Math 230a Problem Set 11

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Due: December 4, 2024

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Problem 1. Let X be a Riemannian manifold and suppose $\{x_1, \dots, x_N\} \subset X$ is a finite subset. In lecture, I constructed a manifold with boundary $\widehat{X}$ and map $p : \widehat{X} \to X$ that is a diffeomorphism on the complement of $\bigsqcup_i p^{-1}(x_i)$. Furthermore, if ξ is a vector field on X with a transverse zero at each x_i and no other zero, then the normalization $s = \xi/\|\xi\| : X \setminus \{x_1, \dots, x_N\} \to S(TX)$ extends over $\widehat{X}$. Explore this construction in detail when $\dim X = 1$.

Problem 2. Consider Chern-Simons-Weil forms for $G = \mathrm{U}_1$ the circle group of unit norm complex numbers. The Lie algebra is $\mathfrak{g} = i\mathbb{R}$.

(a). For each $k \in \mathbb{Z}^{>0}$ identify the vector space of degree k symmetric G-polynomials on $\mathfrak{g}$.

Since U_1 is abelian, the adjoint representation is trivial. Now as $\mathfrak{u}_1 = i\mathbb{R}$, the vector space of degree k symmetric G-polynomials is

$$(\mathrm{Sym}^k \mathfrak{u}_1^*)^{\mathrm{U}_1} = \mathrm{Sym}^k(\mathfrak{u}_1)^* = \mathbb{R} \cdot (-i \cdot x)^k.$$

We'll represent any $h \in (\mathrm{Sym}^k \mathfrak{u}_1^*)^{\mathrm{U}_1}$ as the monomial $h(x) = (-i)^k \kappa \cdot x^k$ for some $\kappa \in \mathbb{R}$.

(b). If $\pi : P \to X$ is a principal G-bundle with connection Θ and h is a G-invariant polynomial of degree k, then the Chern-Simons form restricts to a closed $(2k-1)$-form on each fiber of π, and this form may be identified with a bi-invariant form on G. Identify this form for $k = 1$. For which h is the integral of this form an integer?

Recall that the Chern-Simons form of a connection Θ is given by

$$\omega = k \cdot h(\Theta \wedge \Omega \wedge \cdots \wedge \Omega) \in \Omega^{2k-1}(P).$$

Now since U_1 is abelian, $\Omega = d\Theta + [\Theta \wedge \Theta]/2 = d\Theta$ so the Chern-Simons form is $\omega = k \cdot h(\Theta \wedge d\Theta \wedge \cdots \wedge d\Theta)$. Let $x \in X$ be a point in the base space, and let $\iota_x : P_x \to P$ be the inclusion of the fiber over x. By the definition of a connection form, we know that the pullback $\iota_x^* \Theta$ along this inclusion is the Maurer-Cartan form $\theta_{P_x} \in \Omega^1(P_x; \mathfrak{u}_1)$ on the fiber. Pulling back the Chern-Simons form gives

$$\iota_x^* \omega = k \cdot h(\iota_x^* \Theta \wedge \iota_x^* d\Theta \wedge \cdots \wedge \iota_x^* d\Theta) = k \cdot h(\theta_{P_x} \wedge d\theta_{P_x} \wedge \cdots \wedge d\theta_{P_x}).$$

However the fibers are each diffeomorphic to U_1, and $d\theta_{P_x}$ is a 2-form so it must vanish. Thus the only non-vanishing Chern-Simons forms are those for $k = 1$. In this case, any $h \in (\mathrm{Sym}^1 \mathfrak{u}_1^*)^{\mathrm{U}_1}$ is given by $h(x) = -i\kappa \cdot x$, so the restricted Chern-Simons form is

$$\iota_x^* \omega = -i\kappa \cdot \theta_{P_x} \in \Omega^1(P_x).$$

This can be identified with the bi-invariant form $-i\kappa \cdot \theta_{U_1} = \kappa \cdot d\theta \in \Omega^1(\mathrm{U}_1)$. The integral of this form is

$$\int_{P_x} \iota_x^* \omega = \int_{\mathrm{U}_1} -i\kappa \cdot \theta_{\mathrm{U}_1} = \int_0^{2\pi} \kappa \cdot d\theta = 2\pi\kappa.$$

For this to be an integer, κ must be equal to $n/2\pi$ for some $n \in \mathbb{Z}$.

(c). Consider the Hopf bundle $\pi : S^3 \to S^2$, which is a principal G-bundle. Construct a connection. Compute the integral of the Chern-Weil form over S^2. For which h is this an integer?

