Math 230a Problem Set 11

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Problem 2. Consider Chern-Simons-Weil forms for $G = \mathrm{U}_1$ the circle group of unit norm complex numbers. The Lie algebra is $\mathfrak{g} = i\mathbb{R}$.

(a). For each $k \in \mathbb{Z}^{>0}$ identify the vector space of degree k symmetric G-polynomials on $\mathfrak{g}$.

Since U_1 is abelian, the adjoint representation is trivial. Now as $\mathfrak{u}_1 = i\mathbb{R}$, the vector space of degree k symmetric G-invariant polynomials is just the space of symmetric polynomials

$$\mathbb{R}[-ix_1, \ldots, -ix_k]^{\mathfrak{S}_k}.$$

(b). If $\pi : P \to X$ is a principal G-bundle and h is a G-invariant polynomial of degree k, then the Chern-Simons form restricts to a closed $(2k-1)$-form on each fiber of π, and this form may be identified with a bi-invariant form on G. Identify this form for $k = 1$. For which h is the integral of this form an integer?

Recall from the notes that the pullback of the Chern-Simons form α to a fiber is given by

$$\iota_x^*\alpha = c_k \cdot h(\theta_{P_x} \wedge [\theta_{P_x} \wedge \theta_{P_x}]^{\wedge(k-1)}) \quad \text{where} \quad c_k = k\int_0^1 dt \left(\frac{t^2 - t}{2}\right)^{k-1}$$

and θ_{P_x} is the Maurer-Cartan form. In out case of $k = 1$, we have $c_k = 1$ and $h(x) = -i\kappa \cdot x$ so the restricted Chern-Simons form is $\iota_x^*\alpha = -i\kappa\theta_{P_x}$. This can be identified with the bi-invariant form $-i\kappa \cdot \theta_{U_1} = \kappa\, d\theta \in \Omega^1(\mathrm{U}_1)$. The integral of this form is

$$\int_{P_x} \iota_x^*\alpha = \int_{\mathrm{U}_1} -i\kappa \cdot \theta_{\mathrm{U}_1} = \int_0^{2\pi} \kappa \cdot d\theta = 2\pi\kappa.$$

For this to be an integer, the polynomial h must take the form

$$h(x) = \frac{n}{2\pi i} \cdot x \quad \text{for} \quad n \in \mathbb{Z}.$$

(c). Consider the Hopf bundle $\pi : S^3 \to S^2$, which is a principal G-bundle. Construct a connection. Compute the integral of the Chern-Weil form over S^2. For which h is this an integer?

Recall that SU_2 consists of unitary complex 2×2 matrices with determinant 1, and the Lie algebra $\mathfrak{su}_2$ consists of 2×2 skew-Hermitian traceless matrices. We can explicitly parametrize them by

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \begin{array}{l} \alpha,\beta \in \mathbb{C} \\ |\alpha|^2 + |\beta|^2 = 1 \end{array} \right\} \quad \text{and} \quad \mathfrak{su}_2 = \left\{ \begin{pmatrix} -i\theta & \overline{z} \\ -z & i\theta \end{pmatrix} : \theta \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

The Lie algebra $\mathfrak{su}_2$ can be given a basis of Pauli matrices

$$\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Note that the commutators are $[\sigma_1,\sigma_2] = -\sigma_3$, $[\sigma_2,\sigma_3] = -\sigma_1$, and $[\sigma_3,\sigma_1] = -\sigma_2$. We include the normalization factor of $1/\sqrt{2}$ so that we have

$$\mathrm{Tr}(\sigma_1^2) = \mathrm{Tr}(\sigma_2^2) = \mathrm{Tr}(\sigma_3^2) = 1$$

or in other words, $\sigma_1, \sigma_2, \sigma_3$ is an orthonormal basis for $\mathfrak{su}_2$ under the inner product $\langle X, Y\rangle = \mathrm{Tr}(X^\dagger Y)$.

