Math 132 Problem Set 3

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Problem 1. Show using the preimage theorem that the tangent space to the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^n$ at a point $[v_1, v_2]$, is the vector space of vectors $(v, w) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $v_1 \cdot v = 0$, $v_2 \cdot w = 0$, and $v_1 \cdot v + v_2 \cdot w = 0$.

Recall that the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^n$, denoted $S_{n,2}$, is the preimage of the point $(1, 1, 0) \in \mathbb{R}^3$ under the map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^3$ given by $f(v, w) = (v \cdot v, w \cdot w, v \cdot w)$. Taking partial derivatives, we get

$$\frac{\partial f_0}{\partial v_i} = 2v_i,\ \frac{\partial f_1}{\partial w_i} = 2w_i,\ \frac{\partial f_2}{\partial w_i} = v_i,\ \frac{\partial f_2}{\partial v_i} = w_i$$

with all others set to 0. So the derivative map $df_{[v_1,v_2]} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^3$ sends (v, w) to $(2v_1 \cdot v, 2v_2 \cdot w, v_1 \cdot w + v_2 \cdot w)$. Thus by the preimage theorem, the tangent space $T_{[v_1,v_2]} S_{n,2}$ is the kernel of this derivative map, which is exactly the space described in the problem statement.

Problem 2. GP §5, Problem 1

Transversality of linear subspaces.

a. Suppose that $A : \mathbb{R}^k \to \mathbb{R}^n$ is a linear map and V is a vector subspace of $\mathbb{R}^n$. Check that $A \pitchfork V$ means just that $A(\mathbb{R}^k) + V = \mathbb{R}^n$.

Recall that the derivative of a linear map at a vector $v \in \mathbb{R}^k$, i.e. $dA_v : T_v\mathbb{R}^k \to T_{Av}R^n$ is just $A : \mathbb{R}^k \to \mathbb{R}^n$, and does not depend on the choice of v. Thus, for A to be transverse to a linear subspace $V \subset \mathbb{R}^n$, we need that for every $v \in \mathbb{R}^k$ such that $Av \in V$, the map

$$T_v\mathbb{R}^k \oplus T_{Av}V \to T_{Av}\mathbb{R}^n$$

is surjective. On the first coordinate, this map is just A, and on the second it is the inclusion $i_V : V \to \mathbb{R}^n$ since the tangent space of a linear subspace is the space itself. Thus we want the map $A \oplus i_V$ to be surjective. This just means that every vector $w \in \mathbb{R}^n$ can be expressed as a sum of some Av_1 and $v_2 \in V$, which is exactly the condition $A(\mathbb{R}^k) + V = \mathbb{R}^n$.

b. If V and W are linear subspaces of $\mathbb{R}^n$, then $V \pitchfork W$ means just that $V + W = \mathbb{R}^n$.

This follows from the previous part by letting A be the injective map $\phi_V : \mathbb{R}^{\dim V} \to \mathbb{R}^n$. Then the condition of $V \pitchfork W$ is equivalent to $\phi_V(\mathbb{R}^{\dim V}) + W = \mathbb{R}^n$, and $\phi_V(\mathbb{R}^{\dim V}) = V$ by construction.

Problem 3. (GP, §5, Problem 2) Which of the following spaces intersect transversally?

(a) The xy plane and the z-axis in $\mathbb{R}^3$.

(b) The xy plane and the plane spanned by $\{(3,2,0),(0,4,-1)\}$ in $\mathbb{R}^3$.

(c) The plane spanned by $\{(1,0,0),(2,1,0)\}$ and the y axis in $\mathbb{R}^3$.

(d) $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^\ell$ in $\mathbb{R}^n$.

(e) $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^\ell \times \{0\}$ in $\mathbb{R}^n$.

(f) $V \times \{0\}$ and the diagonal in $V \times V$.

(g) The symmetric and skew symmetric matrices in $M(n)$.

a: Yes, since $xy + z = \mathbb{R}^3$.

b: Yes, since $xy + \{(3,2,0),(0,4,-1)\} = \mathbb{R}^3$.

c: No, because $\{(1,0,0),(2,1,0)\} + y = xy$.

d: Yes, only if $k + \ell \geq n$.

e: Yes, only if $\max(k,\ell) = n$.

f: Yes, since $V \times \{0\} + \Delta = V \times V$.

g: Yes, since any matrix can be expressed as a sum of a symetric and skew symmetric matrix.

