Math 114 Problem Set 3

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Problem 6. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(a) There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

(b) However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim\limits_{|x|\to\infty} f(x) = 0$.

(a) For any interval $I = (a,b) \subset \mathbb{R}$ and $h > 0$, pick some function $F_{I,h}$ which is nonzero only inside (a,b) and satisfies

$$F_{I,h}\left(\frac{b-a}{2}\right) = h \quad \text{and} \quad \int_{\mathbb{R}} F_{I,h}\, dm = \int_I F_{I,h}\, dm = \frac{h\cdot m(I)}{2}.$$

In other words, $F_{I,h}$ is some sort of "pointy" function on I which touches $y = h$. Now letting f be:

$$f = \sum_{n=1}^{\infty} F_{(n,n+1/n^3),2n} \implies \int_{\mathbb{R}} f\, dm = \sum_{n=1}^{\infty}\int_{\mathbb{R}} F_{(n,n+1/n^3),2n} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

by monotone convergence. This means that f is integrable, yet this function is unbounded as $x \to \infty$ so $\limsup_{x\to\infty} f(x) = \infty$.

(b) We can assume WLOG that $f \geq 0$, since if $\int_{\mathbb{R}} f\, dm < \infty$ then $\int_{\mathbb{R}} |f|\, dm < \infty$ and if $\lim\limits_{|x|\to\infty} |f| = 0$ then $\lim\limits_{|x|\to\infty} f = 0$. We can also split the integral in half, so that we only need to consider the behavior of f as $x \to \infty$. Suppose for the sake of contradiction that $\lim\limits_{x\to\infty} f(x) \neq 0$, so there exists some sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim\limits_{n\to\infty} f(x_n) = \ell$ for some $\ell > 0$. We can assume WLOG that $\{x_n\}$ is increasing, and that $|x_{n+1} - x_n| > \epsilon$ for all n and for some ϵ. (i.e. x_n doesn't cluster around any point) Then there must exist some N such that for all $n \geq N$ we have $|f(x_n) - \ell| < \ell/4$. Since f is uniformly continuous, there must exist some δ such that $|x_n - x| < \delta \implies |f(x_n) - f(x)| < \ell/4$. This latter expression is equivalent to $|f(x) - \ell| < \ell/2$. So putting this all together, we get

$$\int_{x_n-\delta}^{x_n+\delta} f(x)\, dm > \delta\ell,\ \forall n \geq N.$$

However this is a contradiction to the assumption that $\int_{\mathbb{R}} f(x)\, dm < \infty$.

Problem 9. Tchebychev Inequality. Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha}\int_{\mathbb{R}} f\, dm.$$

Observe that $\alpha\chi_{E_\alpha} \leq f$. This means that $\int \alpha\chi_{E_\alpha} \leq \int f$ and so we get

$$m(E_\alpha) \leq \frac{1}{\alpha}\int_{\mathbb{R}} f\, dm.$$

Problem 12. Show that there are $f \in L^1(\mathbb{R}^d)$ and a sequence $\{f_n\}$ with $f_n \in L^1(\mathbb{R}^d)$ such that

$$\|f - f_n\|_{L^1} \to 0,$$

but $f_n(x) \to f(x)$ for no x.

Let $\{q_r\}_{r=1}^{\infty}$ be an enumeration of the rationals $\mathbb{Q}$ in $\mathbb{R}$. Consider the sequence of functions

$$f_n = \chi_{I_n} \quad \text{where} \quad I_n = (q_n - 2^{-n}, q_n + 2^{-n}).$$

These functions are clearly L^1, with $\|f_n\|_{L^1} = 2^{-n+1}$. Then if $f = 0$, we have $\|f - f_n\|_{L^1} = 2^{-n+1} \to 0$, yet $f_n(x) \not\to f(x)$ for any x since the rationals are dense in $\mathbb{R}$ so for any $x \in \mathbb{R}$ there will be an infinite number of n for which $f_n(x) = 1$.

Problem 15. Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\widetilde{F}$ that agrees with F a.e is unbounded in any interval.

Since $2^{-n} f(x - r_n) \geq 0$, by monotone convergence we have

$$\begin{aligned} \int_{\mathbb{R}} F \, dm = \int_{\mathbb{R}} \sum_{n=1}^{\infty} 2^{-n} f(x - r_n) \, dm = \sum_{n=1}^{\infty} 2^{-n} \int_{\mathbb{R}} f(x - r_n) \, dm &= \sum_{n=1}^{\infty} 2^{-n} \int_{(0,1)} x^{-1/2} \, dm \\ &= \sum_{n=1}^{\infty} 2^{-n+1} = 2 < \infty. \end{aligned}$$

So F is integrable. We claim that it is unbounded on any interval. Indeed, note that for any $I \subset \mathbb{R}$, there is a $r_n \in I$, so $\lim\limits_{x \to r_n} F(x) \geq \lim\limits_{x \to r_n} 2^{-n} f(x - r_n) = \infty$ so F is unbounded on I. This same argument would hold for any function $\widetilde{F}$ which is equal to F almost everywhere since intervals have positive measure.