Math 230a Problem Set 3

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Problem 1. Distributions of rank 2 on 3-manifolds.

a. Let M be a 3-manifold and α a non-zero 1-form. Prove that the 2-dimensional distribution determined by α is integrable if and only if $\alpha \wedge d\alpha = 0$.

Suppose α is a nowhere zero 1-form on M. We can consider this a section $\alpha \in \Gamma(T^*M)$. Since α is nowhere zero, the kernel bundle $\ker \alpha$ is a 2-dimensional subbundle of TM. Let's call this distribution E_α. Clearly, a vector field $\xi \in \Gamma(TM)$ belongs to E_α if and only if $\alpha(\xi) = 0$. The distribution is integrable if and only if for all vector fields ξ_1, ξ_2 belonging to E_α, their commutator $[\xi_1, \xi_2]$ also belongs to E_α. This means that we can we must show that:

$$\{\alpha \wedge d\alpha = 0\} \quad \iff \quad \{\alpha([\xi_1, \xi_2]) = 0 \quad \text{whenever} \quad \alpha(\xi_1) = \alpha(\xi_2) = 0\}.$$

By the standard commutator relations between $d, \iota, \mathcal{L}$, for any vector fields ξ_1, ξ_2 belonging to E_α, we have

$$\begin{aligned}
\alpha([\xi_1, \xi_2]) &= \iota_{[\xi_1,\xi_2]}\alpha \\
&= \mathcal{L}_{\xi_1}\iota_{\xi_2}\alpha + \iota_{\xi_2}\mathcal{L}_{\xi_1}\alpha \\
&= \iota_{\xi_2}\mathcal{L}_{\xi_1}\alpha \\
&= \iota_{\xi_2}(d\iota_{\xi_1} + \iota_{\xi_1}d)\alpha \\
&= \iota_{\xi_2}d\iota_{\xi_1}\alpha + \iota_{\xi_1}d\alpha \\
&= \iota_{\xi_2}\iota_{\xi_1}d\alpha \\
&= d\alpha(\xi_2, \xi_1)
\end{aligned}$$

Next, suppose ξ_1, ξ_2, ξ_3 are any vector fields, not necessarily in E_α. Then, we have

$$\begin{aligned}
(\alpha \wedge d\alpha)(\xi_1, \xi_2, \xi_3) &= \iota_{\xi_1}\iota_{\xi_2}\iota_{\xi_3}(\alpha \wedge d\alpha) \\
&= \iota_{\xi_1}\iota_{\xi_2}(\iota_{\xi_3}\alpha \wedge d\alpha - \alpha \wedge \iota_{\xi_3}d\alpha) \\
&= \iota_{\xi_1}(-\iota_{\xi_3}\alpha \wedge \iota_{\xi_2}d\alpha - \iota_{\xi_2}\alpha \wedge \iota_{\xi_3}d\alpha + \alpha \wedge \iota_{\xi_2}\iota_{\xi_3}d\alpha) \\
&= \iota_{\xi_3}\alpha \wedge \iota_{\xi_1}\iota_{\xi_2}d\alpha + \iota_{\xi_2}\alpha \wedge \iota_{\xi_1}\iota_{\xi_3}d\alpha + \iota_{\xi_1}\alpha \wedge \iota_{\xi_2}\iota_{\xi_3}d\alpha \\
&= \alpha(\xi_3) \cdot d\alpha(\xi_1, \xi_2) + \alpha(\xi_2) \cdot d\alpha(\xi_1, \xi_3) + \alpha(\xi_1) \cdot d\alpha(\xi_2, \xi_3).
\end{aligned}$$

Here, we make use of the identity $\iota_\xi(\alpha \wedge \beta) = \iota_\xi\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota_\xi\beta$, and cancel terms such as $\iota_{\xi_i}\iota_{\xi_j}\alpha$ and $\iota_{\xi_i}\iota_{\xi_j}\iota_{\xi_k}d\alpha$ since all negative degree forms are zero. With the two identities we derived, we can now get our result.

