Math 114 Problem Set 4

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October 16, 2022

Problem 17. Suppose f is defined on $\mathbb{R}^2$ as follows: $f(x,y) = a_n$ if $n \le x < n+1$ and $n \le y < n+1$, $(n \ge 0)$; $f(x,y) = -a_n$ if $n \le x < n+1$ and $n+1 \le y < n+2$, $(n \ge 0)$; while $f(x,y) = 0$ elsewhere. Here $a_n = \sum_{k \le n} b_k$, with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.

(a) Verify that each slice f_y and f_x is integrable. Also for all x, $\int f_x(y)\ dy = 0$, and hence $\iint f(x,y)\ dy\ dx = 0$.

(b) However, $\int f_y(x)\ dx = a_0$ if $0 \le y < 1$, and $\int f^y(x)\ dx = a_n - a_{n-1}$ if $n \le y < n+1$ with $n \ge 1$. Hence $y \mapsto \int f^y(x)\ dx$ is integrable on $(0, \infty)$ and

$$\iint f(x,y)\ dx\ dy = s.$$

(c) Note that $\iint |f(x,y)|\ dx\ dy = \infty$.

(a) First note that for every $y \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} |f_y(x)|\ dx = |a_n| + |a_{n-1}| \quad \text{where } n \le y < n+1.$$

Similarly, for every $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} |f_x(y)|\ dy = 0$$

since the a_n and $-a_n$ terms cancel out. This implies that

$$\iint f(x,y)\ dy\ dx = 0.$$

(b) Recall that in (a) we proved that

$$\int_{\mathbb{R}} f_y(x)\ dx = \begin{cases} a_0 & 0 \le y < 1, \\ a_n - a_{n-1} & n \le n+1, n \ge 1. \end{cases}$$

Thus we have

$$\iint f(x,y)\ dx\ dy = a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) = s.$$

(c) Again using (a), we have

$$\iint |f(x,y)|\ dx\ dy \ge \sum_{n=1}^{\infty} |a_n| + |a_{n-1}| \ge \sum_{k=1}^{\infty} s = \infty.$$

Problem 22. Prove that if $f \in L^1(\mathbb{R}^d)$ and

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \zeta}\, dx,$$

then $\widehat{f}(\zeta) \to 0$ as $|\zeta| \to \infty$. (This is the Riemann-Lebesgue lemma.)

Note that

$$\widehat{f}(\zeta) = \frac{1}{2} \int_{\mathbb{R}^d} \left(f(x) - f(x - \zeta') \right) e^{-2\pi i x \zeta} dx \quad \text{where} \quad \zeta' = \frac{\zeta}{2|\zeta|^2}.$$

Then since $\zeta = \zeta'/2|\zeta'|^2$ we get

$$\begin{aligned}
\lim_{|\zeta| \to \infty} \widehat{f}(\zeta) &= \frac{1}{2} \lim_{|\zeta| \to \infty} \int_{\mathbb{R}^d} \left(f(x) e^{-2\pi i x \zeta} - f(x - \zeta') e^{-2\pi i x \zeta} \right) dx \\
&= \frac{1}{2} \lim_{|\zeta'| \to 0} \int_{\mathbb{R}^d} \left(f(x) e^{-2\pi i x \zeta} - f(x - \zeta') e^{-2\pi i x \zeta} \right) dx \\
&= \frac{1}{2} \lim_{|\zeta'| \to 0} \left\| f(x) e^{-2\pi i x \zeta}, f(x - \zeta') e^{-2\pi i x \zeta} \right\|_1 = 0,
\end{aligned}$$

where the last equality follows because $f(x)e^{-2\pi i x \zeta}$ is integrable.

Problem 24. Consider the convolution

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - t) g(t)\, dt.$$

(a) Show that $f * g$ is uniformly continuous when f is integrable and g bounded.

(b) If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

(a) Let $B > 0$ be a bound for g, i.e. $|g| < B$. To prove uniform continuity, let $\epsilon > 0$. Recall that since f is integrable, we have $\lim_{x \to y} \|f(x - t) - f(y - t)\|_1 = 0$ for any $t \in \mathbb{R}^d$. Let $\delta > 0$ be such that $|x - y| < \delta$ implies $\|f(x - t) - f(y - y)\| < \epsilon$. But then

$$\begin{aligned}
|(f * g)(x) - (f * g)(y)| &= \left| \int_{\mathbb{R}^d} f(x - t) g(t)\, dt - \int_{\mathbb{R}^d} f(y - t) g(t)\, dt \right| \le \int_{\mathbb{R}^d} |[f(x - t) - f(y - t)]\, g(t)|\, dt \\
&\le B \int_{R^d} |f(x - t) - f(y - t)|\, dt = B \| f(x - t) - f(y - t) \| |_1 \le B\epsilon
\end{aligned}$$

We can readjust this by setting $\epsilon = \epsilon/B$, this gives us uniform continuity.

