Math 231b Problem Set 5

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Problem 1. Recall that the homotopy fiber of a map $f : X \to Y$ over some point $* \in Y$ as the space $F(f) = X \times_Y Y^I_*$. Assuming that Y is path connected, the loop space ΩY can "act" on this space on the right by sending $(x, \sigma) \cdot \omega = (x, \sigma \cdot \omega)$.

By passing to π_0, the action described provides a right action of the group $\pi_1(Y)$ on $\pi_0(F(f))$.

a. Show that two elements in $\pi_0(F(f))$ map to the same element of $\pi_0(X)$ if and only if they are in the same orbit under this action.

First observe that a path $\omega : I \to F(f)$ is uniquely deterimed by paths $\omega_x : I \to X$ and a homotopy $h_\omega : I \times I \to Y$ with $h_\omega(0, t) = *$, $h_\omega(1, t) = \omega_x(t)$. Thus, there is a path between $(x_1, \omega_1), (x_2, \omega_2) \in F(f)$ if and only if there is a path $\omega : x_1 \to x_2$, and a homotopy $h : \omega_1 \to \omega_2$ with $h_0 = c_*$ and $h_1 = f(\omega)$. By a reparametrization of the unit square, this is equivalent to constructing a nullhomotopy of the loop $\omega_1 \cdot \omega_x \cdot \overline{\omega_2}$ by a homotopy which fixes $*$.

Recall that the map $\pi(f) : F(f) \to X$ is simply a projection onto the X component, so two path components $[(x_1, \omega_1)]$ and $[(x_2, \omega_2)]$ in $\pi_0(F(f))$ will map to the same element of $\pi_0(X)$ if and only if x_1 and x_2 are in the same path component on X. Now if x_1 and x_2 are in different path components, then clearly the classes $[(x_1, \omega_1)]$ and $[(x_2, \omega_2)]$ must be in different orbits, since the action can not affect the path component of x_1, x_2.

Suppose instead that there is some path $\omega : x_1 \to x_2$. We want to find a loop $\sigma \in \pi_1(Y)$ such that we can construct a path between $(x_1, \sigma \cdot \omega_1)$ and (x_2, ω_2). As we discussed before, this involves finding a nullhomotopy $(\sigma \cdot \omega_1) \cdot f(\omega) \cdot \overline{\omega_2}$ which preserves the basepoint. However, since we can choose σ, we can simply let $\sigma = \omega_2 \cdot \overline{f(\omega)} \cdot \overline{\omega_1}$. This nullhomotopy thus shows that $(x_1, \sigma \cdot \omega_1)$ and (x_2, ω_2) are equal in $\pi_0(F(f))$ so (x_1, ω_1) and (x_2, ω_2) are in the same orbit.

b. Suppose ω is a path in Y from $*$ to y. Write $\omega_\# : \pi_1(Y, *) \to \pi_1(Y, y)$ for the group isomorphism sending σ to $\omega\sigma\omega^{-1}$. Show that the isotropy group of the component of (x, ω) in $F(f, *)$ is

$$\omega_\#^{-1}\mathrm{Im}(f_*) \subset \pi_1(Y, *)$$

where $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$.

Let $\pi_1(Y, *)_{[(x,\omega)]}$ be the described isotropy group. This means that if $\sigma \in \pi_1(Y, *)_{[(x,\omega)]}$, we have a path from $(x, \sigma \cdot \omega)$ to (x, ω). By the argument in the previous part, this is equivalent to saying that we have a (based) nullhomotopy of $(\sigma \cdot \omega) \cdot f(\zeta) \cdot \overline{\omega}$ for some loop $\zeta \in \pi_1(X, x)$. So

$$\sigma \cdot (\omega \cdot f(\zeta) \cdot \overline{\omega}) = c_* \qquad \Longrightarrow \qquad \sigma = \overline{\omega} \cdot \overline{f(\zeta)} \cdot \omega = \omega_\#^{-1}(f(\zeta)) \in \omega_\#^{-1}\mathrm{Im}(f_*).$$

Conversely, given $\sigma = \overline{\omega} \cdot \overline{f(\zeta)} \cdot \omega$ for some $\zeta \in \pi_1(X, x)$, we get a nullhomotopy of $(\sigma \cdot \omega) \cdot f(\zeta) \cdot \overline{\omega}$ so $(x, \sigma \cdot \omega)$ to (x, ω) are in the same path component and hence $\sigma \in \pi_1(Y, *)_{[(x,\omega)]}$. Thus $\pi_1(Y, *)_{[(x,\omega)]} = \omega_\#^{-1}\mathrm{Im}(f_*)$.

c. Suppose that X is path connected, and pick $* \in X$. Conclude from (a) that the evident surjection $\pi_n(X,*) \to [S^n, X]$ can be identified with the orbit projection for the action of $\pi_1(X,*)$ on $\pi_n(X,*)$.

