# Math 132
# Differential Topology
# March 3, 2023
# Spring 2023

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# Contents

# Part 1

# Basics of Manifold Theory

CHAPTER I

# The tangent space, locality and colloquialisms

The definition of the tangent space $T_xM$ of a smooth manifold $M$ at a point $x$ is a first encounter with a point of view that is at the heart of this course. Before turning to the tangent space in differential topology, we establish some basic terminology and then review the notion of derivative in multivariable calculus

## 1. Coordinate systems and parameterizations

**1.1. Local Coordinates.** Suppose that $X \subset \mathbb{R}^N$ is a smooth manifold of dimension $n$, and $x \in X$ is a point.

**Definition 1.1.** A *coordinate chart* (or a *coordinate neighborhood*) is a pair $(U, \Phi)$ consisting of an open set $U \subset X$ and a smooth function

$$\Phi = (\Phi_1, \dots, \Phi_n) : U \to \mathbb{R}^n$$

which is a diffeomorphism with an open subset $W = \Phi(U) \subset \mathbb{R}^n$. The components $\Phi_i$ of $\Phi$ are called *local coordinate functions.*

**Definition 1.2.** A *coordinate chart around* $x \in X$ (or a *coordinate neighborhood of* $x$) is a coordinate chart $(U, \Phi)$ having the property that $x$ is an element of $U$.

The data of a coordinate chart around $x$ is summarized in

$$\begin{array}{ccccc} U & \subset & X & \subset & \mathbb{R}^N \\ \downarrow{\scriptstyle \Phi} & & & & \\ W & \subset & \mathbb{R}^n & & \end{array}$$

A coordinate chart is sometime just called a *chart.*

**Definition 1.3.** A coordinate chart $(U, \Phi)$ is *centered at* $x$ if $\Phi(x) = 0$.

**Remark 1.4.** Suppose that $(U, \Phi)$ is any coordinate chart and write

$$v = \Phi(x) \in \mathbb{R}^n.$$

Let

$$T : \mathbb{R}^n \to \mathbb{R}^n$$
$$w \mapsto w - v$$

be translation by $-v$. The composition

$$T \circ \Phi : U \to \mathbb{R}^n$$

satisfies

$$T \circ \Phi(x) = 0$$

and is a diffeomorphism of $U$ with the translation

$$T(W) = \{v \in \mathbb{R}^n \mid v + \Phi(x) \in W\}$$

of $W$ by $-v$. Thus $(U, T(W))$ is a coordinate neighborhood centered at $x$.

**1.2. Shrinking and coordinate models for maps.** Suppose that $(U, \Phi)$ is a coordinate chart and $U' \subset U$ is an open subset. The pair $(U', \Phi|_{U'})$ is also a coordinate chart. We say that $(U', \Phi|_{U'})$ is obtained from $(U, \Phi)$ by *shrinking.* If $(U, \Phi)$ is a coordinate neighborhood of $x \in X$ and $U'$ contains $X$ then $(U', \Phi|_{U'})$ is also a coordinate neighborhood of $x$.

**Definition 1.5.** Suppose that $f : X \to Y$ is a smooth map of smooth manifolds of dimensions $n$ and $m$ respectively, $x \in X$ is a point and $y = f(x)$. A *coordinate model for $f$ near $x$* consists of a coordinate chart $(U_x, \Phi_x)$ near $x \in X$ and a coordinate chart $(U_y, \Phi_y)$ having the property that $f(U_x) \subset U_y$.

A coordinate model for $f$ near $x$ determines a diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & \supset & U_x \arrow[r, "\Phi_x"] \arrow[d, "f"'] & W_x \arrow[d, "g"] & \subset & \mathbb{R}^n \\ Y & \supset & U_y \arrow[r, "\Phi_y"'] & W_y & \subset & \mathbb{R}^m \end{tikzcd}$$

in which

$$W_x = \Phi_x(U_x)$$
$$W_y = \Phi_y(U_y)$$

and

$$g = \Phi_y \circ f \circ \Phi_x^{-1}.$$

A coordinate model for $f$ allows one to study $f$ near $x$ in terms of a smooth map $g$ between open subsets of Euclidean space.

**Example 1.6.** In the situation above suppose that

$$(U_x, \Phi_x) \quad \text{and} \quad (U_y, \Phi_y)$$

are coordinate charts near $x$ and $y$ respectively. Shrinking $U_x'$ to $U_x \cap f^{-1}(U_y)$ gives a new coordinate neighborhood $(U_x', \Phi_x)$ of $x$ having the property that $f(U_x') \subset U_y$, so that $(U_x', \Phi)$ and $(U_y, \Phi_y)$ form a coordinate model for $f$.

**1.3. Parameterizations.** For some purposes it is more convenient to focus on the diffeomorphism

$$\Phi^{-1} : \Phi(U) \to U$$

going from an open subset of $\mathbb{R}^n$ to an open subset of $X$. There is a different terminology associated to this situation.

**Definition 1.7.** Suppose that $X$ is a smooth manifold of dimension $n$. A *local parameterization* of $X$ is a smooth function

$$\Psi : W \to X$$

from an open subset $W \subset \mathbb{R}^n$ of $\mathbb{R}^n$ to $X$, having the property that it gives a diffeomorphism with an open subset $\Psi(W) \subset X$ of $X$.

**Definition 1.8.** Suppose that $X$ is a smooth manifold of dimension $n$ and $x \in X$ is a point. A *local parameterization of $X$ near $x$* is a local parameterization $(W, \Psi)$ having the property that $x$ is an element of $\Psi(W)$. A local parameterization of $X$ near $x$ is *centered* at $x$ if $W$ contains 0 and $\Psi(0) = x$.

If $\Psi : W \to X$ is a local parameterization and $W' \subset W$ is an open subset, then

$$\Psi' = \Psi|_{W'} : W' \to X$$

is a local parameterization. We say $(W', \Psi')$ is obtained from $(W, \Psi)$ by *shrinking*.

**Definition 1.9.** Suppose that $f : X \to Y$ is a smooth map of smooth manifolds of dimensions $n$ and $m$ respectively, $x \in X$ is a point and $y = f(x)$. A *local parameterization of $f$ near $x$* consists of a local parameterization $(W_x, \Psi_x)$ near $x \in X$ and a local parameterization $(W_y, \Psi_y)$ near $y \in Y$ having the property that that $f(\Psi_x(W_x)) \subset \Psi_y(W_y)$.

A local parameterization of $f$ near $x$ determines a diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & W_x \arrow[l, "\Psi_x"'] \arrow[d, "g"] & \subset & \mathbb{R}^n \\ Y & W_y \arrow[l, "\Psi_y"] & \subset & \mathbb{R}^m \end{tikzcd}$$

in which $g(w)$ is the unique element $w' \in W_y$ satisfying

$$\Psi_y(w') = f(\Psi_x(w)).$$

As with local coordinate models, a local parameterization for $f$ allows one to study $f$ near $x$ in terms of a smooth map $g$ between open subsets of Euclidean space. Given any local parameterizations

$$(W_x, \Psi_x) \quad \text{and} \quad (W_y, \Psi_y)$$

near $x \in X$ and $y \in Y$ one can construct a local parameterization of $f$ by shrinking $W_x$ to

$$W'_x = W_x \cap \Psi_x^{-1}(f^{-1}(\Psi_y(W_y))).$$

**Example 1.10.** In the situation above suppose that

$$(U_x, \Phi_x) \quad \text{and} \quad (U_y, \Phi_y)$$

are coordinate charts near $x$ and $y$ respectively. Shrinking $U'_x$ to $U_x \cap f^{-1}(U_y)$ gives a new coordinate neighborhood $(U'_x, \Phi_x)$ of $x$ having the property that $f(U'_x) \subset U_y$, so that $(U'_x, \Phi)$ and $(U_y, \Phi_y)$ form a coordinate model for $f$.

**Example 1.11.** Suppose that $(U, \Phi)$ is a local coordinate system on $X$. Set

$$W = \Phi(U)$$

and define

$$\Psi : W \to X$$

by $\Psi(w) = \Phi^{-1}(w)$. The pair $(W, \Psi)$ is then a local parameterization of $X$.

**Example 1.12.** Recall that the $(n-1)$-sphere is the subset $S^{n-1} \subset \mathbb{R}^n$ consisting of points $x = (x_1, \cdots, x_n)$ with $|x| = 1$. For convenience let's write points $x \in \mathbb{R}^n$ as

$$\begin{aligned} x &= (v, x_n) \\ v &= (x_1, \ldots, x_{n-1}). \end{aligned}$$

Now consider the functions

$$\begin{aligned} \tilde{f}(v, x_n) &= \frac{v}{\sqrt{1-|v|^2}} \\ g(w) &= \left( \frac{w}{\sqrt{1+|w|^2}}, \frac{1}{\sqrt{1+|w|^2}} \right) \qquad w \in \mathbb{R}^{n-1} \end{aligned}$$

The function $\tilde{f}$ is defined and smooth on the complement of the cylinder $|v| = 1$, while $g$ is defined for all $w \in \mathbb{R}^{n-1}$. The function $\tilde{f}$ restricts to a smooth function on the upper hemisphere

$$\begin{aligned} f &: U \to \mathbb{R}^k \\ U &= \{(v, x_n) \in S^{n-1} \mid x_n > 0\} \end{aligned}$$

and is a diffeomorphism, with inverse $g$. This $f$ is a *local coordinate system* on the upper hemisphere, centered at $(0,1)$. The function $g$ is a *local parameterization* of $S^{n-1}$.

### Exercises

**1.1.** Suppose that

$$X_0 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

is a sequence of smooth maps between smooth manifolds. For a point $x_0 \in X$ define a sequence of points $x_i \in X_i$, $i = 0, \ldots, n$ inductively by

$$\begin{aligned} x_0 &= x_0 \\ x_i &= f_i(x_{i-1}). \end{aligned}$$

Show that there are coordinate neighborhoods $(U_i, \Phi_i)$ of $x_i$, $i = 0, \cdots n$ having the property that for all $i = 1, \ldots, n$ the coordinate neighborhoods $(U_{i-1}, \Phi_{i-1})$ and $(U_i, \Phi_i)$ form a local coordinate model for $f_i$.

## 2. Derivatives and the tangent space

**2.1. Derivatives in multivariable calculus.** If we have maps

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R},$$

the chain rule tells us that

$$(g \circ f)'(x) = g'(y) \cdot f'(x)$$

where

$$y = f(x).$$

For a function $f : \mathbb{R}^k \to \mathbb{R}$ we have the set of partial derivatives

$$\frac{\partial f}{\partial x_k}$$

and so for a function

$$f = (f_1, \ldots, f_k) : \mathbb{R}^k \to \mathbb{R}^n$$

we have the partials

$$\frac{\partial f_i}{\partial x_j}.$$

It turns out that the best way to organize these is as a matrix

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_k} \end{pmatrix} \tag{2.1}$$

As stated, this is a matrix of functions. If we want to specify the value of this matrix of functions at a point $x = (x_1, \ldots, x_k)$ we will write

$$J_f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_k} \end{pmatrix}_x .$$

Now, the point of doing this is that the chain rule for the composition has a really nice expression. If we have a composition

$$\mathbb{R}^k \xrightarrow{f} \mathbb{R}^\ell \xrightarrow{g} \mathbb{R}^m$$

then the derivative of $g \circ f$ is the matrix product of the derivatives of $f$ and the derivatives of $g$:

$$\begin{pmatrix} \frac{\partial (g\circ f)_1}{\partial x_1} & \cdots & \frac{\partial (g\circ f)_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial (g\circ f)_n}{\partial x_1} & \cdots & \frac{\partial (g\circ f)_n}{\partial x_k} \end{pmatrix}_x = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_\ell} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_\ell} \end{pmatrix}_y \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_\ell}{\partial x_1} & \cdots & \frac{\partial f_\ell}{\partial x_k} \end{pmatrix}_x .$$

with $y = f(x)$, or more compactly

$$\begin{aligned} J_{g\circ f}(x) &= J_g(y) J_f(x) \\ y &= f(x) . \end{aligned}$$

Using the chain rule one can compare the derivative of a function when it is computed in different coordinate systems, such as cylindrical or spherical coordinates.

**2.2. Derivatives in differential topology.** Differential topology provides a point of view in which the notion of derivative can be formulated without reference to a choice of coordinates.

The first thing to notice is that a matrix represents a linear transformation from one vector space to another vector space, once one has chosen a basis for both vector spaces. One of the fundamental ideas of the theory of smooth manifolds is that this vector space can be constructed for a point in a smooth manifold, and that it gets a basis when one chooses a coordinate system in a neighborhood of that point, but it doesn't otherwise come equipped with a basis.

It's worth thinking this through abstractly, as one meets an important principle. Let's imagine we wish to associate to each point $x \in X$ in a smooth manifold $X$ of dimension $k$ a $k$-dimensional vector space $T_xX$ which we will call the *tangent space to $X$ at $x$*. We also suppose that for each smooth map

$$f : X \to Y$$

between smooth manifolds $X$ and $Y$ (not necessarily of the same dimension), there is, for each $x \in X$ a map

$$df = df_x : T_xX \to T_yY$$
$$y = f(x)$$

which we call the *derivative of $f$ at $x$*. Let's also ask that this notion of derivative satisfies the *chain rule* in the sense that given

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

one has

$$d(g \circ f)_x = dg_y \circ dg_x$$

where $y = f(x)$. In terms of commutative diagrams, the chain rule is that for every commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

of maps between smooth manifolds, and every $x \in X$ the diagram of derivatives

$$\begin{tikzcd} T_xX \arrow[r, "df_x"] \arrow[dr, "dh_x"'] & T_yY \arrow[d, "dg_y"] \\ & T_zZ \end{tikzcd}$$

commutes. Here $y = f(x)$ and $z = g(y) = g(f(x))$.

The next principle is that the derivative of $f$ and $x$ depends only on the values of $x$ in any open neighborhood of $x$. We can express this by requiring that if $f : U \to X$ is the inclusion of an open subset $U$ of $X$ then for $x \in U$ the map

$$df : T_xU \to T_xX$$

is an isomorphism.