First, suppose that $\alpha \wedge d\alpha = 0$. Whenever we have vector fields ξ_1, ξ_2 belonging to E_α, and a vector field η not necessarily belonging to E_α, note that

$$\begin{aligned} 0 = (\alpha \wedge d\alpha)(\xi_1, \xi_2, \eta) &= \alpha(\eta) \cdot d\alpha(\xi_1, \xi_2) + \alpha(\xi_2) \cdot d\alpha(\xi_1, \eta) + \alpha(\xi_1) \cdot d\alpha(\xi_2, \eta) \\ &= \alpha(\eta) \cdot \alpha([\xi_2, \xi_1]). \end{aligned}$$

Since α is nonzero, we can find some vector field η such that $\alpha(\eta)$ is nonzero. This means that $\alpha([\xi_2, \xi_1]) = 0$, and so the distribution is integrable.

In the converse direction, suppose that for any vector fields ξ_1, ξ_2 belonging to E_α, we have $\alpha([\xi_1, \xi_2]) = 0$. Let U be an open set on which we can find such fields $\xi_1, \xi_2 \in \Gamma(E_\alpha; U)$ which are also linearly independent. Finally, suppose that there is a third field $\eta \in \Gamma(TM; U)$, which is linearly independent to ξ_1, ξ_2 so that these fields span the tangent bundle TM restricted to U. On U, we have

$$(\alpha \wedge d\alpha)(\xi_1, \xi_2, \eta) = \alpha(\eta) \cdot \alpha([\xi_2, \xi_1]) = 0.$$

However, these fields formed a local frame the tangent bundle, so it follows that $\alpha \wedge d\alpha$ vanishes on U. Since our choice of U was arbitrary, we can do this over the entire manifold to show that $\alpha \wedge d\alpha$ vanishes globally.

b. The Hopf fibration $\pi : S^3 \to S^2$ may be constructed by identifying S^3 as the unit sphere in $\mathbb{C}^2$ and S^2 as $\mathbb{CP}^1$; then the map is a restriction of the canonical projection $(\mathbb{C}^2)^\times \to \mathbb{CP}^1$. The kernel $E' = \ker d\pi$ is an (integrable) one-dimensional distribution on S^3. Let $E \subset TS^3$ be the 2-dimensional distribution given by the orthogonal complement of E' with respect to the standard round metric. Is E integrable? Find a nonzero 1-form α which generates the ideal $\mathcal{I}(E)$ associated to E. Compute $d\alpha$ and $\alpha \wedge d\alpha$.

We can consider S^3 as the subset of $\mathbb{C}^2$ given by points (z_1, z_2) with $|z_1|^2 + |z_2|^2 = 1$. Choosing coordinates:

$$\begin{cases} z_1 = \cos(\theta) \cdot e^{i(\phi+\psi)} \\ z_2 = \sin(\theta) \cdot e^{i(\phi-\psi)} \end{cases} \qquad \text{where} \quad (\theta, \phi, \psi) \in [0, \pi/2] \times [0, \pi]^2.$$

In these coordinates, the Hopf fibration becomes $\pi(\theta, \phi, \psi) = (\theta, \phi)$ where we use the standard spherical coordinate system on S^2.

Next, let's see what form the round metric takes on S^3 using the coordinate system we provided. Recall that the round metric on S^3 is the pullback of the Euclidean metric on $\mathbb{R}^4$ from the canonical embedding $\mathbb{C}^2 \to \mathbb{R}^4$. Expanding our coordinate system in $\mathbb{R}^4$, and using standard trigonometric identities, up to scaling, the metric g on S^3 has matrix form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\cos(2\theta) \\ 0 & -\cos(2\theta) & 1 \end{pmatrix} \quad \Longleftrightarrow \quad g = d\theta^2 + d\phi^2 - 2\cos(2\theta)\, d\phi\, d\psi + d\psi^2.$$