The orbit projection is the map $\pi_n(X,*) \to \pi_n(X,*)/\pi_1(X,*)$. However by a similar argument employed in (a) and the last problem of the previous pset, it's fairly clear to see that $\pi_n(X,*)/\pi_1(X,*)$ can be naturally identified with $[S^n, X]$ in a canonical way.

Problem 2. Given a map $f : X \to Y$ and a point $y \in Y$, let $F(f, y)$ denote the homotopy fiber of f above the point y. Given a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "g"] & Y_2 \end{tikzcd}$$

prove that if $Y_1 \to Y_2$ is an n-equivalence and $F(f_1, y) \to F(f_2, g(y))$ is an n-equivalence for all $y \in Y_1$, then $X_1 \to X_2$ is an n-equivalence.

Extending the fiber sequence one step further, deduce that if $X_1 \to X_2$ is an n-equivalence and $Y_1 \to Y_2$ is an $(n+1)$-equivalence, then $F(f_1, y) \to F(f_2, g(y))$ is an n-equivalence for all $y \in Y_1$.

Recall that the following commutative square square commutes up to homotopy:

$$\begin{tikzcd} \cdots \arrow[r] & \Omega_y Y_1 \arrow[r] \arrow[d] & F(f_1, y) \arrow[r] \arrow[d] & X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[r] \arrow[d, "g"] & \cdots \\ \cdots \arrow[r] & \Omega_{g(y)} Y_2 \arrow[r] & F(f_2, g(y)) \arrow[r] & X_2 \arrow[r, "f_2"] & Y_2 \arrow[r] & \cdots \end{tikzcd}$$

Thus, this diagram passes to the following diagram with exact rows for all k:

$$\begin{tikzcd} \pi_{k+1}(Y_1) \arrow[r, "i(f_1)_*"] \arrow[d] & \pi_k(F(f_1, y)) \arrow[r, "\pi(f_1)_*"] \arrow[d] & \pi_k(X_1) \arrow[r, "(f_1)_*"] \arrow[d] & \pi_k(Y_1) \arrow[r, "i(f_1)_*"] \arrow[d] & \pi_{k-1}(F(f_1, y)) \arrow[d] \\ \pi_{k+1}(Y_2) \arrow[r, "i(f_2)_*"] & \pi_k(F(f_2, g(y))) \arrow[r, "\pi(f_2)_*"] & \pi_k(X_2) \arrow[r, "(f_2)_*"] & \pi_k(Y_2) \arrow[r, "i(f_2)_*"] & \pi_{k-1}(F(f_2, g(y))) \end{tikzcd}$$

We have a couple cases to consider. When $0 < k < n$, the second and fourth vertical arrows are isomorphisms, the fifth arrow is an injection, and the first arrow is a surjection, hence by the five lemma, the middle arrow is an isomorphism. In the case when $k = 0$, it's clear to see that path components are preserved. Finally, in the case when $k = n$, we get a diagram:

$$\begin{tikzcd} \pi_n(F(f_1, y)) \arrow[r] \arrow[d] & \pi_n(X_1) \arrow[r, "(f_1)_*"] \arrow[d] & \pi_n(Y_1) \arrow[r] \arrow[d, "g_*"] & \pi_{n-1}(F(f_1, y)) \arrow[d] \\ \pi_n(F(f_2, g(y))) \arrow[r] & \pi_n(X_2) \arrow[r, "(f_2)_*"] & \pi_n(Y_2) \arrow[r] & \pi_{n-1}(F(f_2, g(y))) \end{tikzcd}$$

with exact rows. Now we have a surjective first column, surjective third column, and isomorphic last column, so by the four lemma, the second column is surjective, proving that $X_1 \to X_2$ is an n-equivalence. By an identical argument (really, we use the same four/five lemma argument on a slightly extended version of the diagram) we can deduce the second part. Here we need $F(f_1, y) \to F(f_2, g(y))$ to be an $(n+1)$ equivalence because the induced vertical arrows are to the right, so we need the extra surjectivity of $\pi_{n+1}(F(f_1, y)) \to \pi_{n+1}(F(f_2, g(y)))$ to use the four lemma in order to deduce surjectivity of $\pi_n(Y_1) \to \pi_n(Y_2)$.