Now let's construct a connection $\Theta \in \Omega^1(\mathrm{SU}_2; \mathfrak{u}_1)$ on $\pi : S^3 \to S^2$. Let's start with the Maurer-Cartan form $\theta_{\mathrm{SU}_2} \in \Omega^1(\mathrm{SU}_2; \mathfrak{su}_2)$. The Maurer-Cartan form can be written as $\theta_{\mathrm{SU}_2} = \theta^1\sigma_1 + \theta^2\sigma_2 + \theta^3\sigma_3$ for some forms $\theta^i \in \Omega^1(\mathrm{SU}_2)$. These forms must satisfy the Maurer-Cartan equations:

$$d\theta^i + \frac{1}{2}\sum_{j,k} c^i_{jk}\theta^j \wedge \theta^k = 0 \quad \Longrightarrow \quad \begin{aligned} d\theta^1 &= \theta^2 \wedge \theta^3 \\ d\theta^2 &= \theta^3 \wedge \theta^1 \\ d\theta^3 &= \theta^1 \wedge \theta^2 \end{aligned}$$

Let's split $\mathfrak{su}_2 = \mathfrak{m} \oplus \mathfrak{u}_1$ by letting $\mathfrak{m}$ be the span of σ_1 and σ_2, and with $\mathfrak{u}_1$ the span of σ_3. This is a reductive structure on the symmetric space $\mathrm{SU}_2 / \mathrm{U}_1$. Let's now let Θ be the projection of θ_{SU_2} onto $\mathfrak{u}_1$. We can thus write $\Theta = \theta^3\sigma_3 = i\cdot\theta^3$. Since U_1 is abelian, the curvature is

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta] = d\Theta = i(d\theta^3) = i\cdot(\theta^1 \wedge \theta^2) \in \Omega^2(\mathrm{SU}_2; \mathfrak{u}_1).$$

Note that all Chern-Weil forms vanish for $k > 1$, since $\Omega \wedge \Omega = 0$, as it is a 4-form on the 3-manifold SU_2. When $k = 1$, a polynomial $h \in (\mathrm{Sym}^k \mathfrak{u}_1^*)^{\mathrm{U}_1}$ can be written as $h(x) = -i\kappa \cdot x$ for some $c_k \in \mathbb{R}$. In this case, the Chern-Weil form is

$$\omega = h(\Omega) = \kappa \cdot (\theta^1 \wedge \theta^2).$$

This is a closed 2-form on SU_2 since $\omega = d(\kappa\cdot\theta^3)$. It also must descend to some 2-form $\widetilde{\omega} \in \Omega^2(S^2)$, i.e. we have $\omega = \pi^*\widetilde{\omega}$. We would like to calculate the integral of $\widetilde{\omega}$ over S^2. It's clear that $\widetilde{\omega}$ is an area form on S^2, so all that remains is to compute the scale factor with respect to the standard area form on the unit sphere.

Let's first write the Hopf fibration as

$$\pi \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} 2\cdot\Re(\alpha\overline{\beta}) \\ 2\cdot\Im(\alpha\overline{\beta}) \\ |\alpha|^2 - |\beta|^2 \end{pmatrix} \in S^2 \subset \mathbb{R}^3$$

Writing $\alpha = x_1 + iy_1$ and $\beta = x_2 + iy_2$, the differential of this map now takes the form

$$d\pi_{(\alpha,\beta)} = \begin{pmatrix} 2x_2 & 2y_2 & 2x_1 & 2y_1 \\ -2y_2 & 2x_2 & 2y_1 & -2x_1 \\ 2x_1 & 2y_1 & -2x_2 & -2y_1 \end{pmatrix}.$$

The differential of the basis for $T_e\,\mathrm{SU}_2$ give by the Pauli matrices thus becomes:

$$d\pi_{(1,0)}\sigma_1 = \begin{pmatrix} -\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad d\pi_{(1,0)}\sigma_2 = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}, \quad d\pi_{(1,0)}\sigma_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since these tangent vectors have length $\sqrt{2}$, it follows that $\theta^1 \wedge \theta^2$ descends to $(1/\sqrt{2})(1/\sqrt{2})dA$ where dA is the area form on the unit sphere S^2. In particular, this implies that

$$\int_{S^2} \widetilde{\omega} = \frac{\kappa}{2} \int_{S^2} dA = \kappa \cdot 2\pi.$$

Thus, the only polynomials for which the integral is an integer are

$$h(x) = \frac{n}{2\pi i} \cdot x \quad \text{for} \quad n \in \mathbb{Z}.$$

(d). Now consider $k = 2$, so the Chern-Weil form has degree 4. Construct a nontrivial principal G-bundle over $S^2 \times S^2$ by first taking the Cartesian product of the Hopf bundle with itself to form a principal $(G \times G)$-bundle over $S^2 \times S^2$, then use the homomorphism $G \times G \to G$ to form the associated principal G-bundle. Compute the integral of the Chern-Weil form. For which h is the answer an integer?

