Math 114 Problem Set 1

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Problem 1. The Cantor set can also be described in terms of ternary expansions:

(a) Every number in $[0,1]$ has the ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where } a_k = 0, 1, \text{ or } 2.$$

Prove that $x \in \mathcal{C}$ if and only if x has a representation as above with $a_k = 0$ or 2.

(b) The *Cantor-Lebesgue function* is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad \text{if } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \text{ where } b_k = a_k/2.$$

Here we choose the expansion from (a).

Show that F is well defined and continuous on $\mathcal{C}$, and moreover $F(0) = 0$ and $F(1) = 1$.

(c) Prove that $F : \mathcal{C} \to [0,1]$ is surjective.

(d) Extend F to a continuous function on $[0,1]$.

(a) Let's define the Cantor set recursively as follows. Define $\mathcal{C} = \bigcap_{k\geq 0} C_k$ where

$$C_k = \begin{cases} [0,1] & k = 0 \\ C_{k-1}/3 + (C_{k-1}+2)/3 & k > 0 \end{cases}.$$

For any $x \in \mathcal{C}$, define a_k to be 0 if $x \in C_{k-1}/3$ and 2 if $x \in (C_{k-1}+2)/3$. Note that by induction, since $\sum_{k=1}^{N} a_k/3^k \in C_N$ and $x \in C_N$, we have

$$\left| x - \sum_{k=1}^{N} a_k/3^k \right| < \frac{1}{3^{N+1}} \implies x = \sum_{k=1}^{\infty} a_k/3^k.$$

Clearly, such a representation is unique since the sets $C_k/3$ and $(C_k+2)/3$ are disjoint, and $C_{k+1} \subset C_k$. This proves the backward direction. For the forward direction, we'll prove the contrapositive. Suppose x only has representations which have some $a_k = 1$. Then $\sum_{i=1}^{k-1} a_i/3^i \in C_{k-1}$, yet $\sum_{i=1}^{k} a_i/3^i \in (C_{k-1}+1)/3 \not\subset \mathcal{C}$. This concludes the proof.

(b) First of all, this function is well defined by the uniqueness proof in (a). Similarly, its clear that $F(0) = 0$ and $F(1) = 1$ because $0 = 0/3^1 + 0/3^2 + \cdots$ and $1 = 2/3^1 + 2/3^3 + \cdots$. To prove continuity, first consider the metric space of binary sequences $\{0,1\}^{\mathbb{N}}$ with the metric

$$d_3(a,b) = \sum_{k=1}^{\infty} \frac{2|a_k - b_k|}{3^k}.$$

This can be easily checked to be a metric. Note that there is a canonical homeomorphism

$$\mu : (\{0,1\}^{\mathbb{N}}, d_3) \to \mathcal{C}$$

which sends a to $\sum_{k=1}^{\infty} a_k/3^k$. (Here the metric d_3 coincides with the standard Euclidean metric on $\mathcal{C}$.) Open sets in $(\{0,1\}^{\mathbb{N}}, d_3)$ are generated by "open intervals" of the form $[a_1, \dots, a_k, *, *, \cdots]$ for some fixed $[a_1, \dots, a_k]$. Next, recall that we have the basis of $[0,1]$ given by intervals of the form $(\alpha - 1/2^N, \alpha + 1/2^N)$ where $\alpha = \sum_{k=1}^{N-1} \alpha_k/2^k$. The preimage of this under F is $\mu([\alpha_1, \dots, \alpha_{N-1}, *, *, \cdots])$. This is exactly the open set $(\beta, \beta + 2/3^N + \epsilon) \cap \mathcal{C}$ where $\beta = \sum_{k=1}^{N-1} 2\alpha_k/3^k$ so F is continuous.

(c) This follows immediately from the fact that every number has at least one binary expansion.

(d) Consider the function $\widetilde{F} : [0,1] \to [0,1]$ given by

$$\widetilde{F}(x) = \begin{cases} F(x) & x \in \mathcal{C} \\ \sup_{y \le x, y \in \mathcal{C}} F(x) & \text{otherwise} \end{cases}.$$

We claim this is continuous. To prove this, note that $F(x)$ is clearly a (non-strictly) monotonically increasing function, so by definition $\widetilde{F}$ is as well. In fact, $\widetilde{F}(x)$ must be the unique monotonically increasing function extending F. Since $\widetilde{F}(x)$ is surjective and increasing, it must be continuous.

Problem 2. **(The Borel-Cantelli Lemma)** Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\} \\ &= \limsup_{k \to \infty}(E_k). \end{aligned}$$

(a) Show that E is measurable.

(b) Prove $m(E) = 0$.

(a) Observe that $E = \bigcap_{k=1}^{\infty} \bigcup_{n \ge k} E_n$. Since all of the E_k were assumed measurable, it follows that E is as well since it is a countable union/intersection of measurable sets.

(b) We have

$$m(E) = m\left(\bigcap_{k=1}^{\infty} \bigcup_{n \ge k} E_n\right) \le m\left(\bigcup_{n \ge k} E_n\right) \le \sum_{n \ge k} m(E_n).$$

Since $\sum_{k=1}^{\infty} m(E_k)$ converges, it follows that this upper bound must approach zero as $k \to \infty$, and so $m(E) = 0$.

Problem 3. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there exists a sequence c_n of positive real numbers such that

$$\frac{f_n(x)}{c_n} \to 0 \quad \text{a.e. } x.$$

First consider the function:

$$\lambda_n(c) = m\left(f_n^{-1}\left(\overline{\mathbb{R}} \setminus [-c/n, c/n]\right)\right) \quad \forall c \in \overline{\mathbb{R}}.$$

Since f_n are all measurable functions, this is well defined since $\overline{\mathbb{R}} \setminus [-c/n, c/n]$ is a Borel set. Note that $\lambda_n(0) \le m([0,1]) = 1$. Next we claim that $\lim_{c \to \infty} \lambda(c) = 0$. Note that $\{\pm\infty\} \subset \overline{\mathbb{R}}$ is a measurable set, and by assumption

$m(f_n^{-1}(\{\pm\infty\})) = 0$. Since $\overline{\mathbb{R}} = \{\pm\infty\} \cup \bigcup_c [-c/n, c/n]$, it follows that $\lim_{c\to\infty} m(f_n^{-1}([-c/n, c/n])) = 1$. Then

$$\lim_{c\to\infty} \lambda(c) = \lim_{c\to\infty} m(f_n^{-1}(\overline{\mathbb{R}} \setminus [-c/n, c/n])) = \lim_{c\to\infty} \left(1 - m(f_n^{-1}([-c/n, c/n]))\right) = 0$$

as desired. Now for each n, choose some c_n such that $\lambda(c_n) < 2^{-n}$. Let $E_k = f_n^{-1}(\overline{\mathbb{R}} \setminus [-c_n/n, c_n/n])$ so that $m(E_k) = \lambda(c_n) < 2^{-n}$. Note that

$$\sum_{k=1}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty$$

so we can apply the Borel-Cantelli lemma. Let $E = \limsup_{k\to\infty} E_k$. Then $x \in E$ if $|f_n(x)/c_n| > 1/n$ for infinitely many n so $x \in [0,1] \setminus E$ if $|f_n(x)/c_n| \leq 1/n$ for infinitely many n. This means that $f_n(x)/c_n \to 0$ for every point $x \in [0,1] \setminus E$. Since $m(E) = 0$ by the Borel-Cantelli lemma we have $m([0,1] \setminus E) = 1$, completing the proof.