Math 114 Problem Set 2

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Problem 18. Prove the following assertion: Every measurable function is the limit a.e. of a sequence of continuous functions

Consider the sequence of nested closed balls $B_1 \subset B_2 \subset \cdots$. Recall that by Lusin's theorem, for each $n \geq 1$ and $\epsilon > 0$ there is some compact set $K_{n,\epsilon} \subset B_n$ such that $m(B_n \setminus K_{n,\epsilon}) < \epsilon$ and $f\big|_{K_{n,\epsilon}}$ is continuous. Let's define f_n be the extension of $f\big|_{K_{n,2^{-n}}}$ to all of $\mathbb{R}^d$ by the Tietze extension theorem.

We claim that $\lim\limits_{n\to\infty} f_n = f$ almost everywhere. Clearly $f_n = f$ on a set $K_{n,2^{-n}}$ with $m(B_n \setminus K_{n,2^{-n}}) < 2^{-n}$. Let $E_n = B_n \setminus K_{n,2^{-n}}$. Then

$$\sum_{k=1}^{\infty} m(E_n) = \sum_{k=1}^{\infty} m(B_n \setminus K_{n,2^{-n}}) \leq \sum_{k=1}^{\infty} 2^{-n} = 1$$

so applying the Borel-Cantelli lemma, it follows that $m\left(\limsup_{n\to\infty} E_n\right) = 0$. Note that by construction, $\lim\limits_{n\to\infty} f_n(x) = f(x)$ if and only if $x \in K_{n,2^{-n}}$ for all $n \geq N$ for some N. Thus the set of points for which $\lim\limits_{n\to\infty} f_n(x) \neq f(x)$ is a subset of $\limsup_{n\to\infty} E_n$. But this set has measure zero so we are done.

Problem 23. Suppose $f(x, y)$ is a function on $\mathbb{R}^2$ that is separately continuous: for each fixed variable, f is continuous in the other variable. Prove that f is measurable on $\mathbb{R}^2$.

Let ψ_n be a sequence of step functions converging to the identity function on $\mathbb{R}$; for instance take $\psi_n(x) = \lfloor nx \rfloor / n$. For any n, consider the function $g_n(x, y) = f(\psi_n(x), y)$. We claim that for any $n \geq 1$, g_n is measurable. Let $h_{n,m}(x, y) = f(\psi_n, \psi_m(y))$. Clearly $h_{n,m}$ is measurable because it has countable image, and the inverse image of any point in the image is a countable union of cubes. Then since f is separately continuous,

$$\lim_{m\to\infty} h_{n,m}(x, y) = \lim_{m\to\infty} g_n(x, \psi_m(y)) = g_n\left(x, \lim_{m\to\infty} \psi_m(y)\right) = g_n(x, y)$$

so g_n is measurable. Then,

$$\lim_{n\to\infty} g_n(x, y) = \lim_{n\to\infty} f(\psi_n(x), y) = f\left(\lim_{n\to\infty} \psi_n(x), y\right) = f(x, y)$$

and so f must be measurable as well.

Problem 36. Here we will construct a measurable function f on $[0, 1]$ such that every function almost equal to f is discontinuous everywhere.

(a) Construct a measurable set $E \subset [0, 1]$ such that for any open interval $I \subset [0, 1]$ satisfies $m(E \cap I) > 0$ and $m(E^c \cap I) > 0$.

(b) Show that $f = \chi_E$ has the property that whenever $g = f$ a.e, then g must be discontinuous at every point in $[0, 1]$.

(a) *Skipped to prevent unreasonable suffering*

(b) Suppose g is a function on $[0,1]$ with $A = \{x \in [0,1] : f(x) \neq g(x)\}$ satisfying $m(A) = 0$. For any open interval $I \subset [0,1]$, we have $m((E \cap I) \setminus A) > 0$ and $m((E^c \cap I) \setminus A) > 0$. Then $g(E \cap I \setminus A) = \chi_E(E \cap I) = 1$ and $g(E^c \cap I \setminus A) = \chi_E(E^c \cap I) = 0$. This is a clear violation of continuity at every point in the image.