Let's begin with the principal $(\mathrm{U}_1 \times \mathrm{U}_1)$-bundle $\pi \times \pi : \mathrm{SU}_2 \times_{S^2 \times S^2} \mathrm{SU}_2 \to S^2 \times S^2$ – for brevity let's call the total space E. Let $\pi_1, \pi_2 : E \to \mathrm{SU}^2$ be respective projection maps, and let $\widetilde{\pi}_i = \pi \circ \pi_i$ be the composition with the Hopf fibration. The induced connection on this bundle can be written as

$$\Theta = (i \cdot \pi_1^* \theta^3) \oplus (i \cdot \pi_2^* \theta^3) \in \Omega^1(E; \mathfrak{u}_1 \oplus \mathfrak{u}_1)$$

By the results of the previous problem, the curvature of this connection is then

$$\Omega = (i \cdot \widetilde{\pi}_1^* \, dA) \oplus (i \cdot \widetilde{\pi}_2^* \, dA) \in \Omega^2(E; \mathfrak{u}_1 \oplus \mathfrak{u}_1),$$

where dA is the area form on S^2.

Now we want to understand the pushforward of Θ to the principal U_1-bundle $E' = E \times_{\mathrm{U}_1 \times \mathrm{U}_1} \mathrm{U}_1$ associated to E under the multiplication homomorphism $\mu : \mathrm{U}_1 \times \mathrm{U}_1 \to \mathrm{U}_1$. The differential $\mu_* : \mathfrak{u}_1 \oplus \mathfrak{u}_1 \to \mathfrak{u}_1$ is the addition map. Let $\overline{\theta}_i^j \in \Omega^1(E')$ be the pushforwards of $\theta_j^i \in \Omega^1(E)$ to the associated bundle. Then, the connection and curvature of the associated bundle is

$$\Theta' = i \cdot (\overline{\theta}_1^3 + \overline{\theta}_2^3) \quad \text{and} \quad \Omega' = 2i \cdot (\overline{\theta}_1^1 \wedge \overline{\theta}_1^2 + \overline{\theta}_2^1 \wedge \overline{\theta}_2^2).$$

A polynomial $h \in (\mathrm{Sym}^2 \mathfrak{u}_1^*)^{\mathrm{U}_1}$ can be written as $h(x) = -\kappa \cdot xy$ so the Chern-Weil form is

$$\begin{aligned} \omega = h(\Omega' \wedge \Omega') = -\kappa \cdot \Omega' \wedge \Omega' &= 4\kappa \cdot (\overline{\theta}_1^1 \wedge \overline{\theta}_1^2 + \overline{\theta}_2^1 \wedge \overline{\theta}_2^2)(\overline{\theta}_1^1 \wedge \overline{\theta}_1^2 + \overline{\theta}_2^1 \wedge \overline{\theta}_2^2) \\ &= 8\kappa \cdot \overline{\theta}_1^1 \wedge \overline{\theta}_1^2 \wedge \overline{\theta}_2^1 \wedge \overline{\theta}_2^2. \end{aligned}$$

Recall that $\overline{\theta}_i^1 \wedge \overline{\theta}_i^2$ descends to a form $\widetilde{\omega}_i$ on the i-th S^2 term in $S^2 \times S^2$ with total integral 2π. Since ω descends to $8\kappa \cdot \widetilde{\omega}_1 \wedge \widetilde{\omega}_2$ on $S^2 \times S^2$, we can compute

$$\int_{S^2 \times S^2} 8\kappa \cdot \widetilde{\omega}_1 \wedge \widetilde{\omega}_2 = 8\kappa \cdot \left(\int_S^2 \widetilde{\omega}_1 \right)^2 = 32\pi^2 \kappa.$$

Thus, the only h for which the integral is an integer are

$$h(x) = -\frac{n}{32\pi^2} \cdot x^2 \quad \text{for} \quad n \in \mathbb{Z}.$$

(e). Continuing with $k = 2$, take the base 4-manifold to be $\mathbb{CP}^2$. For which h do you find an integer when you integrate the Chern-Weil form?

