r : U \to Z$.

Suppose X is embedded in $\mathbb{R}^n$. The tubular neighborhood theorem gives us a tubular neighborhood $V \subset Z$ of Z with a smooth retraction $r : V \to Z$. Intersecting with X gives us $U = V \cap X$ which is the desired open neighborhood, and $r|_U$ the desired smooth retraction.

b. If X is a compact manifold with non-empty boundary, ∂X, there is an open neighborhood $\partial X \subset U \subset X$ and a retraction $r : U \to \partial X$.

Since ∂X is a compact submanifold of X (if X is compact) this follows immediately from the previous part, since we never used the assumption that X had to be boundaryless in (a).

Problem 4. This problem proves the existence of a *collar* neighborhood of the boundary of a manifold.

Suppose that X is a compact smooth manifold with non-empty boundary ∂X.

a. Show that there is a smooth function $f : X \to [0, \infty)$ having the properties that $f^{-1}(0) = \partial X$ and $df_x : T_x X \to T_0\mathbb{R}$ is surjective for all $x \in \partial X$.

Let $\{(U_i, \varphi_i)\}$ be a finite set of atlases, i.e. diffeomorphisms $\varphi_i : U_i \to V_i \subset \mathbb{H}^n$. Letting π be the projection map $\mathbb{H}^n \to [0, \infty)$, we can then set

$$f = \sum_i \psi_i \cdot \pi(\varphi_i)$$

where ψ_i is some partition of unity subordinate to $\{U_i\}$. By the positivity conditions of π and ψ_i, note that $f(x) = 0$ if and only if $\pi(\varphi_i(v)) = 0$ for all U_i which contain v. This means that $\varphi_i(v) \in \partial\mathbb{H}^n$ and thus $v \in \partial X$.

To show that $df_x : T_x X \to T_0\mathbb{R}$ is surjective, note that

$$\begin{aligned} df_x(v) = \sum_i d(\psi_i \cdot \pi(\varphi_i))_x(v) = \sum_i d(\psi_i \cdot \pi(\varphi_i))_x(v) &= \sum_i (d\psi_i)_x(v)\pi(\varphi_i)(v) + \psi_i(v) d(\pi(\varphi_i))_x(v) \\ &= \sum_i \psi_i(v) \cdot d(\pi(\varphi_i))_x(v). \end{aligned}$$

Thus it suffices to show that there is a vector $v \in T_x X$ with $d(\pi(\varphi_i))_x(v) > 0$ for all i such that $x \in U_i$. Recall that we have an isomorphism $T_x X \to T_x\partial X \oplus N$ where $N = (T_x\partial X)^\perp$. Since ∂X has codimension 1, and represents the normal vectors to $x \in \partial X$, we can pick some inward pointing normal vector $v \in \partial X$, and then $d(\pi(\varphi_i))_x(v) > 0$ for all i such that $x \in U_i$. This means that $df_x(v) > 0$ and so df_x is a surjection.

b. Now let (U, r) be the neighborhood of ∂X and the retraction $r : U \to \partial X$ constructed in the previous problem. Show that the map $(r, f) : U \to \partial X \times [0, \infty)$ restricts to a diffeomorphism $U' \to \partial X \times [0, \epsilon)$ for some neighborhood $U' \subset U$ of ∂X. Such a neighborhood is called a *collar neighborhood.*

Note that by the inverse function theorem, it suffices to show that $dr \times df$ is an isomorphism at each ∂X to show that (r, f) is a local diffeomorphism on ∂X. To do this, we'll show that $\ker(dr_x \times df_x) = 0$ for all $x \in \partial X$. Since $f|_{\partial X} = 0$, $T_x\partial X \subset \ker(df_x)$ but by the rank-nullity theorem, $\dim\ker(df_x) = n - 1$ so $\ker(df_x) = T_x\partial X$. Thus $\ker(dr_x \times df_x) = \ker(dr_x) \cap T_x\partial X$, which is trivial because $r|_{\partial X} = 1_{\partial X}$.

Next, let $\{\mathcal{U}_i\}$ be an open cover of ∂X on which (r, f) restricts to a diffemorphism. Let $\mathcal{V} = \bigcup_i \mathcal{U}_i$ and $A_\epsilon = f^{-1}([0, \epsilon))$ and $B_\epsilon = f^{-1}((\epsilon, \infty))$. $\mathcal{V}$ must contain some A_ϵ, so since V^c is compact, there must be a finite set of ϵ_i with $V^c \subset B_{\epsilon_1} \cup \cdots \cup B_{\epsilon_k}$. Letting $\epsilon = \min(\epsilon_1, \ldots, \epsilon_k)/2$ we see that $V \subset A_\epsilon$, and so (r, f) restricts to a local diffeomorphism on A_ϵ.

Finally, let us show show that we can find some $\epsilon' > 0$ for which (r, f) is injective on $U' = U_{\min(\epsilon,\epsilon')}$. Assume for the sake of contradiction, that we can't find such an ϵ'. For every positive integer n, we can find distinct $x_n, y_n \in U_{1/n}$ such that $(r, f)(x_n) = (r, f)(y_n)$. By compactness of X, we can find convergent subsequences of $\{x_{n_k}\}$ and $\{y_{n_k}\}$. Let $x = \lim_{k\to\infty} x_{n_k}$ and $y = \lim_{k\to\infty} y_{n_k}$. Since (r, f) is continuous and $(r, f)(x_{n_k}) = (r, f)(y_{n_k})$ for all k, it follows that $(r, f)(x) = (r, f)(y)$. By construction $x, y \in \partial X$, so since r restricts to the identity on ∂X, we have $x = y$. This contradicts the injectivity of (r, f) in a neighborhood of x, which contradicts the fact that (r, f) is a local diffeomorphism at x, as desired.

Problem 5. Let X and Y be submanifolds of $\mathbb{R}^N$. Show that for almost all $a \in \mathbb{R}^N$, the translate $X + a = \{x + a : x \in X\}$ intersects Y transversally.

Consider the translation map $t : \mathbb{R}^N \times X \to \mathbb{R}^N$ which sends $t(a, x) = a + x$. This is clearly smooth and a submersion, since $dt_{(a,x)}(v, w) = v + w$. Clearly $t \pitchfork Y$, so by the transversality theorem we get $t(a, -) \pitchfork Y$ for almost all $a \in \mathbb{R}^N$. For any of these $a \in \mathbb{R}^N$, note that we have $T_x(X)+T_{a+x}(Y) = T_{a+x}(a+X)+T_{a+x}Y = \mathbb{R}^N$ because $t(a, -)$ is a diffeomorphism when restricted to X. Thus $(a + X) \pitchfork Y$ as desired.