Problem 4. (GP, §5, Problem 9). Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A : V \to V$, consider the graph $W = \{(v, Av)\}$. Show that $W \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of A.

If $W \pitchfork \Delta$, this means that for every $v \in V$ such that $(v, Av) = (v, v)$, the natural map

$$\psi : T_{(v,Av)}W \oplus T_{(v,v)}\Delta \to T_{(v,v)}V \times V$$

is onto. If A does not have $+1$ as an eigenvalue, this preimage is empty so the spaces are vacuously transverse. Conversely, if we only know that $W \pitchfork \Delta$, we know this map must be onto. By the first problem set, the tangent space at the graph of a function is the graph of it's derivative, so $T_{(v,Av)}W = W$, since $A \cdot v$ is a linear function. Similarly Δ is the graph of the identity function, so overall the map ψ takes $W \oplus \Delta \to V \times V$ by sending $((v, Av),(v,v)) \mapsto (2v, Av + v)$. This is a contradiction since there can't be a preimage of $(0,1)$ for instance. Thus there can be no vector for which $v = Av$, and so A has no eigenvalue $+1$.

Problem 5. (GP, §5, Problem 10). Let $f : X \to X$ be a map with fixed point x; that is, $f(x) = x$. If $+1$ is not an eigenvalue of $df_x : T_xX \to T_xX$ then x is called a *Lefschetz fixed point* of f. A map f is called a *Lefschetz map* if all of its fixed points are Lefschetz. Prove that if X is compact and f is Leschetz, then f has only finitely many fixed points.

First we notice that the proof of the previous problem only relied on the fact that the derivative of the map $A \cdot v$ had no eigenvalue $+1$. Thus we can generalize to the following lemma:

Claim. Let $f : X \to X$ be a map, and let $F = \{(x, f(x)) \in X \times X : x \in X\}$ be the graph of f. Then $F \pitchfork \Delta$ if and only if $+1$ isn't an eigenvalue of df_x for all fixed points $x \in X$ of f.

Proof. Follows from the proof of previous problem and the fact that transversality is a local property. $\square$

Now let $x \in X$ be a fixed point of a Lefschetz map f. This means that $F \pitchfork \Delta$ in $X \times X$, where F is the graph of f. By the generalization of the preimage theorem, this means that the pullback $W = F \times_{X \times X} \Delta$ is a smooth manifold of dimension 0. However there is a homeomorphism between W and the set of fixed points of f, viewed as a subspace of X. (This is just by construction.) So the set of fixed points of f is a 0-submanifold of a compact manifold, and hence compact itself. Thus it must be finite.

Problem 6. If $f : M \to N$ is a diffeomorphism of smooth manifolds of dimension n and $x \in M$ is a point, then there are coordinate neighborhoods

$$\begin{aligned} \Phi_1 : U_1 \to \mathbb{R}^n \quad & U_1 \subset M \\ \Phi_2 : U_2 \to \mathbb{R}^n \quad & U_2 \subset N \end{aligned}$$

around x and $f(x)$ respectively, having the property that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, dashed, no head] & U_1 \arrow[l, "\Phi_1"'] \arrow[r, hook] \arrow[d, "f"] & M \arrow[d, "f"] \\ \mathbb{R}^n & U_2 \arrow[l, "\Phi_2"'] \arrow[r, hook] & N \end{tikzcd}$$

Let U_1 be a neighborhood of x with $\Phi_1 : U_1 \to \mathbb{R}^n$ a diffeomorphism. Since f is a diffeomorphism, the restriction $f\big|_{U_1} : U_1 \to f(U_1)$ is also a diffeomorphism, and $f(U_1)$ is an open neighborhood of $f(x)$ since diffeomorphisms are open maps. We can then let $\Phi_2 : f(U_2) \to \mathbb{R}^n$ be the composition $\Phi_1 \circ f^{-1}\big|_{U_1}$. This clearly makes the diagram commute.

Problem 7. This problem is from the section *Colloquialisms in differential topology* in the lecture notes.