Finally, we want our coordinate free notion of derivative to coincide with the multivariable calculus notion of derivative. We therefore ask that for $x \in \mathbb{R}^k$ we have $T_x\mathbb{R}^k = \mathbb{R}^k$, and that under this identification derivative

$$df_x : T_x\mathbb{R}^k \to T_{f(x)}\mathbb{R}^\ell$$

of a smooth map $f : \mathbb{R}^k \to \mathbb{R}^\ell$ is the linear transformation given by the matrix (2.1).

So they are easy to reference, here is a list of our assumptions.

(1) Associated to each point $x \in X$ in a smooth manifold $X$ of dimension $k$ is a $k$-dimensional vector space $T_xX$ called the *tangent space to $X$ at $x$*.

(2) A smooth map $f : X \to Y$ induces, for each $x \in X$ a linear map

$$df : T_xX \to T_yY.$$

(3) (Chain Rule) If $f : X \to Y$ and $g : Y \to Z$ are smooth maps, and $g = g \circ f$ then

$$dh_x = dg_{f(x)} \circ df_x.$$

(4) (Locality of the tangent space) If $U \subset X$ is an open neighborhood of $x$, and $i : U \to X$ is the inclusion, then

$$di : T_xU \to T_xX$$

is an isomorphism.

(5) If $X = \mathbb{R}^n$ then $T_xX = \mathbb{R}^n$.

(6) If $f : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth map then, under the identification given by (5), the map

$$df_x : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^k$$

is the matrix $J_f(x)$.

**Remark 2.2.** It follows from locality and the chain rule that if

$$f : X \to Y$$

is a a local diffeomorphism at $x$ (meaning a diffeomorphism in a neighborhood of $x$) then

$$df : T_xX \to T_{f(x)}Y$$

is an isomorphism.

**Remark 2.3.** The locality of the tangent space means that if $i : X \subset \mathbb{R}^n$ is the inclusion of an open set, and $x \in X$ then

$$di_x : T_xX \to T_x\mathbb{R}^n = \mathbb{R}^n$$

is an isomorphism. In this way we can generalize (5) to the case in which $X$ is an open subset of $\mathbb{R}^n$. A similar remark applies to (6).

Under all of these assumptions, the definition of tangent space $T_xX$ is forced on us. Indeed suppose that $X \subset \mathbb{R}^n$ is a smooth manifold of dimension $k$, and $x \in X$ is a point. Choose a local parameterization centered at $x$ (Definition 1.8)

$$\mathbb{R}^k \supset W \xrightarrow{\Psi} X.$$

Since $\Psi$ is a local diffeomorphism, Remarks 2.2 and 2.3 imply that $d\Psi$ gives an isomorphism

$$\mathbb{R}^k \approx T_0W \xrightarrow{d\Psi} T_xX,$$

so we at least have some control on $T_xX$. We can do a little better. Write $i : X \to \mathbb{R}^n$ for the inclusion map, so that $i \circ \Psi$denotes the composition

$$W \xrightarrow{\Psi} X \subset \mathbb{R}^n.$$

By Lemma 2.7 below, the map

$$d(i \circ \Psi) : T_0W \to T_x\mathbb{R}^n \tag{2.4}$$

is a momomorphism and so its image has dimension $k$. Since $T_xX$ also has dimension $k$ the map

$$d_i : T_xX \to T_x\mathbb{R}^n$$

is an isomoprhism of $T_x$ with the image of 2.4. We conclude that if the properties above are all to hold, then we are forced to make the following definition

**Definition 2.5.** Suppose that $X \subset \mathbb{R}^n$ is a smooth manifold of dimension $k$, and $x \in X$ is a point. Let $(W, \Psi)$ be a local parameterization centered at $x$. The *tangent space to* $X$ *and* $x$ is the vector space

$$T_x X = \text{image of } d(i \circ \Psi),$$

in which $i : X \to \mathbb{R}^n$ is the inclusion map.

We need to show that this definition is independent of the choice of local parameterization.

**Proposition 2.6.** *Suppose that $X \subset \mathbb{R}^n$ is a smooth manifold of dimension $k$, and $x \in X$ is a point. If $(U, V\Phi)$ and $(U', V', \Phi')$ are two local coordinate systems centered at $x$ then the image of*

$$d\phi : T_0 U \to T_x \mathbb{R}^n$$

*is equal to the image of*

$$d\phi' : T_0 U' \to T_x \mathbb{R}^n$$

*where $\phi$ and $\phi'$ denote the compositions*

$$U \xrightarrow{\Phi^{-1}} V \subset X \subset \mathbb{R}^n \quad \textit{and}$$
$$U' \xrightarrow{(\Phi')^{-1}} V' \subset X \subset \mathbb{R}^n$$

*respectively.*

*Proof:* Replacing $U$ by $U \cap (\phi)^{-1}(V')$. and appealing to the locality of the tangent space, we may assume that $\phi(U) \subset \phi(U')$. This gives a commutative diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "\phi"'] & U' \arrow[d, "\phi'"] \\ & \mathbb{R}^n \end{tikzcd}$$

with $g(u) = \Phi'(\phi(u))$. The chain rule now tells us that the image of $d\phi(0)$ is contained in the image of $d\phi'(0)$. Reversing the roles of $\phi$ and $\phi'$ we conclude that the image of $d\phi(0)$ is equal to the image of $d\phi'(0)$. $\square$

This provides a definition of $T_x X$. It remains to define the derivative of a smooth map $f : X \to Y$, and check that it satisfies the chain rule, and reduces to the Jacobian matrix when $X$ and $Y$ are open subsets of Euclidean space.

We have used

**Lemma 2.7.** *Suppose that $X \subset \mathbb{R}^n$ is a manifold of dimension $k$ and $\phi : U \to X$ is a diffeomorphism of an open subset $U \subset \mathbb{R}^k$ with $X$. For each $u \in U$ the composition*

$$d\phi : T_u U \to T_{\phi(u)} X \xrightarrow{di_X} T_{\phi(u)} \mathbb{R}^n$$

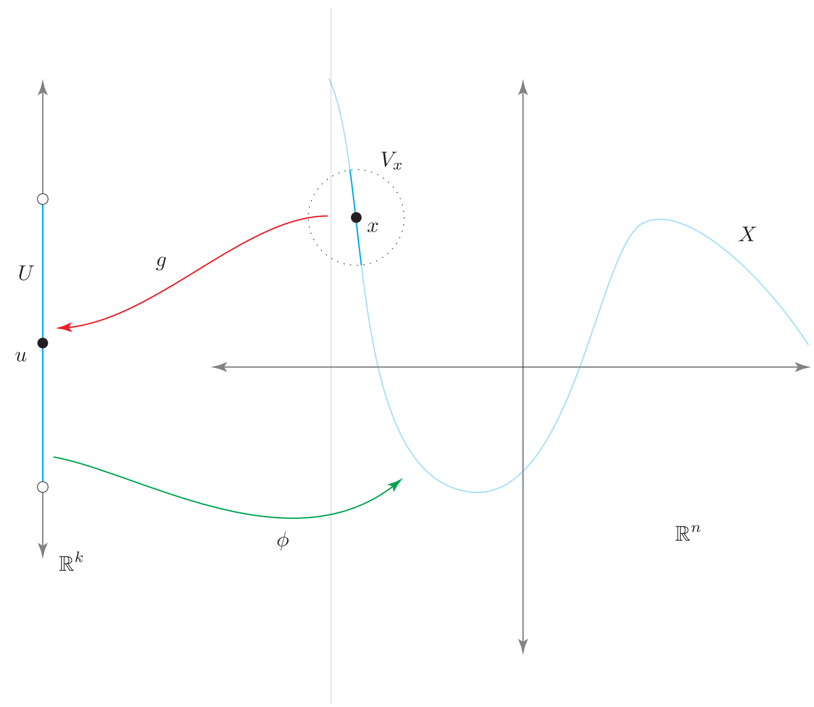
*is a monomorphism, and hence an isomorphism of $T_u U$ with a $k$-dimensional subspace of $T_{\phi(u)} \mathbb{R}^n$.*

*Proof:* The definition of diffeomorphism provides a smooth map $\phi^{-1} : X \to U$ satisfying

$$\phi^{-1}(\phi(u)) = u$$
$$\phi(\phi^{-1})(x) = x$$

for all $u \in U$ and $x \in X$. The definition of smooth function means that for each $x \in X$ there is an open neighborhood $V_x \subset \mathbb{R}^n$ of $x$, and a smooth function $g : V_x \to U$ having the property that for all $x' \in V_x \cap X$

$$g(x') = \phi^{-1}(x').$$




Now consider the diagram

$$\begin{tikzcd}
T_uU \arrow[r, "d\phi", "\approx"'] \arrow[d, equal] & T_xX \arrow[d] & \\
T_uU & T_xV_x \arrow[l, "dg"] \arrow[r, "\approx"] & T_x\mathbb{R}^n .
\end{tikzcd}$$

The bottom right map is an isomorphism by locality of the tangent space, and the composition in the square shows that the map $T_uU \to T_xV_x$ is a monomorphism. The claim follows. $\square$

## Exercises

**2.1.** Suppose that $f(y_1, \ldots, y_n)$ is a smooth function of $n$ variables, each of which is, in turn, a smooth function of $k$ other variables

$$y_i = y_i(x_1, \ldots, x_k).$$

In calculus you learn the formula

$$\frac{\partial f}{\partial x_i} = \sum \frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial x_i}.$$

How does this become the chain rule as expressed in §2.1 above?

**2.2.** Deduce the claim of Remark 2.2 from the axioms (2) to (6) above.

## 3. Colloquialisms in differential topology

By now we've accumulated quite a few concepts and techniques which make for some simple ways of stating some rather elaborate things. The point of this section is to make this clearer by expanding out the technical assertions some of these brief statements make.

First let's recall one of our basic definitions (Definition 1.1). Suppose that $M$ is a smooth manifold of dimension $n$ and $x \in M$ is a point.

**Definition 3.1.** A *coordinate chart* (or a *coordinate neighborhood*) is a pair $(U, \Phi)$ consisting of an open set $U \subset X$ and a smooth function

$$\Phi = (\Phi_1, \dots, \Phi_n) : U \to \mathbb{R}^n$$

which is a diffeomorphism with an open subset $W = \Phi(U) \subset \mathbb{R}^n$. The components $\Phi_i$ of $\Phi$ are called *local coordinate functions.*

Sometimes it is convenient to make some abbreviations and not overload the discussion with notation. If

$$M \supset U \xrightarrow{\Phi} \mathbb{R}^n$$

is a coordinate chart on $M$, then for every $a \in U$ the point $\Phi(a)$ has the form

$$\Phi(a) = (x_1, \dots, x_n)$$

and it's convenient to replace $\Phi_i(a)$ with $x_i = x_i(a)$. The functions $x_i$ are smooth functions on $U$ and give the "coordinates" of the points $a \in U$. Sometimes one drops $\Phi$ from the notation completely, and says, "let $x_1, \dots, x_n$ be a system of coordinates on $U \subset M$." If it doesn't cause trouble one might even drop $U$ from the discussion and say "Let $x_1, \dots, x_n$ be a system of local coordinates on $M$." This is short for the assertion that there is an open set $U \subset M$ and smooth functions $x_i : U \to \mathbb{R}$ with the property that the function

$$\Phi : U \to \mathbb{R}^n$$
$$a \mapsto (x_1(a), \dots, x_n(a))$$

is a diffeomorphism of $U$ with an open subset of $\mathbb{R}^n$. Other colloquialisms along these lines are "let $x_1, \dots, x_n$ be a local coordinate system near $a \in M$" meaning $a$ is in $U$, and "let $x_1, \dots, x_n$ be a local coordinate system centered at $a \in M$" meaning $a \in U$ and $x_i(a) = 0$ for all $i$.

Here is another example of a very compressed mathematical statement (which is kind of trivial to prove once you expand it out).

**Assertion 3.2.** Locally every diffeomorphism of smooth $n$-manifolds looks like the identity map of $\mathbb{R}^n$.

Expanded out it becomes

**Assertion 3.3.** $f : M \to N$ is a diffeomorphism of smooth manifolds of dimension $n$ and $x \in M$ is a point, then there are coordinate neighborhoods

$$\begin{aligned} \Phi_1 &: U_1 \to \mathbb{R}^n \qquad U_1 \subset M \\ \Phi_2 &: U_2 \to \mathbb{R}^n \qquad U_2 \subset N \end{aligned}$$

around $x$ and $f(x)$ respectively, having the property the the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, "\text{identity}"'] & U_1 \arrow[l, "\Phi_1"'] \arrow[r, "\subset"] \arrow[d, "f"] & M \arrow[d, "f"] \\ \mathbb{R}^n & U_2 \arrow[l, "\Phi_2"'] \arrow[r, "\subset"] & N \end{tikzcd} .$$

We've seen quite a few of these statements. The exercises below ask you to think about some of them.

**Remark 3.4.** The term *transverse pullback square* appears in the exercises below, but is not mentioned in our book. We say that two smooth maps

$$\begin{tikzcd} & X \arrow[d, "f"] \\ W \arrow[r, "g"'] & Y \end{tikzcd} \tag{3.5}$$

are *transverse* if for ever $x \in X$, and $w \in W$ with $f(x) = g(w)$ the map

$$d_f + d_g : T_xX \oplus T_wW \to T_yY \qquad (y = f(x) = g(w)) \tag{3.6}$$

is surjective. In this case the subset

$$Z = \{(w, x) \in W \times X \mid g(w) = f(x)\}$$

is a smooth submanifold of dimension

$$\dim X + \dim W - \dim Y$$

and writing $z = (w, x)$, the tangent space $T_zZ$ is the kernel of (3.6). The space $Z$ is called the pullback of (3.5), and the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & Y \end{tikzcd}$$

is said to be a *transverse pullback square*. The result can be deduced from what is stated in Guillemin and Pollack by considering

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "p=(f,g)"] \\ Y \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd}$$

and observing that $Z = p^{-1}\Delta(Y)$.

