Math 231a Sample Solutions 1

Lev Kruglyak

Due: Wednesday, September 27

Problem 1.1. Let p be a polynomial function on $\mathbb{C}$ which has no root on S^1. Show that the number of roots of $p(z) = 0$ with $|z| < 1$ is the degree of the map $\hat{p} : S^1 \to S^1$ specified by $\hat{p}(z) = p(z)/|p(z)|$.

Problem 1.2. Show that any map $f : S^1 \to S^1$ such that $\deg(f) \neq 1$ has a fixed point.

Problem 1.3. Let G be a topological group and take its identity element e as its base-point. Define the pointwise product of loops α and β by $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Prove that $\alpha\beta$ is equivalent to the composition of paths $\beta \cdot \alpha$. Deduce that $\pi_1(G, e)$ is abelian.

(a) This clearly makes the set of based loops in (G, x_0) into a group. It clearly satisfies associativity, identity (by taking the constant loop c_1, where 1 is the identity element), and inverses (given a loop $f : I \to G$ consider $f^{-1} : I \to G$ defined by taking the inverse piecewise). To show that it induces an operation on $\pi_1(G, x_0)$ it suffices to show that $\cdot$ is well defined with regards to homotopy. Indeed, suppose $f \simeq f'$ and $g \simeq g'$ are homotopic loops with path homotopies H_f and H_g respectively. Then $H_f \cdot H_g$ is a path homotopy between $f \cdot g$ and $f' \cdot g'$.

(b) Let $f, g : I \to G$ be loops in G based at x_0. Then $f \cdot g = (f * e_{x_0}) \cdot (e_{x_0} * g)$. However

$$(f * e_{x_0}) \cdot (e_{x_0} * g) = \begin{cases} f(2t) \cdot x_0 & 0 \le t \le \frac{1}{2} \\ x_0 \cdot g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} = f * g.$$

(c) Note that by a similar argument to (b), we have for any loops $f, g : I \to G$,

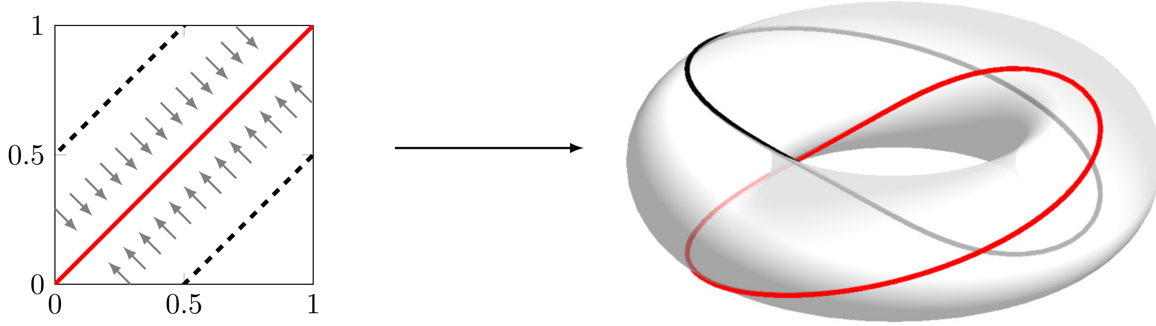
$$f * g = f \cdot g = (e_{x_0} * f) \cdot (g * e_{x_0}) = (e_{x_0} \cdot g) * (f \cdot e_{x_0}) = g * f.$$

Thus $\pi_1(G, x_0)$ is abelian.

Problem 2.1. Compute the fundamental group of the two-holed torus (the compact surface of genus 2 obtained by sewing together two tori along the boundaries of an open disk removed from each).

Problem 2.2. The Klein bottle K is the quotient space of $S^1 \times I$ obtained by identifying $(z, 0)$ with $(z^{-1}, 1)$ for $z \in S^1$. Compute $\pi_1(K)$.

Problem 2.3. Let $X = \{(p, q) \ : \ p \neq -q\} \subset S^n \times S^n$. Define a map $f : S^n \to X$ by $f(p) = (p, p)$. Prove that f is a homotopy equivalence.



Problem 2.4. Let $\mathscr{C}$ be a category that has all coproducts and coequalizers. Prove that $\mathscr{C}$ is cocomplete (has all colimits). Deduce formally, by use of opposite categories, that a category that has all products and equalizers is complete.

Let I be a small category, and $\mathcal{C}$ be a complete category, and suppose $F : I \to \mathcal{C}$ is some functor. Consider the products in $\mathcal{C}$:

$$F_S = \prod_{i \in I} F(i) \quad \text{and} \quad F_D = \prod_{f \in I(i,j)} F(j).$$

Now consider the two maps $s, t : F_S \to F_D$ given by:

$$s(x_i)_{x_i \in F(i)} = \prod_{f \in I(i,j)} F(f)(x_i) \quad \text{and} \quad t(x_i)_{x_i \in F(i)} = \prod_{f \in I(i,j)} x_j.$$

We claim that $\lim_{i \in I} F(i) \cong \mathrm{Eq}(s,t)$. Letting $\pi_i : \prod_{k \in I} F(k) \to F(i)$ and $p_f : \prod_{f \in I(i,j)} F(j) \to F(j)$ be the natural projection maps, notice that the maps s and t are uniquely characterized by the relations $p_f \circ s = F(f) \circ \pi_i$ and $p_f \circ t = \pi_j$ for all $f \in I(i,j)$.

Now for any $i \in I$, consider the composition $r_i = \pi_i \circ \iota_{\mathrm{Eq}} : \mathrm{Eq}(s,t) \to F(i)$. By definition of equalizer, the map ι_{Eq} satisfies $s \circ \iota_{\mathrm{Eq}} = t \circ \iota_{\mathrm{Eq}}$. Thus for any function $f \in I(i,j)$, we have

$$r_j = \pi_j \circ \iota_{\mathrm{Eq}} = p_f \circ t \circ \iota_{\mathrm{Eq}} = p_f \circ s \circ \iota_{\mathrm{Eq}} = F(f) \circ \pi_i \circ \iota_{\mathrm{Eq}} = F(f) \circ r_i.$$

This identity $r_j = F(f) \circ r_i$ shows that $\mathrm{Eq}(s,t)$, together with the maps $\{r_i\}_{i \in I}$ form a cone over F.

To prove that this cone is in fact universal, suppose we had another cone Y with maps $q_i : Y \to F(i)$. By the universal property of product, this extends to a map $q : Y \to F_S$, and since it is a cone over F, it naturally follows that $s \circ q = t \circ q$. Thus by the universal property of the equalizer, q extends to a map $\iota_Y : Y \to \mathrm{Eq}(s,t)$ which commutes with s, t. This in turn means that the maps q_i satisfy $q_i = r_i \circ \iota_Y$ so we are done. Since the limit is unique up to isomorphism, we thus get:

$$\lim_{i \in I} F(i) \cong \mathrm{Eq}(s,t).$$