Math 132 Problem Set 4

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Problem 1. Suppose that $X \subset \mathbb{R}^N$ is a smooth n-manifold, and let k be a non-negative integer. Show that the set of $((v_1, \dots, v_k), x) \in \mathbb{R}^{Nk} \times \mathbb{R}^N$ with $x \in X$ and $v_i \in T_xX$ is a smooth manifold of dimension $n(k+1)$.

Let's call this manifold W. First, observe that this manifold W is exactly the pullback of the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & (TX)^k \arrow[d, "p^k"] \\ X \arrow[r, "\Delta"'] & X^k \end{tikzcd}$$

where $p : TX \to X$ is the standard tangent bundle projection, and Δ is the diagonal map. We claim that $p^k \pitchfork \Delta$, this would prove that W is a smooth manifold of dimension $\dim TX^k - \dim X^k + \dim X = n(k+1)$. To prove transversality, notice that on the preimage of a point $\Delta(x) \in X^k$, dp^k must be onto because p and therefore p^k is a submersion. This is by the fiber bundle property proved in 3b, since locally p is a composition of a diffeomorphism and the submersion $U \times F \to U$.

Problem 2. Suppose that $k \leq \ell$. Show that the set of linear transformations $T : \mathbb{R}^k \to \mathbb{R}^\ell$ having rank less than k has measure zero.

Let's call this space $H \subset \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^\ell)$. First consider the space $W_k(S^{k-1})$ from the previous problem, where $X = S^{n-1}$. This space has dimension $(k-1)(k+1)$. Now note that we have a smooth (linear) map $W_k(S^{k-1}) \to \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^\ell)$, where a point $((v_1, \dots, v_k), x)$ is sent to the matrix T with columns v_i. Here note that $Tx = 0$ and $|x| = 1$. So the image of $W_k(S^{k-1})$ is the space of maps that have a nontrivial kernel, which is exactly the space we're looking for. Since $k \leq \ell$, $(k-1)(k+1) < k\ell$, so by Sard's theorem the desired set of transformations has measure zero.

Problem 3. Suppose that F is a smooth manifold of dimension k and M is a smooth manifold of dimension ℓ. A *smooth fiber bundle* is a subspace $E \subset \mathbb{R}^n$ and a smooth map $p : E \to M$ with the property that for each $x \in M$ there is an open neighborhood $U \subset M$ of x and a diffeomorphism $p^{-1}(U) \to U \times F$ having the property that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "p"'] & U \times F \arrow[d, "\pi_U"] \\ U \arrow[r, dashed, no head] & U \end{tikzcd}$$

commutes.

If F and $Mkik$ are k and ℓ manifolds respectively:

a. Show that E is a smooth manifold of dimension $k + \ell$.

For any $x \in E$, we can find some open neighborhood $p^{-1}(U)$ of $p(x) \in M$ such that $x \in p^{-1}(U) \to U \times F$ is a diffeomorphism. Since $U \times F$ is a $k + \ell$ dimensional manifold (being a product of an open subset of a k-manifold with an ℓ manifold), we can pick some chart around the image of x and precompose it with the map to get a $k + \ell$ chart at $x \in E$.

b. Show that the tangent bundle of a manifold is a smooth fiber bundle.

To be fully rigorous, we'll state and prove several "obvious" claims.

Claim. Let $\mathcal{U} \subset \mathbb{R}^k$ be an open subset. Then $T\mathcal{U} = \mathcal{U} \times \mathbb{R}^k$.

Proof. This follows from the fact that $T_u\mathcal{U} = \mathbb{R}^k$ for any $u \in \mathcal{U}$. □

Claim. Let $f : X \to Y$ be a diffeomorphism. Then $df : TX \to TY$ is also a diffeomorphism.

Proof. This follows from the fact that $d(-)$ is "functorial", i.e. if $g : Y \to X$ is a smooth inverse, dg will be a smooth inverse for df, and vice versa. □

Now suppose $x \in B$ is any point. Let $\Phi : \mathcal{U} \to V \subset \mathbb{R}^k$ be a chart. (i.e. a diffeomorphism) Then $d\Phi : T\mathcal{U} \to V \times \mathbb{R}^k$ is also a diffeomorphism, and composing with $\Phi^{-1} \times 1_{\mathbb{R}^k}$ gives us a diagram

$$\begin{tikzcd} p^{-1}(\mathcal{U}) = T\mathcal{U} \arrow[r, "d\Phi"] \arrow[d, "p"'] & V \times \mathbb{R}^k \arrow[r, "\Phi^{-1}\circ 1"] & \mathcal{U} \times \mathbb{R}^k \arrow[d, "\pi_{\mathcal{U}}"] \\ \mathcal{U} \arrow[rr, dashed, no head] & & \mathcal{U} \end{tikzcd}$$

This square commutes since $(\Phi^{-1} \circ 1)(d\Phi)(u, v) = (\Phi^{-1} \circ 1)(\Phi(u), d\Phi_u(v)) = (u, d\Phi_u(v))$. Projecting onto $\mathcal{U}$ then just gives us u so we have the identity.

