Math 231b Problem Set 2

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Problem 1. Let W be a pointed k-space. Show that the functors

$$W \wedge - : k\mathbf{Top}_* \to k\mathbf{Top}_* \quad \text{and} \quad (-)^W_* : k\mathbf{Top}_* \to k\mathbf{Top}_*$$

are *homotopy functors*: they descend to well-defined functors

$$W \wedge - : \mathrm{ho}(k\mathbf{Top}_*) \to \mathrm{ho}(k\mathbf{Top}_*) \quad \text{and} \quad (-)^W_* : \mathrm{ho}(k\mathbf{Top}_*) \to \mathrm{ho}(k\mathbf{Top}_*).$$

For the functor $W \wedge -$, suppose we had pointed k-spaces X and Y with maps $f, g : X \to Y$. By a simple construction, a homotopy $f \simeq g$ is equivalent to a map $H : X \wedge I_+ \to Y$, where I_+ is the interval disjoint union a basepoint. Applying the $W \wedge -$ functor gives us a map $W \wedge H : W \wedge (X \wedge I_+) \to Y$. If we use the natural associativity isomorphism on smash products, this can be considered as a map $W \wedge H : (W \wedge X) \wedge I_+ \to Y$, which by naturality must be a homotopy between the maps $W \wedge f$ and $W \wedge g$. This is sufficient to show that $W \wedge -$ is a well defined functor on homotopy categories.

For the other functor $(-)^W_*$, we first will construct a map $\Psi : A \wedge X^W_* \to (A \wedge X)^W_*$. Using the adjunction, associativity, natural isomorphisms, we get:

$$\begin{aligned} k\mathbf{Top}_*(X^W_*, X^W_*) &\to k\mathbf{Top}_*(W \wedge (X^W_*), X) \to k\mathbf{Top}_*(A \wedge (W \wedge X^W_*), A \wedge X) \\ &\to k\mathbf{Top}_*(W \wedge (A \wedge X^W_*), A \wedge X) \to k\mathbf{Top}_*(A \wedge X^W_*, (A \wedge X)^W) \end{aligned}$$

So natural choice of map $A \wedge X^W_* \to (A \wedge X)^W_*$ is thus to just take the image of the identity $1_{X^W_*}$ under this sequence of compositions. A simpler presentation of this map is given by $(a, f) \in A \wedge X^W_* \mapsto ((a', x) \mapsto f(x)) \in (A \wedge X)^W_*$. Now suppose $f, g : X \to Y$ are two maps, and $H : X \wedge I_+ \to Y$ is a homotopy between them. By functoriality, we then have a map $H^W_* : (X \wedge I_+)^W_* \to Y^W_+$. Precomposing with Ψ, we get a map $H^W_* \circ \Psi : X^W_* \wedge I_+ \to Y^W_+$. It's clear to see that this is a homotopy between f^W_* and g^W_* by the expanded definition of Ψ, so this completes the proof.

Problem 2. Cofibration sequences and co-exactness.

Homotopy equivalence of mapping cones:

a. Use a homotopy $h : A \times I \to Y$ between the branches of the first diagram to construct a map $C(i) \to C(j)$ such that in the second diagram, the left square commutes and the right one commutes up to homotopy:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r, "j"] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X \arrow[r] \arrow[d, "g"] & C(i) \arrow[r] \arrow[d] & \Sigma A \arrow[d, "\Sigma f"] \\ Y \arrow[r] & C(j) \arrow[r] & \Sigma B \end{tikzcd}$$