**Exercises**

**3.1.** Prove Assertion 3.3.

**3.2.** Write expanded versions of the following assertions

(a) Locally every immersion looks like the standard immersion $\mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^\ell$ which sends $x$ to $(x, 0)$.

(b) Locally every submersion looks like the standard submersion $\mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^k$ sending $(x, y)$ to $x$.

(c) Every transverse pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\pitchfork"] & Y \arrow[d, "\subset"] \\ X \arrow[r, "\subset"'] & M \end{tikzcd}$$

in which $X$ and $Y$ submanifolds of $M$, looks, near every $w \in W$, like

$$\begin{tikzcd} \mathbb{R}^\ell \arrow[r, "x \mapsto (x,0)"] \arrow[d, "x \mapsto (0,x)"'] & \mathbb{R}^\ell \times \mathbb{R}^m \arrow[d, "(x,y) \mapsto (0,x,y)"] \\ \mathbb{R}^k \times \mathbb{R}^\ell \arrow[r, "(a,b) \mapsto (a,b,0)"'] & \mathbb{R}^k \times \mathbb{R}^\ell \times \mathbb{R}^m \end{tikzcd} .$$

**3.3.** Suppose that $M$ is a smooth manifold of dimension 2, that $X$ and $Y$ are submanifolds of $M$ of dimension 1 intersecting transversally and that $x$ is a point of $X \cap Y$. Show that locally near $x$ in $M$, the submanifolds $X$ and $Y$ look like the coordinate axes of $\mathbb{R}^2$.

CHAPTER II

# Measure zero and Sard's Theorem.

Cantor's Lollipop (created by Jeremy Hahn and Midjourney AI)

The notion of a *set of measure zero* gives us a mathematically precise way of saying that something almost never happens (the occurrences form a set of measure zero) or that something almost always happens (the complement of the set of occurrences forms a set of measure zero). If you've learned about the *Lebesgue integral* or *measure theory* you'll be familiar with the ins and outs of this stuff. One of the really important things is that sets of measure zero are preserved under smooth maps so that the notion of measure zero extends to subsets of smooth manifolds (see §2).

Anyway, let's get on to the basic definitions.

## 1. Subsets of measure zero

**Definition 1.1.** A *(closed) rectangle* in $\mathbb{R}^n$ is a subset of the form

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

with $a_i < b_i$ for all $i$. The volume of such a rectangle is

$$\mathrm{Vol}(I) = (b_1 - a_1) \times \cdots \times (b_n - a_n).$$

When $n = 1$ I may refer to the volume as the *length* and write $\mathrm{len}(I)$.

**Definition 1.2.** A subset $S \subset \mathbb{R}^n$ has *measure zero* if for every $\epsilon > 0$ there is a sequence $I_1, I_2, \ldots$ of rectangles with

$$S \subset \bigcup_{i=1}^{\infty} I_i$$

and with

$$\sum \mathrm{Vol}(I_i) < \epsilon.$$

Put in plain English, a set in $\mathbb{R}^n$ has measure zero if it can be covered by countably many rectangles whose total volume can be made arbitrarily small. There are several equivalent versions of the definition in which the rectangles can be taken to be arbitrarily small open or closed cubes. Part of the point of this section is to spell this out (Propositions 1.11 and 1.14).

Here is one obvious consequence of the definition

**Lemma 1.3.** *If $S$ has measure zero and $S' \subset S$ is a subset then $S'$ has measure 0.* □

Here is a slightly less obvious consequence.

**Lemma 1.4.** *A countable union of sets of measure zero has measure zero.*

*Proof:* Suppose $S_1, S_2, \ldots$ have measure zero. Given $\epsilon$ cover $S_n$ with a countable number of rectangles $\{I_{n,i}\}_{i=1}^{\infty}$ with total volume $\epsilon/2^n$. Then $\cup S_n$ is covered by the countable collection of rectangles $\{I_{n,i}\}_{n,i=1}^{\infty}$ with total volume less than

$$\epsilon/2 + \epsilon/4 + \cdots = \epsilon.$$

□

**Example 1.5.** A point has measure 0 in $\mathbb{R}^n$.

**Example 1.6.** By Lemma 1.4 any countable set in $\mathbb{R}^n$ has measure 0. Since $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense this shows that sets of measure 0 can be dense.

**Example 1.7.** The *Cantor set* is defined to be the intersection of a sequence of sets $S_i$, each of which is a union of disjoint closed intervals. The sequence $S_i$ is defined inductively by starting with $S_0 = [0, 1]$, and, having constructed $S_{i-i}$, forming $S_i$ by removing the open middle third of each interval in $S_{i-1}$. If you write the numbers in $[0, 1]$ in base 3, the set $S_{\text{Cantor}}$ is the set of all numbers whose base three "decimal" expansion can be written without any 1s. Anyway, the set $S_i$ is a union of closed intervals. What is their total volume? By definition

$$\mathrm{Vol}\, S_i = 2/3\, \mathrm{Vol}\, S_{i-1}.$$

Since $\mathrm{Vol}\, S_0 = 1$ this means that $\mathrm{Vol}\, S_i = (2/3)^i$. Since this can be made arbitrarily small, the Cantor set has measure zero. From the description in terms of the base 3 expansion it is uncountable.

**Lemma 1.8.** *A subset $S \subset \mathbb{R}^n$ has measure zero if and only if every subset $K \subset S$ with compact closure has measure* $0$.

*Proof:* The "only if" assertion is immediate from Lemma 1.3. For the other direction, let $D^r \subset \mathbb{R}^n$ be the closed disk of radius $r$. By the Heine-Borel Theorem, $D^r$ is compact. For $k = 1, 2, \dots$ set $S_k = S \cap D^k$. The closure of $S_k$ is contained in $D^k$ and so is compact. If each $S_k$ has measure 0 then so does $S$ by Lemma 1.4. □

The property of having measure zero is not intrinsic to the set $S$. It depends on the Euclidean space in which it is sitting.

**Corollary 1.9.** *Any $S \subset \mathbb{R}^n$ lying in $\mathbb{R}^{n-1} \times \{0\}$ has measure zero.*

Because of this result it is sometimes helpful to say that a set $S$ has "$n$-measure zero." That being the case, it looks like I never do that in these notes.

*Proof:* By Lemma 1.3 it suffices to prove this in case $S$ is $\mathbb{R}^{n-1} \times \{0\}$ and then by Lemma 1.8 it suffice to prove it for a closed rectangle $I \subset \mathbb{R}^{n-1} \times \{0\}$. But this $I$ is contained in the rectangle $I \times [0, \epsilon]$ with volume $\epsilon\, \mathrm{Vol}\, I$ which can be made arbitrarily small. □

We now turn to some useful variations on the definition of "measure zero."

**Definition 1.10.** A *(closed) cube* in $\mathbb{R}^n$ is a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ for which

$$(b_1 - a_1) = \cdots = (b_n - a_n).$$

The common number $(b_i - a_i)$ is the *edge length* of the cube.

**Proposition 1.11.** *If $S \subset \mathbb{R}^n$ has measure zero, then for each $\epsilon > 0$ and each $\delta > 0$ there is a covering of $S$ by countably many cubes $C_i$ with*

$$\sum \mathrm{Vol}(C_i) < \epsilon$$

*and with each having edge length less than $\delta$.*

**Lemma 1.12.** *Let $\kappa > 1$ and $\delta > 0$ be real numbers. Any rectangle $I \subset \mathbb{R}^n$ can be covered by finitely many cubes $C_i$ with*

$$\sum \mathrm{Vol}(C_i) \le \kappa\, \mathrm{Vol}\, I.$$

*and with the edge length of $C_i$ less than $\delta$.*

*Proof:* We will, for a given $s > 1$, construct a covering with

$$\sum \mathrm{Vol}(C_i) \le s^n\, \mathrm{Vol}\, I.$$

Choosing $s > 1$ with $s^n < \kappa$ then gives the result we were supposed to prove. One more thing. We might as well assume that $a_i = 0$ for all $i$ since we could just translate the whole problem by subtracting $(a_1, \dots, a_n)$ from everything and then adding it back in at the end. This isn't important, but it simplifies the notation slightly, and therefore the processing asked of you.

Now to proceed. Pick $0 < r < \delta$. If it just so happened that $b_i/r$ was an integer, we could simply subdivide the interval $[0, b_i]$ into $b_i/r$ intervals of length $r$. Doing this would result in a subdivision of $I$ into cubes $C_j$ whose edges have size $r$ and whose total volume was exactly the volume of $I$. We're probably not that lucky, but we do have a little bit more room to work with. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ there is a point

$$(b'_1/r, \ldots, b'_n/r) \in \mathbb{Q}^n \cap [b_1/r, s\, b_1/r] \times \cdots \times [b_n/r, s\, b_n/r].$$

Write $b'_i/r = k_i/\ell_i$ with $k, \ell \in \mathbb{N}$. Replacing $r$ by $r/(\ell_1 \cdots \ell_n)$ we have a new $0 < r < \delta$ with the property that for all $i$, $b'_i/r$ is a positive integer. Now the rectangle

$$I' = [0, b'_1] \times \cdots \times [0, b'_n]$$

contains the rectangle $I'$, has volume

$$\operatorname{Vol}(I') = b'_1 \cdots b'_n = s^n b_1 \cdots b_n = s^n \operatorname{Vol}(I),$$

and, as above, *can* be subdivided into cubes $C_j$ of edge length $r$. The cubes $C_j$ form the desired covering. □

*Proof of Proposition 1.11:* Choose a covering of $S$ by a countable collection $\{I_i\}$ of cubes having total volume less than $\epsilon$. Choose $\kappa > 1$ and use Lemma 1.12 to cover each $I_i$ by a finite number of cubes $C_{ij}$ each having edge length less than $\delta$ and with total volume less than $\kappa \operatorname{Vol}(I_i)$. Then the collection of cubes $\{C_{ij}\}$ is countable and has total volume less than $\kappa\epsilon$ which can be made arbitrarily small. □

There is another variation on the definition that can be useful.

**Definition 1.13.** An *open rectangle* in $\mathbb{R}^n$ is a subset of the form

$$J = (a_1, b_1) \times \cdots \times (a_n, b_n).$$

The *volume* of an open rectangle $J$ is $\prod(b_i - a_i)$. An open rectangle is an *open cube* if for all $i, j$, one has $(b_i - a_i) = (b_j - a_j)$. The number $(b_i - a_i)$ is the *edge length* of the open cube.

**Proposition 1.14.** *If $S \subset \mathbb{R}^n$ has measure zero, then for each $\epsilon > 0$ and each $\delta > 0$ there is a covering of $S$ by countably many open cubes $C_i$ with*

$$\sum \operatorname{Vol}(C_i) < \epsilon$$

*and with the edge length of each $C_i$ less than $\delta$.*

*Proof:* For a rectangle $C = [a_1, b_1] \times [a_n, b_n]$ and $t > 0$ let

$$C^{(t)} = (a'_1, b'_1) \times \cdots \times (a'_n, b'_n)$$

be the open rectangle whose $i^{\text{th}}$ edge has length $t(b_i - a_i)$ and whose midpoint is the same as that of $[a_i, b_i]$:

$$\frac{a'_i + b'_i}{2} = \frac{a_i + b_i}{2}.$$

If you work it out you'll see that

$$b'_i = \frac{a_i + b_i}{2} + t\frac{b_i - a_i}{2}$$
$$a'_i = \frac{a_i + b_i}{2} - t\frac{b_i - a_i}{2}.$$

By construction, we have

$$\operatorname{Vol} C^{(t)} = t^n \operatorname{Vol} C.$$

If $t > 1$ then $C \subset C^{(t)}$. Now choose $t > 1$ and suppose that $\{C_i\}$ is a covering of $S$ by cubes of edge length less than $\delta/t$ and total volume less than $\epsilon/t^n$. Since $C_i$ is contained in $C_i^{(t)}$ the collection $\{C_i^{(t)}\}$ is a covering of $S$ by open cubes. The edge length of $C_i^{(t)}$ is less than $\delta$ and

$$\sum \operatorname{Vol} C_i(t) = t^n \sum \operatorname{Vol} C_i < t^n \epsilon/t^n = \epsilon.$$

□

**1.1. Properties of compact sets.** In this section I'm collecting a few things about compact sets we will need. I've mostly written the proofs in the context of topological (Hausdorff) spaces, where the arguments are a bit cleaner. In fact we will only be using these results for metric spaces. If you're not too familiar with topological spaces try and prove the stated theorems for metric spaces.

**Lemma 1.15.** *Suppose that $f : A \to Y$ is a continuous map from a compact space $A$ to a Hausdorff space $Y$, and $y \in Y$ is a point. If $U \subset A$ is a neighborhood of $f^{-1}(y)$ there is a neighborhood $B \subset Y$ of $y$ with the property that $f^{-1}(B) \subset U$.*

*Proof:* Since $A$ is compact the closed subset $A \setminus U$ is also compact. Its image $C = f(A \setminus U) \subset B$ is compact and so closed since $Y$ is Hausdorff. Since $U$ contains all of $f^{-1}(y)$, the point $y$ is not in $C$. We can take $B = Y \setminus C$. □

**Remark 1.16.** The way Lemma 1.15 comes up is as follows. We will have metric spaces $X$ and $Y$, a compact subset $A \subset X \times Y$ and a neighborhood $\tilde{U}$ of $A \times \{y\}$. In this situation Lemma 1.15 gives us a neighborhood $B$ of $y \in Y$ with the property that the set $A_B = \{(x, y) \in A \mid y \in B\}$ is contained in $U$. To apply Lemma 1.15. You take $f$ to be the composition $A \to X \times Y \to Y$ and $U$ to be $\tilde{U} \cap A$. Then you have to think a little to see that the $B$ given by Lemma 1.15 does the job.

**Proposition 1.17.** *Suppose that $X$ is a metric space, $K \subset X$ a compact subspace, and $\mathcal{U} = \{U_i\}$ is an open covering of $K$. There exists an $\epsilon > 0$ with the property that for every $x \in K$ the open ball*

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$$

*is contained in some $U_i$.*

The supremum of all the $\epsilon$ guaranteed by Proposition 1.17 is called the *Lebesgue number* of the covering $\mathcal{U}$. Proposition 1.17 is useful even when $\mathcal{U}$ consists of a single open set $U \subset X$.

