Problem Set # 1

Math262a: Quantum theory from a geometric viewpoint I

These problems are supplementary to the lectures and the lecture notes. I hope you tackle many of them, even if you don't complete them. Many are open-ended. They provide a gateway to engage with the material in the class. I suggest you form small groups to discuss the class material, including these problems. Those getting a grade in the course should regularly hand in some solutions. Please come to office hours and the discussion section to discuss these problems, and also take advantage of Discord as a platform for discussion.

Problems

1. In this problem you will derive the symplectic form on the space of classical trajectories of a particle, for simplicity a particle of mass m moving on the standard Euclidean line $\mathbb{E}^1$ with coordinate x. Let $V\colon \mathbb{E}^1 \to \mathbb{R}$ be a smooth function, which is the potential energy of the particle. Let $\mathcal{F} = \mathrm{Map}(\mathbb{R}, \mathbb{E}^1)$ be the space of smooth possible particle trajectories.

   (a) What is the space $N \subset \mathcal{F}$ of classical trajectories, i.e., trajectories that satisfy Newton's law? Consider the special cases $V = 0$ and $V(x) = kx^2/2$ for some $k > 0$.

   (b) Review the derivation of Newton's law from the calculus of variations (Euler-Lagrange equations) applied to the *action* function

   $$S(\gamma) = \int \Big[\frac{1}{2}\, m\dot{x}(t)^2 - V\big(x(t)\big)\Big]\, dt$$

   where $x\colon \mathbb{R} \to \mathbb{E}^1$ is a smooth motion.

   (c) Rephrase this computation using calculus on $\mathcal{F}\times\mathbb{R}$. Observe that the Euler-Lagrange equations only depend integrand in the action—the *lagrangian*—not its integral. Write δ for the de Rham differential on $\mathcal{F}$ and d for the de Rham differential $d = dt\cdot\partial/\partial t$ on $\mathbb{R}$. Then the total de Rham differential on $\mathcal{F}\times\mathbb{R}$ is $\delta + d$. In particular, $(\delta + d)^2 = 0$. A differential form on $\mathcal{F}\times\mathbb{R}$ has *type* (p, q), $p \in \mathbb{Z}^{\geqslant 0}$, $q \in \{0, 1\}$, if it is nonzero only when evaluated on p vectors tangent to $\mathcal{F}$ and q vectors tangent to $\mathbb{R}$. Calculus on the Cartesian product $\mathcal{F}\times\mathbb{R}$ is depicted in the diagram

   $$\begin{array}{c|ccc} & 0 & 1 & \cdots \\ \hline 1 & & & \\ & d\uparrow & & \\ 0 & & \xrightarrow{\delta} & \\ \mathbb{R} & & & \end{array} \qquad \mathcal{F}$$

   (If one writes the lagrangian and action as a *density* rather than a differential form—use $|dt|$ in place of dt—which we should do since we should not use an orientation on $\mathbb{R}$ to define this mechanical system, then the formatting of the square makes more sense: starting from the top, the vertical degrees are then written as '$|0|$' and '$|-1|$'.) Let

$$\begin{aligned} e\colon \mathcal{F}\times\mathbb{R} &\longrightarrow \mathbb{E}^1 \\ x\ ,\ t &\longmapsto x(t) \end{aligned}$$

be the evaluation function. Show that the integrand above can be written as

$$L = \Big[\frac{1}{2}\, m\dot{e}^2 - V\circ e\Big]\, dt$$

and that it is an element of $\Omega^{0,1}(\mathcal{F}\times\mathbb{R})$. (Be sure to define $\dot{e}$, which is a directional derivative of e along a certain vector field on $\mathcal{F}\times\mathbb{R}$.)

(d) Compute $\delta L\in\Omega^{1,1}(\mathcal{F}\times\mathbb{R})$. Carry out the integration by parts in the following form: find $\theta\in\Omega^{1,0}(\mathcal{F}\times\mathbb{R})$ so that $\delta L + d\theta$ is a function times $\delta e\wedge dt$, i.e., it has no $\delta\dot{e}\wedge dt$ term.

(e) Show that $\delta L + d\theta$ vanishes on N, and in fact that vanishing defines N.

(f) Show that the restriction of $\delta\theta\in\Omega^{2,0}(\mathcal{F}\times\mathbb{R})$ to $N\times\{t_0\}$ is independent of t_0. Show that this restriction is a symplectic form. Identify it with the symplectic form you may have run into before in this context.

2. In the previous problem specialize to $V(x) = kx^2/2$ for fixed $k>0$.

(a) Fix $t\in\mathbb{R}$ and consider the observables (functions on N) given by $\mathcal{O}_1(x) = x(0)$ and $\mathcal{O}_2(x) = x(t)$. Choose a pure state σ, which is a point of N. Compute the expectation values $\langle\mathcal{O}_1\rangle_\sigma$ and $\langle\mathcal{O}_2\rangle_\sigma$. Compute the correlation function $\langle\mathcal{O}_2\mathcal{O}_1\rangle_\sigma$. Investigate the dependence on t and on σ.

(b) Now consider the mixed *Gibbs* state, which is a probability measure on the space N of classical solutions. Its probability density is a constant times $e^{-\beta E}$, where $\beta>0$ is a constant and $E\colon N\to\mathbb{R}$ is the energy function $E = m\dot{x}^2/2 + kx^2/2$. Compute the Gibbs state (a function times the standard measure on the $(x,\dot{x})$-plane) and evaluate the expectation values in part (a) in the Gibbs state.

3. Verify the axioms in §1.2 of the lecture notes for the case of a classical mechanical system.

4. In the context of a general mechanical system, let σ_1,σ_2 be states and let A, A' be observables. Fix $t\in[0,1]$. Prove that for the convex combination $\sigma = t\sigma_1 + (1-t)\sigma_2$ the standard deviations $\Delta_\rho B$ of the probability distributions ρ_B obtained from the various states ρ and observables B satisfy

$$\Delta_\sigma A\,\Delta_\sigma A' \geqslant t(\Delta_{\sigma_1}A)\,(\Delta_{\sigma_1}A') + (1-t)(\Delta_{\sigma_2}A)\,(\Delta_{\sigma_2}A')$$

In particular, conclude that

$$\Delta_\sigma A \geqslant \min(\Delta_{\sigma_1}A, \Delta_{\sigma_2}A).$$

(Recall that the standard deviation of a probability measure on $\mathbb{R}$ is the square root of the integral of $(\lambda - \mu)^2$ times the probability measure on the real line $\mathbb{R}_\lambda$ with coordinate λ, where μ is the *mean* or *expected value*, i.e., μ is the integral of λ times the probability measure.)