In this case, the canonical bundle to consider is the generalized Hopf fibration

$$\mathrm{U}_1 \to S^5 \to \mathbb{CP}^2$$

where the 5-sphere S^5 is considered as the space of triples of complex numbers in $\mathbb{C}^3$ with norm 1. The map sends such a triple (α, β, γ) to the same homogeneous coordinates $[\alpha : \beta : \gamma]$.

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Problem 3. Now consider $G = \mathrm{SU}_2$.

(a). Identify the space of G-invariant polynomials of degree 2 on $\mathfrak{g}$.

Let's use the same bases from the previous problem. First, note that the adjoint representation of SU_2 on $\mathfrak{su}_2$ factors through the double cover $\mathrm{SU}_2 \to \mathrm{SO}_3$. This immediately implies that there are no nonzero SU_2-invariant polynomials of degree $k = 1$. In the next case of $k = 2$, this is the space of symmetric bilinear forms on $\mathfrak{su}_2$ which are invariant under the action of SO_3. Up to scaling, the only such bilinear form is $\langle X, Y \rangle = \mathrm{Tr}(X^\dagger Y)$. Thus, we can write any $h \in (\mathrm{Sym}^2\, \mathfrak{su}_2^*)^{\mathrm{SU}_2}$ as $h(X \wedge Y) = \kappa \cdot \mathrm{Tr}(X^\dagger Y)$ for some $\kappa \in \mathbb{R}$. (Technically this Hermitian transpose is unnecessary since all elements in $\mathfrak{su}_2$ are skew-Hermitian and so this is a scalar multiple of the bilinear form $\mathrm{Tr}(XY)$)

(b). Write an explicit formula for the Chern-Simons 3-form of a connection. Use matrix multiplication in your formula rather than the Lie bracket.

If $h \in (\mathrm{Sym}^2\, \mathfrak{su}_2^*)^{\mathrm{SU}_2}$, then the Chern-Simons 3-form of a connection $\Theta \in \Omega^1(P; \mathfrak{su}_2)$ is:

$$\mathrm{CS}_3(\Theta) = 2\int_0^1 dt\, h(\Theta \wedge \Omega_t) \quad \text{where} \quad \Omega_t = t\, d\Theta + \frac{t^2}{2}[\Theta \wedge \Theta].$$

Since h is a symmetric bilinear form, we get

$$\mathrm{CS}_3(\Theta) = 2\int_0^1 t \cdot h(\Theta \wedge d\Theta) + \frac{t^2}{2} \cdot h\left(\Theta \wedge [\Theta \wedge \Theta]\right)\, dt = h(\Theta \wedge d\Theta) + \frac{1}{3} h(\Theta \wedge [\Theta \wedge \Theta])$$

Let's suppose $h(X \wedge Y) = \mathrm{Tr}(X^\dagger Y)$, since all other forms under consideration are scalar multiples of this bilinear form. Since $\mathfrak{su}_2$ is skew-hermitian, it follows that $h(\Theta \wedge d\Theta) = \mathrm{Tr}(\Theta^\dagger d\Theta) = -\mathrm{Tr}(\Theta d\Theta)$. For the term involving the Lie bracket, we write $\Theta = \theta^i \sigma_i$ where σ_i are the Pauli matrices defined in the previous problem. Then by applying the commutator rules we get

$$h(\Theta \wedge [\Theta \wedge \Theta]) = \theta^k \wedge \theta^i \wedge \theta^j\, \mathrm{Tr}(\sigma_k^\dagger [\sigma_i, \sigma_j]) = 6 \cdot \theta^1 \wedge \theta^2 \wedge \theta^3.$$

Similarly, expanding the trace of a power of the connection, we get

$$\mathrm{Tr}(\Theta^3) = \theta^i \wedge \theta^j \wedge \theta^k\, \mathrm{Tr}(\sigma_i \sigma_j \sigma_k) = \theta^1 \wedge \theta^2 \wedge \theta^3.$$

Combining everything together, we get the Chern-Simons 3-form of an SU_2-connection solely in terms of matrix multiplication:

$$\mathrm{CS}_3(\Theta) = \mathrm{Tr}(\Theta^\dagger d\Theta + \Theta^\dagger [\Theta \wedge \Theta]/3) = \mathrm{Tr}(-\Theta d\Theta) + 2\,\mathrm{Tr}(\Theta^3) = \mathrm{Tr}(2\Theta^3 - \Theta d\Theta).$$

Problem 4. Let $\Sigma \subset \mathbb{E}^3$ be a closed surface that is a submanifold of Euclidean 3-space. Prove that Σ has a point of positive Gauss curvature.

