Math 55b Problem Set 12

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I collaborated with AJ LaMotta on this problem set.

Problem 1. Let D is a bounded domain with (piecewise) smooth boundary $\partial D = \gamma$, $f(z)$ an analytic function on an open set containing $\overline{D}$, and assume that f does not vanish at any point of γ. Denote by z_i the zeroes of f inside D and m_i their multiplicities. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\,dz = \sum m_i z_i.$$

First we observe that the zeroes z_i are exactly the poles of zf'/f. For any zero z_i of order m_i, there is some open neighborhood U of z_i such that $f(z) = (z-z_i)^{m_i}g(z)$ on U for some function g, analytic on U. By the quotient rule, we then have (on U)

$$\frac{zf'(z)}{f(z)} = \frac{m_i z(z-z_i)^{m_i-1}g(z) + z(z-z_i)^{m_i}g'(z)}{(z-z_i)^{m_i}g(z)} = \frac{m_i z}{z-z_i} + \frac{zg'(z)}{g(z)}.$$

Since $g \neq 0$, zg'/g is analytic on U, so by the residue theorem we have

$$\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\,dz = \sum \operatorname{Res}\left(\frac{zf'(z)}{f(z)}, z_i\right) = \sum m_i z_i.$$

Problem 2. Evaluate the following integrals by the method of residues:

$$\text{(a)} \int_0^{\pi/2} \frac{dx}{a+\sin^2 x}\ (a>1), \qquad \text{(b)} \int_0^\infty \frac{x^2\,dx}{x^4+5x^2+6}, \qquad \text{(c)} \int_0^\infty \frac{\cos x}{x^2+a^2}\,dx\ (a>0)$$

(a) Let $z = e^{ix}$, so that $\sin(x) = (z+z^{-1})/2$ and $dz = ie^{ix}dx$. Then

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = \frac{1}{4}\int_0^{2\pi} \frac{dx}{a+\sin^2 x} = \frac{1}{4}\int_{S^1} \frac{-4i\,dz}{z^4+(4a+2)z^2+1}.$$

Solving $z^4+(4a+2)z^2+1=0$, we get $z^2 = -2a-1\pm 2\sqrt{a^2+a})$. The only such z which satisfy $|z|^2 \le 1$ are the two roots $z^2 = -2a-1+2\sqrt{a^2+a}$. These roots give rise to simple poles the residue at each z_0 satisfying $z_0^2 = -2a-1+2\sqrt{a^2+a}$ is

$$\operatorname{Res}\left(\frac{z}{z^4+(4a+2)z^2+1}, z_0\right) = \lim_{z\to z_0} \frac{z(z-z_0)}{z^4+(4a+2)z^2+1} = \frac{z_0}{2z_0(z_0^2+2a+1+2\sqrt{a^2+a}} = \frac{1}{8\sqrt{a^2+a}}.$$

By the residue theorem, the integral is equal to

$$2\pi i\cdot\frac{1}{i}\left(\operatorname{Res}\left(\frac{z}{z^4+(4a+2)z^2+1}, z_0\right), \operatorname{Res}\left(\frac{z}{z^4+(4a+2)z^2+1}, -z_0\right)\right) = \frac{\pi}{2\sqrt{a^2+a}}.$$

(b) Solving $x^4 + 5x^2 + 6 = 0$, we get $x = \pm i\sqrt{2}, \pm i\sqrt{3}$. Furthermore, since $f(x)$ is an even function, we solve

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\,dx = \frac{1}{2}\left(2\pi i\left(\text{Res}\left(f, i\sqrt{2}\right) + \text{Res}\left(f, i\sqrt{3}\right)\right)\right)$$

Calculating these simple residues, we finally get

$$\begin{aligned}\int_0^\infty f(x)\,dx &= \pi i\left(\frac{(i\sqrt{3})^2}{((i\sqrt{3})^2+2)(2i\sqrt{3})} + \frac{(i\sqrt{2})^2}{((i\sqrt{2})^2+3)(2i\sqrt{2})}\right)\\ &= \pi i\left(\frac{\sqrt{3}}{2i} - \frac{\sqrt{2}}{2i}\right) = \frac{\pi(\sqrt{3}-\sqrt{2})}{2}.\end{aligned}$$

(c) Using the same tricks as in (a) and (b), we have

$$\int_0^\infty \frac{\cos(x)}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}\,dx = \frac{1}{2}\Re\left(\int_{-\infty}^\infty \frac{e^{ix}\,dx}{x^2+a^2}\right).$$

Note that $1/(x^2+a^2)$ is rational, the degree of the denominator is 2 more than the denominator of the numerator, and the denominator has no real roots, we have

$$\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx = \frac{1}{2}\Re\left(2\pi i\text{Res}\left(\frac{e^{iz}}{z^2+a^2}, ai\right)\right) = \Re\left(\pi i\frac{e^{-a}}{2ai}\right) = \frac{\pi e^{-a}}{2a}.$$

Problem 3. Use the method of residues to evaluate the integrals $\int_0^\infty \frac{\log x}{1+x^2}\,dx$ and $\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx$.