Recall that SU_2 consists of unitary complex 2×2 matrices with determinant 1, and the Lie algebra $\mathfrak{su}_2$ consists of 2×2 skew-Hermitian traceless matrices. We can explicitly parametrize them by

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \begin{array}{c} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 + |\beta|^2 = 1 \end{array} \right\} \quad \text{and} \quad \mathfrak{su}_2 = \left\{ \begin{pmatrix} -i\theta & \overline{z} \\ -z & i\theta \end{pmatrix} : \theta \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

The Lie algebra $\mathfrak{su}_2$ can be given a basis of Pauli matrices

$$\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Note that the commutators are $[\sigma_1, \sigma_2] = -\sigma_3$, $[\sigma_2, \sigma_3] = -\sigma_1$, and $[\sigma_3, \sigma_1] = -\sigma_2$. We include the normalization factor of $1/\sqrt{2}$ so that we have

$$\mathrm{Tr}(\sigma_1^2) = \mathrm{Tr}(\sigma_2^2) = \mathrm{Tr}(\sigma_3^2) = 1$$

or in other words, $\sigma_1, \sigma_2, \sigma_3$ is an orthonormal basis for $\mathfrak{su}_2$ under the inner product $\langle X, Y \rangle = \mathrm{Tr}(XY)$.

Now let's construct a connection $\Theta \in \Omega^1(\mathrm{SU}_2; \mathfrak{u}_1)$ on $\pi : S^3 \to S^2$. Let's start with the Maurer-Cartan form $\theta_{\mathrm{SU}_2} \in \Omega^1(\mathrm{SU}_2; \mathfrak{su}_2)$. The Maurer-Cartan form can be written as $\theta_{\mathrm{SU}_2} = \theta^1 \sigma_1 + \theta^2 \sigma_2 + \theta^3 \sigma_3$ for some forms $\theta^i \in \Omega^1(\mathrm{SU}_2)$. These forms must satisfy the Maurer-Cartan equations:

$$d\theta^i + \frac{1}{2} \sum_{j,k} c^i_{jk} \theta^j \wedge \theta^k = 0 \quad \Longrightarrow \quad \begin{aligned} d\theta^1 &= \theta^2 \wedge \theta^3 \\ d\theta^2 &= \theta^3 \wedge \theta^1 \\ d\theta^3 &= \theta^1 \wedge \theta^2 \end{aligned}$$

Let's split $\mathfrak{su}_2 = \mathfrak{m} \oplus \mathfrak{u}_1$ by letting $\mathfrak{m}$ be the span of σ_1 and σ_2, and with $\mathfrak{u}_1$ the span of σ_3. This is a reductive structure on the symmetric space $\mathrm{SU}_2 / \mathrm{U}_1$. Let's now let Θ be the projection of θ_{SU_2} onto $\mathfrak{u}_1$. We can thus write $\Theta = \theta^3 \sigma_3 = i \cdot \theta^3$. Since U_1 is abelian, the curvature is

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta] = d\Theta = i(d\theta^3) = i \cdot (\theta^1 \wedge \theta^2) \in \Omega^2(\mathrm{SU}_2; \mathfrak{u}_1).$$

Note that all Chern-Weil forms vanish for $k > 1$, since $\Omega \wedge \Omega = 0$, as it is a 4-form on the 3-manifold SU_2. When $k = 1$, a polynomial $h \in (\mathrm{Sym}^k \mathfrak{u}_1^*)^{\mathrm{U}_1}$ can be written as $h(x) = -i\kappa \cdot x$ for some $c_k \in \mathbb{R}$. In this case, the Chern-Weil form is

$$\omega = h(\Omega) = \kappa \cdot (\theta^1 \wedge \theta^2).$$

This is a closed 2-form on SU_2 since $\omega = d(\kappa \cdot \theta^3)$. It also must descend to some 2-form $\widetilde{\omega} \in \Omega^2(S^2)$, i.e. we have $\omega = \pi^* \widetilde{\omega}$. We would like to calculate the integral of $\widetilde{\omega}$ over S^2.

(d). Now consider $k = 2$, so the Chern-Weil form has degree 4. Construct a nontrivial principal G-bundle over $S^2 \times S^2$ by first taking the Cartesian product of the Hopf bundle with itself to form a principal $(G \times G)$-bundle over $S^2 \times S^2$, then use the homomorphism $G \times G \to G$ to form the associated principal G-bundle. Compute the integral of the Chern-Weil form. For which h is the answer an integer?