*Proof:* First a couple of reductions. Since $K$ is compact the cover $\mathcal{U} = \{U_i\}$ has a finite subcover. Replacing $\mathcal{U}$ by this finite subcover we might as well suppose that

$$\mathcal{U} = \{U_1, \dots, U_\ell\}$$

is finite. Next note that if some $U_i$ contains all of $X$ we can just take $\epsilon$ to be any positive number. So we might as well suppose that no $U_i$ contains all of $X$ or, equivalently, that for all $i$ the set

$$C_i = X \setminus U_i$$

is non-empty. A straightforward application of the triangle inequality shows that the function

$$g_i(x) = d(x, C_i) = \inf\{d(x, y) \mid y \in C_i\}$$

is continuous. Since $C_i$ is closed, one has

$$d(x, C_i) = 0 \Leftrightarrow x \in C_i.$$

Now let $g(x)$ be the average of the $g_i$

$$g(x) = \frac{g_1(x) + \cdots + g_\ell(x)}{\ell}.$$

Note that for all $x$, $g(x) > 0$. Since $g(x)$ is continuous and $K$ is compact, $g$ has a minimum $\bar{\epsilon} > 0$ on $K$. Take $\epsilon$ to be any number $0 < \epsilon < \bar{\epsilon}$. □

Another important thing related to compact sets is the notion of a proper map.

**Definition 1.18.** A map $f : X \to Y$ is a *proper map* if the inverse image of a compact subset of $Y$ is a compact subset of $X$.

**Example 1.19.** If $K$ is compact and $X$ is any space then the projection map

$$K \times X \to X$$

is proper.

**Example 1.20.** If $f : X \to Y$ is proper and $Z \subset Y$ is any subspace then the restriction of $f$ to

$$f^{-1}(Z) \to Z$$

is proper. Indeed if $A \subset Z$ is compact then $A$ is compact in $Y$ and $f^{-1}(A)$ is compact since $f$ is proper.

**Example 1.21.** Recall that a compact subset of a metric space is closed. This implies that if $X$ and $Y$ are metric spaces and $X$ is compact then any continuous $f : X \to Y$ is proper. Indeed if $A \subset Y$ is compact it is closed and so $f^{-1}(A)$ is a closed subset of the compact space $X$ and hence compact. This isn't special to metric spaces. The same proof would apply to a continuous map from a compact space to a Hausdorff space, since compact subsets of Hausdorff spaces are closed.

**Proposition 1.22.** *Any proper map $f : X \to \mathbb{R}^n$ is* closed*: if $A \subset X$ is closed then $f(A)$ is closed in $Y$.*

Before turning to the proof let's recall a property of $\mathbb{R}^n$.

**Lemma 1.23.** *A subset $C \subset \mathbb{R}^n$ is closed if and only if its intersection with every compact subset is closed.*

*Proof:* Suppose that the intersection of $C$ with every compact set is closed and $y \notin C$. Let $D$ be the closed unit disk centered at $y$ and $D^0 \subset D$ its interior (the open unit disk). Since $D$ is compact $C \cap D$ is closed and so

$$D^0 \cap \big(\mathbb{R}^n \setminus C \cap D\big)$$

is an open set containing $y$ and not meeting $C$. This gives one implication. The other is trivial. □

*Proof of Proposition 1.22:* Suppose that $f : X \to \mathbb{R}^n$ is proper and $A \subset X$ is closed. By the remark above it suffices to show that if $D \subset \mathbb{R}^n$ is compact then

$f(A) \cap D$ is closed in $D$. Let $B = f^{-1}(D)$ and Since $D$ is compact and $f$ is proper, $B$ is compact. Since $A$ is closed, $A \cap B$ is also compact. This implies that

$$A \cap D = f(A \cap B)$$

is a compact subset of $\mathbb{R}^n$ hence closed. $\square$

**Remark 1.24.** The theorem is not special to the range being $\mathbb{R}^n$. The same proof would apply with $\mathbb{R}^n$ replaced by any Hausdorff space $Y$ having the property that a subset of $Y$ is closed if and only if its intersection with every compact subset is closed.

**1.2. Diameter, volumes and cubes.** The *diameter* of a subset $K \subset \mathbb{R}^n$ is the supremum of the distances between points of $K$

$$\operatorname{diam}(K) = \sup(|x - y| \mid x, y \in K).$$

If $K$ is contained in a compact set then the diameter of $K$ is finite.

**Example 1.25.** The diameter of a rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is

$$\sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}.$$

**Example 1.26.** The diameter, edge length and volume of a cube in $\mathbb{R}^n$ are related in a simple manner. If the edge length is $r$ then the diameter is $r\sqrt{n}$. For most purposes the specific value $\sqrt{n}$ is not needed and what one uses is

$$r = \mathbf{const} \cdot \operatorname{diam}$$
$$\operatorname{Vol} = \mathbf{const} \cdot r^n = \mathbf{const} \cdot \operatorname{diam}^n$$

where "**const**" refers to a generic quantity that is independent of $r$.

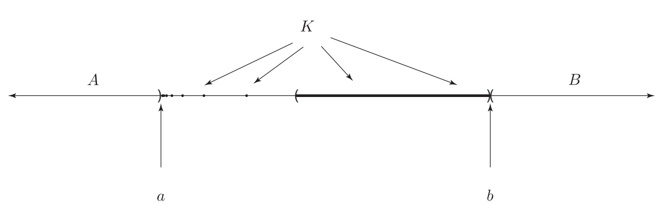
**Example 1.27.** Suppose $K \subset \mathbb{R}^1$ is a subset with finite diameter $d$. Then the sets

$$A = \{x \in \mathbb{R} \mid \forall k \in K, x < k\}$$
$$B = \{x \in \mathbb{R} \mid \forall k \in k, x > k\}.$$

are non-empty. Set

$$a = \sup A$$
$$b = \inf B.$$

Then $K$ is a subset of $[a, b]$ and in fact $\operatorname{diam} K = b - a$.




As a reminder of some basic analysis here is a proof. By definition, if $k \in K$ then $k > A$. It follows that $k \geq a = \sup A$. Similarly $k \leq b$. This shows that $K \subset [a, b]$ and so $\operatorname{diam} K \leq b - a$. Given $\epsilon > 0$ there is an $x \in K$ with $a \leq x < \epsilon/2$ and a $y \in K$ with $b - \epsilon/2 < y \leq b$. It follows that if $\epsilon$ is small enough ($\epsilon < (b-1)/2$) then $|y - x| = (y - x) < \epsilon + (b - a)$. Since $\epsilon$ is arbitrary this shows that $\operatorname{diam} K \geq b - a$.

**Proposition 1.28.** *If $K \subset \mathbb{R}^n$ is compact and has diameter $d$ there is a (closed) cube $C \subset \mathbb{R}^n$ of edge length $d$ with $K \subset C$.*

*Proof:* Let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the map given by projection to the $i^{\text{th}}$ coordinate

$$p_i(x_1, \dots, x_n) = x_i.$$

Note that for $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ we have

$$|x - y|^2 = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 \leq (x_i - y_i)^2 = (p_i(x) - p_i(y))^2.$$

This means that the set $p_i(K) \subset \mathbb{R}$ has diameter less than or equal to $d$ and, by Example 1.27 is therefore contained in an interval $I_i$ of length less than or equal to $d$, and hence in a possibly bigger interval of length actually equal to $d$. The region $K$ is contained in $I_1 \times \cdots \times I_n$. $\square$

**1.3. Sets that *don't* have measure zero.** Well, we've been doing pretty well showing that things have measure zero, but at the moment for all we know, *every set* has measure zero. Let's remedy that.

**Proposition 1.29.** *Suppose that $S \subset \mathbb{R}^n$ is a closed cube. If $I_i$ is a sequence of open cubes with $S \subset \bigcup I_i$ then*

$$\operatorname{Vol} S \leq \sum \operatorname{Vol} I_i.$$

This result means that no cube in $\mathbb{R}^n$ has measure zero, and no set containing a cube has measure zero. In particular

**Corollary 1.30.** *No subset of $\mathbb{R}^n$ which contains an open set has measure zero.* $\square$

*Proof of Proposition 1.29:* Subdivide $S$ into small cubes $S_k$ with $\operatorname{diam} S_k$ smaller than the Lebesgue number of the covering $\{I_i\}$ (see Proposition 1.17). For each $S_k$ choose a cube $I_{i_k}$ containing $S_k$. Then

$$\operatorname{Vol} S = \sum \operatorname{Vol} S_k = \sum_i \sum_{i_k = i} \operatorname{Vol} S_k \leq \sum_i \operatorname{Vol} I_i.$$

$\square$

This is good to know. It means that the complement of a set of measure zero in $\mathbb{R}^n$ is non-empty and, in fact uncountable.

**Remark 1.31.** Our book ([**1**, p. 203]) contains a clever proof of this due to Von Neumann. The above seems more straightforward, though to be fair, one should write out the proof of the assertion giving the last inequality: if $\{S_i\}$ is a finite collection of disjoint open rectangles in a rectangle $I$ then

$$\sum \operatorname{Vol} S_i < \operatorname{Vol} I.$$

I suspect that to really write this out carefully could get a bit elaborate.

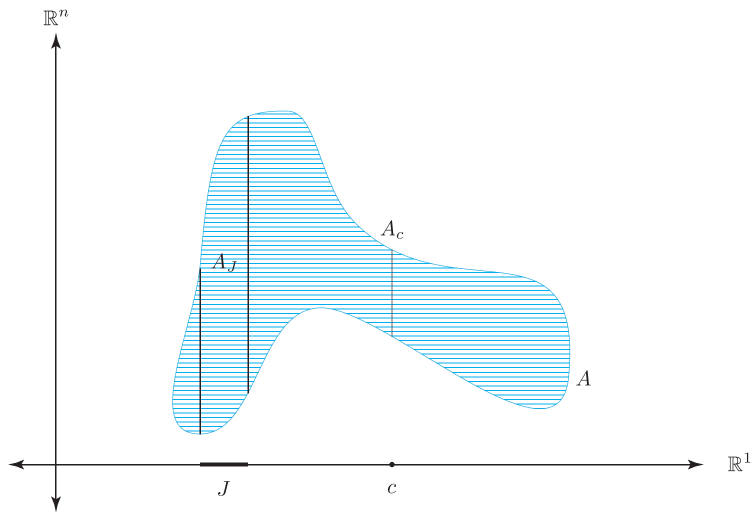



FIGURE 1.

**1.4. Fubini.** Lemma 1.4 tells us that every countable union of sets of measure zero has measure zero. Suitably interpreted Fubini's Theorem gives us an extension of this situation to an uncountable union. The twist is that the uncountable union sits in a bigger Euclidean space.

Suppose that $A$ is a subset of $\mathbb{R}^n \times \mathbb{R}^k$. For $J \subset \mathbb{R}^k$ set

$$\begin{aligned} A_J &= \{(x, y) \in A \mid y \in J\} \\ &= A \cap \big(\mathbb{R}^n \times J\big). \end{aligned}$$

The subset $A_J \subset \mathbb{R}^n \times \mathbb{R}^k$ is the inverse image of $J$ under the projection mapping $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$. When $J$ is the one element set $\{c\}$ we will write $A_c$ instead of $A_{\{c\}}$ (see Figure 1). We may think of $A_c$ as a subset of $\mathbb{R}^n$ by using the projection map $(x, c) \mapsto x$.

In calculus you learn that for nice enough $A$ you can compute the $(n + k)$ volume of $A$ by integrating the $n$-volume of $A_c$ over all $c \in \mathbb{R}^k$. This suggests that if each $A_c$ has measure zero in $\mathbb{R}^n$ then $A$ has measure zero in $\mathbb{R}^n \times \mathbb{R}^k$. This is in fact the case. The calculus argument can be made to work with a fancy enough notion of integration (the Lebesgue integral). Since I'm not assuming you know about that we will give a different proof, in the special case in which $A$ is closed.

The proof below of Fubini's Theorem is a slightly expanded version of the one in the Appendix 1 of Guillemin-Pollack [**1**]. That proof seems to be the one indicated by Milnor [**8**, p 17] who refers the reader to Sternberg [**9**, p. 51] who, in turn, attributes the argument to Furstenberg.

**Theorem 1.32** (Fubini's Theorem)**.** *If $A \subset \mathbb{R}^n \times \mathbb{R}^k$ is a* closed set *and for each $c \in \mathbb{R}^k$ the set $A_c$ has measure zero in $\mathbb{R}^n$ then $A$ has measure zero in $\mathbb{R}^n \times \mathbb{R}^k$.*

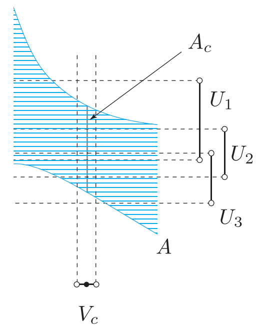



FIGURE 2.

I want to emphasize that this result is proved under the assumption that $A$ is closed. This means that when we apply Fubini's Theorem we have to make sure we are working with a *closed* subset. Some hypothesis is necessary as there are counterexamples for general $A$. In a course on measure theory you will learn that the actual assumption is that $A$ be *measurable.*

*Proof:* By Lemma 1.8 we may assume that $A$ is compact. By induction on $k$ we reduce to the case $k = 1$. Since $A$ is compact, its image in $\mathbb{R}^1$ is bounded and so contained in an open interval $(a, b)$ with $(b - a) < \infty$. We're actually going to work with the closed interval $[a, b]$ since it is compact, but we want the image of $A$ to be in its interior. Fix $\epsilon > 0$ and for each $c \in [a, b]$ choose a covering $\mathcal{U}^c = \{U_i^c\}$ of $A_c$ by open $n$-cubes in whose total volume is less than $\epsilon$. The union

$$\bigcup U_i^c \times (a, b)$$

is an open subset of $\mathbb{R}^n \times (a, b)$ containing $A_c$. Since $A$ is compact there is an interval $V_c$ around $c$ for which $A \cap \mathbb{R}^n \times V_c = A_{V_c}$ is contained in $U$ (Lemma 1.15 and Remark 1.16). The situation is illustrated in Figure 2. Write $\bar{U}_i$ for the closure of $U_i$. Note that if $J$ is any closed interval contained in $V_c$ then the collection $\{\bar{U}_i^c \times J\}$ is a covering of $A_J$ by (closed) rectangles with total volume

$$\sum \mathrm{Vol}(U_i \times J) < \epsilon \; \mathrm{len}(J) \tag{1.33}$$

The collection of intervals $V_c$ forms an open covering of $[a, b]$. Let $\delta > 0$ be the number provided by Proposition 1.17 (the Lebesgue number), so that any closed interval $[s, t] \subset [a, b]$ of length less than $\delta$ is contained in some $V_c$. By the above, if $J \subset (a, b)$ is any closed interval with $\mathrm{len}\, J < \delta$ then $A_j$ has a covering by open rectangles of total volume less than $\epsilon \cdot \mathrm{len}(J)$.