First of all, to disambiguate the homotopy $h : A \times I \to Y$, let $h(-,0) = g \circ i$ and $h(-,1) = j \circ f$. Now by the universal property of the pushout, to construct a map $X \cup_i CA \to Y \cup_j CB$, it suffices to construct maps $X \to Y \cup_j CB$ and $CA \times I \to Y \cup_j CB$ which agree on the map $i : A \to X$ and inclusion $A \to CA$. To get from $X \to Y \cup_j CB$ we can simply compose $g : X \to Y$ with the natural inclusion $i(j) : Y \to Y \cup_j CB$. For the map $H : CA \times I \to Y \cup_j CB$, let's define it as

$$H(a,t) = \begin{cases} i(j)(h(a,2t)) & 0 \leq t \leq 1/2 \\ i_{CB}(Cf(a,2t-1)) & 1/2 \leq t \leq 1 \end{cases}$$

where $i(j) : Y \to Y \cup_f CB$ and $i_{CY} : CB \to Y \cup_f CB$ are the natural maps. Note that this is well defined since at $t = 1/2$, we have $i(j)(h(a,1)) = (i(j) \circ j \circ f)(a) = i_{CB}(f(a)) = i_{CB}(Cf(f(a),0)) = i_{CB}(Cf(a,0))$. Furthermore, H is a well defined map on the cone, since $H(a,1) = i_{CB}(Cf(a,1))$ is a constant map. Next, we claim that $i(j) \circ g \circ i = H \circ \iota_A$ where $\iota : A \to CA$ is the natural inclusion. This is because for any $a \in A$, we have $i(j) \circ g \circ i(a) = i(j)(h(a,0)) = H(a,0) = H \circ \iota_A(a)$. Putting everything together, we have a map $\lambda_h : C(i) \to C(j)$ by the universal property of the coproduct. We thus have two things to check.

1. <u>The left square commutes:</u> The bottom side is the map $X \to Y \cup_j CB = C(j)$ given by g composed with the inclusion $Y \to Y \cup_j CB$. However by construction of λ_h, the top square is also the composition of g with the inclusion $Y \to Y \cup_j CB$.

2. <u>The right square commutes up to homotopy:</u> Both maps $C(i) \to \Sigma B$ are uniquely determined by maps $X \to \Sigma B$ and $CA \to \Sigma B$ that agree on the map $A \to X$ and inclusion $A \to CA$. Since mapping from the mapping cones to the suspension collapses X, Y to a point, it suffices to just specify maps $CA \to \Sigma B$. The top map is the standard map $CA \to \Sigma B$ which sends $A \times I \to B \times I$ by $f \times I$ and passes to the quotient in both cases. The bottom map is the map which takes CA to the top half of ΣB. However these maps are clearly homotopic by linear interpolation.

b. Use a homotopy $f \simeq g : X \to Y$ to construct a homotopy equivalence $C(f) \simeq C(g)$.

Letting h be the homotopy $f \simeq g$ and $\overline{h}$ be the homotopy $g \simeq f$. Note that the functions $\lambda_h : C(f) \to C(g)$ and $\lambda_{\overline{h}} : C(g) \to C(f)$ are given on CX by:

$$\lambda_h(x,t) = \begin{cases} h(x,2t) & 0 \leq t \leq \frac{1}{2}, \\ (x, 2t-1) & \frac{1}{2} \leq t \leq 1, \end{cases} \quad \text{and} \quad \lambda_{\overline{h}}(x,t) = \begin{cases} h(x,1-2t) & 0 \leq t \leq \frac{1}{2}, \\ (x, 2t-1) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

Then the composition $\lambda_{\overline{h}} \circ \lambda_h$ is given by:

$$\lambda_{\overline{h}} \circ \lambda_h(x,t) = \begin{cases} h(x,2t) & 0 \leq t \leq \frac{1}{2}, \\ h(x,3-4t) & \frac{1}{2} \leq t \leq \frac{3}{4}, \\ (x,4t-3) & \frac{3}{4} \leq t \leq 1. \end{cases}$$

We can then define a homotopy $\lambda_{\overline{h}} \circ \lambda_h \simeq \mathrm{id}_{C(f)}$ by

$$H(x,t,s) = \begin{cases} h(x,2t(1-s)) & 0 \leq t \leq \frac{1-s}{2}, \\ h(x,(3-4t)(1-s)) & \frac{1-s}{2} \leq t \leq \frac{3(1-s)}{4}, \\ (x,(4t-3)+st) & \frac{3(1-s)}{4} \leq t \leq 1. \end{cases}$$

We can do a very similar thing for $\lambda_h \circ \lambda_{\overline{h}}$, so we have a homotopy equivalence $C(f) \simeq C(g)$.

