MATH 231A: ALGEBRAIC TOPOLOGY
HOMEWORK 6
DUE: WEDNESDAY, OCTOBER 19 AT 10:00PM ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Lens spaces (10 points)

Do Exercise 17.2 of LAT.

2. Problem 2: Euler characteristic of a union (5 points)

Do Exercise 18.7 of LAT.

3. Problem 3: Even maps of spheres (10 points)

A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an *even map*. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show that there exist even maps of any given even degree.

4. Problem 4: Generalized Jordan curve theorem (20 points)

The goal of this problem is to prove the following theorem, which computes the homology of the complements of embedded spheres and disks in a sphere.

(a) For an embedding[1] $h : D^k \hookrightarrow S^n$, $\widetilde{H}_*(S^n - h(D^k)) \cong 0$.

(b) For an embedding $h : S^k \hookrightarrow S^n$, $\widetilde{H}_*(S^n - h(S^k)) \cong \mathbb{Z}[n-k-1]$.

In particular, part (b) contains the Jordan curve theorem as the special case $n = 2$, $k = 1$: the complement of any embedding $S^1 \hookrightarrow S^2$ has two (path)[2] components. It also contains the following generalization: the complement of any embedding $S^{n-1} \hookrightarrow S^n$ has two (path) components.

We begin by proving (a). The proof will be by induction on k. It will be convenient to replace D^k by the cube I^k.

(i) Prove statement (a) above in the case $k = 0$.

(ii) Suppose that we are given an embedding $h : I^k \hookrightarrow S^n$ and assume that statement (a) is true for $k - 1$. Using the Mayer–Vietoris sequence, prove that if there exists a nonzero class $\alpha \in \widetilde{H}_i(S^n - h(I^k))$ then it maps to a nonzero element in $\widetilde{H}_i(S^n - h([0, \frac{1}{2}] \times I^{k-1}))$ or $\widetilde{H}_i(S^n - h([\frac{1}{2}, 1] \times I^{k-1}))$.

(iii) Conclude by iterating (ii) that there is a sequence of closed intervals $I \supset I_1 \supset I_2 \supset \dots$ where I_ℓ has length $2^{-\ell}$ such that the image of α in $\widetilde{H}_i(S^n - h(I_\ell \times I^{k-1}))$ is nonzero for all $\ell \geq 1$.

(iv) Let $\{x\} = \bigcap_{\ell=1}^{\infty} I_\ell$. Prove that the image of α in $\widetilde{H}_i(S^n - h(\{x\} \times I^{k-1}))$ is nonzero. Conclude that (a) holds by induction on k.

(v) Using (a) and the Mayer–Vietoris sequence, prove (b) by induction on k.

[1] In this context, an embedding is just a continuous injection which is a homeomorphism onto its image.

[2] There is no difference between componenets and path components here because any open subset of S^n is locally path connected.

Remark: This argument is a remarkable demonstration of the power of homology as a tool to study topology. Even if one is only interested in the statement that the complement of an embedded S^{n-1} in S^n has two connected components, it is homology that makes it possible to prove this by induction starting with the trivial case of S^0 embedded in S^n.

5. Problem 5: Invariance of domain (5 points)

Prove that if U is an open set in $\mathbb{R}^n$ and $h : U \to \mathbb{R}^n$ is a continuous injection, then the image $h(U)$ is an open set in $\mathbb{R}^n$ and h is a homeomorphism onto $h(U)$. (Hint: It suffices to prove that $h(D^n - \partial D^n)$ is open in $\mathbb{R}^n$ for any closed disk $D^n \subset U$. Enlarge $\mathbb{R}^n$ to S^n and prove this using Problem 4 above.)