(b) We'll begin by proving $f * g$ is integrable. Note that

$$\begin{aligned}
\int_{\mathbb{R}^d} |(f * g)(x)|\, dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x - t) g(t)\, dt \right| dx \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x - t) g(t)|\, dx\, dt \\
&= \int_{\mathbb{R}^d} |g(t)| \int_{\mathbb{R}^d} |f(x - t)|\, dx\, dt = \left(\int_{\mathbb{R}^d} |g(t)|\, dt \right) \left(\int_{\mathbb{R}^d} |f(x)|\, dx \right) < \infty.
\end{aligned}$$

Since $f * g$ is integrable, we can use the argument from the previous problem set to see that $\lim_{|x| \to \infty} (f * g)(x) = 0$. (Note that the proof can be modified to work for $\mathbb{R}^d$ by replacing intervals with balls.)

Problem 25. Show that for each $\epsilon > 0$ the function $F(\zeta) = \frac{1}{(1 + |\zeta|^2)^\epsilon}$ is the Fourier transform of an L^1 function.

Let $K_\delta(x) = e^{-\pi|x|^2/\delta}\delta^{-d/2}$, and for any $\epsilon > 0$ consider the function

$$f(x) = \int_0^\infty K_\delta(x) e^{-\pi\delta}\delta^{\epsilon-1}\, d\delta.$$

Notice that by Fubini's theorem, we get

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\zeta}dx = \int_{\mathbb{R}^d}\int_0^\infty K_\delta(x)e^{-\pi\delta}\delta^{\epsilon-1}\, d\delta\; e^{-2\pi i x\zeta}dx = \int_0^\infty \int_{\mathbb{R}^d} K_\delta(x)e^{-2\pi i x\zeta}\, dx\; e^{-\pi\delta}\delta^{\epsilon-1}d\delta.$$

But the "elementary calculation" in Stein shows that

$$\int_{\mathbb{R}^d} K_\delta(x)e^{-2\pi i x\zeta}dx = \delta^{-d/2}\int_{\mathbb{R}^d} e^{-\pi|x|^2/\delta}e^{-2\pi i x\zeta}dx = e^{-\pi\delta|\zeta|^2}.$$

Then combining this with the previous calculation, we get

$$\widehat{f}(\zeta) = \int_0^\infty e^{-\pi\delta(1+|\zeta|^2)}\delta^{\epsilon-1}d\delta.$$

Now let $v = \pi(1+|\zeta|^2)\delta$ so $dv = \pi(1+|\zeta|^2)d\delta$. Then

$$\int_0^\infty e^{-\pi\delta(1+|\zeta|^2)}\delta^{\epsilon-1}d\delta = \frac{1}{\pi(1+|\zeta|^2)}\int_0^\infty e^{-v}\left(\frac{1}{\pi(1+|\zeta|^2)}\right)^{\epsilon-1}\delta^{\epsilon-1}d\delta = \pi^{-\epsilon}\Gamma(\epsilon)\frac{1}{(1+|\zeta|^2)^\epsilon} < \infty.$$

Now let $G(x) = \pi^\epsilon/\Gamma(\epsilon) f(x)$. Then $\widehat{G}(\zeta) = F(\zeta)$. To prove that $G \in L^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |G(x)|\, dx = \int_{\mathbb{R}^d} |G(x)| \cdot |e^{-2\pi i x\zeta}|\, dx < \infty$$

since $\widehat{G}(\zeta)$ is finite.

Problem 4 (Rudin). Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p\, d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

(b) Prove that $\log\varphi$ is convex in the interior of E and that φ is continuous on E.

(c) ~~By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0,\infty)$?~~

(d) If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Prove that if $f \in L_q(\mathbb{R}^d)$ for some $q \ge 1$ then $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

(a) Recall the inequality for any $r < p < s$ as above: $|y|^r + |y|^s \ge |y|^p$ for all $y \in \mathbb{R}$. Then we have

$$\int_X |f|^p\, d\mu \le \int_X |f|^r + |f|^s\, d\mu = \int_X |f|^r\, d\mu + \int_X |f|_s\, d\mu < \infty,$$

so $p \in E$ as desired.