Now suppose at a point $(\theta, \phi, \psi) \in S^3$, the tangent vector $v \in T_{(\theta,\phi,\psi)} S^3$ is in the kernel of $d\pi$. This means that v must have no $\partial/\partial\phi$ or $\partial/\partial\psi$ components. The space of vectors $v = (x, y, z)$ in the complement of this kernel must then satisfy, for every $t \in \mathbb{R}$,

$$\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}^\intercal \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\cos(2\theta) \\ 0 & -\cos(2\theta) & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \Longrightarrow \quad -\cos(2\theta) y + z = 0.$$

In other words, E is spanned by the vector fields

$$\xi_1 = \frac{\partial}{\partial\theta} \quad \text{and} \quad \xi_2 = \frac{\partial}{\partial\psi} - \cos(2\theta) \frac{\partial}{\partial\phi}.$$

This distribution is thus **not integrable**, since a simple calculation shows that the commutator of these vector fields is $[\xi_1, \xi_2] = 2\sin(2\theta)\partial/\partial\phi$, which does not belong to E.

Next, we need to find a 1-form α which generated the ideal associated to the distribution. Since we have vector fields which already span E, it's clear that the 1-form $\alpha = \cos(2\theta)\, d\psi + d\phi$ vanishes on E, and clearly must be a generator for $\mathcal{I}(E)$ since it has minimal degree. Computing $d\alpha$ and $\alpha \wedge d\alpha$, we get

$$d\alpha = -2\sin(2\theta)\, d\theta \wedge d\psi \quad \Longrightarrow \quad \alpha \wedge d\alpha = (2\cos(2\theta)d\psi - d\phi) \wedge (-2\sin(2\theta)d\theta \wedge d\psi) = 2\sin(2\theta)d\phi \wedge d\theta \wedge d\psi$$

Since $\alpha \wedge d\alpha$ is non-zero, this agrees with the previous part of the problem.

Problem 2. Suppose M is a smooth manifold and $E \subset TM$ is a distribution. Define:

$$\mathcal{I}(E) = \{\omega \in \Omega^\bullet(M) : \omega|_E = 0\}.$$

a. Prove that $\mathcal{I}(E) \subset \Omega^\bullet_M$ is an ideal.

Clearly $\mathcal{I}(E)$ is additively closed. For any $\omega, \eta \in \Omega^\bullet(M)$ with $\omega \in \mathcal{I}(E)$ and $\xi_i \in \Gamma(E)$, recall that

$$(\omega \wedge \eta)(\xi_1, \ldots, \xi_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \cdot \omega(\xi_{\sigma(1)}, \ldots, x_{\sigma(k)}) \cdot \eta(x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}),$$

where $|\omega| = k$ and $n = |\omega| + |\eta|$.

Clearly, every term in this sum will vanish since $\omega|_E = 0$ and all vector fields belong to E.

b. Prove that if E has corank r – that is, if $\dim E_x + r = \dim_x M$ for all $x \in M$ – then E is locally generated by r independent 1-forms.

It suffices to consider connected manifolds when the dimension is a fixed n, since the result is local anyways. Let's work in some chart U with a local frame $\xi_1, \ldots, \xi_{n-r}, \eta_1, \ldots \eta_r$, and $\xi_1, \ldots, \xi_{n-r}$ a local frame for E. Let the dual coframe be $\xi^1, \ldots, \xi^{n-r}, \ldots, \eta^1, \ldots, \eta^r$, and consider the distribution

$$V = \bigcap_{1 \le i \le r} \ker \eta^i.$$

At any point $p \in U$, the $n-r$ linearly independent vectors $(\xi_1)_p, \ldots, (\xi_{n-r})_p$ lie in V_p and $(\eta_1)_p, \ldots, (\eta_r)_p$ don't. Since these vectors form a basis for T_pM, it follows that $V_p = E_p$ and so the distribution E is generated on U by the independent 1-forms $\eta^1\, ldots, \eta^r$.

c. Prove that $\mathcal{I}(E)$ is closed under d if and only if E is integrable.