Since Σ is compact, it must be bounded by a sphere in $\mathbb{E}^3$. Without loss of generality, we can shrink the sphere so that it intersects Σ tangentially at some point $p \in \Sigma$. Note that a non-tangential intersection point of this type is impossible, since the sphere would no longer bound the surface. Then the Gauss curvature at this point K_p must be positive, since otherwise following a geodesic path would allow one to reach points of Σ outside of the sphere – a contradiction.

Problem 5. Let G be a Lie group, let $H \subset G$ be a closed Lie subgroup, and assume that the homogeneous manifold G/H has a reductive structure, i.e. an H-invariant splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ as vector spaces.

(a). Recall the definition of the canonical connection on the principal H-bundle $\pi : G \to G/H$.

A connection on π is a form $\Theta \in \Omega^1(G; \mathfrak{h})$. The canonical choice is to start with the Maurer-Cartan form $\theta_G \in \Omega^1(G; \mathfrak{g})$ and set $\Theta = \pi_{\mathfrak{h}} \theta_G$ where $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ is the projection with kernel $\mathfrak{p}$.

(b). Compute its curvature.

The Maurer-Cartan equation states that

$$d\theta_G + \frac{1}{2}[\theta_G \wedge \theta_G] = 0.$$

Combining this with the curvature of the connection and the facts that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$, we get:

$$\begin{aligned}
\Omega &= d\Theta + \frac{1}{2}[\Theta \wedge \Theta] \\
&= d\pi_{\mathfrak{h}}\theta_G + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= \pi_{\mathfrak{h}} d\theta_G + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[\theta_G \wedge \theta_G] + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[(\pi_{\mathfrak{h}}\theta_G + \pi_{\mathfrak{p}}\theta_G) \wedge (\pi_{\mathfrak{h}}\theta_G + \pi_{\mathfrak{p}}\theta_G)] + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] - \frac{1}{2}\pi_{\mathfrak{h}}[\pi_{\mathfrak{p}}\theta_G \wedge \pi_{\mathfrak{p}}\theta_G] + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \qquad (\text{since } \pi_{\mathfrak{h}}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{p}}\theta_G] = 0) \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[\pi_{\mathfrak{p}}\theta_G \wedge \pi_{\mathfrak{p}}\theta_G].
\end{aligned}$$

(c). What is the meaning of the torsion of this connection? Compute it.

The torsion of a connection on a principal bundle associated to a reductive homogenous space G/H is the obstruction to horizontals being integrable. Let $\omega = \pi_{\mathfrak{p}}\theta_G \in \Omega^1(G; \mathfrak{p})$ be the solder form. Note that it is horizontal and H-equivariant descends to a form in $\Omega^1(G/H; \mathfrak{p})$. Recall that the torsion is given by

$$\tau = d\omega + [\Theta \wedge \omega] = d(\pi_{\mathfrak{p}}\theta_G) + [\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{p}}\theta_G].$$

By the Muarer-Cartan equations, this simplifies to

$$\tau = -\frac{1}{2}\pi_{\mathfrak{p}}[\pi_{\mathfrak{p}}\theta_G, \pi_{\mathfrak{p}}\theta_G].$$

In particular, symmetric spaces admit a canonical torsion-free connection.

(d). What is the meaning of geodesics of this connection? Compute them.

Any curve $\gamma : \mathbb{R} \to G/H$ can be lifted to a curve $\widetilde{\gamma} : \mathbb{R} \to G$. Geodesic curves must satisfy $\pi_{\mathfrak{h}}\theta_G(\widetilde{\gamma}'(t)) = 0$. The only explicit solutions to this are $\widetilde{\gamma}(t) = \exp(tX)$ for some $X \in \mathfrak{p}$. Thus every geodesic of the canonical connection on G/H must be of the form $\gamma(t) = \exp(tX)H$ for some $X \in \mathfrak{p}$.

(e). Consider the transitive action of $G \times G$ on G by left and right multiplication. Is this homogeneous space reductive?