Let's consider the branch of the complex log with imaginary part in $(-\pi, \pi)$ and integrate along the contour γ, which goes from $\epsilon \in \mathbb{R}$ to $R \in \mathbb{R}$ along $\mathbb{R}$, then travels counterclockwise via an arc C_R to $-R \in R$, then from $-R$ to $-\epsilon$ along $\mathbb{R}$, and finally by a small arc C_ϵ back to ϵ. We'll see what happens as we take the limits $R \to \infty$ and $\epsilon \to \infty$. Notice that

$$\left|\int_{C_R} \frac{\log z}{1+z^2}\,dz\right| \leq \frac{\sqrt{\log(R)^2 + (\pi i)^2}}{1+R^2}(\pi R) \leq \pi\frac{|\log(R)| + \pi i}{R} \to 0$$

as $R \to \infty$ since $\log(R)/R$ and $1/R$ both tend to 0. (This result holds true in the second case as well by the same reasoning.) Next, note that

$$\left|\int_{C_\epsilon} \frac{\log z}{1+z^2}\,dz\right| \leq \frac{\sqrt{\log(\epsilon)^2 + (\pi i)^2}}{1+\epsilon^2}(\pi\epsilon) \leq \pi\frac{\epsilon(|\log(\epsilon)| + \pi i)}{1+\epsilon^2} \to 0$$

as $\epsilon \to 0$ because $\epsilon/(1+\epsilon^2)$ and $\epsilon\log(\epsilon)/(1+\epsilon^2)$ both go to zero. (Again, this result holds true in the second case by the same reasoning.) Combining these results, we et

$$\int_\gamma f(z)\,dz = \int_\epsilon^R f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{-R}^{-\epsilon} f(z)\,dz + \int_{C_\epsilon} f(z)\,dz \to \int_\infty^\infty f(z)\,dz.$$

as $R \to \infty$ and $\epsilon \to 0$. Note that we can apply the residue theorem to the left hand side in order to calculate the right side. We can now look at the cases separately.

When $f(z) = \log(z)/(1+z^2)$, we get

$$2\pi i\lim_{z\to i} f(z)(z-i) = \frac{i\pi^2}{2} = \int_0^\infty \frac{\log x}{1+x^2}\,dx + \int_{-\infty}^0 \frac{\log|x| + \pi i}{1+x^2}\,dx.$$

Using the fact that $\int_0^\infty dx/(1+x^2) = \pi/2$ and using a simple change of variables, we can then calculate

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx = \frac{1}{2}\left(\frac{i\pi^2}{2} - \pi i \int_0^\infty \frac{dx}{1+x^2}\right) = 0.$$

Now in the second case when $f(z) = \log(z)^2/(1+z^2)$ we similarly calculate

$$2\pi i \lim_{z\to i} f(z)(z-i) = -\frac{\pi^3}{4} = \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx + \int_{-\infty}^0 \frac{(\log|x| + \pi i)^2}{1+x^2}\,dx.$$

Using change of variables, as well as the integrals $\int_0^\infty \log x/(1+x^2)\,dx = 0$ and $\int_0^\infty dx/(1+x^2) = \pi/2$ we have

$$\int_0^\infty \frac{(\log|x| + \pi i)^2}{1+x^2}\,dx = \int_0^\infty \frac{(\log|x|)^2 + 2\pi i \log x - \pi^2}{1+x^2}\,dx$$

so we can solve for the desired integral to get

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{1}{2}\left(-\frac{\pi^3}{4} + \frac{\pi^3}{2}\right) = \frac{\pi^3}{8}.$$

Problem 4. Let $f(z) = \pi\cot(\pi z)$. We have seen in class that $f(z)$ has simple poles at all integers, with residues all equal to 1. Let $k \geq 1$ be a positive integer.

(a) For $n = 1, 2, \ldots$, let $R_n = \{z \in \mathbb{C},\ |\Re(z)| \leq n + \frac{1}{2} \text{ and } |\Im(z)| \leq n\}$. Show that

$$\lim_{n\to\infty} \int_{\partial R_n} \frac{f(z)}{z^{2k}}\,dz = 0.$$

(Hint: do this directly, not using residues: bound the integrand over the horizontal edges by showing that $|\cot(\pi z)| \to 1$ as $|\Im(z)| \to \infty$, and over the vertical edges by showing that $|\cot(\pi z)|$ is uniformly bounded by a constant (in fact, by 1) for all z such that $\Re(z) \in \mathbb{Z} + \frac{1}{2}$.)