Problem 3. Prove that a map $X \to Y$ of path-connected spaces may be factored as $X \to Z_n \to Y$ with $X \to Z_n$ an isomorphism on π_i for $i \leq n$ and $Z_n \to Y$ an isomorphism on π_i for $i > n$.

We'll construct the space Z_n by successive approximations W_k and then $Z_n = \varinjlim_k W_k$. At each stage, we should have maps $\omega_k : X \to W_k$ and $\sigma_k : W_k \to Y$ with a factorization of f through $\sigma_k \circ \omega_k : X \to W_k \to Y$. Furthermore, there should also be maps $\iota_k : W_k \to W_{k+1}$ which are consistent with the ω_k and σ_k. We also should have $\pi_i(\omega_k)$ an isomorphism for all $i \leq k$ and $i \leq n$, and $\pi_i(\sigma_k)$ an isomorphism for all $i \leq k$ and $i > n$. Then by construction, Z_n would be a desired factorization.

To construct such a space for a given n, for any $k \leq n$ let's start by setting $W_k = X$, with $\omega_k = 1_X$, $\sigma_k = f$, and $\iota_{k-1} = 1_X$. This satisfies all our desired properties. Once $k = n+1$, we require only that we have a factorization and that $\pi_k W_k \to \pi_k Y$ is an isomorphism. We'll present a construction that works by induction to generate the rest of the W_k . Since X, Y are path connected, let's choose some arbitrary consistent basepoint for both, i.e. $* \in X, f(*) \in Y$, and make all maps pointed. Starting with W_{k-1}, consider the space

$$W_k' = W_{k-1} \vee \bigvee_{\alpha \in \pi_k(Y)} S^k$$

with $X \to W_k'$ the composition of σ_{k-1} with the inclusion $W_{k-1} \to W_{k-1} \vee -$. To define $\sigma_k' : W_k' \to Y$, let it be the map which sends $W_{k-1} \to Y$ along σ_{k-1}, and each S^k component corresponding to an α by $\alpha : S^k \to Y$ to Y. Then the map $\pi_k(W_k') \to \pi_k(Y)$ is surjective, since the trivial map of S^k into a component α maps to $\alpha \in \pi_k(Y)$ by $(\sigma_k')_*$.

Next, we make this map injective, which completes the proof. Let's define W_k as the space

$$W_k = W_k' \cup_\beta \bigsqcup_{\beta \in \ker(\sigma_k')_*} D^{k+1}$$

where for every map $\beta : S^k \to X$ in the kernel, we glue a $(k+1)$-cell to W_k' by attaching its boundary via β. Now this map has trivial kernel, since any map in the kernel can now be nullhomotped via the atached $(k+1)$-cell. Thus we have an isomorphism $\pi_k(W_k) \to \pi_k(Y)$ so by induction we are done. Note that these maps don't change the previous homotopy groups since we attach cells of codimension greater than 1 at each step.

Problem 4. Suppose that X and Y are pointed CW complexes with X m-connected and Y n-connected. Prove that the inclusion $X \vee Y \to X \times Y$ is an $(m+n+1)$-equivalence and $X \wedge Y$ is $(m+n+1)$-connected.

By cellular approximation, we can reduce homotopically to the case when X (resp. Y) are complexes with a single basepoint in 0-dimensions, and no cells in dimensions $k \leq m$. (resp. $k \leq n$). In this case, note that $S^k \wedge S^\ell \simeq S^{k+\ell}$, which in turn implies that $X \wedge Y$ has no cells of dimensions less than $(m+1)+(n+1)$, so $X \wedge Y$ is $(m+n+1)$-connected.

By the same argument, $X \wedge Y$ consists of a 0-cell, a $(n+1)$-cell, and a $(m+1)$-cell, while $X \times Y$ consits of a 0-cell, a $n+1$-cell, a $(m+1)$-cell, and $(n+m+2)$-cells and above. Thus $\pi_i(X \wedge Y) \to \pi_i(X \times Y)$ is an isomorphism for all $i \leq n+m$ (since the next biggest cell is codimension 2). Finally, $\pi_{n+m+1}(X \wedge Y) \to \pi_{n+m+1}(X \times Y)$ is surjective because it is the induced n-th homotopy map of the an n-skeleton into an $(n+1)$-skeleton. Thus $X \vee Y \to X \times Y$ is an $(n+m+1)$-equivalence.