Astron 140 Problem Set 10

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Problem 1. Following the lectures,

(a). Show how $\overline{h}_{\mu\nu}$, the trace-reverse of the metric pertubation, transforms under the gauge transformation $x^\mu \mapsto x^\mu_{\text{new}} = x^\mu + \xi^\mu(x)$.

Under the assumption of a weak field limit, we can write the metric of a solution to the Einstein equations as a small perturbation of the flat Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{where} \quad |h_{\mu\nu}| \ll 1.$$

Given a gauge transformation $x^\mu \mapsto x^\mu_{\text{new}} = x^\mu + \xi^\mu(x)$, we get

$$\begin{aligned}
g_{\mu\nu} \mapsto g^{\text{new}}_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x^\mu_{\text{new}}} \frac{\partial x^\beta}{\partial x^\nu_{\text{new}}} g_{\alpha\beta} \\
&= \left(\frac{\partial x^\alpha_{\text{new}}}{\partial x^\mu_{\text{new}}} - \frac{\partial \xi^\alpha}{\partial x^\mu_{\text{new}}} \right) \left(\frac{\partial x^\beta_{\text{new}}}{\partial x^\nu_{\text{new}}} - \frac{\partial \xi^\beta}{\partial x^\nu_{\text{new}}} \right) (\eta_{\alpha\beta} + h_{\alpha\beta}) \\
&= \left(\delta^\alpha_\mu - \partial_\mu \xi^\alpha \right) \left(\delta^\beta_\nu - \partial_\nu \xi^\beta \right) (\eta_{\alpha\beta} + h_{\alpha\beta}) \\
&\approx \eta_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + h_{\mu\nu}
\end{aligned}$$

where the last equality only holds up to first order. This implies that $h_{\mu\nu}$ undergoes the gauge transformation

$$h_{\mu\nu} \mapsto h^{\text{new}}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu.$$

It is however useful to consider the trace reverse of $h_{\mu\nu}$, namely

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \quad \text{where} \quad h = h^\alpha_\alpha = \eta^{\alpha\beta} h_{\beta\alpha}.$$

This trace reversed perturbation undergoes the gauge transformation

$$\begin{aligned}
\overline{h}_{\mu\nu} \mapsto \overline{h}^{\text{new}}_{\mu\nu} &= h^{\text{new}}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\text{new}} \\
&= h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}(h_{\beta\alpha} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta)) \\
&= h^{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha \xi_\beta \\
&= \overline{h}^{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\alpha \xi_\alpha.
\end{aligned}$$

Thus, the gauge transformation for the trace reversed perturbation is

$$\overline{h}_{\mu\nu} \mapsto \overline{h}^{\text{new}}_{\mu\nu} = \overline{h}^{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\alpha \xi_\alpha.$$

(b). Explain why the Lorentz gauge $\partial^\mu \overline{h}_{\mu\nu} = 0$ restricts almost all the degrees of freedom in this gauge transformation; but still leaves some residual gauge degrees of freedom.

The Lorentz gauge $\partial^\mu \overline{h}_{\mu\nu} = 0$ has the following effect:

$$\partial^\mu \overline{h}^{\text{new}}_{\mu\nu} = \partial^\mu \overline{h}_{\mu\nu} - \Box\xi_\nu - \partial_\nu \partial^\mu \xi_\mu + \partial_\nu \partial^\alpha \xi_\alpha = 0$$

Since $\partial^\mu \overline{h}^{\text{new}}_{\mu\nu} = 0$ as well, and the last two terms cancel, we get $\Box\xi_\nu = 0$. This is a massive restriction on the space of possible perturbations of ξ_ν, and makes it finite dimensional.

(c). What are these residual gauge degrees of freedom?

The restriction we derived is a wave equation, with solutions $\xi_\mu = B_\mu e^{ik_\alpha x^\alpha}$, where $k^\mu k_\mu = 0$. This gives us 4 degrees of freedom in B_μ, and k_α shared with the wave solutions to the Einstein equation.

(d). What other conditions need to be introduced to fix them, and then why are they fixed?

Earlier in the lectures, we derived that the Einstein equation gives us $\Box \overline{h}_{\mu\nu} = 0$, another wave equation with solutions $\overline{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha}$ with $k_0^2 = k_i^2$. Combined with e Lorentz gauge, we get $k^\mu A_{\mu\nu} = 0$ which leaves 6 degrees of freedom. Note that

$$A_{\mu\nu} \mapsto A^{\text{new}}_{\mu\nu} = A_{\mu\nu} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu} k^\alpha b_\alpha.$$

Thus, adding the traceless condition $\eta^{\mu\nu} A_{\mu\nu} = 0$ removes 1 degree of freedom leaving us with 5, and adding a transversality condition $A_{0\mu} = 0$ gives us 3 conditions (one redundant with the Lorenz gauge) so we are left with 2 degrees of freedom. In particular, all of the B_μ are set to zero since they correspond to non-physical transformations. The remaining degrees of freedom correspond to polarizations of gravitational waves in the periodic solutions $\overline{h}_{\mu\nu}$.