(b) Use the residue theorem to show that $\mathrm{Res}_{z=0}\Big(f(z)/z^{2k}\Big) + 2\sum_{n=1}^\infty \frac{1}{n^{2k}} = 0.$

(c) By calculating the Laurent series of $f(z)$ near $z = 0$, deduce the values of $\sum_{n=1}^\infty \frac{1}{n^2}$ and $\sum_{n=1}^\infty \frac{1}{n^4}$.

(a) Recall the hyperbolic trigonometric identity

$$|\cot(x+iy)|^2 = \frac{\cosh^2(y) - \sin^2(x)}{\cosh^2(y) - \cos^2(x)}.$$

Then for all $x = \pi(n + 1/2)$ for integer n, we see that

$$|\cot(x+iy)|^2 = \frac{\cosh^2(y) - 1}{\cosh^2(y)},$$

so since $\cosh^2(y) \geq 1$ for all $y \in \mathbb{R}$, the above identities show that $|\cot(\pi z)| \leq 1$ on the vertical edges of ∂R_n. For the horizontal edges, we see that $|\cot(\pi z)| \to 1$ as $|\mathrm{Im}(z)| \to \infty$ since $\cos^2(x)$ and $\sin^2(x)$ are bounded real functions while $\cosh^2(y) \to \infty$. Now let C_1 and C_2 be the top and left edges of the boundary ∂R_n respectively. Since $|z| \geq |\Re(z)|$ and $|z| \geq |\mathrm{Im}(z)|$, we see that

$$\lim_{n\to\infty} \left|\int_{C_1} \frac{f(z)}{z^{2k}}\,dz\right| \leq \lim_{n\to\infty} \frac{\pi \sup_{C_1} |\cot(\pi z)|}{n^{2k}}(2n+1) = 0.$$

and similarly for the left edges we get

$$\lim_{n\to\infty}\left|\int_{C_2}\frac{f(z)}{z^{2k}}\,dz\right| \le \lim_{n\to\infty}\frac{\pi\sup_{C_2}|\cot(\pi z)|}{\left(n+\frac{1}{2}\right)^{2k}} = 0.$$

We can do the same thing for the bottom and right edges of ∂R_n so we can conclude that

$$\lim_{n\to\infty}\int_{\partial R_n}\frac{f(z)}{z^{2k}}\,dz = 0.$$

(b) Any nonzero integer $n \in \mathbb{Z}$ is a simple pole of $f(z)/z^{2k}$ with residue 1, so

$$\operatorname{Res}\left(\frac{f(z)}{z^{2k}}, n\right) = \lim_{z\to n}\frac{f(z)(z-n)}{z^{2k}} = \frac{1}{n^{2k}}.$$

Since the only other pole is 0, the residue theorem tells us that

$$\frac{1}{2\pi i}\int_{\partial R_N}\frac{f(z)}{z^{2k}}\,dz = \operatorname{Res}\left(\frac{f(z)}{z^{2k}}, 0\right) + 2\sum_{n=1}^{N}\frac{1}{n^{2k}}.$$

So as $N\to\infty$ and using (a) we get

$$\operatorname{Res}\left(\frac{f(z)}{z^{2k}}, 0\right) + 2\sum_{n=1}^{N}\frac{1}{n^{2k}} = 0$$

(c) First, using the Taylor series $\sin(\pi z) = \pi z - \frac{\pi^3}{6}z^3 + \frac{\pi^5}{120}z^5 + \cdots$, we then use geometric series to get

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + \frac{\pi^2}{6}z + \left(\frac{\pi^4}{36} - \frac{\pi^4}{120}\right)z^3 + \cdots = \frac{1}{z} + \frac{\pi^2}{6}z + \frac{7\pi^4}{360}z^3 + \cdots.$$

Lastly, we multiply with the Taylor series for $\cos(\pi z)$, we get

$$f(z) = \left(\frac{1}{z} + \frac{\pi^2}{6}z + \frac{7\pi^4}{360}z^3 + \cdots\right)\left(1 - \frac{\pi^2}{2}z^2 + \frac{\pi^4}{24}z^4 + \cdots\right) = \frac{1}{z} - \frac{\pi^2}{3}z - \frac{\pi^4}{45}z^3 + \cdots.$$

So the residues of 0 of f/z^2 and f/z^4 are $-\pi^2/3$ and $-\pi^4/45$. Plugging into the formula, we get

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Problem 5.

