# Math 231a Problem Set 1

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**Problem 1.** Let $\Delta$ be the category of totally ordered finite sets with weakly order preserving maps, and let $\Delta_{\text{inj}}$ be the subcategory of $\Delta$ with injective order preserving maps. Show the following equivalences of categories:

1. $\mathbf{sSet} \cong \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})$.
2. $\mathbf{ssSet} \cong \text{Fun}(\Delta^{\text{op}}_{\text{inj}}, \mathbf{Set})$.

Recall that a simplicial set is a sequence of sets $\{K_i\}_{i\geq 0}$ with face/degeneracy maps $s_i, d_j$. Given some contravariant functor $F : \Delta \to \mathbf{Set}$, consider the simplicial set $\{F([i])\}_{i=0}$ with face/degeneracy maps $F(d_i)$ and $F(s_i)$, where $d_i$ is the order preserving map skipping $i$ and $s_i$ is the order preserving map doubling up on $i$. Note that contravariance sends $d_i : [n] \to [n+1]$ to $F(d_i) : K_{n+1} \to K_n$ and similarly for $s_i$. Since $F$ preserves composition, Problem 1 on Problem Set 1 shows that the maps $F(d_i)$ and $F(s_i)$ follow the axioms of a simplicial set. So $\text{Fun}(\Delta^{\text{op}}, \mathbf{Set}) \subset \mathbf{sSet}$. Natural transformations of two functors $F, D : \Delta \to \mathbf{Set}$. Correspond to maps of simplicial sets in the following way: For each $i$, consider the map $F([i]) \to D([i])$ given by the natural transformation. This is a simplicial set basically by definition.

Conversely, given a simplicial set $\{K_i\}_{i\geq 0}$, we can consider the contravariant functor $F : \Delta \to \mathbf{Set}$ which sends $[n]$ to $K_n$, sends the face and degeneracy maps to $d_i$ and $s_i$ respectively. Since every order preserving map factors uniquely as a composition of a $d_i$ and an $s_i$, this fully defines a set map for every order preserving map $[n] \to [m]$. Again, the identities are preserved on both sides, so this gives us a unique functor. Thus $\mathbf{sSet} \cong \text{Fun}(\Delta^{\text{op}}, \mathbf{Set})$.

The same thing happens for $\mathbf{ssSet} \cong \text{Fun}(\Delta^{\text{op}}_{\text{inj}}, \mathbf{Set})$.

**Problem 2.** Let $\pi_0(X)$ denote the set of path components of a space $X$. Prove that there is a natural isomorphism $\mathbb{Z}\pi_0(X) \xrightarrow{\sim} H_0(X)$.

For any space $X$, lets define the map $\phi_X : \mathbb{Z}\pi_0(X) \to H_0(X)$ by sending any path component $X_i$ to $[c^0_{x_i}]$ for any $x_i \in X_i$. Clearly this is well defined, since if $y_i \in X_i$ is any other point in the same path component, with $\gamma : \Delta^1 \to X_i$ a path connecting them, them $d\gamma = c^0_{x_i} - c^0_{y_i}$ in $H_0(X)$ and so $[c^0_{x_i}] = [c^0_{y_i}]$ in $H_0(X)$. We can define the map on all of $\mathbb{Z}\pi_0(X)$ by extending linearly in the natural way. This map is clearly injective since $\Delta^1$ is connected, and surjective because the inverse image of any $c^0_x$ is the path component of $X$.

Now suppose $f : X \to Y$ is a continuous map. We have a natural induced map $\mathbb{Z}\pi_0(f) : \mathbb{Z}\pi_0(X) \to \mathbb{Z}\pi_0(Y)$, which sends the path component of $x$ to the path component of $f(x)$. To prove naturality, notice that $H_0(f)(\phi_X(x)) = H_0(f)([c^0_x]) = [f \circ c^0_x] = [c^0_{f(x)}]$. On the other side, we have $\phi_Y(\mathbb{Z}\pi_0(f)(x)) = \phi_Y(f(x)) = [c^0_{f(x)}]$. This means that extended linearly, the maps must be equal and so the transformation is natural.

**Problem 3.** Let $X$ be a path-connected space and let $x \in X$. We will show a natural isomorphism

$$\pi_1(X, x)^{\text{ab}} \xrightarrow{\sim} H_1(X).$$

Given a path $f : I \cong \Delta^1 \to X$, let $[f] \in S_1(X)$ denote the corresponding singular 1-chain.

