Math 222 Problem Set 3

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Problem 2. Let G be a connected Lie group and suppose $\pi : \widetilde{G} \to G$ is a covering map. Fix $\widetilde{e} \in \pi^{-1}(e)$. Construct a Lie group structure on $\widetilde{G}$ such that $\widetilde{e}$ is the identity element and π is a homomorphism of Lie groups. Is this Lie group structure unique?

The Lie group structure is unique of $\widetilde{G}$ is connected. Otherwise, you could have a situation like in the case of the coverings $\mathrm{U}_1 \times S_3 \to \mathrm{U}_1$ and $\mathrm{U}_1 \times \mathbb{Z}/6 \to \mathrm{U}_1$. Topologically, these covering maps are identical, but of course the total spaces do not have the same group structure.

Let's now suppose that $\widetilde{G}$ is connected and $\widetilde{e} \in \pi^{-1}(e)$ is fixed. We can find some diffeomorphism $\pi^{-1}|_{\widetilde{U}} : \widetilde{U} \to U$ where $\widetilde{U}$ is a neighborhood of $\widetilde{e}$. We can define a group structure on $\widetilde{U}$ by setting $g \cdot h = \pi^{-1}(\pi(g) \cdot \pi(h))$. We can cover $\widetilde{G}$ by open sets on which π is a diffeomorphism, and can make this a good cover so that the intersections only contain a finite number of points. Then, starting at the original $\widetilde{U}$ we can continue to extend the multiplication in this way, eventually covering the whole connected manifold. This construction necessarily requires connectedness, and a good cover to apply the glueing lemma.

Problem 4. Let $T \subset \mathrm{U}_3$ be the subgroup of diagonal matrices. Identify its normalizer $N(T) \subset \mathrm{U}_3$. Identify the quotient group $N(T)/T$. Points of U_3/T parametrize a certain geometric structure on $\mathbb{C}^3$; what is that geometric structure? Do the same for $\mathrm{U}_3/N(T)$. Generalize to U_n for all $n \in \mathbb{Z}^{>0}$. Specialize to SU_2, where again $T \subset \mathrm{SU}_2$ is the subgroup of diagonal matrices. Do you recognize the group $N(T)$? What is its identity component?

Let's begin by determining the normalizer $N(T) \subset \mathrm{U}_3$. Recall that a matrix $u \in \mathrm{U}_3$ is in the normalizer of T if and only if $uTu^{-1} = T$. The only matrices which preserve diagonal matrices are those which permute diagonal entries and those which multiply each basis vector by a phase. It follows that every matrix in the normalizer can be written as $P_\sigma D$ with P_σ the permutation matrix for a permutation $\sigma \in S_3$ and $D \in T$ is a diagonal matrix which shifts the phase of each vector. The quotient $N(T)/T$ is then isomorphic to the symmetry group S_3. These results generalize for all U_n, we would then have $N(T)/T \cong S_n$.

Next, let's identify the geometric structure induced on $\mathbb{C}^n$ by points of U_n/T. Let's suppose we have some matrix $u \in \mathrm{U}_n/T$. Let $u_1, \dots, u_n$ be the column vectors of this matrix, unique up to multiplication by a phase. Each of these defines a unique line and they must be orthogonal by the unitary condition. Thus, U_n/T parametrizes the space of ordered sequences of orthogonal complex lines in $\mathbb{C}^n$. By similar logic, $\mathrm{U}_n/N(T)$ parametrizes unordered sequences of orthogonal complex lines in $\mathbb{C}^n$ since we only get a permutation class of column vectors.

Finally, for SU_2, the diagonal matrices must be be of the form

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$$

The normalizer is generated by T with a matrix that sends $\theta \mapsto -\theta$. This is the transposition generating S_2. It follows that $N(T)$ is isomorphic to O_2. These results fit with the exceptional isomorphism $\mathrm{SU}_2 \cong \mathrm{Spin}_3$ which double covers SO_2, the normalizer $N(T)$ which consists of two circles maps onto the single circle SO_2.

Problem 5. Continuing with the previous problem, prove that every conjugacy class in U_3 has nonempty intersection with T. What is that intersection? Are those intersections the orbits of a group action on T? What is the analog for SU_3? Can you draw pictures for SU_3? Topologize the space of conjugacy classes. What can you say about this space?

Problem 6.

(a). Let G be a Lie group and let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ be its adjoint representation. The center $Z(G) \subset G$ is contained in the kernel of Ad. Find a Lie group G for which the kernel of Ad is strictly larger than $Z(G)$.

A simple set of examples occurs for non-abelian discrete Lie groups. If a group G is discrete, then $\mathfrak{g}$ is 0-dimensional and so $\ker \mathrm{Ad} = G$. Picking any non-abelian discrete group thus has $Z(G) \subset \ker \mathrm{Ad}$ as a proper subgroup.

(b). Find an example of the following: Lie groups G', G with G' connected, and a homomorphism $\phi : \mathfrak{g}' \to \mathfrak{g}$ between their Lie algebras such that there does *not* exist a homomorphism $\psi : G' \to G$ with $\dot{\psi} = \phi$.

Let $G' = \mathrm{SO}_n$ and $G = \mathrm{GL}_m$. Letting $\phi : \mathfrak{so}_n \to \mathfrak{gl}_m$ be a spinor representation (arising from a map $\mathrm{Spin}_n \to \mathrm{GL}_m$) which does not arise from a representation of SO_n, it follows by assumption ϕ does not come from a map $\psi : \mathrm{SO}_n \to \mathrm{GL}_m$. This requires a little bit of knowledge about the representation theory of Spin_n and SO_n.