(b) Let $x \in E$ be an interior point. This means that there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset E$. Suppose $r, s \in B_\epsilon(x)$. We want to show that $(1-t)\log\varphi(r) + t\log\varphi(s) \geq \log\varphi((1-t)r+ts)$ for any $t \in [0,1]$. This is equivalent to proving $\varphi(r)^{1-t}\varphi(s)^t \geq \varphi((1-t)r+ts)$ which in turn means

$$\left(\int_X |f|^r \, d\mu\right)^{1-t} \left(\int_X |f|^s \, d\mu\right)^t \geq \int_X |f|^{(1-t)r+ts} \, d\mu.$$

By Hölder's inequality, we get

$$\|f^{(1-t)r} \cdot f^{ts}\|_1 = \int_X |f|^{(1-t)r+ts} \, d\mu \leq \left(\int_X |f|^r \, d\mu\right)^{1-t} \left(\int_X |f|^s\right)^t = \|f^{(1-t)r}\|_{1/(1-t)} \|f^{ts}\|_{1/t}$$

which is what we wanted. To prove continuity, first notice that φ is continuous on $\mathrm{Int}(E)$ since it is log convex. To prove continuity on the whole of E, let $\{v_n\}_{n\geq 1} \subset E$ be a strictly increasing or decreasing sequence converging to some $v \in \mathbb{R}$. We'll show that $v \in E$ as well. By the monotone convergence theorem,

$$\lim_{n\to\infty} \varphi(v_n) = \lim_{n\to\infty} \int_X |f|^{v_n} \, d\mu = \int_X \lim_{n\to\infty} |f|^{v_n} \, d\mu = \int_X |f|^v \, d\mu = \varphi(v).$$

This suffices to prove continuity on the closure $\overline{\mathrm{Int}(E)} = E$.

(d) Recall the inequality for any function f and $r, s \in \mathbb{R}$ we have

$$\frac{(1-t)f(r) + tf(s)}{(1-t)r+ts} \leq \max\left(\frac{f(r)}{r}, \frac{f(s)}{s}\right) \quad \forall t \in [0,1].$$

Then for any $p \in (r,s)$, we can write $p = (1-t)r + ts$ for a $t \in [0,1]$. By log convexity of φ, we have

$$\frac{\log\varphi(p)}{p} = \frac{\log\varphi((1-t)r+ts)}{(1-t)r+ts} \leq \frac{(1-t)\log\varphi(r) + t\log\varphi(s)}{(1-t)r+ts} \leq \max\left(\frac{\log\varphi(r)}{r}, \frac{\log\varphi(s)}{s}\right)$$
$$\implies \log\|f\|_p \leq \max(\log\|f\|_r, \log\|f\|_s) \implies \|f\|_p \leq \max(\|f\|_r, \|f\|_s).$$

This is what we were trying to prove. Note that if $\|f\|_r < \infty$ and $\|f\|_s < \infty$ we get $\|f\|_p < \infty$ so we have $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Recall that for any $f : \mathbb{R}^d \to \mathbb{R}$, we define the L^∞ norm as

$$\|f\|_\infty = \inf\{B \in \mathbb{R} : \mu(\{x \in \mathbb{R}^d : f(x) \geq B\}) > 0\}.$$

Now we claim that $\|f\|_p \to \|f\|_\infty$ on a set $X \subset \mathbb{R}^d$ of finite measure. For any $\|f\|_\infty \geq \delta > 0$, let

$$S_\delta = \{x \in X : |f(x)| \geq \|f\|_\infty - \delta\}.$$

Then we have

$$\|f\big|_X\|_p = \left(\int_X f^p \, d\mu\right)^{1/p} \geq \left(\int_{S_\delta} (\|f\|_\infty - \delta)^p \, d\mu\right)^{1/p} = (\|f\|_\infty - \delta)\mu(S_\delta)^{1/p}.$$

This in turn implies that $\liminf_{p\to\infty} \|f\big|_X\|_p \geq \|f\big|_X\|_\infty - \delta$. Now since $|f(x)| \leq \|f\|_\infty$ for a.e. x, we get for any $p > q$ that

$$\|f\big|_X\|_p \leq \left(\int_X |f(x)|^{p-q}|f(x)|^q\right)^{1/p} \leq \|f\big|_X\|_\infty^{1-p/q}\|f\big|_X\|_q^{q/p}.$$

This gives the reverse inequality so we have $\lim_{p\to\infty} \|f\big|_X\|_p = \|f\big|_X\|_\infty$. Since $\mathbb{R}^d$ is σ-finite, we can let X increase to get $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.