Math 230a Problem Set 5

Lev Kruglyak

Due: October 9, 2024

Collaborators: *AJ LaMotta, Ignasi Vicente*

Problem 1. Let n be a positive integer, V be an n-dimensional vector space, A an affine space over V, and $U \subset A$ an open subset.

(a). Let $n = 2$. Suppose $\theta^1, \theta^2 \in \Omega^1(U)$ are 1-forms so that for each $p \in U$ the values θ^1_p, θ^2_p form a basis of V^*. Prove that there exists a unique $\Theta \in \Omega^1(U)$ such that

$$d\theta^1 + \Theta \wedge \theta^2 = 0,$$
$$d\theta^2 - \Theta \wedge \theta^1 = 0.$$

Suppose first that there were two such forms, Θ, Θ' satisfying the equations. Let $\delta = \Theta - \Theta'$. By taking the differences of the two equations for Θ and Θ', we would get

$$\delta \wedge \theta^2 = 0 \quad \text{and} \quad \delta \wedge \theta^1 = 0.$$

However since θ^1, θ^2 form a basis, we can write $\delta = f\theta^1 + g\theta^2$ for some functions $f, g \in \Omega^0(U)$. However $0 = \delta \wedge \theta^2 = f\theta^1 \wedge \theta^2$ and $0 = \delta \wedge \theta^1 = -g\theta^1 \wedge \theta^2$ so $f = g = 0$ and hence $\delta = 0$. This proves uniqueness.

To prove existence, let's express a hypothetical solution Θ as $\Theta = f\theta^1 + g\theta^2$. Then the conditions for it to be a solution are

$$d\theta^1 = -f\theta^1 \wedge \theta^2$$
$$d\theta^2 = -g\theta^1 \wedge \theta^2$$

However, since θ^1, θ^2 form a basis, $\theta^1 \wedge \theta^2$ is a generator for $\Omega^2(U)$. So functions $-f$ and $-g$ exist by virtue of $d\theta^1$ and $d\theta^2$ being 2-forms.

(b). Repeat for arbitrary n. Hence $\theta^1, \dots, \theta^n$ is a moving coframe on U and we seek unique 1-forms $\Theta^i_j \in \Omega^1(U)$ such that for all $1 \le i, j \le n$ we have

$$d\theta^i + \Theta^i_j \wedge \theta^j = 0,$$
$$\Theta^i_j + \Theta^j_i = 0.$$

As before, let's begin by proving uniqueness. Suppose Θ^i_j and $\overline{\Theta}^i_j$ are two sets of 1-forms which satisfy the equations. Letting $\delta^i_j = \Theta^i_j - \overline{\Theta}^i_j$, note that as before we get

$$\delta^i_j \wedge \theta^j = 0 \quad \text{and} \quad \delta^i_j + \delta^j_i = 0.$$

If we write $\delta^i_j = \alpha^i_{j,k}\theta^k$ for some $\alpha^i_{j,k} \in \Omega^0(U)$, then by the first equation we get $0 = \delta^i_j \wedge \theta^j = \alpha^i_{j,k}\theta^k \wedge \theta^j$ which means that $\alpha^i_{j,k} = -\alpha^i_{k,j} = 0$ for all $k \neq j$. This implies that $\delta^i_j = \alpha^i_{j,j}\theta^j$. However, by the second equation, we can derive that $\alpha^i_{j,j} = 0$ and so $\delta^i_j = 0$. Thus, Θ^i_j are unique.

To show existence, let's write $\Theta^i_j = \alpha^i_{j,k}\theta^k$ for functions $\alpha^i_{j,k} \in \Omega^0(U)$. Plugging this into the required equations, we get

$$d\theta^i = -\alpha^i_{j,k}\theta^k \wedge \theta^j \quad \text{and} \quad \alpha^i_{j,k}\theta^k = -\alpha^j_{i,k}\theta^k.$$

Note that $\theta^k \wedge \theta^j$ form a basis of $\Omega^2(U)$ for $k < j$. Thus, we can let $\alpha^i_{j,k}$ be the coefficient of $\theta^j \wedge \theta^k$ in $d\theta^i$ for $k < j$, and extend for $k \geq j$ by setting $\alpha^i_{j,j} = 0$ and $\alpha^i_{j,k} = -\alpha^i_{k,j}$. This clearly satisfies the equations so $\Theta = \alpha^i_{j,k}\theta^k$ is a solution.