(a) Show that $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

(b) Show that, for $|z| < 1$, $\displaystyle\prod_{n=0}^{\infty}(1 + z^{2^n}) = \frac{1}{1-z}$.

(a) Rearranging some terms,

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\prod_{n=2}^{\infty}\left(\frac{(n-1)(n+1)}{n^2}\right).$$

This telescopes, with partial product equal to $(N+1)/2N$ so as $N\to\infty$, this approaches $1/2$.

(b) Let p_N be the partial product up to the N-th term of the product. Starting with the base case p_0, its clear that $p_0=1+z$. We claim that $p_N=1+z+z^2+\cdots+z^{2^{N+1}-1}$. This agrees with our base case and inductively it is clear that

$$\left(1+z+z^2+\cdots+z^{2^{N+1}-1}\right)\left(1+z^{2^{N+1}}\right)=1+z+z^2+\cdots+z^{2^{N+1}}.$$

Since $\sum z^n=1/(1-z)$ for all $|z|<1$, and since p_N is a subsequence of this, it follows that the infinite product also converges to $1/(1-z)$.

Problem 6.

(a) What is the value of $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2+a^2}$?

(b) Optional: deduce that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{\pi}{2}\coth(\pi)-\frac{1}{2}$ and $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2-\frac{1}{16}}=\frac{-4\pi}{\cos(2\pi z)}$.

(a) We can substitute $-n$ for n, so the only two cases we have is whether or not a is zero or not. If a is zero, then

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2}=\frac{\pi^2}{\sin^2(\pi z)}$$

as we've seen in class. When $a\neq 0$, we use partial fraction decomposition, writing

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2+a^2}=\frac{1}{2ai}\sum_{n\in\mathbb{Z}}\left(\frac{1}{(z-ai)-n}-\frac{1}{(z+ai)-n}\right).$$

Using the identity $\pi\cot(\pi z)-1/z=\sum_{n\in\mathbb{Z}}(1/(z-n)+1/n)$, we also write

$$\begin{aligned}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{(z+n)^2+a^2}&=\frac{1}{2ai}\left(\frac{1}{z+ai}-\pi\cot(\pi(z+ai))-\frac{1}{z-ai}+\pi\cot(\pi(z-ai))\right)\\&=-\frac{1}{z^2+a^2}+\frac{\pi^2}{2ai}\left(\cot(\pi(z-ai))-\cot(\pi(z+ai))\right)\end{aligned}$$

Now when we add the $n=0$ term back in, we get

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2+a^2}=\frac{\pi}{2ai}\left(\cot(\pi(z-ai))-\cot(\pi(z+ai))\right).$$

Of course, this only makes sense assuming $z\neq n\pm ai$.

(b) Using the identity from (a) and setting $z=0$, $a=1$, notice that

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{1}{2}\left(\sum_{n\in\mathbb{Z}}\frac{1}{n^2+1}-1\right)=\frac{1}{2}\left(\frac{\pi}{2i}(\cot(-\pi i)-\cot(\pi i)-1\right)=\frac{\pi}{2}\coth(\pi)-\frac{1}{2}.$$

Similarly setting $a=i/4$, we get

$$\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2-\frac{1}{16}}=-2\pi(\cot(\pi z+\pi/4)-\cot(\pi z-\pi/4))=\frac{-4\pi}{\cos(2\pi z)}.$$

Problem 7.

(a) Show that there exists a continuous complex valued function $F(z)$ on $\overline{\mathbb{H}} = \{\Im z \geq 0\}$ such that F is analytic on $\mathbb{H} = \{\Im z > 0\}$ and $F(x) = \int_0^x \frac{dt}{\sqrt{t(1-t^2)}}$ for all $x \in [0, 1]$. *(Hint: Find a suitable open subset $U \subset \mathbb{C}$ over which the quantity $1/\sqrt{z(1-z^2)}$ and its antiderivative are well-defined and analytic. It is helpful to choose U so that it contains as much of $\overline{\mathbb{H}}$ as possible.)*

(b) Show that $S = F(\mathbb{H})$ is the interior of a square in $\mathbb{C}$, and that $F : \mathbb{H} \to S$ is a biholomorphism (i.e., an analytic bijection with analytic inverse). *(Hint: Using the argument principle, the image of $\mathbb{H}$ under F is determined by the image of the real axis and the behavior of F near infinity. Hence, the key step is to determine $\arg(F'(z))$ on the various subintervals of the real line over which it is defined, as well as the existence of a limit of $F(z)$ as $|z| \to \infty$.)*

(a) :(

(b) :(