Problem 3. Let p denote a prime number and let $n \geq 2$. Let $M(\mathbb{Z}/p, n) = S^{n-1} \cup_p D^n$ denote the n-dimensional mod p Moore space, and define the *mod p homotopy groups* of a pointed space X to be $\pi_n(X; \mathbb{Z}/p) = [M(\mathbb{Z}/p, n), X]_*$. Since $M(\mathbb{Z}/p, n) \simeq \Sigma^{n-2} M(\mathbb{Z}/p, 2)$, this is a group for $n \geq 3$, and abelian for $n \geq 4$. When $n \geq 3$, prove that there is a short exact sequence

$$0 \to \pi_n(X)/p \to \pi_n(X; \mathbb{Z}/p) \to \operatorname{tors}_p \pi_{n-1}(X) \to 0.$$

This is the analogue of the universal coefficients theorem for homotopy groups.

Before constructing this exact sequence, we'll first prove some relevant properties of the degree of a map of spheres. (We've only defined it in terms of homology.)

Claim. For any $n \geq 1$, suppose $f : S^n \to S^n$ is a continuous map. We claim that $\deg f = \pm \deg \Sigma f$ where Σf is the induced map $\Sigma S^n \to \Sigma S^n$ and we use the homeomorphism $\Sigma S^n \cong S^{n+1}$.

Proof. We proved last semester that for any $n \geq 1$ there is a natural isomorphism $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X)$ for any space X. Since this is natural in X, we can form a commutative square:

$$\begin{tikzcd} \widetilde{H}_n(S^n) \arrow[r, "f_*"] \arrow[d] & \widetilde{H}_n(S^n) \arrow[d] \\ \widetilde{H}_{n+1}(\Sigma S^n) \arrow[r, "\Sigma f_*"'] & \widetilde{H}_{n+1}(\Sigma S^n) \end{tikzcd}$$

All the groups involved are $\mathbb{Z}$, and the vertical maps must be the ± 1 maps since they are isomorphisms, thus we have $\deg f = \pm \deg \Sigma f$. $\square$

Claim. Let $n \geq 1$, and suppose $f : S^n \to S^n$ is a continuous map. For any space X, we defined the n-th homotopy group of X as $\pi_n(X) = [S^n, X]_*$. Thus we have an induced pullback map $f^* : \pi_n(X) \to \pi_n(X)$ given by precomposition with f. We claim that $f^*(\sigma) = \deg f \cdot \sigma$.

Proof. To prove this fully, we need a couple of facts from homotopy theory that we haven't proved yet. To make our life easier, let's assume that we have the natural group composition map $* : [S^n, X]_* \times [S^n, X]_* \to [S^n, X]_*$ which is associative, unital, and invertible. Furthermore, we assume that the map $f \in [S^n, S^n]_*$ is given by $p \cdot 1_{S^n} = 1_{S^n} * \cdots * 1_{S^n}$.

Now for any $\sigma \in \pi_n(X)$, the pullback map $f^*(\sigma)$ is given by $\sigma \circ f$, which under homotopy is $\sigma \circ (1_{S^n} * \cdots * 1_{S^n})$. By naturality of the group operation, this is $(\sigma \circ 1_{S^n}) * \cdots * (\sigma \circ 1_{S^n}) = \sigma * \cdots * \sigma$ as desired. $\square$