Now choose an integer $m \gg 0$ with the property that $(b - a)/m < \delta$ and subdivide $[a, b]$ into $m$ closed intervals $I_1, \ldots, I_m$of size $(b - a)/m < \delta$. Then each $A_{I_i}$ has a covering by closed rectangles with total volume less than $\epsilon \cdot (b - a)/m$. Taking the union of these gives $A = A_{[a,b]}$ a covering by closed rectangles of total volume

$$m\epsilon(b - a)/m = \epsilon\,(b - a).$$

Since this can be made arbitrarily small this shows that $A$ has measure 0. ☐

**Remark 1.34.** In the sources I was following, some fussing around was done to arrange that the intervals $I_i$ were open and overlapping. It seemed easier not to do that. In case I've overlooked some subtlety, one could choose a constant $\kappa$ with

$$1 < \kappa < \delta / \big((b-a)/m\big)$$

so that $\kappa\,(b-a)/m$ is still less than the Lebesgue number $\delta$. As in the proof of Proposition 1.14 intervals $I_i$ are contained in open intervals $J_i$ with the same center, but with length $\kappa \operatorname{len} I_i$. Now one can work entirely with open cubes and see that each $A_{J_i}$ is covered by open cubes with total volume less than $\epsilon\kappa(b-a)/m$ and that $A$ is then covered by cubes with total volume less than $\epsilon\kappa(b-a)$.

## 2. Measure zero and smooth maps

We now come to an important result.

**Proposition 2.1.** *Suppose that $S \subset \mathbb{R}^n$ has measure zero. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map then the image $f(S)$ has measure zero.*

Proposition 2.1 is what allows us to extend the notion of measure zero $S \subset \mathbb{R}^n$ to measure zero $S \subset M$ with $M$ an $n$-manifold: one defines $S \subset M$ to have measure zero if the intersection of $S$ with any coordinate chart has measure zero in $\mathbb{R}^n$. It's easy to see that it is enough to check this condition for a collection of coordinat charts forming a covering of $M$ (an *atlas* in the terminology of my notes on the definition of manifold). From Corollary 1.30 no subset subset of a manifold containing a non-empty open set can have measure zero. In particular a non-empty manifold $M$ is non-empty it does not have measure zero when regarded as a subset of itself.

Note that the dimension of the domain of $f$ is equal to the dimension of the range of $f$. That is important. For instance the projection map

$$\mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x$$

sends the $x$-axis (which has measure 0 in $\mathbb{R}^2$ to all of $\mathbb{R}^1$). It is also important that $f$ is smooth (or at least has a derivative). The result is false for continuous functions.

**Example 2.2.** The *Cantor function*, (or the *Devil's staircase*) is a continuous function

$$f : [0, 1] \to [0, 1]$$

with the property that $f$ is strictly increasing, and takes the complement of the Cantor set to the set of rational numbers in $[0, 1]$ whose denominator is a power of 2. The image of the Cantor set (which has measure zero) is the complemnt of a countable set in $[0, 1]$ and so does not have measure zero. I'll let you look up this function in the Wikipedia.

Since we need to make use of the fact that $f$ is smooth I'd like to take some time and develop a useful property of smooth functions. The first result is one we will use later in proving the Morse Lemma.

**Proposition 2.3.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ is a point. There are smooth real valued functions functions $g_1, \dots, g_n$ defined in a neighborhood $U$ of $a$ with the property that for $x = (x_1, \dots, x_n) \in U$ one has*

$$f(x) = f(a) + (x_1 - a_1)\, g_1(x) + \cdots + (x_n - a_n)\, g_n(x). \tag{2.4}$$

*Proof:* This is a standard calculus move. Write

$$\begin{aligned} f(x) = \int_0^1 \frac{d}{dt} f(a + t(x-a)\, dt &= \int_0^1 \sum \frac{\partial f(a + t(x-a))}{\partial x_i}(x_i - a_i)\, dt \\ &= \sum (x_i - a_i) \int_0^1 \frac{\partial f(a + t(x-a))}{\partial x_i}\, dt, \end{aligned}$$

so we may take

$$g_i(x) = \int_0^1 \frac{\partial f(a + t(x-a))}{\partial x_i}\, dt.$$

□

Now the proof of the above result proves something a little stronger. From the construction, the function $g_i$ is smooth as a function of *both* $x$ and $a$. Furthermore, an analogous result holds for functions to $\mathbb{R}^k$ with $k \geq 1$. To formulate this it's helpful to re-write the formula above. Let $g(x)$ be the row matrix $[g_1(x), \dots, g_n(x)]$. Then (2.4) becomes

$$f(x) = f(a) + g(x) \cdot (x - a).$$

This now makes it easy to state what happens when $f$ is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^k$. Applying the above to each component of $f$ produces a $k \times n$ matrix $g(x)$ of smooth functions, with the property that

$$f(x) = f(a) + g(x) \cdot (x - a). \tag{2.5}$$

Summarizing, we have

**Proposition 2.6.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^k$ is smooth, and $a \in \mathbb{R}^n$ is a point. There is a neighborhood $U$ of $a$ and $k \times n$ matrix $g(x, y)$ of smooth functions on $U \times U$ with the property that for $x, y \in U$ one has*

$$f(y) = f(x) + g(x, y) \cdot (y - x).$$

□

Next we need a fact from linear algebra. Remember the Schwarz inequalty? It says that for vectors $v, w \in \mathbb{R}^n$ there is an inequality

$$|v \cdot w| \leq |v||w|.$$

This generalizes to the case in which $v$ is a matrix $A$. For $A = (a_{ij})$ define

$$|A|^2 = \sum a_{ij}^2$$

and let $|A|$ be the positive square root of $|A|^2$. The Schwarz inequality for matrices is

$$|A \cdot w| \leq |A||w|. \tag{2.7}$$

One deduces this from the Schwarz inequality. If $A$ has rows $r_1, \dots, r_n$ then $A \cdot w$ has entries $r_i \cdot w$. By the usual Schwarz inequality

$$|r_i . w|^2 \leq |r_i|^2\, |w|^2$$

and so

$$|A \cdot w|^2 = \sum |r_i.w|^2 \leq \sum |r_i|^2 |w|^2 = |A|^2 |w|^2.$$

We now get to a key property of smooth functions

**Lemma 2.8.** *If that $f : U \to \mathbb{R}^k$ is a smooth function defined on an open set $V \subset \mathbb{R}^n$ then for each $a \in V$ there is a neighborhood $U$ of $a$ and a constant $c$ such that for all $x \neq y \in U$*

$$\frac{|f(x) - f(y)|}{|x - y|} < c.$$

*Proof:* By Proposition 2.6 there is a matrix $g(x, y)$ of smooth functions defined in a neighborhood $U'$ of $(a, a)$ with the property that in this neighborhood

$$f(y) - f(x) = g(x, y) \cdot (y - x).$$

Applying the Schwarz inequality gives

$$|f(y) - f(x)| \leq |g(x, y)|\,|y - x|$$

or, for $x \neq y$

$$\frac{|f(y) - f(x)|}{|y - x|} \leq |g(x, y)|.$$

Taking $U$ to be a neighborhood of $a$ with compact closure $\bar{U} \subset U'$ (for instance the interior of a small closed ball around $a$) we may arrange that $|g(x, y)|$ is bounded on $U \times U$. This completes the proof. □

*Proof of Proposition 2.1:* By Lemma 1.8 it is enough to establish the case in which $S$ is contained in a compact set $K$. Since $f$ is smooth, Proposition 2.6 give for every $a \in K$ a neighborhood $U \subset \mathbb{R}^n$ of $a$ and a constant $c$ with the property that for all $x, y \in U$

$$|f(x) - f(y)| \leq c\,|x - y|. \tag{2.9}$$

Since $K$ is compact it is covered by finitely many of such $U$ and so there is a constant $c$ and an open set $U \subset \mathbb{R}^n$ containing $K$ such for all $x, y \in U$, the inequality (2.9) holds. Choose a covering of $S$ by cubes $I_i$ with total volume less than $\epsilon$, and with edge length small enough so that $I_i$ is contained in $U$ (Proposition 1.17 with the covering $U$ consisting of the single set $U$). By Example 1.26 the diameter $I_i$ is smaller **const** $\mathrm{Vol}(I_i)^{1/n}$, and so the diameter of $f(I_i)$ is smaller than **const** $\mathrm{Vol}(I_i)^{1/n}$ (for some new constant). By Proposition 1.28 $f(I_i)$ is contained in a cube $J_i$ with edge length **const** $\mathrm{Vol}(I_i)^{1/n}$ (newer constant) and volume **const** $\mathrm{Vol}(I_i)$ (newer newer constant). This final constant is independent of $I_i$ (it just involved $n$ and the constant $c$ from (2.9)). Putting this together we see that there is a constant $c'$ such that for all $i$

$$\mathrm{Vol}(J_i) \leq c' \,\mathrm{Vol}(I_i).$$

By construction $f(K)$ is covered by the cubes $J_i$. We have

$$\sum \mathrm{Vol}(J_i) \leq c' \sum \mathrm{Vol}(I_i) = c'\,\epsilon.$$

Since this can be made arbitrarily small this shows that $f(K)$ has measure zero. □

Here is a simple application of this result

**Proposition 2.10.** *If $k < n$ the image of any smooth map $f : \mathbb{R}^k \to \mathbb{R}^n$ has measure zero.*

*Proof:* Let $g : \mathbb{R}^k \times \mathbb{R}^{n-k}$ be the composition of the projection to $\mathbb{R}^k$ followed by $f$. Since $n - k > 0$, the subset $\mathbb{R}^k \times \{0\}$ has measure zero in $\mathbb{R}^k \times \mathbb{R}^{n-k}$ so by Proposition 2.1 its image under $g$ in $\mathbb{R}^n$ has measure zero. But this is the same as its image under $f$. $\square$

The same result extends in the obvious way to prove

**Proposition 2.11.** *If $k < n$ the image of a smooth map $f : M \to N$ from a $k$-manifold to an $n$-manifold has measure zero. In particular there is a point $y \in N$ which is not in the image of $f$.* $\square$

## 3. Sard's Theorem

We now turn to the important theorem of Sard.

**Theorem 3.1** (Sard's Theorem)**.** *Suppose that $f : X \to Y$ is a smooth map between smooth manifolds, and let $C \subset X$ denote the set of critical points of $f$. The set of critical values $f(C)$ measure zero in $Y$.*

*Proof:* We begin with some reductions. Since $X$ and $Y$ are subsets of $\mathbb{R}^N$ for some $N$, every open cover has a countable subcover. We may therefore find a countable set of coordinate models for $f$ whose domains and ranges cover $X$ and $Y$ respectively. This means it suffices to prove the theorem when $X$ and $Y$ are opens subsets of $\mathbb{R}^k$ and $\mathbb{R}^\ell$ respectively. Under construction. $\square$

CHAPTER III

# Morse Functions

The theory of Morse functions has a very large number of applications in geometry. In a way, the basic ideas originate in understanding the second derivative test from calculus.

## 1. Definition and existence of Morse functions

Recall that a *critical point* of a function $f : M \to \mathbb{R}$ is a point $x \in M$ for which $df : T_xM \to \mathbb{R}$ is zero. A critical point is $x$ is *non-degnerate* if in some local coordinate system the determinant of the *Hessian matrix*

$$H_f = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

is non-zero at the point corresponding to $x$. One can check that this condition is independent of the choice of local coordinate system.

**Definition 1.1.** A *Morse function* $f : M \to \mathbb{R}$ is a smooth function with no non-degenerate critical points.

In a suitable sense, most functions on $f$ are Morse functions. The point of this section is to establish the following expression of this idea.

**Proposition 1.2.** *Suppose that $M \subset \mathbb{R}^n$ is a smooth manifold and $f : M \to \mathbb{R}$ is any smooth function. For $1 \leq i \leq n$ let $\phi_i : M \to \mathbb{R}$ be the restriction to $M$ of the $i^{th}$ coordinate function on $\mathbb{R}^n$. For almost all $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ the function*

$$f_a = f + a_1\phi_1 + \cdots + a_n\phi_n$$

*is a Morse function.*

I think the proof is easier to understand if one spends a bit of time messing around with the statement, and generally just setting the table. So let's do that. First of all, the assumptions in Proposition 1.2 are stronger than one really needs. Here is a slightly leaner statement.

**Proposition 1.3.** *Suppose that $M$ is a smooth manifold and $f : M \to \mathbb{R}$ is a smooth function. If*

$$\phi = (\phi_1, \dots, \phi_n) : M \looparrowright \mathbb{R}^n$$

*is an immersion, then for almost all $a = (a_1, \dots, a_n)$ the function*

$$f_a = f + a_1\phi_1 + \cdots + a_n\phi_n$$

*is a Morse function.*

Actually there's a hidden choice here. The vector $a$ is really an element of the *dual* of $\mathbb{R}^n$, and to write it this way we've chosen a basis in order to have an isomorphism of some vector space with its dual. This can be totally avoided. An even cleaner statement would be the following.

