Math 132
Problem Set 2
Spring, 2023

This problem set is due on Friday, Feb. 9th. Please make your answers as complete and clear as possible. You are allowed to discuss these problems with others in the class, but your writing should be your own.

1. Using the *Preimage Theorem*, show that the Stiefel Manifold from Problem 2 of Problem Set 1 is a smooth manifold of dimension $2n-3$. Can you generalize your argument to the Stiefel manifold of orthornormal k-frames in $\mathbb{R}^n$?

2. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix, and define

$$f_A : \mathbb{R}^n \to \mathbb{R}$$

by

$$\begin{aligned} f_A(v) &= v^T \cdot A \cdot v \\ &= \sum a_{ij} v_i v_j \end{aligned}$$

in which we are interpreting a vector $v \in \mathbb{R}^n$ as a column vector, with i^{th} coordinate v_i. Set

$$S_A = \{v \in \mathbb{R}^n \mid f_A(v) = 1\}.$$

Show that if $\det A \neq 0$, then S is a smooth manifold of dimension $(n-1)$. Describe the tangent space to S at a point v. (Recall that in the Preimage Theorem, the tangent space to $f^{-1}(y)$ at a point x is the kernel of df_x.)

3. (This is GP, Problem 18 of Chapter 1, §1. We will be making quite a lot of use of this in the days ahead).

(a) An extremely useful function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Prove that f is smooth.

(b) Suppose that $a < b$ are real numbers. Show that $g(x) = f(x-a)f(b-x)$ is a smooth function, positive on (a,b) and zero elsewhere. It follows that

$$h(x) = \frac{\int_{-\infty}^{x} g\,dx}{\int_{-\infty}^{\infty} g\,dx}$$

is a smooth function satisfying

$$h(x) = \begin{cases} 0 & x < a \\ 1 & x > b \\ 0 < h(x) < 1 & x \in (a,b). \end{cases}$$

In fact h is monotone increasing on (a,b).

(c) Now construct a smooth function on $\mathbb{R}^k$ that equals 1 on the ball of radius a, is zero outside the ball of radius b and is strictly between 0 and 1 at points x with $a < |x| < b$. Such a function is called a *bump* function, and will play a very important role in our later work.

4. (GP, §4, Problem 7. You'll need some point set topology for this one. Look at this one in the book. It has a good picture). Suppose that y is a regular value of $f : X \to Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$. Prove there exists a neighborhood U of $y \in Y$ such that $f^{-1}(U)$ is a disjoint union $V_1 \coprod \cdots \coprod V_N$, where V_i, is an open neighborhood of x_i, and f maps each V_i, diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W, of X, that are mapped diffeomorphically. Show that $f(X - \cup W_i)$ is compact and does not contain y.]

5. (GP, §4, Problem 12). Prove that the set of all 2×2 matrices of rank 1 is a three-dimensional submanifold of $\mathbb{R}^4 = M(2)$. [HINT: Show that the determinant function is a submersion on the manifold of nonzero 2×2 matrices $M(2) - 0$]. (REMARK: This can be generalized to the space of $(n+1) \times (n+1)$ rank 1 matrices, and also to the space of *projection matrices* (a projection matrix is a matrix P satisfying $P^2 = P$). The space of all rank 1-matrices is a smooth manifold of dimension $2n + 1$. The space of rank 1 projection matrices is a smooth manifold of dimension $2n$ called a *Jouanolou device* for $\mathbf{RP}^n$. For the general case it is easier to show directly that it is a manifold than it is to use the Preimage Theorem.)