MATH 231A: ALGEBRAIC TOPOLOGY
HOMEWORK 7
DUE: WEDNESDAY, OCTOBER 26 AT 10:00PM ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Euler characteristic and homology with coefficients in a field (5 points)

Do Exercise 19.2 of LAT.

2. Problem 2: $\mathbb{F}_p$ sometimes knows more than $\mathbb{Z}$ (10 points)

Do Exercise 19.3 of LAT.

3. Problem 3: Tensor product of cyclic groups (5 points)

Do Exercise 20.12 of LAT.

4. Problem 4: The Borsuk–Ulam theorem (20 points)

The goal of this problem is to prove the Borsuk–Ulam theorem, which states that for every map

$$g : S^n \to \mathbb{R}^n,$$

there is a point $x \in S^n$ with $g(x) = g(-x)$. Along the way, we will prove that any *odd* map $f : S^n \to S^n$, i.e. any map satisfying $f(-x) = -f(x)$ for all $x \in S^n$, has odd degree.

Let $p : \widetilde{X} \to X$ denote a two-sheeted covering space.

(i) Prove that each singular n-simplex $\sigma : \Delta^n \to X$ admits exactly two lifts $\sigma_1, \sigma_2 : \Delta^n \to \widetilde{X}$. (You may freely use standard facts about covering spaces.)

(ii) Prove that there is a short exact sequence of chain complexes

$$0 \to S_*(X; \mathbb{F}_2) \xrightarrow{\tau} S_*(\widetilde{X}; \mathbb{F}_2) \xrightarrow{p_*} S_*(X; \mathbb{F}_2) \to 0$$

where the *transfer map* τ is defined on n-simplices σ by $\tau(\sigma) = \sigma_1 + \sigma_2$. This gives rise to the long exact *transfer sequence*

$$\cdots \to H_n(X; \mathbb{F}_2) \xrightarrow{\tau_*} H_n(\widetilde{X}; \mathbb{F}_2) \xrightarrow{p_*} H_n(X; \mathbb{F}_2) \to H_{n-1}(X; \mathbb{F}_2) \to \dots.$$

(iii) Given an odd map $f : S^n \to S^n$, there is an induced map $\overline{f} : \mathbb{RP}^n \to \mathbb{RP}^n$. Prove that there is a commutative diagram of transfer sequences of the form

$$\begin{tikzcd}
\dots \arrow[r] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r, "\tau_*"] \arrow[d, "\overline{f}_*"] & H_k(S^n; \mathbb{F}_2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] \arrow[d, "\overline{f}_*"] & H_{k-1}(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] \arrow[d, "\overline{f}_*"] & \dots \\
\dots \arrow[r] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r, "\tau_*"] & H_k(S^n; \mathbb{F}_2) \arrow[r, "p_*"] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] & H_{k-1}(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] & \dots
\end{tikzcd}$$

(iv) Using (iii), prove that any odd map $f : S^n \to S^n$ has odd degree.

(v) Given a map $g : S^n \to \mathbb{R}^n$, prove that the odd map $f : S^n \to \mathbb{R}^n$ given by $f(x) = g(x) - g(-x)$ must have a zero. Deduce the Borsuk–Ulam theorem. (Hint: Consider the equatorial inclusion $S^{n-1} \subset S^n$)

An immediate corollary is the fact that S^n is not homeomorphic to a subset of $\mathbb{R}^n$. Another important consequence of the Borsuk–Ulam theorem is the *ham sandwich theorem*, which states that for any n compact sets $A_1, \dots A_n$ in $\mathbb{R}^n$ (note that the ns are the same!), there is a *single* hyperplane dividing each A_i into two sets of equal measure.

5. Problem 5: Homology of $\mathbb{RP}^n$ using the transfer sequence (10 points)

The computation of the homology of $\mathbb{RP}^n$ via cellular homology presented in class depended on a careful analysis of orientations and signs. In this problem, you will use *transfer sequence* introduced in Problem 4 to recompute the homology of $\mathbb{RP}^n$ in a way that is less vulnerable to sign errors.

(i) Given the fact that $\mathbb{RP}^n$ is an n-dimensional CW complex, use the transfer sequence associated to the cover $p : S^n \to \mathbb{RP}^n$ to compute $H_*(\mathbb{RP}^n; \mathbb{F}_2)$.

(ii) The transfer map τ may be defined at the level of integral chains by the same formula as in Problem 4(ii). Verify that the induced composite

$$H_n(X; \mathbb{Z}) \xrightarrow{\tau_*} H_n(\widetilde{X}; \mathbb{Z}) \xrightarrow{p_*} H_n(X; \mathbb{Z})$$

is multiplication by 2.

(iii) Using the pushout squares

$$\begin{tikzcd} S^{n-1} \arrow[r, "p"] \arrow[d] & \mathbb{RP}^{n-1} \arrow[d] \\ D^n \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

and induction on n, reduce the computation of $H_*(\mathbb{RP}^n; \mathbb{Z})$ to the statement that, when n is odd, $p_* : H_n(S^n; \mathbb{Z}) \to H_n(\mathbb{RP}^n; \mathbb{Z})$ sends a generator to ± 2 times a generator.

(iv) Using (i) and (ii), prove this statement.