Write expanded versions of the following assertions.

a. Locally every immersion looks like the standard immersion $\mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^\ell$ which sends x to $(x, 0)$.

Let $f : X \to Y$ be an immersion of a k-manifold into an n-manifold. Then for any $x \in X$, there exist open neighborhoods $x \in U_1$ and $f(x) \in U_2$ with diffeomorphisms $\Psi_1 : U_1 \to \mathbb{R}^k$ and $\Psi_2 : U_2 \to \mathbb{R}^n$ such that $\Psi_2 \circ f \circ \Psi_1^{-1} : \mathbb{R}^k \to \mathbb{R}^n$ is the map which sends x to $(x, 0)$.

b. Locally every submersion looks like the standard submersion $\mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^k$ sending $(x, y) \to x$.

Let $f : X \to Y$ be a submersion of an n-manifold to a k-manifold. Then for any $x \in X$, there exist open neighborhoods $x \in U_1$ and $f(x) \in U_2$ with diffeomorphisms $\Psi_1 : U_1 \to \mathbb{R}^n$ and $\Psi_2 : U_2 \to \mathbb{R}^k$ such that $\Psi_2 \circ f \circ \Psi_1^{-1} : \mathbb{R}^n \to \mathbb{R}^k$ is the map which sends (x, y) to x.

c. Every transverse pullback square W of $X \pitchfork Y \subset M$ in which X and Y are submanifolds of M looks, near every $w \in W$, like:

$$\begin{tikzcd}
\mathbb{R}^\ell \arrow[r, "x \mapsto (x,0)"] \arrow[d, "x \mapsto (x,0)"'] & \mathbb{R}^\ell \times \mathbb{R}^m \arrow[d, "(x,y) \mapsto (0,x,y)"] \\
\mathbb{R}^k \times \mathbb{R}^\ell \arrow[r, "(a,b) \mapsto (a,b,0)"'] & \mathbb{R}^k \times \mathbb{R}^\ell \times \mathbb{R}^m
\end{tikzcd}$$

Suppose X and Y are transverse submanifolds of M with pullback W. Let $n = \dim X, m = \dim Y$, $\ell = \dim W$, and $q = \dim M$. Then for any point $w \in W$, there are neighborhoods U_W, U_X, U_Y, and U_M that make the following diagram commute:

$$\begin{tikzcd}
U_W \arrow[rrr] \arrow[ddd] & & & U_Y \arrow[ddd] \\
& \mathbb{R}^\ell \arrow[r] \arrow[d] \arrow[ul] & \mathbb{R}^m \arrow[d] \arrow[ur] & \\
& \mathbb{R}^n \arrow[r] \arrow[dl] & \mathbb{R}^q \arrow[dr] & \\
U_X \arrow[rrr] & & & U_M
\end{tikzcd}$$

where the central square is the one given in the problem description.

Problem 8. Suppose that M is a smooth manifold of dimension 2, that X and Y are submanifolds of M of dimension 1 intersecting transversally and that x is a point of $X \cap Y$. Show that there is a coordinate neighborhood $\Phi : U \to \mathbb{R}^2$ centered at $x \in M$ under which $\Phi(X \cap U)$ is the x-axis and $\Phi(Y \cap U)$ is the y-axis.

Starting with some point $x \in X \cap Y$, let's pick a neighborhood $U_{X \cap Y}$ of x. Since the pullback of $X \pitchfork Y$ is a 0 manifold, the set $X \cap Y$ doesn't have any limit points. Thus we can shrink $U_{X \cap Y}$ so that it only contains one intersection point of X and Y. Since we're investigating a local property, it suffices to just consider the case when $X \cap Y = \{x\}$, and replace X and Y with their intersections with $U_{X \cap Y}$. Then we get the transverse pullback square $\{x\} \subset X, Y \subset M$ which gives a diagram on charts:

$$\begin{tikzcd}
\{x\} \arrow[r] \arrow[d] & \{x\} \times \mathbb{R} \arrow[d] \\
\mathbb{R} \times \{x\} \arrow[r] & \mathbb{R} \times \mathbb{R}
\end{tikzcd}$$

This is exactly the axis embedding we are looking for.