Math 114 Problem Set 6

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Problem 1. The following exercise illustrates the principle that the decay of $\widehat{f}$ is related to the continuity properties of f.

Let f be a continuous function on $\mathbb{R}$.

a. Suppose f is a function of moderate decrease whose Fourier transform is continuous and satisfies:

$$\widehat{f}(\zeta) = O\left(\frac{1}{|\zeta|^{1+\alpha}}\right) \qquad \text{as } |\zeta| \to \infty.$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hölder condition of order α, that is that

$$|f(x+h) - f(x)| \leq M|h|^\alpha \qquad \text{for some } M > 0 \text{ and all } x, h \in \mathbb{R}.$$

First, let's prove that $\widehat{f}$ is a function of moderate descent. First, we know by the asymptotic bound that there is some constant C and $\ell \in \mathbb{R}$ such that $|\widehat{f}(\zeta)| \leq C/|\zeta|^{1+\alpha}$ for all $|\zeta| \geq \ell$. Then assuming $\ell \geq 1$, we have

$$|\widehat{f}(\zeta)| \leq \frac{C}{|\zeta|^{1+\alpha}} = \frac{2C}{2|\zeta|^{1+\alpha}} \leq \frac{2C}{1+|\zeta|^{1+\alpha}}$$

since $|\zeta|^{1+\alpha} \geq 1$. For ζ lying inside the disk D_ℓ of radius ℓ, we have

$$|\widehat{f}(\zeta)| \leq \max_{\zeta \in D_\ell} |\widehat{f}(\zeta)| = \frac{\max_{\zeta \in D_\ell} |\widehat{f}(\zeta)|(1+\ell^{1+\alpha})}{1+\ell^{1+\alpha}} \leq \frac{\max_{\zeta \in D_\ell} |\widehat{f}(\zeta)|(1+\ell^{1+\alpha})}{1+|\zeta|^{1+\alpha}}.$$

Thus for all ζ we have

$$|\widehat{f}(\zeta)| \leq \frac{C'}{1+|\zeta|^{1+\alpha}} \quad \text{where} \quad C' = \max\left(\max_{\zeta \in D_\ell} |\widehat{f}(\zeta)|(1+\ell^{1+\alpha}), 2C\right)$$

and so $\widehat{f}$ is of moderate decrease. Using the Fourier inversion formula, we get

$$f(x+h) - f(x) = \int_{-\infty}^{\infty} \widehat{f}(\zeta) e^{2\pi i \zeta x}(e^{2\pi i \zeta h} - 1) d\zeta.$$

Since $|e^{2\pi i \zeta h} - 1| \leq 2|\sin(2\pi \zeta h)|$, this can be reexpressed as

$$\begin{aligned} |f(x+h) - f(x)| &\leq \int_{-\infty}^{\infty} 2|\widehat{f}(\zeta)||\sin(2\pi\zeta h)| d\zeta \leq 2C' \int_{-\infty}^{\infty} \frac{|\sin(2\pi\zeta h)|}{1+|\zeta|^{1+\alpha}} d\zeta \\ &\leq 4C'(2\pi|h|)^\alpha \int_0^\infty \frac{|\sin x|}{x^{1+\alpha}} dx = 4C'(2\pi)^\alpha |h|^\alpha \left(\int_0^\infty |\sin(x)|/|x|^{1+\alpha} dx\right). \end{aligned}$$

Thus, letting $M = 4C'(2\pi)^\alpha \int_0^\infty |\sin(x)|/|x|^{1+\alpha} dx$ completes the proof.

b. Suppose f vanishes for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of the origin. Prove that $\widehat{f}$ is not of moderate decrease. In fact, there is no $\epsilon > 0$ so that $\widehat{f}(\zeta) = O(1/|\zeta|^{1+\epsilon})$ as $|\zeta| \to \infty$.

Assume that there exists an $\epsilon > 0$ with $\widehat{f}(\zeta) = O(1/|\zeta|^{1+\epsilon})$ as $|\zeta| \to \infty$, say WLOG that $\epsilon < 1$. As a compactly supported continuous function, f is of moderate decrease and $\widehat{f}$ is continuous because $f \in L^1$.

By the first part, we have some $M > 0$ such that $|f(x+h) - f(x)| \leq M|h|^\epsilon$ for all $x, h \in \mathbb{R}$. In particular, we have

$$\frac{|f(h) - f(0)|}{h^\epsilon} = \frac{1}{\log(1/h)h^\epsilon} \leq M < \infty.$$

This is a contradiction, since by L'Hospital's rule we can evaluate the limit as $h \to 0$ as

$$\lim_{h\to 0}\left(\frac{h^{-\epsilon}}{\log(1/h)}\right) = \lim_{h\to 0}\left(\frac{\epsilon}{h^\epsilon}\right) = \infty.$$

Problem 2. Below is an outline of a different proof of the Weierstrauss approximation theorem. Define the *Landau* kernels by

$$L_n(x) = \begin{cases} \frac{(1-x^2)^n}{c_n} & |x| \leq 1, \\ 0 & |x| \geq 1, \end{cases}$$

where c_n is chosen so that $\int_{-\infty}^{\infty} L_n(x)\, dx = 1$. Prove that $\{L_n\}_{n\geq 0}$ is a family of good kernels as $n \to \infty$. As a result, show that if f is a continuous function supported in $[-1/2, 1/2]$, then $(f * L_n)(x)$ is a sequence of polynomials on $[-1/2, 1/2]$ which converges uniformly to f.

