MATH 231A: ALGEBRAIC TOPOLOGY
HOMEWORK 8
DUE: WEDNESDAY, NOVEMBER 2 AT 10:00PM ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Coefficient exact sequence (10 points)

Let X denote a space. Using the long exact sequence associated to the short exact sequence of chain complexes

$$0 \to S_*(X;\mathbb{Z}) \xrightarrow{n} S_*(X;\mathbb{Z}) \to S_*(X;\mathbb{Z}/n\mathbb{Z}) \to 0,$$

prove that there are short exact sequences

$$0 \to H_k(X;\mathbb{Z})/nH_k(X;\mathbb{Z}) \to H_k(X;\mathbb{Z}/n\mathbb{Z}) \to \mathrm{tors}_n(H_{k-1}(X;\mathbb{Z})) \to 0,$$

where, given an abelian group A, $\mathrm{tors}_n(A)$ denotes the subgroup of n-torsion elements of A. This is a special case of the universal coefficients theorem!

Use this to recompute $H_*(\mathbb{RP}^n;\mathbb{F}_2)$ from $H_*(\mathbb{RP}^n;\mathbb{Z})$.

2. Problem 2: Sequential colimits (15 points)

Let $\mathbb{N}_{\leq}$ denote the category whose objects are the natural numbers $\{0, 1, 2, \dots\}$ and for which there exists a unique morphism $i \to j$ if and only if $i \leq j$. Given a category $\mathcal{C}$, a *sequence of objects in* $\mathcal{C}$ is a functor $X_\bullet : \mathbb{N}_{\leq} \to \mathcal{C}$. In other words, a sequence of objects in $\mathcal{C}$ is a diagram of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots,$$

where X_i are objects in $\mathcal{C}$ and f_i are morphisms in $\mathcal{C}$.

The *sequential colimit* $\varinjlim_n X_n$ of a diagram $X_\bullet$ is defined via the following universal property: there is a commuting diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[drr, "g_0"'] & X_1 \arrow[r, "f_1"] \arrow[dr, "g_1"] & X_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & \dots \\
 & & \varinjlim_n X_n, &
\end{tikzcd}$$

and given any other commuting diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[drr, "h_0"'] & X_1 \arrow[r, "f_1"] \arrow[dr, "h_1"] & X_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & \dots \\
 & & Y, &
\end{tikzcd}$$

there exists a unique morphism $\alpha : \varinjlim_n X_n \to Y$ such that $h_n = \alpha \circ g_n$. For example, regard an increasing sequence of subsets $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ of a set X as a sequence of spaces. Then it is simple to verify that $\varinjlim_n X_n = \bigcup_{n=0}^\infty X_n$.

Fix a commutative ring R.

(a) Given a sequence $M_\bullet$ of R-modules, prove that

$$\varinjlim_n M_n \cong \frac{\bigoplus_{n\in\mathbb{N}} M_n}{(f_n(x_n) - x_n \text{ for } n \in \mathbb{N}, x_n \in M_n)}.$$

In particular, $\varinjlim_n M_n$ exists.

(b) Suppose that we are given sequences $M_\bullet$, $N_\bullet$, and $P_\bullet$ of R-modules and an exact sequence $M_\bullet \to N_\bullet \to P_\bullet$, i.e. natural transformations of functors with the property that each $M_n \to N_n \to P_n$ is exact. Prove that $\varinjlim_n M_n \to \varinjlim_n N_n \to \varinjlim_n P_n$ is an exact sequence. (Hint: along the way, you will need to show that if the image of $x \in M_n$ in $\varinjlim_n M_n$ is zero, then the image of x in M_k is zero for some $k \geq n$.)

(c) Given a sequence of R-modules $M_\bullet$ and an R-module N, prove that there is a natural isomorphism

$$(\varinjlim_n M_n) \otimes_R N \simeq \varinjlim_n (M_n \otimes_R N).$$

(Hint: use the natural isomorphism $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \cong \mathrm{Hom}_R(M \otimes_R N, P)$.)

Remark: More generally, one may prove analogous results for *filtered diagrams* and *filtered colimits*. These are basically equivalent to *directed systems* and *directed limits*, which are studied in Lecture 23 of LAT.[1] Note that directed limits are in fact colimits and not limits: the terminology is classical and predates the colimit/limit terminology.

3. Problem 3: Flatness of $\mathbb{Q}$ (15 points)

(a) Prove that $\mathbb{Q}$ is isomorphic to the sequential colimit of the following diagram:

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \dots.$$

(b) Using part (a) and Problem 2, prove that $\mathbb{Q}$ is a *flat* $\mathbb{Z}$-module, i.e. that the functor $- \otimes_{\mathbb{Z}} \mathbb{Q} : \mathrm{Ab} \to \mathrm{Vect}_{\mathbb{Q}}$ is exact.

(c) Given a finitely generated abelian group $A \cong \mathbb{Z}^{\oplus r} \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$, prove that $A \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\oplus r}$.

(d) By tensoring with $\mathbb{Q}$ and using basic facts from linear algebra, prove that the rank of finitely generated abelian groups is additive in short exact sequences. That is, given a short exact sequence

$$0 \to A \to B \to C \to 0$$

of finitely generated abelian groups, prove that

$$\mathrm{rk}(B) = \mathrm{rk}(A) + \mathrm{rk}(C).$$

Remark: This is generalized by the notion of *localization* in commutative algebra. Given a multiplicatively closed set $S \subset R$, the localization $R[S^{-1}]$ is defined by formally inverting the elements in S. Then $R[S^{-1}]$ is always flat as an R-module.

4. Problem 4: Tor computations (10 points)

Let R denote a commutative ring.

(a) Given a nonzerodivisor $x \in R$ and an R-module M, compute $\mathrm{Tor}_*^R(R/x, M)$.

(b) Given two ideals $I, J \subset R$, prove that $\mathrm{Tor}_1^R(R/I, R/J) = (I \cap J)/IJ$.

(c) Compute $\mathrm{Tor}_*^{R[x]/x^n}(R, R)$ for any $n \geq 2$.

[1] The only difference is that one assumes that directed limits are indexed over posets, while one works with more general indexing categories when defining filtered colimits.