Lecture 5: The local Frobenius theorem

Contents

In this lecture we discuss the local version of the Frobenius theorem and the integrability condition which pertains. In the next lecture we turn to global aspects and applications.

Line fields

(5.1) *Recollection.* Let X be a smooth manifold. We recall the notions of an injective immersion and of a submanifold. (Some authors use 'submanifold' for the former; we reserve the term for a subset of a smooth manifold which is locally diffeomorphic to a standard normal form.)

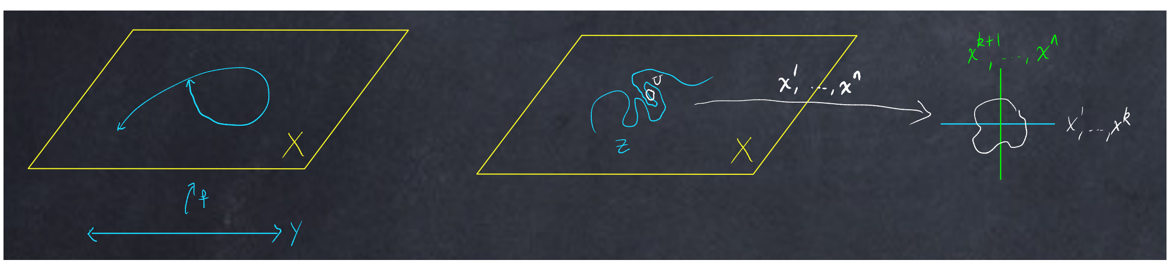

Figure 11. An injective immersion which is not an embedding; a submanifold

Definition 5.2.

(1) A smooth injective map $f\colon Y\to X$ is an *injective immersion* if df_y is injective for all $y\in Y$.
(2) A subset $Z\subset X$ is a *submanifold* if every point $z\in Z$ belongs to a chart $(U;x^1,\dots,x^n)$ such that $Z\cap U=\{x^{k+1}=\cdots=x^n=0\}$ for some $0\le k\le n$.

Theorem 5.3. *The image $f(Y)\subset X$ of an injective immersion $f\colon Y\to X$ is a submanifold iff f factors through a homeomorphism $Y\to f(Y)$ (in which $f(Y)$ has the subspace topology).*

An injective immersion which satisfies this condition is called an *embedding*. The left side of Figure 11 illustrates an injective immersion which is not an embedding: the subspace topology on $f(Y)$ pulls back via f to a topology on Y which is coarser than the given topology. Theorem 5.3 is Theorem 7.1 in the Differential Topology notes, and I defer to that reference for the proof.

Differential Geometry (Math230a, Fall '24), Dan Freed, September 17, 2024

(5.4) *Vector fields and line fields.* A vector field $\xi \in \mathfrak{X}(X)$ on a smooth manifold X induces an ordinary differential equation (2.3) whose solutions are called integral curves. Recall that there is local and global existence and uniqueness (Theorem 2.4). An integral curve is a function with domain an interval in $\mathbb{R}$, i.e., a *motion* in X. Now we switch our focus to the *image* of the motion—a subset of X—and its tangent lines. Suppose $L \subset TX$ is a line subbundle of the tangent bundle of X. We seek an injective immersion $f\colon Y \to X$ of a connected 1-manifold Y such that

$$f_*(T_yY) = L_{f(y)}, \qquad y \in Y. \tag{5.5}$$

This is the analog of (2.3) for a line field, and is called an *integral manifold* of L.

Remark 5.6. The existence of a line field on X imposes a topological constraint: the Euler number of X must vanish. For example, S^2 does not admit a line field.

A section of $L \to X$ is a vector field. At least locally we can find a *nonzero* section ξ. Then an integral curve of ξ is an injective immersion that satisfies (5.5). By this device we can parlay the local and global theory of integral curves into theorems about integral manifolds for line fields. But we resist in favor of a general theory for higher rank distributions.

