Problem Set # 4

Math 262a: Quantum theory from a geometric viewpoint I

Due: October 6

Problems

1. The construction of the principal $\mathbb{R}$-bundle with connection $T \to N$ in §3.4 of the notes is stated poorly; here is another attempt.

   Quite generally, if $\pi\colon M \to N$ is a map and if $\mathcal{O}$ is a contravariant local mathematical object on M—a cocycle for a cohomology class, a differential form, fiber bundle over M, a connection on a principal bundle over M, etc.—then there are two general ways we can think of obtaining a mathematical object $\overline{\mathcal{O}}$ on N. First, there might be a *pushforward* or *umker map* or *integration* that produces $\pi_*\overline{\mathcal{O}}$. In general, the geometric nature of $\overline{\mathcal{O}}$ is different from that of $\mathcal{O}$. For example, there might be a degree shift (down by the relative dimension of the map π). The second general construction is *descent*: an object $\overline{\mathcal{O}}$ on N and an isomorphism of $\mathcal{O}$ with the pullback $\pi^*\overline{\mathcal{O}}$. The descended object $\overline{\mathcal{O}}$ has the same geometric nature as $\mathcal{O}$. It is this second scenario that applies here. We put ourselves in the situation of §3.4.

   (a) Consider the product principal $\mathbb{R}$-bundle

   $$p\colon P = (N \times \mathbb{R}_t) \times \mathbb{R}_s \longrightarrow N \times \mathbb{R}_t$$

   Prove that the real-valued 1-form

   $$\Theta = ds + p^*\gamma \;\in \Omega^1_P$$

   is a connection. Compute its curvature $\Omega \in \Omega^2_{N\times\mathbb{R}}$.

   (b) Prove that the curvature descends under the projection

   $$\pi\colon N \times \mathbb{R} \longrightarrow N$$

   In other words, construct a 2-form $\omega \in \Omega^2_{N\times\mathbb{R}}$ such that $\Omega = \pi^*\omega$.

   (c) Now descend $p\colon P \to N \times \mathbb{R}$ and its connection Θ to a principal $\mathbb{R}$-bundle $T \to N$ with connection. Be sure to produce the requisite isomorphism that proves you have a descent. (Hint: the fiber T_n over $n \in N$ is the $\mathbb{R}$-torsor of parallel sections of the restriction of $p\colon P \to N \times \mathbb{R}$ to $\pi^{-1}(n) = \{n\} \times \mathbb{R}$.) How does this relate to the construction in the notes?

   (d) Consider the translation action of $\mathbb{R}_t$ on $N \times \mathbb{R}_t$. Define a lift to P such that that (i) the connection 1-form Θ is invariant, and (ii) if $\xi = \partial/\partial t$ denotes the vector field that generates this lifted translation action, then the contraction $\iota_\xi\Theta$ vanishes. Conclude that the 1-form Θ descends to the quotient of P by the translation action.

   (e) More generally, suppose $\pi\colon Q \to N$ is a principal G-bundle for some Lie group G. What are the descent conditions on a differential form $\Omega \in \Omega^\bullet_Q$? In other words, characterize the image of the injective linear map $\pi^*\colon \Omega^\bullet_N \to \Omega^\bullet_Q$.

2. Consider the simple harmonic oscillator: a mass one particle on $\mathbb{E}^1_x$ with potential $V(x) = x^2/2$. The Hilbert space is $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$ and the Hamiltonian is

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$$

   (a) Compute eigenfunctions for the four smallest eigenvalues of H.

   (b) State and prove the spectral decomposition for H, namely, $\mathcal{H}$ is the Hilbert space completion of the direct sum of the eigenspaces.

   (c) Recall that the classical phase space is $N = \mathbb{E}^1_x \times \mathbb{R}^1_y$ with symplectic form $\omega = dy \wedge dx$. Verify that the 3-dimensional real vector space of affine functions on N is closed under Poisson bracket. Verify the same for the 6-dimensional subspace of affine quadratic functions. What about the affine cubic functions?

   (d) The affine functions under Poisson bracket form the Heisenberg Lie algebra. Construct a representation by unbounded operators on $\mathcal{H}$ (which are skew-adjoint.) Can you extend the representation to the affine quadratic functions?

   (e) Construct an antiunitary operator on $\mathcal{H}$ which implements the time-reversal symmetry that preserves the operator x and sends $\partial/\partial x \mapsto -\partial/\partial x$. Show that the induced operator on $\mathbb{P}\mathcal{H}$ has order 2. What about the operator on $\mathcal{H}$?

3. Let $V \colon \mathbb{E}^1 \to \mathbb{R}$ be a smooth function and consider a quantum mass $m > 0$ particle moving on $\mathbb{E}^1$ with potential energy V. An eigenfunction with eigenvalue $E \in \mathbb{R}$ is an L^2 function $\psi \colon \mathbb{E}^1 \to \mathbb{C}$ that satisfies the *Schrödinger equation*

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \big(V(x) - E\big)\psi(x) = 0.$$

   (a) Prove that if ψ_1, ψ_2 are nonzero eigenfunctions with the same eigenvalue E, then ψ_2 is proportional to ψ_1. In other words, eigenspaces of the Hamiltonian are 1-dimensional. Observe that, depending on V, there may be no eigenspaces, the eigenspaces may span the space of L^2 functions, or anything in between. (Hint: Consider the *Wronskian* $\psi_1\psi_2' - \psi_1'\psi_2$. You may need to use some facts about elliptic second-order ordinary differential equations. For example, solutions are smooth functions and their zero set contains no accumulation points.)

   (b) Prove that each eigenspace contains a real-valued function.

   (c) Prove that if the Hamiltonian is gapped, then the unique up to a constant eigenfunction for the minimal eigenvalue is a nonvanishing function.

4. Let W be a finite dimensional real vector space. Recall that in lecture we discussed the bosonic harmonic oscillator as a representation of an algebraic operator algebra generated by $W \oplus W^*$ on the complex vector space $\mathrm{Sym}^\bullet W^*_{\mathbb{C}}$. The goal now is to introduce inner products and complete to a Hilbert space. For this, suppose W is endowed with a real inner product.

   (a) Induce a hermitian inner product on $W^*_{\mathbb{C}}$.

   (b) Induce a hermitian inner product on $\mathrm{Sym}^\bullet W^*_{\mathbb{C}}$.

   (c) What is the adjoint of the operator $A(w)$, $w \in W$, defined in lecture?

   (d) Is the Hamiltonian (formally) self-adjoint?

5. The abelian group $\mathbb{T} \subset \mathbb{C}$ of unit norm complex numbers has an automorphism $\lambda \mapsto \lambda^{-1}$. Let $\mu\!\!\!/_2 \subset \mathbb{T}$ be the group $\{\pm 1\}$ of square roots of unity.

   (a) Classify group extensions (up to isomorphism)

   $$1 \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow \mu\!\!\!/_2 \longrightarrow 1$$

   in which $\mu\!\!\!/_2$ acts on $\mathbb{T}$ by the nontrivial automorphism.

   (b) Classify group extensions

   $$1 \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow \mathrm{O}_2 \longrightarrow 1$$

   in which O_2 acts nontrivially on $\mathbb{T}$ via the determinant $\mathrm{O}_2 \longrightarrow \mu\!\!\!/_2$.