Physics 212 Problem Set 3

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Problem 1. Given the first two slow-roll parameters defined as

$$\epsilon = -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta = \frac{|\dot{\epsilon}|}{H\epsilon}$$

where overdots represent derivatives with respect to physical time, determine the predictions of an inflationary model with a quadratic potential $V(\phi) = m^2\phi^2$. Under the slow-roll approximation:

(a). Compute the slow-roll parameters ϵ and η in terms of ϕ.

From the Friedmann equations, we can derive the following expressions for the slow-roll parameters:

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \quad \text{and} \quad \eta = M_{pl}^2\left(\frac{V''(\phi)}{V(\phi)}\right).$$

Plugging our quadratic potential into these equations, we get

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{2m^2\phi}{m^2\phi^2}\right)^2 = \frac{2M_{pl}^2}{\phi^2} \quad \text{and} \quad \eta = M_{pl}^2\left(\frac{2m^2}{m^2\phi^2}\right) = \frac{2M_{pl}^2}{\phi^2}.$$

(b). Determine ϕ_{end}, the value of the field at which inflation ends. What is the amplitude of the potential at that point?

At the end of inflation, we have $\epsilon = 1$, which means that $\phi_{\text{end}}^2 = 2M_{pl}^2$ and hence $\phi_{\text{end}} = M_{pl}\sqrt{2}$. The amplitude of the potential at this point is $V(\phi_{\text{end}}) = 2m^2M_{pl}^2$.

(c). What is the value of the field (in units of M_{pl}) when the field is 60 e-folds away from the end of inflation? What is the amplitude of the potential at that point?

The number of e-folds between times t_i and t_f is given by the integral $N = \int_{t_i}^{t_f} H\,dt$, or in other words $dN = H\,dt$. However, the slow-roll equation of motion and Friedmann equation state

$$3H\dot{\phi} \approx -V'(\phi) \quad \text{and} \quad H^2 \approx \frac{V(\phi)}{3M_{pl}^2}.$$

These equations allow us to rewrite dN as

$$dN = H\,dt = H\frac{d\phi}{\dot{\phi}} \approx -\frac{3H^2}{V'(\phi)}\,d\phi \approx -\frac{V(\phi)}{M_{pl}^2V'(\phi)}\,d\phi.$$

Integrating both sides, we get the e-fold equation in terms of the potential

$$N = \frac{1}{M_{pl}^2} \int_{\phi_f}^{\phi_i} \frac{V(\phi)}{V'(\phi)} \, d\phi.$$

Plugging in our quadratic potential, setting $\phi_f = \phi_{\text{end}}$, and $N = 60$, we get

$$60 = \frac{1}{M_{pl}^2} \int_{M_{pl}\sqrt{2}}^{\phi_i} \frac{m^2\phi^2}{2m^2\phi} \, d\phi = \frac{1}{M_{pl}^2} \left(\frac{\phi_i^2 - 2M_{pl}^2}{4} \right) \quad \Longrightarrow \quad \phi_i = M_{pl}^2\sqrt{242} \approx 15.56\,\mathrm{M}_{\mathrm{pl}}^2.$$

At this point, the amplitude of the potential is $V(\phi_i) = 242m^2 M_{pl}$.

Problem 2. In a flat $\Omega_m = 1$ universe with no radiation, calculate the physical size of the horizon at $z = 1100$. What is the angular scale subtended by this scale today? Express your result in degrees.

In a flat matter-dominated universe, the physical horizon size is

$$d_{\text{horiz}}(z) = \frac{1}{H_0} \frac{1}{(1+z)^{3/2}}.$$

At $z = 1100$, this horizon size is

$$d_{\text{horiz}}(1100) = \frac{2 \times 4300\,\text{Mpc}}{\sqrt{1101^3}} = 0.2354\,\text{Mpc}.$$

Meanwhile, the angular distance in a flat matter-dominated universe is given by

$$d_A(z) = \frac{2}{H_0(1+z)} \left(1 - \frac{1}{\sqrt{1+z}} \right).$$

At $z = 1100$, this becomes

$$d_{\text{horiz}}(1100) = \frac{2 \times 4300\,\text{Mpc}}{1101} \left(1 - \frac{1}{\sqrt{1+1100}} \right) = 7.576\,\text{Mpc}.$$

The angular scale is the ratio of the physical horizon length by the

$$\theta_{\text{horiz}}(1100) = \frac{d_{\text{horiz}}(1100)}{d_A(1100)} = 0.03107\,\text{rad} \times \frac{180^\circ}{\pi\,\text{rad}} = 1.78^\circ.$$

Problem 3. After a GUT phase transition, we expect an abundance of monopoles corresponding to one monopole per Hubble volume at the transition time. Assume that the phase transition happens at temperature $T \sim 10^{15}$ GeV and that no mechanism exists to dilute/reduce the abundance of monopoles produced.

(a). Consider a monopole with mass $m_M = 10^{15}$ GeV and calculate the monopole density paramter today ($\Omega_{\text{mon}} = \rho_{\text{mon}}/\rho_{\text{crit}}$). Assume that the monopole density ρ_{mon} scales as $\propto T^3$ and the Hubble volume scales as $\propto H^{-3}$.

We know that the number density at the phase transition $n_{\text{mon}}^{\text{pt}} \approx 1/H^{-3} \sim H^3$. However, the Hubble factor is of order $H \sim T_{\text{pt}}^2/M_{\text{pl}}$. It follows that

$$\rho_{\text{mon}}^{\text{pt}} \approx m_M n_{\text{mon}}^{\text{pt}} \approx m_M \frac{T_{\text{pt}^6}}{M_{\text{pl}}^3}.$$

Monopoles are not relativistic by the assumption of the problem, and their number density scales with $a^{-3} \sim T^3$. Therefore, their energy density today is given by

$$\rho^0_{\text{mon}} \approx m_M n^0_{\text{mon}} \sim m_M \frac{T^6_{\text{pt}}}{M^3_{\text{pl}}} \left(\frac{T_0}{T_{\text{pt}}} \right)^3 .$$

Given that the current CMB temperature is $\approx 2.7\,\text{K} \approx 2.3 \times 10^{-13}\,\text{GeV}$, we get

$$\rho^0_{\text{mon}} \approx 1.2 \times 10^{-35}\,\text{GeV}^3 .$$

Compared to the critical density of the universe today which is $10^{-47}\,\text{GeV}^3$, we get $\Omega_{\text{mon}} \approx 1.2 \times 10^12$, an absurdly large density.

(b). There is a bound, known as the *Parker bound*, on the density of monopoles today which restricts $\Omega_{\text{mon}} < 10^{-6}$. Calculate the number of e-folds of inflation required to dilute the monopole abundance to a level consistent with the Parker bound.

To get Ω_{mon} below 10^{-6}, we need a dilution of at least $10^{12}/10^{-6} = 10^{18}$. Since $n_{\text{mon}} \propto a^{-3}$, we want $e^{-3N} \leq 10^{-18}$. Solving for the number of e-folds gives us $N \geq 13.8$ e-folds required for the dilution of monopoles to agree with observation. This is consistent with many models of inflation.