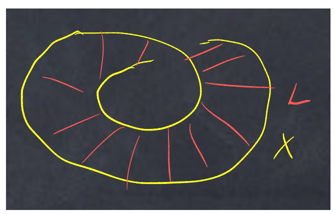

FIGURE 12. A line field on the Möbius band

Example 5.7 (A line field with no nonzero global section)**.** Figure 12 illustrates a line field on the Möbius band which has the property that every global section has a zero.

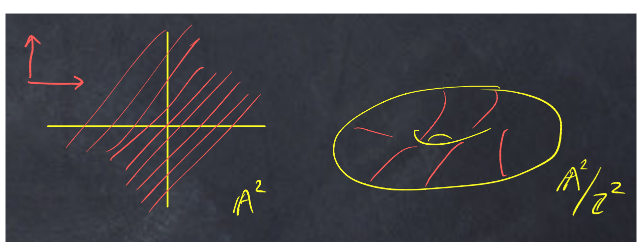

FIGURE 13. A parallel line field on the 2-torus

Example 5.8 (Line fields on the 2-torus)**.** The standard affine plane $\mathbb{A}^2$ has an action of the vector group $\mathbb{R}^2$ by translation. The countable discrete subgroup $\mathbb{Z}^2 \subset \mathbb{R}^2$ acts freely (and properly discontinuously). The quotient $X = \mathbb{A}^2/\mathbb{Z}^2$ inherits the structure of a smooth manifold; it is diffeomorphic to a 2-torus. The global parallelism of affine space descends to a global parallelism of X. A parallel line field is determined by a 1-dimensional subspace of $\mathbb{R}^2$, for example

$$L^{(a)} = \{(\xi^1, \xi^2) \in \mathbb{R}^2 : \xi^2 = a\xi^1\}, \qquad a \in \mathbb{R}. \tag{5.9}$$

The integral manifolds of the resulting parallel line field on $\mathbb{A}^2$ are affine lines, and there is an unique integral manifold through each point; see Figure 13. Upon descent to the 2-torus X, the picture depends strongly on whether $a \in \mathbb{R}$ is rational or irrational. If a is rational, then $L^{(a)} \cap \mathbb{Z}^2$ is an infinite cyclic group, each integral line in $\mathbb{A}^2$ parallel to $L^{(a)}$ is fixed by translations in this subgroup, and the integral manifolds in $\mathbb{A}^2$ descend to integral manifolds in X, each of which is a submanifold diffeomorphic to a circle. If a is irrational, then translation by $L^{(a)}$ acts freely on each integral manifold in X, and each integral manifold is dense and is the image of an injective immersion, but it is not a submanifold. If $U \subset X$ is the image of a ball in $\mathbb{A}^2$ of radius $< 1/2$, then the intersection of each integral manifold with U has countably many components.

Distributions and the Frobenius tensor

We begin with the definition of a higher dimensional line field.

Definition 5.10. Let X be a smooth manifold.

(1) A *distribution* is a vector subbundle $E \subset TX$ of the tangent bundle.

(2) An injective immersion $f\colon Y \hookrightarrow X$ is an *integral manifold* of E if

$$f_*(T_yY) = E_y, \qquad y \in Y. \tag{5.11}$$

(3) A vector field $\xi \in \mathfrak{X}(X)$ *belongs to* E if $\xi_x \in E_x$ for all $x \in X$.

(4) E is *integrable* (or *involutive*) if whenever ξ and η belong to E, then so too does $[\xi, \eta]$.

So a vector field ξ belongs to E iff the map $\xi\colon X \to TX$ factors through the inclusion $E \hookrightarrow TX$:

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr, shift left] & & TX \arrow[dl, shift left] \\ & X \arrow[ul, dashed, bend left] \arrow[ur, bend right, "\xi"'] & \end{tikzcd} \tag{5.12}$$

There are different standard notations for the vector subspace of vector fields that belong to E:

$$\Gamma(E) = \Gamma_X(E) = C^\infty_X(E) = \Omega^0_X(E) = \{\xi \in \mathfrak{X}(X) :\ \xi \text{ belongs to } E\}. \tag{5.13}$$

The following lemma allows us to reexpress Definition 5.10(4) as the vanishing of a tensor.

