Math 262a Problem Set 2

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Due: Wednesday, September 20

Problem 1. Let N be a smooth manifold of dimension $n = 2m$ and suppose $\omega \in \Omega^2(N)$.

a. Prove that ω is non-degenerate at each point of N if and only if $\omega^{\wedge m} \in \Omega^{2m}(N)$ is everywhere nonzero.

b. Now assume that ω is a symplectic form. Prove that the subspace $\mathfrak{X}^\omega(N) \subset \mathfrak{X}(N)$ of symplectic vector fields is closed under the Lie bracket of vector fields.

c. Define a Lie algebra structure on $\Omega^0(N)$ by transport from the Lie bracket on $\mathfrak{X}^\omega(N)$ via the symplectic gradient map $f \mapsto \zeta_f$. Prove the formula:

$$\{f, g\} = \zeta_f \cdot g = \omega(\zeta_f, \zeta_g), \quad f, g \in \Omega^0(N).$$

d. A *Hamiltonian vector field* is a symplectic gradient of a function. Is the subspace of Hamiltonian vector fields invariant under the Lie bracket?

Problem 2. Let V be a real symplectic vector space, and suppose A is an affine space over V. Feel free to work with the model spaces.

a. Define the space of affine polynomial functions on A and prove that it is closed under the Poisson bracket on smooth functions.

b. Prove that the space of affine linear functions is also invariant. Do you recognize the resulting Lie algebra?

c. Prove that the space of affine quadratic functions is also invariant. Do you recognize the resulting Lie algebra?

d. Does the pattern continue?

Problem 3. An almost symplectic structure on a smooth manifold N is a section of $\mathrm{Symp}(TN) \to N$. An almost complex structure on a smooth manifold N is a section I of $\mathrm{End}(TN) \to TN$ such that $I^2 = -1$.

a. Prove that an almost symplectic manifold admits an almost complex structure.

b. Prove that S^4 does not admit an almost symplectic structure.

Problem 4. Let (M, g) be a Riemannian manifold and suppose $V : M \to \mathbb{R}$ is a smooth potential function. Fix some mass $m > 0$. Now let $\mathcal{F} = C^\infty(\mathbb{R}, M)$ be the infinite dimensional manifold of particle trajectories. Consider the Lagrangian:

$$L = \left\{ \frac{m}{2} \langle \dot{x}, \dot{x} \rangle - V(x) \right\} |dt|$$

where $x : \mathcal{F} \times \mathbb{R} \to M$ is the evaluation map. Compute the Euler-Lagrange equations, the symplectic form, and the Hamiltonian.

Problem 5. Define the notion of a symmetry of a general mechanical system. Does a symmetry have to preserve the direction of time or can a symmetry reverse time flow?