Math 213a Problem Set 1

Lev Kruglyak

Due: Tuesday, September 12

Problem 1. Let $T \subset \mathbb{R}^3$ be the spherical triangle defined by $x^2 + y^2 + z^2 = 1$ and $x, y, z \geq 0$.

Let $\alpha = z\ dx\ dz$.

a. Find a smooth 1-form β on $\mathbb{R}^3$ such that $\alpha = d\beta$.

Consider the 1-form $\beta = xz\ dz$. Then $d\beta = (x\ dz + z\ dx) \wedge dz = z\ dx \wedge dz = \alpha$.

b. Define consistent orientations for T and ∂T.

Let ψ be the graph parametrization of T, i.e. the diffeomorphism $\psi : Q_1 \to T$ given by

$$\psi(u, v) = (u, v, \sqrt{1 - u^2 - z^2}),$$

and Q_1 the first quadrant of the unit disk. The standard orientation on $\mathbb{R}^2$ induces an orientation on Q_1, which in turn can be pulled back to get an orientation on T and ∂T. This looks like the counterclockwise orientation when viewed from above,i.e. by collapsing z.

c. Using your choices in (b), compute $\int_T \alpha$ and $\int_{\partial T} \beta$ directly, and check that they agree. (Why should they agree?)

First note that the pullback form $\psi^*\alpha$ is given by

$$\psi^*\alpha = z(u, v)\ du \wedge d(z(u, v)) = z(u, v)\ du \wedge \left(-\frac{u}{z(u, v)}\ du - \frac{u}{z(u, v)}\ dv \right) = -v\ du \wedge dv.$$

Our orientation on T was induced by this pullback, so this form is $-v\ dA$ where dA is the standard area form on $\mathbb{R}^2$. Then standard calculus gives us:

$$\int_T \alpha = \int_{Q_1} -v\ du \wedge dv = \int_0^{\pi/2} \int_0^1 -r^2 \sin\theta\ dr \wedge d\theta = -\frac{1}{3}.$$

On the other hand, to calculate the integral of β over ∂T, we split it into disjoint segments. Since $\beta = xz\ dz$ vanishes along the lines $x = 0$ and $z = 0$, we only need to integrate it over the intersection of ∂T with the line $y = 0$. This can be parametrized by some $\phi : [0, 1] \to \partial T$ given by

$$\phi(t) = \left(t, 0, \sqrt{1 - t^2}\right),$$

and this clearly respects the standard orientation when $[0, 1]$ is considered as a subspace $[0, 1] \times \{0\} \subset Q_1$. The pullback form $\phi^*\beta$ is now given by

$$\phi^*\beta = tz(0, t)\ dz(0, t) = -t^2\ dt.$$

Thus, we can evaluate the integral

$$\int_{\partial T} \beta = \int_{[0,1]} -t^2 \, dt = -\frac{1}{3}.$$

This agrees with the earlier theorem since Stoke's theorem holds true for compact smooth manifolds with boundary.

Problem 2. Let $f(z) = (az+b)/(cz+d)$ be a Möbius transformation. Show that the number of rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$g(g(g(g(g(z))))) = f(z)$$

is $1, 5$, or ∞. Explain how to determine which alternative holds for a given f.

We will use the following classic facts about Möbius transformations:

Claim. Let f be a Möbius transformation. Exactly one of the following holds true:

1. f is the identity.
2. f has exactly one fixed point, and must be conjugate to a map of the form $f(z) = z + b$.
3. f has exactly two fixed points, and must be conjugate to a map of the form $f(z) = az$.

Claim. Every (rational) automorphism of the Riemann sphere is a Möbius transformation.

Now first of all, it's clear that g must be an automorphism of the Riemann sphere, so g is a Möbius transformation. We have three cases to check:

Case 1: f is the identity.

In this case, let $\zeta \neq 1$ be a fifth root of unity and $a, b \in \mathbb{C}$ with $|a|, |b| > 1$. Then the map

$$g = \left(\frac{z-a}{z-b}\right) \cdot \zeta z \cdot \left(\frac{z-a}{z-b}\right)^{-1}$$

satisfies $g^{\circ 5} = f$, so there are infinitely many solutions.

Case 2: f has one fixed point.

In this case, f must be conjugate to some map of the form $z + b$ for $b \neq 0$. Since repeated composition preserves conjugation, we can assume without loss of generality that $f(z) = z + b$. Since repeated composition also preserves fixed points, we know also that g can have at most one fixed point, so it has a fixed point as well, and must also be of the form $z + b'$. The only possible b' is $b/5$, so we have one solution.

Case 3: f has two fixed points.

In this case, g must also have two fixed points since if it had one, then f would as well. Thus both maps are conjugate to $f(z) = az$. The only solutions are thus $g(z) = a'z$ where a' are fifth roots of a, which is exactly five since $a \neq 1$.

Problem 3. Let $\tanh(z)$ be the hyperbolic tangent.