First of all, the fact that $\int_{-\infty}^{\infty} |L_n(x)|\, dx = 1$ by definition. So to prove that L_n are good kernels, it suffices to show for any $\mathfrak{y} > 0$ that

$$\int_{|x|>\mathfrak{y}} L_n(x)\, dx \to 0 \quad \text{as } n \to \infty.$$

Assume without loss of generality that $\mathfrak{y} < 1$. Since $(1-x^2)^n$ is an even function, so we have

$$\int_{-\infty}^{\infty} \frac{(1-x^2)^n}{c_n}\, dx = 1 \implies c_n = \int_{-1}^{1} (1-x^2)^n\, dx = 2\int_0^1 (1-x^2)^n\, dx \geq 2\int_0^1 (1-x)^n dx = \frac{2}{n+1}.$$

Since $\mathfrak{y} \leq x \leq 1$, we have $(1-x^2)^n \leq (1-\mathfrak{y}^2)^n$, so we have

$$\int_{|x|>\mathfrak{y}} L_n(x)\, dx = \frac{2}{c_n}\int_{\mathfrak{y}}^1 (1-x^2)^n\, dx \leq (n+1)\int_{\mathfrak{y}}^1 (1-\mathfrak{y}^2)^n\, dx = (n+1)(1-\mathfrak{y})(1-\mathfrak{y}^2)^n.$$

This right hand side goes to 0 as $n \to \infty$ since $|1-\mathfrak{y}^2| < 1$.

Now suppose f is a continuous function supported in $[-1/2, 1/2]$. We know that $f * L_n$ converges to f uniformly on $[-1/2, 1/2]$ because L_n are good kernels. Furthermore, we can express $f * L_n$ as a polynomial of degree at most $2n$ on the interval since $(f * L_n)^{2n+1} = 0$. (These facts can be found in Auroux's lecture notes from Math 55.)

Problem 3. Let A be a real symmetric $n \times n$ matrix whose eigenvalues are all positive. Prove that

$$\int_{\mathbb{R}^n} f(x)\, d\mu = 1, \quad f(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det A}} e^{-\left\langle x, Ax \right\rangle/2}.$$

Find the Fourier transform of the function f.

By the Cholesky decomposition theorem, A can be decomposed as a product $B^T B$ where $\det B = \sqrt{\det A}$. For any vector v, we have $v^T A v = |Av|^2$. Now if we perform a change of variables $w = Bv/\sqrt{2}$, we get

$$\int_{\mathbb{R}^n} f(v)\, dv = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|w|^2}\, dw = 1.$$

We can make the same variable change in the Fourier transform:

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \zeta \cdot v}\, dv = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|w|^2} e^{-2\pi i (\sqrt{2}(B^{-1})^T \zeta)\cdot w}\, dw.$$

Using the last integral from the proof of Lemma 4.4 in Stein Chapter 2, we can evaluate this integral as

$$\widehat{f}(\zeta) = \pi^{-n/2}(\pi^{n/2} e^{-2\pi^2 |(B^{-1})^T \zeta|^2}) = e^{-2\pi^2 \zeta^T A^{-1} \zeta}.$$

Problem 4. Find the Fourier transform of the function $f(x) = e^{-|x|}$ with $x \in \mathbb{R}$.

Basic calculus gives us:

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i \xi x} dx = \int_{-\infty}^{0} e^{(-2\pi i \xi + 1)x} dx + \int_{0}^{\infty} e^{(-2\pi i \xi - 1)x} dx \\
&= \frac{e^{(-2\pi i \xi + 1)x}}{1 - 2\pi i \xi}\bigg|_{-\infty}^{0} - \frac{e^{(-2\pi i \xi - 1)x}}{1 + 2\pi i \xi}\bigg|_{0}^{\infty} = \frac{1}{1 - 2\pi i \xi} + \frac{1}{1 + 2\pi i \xi} = \frac{2}{1 + 4\pi^2 \xi^2}.
\end{aligned}$$

Problem 5. Find the Fourier transform of the function on $\mathbb{R}^3$:

$$f(x) = \frac{1}{m^2 + |x|^2}, \quad x \in \mathbb{R}^3.$$

Consider the function on $\mathbb{R}^3$ given by $g(x) = e^{-|x|}/|x|$. Since g is invariant under the change of variables $x \mapsto e^{i\theta} x$, it suffices to compute $\widehat{g}(z, 0, 0)$ for $z \in \mathbb{R}$. Note that

$$\begin{aligned}
\widehat{g}(z,0,0) = \int_{\mathbb{R}^3} \frac{e^{-|x|}}{|x|} e^{-2\pi i x_1 z}\, dx &= \int_{-\infty}^{\infty} e^{-2\pi i x_1 z} \left(\int_{\mathbb{R}^2} \frac{e^{-\sqrt{x_1^2 + x_2^2 + x_3^3}}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\, dx_2\, dx_3 \right) dx_1 \\
&= \int_{-\infty}^{\infty} e^{-2\pi i x_1 z} \left(2\pi \int_0^{\infty} \frac{r}{\sqrt{x_1^2 + r^2}} e^{-\sqrt{x_1^2 + r^2}}\, dr \right) dx_1 \\
&= \int_{-\infty}^{\infty} e^{-2\pi i x_1 z} 2\pi e^{-|x_1|}\, dx_1 = \frac{4\pi}{1 + 4\pi^2 z^2}.
\end{aligned}$$

Let $y = 2\pi m \zeta$. Then $f(y) = \widehat{g}(\zeta)$ so by Fourier inversion and some minor calculations we get

$$\hat{f}(\xi) = h(\xi) = \frac{\pi}{|\xi|} e^{-2\pi m |\xi|}.$$