# Math 222 Problem Set 1

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**Problem 2.** Let $G$ have the structures of a group and a smooth manifold, and suppose that multiplication $G \times G \to G$ is smooth. Prove that inversion $G \to G$ is smooth map.

Let $m : G \times G \to G$ be the multiplication map. For any fixed element $g \in G$, the left multiplication map $L_g : G \to G$ is a diffeomorphism. Consider the map

$$\begin{array}{rrcl} f \ : & G \times G & \longrightarrow & G \times G \\ & (g, h) & \longmapsto & (g, gh). \end{array}$$

At the identity $(e, e)$, the differential of this map takes the form

$$\begin{array}{rrcl} df_{(e,e)} \ : & \mathrm{T}_e G \oplus \mathrm{T}_e G & \longrightarrow & \mathrm{T}_e G \oplus \mathrm{T}_e G \\ & \xi \oplus \eta & \longmapsto & \xi \oplus (\xi + \eta). \end{array}$$

By the inverse function theorem, this means that $f$ is a diffeomorphism in a neighborhood of $(e, e)$. By composing with the left multiplication diffeomorphisms, we can see that $f$ is a diffeomorphism over the whole manifold. Since the inversion map $i : G \to G$ can be written as $i(g) = \pi_2 \circ f^{-1}(g, e)$, a composition of smooth maps, the inversion map must be a smooth map as well.

**Problem 4.** Classify 1-dimensional Lie groups $G$ which have two components and whose identity component is diffeomorphic to the circle.

Pick any nonzero vector $\xi \in \mathfrak{g}$ and consider the exponential map

$$\begin{array}{rrcl} \exp \ : & \mathbb{R} & \longrightarrow & G \\ & t & \longmapsto & \exp(t\xi). \end{array}$$

This is a homomorphism of Lie groups. Since the identity component is diffeomorphic to $S^1$ and the exponential map is a homomorphism of Lie groups, it follows that there is a non-empty lattice $\ker(\exp) = \Lambda \subset \mathbb{R}$ so that we have an isomorphism $\mathbb{R}/\Lambda \to G$ of Lie groups. This let's us construct isomorphism with the circle group $\mu \to G_e$ so $G_e$ is not only diffeomorphic to a circle but has the group structure of the circle group.

Next, note that $G/G_e$ is a group of order two and so must be isomorphic to $\mu_2$. From a group theoretic perspective, the only group structures on $G$ are then $G \times \mu_2$ and $G \rtimes \mu_2$. These correspond to the Lie groups $\mu_2 \times \mu$ and $\mathrm{O}_2$ respectively.

**Problem 5.**

**(a).** Identify the group of rotations in $\mathbb{R}^3$ with the matrix group $\mathrm{SO}_3$.

Any rotation in $\mathbb{R}^3$ must preserve distances, preserve orientation, and fixes a line. Let $G \subset \mathrm{GL}_3(\mathbb{R})$ be the group of matrices representing rotations. The requirement that rotations preserve distances restricts us to $\mathrm{O}_3 \subset \mathrm{GL}_3(\mathbb{R})$ and the requirement that rotations preserve orientation restricts us to $\mathrm{SO}_3 \subset \mathrm{O}_3$. Therefore $G \subset \mathrm{SO}_3$ so it suffices to show that every special orthogonal matrix corresponds to a rotation.

Let $A \in \mathrm{SO}_3$ be a special orthogonal matrix. The characteristic polynomial of $A$ is a real polynomial is a degree 3 real polynomial and so it must have a real root $\lambda$. We can thus write $A$ as

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}$$

for some $2 \times 2$ matrix $M$ in a basis $\{e_1, e_2, e_3\}$ It follows that $(\lambda)$ and $M$ must be orthogonal matrices, which in particular implies that $\lambda = \pm 1$. Therefore, $A$ represents rotation about the $e_1$ axis.

**(b).** Embed $\mathrm{O}_2$ as a Lie subgroup of $\mathrm{SO}_3$.

For any basis, consider the inclusion map

$$\begin{aligned} f\,:\, \mathrm{O}_2 &\longrightarrow \mathrm{SO}_3 \\ M &\longmapsto \begin{pmatrix} \det(M) & 0 \\ 0 & M \end{pmatrix}. \end{aligned}$$

Since the determinant of the matrix $f(M)$ is $\det(M)^2$, it follows that $f(M)$ is a special orthogonal matrix. This is injective and a Lie group homomorphism because

$$\begin{pmatrix} \det(M_1) & 0 \\ 0 & M_1 \end{pmatrix} \begin{pmatrix} \det(M_2) & 0 \\ 0 & M_2 \end{pmatrix} = \begin{pmatrix} \det(M_1 M_2) & 0 \\ 0 & M_1 M_2 \end{pmatrix}.$$