Lemma 5.14. *The map*

$$\begin{aligned} \Gamma(E) \times \Gamma(E) &\longrightarrow \Gamma(TX/E) \\ \xi \quad , \quad \eta \quad &\longmapsto \quad [\xi, \eta] \pmod{E} \end{aligned} \tag{5.15}$$

is linear over functions.

See **(19.11)** in the Differential Topology notes for the concept of "linear over functions", especially Proposition 19.16 for the relationship to tensors.

Proof. Let $\xi, \eta \in \Gamma(E)$ and $f, g \in \Omega^0_X$. Then

$$[f\xi, g\eta] = fg[\xi,\eta] + f(\xi g)\eta - g(\eta f)\xi. \tag{5.16}$$

Hence

$$[f\xi, g\eta] \pmod{E} = fg\,[\xi,\eta] \pmod{E}, \tag{5.17}$$

as desired. $\square$

Definition 5.18. Let $E \subset TX$ be a distribution on a smooth manifold X. The *Frobenius tensor* of E is

$$\begin{aligned} \phi_E \colon E \times E &\longrightarrow TX/E \\ \xi_x\,,\eta_x &\longmapsto [\xi,\eta]_x \pmod{E}, \end{aligned} \tag{5.19}$$

where $\xi, \eta \in \mathfrak{X}(X)$ are vector fields that extend ξ_x, η_x.

Remark 5.20.

(1) E is integrable iff ϕ_E vanishes.
(2) The Frobenius tensor ϕ_E is a section of

$$\textstyle\bigwedge^2 E^* \otimes TX/E \longrightarrow X. \tag{5.21}$$

The following example gives some insight into the partial differential equations that underlie the geometric condition (5.11) for an integral manifold, and the integrability condition Definition 5.10(4), or equivalently the vanishing of the Frobenius tensor (5.19).

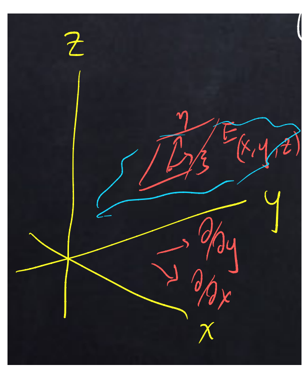

FIGURE 14. A distribution on $\mathbb{A}^2 \times \mathbb{R}$ whose integral manifold is a graph

Example 5.22. Let $P, Q\colon \mathbb{A}^2 \to \mathbb{R}$ be smooth functions, and $\ell_{(x,y)}\colon \mathbb{R}^2 \to \mathbb{R}$ the linear function defined by the matrix $\begin{pmatrix} P(x,y) & Q(x,y) \end{pmatrix}$. Let E be the distribution on $\mathbb{A}^2_{x,y} \times \mathbb{R}_z$ whose value at (x, y, z) is the graph of the linear function $\ell_{(x,y)}$; it is translationally invariant in the z-direction. Then E has a global basis of vector fields

$$\begin{aligned} \xi &= \frac{\partial}{\partial x} + P\frac{\partial}{\partial z} \\ \eta &= \frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} \end{aligned} \tag{5.23}$$

If $f\colon \mathbb{A}^2 \to \mathbb{R}$ is a smooth function, and $\Gamma(f) \subset \mathbb{A}^2 \times \mathbb{R}$ is its graph, then $T_{(x,y,f(x,y))}\Gamma(f)$ has basis

$$\begin{aligned} &\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial z} \\ &\frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z} \end{aligned} \tag{5.24}$$

Hence the condition (5.11) that $\Gamma(f)$ be an integral manifold is the system of PDEs (partial differential equations)

$$\begin{aligned} \frac{\partial f}{\partial x} &= P \\ \frac{\partial f}{\partial y} &= Q \end{aligned} \tag{5.25}$$

These are familiar equations. Any C^2 solution f, much less any C^∞ solution, must have mixed second partials that are equal:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}. \tag{5.26}$$

Combining (5.26) and (5.25) we deduce the integrability condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{5.27}$$

This familiar integrability condition for (5.26) is in fact the integrability condition for the distribution E. In fact, for the vector fields (5.23) belonging to E we compute

$$[\xi, \eta] = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}, \tag{5.28}$$

and so $[\xi, \eta] \pmod{E} = 0$ iff (5.27) holds. It is not too difficult to see that this is true iff $\phi_E = 0$.

