Define the **Landau** kernels by

$$L_n(x) = \begin{cases} \dfrac{(1-x^2)^n}{c_n} & \text{if } -1 \le x \le 1, \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where c_n is chosen so that $\int_{-\infty}^{\infty} L_n(x)\,dx = 1$. Prove that $\{L_n\}_{n\ge 0}$ is a family of good kernels as $n \to \infty$. As a result, show that if f is a continuous function supported in $[-1/2, 1/2]$, then $(f * L_n)(x)$ is a sequence of polynomials on $[-1/2, 1/2]$ which converges uniformly to f.

[Hint: First show that $c_n \ge 2/(n+1)$.]

11. Suppose that u is the solution to the heat equation given by $u = f * \mathcal{H}_t$ where $f \in \mathcal{S}(\mathbb{R})$. If we also set $u(x, 0) = f(x)$, prove that u is continuous on the closure of the upper half-plane, and vanishes at infinity, that is,

$$u(x,t) \to 0 \quad \text{as } |x| + t \to \infty.$$

[Hint: To prove that u vanishes at infinity, show that (i) $|u(x,t)| \le C/\sqrt{t}$ and (ii) $|u(x,t)| \le C/(1+|x|^2) + Ct^{-1/2}e^{-cx^2/t}$. Use (i) when $|x| \le t$, and (ii) otherwise.]

12. Show that the function defined by

$$u(x,t) = \frac{x}{t}\mathcal{H}_t(x)$$

satisfies the heat equation for $t > 0$ and $\lim_{t\to 0} u(x,t) = 0$ for every x, but u is *not* continuous at the origin.

[Hint: Approach the origin with (x,t) on the parabola $x^2/4t = c$ where c is a constant.]

13. Prove the following uniqueness theorem for harmonic functions in the strip $\{(x,y) : 0 < y < 1,\ -\infty < x < \infty\}$: if u is harmonic in the strip, continuous on its closure with $u(x,0) = u(x,1) = 0$ for all $x \in \mathbb{R}$, and u vanishes at infinity, then $u = 0$.

14. Prove that the periodization of the Fejér kernel $\mathcal{F}_N$ on the real line (Exercise 9) is equal to the Fejér kernel for periodic functions of period 1. In other words,

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x),$$

when $N \ge 1$ is an integer, and where

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N}\frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

15. This exercise provides another example of periodization.

(a) Apply the Poisson summation formula to the function g in Exercise 2 to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}$$

whenever α is real, but not equal to an integer.

(b) Prove as a consequence that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)} = \frac{\pi}{\tan \pi\alpha} \tag{15}$$

whenever α is real but not equal to an integer. [Hint: First prove it when $0 < \alpha < 1$. To do so, integrate the formula in (b). What is the precise meaning of the series on the left-hand side of (15)? Evaluate at $\alpha = 1/2$.]

16. The Dirichlet kernel on the real line is defined by

$$\int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = (f * \mathcal{D}_R)(x) \quad \text{so that} \quad \mathcal{D}_R(x) = \widehat{\chi_{[-R,R]}}(x) = \frac{\sin(2\pi R x)}{\pi x}.$$

Also, the modified Dirichlet kernel for periodic functions of period 1 is defined by

$$D_N^*(x) = \sum_{|n| \le N-1} e^{2\pi i n x} + \frac{1}{2}(e^{-2\pi i N x} + e^{2\pi i N x}).$$

Show that the result in Exercise 15 gives

$$\sum_{n=-\infty}^{\infty} \mathcal{D}_N(x+n) = D_N^*(x),$$

where $N \ge 1$ is an integer, and the infinite series must be summed symmetrically. In other words, the periodization of $\mathcal{D}_N$ is the modified Dirichlet kernel D_N^*.

17. The **gamma function** is defined for $s > 0$ by

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, dx.$$

(a) Show that for $s > 0$ the above integral makes sense, that is, that the following two limits exist:

$$\lim_{\substack{\delta \to 0 \\ \delta > 0}} \int_{\delta}^{1} e^{-x} x^{s-1}\, dx \quad \text{and} \quad \lim_{A\to\infty} \int_1^A e^{-x} x^{s-1}\, dx.$$