For any k-form $\omega \in \Omega^\bullet(M)$ and vector fields $\xi_1, \ldots, \xi_{k+1} \in \Gamma(TM)$, we have

$$\begin{aligned} d\omega(\xi_1, \ldots, \xi_{k+1}) = &\sum_{1 \le i \le k+1} (-1)^{i+1} \xi_i \omega(\xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_{k+1}) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j+1} \omega([\xi_i, \xi_j], \xi_1, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_{k+1}), \end{aligned}$$

where $\widehat{\xi_i}$ denotes omission of the i-th entry. This identity can be proved inductively using the standard commutator relations between ι, d, and $\mathcal{L}$.

Now, recall that E is integrable if and only if for any vector fields $\xi_1, \xi_2 \in \Gamma(E)$ belonging to E, their commutator $[\xi_1, \xi_2]$ belongs to E as well. Supposing E is integrable and that ω vanishes on E, it's clear that

$d\omega$ vanishes on E since all the vector fields in the terms of the sum are in E, and ω vanishes on vector fields in E.

Conversely, suppose that $\mathcal{I}(E)$ is closed under d. For any forms $\xi, \eta \in \Gamma(E)$, we can locally find 1-forms $\alpha_1, \dots, \alpha_r$ which locally generate E. These forms can be extended to $\Gamma(E^*)$. Then $\alpha_i \in \mathcal{I}(E)$ and hence $d\alpha_i \in \mathcal{I}(E)$ by assumption. However,

$$\alpha_i([\xi, \eta]) = \eta\alpha_i(\xi) - \xi\alpha_i(\eta) - d\alpha_i(\xi, \eta).$$

All the terms on the right hand side must vanish by assumption, and so $\alpha_i([\xi, \eta]) = 0$ for all i. From this it's immediately implied that $[\xi, \eta] \in \Gamma(E)$ locally. We can do this across the entire manifold so E is integrable.

d. Consider the distribution E on $\mathbb{A}^3_{x,y,z}$ spanned by the vector fields $\partial/\partial x$ and $x\partial/\partial y + \partial/\partial z$. Show that E is not integrable. Show that any point $(x, y, z) \in \mathbb{A}$ may be joined to the origin by a piecewise smooth curve whose tangent line belongs to E.

Letting $\xi = \partial/\partial x$ and $\eta = x\partial/\partial y + \partial/\partial z$, we can compute the commutator $[\xi, \eta]$ as

$$[\xi, \zeta] = \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right] = \frac{\partial}{\partial y}.$$

This proves that E cannot integrable since this commutator can clearly not be expressed as a linear combination of ξ and η.

Next, let's show that any point in $\mathbb{A}^3$ can be joined to the origin by a piecewise smooth curve whose tangent lines form a subbundle of E. The flows associated to the vector fields ξ and η are

$$\varphi_t(x, y, z) = (x + t, y, z) \quad \text{and} \quad \psi_t(x, y, z) = (x, xt + y, t + z).$$

Thus, the piecewise smooth integral curve is given by travelling along ξ for time $y/z - x$ and then travelling along η for time $-y/z$. If $z = 0$, we can follow ξ for time $-x$ and then ψ for time 1. Then, we repeat the procedure for the nonzero z case.

Problem 3. Example or proof of nonexistence: A codimension 1 foliation on the sphere S^4.

Suppose for the sake of contradiction that exists a foliation $\mathcal{F}$ of codimension 1 on S^4. Let $E_{\mathcal{F}}$ be the associated distribution of codimension 1. Using the standard metric on S^4, the orthogonal complement $E_{\mathcal{F}}^{\perp}$ gives us a line field on S^4. However, $\chi(S^4) = 2$, so it cannot admit a line field. This is a contradiction, so a codimension 1 foliation does not exist.