**Proposition 1.4.** *Suppose that $M$ is a smooth manifold $f : M \to \mathbb{R}$ is a smooth function and $V$ is a finite dimensional vector space. If*

$$\phi : M \to V$$

*is an immersion, then for almost all linear $\ell : V \to R$ the function*

$$f_\ell = f + \ell \circ \phi$$

*is a Morse function.*

For me, it is cleaner to state things without choosing a basis of $V$. For one thing this lets us get away from using the dot product. There's another place in the proof where you have to "project to a coordinate plane." This is also cleaner in a basis free presentation. But unless you are pretty fluent in "basis free" linear algebra, this will probably make things more confusing. So I'll go back to the version expressed in Proposition 1.3. If you're comfortable with abstract vector spaces you might think it all through in a coordinate free manner.

Now this "almost all" business is all well and good, but it means that we are supposed to be proving things about the complement of the set of things we want to study. So we'd better focus on that. Given $f : M \to \mathbb{R}$ let

$$\tilde{C} \subset M \times \mathbb{R}^n$$

be the set of all pairs $(x, a)$ for which $x$ is a *degenerate* critical point of $f_a$ (this means that $x$ is a critical point of $f_a$ and the Hessian of $f_a$ has zero determinant). The set of $a$ for which $f_a$ is a Morse function is the complement in $\mathbb{R}^n$ of the image $C$ of $\tilde{C}$ under the projection $\pi : M \times \mathbb{R}^n \to \mathbb{R}^n$. We are therefore trying to show that $C$ has measure zero. Now eventually we will be applying Fubini, so we need to say some things in advance about $\pi(C)$.

**Lemma 1.5.** *The set $\tilde{C} \subset M \times \mathbb{R}^n$ described above is closed.*

*Proof:* It suffice to show that the intersection of $\tilde{C}$ with $U \times \mathbb{R}^n$ is closed when $U$ is a coordinate neighborhood. In other words, it suffices to prove the Proposition when $M = \mathbb{R}^k$. Writing $\phi = (\phi_1, \ldots, \phi_n)$ and

$$f_a = f + a \cdot \phi$$

we are trying to show that the set of all $(x, y)$ for which

$$df + a \cdot d\phi = 0$$
$$\det(H_f + a \cdot H_\phi) = 0$$

is closed. But this is obvious since both functions above are continuous. $\square$

For a subset $U \subset M$ let $\tilde{C}_U = \tilde{C} \cap U \times \mathbb{R}^n$ and $C_U = \pi(\tilde{C}_U) \subset \mathbb{R}^n$. In other words $C_U$ is the set of $a \in \mathbb{R}^n$ for which $f_a$ has a degenerate critical point in $U$. Note that for two subset $U, V \subset M$

$$C_{U \cup V} = C_U \cup C_V.$$

**Lemma 1.6.** *If $K \subset M$ is compact, the set $C_K \subset \mathbb{R}^n$ is closed.*

*Proof:* The set $\tilde{C}_K \subset K \times \mathbb{R}^n$ is closed since it is the intersection of $K \times \mathbb{R}^n$ with the closed subset $\tilde{C}$ (Lemma 1.6). Since $K$ is compact, the map $K \times \mathbb{R}^n \to \mathbb{R}^n$ is proper (Example II.1.19), hence closed by Proposition II.1.22. $\square$

In case you've forgotten, we're still setting the table. Let's continue. We want to consider the sets $C_U$ but we want to consider more than one $f : M \to \mathbb{R}$ and more than one immersion $\phi : M \looparrowright \mathbb{R}^n$. So let's put these into the notation and write

$$C_U = C_U^{\phi,f},$$

and think about the dependence on these pieces of data. We've already seen that

$$C_{U \cup V}^{\phi,f} = C_U^{\phi,f} \cup C_V^{\phi,f}.$$

We now turn to the dependence on the pair $(\phi, f)$.

Note that if

$$\phi : M \looparrowright \mathbb{R}^n$$

is an immersion and $g : M \to \mathbb{R}^k$ is any smooth function, then

$$\phi_g = (\phi, g) : M \to \mathbb{R}^n \times \mathbb{R}^k$$

is also immersion. How does the set

$$C_U^{\phi_g,f} \subset \mathbb{R}^n \times \mathbb{R}^k$$

relate to the set

$$C_U^{\phi,f} \subset \mathbb{R}^n?$$

To work this out write

$$\begin{aligned} \phi &= (\phi_1, \ldots, \phi_n) \\ g &= (g_1, \ldots, g_k) \end{aligned}$$

so that for $(a, b) \in \mathbb{R}^n \times \mathbb{R}^k$, we have

$$f_{(a,b)} = f + \phi \cdot a + g \cdot b = (f + g \cdot b) + \phi \cdot a = (f + g \cdot b)_a,$$

so that $f_{(a,b)}$ has a degerate critical point at $x$ if and only if $(f + g \cdot b)_a$ does. So for fixed $b \in \mathbb{R}^k$ we have

$$C_U^{\phi_g,f} \cap \mathbb{R}^n \times \{b\} = C_U^{\phi,f+g\cdot b}, \tag{1.7}$$

or in other words, the slice $\left(C_U^{\phi_g,f}\right)_b$ of $C_U^{\phi_g,f}$ through $\mathbb{R}^n \times \{b\}$ is $C_U^{\phi,\tilde{f}}$ for some different function $\tilde{f} : M \to R$. This makes it look like we should use Fubini.

**Lemma 1.8.** *Suppose that $K \subset M$ is compact, $\phi : M \looparrowright \mathbb{R}^n$ is an immersion and $g : M \to \mathbb{R}^k$ is any smooth function. If the set $C_K^{\phi,f} \subset \mathbb{R}^n$ has measure zero for every $f : M \to \mathbb{R}$ then $C_K^{\phi_g,f} \subset \mathbb{R}^n \times \mathbb{R}^k$ has measure zero for every $f : M \to \mathbb{R}$.*

*Proof:* Using Lemma 1.6 and (1.7) this follows from Fubini's Theorem. $\square$

Since any manifold $M$ is a countable union of compact sets, $C_M$ is a countable union of $C_K$ with $K \subset M$ compact. Since countable unions of sets of measure zero have measure zero (Lemma II.1.4) we have

**Corollary 1.9.** *Suppose that $\phi : M \looparrowright \mathbb{R}^n$ is an immersion of a manifold $M$ into $\mathbb{R}^n$, and $g : M \to \mathbb{R}^k$ is any smooth function. If the set $C_M^{\phi,f} \subset \mathbb{R}^n$ has measure zero for every $f : M \to \mathbb{R}$ then $C_M^{\phi_g,f} \subset \mathbb{R}^n \times \mathbb{R}^k$ has measure zero for every $f : M \to \mathbb{R}$.* $\square$

So once we've proved our result for one immersion it a vector space, we know the result is true for any "bigger" immersion.

Now to prove Proposition 1.3. We begin with a basic case.

**Proposition 1.10.** *Proposition 1.3 holds when the immersion $\phi : M \to \mathbb{R}^n$ is the inclusion of an open set.*

*Proof:* In this case $f$ is a function of a point $x = (x_1, \ldots, x_n)$ and

$$f_a(x) = f(x) + a_1x_1 + \cdots + a_nx_n = f(x) + a \cdot x.$$

Define a function $g : U \to \mathbb{R}^n$ by

$$g(x) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

Note that $x$ is a critical point of $f_a$ if and only if $g(x) = -a$ and, since

$$dg = (\frac{\partial^2 f}{\partial x_i\, \partial x_j}) = H_f = H_{f_a},$$

the point $x$ is a degenerate critical point if and only $dg$ is not surjective. Combining these we see that $x$ is a degenerate critical point of $f_a$ if and only if $-a$ is singular value of $g$. The result now follows from Sard's theorem which tells that the set of singular values of $g$ has measure zero. $\square$

**Lemma 1.11.** *Suppose that $M$ is a smooth manifold of dimension $d$, $\phi : M \looparrowright \mathbb{R}^n$ is an immersion, and $f : M \to \mathbb{R}$ is a smooth function. Each point $x \in M$ has a neighborhood $U$ for which*

$$C_U^{\phi,f} \subset \mathbb{R}^n$$

*has measure zero.*

*Proof:* Since $df : T_xM \to \mathbb{R}^n$ is a monomorphism, there is a projection mapping $\mathbb{R}^n \to \mathbb{R}^k$ to a coordinate $k$-plane with the property that the composition

$$T_xM \xrightarrow{df} \mathbb{R}^n \to \mathbb{R}^d$$

is an isomorphism. In terms of matrices this is the assertion that an $n \times k$ matrix of rank $k$ must have a $k \times k$ submatrix with non-zero determinant. (This point would be somewhat cleaner if we had been working with dual spaces). At any rate this lets us write $\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^k$ and $\phi$ as $(g, \phi_1)$, where $\phi_1 : M \to \mathbb{R}^k$ is an immersion at $x$. By the inverse function theorem $\phi_1$ is a diffeomorphism of a neighborhood $U$ of $x$ with an open set in $\mathbb{R}^k$. The result now follows from Proposition 1.10 and Corollary 1.9. $\square$

*Proof of Proposition 1.3:* By Lemma 1.11 each point $x \in M$ has a neighborhood $U$ for which $C_U$ has measure zero. Being a subset of a Euclidean space, $M$ is covered by a countable set $\{U_i\}$ of these neighborhoods. Since

$$C_M = \bigcup C_{U_i}$$

the set $C_M$ has measure zero. $\square$

## 2. The Morse Lemma

The Morse Lemma can be thought of as the smooth manifold version of the second derivative test from Calculus.

It will make things a lot cleaner to use matrices and other abbreviations to lighten up the notational load. Recall our convention of writing a typical point in $x \in \mathbb{R}^n$ as

$$x = (x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \ddots \\ x_n. \end{bmatrix}$$

In this notation the *transpose* of $x$ is

$$x^T = \begin{bmatrix} x_1 & \cdots x_n \end{bmatrix}.$$

If $A = (a_{ij})$ is an $n \times n$ matrix then

$$\sum a_{ij}\, x_i x_j$$

can be rewritten as

$$\begin{bmatrix} x_1 & \cdots x_n \end{bmatrix} \cdot A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$
$$= x^T \cdot A \cdot x.$$

**Theorem 2.1** (Morse Lemma)**.** *Suppose that* $f : U \to \mathbb{R}$ *is a smooth function defined in a neighborhood of* $0$*, and that* $0$ *is a non-degenerate critical point of* $f$ *and* $f(0) = 0$*. There are coordinates* $(x_1, \ldots, x_n)$ *near* $0$ *in which*

$$f(x) = x^T \cdot A \cdot x$$

*with*

$$A = \frac{1}{2} H_f(0).$$

The case $n = 1$ is relatively easy, and left as an exercise (Exercise 2.4). The general case is a matrix version of the 1-variable case (with a sneaky identity thrown in). You might find the one variable argument a helpful model for the general case.

*Proof of Theorem 2.1:* By Proposition II.2.3 there are smooth function $g_1, \cdots, g_n$ with

$$f(x) = x_1 g_1(x) + \cdots x_n g_n(x).$$

Since

$$g_i(0) = \frac{\partial f(0)}{\partial x_i} = 0$$

we can apply the same result to $g_i$ and write

$$f(x) = \sum x_i x_j h_{ij}(x) \tag{2.2}$$

for smooth functions $h_{ij}(x)$. Replacing $h_{ij}$ by $\frac{1}{2}(h_{ij} + h_{ji})$ we may assume that $h_{ij} = h_{ji}$. Writing $H(x) = (h_{ij}(x))$ we can write (2.2) in the form

$$f(x) = x^T \cdot H(x) \cdot x.$$

Note that $H_f(0) = \frac{1}{2} H(0)$.

The goal is to find a matrix $C(x)$, depending smoothly on $x$, and defined in a possibly smaller neighborhood of zero, for which

$$H(x) = C(x)^T \cdot H(0) \cdot C(x). \tag{2.3}$$

Why? Let's assume we have done that, and define functions

$$y(x) = (y_1(x), \cdots, y_n(x))$$

by

$$y = C(x) \cdot x.$$

Using (2.3) we find

$$\begin{aligned} f(x) &= x^T \cdot H(x) \cdot x \\ &= x^T \cdot C(x)^T H(0) \cdot C(x) \cdot x \\ &= y^T \cdot H(0) \cdot y \\ &= \sum h_{ij}(0)\, y_i y_j \end{aligned}$$

so that in the $y$ coordinates $f$ has the desired form. For this to work we need to know that the $y_i$ form a coordinate system in a neighborhood of 0. By the implicit function theorem we need to compute

$$d\big(C(x) \cdot x\big)$$

at $x = 0$. By the product formula (Exercise 2.1)

$$\begin{aligned} d\big(C(x) \cdot x\big)_0 &= dC(x)_0 \cdot (x(0)) + C(0) \cdot dx_0 \\ &= C(0). \end{aligned}$$

By Equation (2.3) we have

$$\det H(0) = \det C(0)^2 \det H(0)$$

so $\det C(0)^2 = 1$.