Let $\sum a_n z^n$ be the Taylor series for $\tanh(z)$ at $z = 0$.

a. What is the radius of convergence of this power series?

We know that tanh is a meromorphic function, and can be expressed as

$$\tanh(z) = \frac{e^{2z} - 1}{e^{2z} + 1}.$$

Thus, the radius of convergence is the radius of the largest circle which does not include any of the poles. Since the poles of this function are the solutions to $e^{2z} = -1$, this radius of convergence is $\pi/2$.

b. Show that $a_5 = 2/15$.

Recall that by using the Taylor series for $\sinh(x)$ and $\cosh(x)$ at 0, we get the following power series identity:

$$\left(1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \cdots\right) \cdot (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \cdots) = z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots.$$

By expanding and solving for the coefficients, we get:

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{1}{3}, \quad a_4 = 0, \quad a_5 = \frac{2}{15}.$$

c. Give an explicit value of N such that $\tanh(1)$ and $\sum_0^N a_n$ agree to within $\epsilon = 10^{-1000}$. Justify your answer.

Recall that Taylor's theorem implies that for some radius $1 < R < \pi/2$, the error term of the Taylor series up to N is bounded by:

$$|E_N(1)| \leq \frac{\max_{z \in S^1(R)} \tanh(z)}{R^N(R-1)} = \frac{M_R}{R^N(R-1)}.$$

IF we want $|E_N(1)| \leq \epsilon$, we have the inequality:

$$\frac{M_R}{R^N(R-1)} \leq \epsilon \qquad \Longrightarrow \qquad N \geq \frac{\log M_R - \log(R-1) - \log \epsilon}{\log R}.$$

If we set $R = 1.5$, we just need to calculate M_R to get an N which satisfies the error rate. This is quite tedious to do by hand, so I wrote a Rust program which samples 100 million points along the circle of radius R to find the maximum magnitude of $\tanh(z)$. Since neither the function nor its derivative has any poles in an ϵ-neighborhood of this circle, we can rest assured that no significantly higher bounds can slip between our samples.

```
use num::complex::Complex;
use std::f64::consts::PI;

fn main() {
    let radius = 1.5;
    let iterations = 100_000_000;

    let max_magnitude = (0..iterations)
        .map(f64::from)
        .map(|n| 2.0 * PI * n / iterations as f64)
        .map(|theta| Complex::from_polar(radius, theta))
        .map(Complex::tanh)
        .map(Complex::norm)
        .filter(|x| f64::is_finite(*x))
        .fold(f64::NEG_INFINITY, |a, b| a.max(b));

    println!("maximum magnitude = {:?} on R = {radius:?}", max_magnitude);
}
```

Running this code gives:

```
$ maximum magnitude = 14.10141994717166 on R = 1.5
```

For safety, we'll increase this maximum to $M_R = 15$. Plugging this into the above inequality, we get:

$$N \geq \frac{\log 15 - \log 0.5 + 1000}{\log 1.5} > 5688.$$

Problem 4. Let $f : U \to V$ be a proper local homeomorphism between a pair of open sets $U, V \subset \mathbb{C}$. Prove that f is a covering map. (Here *proper* means that $f^{-1}(K)$ is compact whenever $K \subset V$ is compact.)

Assume V is connected, otherwise this problem statement is false as the map won't be surjective. Now let $v \in V$ be an arbitrary point. We can shrink V to be a compact set containing v, and let $U = f^{-1}(V)$ also be compact. Note that since $\{v\}$ is a compact set, $f^{-1}(v)$ is a compact set as well. Since it is a closed, discrete, compact subset of a compact space it must be finite. For each $u_i \in f^{-1}(v)$, choose some open neighborhood U_i of u_i such that $f\big|_{U_i}$ is a homeomorphism. We can further shrink each U_i to be disjoint since there are only a finite number. Finally, it follows by local homeomorphism that $\bigcap f(U_i)$ is an evenly-covered neighborhood of v, and so f is a covering map.

Problem 5. Let $f : \mathbb{C} \to \mathbb{C}$ be given by a polynomial of degree 2 or more. Let

$$V_1 = \{f(z) : f'(z) = 0\} \subset \mathbb{C}$$

be the set of critical values of f, let $V_0 = f^{-1}(V_1)$, and let $U_i = \mathbb{C} - V_i$ for $i = 0, 1$. Prove that $f : U_0 \to U_1$ is a covering map.

First of all, notice that every polynomial is proper because the inverse of any closed and bounded region is closed and bounded. Then it is a local homeomorphism at every regular point because of the inverse function theorem. So it is a covering map by the previous problem.

Problem 6. Give an example where U_0/U_1 is a normal (or Galois) covering, i.e. where $f_*(\pi_1(U_0))$ is a normal subgroup of $\pi_1(U_1)$.

Consider the parabola $f(z) = z^2$. This has one critical point and value, so $\pi_1(U_1) \cong \mathbb{Z}$. This is abelian, so every subgroup is normal.