Problem 38. Prove that $(a+b)^\gamma \geq a^\gamma + b^\gamma$ whenever $\gamma \geq 1$ and $a, b \geq 0$. Also, show that the reverse inequality holds when $0 \leq \gamma \leq 1$.

Note that if $\gamma \geq 1$, we have the inequality $x^{\gamma-1} \leq y^{\gamma-1}$ whenever $x \leq y$. Applying this, we get

$$\begin{aligned}(a+t)^{\gamma-1} \geq t^{\gamma-1} \implies \int_0^b (a+t)^{\gamma-1}\,dt \geq \int_0^b t^{\gamma-1}\,dt &\implies \left(\frac{(a+t)^\gamma}{\gamma}\right)\Bigg|_0^b \geq \left(\frac{t^\gamma}{\gamma}\right)\Bigg|_0^b \\ &\implies \frac{(a+b)^\gamma}{\gamma} - \frac{a^\gamma}{\gamma} \geq \frac{b^\gamma}{\gamma} \\ &\implies (a+b)^\gamma \geq a^\gamma + b^\gamma.\end{aligned}$$

Notice that when $0 \leq \gamma \leq 1$, we have $x^{\gamma-1} \geq y^{\gamma-1}$ whenever $x \leq y$, so in this case we have the reverse inequality.

Problem 39. Establish the inequality

$$\frac{x_1 + \cdots + x_d}{d} \geq (x_1 \cdots x_d)^{1/d} \quad \text{for all } x_j \geq 0, j = 1, \ldots, d$$

by using backward induction as follows:

(a) The inequality is true whenever d is a power of 2 ($d = 2^k, k \geq 1$).

(b) If the inequality holds for some integer $d \geq 2$, then it must hold for $d-1$, that is, one has

$$\frac{y_1 + \cdots + y_{d-1}}{d-1} \geq (y_1 \cdots y_{d-1})^{1/(d-1)}$$

for all $y_j \geq 0$, with $j = 0, \ldots, d-1$.

(a) We'll proceed by induction. First, suppose $k = 1$ so that $d = 2$. Then we have:

$$\begin{aligned}(x-y)^2 \geq 0 \implies x^2 - 2xy + y^2 \geq 0 \implies x^2 + 2xy + y^2 \geq 4xy &\implies \left(\frac{x+y}{2}\right)^2 \geq xy \\ &\implies \frac{x+y}{2} \geq (xy)^{1/2}.\end{aligned}$$

Now suppose for the sake of induction that the claim is true for $d = 2^{k-1}$. Then,

$$\begin{aligned}\frac{(x_1+y_1) + \cdots + (x_d + y_d)}{d} \geq ((x_1+y_1)\cdots(x_d+y_d))^{1/d} &\geq (x_1 \cdots x_d + y_1 \cdots y_d)^{1/d} \\ &\geq \left(2(x_1 \cdots x_d y_1 \cdots y_d)^{1/2}\right)^{1/d} \\ &= 2^{1/d}(x_1 \cdots x_d y_1 \cdots y_d)^{1/2d} \\ \implies \frac{(x_1 + \cdots + x_d + y_1 + \cdots + y_d)}{2d} &\geq (x_1 \cdots x_d y_1 \cdots y_d)^{1/2d}.\end{aligned}$$

This completes the induction.

(b) Suppose that the equality holds for some $d \geq 2$. Then

$$\frac{y_1 + \cdots + y_{d-1}}{d-1} \geq \frac{y_1 + \cdots + y_{d-1}}{d} \geq (y_1 \cdots y_{d-1})^{1/d} \geq (y_1 \cdots y_{d-1})^{1/(d-1)}.$$