Remark 5.29. The integrability condition in Definition 5.10(4) is a geometric form of "mixed partials commute". Recall that the condition $d^2 = 0$ for the Cartan differential on the de Rham complex is another version of "mixed partials commute". The Frobenius tensor measures the failure of this condition for a distribution. This will manifest as *curvature*, as we will see.

Remark 5.30. The distribution E in Example 5.22 is the kernel of the 1-form

$$\theta = P\,dx + Q\,dy - dz. \tag{5.31}$$

We compute

$$d\theta = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy, \tag{5.32}$$

from that we deduce that $\theta \wedge d\theta = 0$ iff (5.27) holds. Quite generally, the condition $\theta \wedge d\theta = 0$ is the integrability condition for a codimension one distribution, and there is a generalization in terms of differential forms for a distribution of any rank.

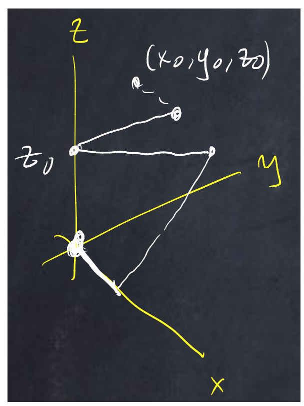

FIGURE 15. A piecewise smooth integral curve of a nonintegrable distribution

Example 5.33 (A nonintegrable distribution)**.** Take the special case of the distribution E in Example 5.22 with $P = 0$ and $Q = x$, so the vector fields (5.23) are

$$\begin{aligned} \xi &= \frac{\partial}{\partial x} \\ \eta &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \end{aligned} \tag{5.34}$$

We claim there is a piecewise smooth path in $\mathbb{A}^2 \times \mathbb{R}$ that begins at $(0,0,0)$, ends at any given (x_0, y_0, z_0), and has velocity in the distribution E. It is illustrated in Figure 15. Here is an explicit path:

(1) Follow ξ to $(1,0,0)$.
(2) Follow η to $(1, z_0, z_0)$.
(3) Follow $-\xi$ to $(0, z_0, z_0)$.
(4) Follow $\pm\eta$ to $(0, y_0, z_0)$.
(5) Follow $\pm\xi$ to (x_0, y_0, z_0).

If E admitted an integral surface S through $(0,0,0)$, then any piecewise smooth integral curve C of E which contains $(0,0,0)$ would lie in S, since at any smooth $c \in C$ we have $T_cC \subset E = T_cS$. So the fact that we can connect $(0,0,0)$ to *any* (x_0,y_0,z_0) indicates that there is no such integral surface, and is consistent with the fact that E is nonintegrable.

Special coordinate systems

As a preliminary to the local Frobenius theorem, we prove two results that allow us to locally specify coordinate vector fields of a chart. Put differently, we prove a local normal form for a set of linearly independent vector fields of cardinality $\le n$ under an commutativity condition.

(5.35) *Normal form for a single nonzero vector field.*

Proposition 5.36. *Let X be a smooth manifold of dimension n, $\xi \in \mathfrak{X}(X)$ a vector field, $p \in X$ a point, and suppose $\xi_p \neq 0$. Then there exists a chart $(U; x^1,\dots,x^n)$ about p such that $\frac{\partial}{\partial x^1} = \xi$.*

The chart $(U; x^1,\dots,x^n)$ is illustrated in Figure 16.