Problem 4. The Frobenius tensor.

a. Let $P, Q : \mathbb{A}^2 \to \mathbb{R}$ be smooth functions. Define the 2-dimensional distribution E on $\mathbb{A}^2_{x,y} \times \mathbb{R}_z$ with

$$E_{(x,y,z)} = \operatorname{span}\left\{\frac{\partial}{\partial x} + P\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + Q\frac{\partial}{\partial z}\right\}.$$

Compute the Frobenius tensor of E.

Since E is spanned by the vector fields

$$\xi_1 = \frac{\partial}{\partial x} + P\frac{\partial}{\partial z}, \quad \text{and} \quad \xi_2 = \frac{\partial}{\partial y} + Q\frac{\partial}{\partial z}$$

it suffices to compute their commutator in order to determine the Frobenius tensor. This computation yields

$$\begin{aligned}\phi_E(\xi_1, \xi_2) &= = \left[\frac{\partial}{\partial x} + P\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + Q\frac{\partial}{\partial z}\right] \mod E \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z} \mod E.\end{aligned}$$

Thus, if vector fields η_1, η_2 in E have coefficients $\eta_1 = \alpha\xi_1 + \beta\xi_2$ and $\eta_2 = \gamma\xi_1 + \kappa\xi_2$, then

$$\phi_E(\eta_1, \eta_2) = (\alpha\kappa - \beta\gamma)\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z} \mod E.$$

b. Suppose X is a manifold and G a Lie group. Let $\{\theta^i\}$ be a basis of left-invariant 1-forms on G and suppose

$$d\theta^i + \frac{1}{2}c^i_{jk}\theta^j \wedge \theta^k = 0$$

for constants c^i_{jk}. Let $\{\theta^i_X\}$ be 1-forms on X. Consider the ideal of differential forms on $X \times G$ generated by $\pi_G^*\theta^i - \pi_X^*\theta^i_X$, where $\pi_X : X \times G \to X$ and $\pi_G : X \times G \to G$ are projections. Prove that this ideal is closed under d if and only if

$$d\theta^i_X + \frac{1}{2}c^i_{jk}\theta^j_X \wedge \theta^k_X = 0.$$

Let $\mathcal{I}$ denote the ideal. To simplify notation, let $x^i = \theta^i_X$ and $X^i = \pi_X^*\theta^i_X$. Similarly, let $g^i = \theta^i$ and $G^i = \pi_G^*\theta^i$. Writing the given relation in this form, we can obtain expressions

$$dg^i + \frac{1}{2}c^i_{jk}g^j \wedge g^k = 0 \quad \Longrightarrow \quad dG^i + \frac{1}{2}c^i_{jk}G^j \wedge G^k,$$

by the naturality of d and $\wedge$. Expanding the differential of a generator, we get

$$\begin{aligned}d(G^i - X^i) &= dG^i - dX^i \\ &= -\frac{1}{2}c^i_{jk}G^j \wedge G^k - dX^i\end{aligned}$$

Now assume that the have the identity $dx^i + (c^i_{jk}/2)\, x^j \wedge x^k = 0$. Then by naturality of d and $\wedge$ we have the identity $dX^i + (c^i_{jk}/2)\, X^j \wedge X^k = 0$. Then expanding the differential of a generator, we have

$$\begin{aligned}d(G^i - X^i) &= dG^i - dX^i \\ &= -\frac{1}{2}c^i_{jk}G^j \wedge G^k + \frac{1}{2}c^i_{jk}X^j \wedge X^k \\ &= -\frac{1}{2}c^i_{jk}\left(G^j \wedge G^k + X^j \wedge X^k\right) \\ &= -\frac{1}{2}c^i_{jk}\left((G^j - X^j) \wedge G^k - (G^k - X^k) \wedge G^j\right)\end{aligned}$$

This is a sum of elements of the ideal, so the ideal is closed under d.

