Math 231a Problem Set 9

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Problem 1. Suppose X is a CW complex of finite type. Find $\dim_{\mathbb{F}_p} H_n(X;\mathbb{F}_p)$ in terms of the Betti numbers and torsion coefficients of $H_*(X;\mathbb{Z})$.

Recall that the Betti numbers and torsion coefficients of $H_*(X;\mathbb{Z})$ are given by

$$H_k(X;\mathbb{Z}) \cong \mathbb{Z}^{\beta_k} \oplus \bigoplus_{i=1}^{t(k)} \mathbb{Z}/n_i(k).$$

By the universal coefficient theorem, we have a short exact sequence:

$$0 \longrightarrow H_k(X;\mathbb{Z}) \otimes \mathbb{F}_p \longrightarrow H_k(X;\mathbb{F}_p) \longrightarrow \mathrm{tors}_p(H_{k-1}(X;\mathbb{Z})) \longrightarrow 0$$

Now let $t_p(k) = |\{1 \le i \le t(k) : p \mid n_i(k)\}|$. By basic properties of tensor product and torsion, we see that the short exact sequence becomes:

$$0 \longrightarrow \mathbb{F}_p^{\beta_k + t_p(k)} \longrightarrow H_k(X;\mathbb{F}_p) \longrightarrow \mathbb{F}_p^{t_p(k-1)} \longrightarrow 0$$

Thus by linear algebra we have:

$$\dim_{\mathbb{F}_p} H_n(X;\mathbb{F}_p) = \beta_n + t_p(n) - t_p(n-1).$$

The special case when $n = 0$ is simple to address, then we have $\dim_{\mathbb{F}_p} H_0(X;\mathbb{F}_p) = \beta_0 + t_p(0)$.

Problem 2. Let I_* denote the following chain complex of abelian groups:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}\{f\} \xrightarrow{f \mapsto e_1 - e_0} \mathbb{Z}\{e_0, e_1\} \longrightarrow 0 \longrightarrow \cdots,$$

where f lies in degree 1 and e_0, e_1 lie in degree 0. Regarding $\mathbb{Z}[0]$ as the chain complex with $\mathbb{Z}$ concentrated at 0, there are chain maps $e_0, e_1 : \mathbb{Z}[0] \to I_*$ given by sending 1 to e_0, e_1. Prove that there is a natural bijection between chain homotopies $f_0 \simeq f_1 : C_* \to D_*$ and chain maps

$$I_* \otimes C_* \to D_*$$

for which the compositions

$$C_* \cong \mathbb{Z}[0] \otimes C_* I_* \otimes C_* \xrightarrow{e_i \otimes \mathrm{id}_{C_*}} D_*$$

are equal to f_i.