Now the observation that $M(\mathbb{Z}/p, n) \simeq \Sigma^{n-2} M(\mathbb{Z}/p, 2)$ follows from the fact that Σ is a left adjoint and thus preserves colimits, including adjunctions, so $\Sigma M(\mathbb{Z}/p, n) = \Sigma(S^{n-1} \cup_p D^n) \cong \Sigma S^{n-1} \cup_{\Sigma p} \Sigma D^n = M(\mathbb{Z}/p, n+1)$. So let's set $n \geq 3$ such that we have a group structure on $\pi_n(X; \mathbb{Z}/p)$. Recall that to define the Moore space, we have a pushout square

$$\begin{tikzcd} S^{n-1} \arrow[r] & S^{n-1} \cup_f D^n \\ S^{n-1} \arrow[u, "f"] \arrow[r] & D^n \arrow[u] \end{tikzcd}$$

where $f : S^{n-1} \to S^{n-1}$ is some map of degree p. We can notice, either by construction or by the universal property that the composition $S^{n-1} \to S^{n-1} \to S^{n-1} \cup_f D^n$ is a homotopy cofiber sequence. Thus for any space X we get the Barratt-Puppe long exact sequence:

$$\begin{tikzcd}
{[S^{n-1}, X]_*} \arrow[d, equal] & {[S^{n-1}, X]_*} \arrow[l, "f_{n-1}^*"'] \arrow[d, equal] & {[S^{n-1} \cup_f D^n, X]_*} \arrow[l, "i(f_{n-1}^*)"'] \arrow[d, equal] & {[\Sigma S^{n-1}, X]_*} \arrow[l, "\pi_{n-1}"'] \arrow[d, equal] & {[\Sigma S^{n-1}, X]_*} \arrow[l, "f_n^*"'] \arrow[d, equal] & \cdots \arrow[l] \\
\pi_{n-1}(X) & \pi_{n-1}(X) \arrow[l] & \pi_n(X; \mathbb{Z}/p) \arrow[l] & \pi_n(X) \arrow[l] & \pi_n(X) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

We can now make a couple of reductions to reduce this long exact sequence into the desired short exact sequence.

First recall that by the second claim, the map $f_{n-1}^* : \pi_{n-1}(X) \to \pi_{n-1}(X)$ is the multiplication by $\pm p$ map, and exactness says that $(f_{n-1}^*)^{-1}(c_*) = \mathrm{Im}(i(f_{n-1}^*))$. However $(f_{n-1}^*)^{-1}(c_*)$ is exactly $\mathrm{tors}_p\, \pi_{n-1}(X)$, so by the first isomorphism theorem we have an exact sequence:

$$\pi_n(X; \mathbb{Z}/p) \to \mathrm{tors}_p\, \pi_{n-1}(X) \to 0.$$

We can do the same thing on the other side of the diagram; by the first claim, the map $f_n^* = \Sigma f_{n-1}^*$ also has degree p, and exactness shows that $\pi_{n-1}^{-1}(c_*) = \mathrm{Im}(f_n^*) = p \cdot \pi_n(X)$. Thus we also have an exact sequence:

$$0 \to \pi_n(X)/p \to \pi_n(X; \mathbb{Z}/p).$$

The center term is exact by the long exact sequence, so we are done, and we get a short exact sequence:

$$0 \to \pi_n(X)/p \to \pi_n(X; \mathbb{Z}/p) \to \mathrm{tors}_p\, \pi_{n-1}(X) \to 0.$$

Problem 4. Let R denote a commutative ring. Given a chain complex of R-modules C and integer $i \in \mathbb{Z}$, let $C[i]$ denote a chain complex with $C[i]_n = C_[n-i]$ and boundary maps $d_n^{C[1]} = (-1)^i d_{n-i}^C$. Given a map $f : C \to D$ of chain complexes of R-modules, define the homotopy cofiber $i(f) : D \to C(f)$ and construct a map $\pi(f) : C(f) \to C[1]$ by analogy with the case of spaces.