Next, let's suppose that the ideal is closed under d. We know that:

$$\begin{aligned}&d(G^i - X^i) = dX^i + \frac{1}{2}c^i_{jk}G^j \wedge G^k \in \mathcal{I} \qquad \text{subtracting } \frac{1}{2}c^i_{jk}(G^j - X^j) \wedge G^k - \frac{1}{2}c^i_{jk}X^j \wedge (G^k - X^k) \\ &\Longrightarrow dX^i + \frac{1}{2}c^i_{jk}X^j \wedge X^k \in \mathcal{I}.\end{aligned}$$

We would like to show that $dx^i + (c^i_{jk}/2)x^j \wedge x^k = 0$, and so far, we've shown that its pullback under π_X is in $\mathcal{I}$. Let's let $\omega = dx^i + (c^i_{jk}/2)x^j \wedge x^k$. Using the generators of $\mathcal{I}$, we can find some 1-forms $\alpha^q \in \Omega^1(X \times G)$ such that:

$$\pi_X^*\omega = \sum_q (G^q - X^q) \wedge \alpha^q.$$

Let's write $\alpha^q = a_{k,q}\, G^k + b_{k,q}\, \beta^{k,q}$ for some forms $\beta^{k,q} \in \Omega^{1,0}(X \times G)$ and 0-forms $a_{k,q}, b_{k,q}$ – we can do this because g^i form a basis for $\Omega^1(G)$ and hence G^i form a basis for $\Omega^{0,1}(X \times G)$. Then we have

$$\begin{aligned}
\pi_X^* \omega = dX^i + \frac{1}{2} c^i_{jk} X^j \wedge X^k &= \sum_{q,k} (G^q - X^q) \wedge \alpha^q \\
&= \sum_{q,k} (G^q - X^q) \wedge a_{k,q} G^q + (G^q - X^q) \wedge b_{k,q} \beta^{k,q} \\
&= \sum_{q,k} G^q \wedge a_{k,q} G^q + (G^q \wedge b_{k,q} \beta^{k,q} - X^q \wedge a_{k,q} G^{k,q}) - X^q \wedge b_{k,q} \beta^{k,q}
\end{aligned}$$

Since the original form $\pi_X^*\omega$ is in $\Omega^{2,0}(X \times G) \subset \Omega^2(X \times G)$, the terms involving G^q must vanish, so in particular $a_{k,q} = 0$ and consequently $b_{k,q} = 0$. However, this means that $\pi_X^*\omega = 0$, which also means that $\omega = 0$. Thus,

$$\omega = d\theta^i_X + \frac{1}{2} c^i_{jk} \theta^j_X \wedge \theta^k_X = 0.$$

c. Compute the Frobenius tensor of the distribution in (b) defined as the simultaneous kernel of the 1-forms $\pi_G^*\theta^i - \pi_X^*\theta^i_X$.

Let E be the distribution on $X \times G$ defined as the intersection

$$E = \bigcap_i \ker(G^i - X^i).$$

The Frobenius tensor is defined as the bilinear map

$$\begin{aligned}
\phi_E : \Gamma(E) \times \Gamma(E) &\longrightarrow \Gamma(T(X \times G)/E) \\
(\xi, \eta) &\longmapsto [\xi, \eta] \mod E.
\end{aligned}$$

Suppose ξ is a vector field in E, which equivalently means that it is a vector field on $X \times G$ with $g^i(\xi_G) = x^i(\xi_X)$, where ξ_G and ξ_X are the projections of ξ to G and X respectively. For any two vector fields ξ and η in E, we have relations

$$dg^i(\xi_G, \eta_G) + \frac{1}{2} c^i_{jk} g^j(\xi_G) \wedge g^k(\eta_G) = 0$$
$$\xi_G g^i(\eta_G) - \eta_G g^i(\xi_G) - g^i([\xi_G, \eta_G]) + \frac{1}{2} c^i_{jk} g^j(\xi_G) \wedge g^k(\eta_G) = 0$$

Unsure how to finish this...