It remains to construct $C(x)$. Consider $G(x) = H(0)^{-1} \cdot H(x)$. Note that

$$G(0) = 1.$$

By 2.2 implies that there is a smooth function

$$G(x)^{\frac{1}{2}}$$

defined in a possibly smaller neighborhood of 0, satisfying

$$(G(x)^{\frac{1}{2}})^2 = G(x).$$

The claim is we can take

$$C(x) = G(x)^{\frac{1}{2}}.$$

This argument is kind of sneaky, and I confess it would have taken me a long time to think of it. We want to show that

$$H(x) = C(x)^T \cdot H(0) \cdot C(x), \tag{2.4}$$

where

$$C(x) = \big(H(x) \cdot H(0)^{-1}\big)^{\frac{1}{2}}.$$

Using

$$C(x)^2 = H(0)^{-1} H(x)$$

we may eliminate $H(x)$ from (2.4), so see that we are trying to prove that

$$H(0)C(x)^2 = C(x)^T \cdot H(0) \cdot C(x).$$

Since $C(x)$ is invertible we can rewrite this as

$$H(0)C(x) = C(x)^T \cdot H(0)$$

or

$$H(0)C(x)H(0)^{-1} = C(x)^T. \tag{2.5}$$

When $x = 0$ both sides are $I$. Since $B \mapsto B^2$ is a diffeomorphism in a neighborhood of $I$ (Exercise 2.2), working in a smaller neighborhood of $x = 0$ it suffices to prove the identity after squaring both sides. Using Exercise 2.2 this becomes

$$H(0)C(x)^2H(0)^{-1} = (C(x)^2)^T. \tag{2.6}$$

From the definition of $C(x)$, the left hand side is

$$H(0)(H(0)^{-1}H(x))H(0)^{-1} = H(x)\,H(0)^{-1},$$

the right hand side is

$$\begin{aligned}\big(H(0)^{-1}H(x)\big)^T &= \big(H(0)^{-1}H(x)\big)^T \\ &= H(x)^T(H(0)^{-1})^T \\ &= H(x)\,H(0)^{-1}\end{aligned}$$

since $H(x)$ is symmetric for all $x$. □

**Remark 2.7.** By the classification of symmetric bilinear forms over $\mathbb{R}$ ( Corollary 3.25) we may make a linear change of coordinates in which $A$ is the diagonal matrix with entries $(1, \cdots, 1, -1, \cdots, -1)$, in wich case

$$f(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2.$$

## Exercises

**2.1.** This exercise concerns the product formula for derivatives.

(a) Suppose that $X$ is a smooth manifold, $x \in X$ is a point and $f, g : X \to R$ are smooth functions. Show that

$$d(fg)_x = df_x g(x) + f(x)dg_x : T_xX \to T_{f(x)g(x)}\mathbb{R}.$$

(b) Now suppose we have functions $f, g : X \to \mathbb{R}^n$, and we think of them as vector valued functions. Show that

$$d(f \cdot g)_x = f(x) \cdot dg_x + df_x \cdot g(x) : T_xX \to T_{f(x)\cdot g(x)}\mathbb{R}.$$

(c) Write $M_{pq}$ for the vector space of $p \times q$ matrices. Suppose

$$\begin{aligned} f &: X \to M_{pq} \\ g &: X \to M_{qr} \end{aligned}$$

are two smooth matrix valued functions. Show that

$$d(f \cdot g)_x = df_x \cdot g(x) + f(x) \cdot dg_x : T_x(x) \to T_{f(x)\cdot g(x)}M_{pr}.$$

**2.2.** Let $M_n$ be the vector space of $n \times n$ matrices, and consider the function

$$f : M_n \to M_n$$
$$f(B) = B^2.$$

(a) Identifying $T_I M_n$ with $M_n$, show that

$$df(I) : T_I M_n \to T_I M_n$$

is the linear map sending $B \in M_n$ to $2B$. Since this has non-zero determinant $f$ is a diffeomorphism in a neighborhood of 1.

(b) Conclude that there is a smooth diffeomorphism

$$B \mapsto B^{\frac{1}{2}}$$

defined in a neighborhood of $I$, satisfying $(B^{\frac{1}{2}})^2 = B$.

**2.3.** It will be useful for some (but not all) parts of this problem to remember that since $B \mapsto B^2$ is a *diffeomorphism* in a neighborhood of $I$, one can show that $S = T$ by showing that $S^2 = T^2$.

(a) Show that $B$ and $B^{\frac{1}{2}}$ commute.

(b) Suppose that $A \in M_n$ is an invertible matrix. Show that in the open neighborhood of $I$ where both $B^{\frac{1}{2}}$ and $(A \cdot B \cdot A^{-1})^{\frac{1}{2}}$ are defined, one has

$$A(B^{\frac{1}{2}})A^{-1} = (ABA^{-1})^{\frac{1}{2}}.$$

(c) Show that in a neighborhood of $I$ one has

$$(A^T)^{\frac{1}{2}} = (A^{\frac{1}{2}})^T.$$

**2.4.** The point of this exercise is to prove Morse (Lemma Theorem 2.1) in dimension 1.

(a) suppose that $f : \mathbb{R} \to \mathbb{R}$ is smooth real valued function with $f(0) = 0$ and that 0 is a non-degerate critical point. Write $a = \frac{1}{2}f''(0)$. Show that there is a smooth function $g(x)$ with $g'(0) = 1$ and with $f(x) = a\, g(x)^2$.

(b) Show that $y = g(x)$ can be used as a local coordinate system near 0 and that $f(y) = ay^2$.

(c) Show that one can find another coordinate system in which $f(y) = \pm y^2$.

## 3. Bilinear forms

The theory of (symmetric) bilinear forms is used in many places in this course, beginning with the notion of non-degenerate critical point. This section contains theory of symmetric bilinear, in enough generality to cover what we need in this course.

### 3.1. Bilinear forms and matrices.

**Definition 3.1.** Suppose that $F$ is a field and $V$ is a vector space over $F$. A *bilinear form on* $V$ is a map

$$B : V \times V \to F$$

having the property that for all $w, x, y \in V$ and all $\lambda \in F$, the following hold

$$\begin{aligned} B(w, x+y) &= B(w,x) + B(w,y) \\ B(w, \lambda x) &= \lambda B(w,x) \\ B(w+x, y) &= B(w,y) + B(x,y) \\ B(\lambda w, x) &= \lambda B(w,x). \end{aligned}$$

**Definition 3.2.** Suppose that $F$ is a field. A *bilinear form over* $F$ is a pair

$$\mathbf{B} = (V, B)$$

consisting of a finite dimensional vector space $V$ over $F$ and a bilinear form $B$ on $V$.

**Example 3.3.** Suppose that $V = F^n$ and for

$$\begin{aligned} x &= (x_1, \dots, x_n) \\ y &= (y_1, \dots, y_n) \end{aligned}$$

we define

$$B(x,y) = x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

Then $B$ is a bilinear form (this is the usual dot product).

**Example 3.4.** More generally suppose we are given $\lambda_1, \dots, \lambda_n \in F$ and we define

$$B(x,y) = \lambda_1 x_1 y_1 + \cdots + \lambda_n x_n y_n.$$

Then $B$ is a bilinear form.

**Example 3.5.** Even more generally suppose we are given an $n \times n$ matrix $M = (\lambda_{ij})$. Then

$$B(x,y) = \sum \lambda_{ij} x_i y_j \tag{3.6}$$

is a bilinear form.

**Example 3.7.** The special case in which $M$ is a $1 \times 1$ matrrix has a special notation. For $\lambda \in F$, the symbol $\langle \lambda \rangle$ denotes the form $(F, B)$ with $B(v,w) = \lambda\, v\, w$.

**Example 3.8.** The most degenerate case of Example 3.5 is when the matrix $M$ is zero. We will call this the *zero* form. There is one zero bilinear form of every dimension over $F$. We denote it by $\mathbf{0}^n = (F^n, 0)$.

We should probably check that (3.6) is actually bilinear. There's a relatively painless way to do this. Write $x$ and $y$ as column vectors. Then (3.6) can be re-written as

$$B(x,y) = x^T \cdot M \cdot y.$$

From this is easy to check the conditions.

Now we will show that every bilinear form arises in this way from a matrix. Suppose that $V$ is finite dimensional of dimension $n$, and that $\alpha = \{v_1, \dots, v_n\}$ is an ordered basis of $V$. Define a matrix $B_\alpha$ by

$$(B_\alpha)_{ij} = B(v_i, v_j).$$

By writing $x \in V$ as $x = x_1 v_1 + \cdots x_n v_n$ we can represent each $x$ uniquely as a column vector

$$x_\alpha = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{3.9}$$

Then

$$B(x, y) = x_\alpha^T M y_\alpha. \tag{3.10}$$

We now have two structures in linear algebra that correspond to square matrices. For a linear transformation $T : V \to V$ a choice of ordered basis $\alpha = \{v_1, \ldots, v_n\}$ of $V$ allows us to identify $V$ with $F^n$ and express $T$ as a matrix $T_\alpha^\alpha$. For another choice of ordered basis $\beta = \{w_1, \ldots, w_n\}$ we get another matrix $T_\beta^\beta$. The matrices $T_\alpha^\alpha$ and $T_\beta^\beta$ are related by

$$T_\beta^\beta = S^{-1} T_\alpha^\alpha S$$

where $S$ is the matrix constructed by solving

$$w_j = \sum s_{ij} v_i.$$

For a bilinear form $B : V \times V \to F$ a choice of ordered basis allows us to represent $B$ by a matrix $B_\alpha$ (your book denotes this as $\psi_\alpha(B)$). The bilinear form may then be computed as (3.10). If $\alpha$ and $\beta$ are two ordered bases, related by a matrix $S$ as above, then

$$B_\beta = S^T B_\alpha S.$$

Two matrices $M_1$ and $M_2$ are *similar* if there is an invertible matrix $S$ for which $M_2 = S^{-1} M_1 S$. Thus linear transformations $T : V \to V$ correspond to matrices up to similarity. Two matrices $M_1$ and $M_2$ are *congruent* if there is an invertible matrix $S$ for which $M_2 = S^T M_1 S$. Bilinear forms correspond to matrices up to congruence.

### 3.2. Symmetric bilinear forms.

**Definition 3.11.** A bilinear form $B$ is *symmetric* if $B(x, y) = B(y, x)$ for all $x, y \in V$

**Definition 3.12.** A bilinear form $B$ is *non-degenerate* for every $0 \neq v \in V$ there exists $w \in V$ such that $B(v, w) \neq 0$.

3.2.1. *Diagonalizability.* We will now restrict our attention to *symmetric* bilinear forms. When the characteristic of $F$ is not equal to 2 it turns out that every symmetric bilinear form can be put into the form of Example 3.5. This fact is called the *diagonalizability of symmetric bilinear forms forms* (over fields of characteristic not equal to 2).

**Theorem 3.13.** *Suppose that $B$ is a symmetric bilinear form on a finite dimensional vector space $V$ over a field $F$. If the characteristic of $F$ is not equal to 2, then there is an ordered basis $\alpha = \{v_1, \ldots, v_n\}$ of $V$ having the property that*

$$B(v_i, v_j) = 0 \quad \textit{if } i \neq j. \tag{3.14}$$

*(equivalently the matrix $B_\alpha$ is diagonal).*

Let's go through the proof. For the rest of this section we will assume that the characteristic of $F$ is not 2.

**Lemma 3.15.** *Suppose that $B$ is a symmetric bilinear form on $V$. If $B$ is non-zero then there is a vector $v \in V$ for which*

$$B(v, v) \neq 0.$$

*Proof:* If $B$ is non-zero there are vectors $x, y \in V$ for which $B(x, y) \neq 0$. Using bilinearity and the fact that $B$ is symmetric, we expand

$$B(x + y, x + y) = B(x, x) + B(y, y) + 2B(x, y).$$

Since $2 \neq 0 \in F$ the rightmost term is non-zero. It follows that at least one of the other terms must be non-zero. We can choose $v$ to be $x$, $y$, or $x+y$ accordingly. $\square$

It will also be useful to have some more terminology. Suppose that $(V, B)$ is a symmetric bilinear form over $F$ and $W \subset V$ is a subspace. We can then define a symmetric bilinear form $(W, B_W)$ by setting

$$B_W(x, y) = B(x, y).$$

**Definition 3.16.** The *restriction of $B$ to $W$* is the bilinear form $(W, B_W)$ constructed above.

**Definition 3.17.** Suppose that $U \subset V$ is a *subset* of $V$. The *$B$-orthogonal complement* (or just *orthogonal complement*) of $U$ is the vector subspace

$$U^{\perp} = \{v \in V \mid B(u, v) = 0 \quad \forall u \in U\}.$$

*Proof of Theorem 3.13:* We prove the result by induction on the dimension of $V$. The result is obvious when $\dim V = 1$. Suppose then that $\dim V = n$ and we have proved the result for all symmetric bilinear forms on vector spaces of dimension less than $n$. If $B(x, y)$ is zero for all $x$ and $y$ then any basis of $V$ will satisfy. We may therefore suppose that $B$ is non-zero. By Lemma 3.15 there is a $v \in V$ for which $B(v, v) \neq 0$. Let

$$W = \{v\}^{\perp} = \{x \in V \mid B(v, x) = 0\},$$

and let $B_W$ be the restriction of $B$ to $W$, so that

$$B_W(x, y) = B(x, y).$$

Note that $W$ is the kernel of the linear transformation

$$B(v, -) : V \to F.$$

Since $B(v, v) \neq 0$, this transformation is surjective, and so its kernel $W$ has dimension $(n - 1)$. We may therefore employ the induction hypothesis and produce a basis $\{v_1, \ldots, v_{n-1}\}$ of $W$ satisfying

$$B_W(v_i, v_j) = B(v_i, v_j) = 0 \qquad \text{if } i \neq j.$$

Now one easily checks that $\{v_1, \ldots, v_{n-1}, v\}$ is a basis of $W$ satisfying (3.14). $\square$

3.2.2. *Isometries, orthogonal sums.* The diagonalizability of symmetric bilinear forms can be stated another way.

Suppose that $\mathbf{B}_1 = (V_1, B_1)$ and $\mathbf{B}_2 = (V_1, B_2)$ are two symmetric bilinear forms over $F$.

**Definition 3.18.** An isometry of $\mathbf{B}_1$ and $\mathbf{B}_2$ is an invertible linear transformation

$$T : V_1 \to V_2$$

having the property that for all $x, y \in V_1$

$$B_2(Tx, Ty) = B_1(x, y).$$

Two symmetric bilinear forms are *isometric* if there is an isometry between them.

If $(V, B)$ is a bilinear form we say that vectors $x, y \in V$ are *orthogonal* (or $B$-*orthogonal*) if $B(x, y) = 0$. Suppose that $\mathbf{B}_1 = (V_1, B_1)$ and $\mathbf{B}_2 = (V_2, B_2)$ are two symmetric bilinear forms over $F$.

**Definition 3.19.** The *orthogonal sum* of $\mathbf{B}_1$ and $\mathbf{B}_2$, is the bilinear form

$$\mathbf{B}_1 \oplus \mathbf{B}_2 = (V_1 \oplus V_2, B_1 \oplus B_2)$$

in which $B_1 \oplus B_2$ is given by

$$(B_1 \oplus B_2)((x_1, y_1), (x_2, y_2)) = B_V(x_1, y_1) + B_W(x_2, y_2).$$

The $n$-fold iterated orthogonal sum of $\mathbf{B}$ with itself will be denoted $\mathbf{B}^{\oplus n}$ (or just $\mathbf{B}^n$ if no confusion is likely).