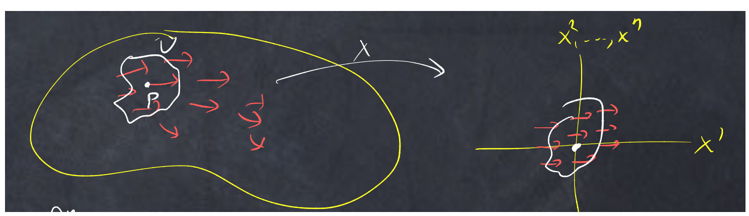

FIGURE 16. Straightening out a single vector field

Proof. Let φ_t be a local flow of ξ near p, and choose a standard chart $(V; y^1,\dots,y^n)$ about p such that the local flow maps a ball about p into V on some time interval $(-\delta,\delta)$ (this is always true); $y^i(p) = 0$, $i = 1,\dots,n$; and $\frac{\partial}{\partial y^1}\Big|_p = \xi_p$. Define

$$F\colon U \longrightarrow X \qquad (5.37)$$
$$(x^1,\dots,x^n) \longmapsto \varphi_{x^1}(0,x^2,\dots,x^n)$$

where we use the y-coordinate system on X and $U \subset \mathbb{A}^n$ is an open neighborhood of $(0,\dots,0) \in \mathbb{A}^n$ on which the map is defined and for which $F(U) \subset V$. Then $dF_{(0,\dots,0)}$ is invertible—in fact, $d(y \circ F)_{(0,\dots,0)} = \mathrm{id}_{\mathbb{R}^n}$ so by the inverse function theorem we can invert F in a neighborhood of the origin. This gives the desired standard chart: in x-coordinates the flow is $\varphi_t\colon (x^1,x^2,\dots,x^n) \to (x^1+t,x^2,\dots,x^n)$. $\square$

(5.38) *Normal form for a set of commuting vector fields.* The next result generalizes Proposition 5.36 to a set of linearly independent vector fields.

Proposition 5.39. *Now suppose $\xi_1,\dots,\xi_k\in\mathcal{X}(X)$ with $\{\xi_1\big|_p,\dots,\xi_k\big|_p\}\subset T_pX$ linearly independent. Then there exists a chart $(U;x^1,\dots,x^n)$ about p with $\frac{\partial}{\partial x^i}=\xi_i$, $i=1,\dots,k$, iff $[\xi_i,\xi_j]=0$ for all $1\le i,j\le k$.*

The commutativity of the vector fields—the vanishing of Lie brackets—is a necessary and sufficient condition for the local normal form.

Proof. If the chart exists, then $[\xi_i,\xi_j]$ is a Lie bracket of coordinate vector fields, hence vanishes.[1] Conversely, suppose the pairwise Lie brackets vanish. Let $\varphi^{(i)}_t$, $i=1,\dots,k$, be local flows of the ξ_i, and let $(V;y^1,\dots,y^n)$ be a standard chart such that $\frac{\partial}{\partial y^i}\big|_p=\xi_i\big|_p$. Define

$$\begin{aligned} F\colon U &\longrightarrow X \\ (x^1,\dots,x^n) &\longmapsto \varphi^{k}_{x^k}\cdots\varphi^{(1)}_{x^1}(0,\dots,0,x^{k+1},\dots,x^n) \end{aligned} \tag{5.40}$$

as in (5.37), for an appropriate domain $U\subset\mathbb{A}^n$. Use the fact that the flows commute (Proposition 2.58) to compute $d(y\circ F)_{(0,\dots,0)}=\mathrm{id}_{\mathbb{R}^n}$ and that, after inverting F, the resulting x-coordinates satisfy $\frac{\partial}{\partial x^i}=\xi_i$. $\square$

The local Frobenius theorem

The main theorem in this lecture is a local normal form—a straightening—for an integrable distribution.

