Math 231b Problem Set 4

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Problem 1. Let $\mathbb{RP}^\infty = \varinjlim_n \mathbb{RP}^n$ and $\mathbb{CP}^\infty = \varinjlim_n \mathbb{CP}^n$ along the usual inclusions. Use the fibrations

$$S^0 \to S^\infty \to \mathbb{RP}^\infty \quad \text{and} \quad S^1 \to S^\infty \to \mathbb{CP}^\infty$$

to compute the homotopy groups of $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$.

First we'll prove that S^∞ is weakly contractible, i.e. has trivial homotopy groups. Note that for any map $f : S^k \to S^\infty$, there must be some $n > k$ such that f factors through the natural CW skeleta inclusion $S^n \to S^\infty$. This is because the image of a compact space in a CW complex must intersect only a finite number of skeleta. However any map $S^k \to S^n$ with $k < n$ is nullhomotopic, so the f is as well.

Now since we have a fibration $S^0 \to S^\infty \to \mathbb{RP}^\infty$, we get a long exact sequence:

$$\cdots \longrightarrow \pi_1(S^0) \longrightarrow \pi_1(S^\infty) \longrightarrow \pi_1(\mathbb{RP}^\infty) \longrightarrow \pi_0(S^0) \longrightarrow \pi_0(S^\infty) \longrightarrow \pi_0(\mathbb{RP}^\infty)$$

Also this has the structure of an exact sequence of groups starting at the $\pi_1(\mathbb{RP}^\infty)$ term. Since $\pi_k(S^\infty) = \pi_k(S^0) = 0$ for all $k > 0$, we get isomorphisms $\pi_k(\mathbb{RP}^\infty) \cong \pi_k(S^\infty) = 0$ for all $k \geq 2$. Clearly $\pi_0(\mathbb{RP}^\infty) = 0$ since $\mathbb{RP}^\infty$ is path connected, so all we have left is to compute π_1.

In this case, we notice that there is a bijective map (as sets) $\pi_1(\mathbb{RP}^\infty) \to \pi_0(S^0)$ since $\pi_0(S^\infty) = \pi_1(S^\infty) = 0$. This means that $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$, since this is the only two element group. Thus we have:

$$\boxed{\pi_k(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z}/2 & k = 1, \\ 0 & \text{otherwise.} \end{cases}}$$

In the case of complex projective space, we have a similar situation, which can be described in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2(S^\infty) \arrow[r] \arrow[d, dashed] & \pi_2(\mathbb{CP}^\infty) \arrow[r] & \pi_1(S^1) \arrow[r] \arrow[d, dashed, no head] & \pi_1(S^\infty) \arrow[r] \arrow[d, dashed, no head] & \pi_1(\mathbb{CP}^\infty) \arrow[r] & \pi_0(S^1) \arrow[r] \arrow[d, dashed, no head] & \cdots \\
 & 0 & & \mathbb{Z} & 0 & & 0 &
\end{tikzcd}$$

Now, the only non-trivial group becomes $\pi_2(\mathbb{CP}^\infty) = \mathbb{Z}$, so

$$\boxed{\pi_k(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & k = 2, \\ 0 & \text{otherwise} \end{cases}}.$$

Problem 2. Use the Hopf fibrations to prove that

$$\pi_n S^2 \cong \pi_n S^3 \oplus \pi_{n-1} S^1, \quad \pi_n S^4 \cong \pi_n S^7 \oplus \pi_{n-1} S^3, \quad \text{and} \quad \pi_n S^8 \cong \pi_n S^1 5 \oplus \pi_{n-1} S^7.$$

For the sake of the problem, let's suppose we had a fibration $S^{n-1} \to S^{2n-1} \to S^n$. The Hopf fibrations give us explicit maps for $n = 1, 2, 4$, or 8. Now we can take the homotopy fiber to get a fibration $F(f) \to S^{n-1} \to S^{2n-1}$, where $f : S^{n-1} \to S^{2n-1}$. This gives the diagram:

$$\begin{tikzcd} F(f) \arrow[r] & S^{n-1} \arrow[r, "f"'] \arrow[r, bend left, "*"] \arrow[r, phantom, "\Updownarrow"] & S^{2n-1} \arrow[d] \\ & & S^n \end{tikzcd}$$

Recall that $F(f)$ is the pullback $S^{n-1} \times_{S^{2n-1}} (S^{2n-1})^I_*$. However since f is a map from a lower dimensional sphere into a high dimensional sphere, it must be nullhomotopic, so $F(f) \simeq F(c_*) = S^{n-1} \times \Omega S^{2n-1}$. However we also have a homotopy equivalence $F(f) \simeq \Omega S^n$ by the fibration. Thus we have an equivalence $S^{n-1} \times \Omega S^{2n-1} \simeq \Omega S^n$. Since every homotopy equivalence is a weak equivalence, by taking π_{k-1}, we get and isomorphism:

$$\pi_{k-1} S^{n-1} \oplus \pi_{k-1} \Omega S^{2n-1} \cong \pi_{k-1} \Omega S^n \quad \Longrightarrow \quad \pi_{k-1} S^{n-1} \oplus \pi_k S^{2n-1} \cong \pi_k S^n.$$

Problem 3. Let $p : E \to B$ and $p' : E' \to B$ be fibrations, and let $f : E \to E'$ be a homotopy equivalnce such that $p' \circ f = p$. Show that f is a fiber-homotopy equivalence.