(a) Let $f, g : I \to X$ denote two paths with $f(0) = g(0)$ and $f(1) = g(1)$. Suppose that $f \simeq g$ rel $\partial I$, i.e. there is a homotopy $h : I \times I \to X$ between $f$ and $g$ which is constant on $\{0\} \times I$ and $\{1\} \times I$. Prove that $[f] \equiv [g] \mod B_1(X)$.

(b) Given two paths $f, g$ such that $f(1) = g(0)$, prove that $[f * g] \equiv [f] + [g] \mod B_1(X)$, where $f * g$ is the composition of $f$ and $g$. Conclude that there is a group homomorphism

$$\phi_* : \pi_1(X, x)^{\text{ab}} \to H_1(X)$$

sending the homotopy class of a loop $f$ to $[f]$.

(c) For each point $y \in X$, fix a path $\lambda_y$ from $x$ to $y$. Define a map

$$\psi : S_1(X) \to \pi_1(x, x)^{\text{ab}}$$

by sending $[f]$ to the image of the homotopy class of $\lambda_{f(0)} * f * \overline{\lambda_{f(1)}}$ in the abelianization. Here, $\overline{\lambda_{f(1)}}$ is the reverse of the path $\lambda_{f(1)}$. Prove that $\psi$ sends all elements of $B_1(X)$ to the identity element, so that $\psi$ induces a map:

$$\psi_* : H_1(X) \to \pi_1(X, x)^{\text{ab}}.$$

(d) Prove that $\psi_* \circ \phi_* = \text{id}_{\pi_1(X,x)^{\text{ab}}}$ and $\phi_* \circ \psi_* = \text{id}_{H_1(X)}$, so that $\phi_*$ and $\psi_*$ are inverse isomorphisms.

**(a)** Let $v_0 = f(0)$ and $v_1 = f(1)$. Consider the simplex $\sigma \in S_2(h(I \times I))$ given by the triangulation

$$\begin{tikzcd}
v_0 \arrow[r, dash, "{[g]}"] \arrow[d, dash] \arrow[dr, dash] & v_1 \arrow[d, dash] \\
v_0 \arrow[r, dash, "{[f]}"'] & v_1
\end{tikzcd}$$

Then $d\sigma = [f] - [g] + c^1_{v_0} + c^1_{v_1}$. But note that any 1-simplex is a boundary because $dc^2_x = c^1_x - c^1_x + c^1_x = c^1_x$. This means that $0 \equiv d\sigma \equiv [f] - [g] + c^1_{v_0} + c^1_{v_1} \equiv [f] - [g] \mod B_1(X)$ and so $[f] \equiv [g] \mod B_1(X)$ as desired.

**(b)** Consider the simplex $\sigma \in S_2(X)$ given by

$$\begin{tikzcd}
v_0 \arrow[d, dash, "{[f]}"'] \arrow[dr, dash, "{[f * g]}"] & \\
v_0 \arrow[r, dash, "{[g]}"'] & v_0
\end{tikzcd}$$

More explicitly, the map can be given by

$$\sigma(x, y) = \begin{cases} f(x + y) & x < y \\ g(x + y) & x \geq y \end{cases}.$$

This map is clearly continuous, and in particular note that $\sigma(0,t) = f(t)$, $\sigma(t,0) = g(t)$, and $\sigma(t,(1-t)) = (f * g)(t)$, as shown in the diagram. So $d\sigma = [f] - [f * g] + [g]$ and so $[f * g] \equiv [f] + [g] \mod B_1(X)$. Thus, we have a well defined map $\phi_*$ from $\pi_1(X,x)^{\mathrm{ab}} \to H_1(X)$ since every loop $f * g * \overline{f} * \overline{g} \in [\pi_1(X,x), \pi_1(X,x)]$ maps to $[f] + [g] - [f] - [g] = 0$.