Let's begin with the principal $(\mathrm{U}_1 \times \mathrm{U}_1)$-bundle $\pi \times \pi : \mathrm{SU}_2 \times_{S^2 \times S^2} \mathrm{SU}_2 \to S^2 \times S^2$ – for brevity let's call the total space E. Let $\pi_1, \pi_2 : E \to \mathrm{SU}^2$ be respective projection maps, and let $\widetilde{\pi}_i = \pi \circ \pi_i$ be the composition with the Hopf fibration. The induced connection on this bundle can be written as

$$\Theta = (i \cdot \pi_1^* \theta^3) \oplus (i \cdot \pi_2^* \theta^3) \in \Omega^1(E; \mathfrak{u}_1 \oplus \mathfrak{u}_1)$$

By the results of the previous problem, the curvature of this connection is then

$$\Omega = (i \cdot \widetilde{\pi}_1^* \, dA) \oplus (i \cdot \widetilde{\pi}_2^* \, dA) \in \Omega^2(E; \mathfrak{u}_1 \oplus \mathfrak{u}_1),$$

where dA is the area form on S^2.

Now we want to understand the pushforward of Θ to the principal U_1-bundle $E' = E \times_{\mathrm{U}_1 \times \mathrm{U}_1} \mathrm{U}_1$ associated to E under the multiplication homomorphism $\mu : \mathrm{U}_1 \times \mathrm{U}_1 \to \mathrm{U}_1$. The differential $\mu_* : \mathfrak{u}_1 \oplus \mathfrak{u}_1 \to \mathfrak{u}_1$ is the addition map. Let $\overline{\theta}_i^j \in \Omega^1(E')$ be the pushforwards of $\theta_j^i \in \Omega^1(E)$ to the associated bundle. Then, the connection and curvature of the associated bundle is

$$\Theta' = i \cdot (\overline{\theta}_1^3 + \overline{\theta}_2^3) \quad \text{and} \quad \Omega' = 2i \cdot (\overline{\theta}_1^1 \wedge \overline{\theta}_1^2 + \overline{\theta}_2^1 \wedge \overline{\theta}_2^2).$$

A polynomial $h \in (\mathrm{Sym}^2 \mathfrak{u}_1^*)^{\mathrm{U}_1}$ can be written as $h(x) = -\kappa \cdot x^2$ so the Chern-Weil form is

$$\begin{aligned}
\omega = h(\Omega' \wedge \Omega') = -\kappa \cdot \Omega' \wedge \Omega' &= 4\kappa \cdot (\overline{\theta}_1^1 \wedge \overline{\theta}_1^2 + \overline{\theta}_2^1 \wedge \overline{\theta}_2^2)(\overline{\theta}_1^1 \wedge \overline{\theta}_1^2 + \overline{\theta}_2^1 \wedge \overline{\theta}_2^2) \\
&= 8\kappa \cdot \overline{\theta}_1^1 \wedge \overline{\theta}_1^2 \wedge \overline{\theta}_2^1 \wedge \overline{\theta}_2^2 \\
&= 2\kappa \cdot d\overline{\theta}_1^3 \wedge d\overline{\theta}_2^3.
\end{aligned}$$

Recall that $d\overline{\theta}_i^3$ descends to a form $\widetilde{\omega}_i$ on the i-th S^2 term in $S^2 \times S^2$ with total integral 2π. This means that ω descends to $2\kappa \cdot \widetilde{\omega}_1 \wedge \widetilde{\omega}_2$ on $S^2 \times S^2$ and so

$$\int_{S^2 \times S^2} 2\kappa \cdot \widetilde{\omega}_1 \wedge \widetilde{\omega}_2 = 2\kappa \cdot \left(\int_S^2 \widetilde{\omega}_1 \right)^2 = 8\pi^2 \kappa.$$

Thus, the only h for which the integral is an integer are

$$h(x) = -\frac{n}{8\pi^2} \cdot x^2 \quad \text{for} \quad n \in \mathbb{Z}.$$

(e). Continuing with $k = 2$, take the base 4-manifold to be $\mathbb{CP}^2$. For which h do you find an integer when you integrate the Chern-Weil form?

Problem 3. Now consider $G = \mathrm{SU}_2$.

(a). Identify the space of G-invariant polynomials of degree 2 on $\mathfrak{g}$.

Let's use the same bases from the previous problem. First, note that the adjoint representation of SU_2 on $\mathfrak{su}_2$ factors through the double cover $\mathrm{SU}_2 \to \mathrm{SO}_3$. **prove this.**

This immediately implies that there are no nonzero SU_2-invariant polynomials of degree $k = 1$. In the next case of $k = 2$, any spherically symmetric polynomial must factor through the bilinear form $\langle X, Y \rangle = \mathrm{Tr}(XY)$. Thus, we can write any $h \in (\mathrm{Sym}^2 \, \mathfrak{su}_2^*)^{\mathrm{SU}_2}$ as $h(X, Y) = \kappa \cdot \mathrm{Tr}(XY)$ for some $\kappa \in \mathbb{R}$.