The action can be defined as $(g,h) \cdot q = gqh^{-1}$. Note that the stabilizer of the identity is the diagonal $\Delta \subset G \times G$ with Lie algebra the diagonal $\Delta_{\mathfrak{g}} \subset \mathfrak{g} \oplus \mathfrak{g}$. The complement can be given by

$$\mathfrak{p} = \{(\xi, -\xi) : \xi \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}.$$

It should be clear that $\mathfrak{p}$ is an invariant complement so the homogeneous space is indeed reductive.

Problem 7. Let G be a Lie group with a left-invariant Riemannian metric, denoted $\langle -, - \rangle$. Suppose ξ, η, ν, ζ are left-invariant vector fields on G.

(a). Prove that

$$\nabla_\xi \eta = \frac{1}{2}([\xi, \eta] - \operatorname{ad}_\xi^* \eta - \operatorname{ad}_\eta^* \xi),$$

where the star denotes the adjoint with respect to the inner product on the Lie algebra.

Recall that the Levi-Civita covariant derivative satisfies the Koszul formula, which here states

$$2\langle \nabla_\xi \eta, \nu \rangle = \langle [\xi, \eta], \nu \rangle - \langle [\xi, \nu], \eta \rangle - \langle [\eta, \nu], \xi \rangle$$

since all of the terms of the form $\xi \langle \eta, \nu \rangle$ vanish due to left-invariance. Recall that the adjoint operator behaves as $\operatorname{ad}_X(Y) = [X, Y]$, so we can write

$$\begin{aligned}
2\langle \nabla_\xi \eta, \nu \rangle &= \langle [\xi, \eta], \nu \rangle - \langle \operatorname{ad}_\xi \nu, \eta \rangle - \langle \operatorname{ad}_\eta \nu, \xi \rangle \\
2\langle \nabla_\xi \eta, \nu \rangle &= \langle [\xi, \eta], \nu \rangle - \langle \nu, \operatorname{ad}_\xi^* \eta \rangle - \langle \nu, \operatorname{ad}_\eta^* \xi \rangle \\
\langle \nabla_\xi \eta, \nu \rangle &= \frac{1}{2}\Big(\langle [\xi, \eta], \nu \rangle - \langle \nu, \operatorname{ad}_\xi^* \eta \rangle - \langle \nu, \operatorname{ad}_\eta^* \xi \rangle\Big) \\
\nabla_\xi \eta &= \frac{1}{2}\Big([\xi, \eta] - \operatorname{ad}_\xi^* \eta - \operatorname{ad}_\eta^* \xi\Big)
\end{aligned}$$

(b). Derive a formula for $\langle R(\xi, \eta)\nu, \zeta \rangle$, where R is the Riemann curvature tensor.

Similar tedious application of formulas as in the last part gives

$$\begin{aligned}
\langle R(\xi, \eta)\nu, \zeta \rangle = &\frac{1}{4}\left(\langle [\xi, \eta], [\nu, \zeta] \rangle - \langle [\xi, \nu], [\eta, \zeta] \rangle - \langle [\eta, \nu], [\xi, \zeta] \rangle\right) \\
&+ \langle B(\xi, \eta), B(\nu, \zeta) \rangle - \langle B(\xi, \nu), B(\eta, \zeta) \rangle - \langle B(\eta, \nu), B(\xi, \zeta) \rangle,
\end{aligned}$$

where $B(X, Y) = (\operatorname{ad}_X^* Y + \operatorname{ad}_Y^* X)/2$.

(c). Derive a formula for $\langle R(\xi, \eta)\eta, \xi \rangle$, which is the sectional curvature.

The previous problem then gives us

$$\langle R(\xi, \eta)\eta, \xi \rangle = \frac{1}{4}\left(\langle [\xi, \eta], [\eta, \xi] \rangle - \langle [\xi, \eta], [\eta, \xi] \rangle\right) + \langle B(\xi, \eta), B(\eta, \xi) \rangle - \langle B(\xi, \eta), B(\eta, \xi) \rangle = 0.$$

(d). How do your formulas simplify for bi-invariant Riemannian metrics?

In the first formula, we have $\mathrm{ad}_\xi^* = -\mathrm{ad}_x i$ and so $B = 0$. Thus, the formula simplifies to

$$\langle R(\xi, \eta)\nu, \zeta \rangle = \frac{1}{4} \left(\langle [\xi, \eta], [\nu, \zeta] \rangle - \langle [\xi, \nu], [\eta, \zeta] \rangle - \langle [\eta, \nu], [\xi, \zeta] \rangle \right).$$