**(c)** Let $\sigma : \Delta^2 \to X$ be a 2-simplex with vertices $x_0, x_1, x_2 \in X$ and edges $e_0, e_1, e_2 : \Delta^1 \to X$ respectively. Then

$$\begin{aligned}\psi(d\sigma) = \psi(e_0 - \overline{e_1} + e_2) = \psi(e_0 + e_2 - \overline{e_1}) &= \psi(e_0) * \psi(e_2) * \overline{\psi(\overline{e_1})} \\ &= [\lambda_{x_0} * e_0 * \overline{\lambda_{x_1}}] * [\lambda_{x_1} * e_2 * \overline{\lambda_{x_2}}] * \overline{[\lambda_{x_0} * \overline{e_1} * \overline{\lambda_{x_2}}]} \\ &= [\lambda_{x_0} * e_0 * \overline{\lambda_{x_1}} * \lambda_{x_1} * e_2 * \overline{\lambda_{x_2}} * \lambda_{x_2} * e_1 * \overline{\lambda_{x_0}}] \\ &= [\lambda_{x_0} * e_0 * e_2 * e_1 * \overline{\lambda_{x_0}}].\end{aligned}$$

Since $e_0 * e_2 * e_1$ is homotopic to a constant map, this is zero, so $\psi$ sends $B_1(X)$to the identity class. This gives us our induced map.

**(d)** For any loop $f \in \pi_1(X,x)$, we have $(\psi_* \circ \phi_*)(f) = [\lambda_{f(0)} * f * \overline{\lambda_{f(1)}}] \equiv [f] \mod [\pi_1(X,x), \pi_1(X,x)]$ and so $\psi_* \circ \phi_* = \mathrm{id}_{\pi_1(X,x)^{\mathrm{ab}}}$. Conversely for any 1-simplex $f \in H_1(X)$, we have $(\phi_* \circ \psi_*)(f) = \lambda_{f(0)} + f - \lambda_{f(1)} = f$ so we are done.

**Problem 4.** Let $A_*$ be a chain complex. It is *acyclic* if $H_*(A_*) = 0$, and *contractible* if it is chain-homotopy-equivalent to the trivial chain complex.

(a) Prove that a chain complex is contractible if and only if it is acyclic and the inclusion $Z_*(A_*) \to A_*$ is a split monomorphism.

(b) Give an example of an acyclic chain complex that is not contractible.

**(a)** Suppose first that $A_*$ is a contractible chain complex; i.e. there is a chain homotopy $h : A_* \to A_*$ between 0 and id. This means $h$ satisfies $dh + hd = \mathrm{id}$. Since chain homotopy equivalence preserves homology, it's clear that $A_*$ is acyclic, since it is chain homotopy equivalent to a trivial chain complex, which has trivial homology groups. To prove that $Z_*(A_*) \to A_*$ is a split monomorphism, note that we have a map $f : A_* \to Z_*(A_*)$ which sends $a$ to $dh(a) = a - hd(a)$. This is clearly a homomorphism, and we have

$$Z_*(A_*) \longrightarrow A_* \longrightarrow Z_*(A_*).$$

Observe that for any cycle $\sigma$, $f(\sigma) = dh(\sigma) = \sigma - hd(\sigma) = \sigma$, so $f$ is a left inverse to the inclusion and we are done. Now conversely suppose that $A_*$ is acyclic and $f : A_* \to Z_*(A_*)$ is a left inverse to the inclusion $Z_*(A_*) \to A_*$. We'll construct a chain homotopy to the trivial chain as follows: for any chain $\sigma \in A_*$, $f(\sigma) \in Z_*(A_*)$ is a cycle. Since the homology groups are trivial, it must also be a boundary. So let $f(\sigma) = d\beta_\sigma$. Let's define $h$ as sending $\sigma$ to $\beta_\sigma$. Then $hd(\sigma) - dh(\sigma) = \sigma - d\beta_\sigma = \sigma$, so $h$ is a homotopy to identity and thus the chain complex is contractible.

**(b)** Consider the short exact sequence

$$0 \longleftarrow \mathbb{Z}/2 \longleftarrow \mathbb{Z} \longleftarrow 2\mathbb{Z} \longleftarrow 0.$$

We can consider this a chain complex with higher degree terms set to 0. Since this is an exact sequence, it must be acyclic since $\ker \partial = \mathrm{Im}\partial$ implies that $H_n = \ker \partial / \mathrm{Im}\partial = 1$. Then it cannot be contractible, since otherwise, by the preceding part we would have a split monomorphism $2\mathbb{Z} \to \mathbb{Z}$. This is impossible, since a composition $2\mathbb{Z} \to \mathbb{Z} \to 2\mathbb{Z}$ which is the identity must have $2 \mapsto 2$ so if $1 \mapsto 2n$ in the second half, we get $2 \mapsto 4n$, a contradiction.