Prove that applying H_0 to the bi-infinite sequence

$$\cdots \longrightarrow C(f)[-1] \xrightarrow{\pi(f)[-1]} C \xrightarrow{f} D \xrightarrow{i(f)} C(f) \xrightarrow{\pi(f)} C[1] \xrightarrow{f[1]} D[1] \xrightarrow{i(f)[1]} C(f)[1] \longrightarrow \cdots$$

gives rise to a long exact sequence

$$\cdots \longrightarrow H_{-1}(C(f)) \longrightarrow H_0(C) \longrightarrow H_0(D) \longrightarrow H_0(C(f)) \longrightarrow H_1(C) \longrightarrow H_1(D) \longrightarrow H_1(C(f)) \longrightarrow \cdots$$

Let $C(f)_\bullet$ be the chain complex given by:

$$C(f)_n = D_n \oplus C_{n-1} \quad \text{with} \quad d^{C(f)}(d, c) = (d^D d + f(c), -d^C c).$$

We have a clear map $i(f) : D \to C(f)$ by taking the inclusion of $D_n \to D_n \oplus C_{n-1}$ and a map $\pi(f) : C(f) \to C[1]$ by taking the projection $D_n \oplus C_{n-1} \to C_{n-1}$. It's clear that these are in fact chain map, by definition of the boundary map $d^{C(f)}$. Now let's prove that we get an exact sequence. Since this exact sequence is completely translation invariant, it suffices to prove three exactness conditions for any $n \in \mathbb{Z}$.

1. $\underline{\mathrm{Im}(H_{n-1}(\pi(f)[-1])) = \mathrm{Ker}(H_n(f))}$: For the forward inclusion, suppose $(\omega_d, \omega_c) \in D_{n+1} \oplus C_n = C(f)_{n-1}$ is a cycle, so $d^D \omega_d + f(\omega_c) = 0$ and $d^C \omega_c = 0$. Then $H_{n-1}(\pi(f)[-1])(\omega_d, \omega_c) = \omega_c$, and $H_n(f)(\omega_c) = 0$ since $f(\omega_c) = -d^D$ is a boundary. For the converse direction, suppose $\omega \in C_n$ and with $H_n(f)(\omega) = 0$. This means $f(\omega) = d^D \sigma$ for some $\sigma \in D_{n+1}$. Then $\omega = H_{n-1}(\pi(f)[-1])(-\sigma, \omega)$ which is a cycle because $d^{H(f)}(-\sigma, \omega) = (-f(\omega) + f(\omega), -d\omega) = 0$.

2. <u>$\mathrm{Im}(H_n(f)) = \mathrm{Ker}(H_n(i(f)))$</u>: For the forward inclusion, suppose $\omega \in C_n$ is a cycle, so $f(\omega) \in D_n$ is a cycle. Then $H_n(i(f))(f(\omega)) = (f(\omega), 0)$. This is also a cycle, since $d^{C(f)}(f(\omega), 0) = (d^D f(\omega), 0) = 0$. But up to boundaries, $0 = d^{C(f)}(0, \omega) = (f(\omega), -d^C\omega) = (f(\omega), 0)$, so $(f(\omega), 0) = 0$. In the reverse direction, suppose $\omega \in D_n$ with $H_n(i(f))(\omega, 0) = 0$. This means that there exists some $(\omega_d, \omega_c) \in D^{n+1} \oplus C_n$ with $d^{C(f)}(\omega_d, \omega_c) = 0$. Thus $d^D\omega_d + f(\omega_c) = \omega$ so $\omega = H_n(f)(\omega_c)$.

3. <u>$\mathrm{Im}(H_n(i(f))) = \mathrm{Ker}(H_n(\pi(f)))$</u>: The forward direction follows trivially because $i(f) \circ \pi(f) = 0$. To prove <u>the converse direction, suppose</u> $(\sigma_d, \sigma_c) \in D_n \oplus C_{n-1}$ for some cycle (σ_d, σ_c), with $H_n(\pi(f))(\sigma_d, \sigma_c) = 0$. This means that σ_c is a boundary, so $H_n(i(f))(\sigma_d) = (\sigma_d, 0)$ is equal to (σ_d, σ_c) relative to a boundary.