Problem 4. We return to the Stiefel manifold $V_k(\mathbb{R}^n)$.

For a unit vector $v \in S^{n-1}$, let $R_v : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal transformation which sends v to $-v$ and fixes all the vectors with $\langle v, w \rangle = 0$. This is the *reflection through the hyperplane perpendicular to* v. It is given by the formula

$$R_v(x) = x - 2\langle x, v \rangle v.$$

Given two vectors $v, w \in S^{n-1}$ with $v \neq -w$, set $m = (v + w)/|v + w|$ and define $R_{v,w} = R_m \circ R_v$. The orthogonal transformation $R_{v,w}$ induces the unique rotation in the 2-plane spanned by v and w sending v to w. Finally, let $p : V_k(\mathbb{R}^n) \to S^{n-1}$ be the map given by

$$p([v_1, \ldots, v_k]) = v_k.$$

a. Given $v \in S^{n-1}$ let $U = S^{n-1} - \{-v\}$, and $H = v^\perp$. Show that the map

$$g : p^{-1}(U) \to U \times V_{k-1}(H)$$

given by

$$g([v_1, \ldots, v_{k-1}, x]) = (x, [Rv_1, \ldots, Rv_{k-1}])$$

where $R = R_{x,v}$ is a diffeomorphism.

This map is smooth because each component map $g_i : p^{-1}(U) \to V_{k-1}(H)_i$ is given by

$$\begin{aligned} R_{x,v}(v_i) &= R_{\frac{x+v}{|x+v|}} \circ R_v = R_{\frac{x+v}{|x+v|}} \left(v_i - 2\langle v_i, v\rangle v\right) \\ &= v_i - 2\langle v_i, v\rangle v - 2\left\langle v_i - 2\langle v_i, v\rangle v, \frac{x+v}{|x+v|}\right\rangle \frac{x+v}{|x+v|} \end{aligned}$$

which is clearly smooth. This also has smooth inverse

$$g^{-1}(x, [w_1, \ldots, v_{k-1}]) = [R_{v,x}w_1, \ldots, R_{v,x}w_{k-1}, x]$$

so g is a diffeomorphism.

b. Show that the map p is a smooth fiber bundle with fiber $V_{k-1}(\mathbb{R}^{n-1})$.

This follows immediately from the previous part by picking any orthonormal basis for H and using the diffeomorphism $V_{k-1}(H) \to V_{k-1}(\mathbb{R}^{n-1})$.

c. Using this, give another proof that the Stiefel manifold is a smooth manifold.

By properties of fiber bundles, we know that if $V_{k-1}(\mathbb{R}^{n-1})$ is a smooth manifold, then so is $V_k(\mathbb{R}^n)$. (The base space is S^{n-1} is always a smooth manifold) By induction, and using the trivial case when $k = 1$, since we can always assume that $k \leq n$.

Problem 5. Sphere bundles and Stiefel manifolds.

Let X be a k-manifold.

a. Let SX be the set of points $(x, v) \in TX$ with $|v| = 1$. Prove that SX is a $(2k-1)$-dimensional submanifold of TX; it is called the *sphere bundle* of X.

Consider the map $f : TX \to \mathbb{R}$ given by $f(x, v) = |v|^2$. This is clearly a smooth map, and $SX = f^{-1}(1)$. If we show that 1 is a regular value of this map, we would be done, since the preimage theorem would show that $\dim SX = 2k - 1$. Recall that at any point $(x, v) \in TX$, there is an identification $T_{(x,v)}TX = T_xX \times \mathbb{R}^k$ and $p : TX \to X$ is a submersion by the argument in problem 1. Thus, locally f looks like the map $\mathcal{U} \times \mathbb{R}^k \to \mathbb{R}$ given by $(u, v) \mapsto |v|^2$, under this identification the derivative map is

$$df_{(x,v)} : T_xX \times R^k \to \mathbb{R} : (y, w) \mapsto w \cdot 2v.$$

Since for any (x, v) with $|v| = 1$, this map is surjective, we are done.

b. Can you think of a relationship between the sphere bundle of S^{n-1} and a Stiefel manifold?

The sphere bundle SS^{n-1} is actually diffeomorphic (technically equal) to the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^n$. Recall that the tangent space TS^{n-1} can be represented as

$$TS^{n-1} = \{(x, v) : x \in S^{n-1}, v \perp x\}.$$

Then the sphere bundle of S^{n-1} would be

$$SS^{n-1} = \{(x, v) : x, v \in S^{n-1}, v \perp x\}.$$

This is exactly the set of orthogonal 2-frames in $\mathbb{R}^n$.