(b). Write an explicit formula for the Chern-Simons 3-form of a connection. Use matrix multiplication in your formula rather than the Lie bracket.

If $h \in (\mathrm{Sym}^2\,\mathfrak{su}_2^*)^{\mathrm{SU}_2}$, then the Chern-Simons 3-form of a principal SU_2-bundle $\pi : P \to X$ with a connection $\Theta \in \Omega^1(P;\mathfrak{su}_2)$ is:

$$\mathrm{CS}_3(\Theta) = 2 \cdot h(\Theta \wedge \Omega) = 2\kappa \cdot \mathrm{Tr}\left(\Theta\, d\Theta + \frac{1}{2}\Theta[\Theta \wedge \Theta]\right) = 2\kappa\left(\mathrm{Tr}(\Theta\, d\Theta) + \frac{1}{2}\mathrm{Tr}(\Theta[\Theta \wedge \Theta])\right)$$

(c). Use the homomorphism $\mathrm{U}_1 \to \mathrm{SU}_2$ to construct SU_2-connections from U_1-connections. How are the Chern-Simons forms related?

(d). Investigate integrality of the integral of Chern-Weil forms for $k = 2$.

Problem 4. Let $\Sigma \subset \mathbb{E}^3$ be a closed surface that is a submanifold of Euclidean 3-space. Prove that Σ has a point of positive Gauss curvature.

Since Σ is compact, it must be bounded by a sphere in $\mathbb{E}^3$. Without loss of generality, we can shrink the sphere so that it intersects Σ tangentially at some point $p \in \Sigma$. Note that a non-tangential intersection point of this type is impossible, since the sphere would no longer bound the surface. Then the Gauss curvature at this point K_p must be positive, since otherwise following a geodesic path would allow one to reach points of Σ outside of the sphere – a contradiction.

Problem 5. Let G be a Lie group, let $H \subset G$ be a closed Lie subgroup, and assume that the homogeneous manifold G/H has a reductive structure, i.e. an H-invariant splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ as vector spaces.

(a). Recall the definition of the canonical connection on the principal H-bundle $\pi : G \to G/H$.

A connection on π is a form $\Theta \in \Omega^1(G;\mathfrak{h})$. The canonical choice is to start with the Maurer-Cartan form $\theta_G \in \Omega^1(G;\mathfrak{g})$ and set $\Theta = \pi_{\mathfrak{h}}\theta_G$ where $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ is the projection with kernel $\mathfrak{p}$.

(b). Compute its curvature.

The Maurer-Cartan equation states that

$$d\theta_G + \frac{1}{2}[\theta_G \wedge \theta_G] = 0.$$

Combining this with the curvature of the connection and the facts that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p}$, we get:

$$\begin{aligned}
\Omega &= d\Theta + \frac{1}{2}[\Theta \wedge \Theta] \\
&= d\pi_{\mathfrak{h}}\theta_G + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= \pi_{\mathfrak{h}} d\theta_G + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[\theta_G \wedge \theta_G] + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[(\pi_{\mathfrak{h}}\theta_G + \pi_{\mathfrak{p}}\theta_G) \wedge (\pi_{\mathfrak{h}}\theta_G + \pi_{\mathfrak{p}}\theta_G)] + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] - \frac{1}{2}\pi_{\mathfrak{h}}[\pi_{\mathfrak{p}}\theta_G \wedge \pi_{\mathfrak{p}}\theta_G] + \frac{1}{2}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{h}}\theta_G] \qquad (\text{since } \pi_{\mathfrak{h}}[\pi_{\mathfrak{h}}\theta_G \wedge \pi_{\mathfrak{p}}\theta_G] = 0) \\
&= -\frac{1}{2}\pi_{\mathfrak{h}}[\pi_{\mathfrak{p}}\theta_G \wedge \pi_{\mathfrak{p}}\theta_G].
\end{aligned}$$

(c). What is the meaning of the torsion of this connection? Compute it.

(d). What is the meaning of geodesics of this connection? Compute them.

(e). Consider the transitive action of $G \times G$ on G by left and right multiplication. Is this homogeneous space reductive?

Problem 6.

(a). Let (Σ, g) be a Riemannian 2-manifold, and suppose $\phi : \Sigma \to \mathbb{R}$ is a smooth function. Compute the Gauss curvature K' of the metric $e^{2\phi}g$ in terms of ϕ and the Gauss curvature K of the metric g.