Problem 2. Let X be a smooth manifold and $\Theta \in \Omega^1(X)$ a nowhere vanishing 1-form. Let $H \in TX$ be the kernel of Θ, which is a codimension one distribution on X. Use Θ to construct a trivialization of the quotient bundle $TX/H \to X$. Compute the Frobenius tensor of H in terms of Θ.

At any point $p \in X$, note that $H_p = \ker \Theta_p$ so Θ induces an isomorphism $\overline{\Theta}_p : T_pX/H_p \to \mathbb{R}$. This is continuous, across $p \in X$, and so we get a diffeomorphism $\overline{\Theta} \times \pi : TX/H \to \mathbb{R} \times X$ where $\pi : TX/H \to X$ is the bundle map. This is exactly a trivialization of TX/H.

Now recall that the Frobenius tensor of H is a bilinear map

$$\begin{aligned} \phi_H : \Gamma(H) \times \Gamma(H) &\longrightarrow \Gamma(TX/H) \\ (\xi, \eta) &\longmapsto [\xi, \eta] \mod H. \end{aligned}$$

If we associate TX/H with $\mathbb{R} \times X$ by our trivialization, the Frobenius tensor becomes

$$\phi_H(\xi, \eta) = \Theta([\xi, \eta]).$$

Problem 3. Let $I \subset \mathbb{R}$ be an interval and suppose $(x, y) : I \to \mathbb{A}^2_{x,y}$ is an immersion into the standard Euclidean plane. Compute the curvature of this immersion as a function of $t \in I$.

Let's first come up with an adapted frame e_1, e_2 and corresponding forms e^1, e^2 in x, y coordinates. Picking a canonical coorientation of the curve, we can choose

$$e_1 = \frac{1}{v}\left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right) \quad \text{and} \quad e_2 = \frac{1}{v}\left(-\dot{y}\frac{\partial}{\partial x} + \dot{x}\frac{\partial}{\partial y}\right)$$

where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$. By solving the system of equations $\langle \theta^i, e_j \rangle = \delta^i_j$, we get the forms

$$e^1 = \frac{1}{v}\left(\dot{x}\,dx + \dot{y}\,dy\right) \quad \text{and} \quad e^2 = \frac{1}{v}\left(-\dot{y}\,dx + \dot{x}\,dy\right).$$

To get the form Θ^2_1, we use the form $\Theta^2_1 = \langle e^2, \dot{e}_1 \rangle$. Using standard differentiation, we get

$$\begin{aligned} \langle e^2, \dot{e}_1 \rangle &= v^{-3}(-\dot{y}(\ddot{x}v - \dot{x}\dot{v}) + \dot{x}(\ddot{y}v - \dot{y}\dot{v}))\,dt \\ &= v^{-2}(\dot{x}\ddot{y} - \dot{y}\ddot{x}). \end{aligned}$$

This means that $\Theta^2_1 = v^{-3}(\dot{x}\ddot{y} - \dot{y}\ddot{x})\theta^1$ so the curvature is

$$\kappa(t) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Problem 4. Compute the curvature of the following curves in the Euclidean plane $\mathbb{A}^2_{x,y}$.

(a). The parabola $y = kx^2$ for $k \in \mathbb{R}$.

Using the previous formula, we get

$$\kappa(t) = \frac{2k}{(1 + 4k^2t^2)^{3/2}}.$$

Thus if $k = 0$ the curvature is zero which makes sense. Also, curvature is maximized at $t = 0$ which makes sense since that's the tightest curving point on the parabola.

(b). The ellipse $x^2/a^2 + y^2/b^2 = 1$, $a, b \in \mathbb{R}^{>0}$.

We can parametrize the ellipse as $x(t) = a\cos(t)$ and $y(t) = b\sin(t)$. We then can compute

$$\dot{x}(t) = -a\sin(t), \quad \ddot{x}(t) = -a\cos(t), \quad \dot{y}(t) = b\cos(t), \quad \ddot{y}(t) = -b\sin(t).$$

Plugging this into the curvature formula and simplifying gives us

$$\kappa(t) = \frac{ab}{(a^2\sin^2(t) + b^2\cos^2(t))^{3/2}}.$$

(c). The cycloid that is the image of $(t - \sin(t), 1 - \cos(t))$.

Computing derivatives, we get:

$$\dot{x}(t) = 1 - \cos(t), \quad \ddot{x}(t) = \sin(t), \quad \dot{y}(t) = \sin(t), \quad \ddot{y}(t) = \cos(t).$$

Using the curvature formula and simplifying gives us

$$\kappa(t) = \frac{\cos(t) - 1}{(2(1 - \cos(t)))^{3/2}}.$$