We'll follow the proof in May, dualising when needed. Let's break up the solution into several parts. For starters, let's denote a fiber-homotopy equivalence between two maps $f, g : E \to E'$ as $h : f \simeq_B f'$. Note that it suffices to find a right fiber-homotopy inverse map $g : E' \to E$ with $f \circ g \simeq_B 1_{E'}$. Once we do this, we can repeat the argument to find a left fiber-homotopy inverse, and it follows by properties of homotopic maps that these two will themselves be homotopic. Now let's break down this proof into a sequence of claims.

Claim. There is a right homotopy inverse $g : E' \to E$ to f satisfying $p \circ g = p'$.

Proof. Since f is a homotopy equivalence, there is some map $g' : E' \to E$ with $f \circ g' \simeq 1_{E'}$. Then $p \circ g' = p' \circ f \circ g' \simeq p' \circ 1_{E'} = p'$. Let's call this homotopy $h : E' \times I \to B$, with $h_0 = p \circ g'$ and $h_1 = p'$. Using the homotopy lifting property we get a lift:

$$\begin{tikzcd} E' \arrow[r, "g'"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ E' \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B \end{tikzcd}$$

It follows that $\widetilde{h}_0 = g'$ and $\widetilde{h}_1$ is some map such that $\widetilde{h}_1 \circ p = h_1 = p'$. Furthermore, $g = \widetilde{h}_1$ is homotopic to g' so $f \circ g \simeq 1_{E'}$. Thus $\widetilde{h}_1$ is our desired map g so we are done. □

Now we have a map $f \circ g : E' \to E'$ that satisfies $p' \circ (f \circ g) = p'$ and $f \circ g \simeq 1_{E'}$. If we could prove that such a map satisfies $f \circ g \simeq_B 1_{E'}$, we would be done. Changing the notation up slightly:

Claim. Given a map $f : E \to E$ with $p \circ f = p$ and $f \simeq 1_E$, there is right fiber homotopy inverse $e : E \to E$ with $f \circ e \simeq_B 1_E$.

Proof. Let $h : E \times I \to E$ be some homotopy $f \simeq 1_E$. By composing with the fibration we get a homotopy $p \circ h : E \times I \to B$, so by the homotopy lifting property, we get a lift:

$$\begin{tikzcd} E \arrow[r, "1_E"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ E \times I \arrow[r, "p \circ h"'] \arrow[ur, dashed, "k"] & B \end{tikzcd}$$

This new homotopy k satisfies $k_0 = 1_E$ and $k_1 = e : E \to E$. The new map $e : E \to E$ satisfies $p \circ e = p$ by the diagram. We claim that $f \circ e \simeq_B 1_E$, we'll prove this by constructing several homotopies. First, let $J : E \times I \to E$ be the homotopy

$$J(x, s) = \begin{cases} f \circ k(x, 1 - 2s) & 0 \leq s \leq \frac{1}{2}, \\ h(x, 2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

This is a homotopy $f \circ e \simeq 1_E$. Next, consider the homotopy of homotopies $K : E \times I \times I \to B$ given by

$$K(x, s, t) = \begin{cases} p \circ f \circ k(x, 1 - 2s(1 - t)) & 0 \leq s \leq \frac{1}{2}, \\ p \circ h(x, 1 - 2(1 - s)(1 - t)) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Now this is a homotopy fbetween $p \circ J$ and p. By the homotopy lifting property, we get a diagram:

$$\begin{tikzcd} E \times I \arrow[r, "J"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ E \times I \times I \arrow[r, "K"'] \arrow[ur, dashed, "L"] & B \end{tikzcd}$$

Since $K(x, 0, t) = K(x, s, 1) = K(x, 1, t) = p(x)$, we can see that by going around the three sides of $I \times I$ other than the J side gives a fiber homotopy $f \circ e \simeq 1_E$. $\square$

Problem 4. Given an H-space $(X, *)$, prove that the action of $\pi_1(X, *)$ on $\pi_n(X, *)$ is trivial for all $n \geq 1$. Conclude that a connected H-space is simple.

One way to view the action of $\pi_1(X, *)$ on $\pi_n(X, *)$ is as follows. Recall that the inclusion $* \subset S^n$ is a cofibration (being a CW inclusion), so given any loop $\omega : (I, *) \to X$ and $\alpha : (S^n, *) \to X$ that agree on the basepoint, we can use the homotopy extension property to get a homotopy $k : S^n \times I \to X$ such that $k_0 = \alpha$ and $k_1 = \omega \cdot \alpha$. (This is how the action is described in Hacher 4A)

Now suppose our space X is in fact an H-space, meaning we have some multiplication map $m : X \times X \to X$ which is homotopy unital, i.e. $m(*, -) \simeq m(-, *) \simeq 1_X$. Let $\Gamma : X \times I \to X$ be the homotopy between $m(*, -)$ and 1_X. We can use this multiplication map to construct a homotopy $H : S^n \times I \to X$ given by $H(x, t) = m(\omega(t), \alpha(x))$. Composing this with Γ gives a homotopy between $\omega \cdot \alpha$ and α, showing that the action is trivial. So the space is simple.