Math 212 Problem Set 5

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Problem 1. Let B denote the $|x| < 1$ ball in $\mathbb{R}^3$. Define a function E on the space $C_c^\infty(B)$ by the rule

$$E(u) = \int_B (|\nabla u|^2 + |u|^4).$$

(a). Explain why E extends from the dense domain $C_c^\infty(B)$ in $L^2_{1,c}(B)$ to define a continuous function on $L^2_{1,c}(B)$ and thus (by restriction) on the subset $S \subset L^2_{1,c}(B)$ of functions with L^2 norm equal to 1.

It is clear that both terms in the integral are finite by the Sobolev embedding theorem, and thus E is a well-defined function on $L^2_{1,c}(B)$. To show continuity in the $L^2_{1,c}$-norm, let $\{u_n\}$ be a sequence of smooth compactly supported approximations $u_n \to u$. It follows that we have convergence of $\nabla u_n \to \nabla u$ in $L^2(B)$ and $u_n \to u$ in $L^4(B)$ so $E(u_n) \to E(u)$. This immediately implies that it restricts to a continuous function on the closed subset S.

(b). Prove that the infimum of E on S is taken on by some function in S (denoted below by u.)

Let $\{u_n\} \subset S$ be a minimizing sequence, i.e. with $E(u_n)$ converging to $\inf_{u\in S} E(u)$. It follows that $\{u_n\}$ is bounded in $L^2_{1,c}(B)$. By a theorem proved on Problem Set 2, since $L^2(B)$ is a Hilbert space, we can choose a subsequence which converges strongly to u in $L^2(B)$. This implies that $\|u\|_{L^2} = 1$ so $u \in S$. All we need to show is that $E(u) \le \liminf E(u_n)$, which would prove that $E(u) = \inf_{u\in S} E(u)$. Note that

$$\liminf_{n\to\infty} E(u_n) = \liminf_{n\to\infty} \left[\int_B |\nabla u_n|^2 + \int_B |u_n|^4\right] \ge \int_B |\nabla u|^2 + \int_B |u|^4 = E(u).$$

This completes the proof.

(c). Prove that u is C^∞ in the interior of Ω and that it obeys the differential equation

$$-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) u + 2u^3 = \lambda u$$

with λ denoting a number which is between $E(u)$ and $2E(u)$.

Introduce a Lagrange multiplier

$$\mathcal{L}(v, \lambda) = E(v) - \lambda \left(\int_B |v|^2 - 1\right).$$

At the minimum, the variation of this functional in any direction $\varphi \in C_c^\infty(B)$ must vanish, so we get:

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(u+t\varphi,\lambda)\Big|_{t=0} &= \int_B (2\nabla u \cdot \nabla \varphi + 4u^3\varphi - 2\lambda u\varphi) \\ &= \int_B (-\Delta u + 2u^3 - \lambda u)\varphi = 0 \end{aligned} \qquad \Longrightarrow \qquad -\Delta u + 2u^3 = \lambda u.$$

To prove that $\lambda \in (E(u), 2E(u))$, note that by multiplying the equation by u and integrating over B, we get

$$\int_B (-\Delta u)u + 2\int_B u^4 = \lambda \int_B u^2 = \lambda,$$

since $\|u\|_{L^2} = 1$. Integration by parts gives $\int_B |\nabla u|^2 + 2\int_B |u|^4 = \lambda$. but recall that $E(u) = \int_B |\nabla u|^2 + \int_B |u|^4$.. Thus, $\lambda = E(u) + \int_B |u|^4$. Since $u \neq 0$, clearly $\int_B |u|^4 > 0$. Hence, $\lambda > E(u)$. Also, we trivially have $\int_B |u|^4 \leq E(u)$, so:

$$\lambda = E(u) + \int_B |u|^4 \leq E(u) + E(u) = 2E(u).$$

Since $\int_B |u|^4 > 0$, strict inequality holds: $\lambda < 2E(u)$. Thus, we have established the desired inequality: $E(u) < \lambda < 2E(u)$.