Math 230a Problem Set 9

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Problem 1. Let $\pi : P \to X$ be a principal G-bundle with connection $\Theta \in \Omega^1(P; \mathfrak{g})$.

(a). Denote the right G-action of G on P as $R : P \times G \to P$. Compute $R^*\Theta$.

Letting R_g denote right multiplication by $g \in G$. By G-equivariance of the connection form, we have the relation $R_g^*\Theta = \mathrm{Ad}_{g^{-1}} \Theta$. Letting θ_{mc} be the Maurer-Cartan form on G, we find that

$$R^*\Theta = \mathrm{Ad}_{g^{-1}} \mathrm{pr}_P^* \Theta + \mathrm{pr}_G^* \theta_{\mathrm{mc}}.$$

(b). Suppose $s : X \to P$ is a section of π and $g : X \to G$ is a function, Compute $\alpha' = (s \cdot g)^*\Theta$ in terms of $\alpha = s^*\Theta$.

Let's write the function $s \cdot g$ as a composition of

$$\begin{array}{rcl} \phi : X & \longrightarrow & P \times G \\ x & \longmapsto & (s(x), g(x)) \end{array} \quad \text{and} \quad \begin{array}{rcl} R : P \times G & \longrightarrow & P \\ (p, g) & \longmapsto & p \cdot g \end{array}$$

We know the pullback $\alpha' = (s \cdot g)^*\Theta = (R \circ \phi)^*\Theta = \phi^* R^*\Theta$. Plugging the map ϕ into the expression from the previous part, we get

$$\alpha' = \mathrm{Ad}_{g^{-1}} \alpha + g^*\theta_{\mathrm{mc}}.$$

(c). Write your results in matrix notation if G is a matrix group.

In a matrix group G and matrix $M \in G$, the adjoint action is conjugation, and the Maurer-Cartan form is $\theta_{\mathrm{mc}} = M^{-1}\, dM$. Thus, we can write the expressions from the previous parts as

$$R^*\Theta = M^{-1}(\mathrm{pr}_1^*\Theta)M + M^{-1}\, dM \quad \text{and} \quad \alpha' = M^{-1}\alpha M + M^{-1}\, dM.$$

Problem 2.

(a). Let G be a Lie group (not necessarily compact) and let $H \subset G$ be a closed Lie subgroup which is compact. Prove that the homogeneous manifold G/H is reductive.

(b). Find an H-invariant complement $\mathfrak{p}$ to $\mathfrak{h} \subset \mathfrak{g}$ for the homogeneous manifolds $\mathrm{O}_{n,1}/\mathrm{O}_n$ (hyperbolic space) and $\mathrm{Sp}_{2n} / \mathrm{U}_n$ (Siegel upper half space).

(c). Compute the curvature of the induced linear connection on hyperbolic space. Compare to the case of the sphere.

(d). Describe the Riemannian metric on hyperbolic space. Identify the total space of the orthonormal frame bundle.

Problem 3. Let $\pi : P \to X$ be a principal G-bundle with connection Θ. Prove that Θ is flat if and only if the holonomy group $\mathrm{Hol}_p(\Theta) \subset G$ is a discrete subgroup for all $p \in P$.

Problem 6.

(a). Fix $m \in \mathbb{Z}^{>0}$. Construct a $\mathbb{C}^\times$-action on $\mathbb{CP}^m$ with precisely $m+1$ fixed points.

Consider the action

$$\begin{array}{rcl} \mathbb{C}^\times & \longrightarrow & \mathrm{End}(\mathbb{CP}^m) \\ \lambda & \longmapsto & \lambda \cdot [z_1 : \cdots : z_{m+1}] = [\lambda z_1 : \lambda^2 z_2 : \cdots : \lambda^{m+1} z_{m+1}]. \end{array}$$

(b).