Theorem 5.41. *Let X be a smooth manifold of dimension n and $E\subset TX$ a distribution of rank k.*

(1) *E is integrable iff for all $p\in X$ there exists a chart $(U;x^1,\dots,x^n)$ about p such that in U we have*

$$E=\operatorname{span}\left\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^k}\right\}. \tag{5.42}$$

(2) *If so, then any connected integral submanifold $f\colon Y\to U$ of E has image a submanifold of the form $\{x^m=c^m\}_{m=k+1,\dots,n}$ for some $c^{k+1},\dots,c^n\in\mathbb{R}$.*

Proof. If E has the form (5.42), then since coordinate vector fields commute it follows that E is integrable. Conversely, suppose that E is integrable. Fix $p\in X$ and choose local coordinates $y^1,\dots,y^n$ about p so that

$$E_p=\operatorname{span}\left\{\frac{\partial}{\partial y^1}\bigg|_p,\dots,\frac{\partial}{\partial y^k}\bigg|_p\right\}. \tag{5.43}$$

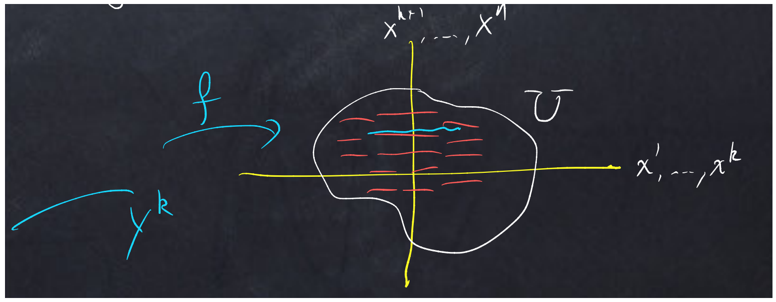

FIGURE 17. The local normal form of an integrable distribution and a connected integral manifold

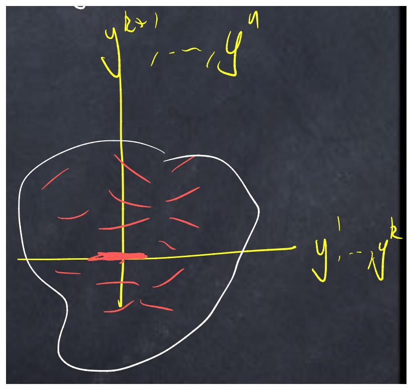

FIGURE 18. A local integrable distribution straightened at a single point

Use these coordinates to transfer to a local problem in $\mathbb{A}^n$; see Figure 18. Let $\pi\colon \mathbb{A}^n \to \mathbb{A}^k$ be affine projection onto the first k coordinates along the last $(n-k)$ coordinates. Choose a neighborhood of p such that $\pi_*\colon \mathbb{R}^n \to \mathbb{R}^k$ restricts to an isomorphism on E. Define vector fields $\xi_1, \dots, \xi_k$ that belong to E and project via π_* to the coordinate vector fields $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}$ on $\mathbb{A}^k$. Then for $1 \le i,j \le n$, the Lie bracket $[\xi_i, \xi_j]$ is π-related to $\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0$, and since by integrability $[\xi_i, \xi_j]$ belongs to E, we conclude that $[\xi_i, \xi_j] = 0$. Now apply Proposition 5.39 to obtain the desired standard chart in (1).

For (2), suppose that $f\colon Y \to U$ is an integral manifold. Then $d(f^*x^m) = f^*(dx^m) = 0$ for $m = k+1, \dots, n$, since E is defined by the simultaneous vanishing of $dx^{k+1}, \dots, dx^n$. It follows that f^*x^m is constant, $m = k+1, \dots, n$, since Y is connected. $\square$

Definition 5.44. Let X be a smooth manifold of dimension n and $E \subset TX$ an integrable distribution of rank k. A chart $(U; x^1, \dots, x^n)$ that satisfies (5.42) is an *E-coordinate system*. For any $c^{k+1}, \dots, c^n \in \mathbb{R}$, the subset $\{x^m = c^m\}_{m=k+1,\dots,n} \subset U$ is a *slice*.

[1] This is another instance of the equality of mixed partials.