Using the notion of orthogonal sums, and the notation of Example 3.7 we may rewrite Theorem 3.13 as

**Theorem 3.20.** *Suppose that $\mathbf{B}$ symmetric bilinear form of dimension $n$ over a field $F$. If the characteristic of $F$ is not equal to 2, then $\mathbf{B}$ is isometric to*

$$\langle \lambda_1 \rangle \oplus \cdots \oplus \langle \lambda_n \rangle$$

□

**3.3. Symmetric Bilinear forms over the reals.** As you may be guessing, we think of a symmetric bilinear form as a "generalized dot product" and borrow the terminology from there. Some things are a bit different in the usual case. For instance the expression $B(v, v)$ is analogous to the "squared norm" of $v$, but there is no reason it has to be a square, or even be non-zero. And there is no generalized sense in which one might talk about it being "positive." Over the real numbers, however, the notion of *positive* does make sense.

**Definition 3.21.** A symmetric bilinear form $(V, B)$ over $\mathbb{R}$ is *positive* if $B(v, v) \geq 0$ for all $v$. It is *positive definite* if it is positive and if

$$B(v, v) = 0 \implies v = 0.$$

Similarly, a symmetric bilinear form over $\mathbb{R}$ is *negative* if $B(v, v) \leq 0$ for all $v$, and *negative definite* if in addition $B(v, v) = 0 \implies v = 0$ .

Consider the symmetric bilinear form $\langle\lambda\rangle$ over $\mathbb{R}$. If $\lambda \neq 0$ we may use the basis vector $v = 1/\sqrt{|\lambda|} \in \mathbb{R}$, and we find

$$\begin{aligned} B_{\langle\lambda\rangle}(v,v) &= \lambda \tfrac{1}{\sqrt{|\lambda|}} \tfrac{1}{\sqrt{|\lambda|}} \\ &= \frac{\lambda}{|\lambda|} = \pm 1. \end{aligned}$$

Write

$$\begin{aligned} \mathbb{R}^{p,0} &= \langle 1\rangle^p \\ \mathbb{R}^{0,q} &= \langle -1\rangle^q \\ R^{p,q} &= \mathbb{R}^{p,0} \oplus \mathbb{R}^{0,1} \\ R^{p,q,r} &= \mathbb{R}^{p,q} \oplus \mathbf{0}^r. \end{aligned}$$

The form $\mathbb{R}^{p,0}$ is $\mathbb{R}^p$ with the usual notion of dot product, and is positive definite. The form $\mathbb{R}^{0,q}$ is $\mathbb{R}^q$ with bilinear form given by the negative of the usual dot product. It is negative definite.

From Theorem Theorem 3.20 we now know that any symmetric bilinear form over $\mathbb{R}$ is isometric to

$$\mathbb{R}^{p,q,r}.$$

A question arises. Are $p$, $q$ and $r$ uniquely determined?

**Proposition 3.22.** *If $V \subset \mathbb{R}^{p,q,r}$ is a positive definite subspace then $\dim V \leq q$. If $V \subset \mathbb{R}^{p,q,r}$ is a negative definite subspace then $\dim V \leq q$.*

*Proof:* We will prove the negative definite assertion. The other follows by replacing $B$ with $-B$. Write $\mathbb{R}^{p,q} \oplus \mathbf{0}^r$ as the orthogonal sum

$$\mathbb{R}^{p,q} \oplus \mathbf{0}^r = V_+ \oplus V_1 \oplus V_0$$

with $V_+ = \mathbb{R}^{p,0}$, $V_- = \mathbb{R}^{0,q}$ and $V_0 = \mathbf{0}^r$. Let

$$\begin{aligned} p : V_+ \oplus V_- \oplus V_0 &\to V_- \\ (v,w) &\mapsto w \end{aligned}$$

for the projection. Note that if $w \in \ker p$ then $B(w,w) \geq 0$. If $w \in V \cap \ker p$ then $B(w,w) \leq 0$ by assumption. This means that $B(w,w) = 0$. Since $V$ was assumed to be negative definitie this implies that $w = 0$. Since $V \cap \ker p = \{0\}$ the restriction of $p$ to $V$ is a monomorphism and so

$$\dim V = \dim p(V) \leq q.$$

□

**Corollary 3.23.** *If $\mathbb{R}^{p,q} \oplus \mathbf{0}^r$ is isometric to $\mathbb{R}^{p',q'} \oplus \mathbf{0}'$ then $p = p'$, $q = q'$, and $r = r'$.*

*Proof:* Since the dimension must be the same, we know that $p+q+r = p'+q'+r'$. It to show that $q = q'$, since replacing the forms by their negatives will show that $q = q'$. Suppose that $T : R^{p',q',r'} \to \mathbb{R}^{p,q,r}$ is an isometry. The subspace

$$V = T(\mathbb{R}^{0,q'}) \subset \mathbb{R}^{p,q,r}$$

is then negative definite, so by Proposition 3.22 we have

$$q' = \dim V \dim T(V) \leq q.$$

The same argument applied to $T^{-1}$ shows that $q \leq q'$. $\square$

**Remark 3.24.** The itegers $p$, $q$, and $r$ in $\mathbb{R}^{p,q,r}$ can therefore be exctracted directily from the bilinear form $(V, B)$, without a choice of diagonalization. The number $p$ is the dimension of a maximal positive definite subspace, $q$ is the dimension of a maximal negative definite subspace, and $r$ is the dimension of the *radical* of $(B, V)$, defined to be the subspace of vectors $v \in V$ for which $B(v, w) = 0$ for all $w$.

**Corollary 3.25.** *If* $\mathbf{B} = (B, V)$ *is a symmetric bilinear form over* $\mathbb{R}$ *then* $\mathbf{B}$ *is isometric to* $\mathbb{R}^{p,q,r}$ *for unique non-negative integers* $p$, $q$, $r$. $\square$

**Remark 3.26.** If $\mathbf{B} = (B, V)$ is non-degerate, then the radical is zero, and so $\mathbf{B}$ is isometric to $\mathbb{R}^{p,q}$. The number $p - q$ is the *signature*, and the number $(p + q)$ is the *rank*. A non-degenerate symmetric bilinear form over $\mathbb{R}$ is there fore determined up to isometry by its rank and signature.

**3.4. Characteristic** 2. Now let's turn to the situation when $F = \mathbb{F}_2$ is the field with two elements. We begin with a few examples.

**Example 3.27.** Let $\mathbf{R} = (F, R)$ be the bilinear form of rank 1 with $R(x, y) = xy$. Note that the $n$-fold orthogonal sum $\mathbf{R}^n$ is just $F^n$ with the usual dot product. The matrix associated to $R \oplus \cdots \oplus R$ in the standard basis is the identity matrix.

**Example 3.28.** Let $\mathbf{H} = (F^2, H)$ be the rank 2 symmetric bilinear form associated to the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$H\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = x_1 y_2 + x_2 y_1. \tag{3.29}$$

The matrix associated to $H \oplus \cdots \oplus H$ in the standard basis is the block diagonal matrix

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

This example gives me an occasion to talk about some further notation. When you're talking about points in $F^n$ it's customary to denote them by

$$(x_1, \ldots, x_n). \tag{3.30}$$

But when you're doing linear algebra it is more useful to denote them by column vectors

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{3.31}$$

This latter expression is inconvenient from the point of view of typography. There are a couple of ways around this. One is to refer to (3.31) as $[x_1, \dots, x_n]^T$. Another is to use (3.30) and (3.31) interchangeably. That's what I will do here. Thus I could write (3.29) as

$$H((x_1, y_1), (x_2, y_2)) = x_1 y_2 + x_2 y_1.$$

This does, however allow for a conflict of notation. The symbol (3.30) can refer to a $1 \times n$ matrix, or to the column vector (3.31), which is an $n \times 1$ matrix. In these notes we're only going to have square matrices, so this will never come up.

We need one more piece of terminology.

**Definition 3.32.** A symmetric bilinear form $(V, B)$ over $\mathbb{F}_2$ is *even* if for all $x \in V$, $B(x, x) = 0$. A symmetric bilinear form is *odd* if there exists $x \in V$ with $B(x, x) = 1$.

Now let's pursue the analogue of Theorem 3.13. We start with a symmetric bilinear form $(V, B)$ over $\mathbb{Z}/2$. Suppose that there is a vector $v \in V$ with $B(v, v) = 1$. As in the proof of Theorem 3.13 we let $W$ be the kernel of

$$B(v, -\,) : V \to F$$

and $B_W$ the restriction of $B$ to $W$, so that $B_W(x, y) = B(x, y)$. Then by **??** the map

$$\begin{aligned} F \oplus W &\to V \\ (t, x) &\mapsto tv + x \end{aligned}$$

is an isometry of $\mathbf{R} \oplus (W, B_W)$ with $(V, B)$. Iterating this we see that $(V, B)$ is isometric to $\mathbf{R}^m \oplus (V', B')$ where $(V', B')$ is even.

Armed with this we now restrict our attention to even symmetric bilinear forms. Suppose that $(V, B)$ is even. Either $B$ is zero, or we can find two vectors $x, y \in V$ with $B(x, y) = 1$. Let $W \subset V$ be the subspace spanned by $x$ and $y$. Note that $x$ cannot equal $y$ since $B$ is even. This means that $W$ has dimension 2, and that the map $F^2 \to W$ sending $e_1$ to $x$ and $e_2$ to $y$ is an isometry of $\mathbf{H}$ with $(W, B_W)$, where, as above, $B_W$ is the restriction of $B$. Let $V' = W^\perp \subset V$ be the orthogonal complement of $W$. Note that $V'$ is the kernel of the map

$$T : V \to F^2$$

sending $v$ to $(B(x, v), B(y, v))$. This map is surjective since $T(x) = (0, 1)$ and $T(y) = (1, 0)$. This implies that $\dim V' = \dim V - 2$.

It now follows from **??** that $(V, B)$ is isometric to $\mathbf{H} \oplus (V', B')$. By Exercise IX.5.2 we know that $(V', B')$ is even. If it is non-zero we may continue. If it is zero we just stop. Putting all of this together gives

**Theorem 3.33.** *Any symmetric bilinear form $\mathbf{B} = (V, B)$ over the field $F = \mathbb{F}_2$ is isometric to one of*

$$\begin{gathered} \mathbf{H}^m \oplus \mathbf{0}^n \\ \mathbf{R} \oplus \mathbf{H}^m \oplus \mathbf{0}^n \\ \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{H}^m \oplus \mathbf{0}^n. \end{gathered}$$

In the above, $(V, B)$ is non-degenerate if and only if $n = 0$. Only the first case is even. This implies

**Corollary 3.34.** *Any even non-degenerate symmetric bilinear form* $(V, B)$ *over* $\mathbb{F}_2$ *is isometric to* $\mathbf{H}^m$ *for some* $m$. *In particular the dimension of* $V$ *must be even.*

**Exercises**

**3.1.** Show that $B$ is symmetric if and only if for every ordered basis $\alpha$, the matrix $B_\alpha$ is a symmetric matrix.

**3.2.** Show that a bilinear form on a finite dimensional vector space $V$ is non-degenerate if and only if for every ordered basis $(v_1 \cdots, v_n)$ the matrix $B_\alpha$ is an invertible matrix.

**3.3.** A bilinear form $B$ on $V$ gives a map $\tilde{B} : V \to V^*$ defined by

$$\tilde{B}(x)(y) = B(x, y).$$

Show that the radical of $B$ is the kernel of $\tilde{B}$ and that $B$ is non-degenerate if and only if $\tilde{B}$ is a monomorphism.

**3.4.** Prove the last two assertions in the above proof: that $\{v_1, \ldots, v_{n-1}, v\}$ is indeed a basis of $W$ and that it satisfies (3.14).

**3.5.** With the notation of the proof of Theorem 3.13, show that if $B$ is non-degenerate then so is $B_W$.

**3.6.** Suppose that the characteristic of $F$ is not 2, and that $B$ is a symmetric bilinear form on a vector space $V$ of dimension $n$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$ satisfying (3.14), and let $\lambda_i = B(v_i, v_i)$. Is the set $\{\lambda_1, \ldots, \lambda_n\}$ determined by $B$? In other words does another basis satisfying (3.14) lead to the same set of $\lambda_i$'s?

**3.7.** Suppose that $(V, B)$ is a non-degenerate symmetric bilinear form, $W \subset V$ is a subspace and write $B_W$ for the restriction of $B$ to $W$. If $(V, B)$ is non-degenerate, must $(W, B_W)$ also be non-degenerate?

**3.8.** Suppose that $(V, B)$ is a symmetric bilinear form and $W \subset B$ is a subspace. Define a map

$$B^T : V \to W^*$$

by $B^T(v)(w) = B(v, w)$. Show that if $B$ is non-degenerate then $B^T$ is surjective and that $\dim W^\perp + \dim W = \dim V$. Find an example of a non-degenerate $B$ for which the map that the map $W \oplus W^\perp \to V$ is not an isomorphism.

**3.9.** Show that $H$ is non-degenerate. Show that for every $x \in F^2$, $H(x, x) = 0$, so that Lemma 3.15 definitely does not hold in characteristic 2.

**3.10.** Consider the symmetric bilinear form $\mathbf{R}^3$. Show that the matrix of this symmetric bilinear form in the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 1)\}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

so that $\mathbf{R}^3$ is isometric to $\mathbf{R} \oplus \mathbf{H}$, and that more generally $\mathbf{R}^{2n+1}$ is isometric to $\mathbf{R} \oplus \mathbf{H}^n$ and $\mathbf{R}^{2n+2}$ is isometric to $\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{H}^n$.

**3.11.** Suppose that $(B, V)$ is a symmetric bilinear form, $W \subset V$ is a subspace, and that the restriction of $B_W$ of $B$ to $W$ is non-degenerate. Let $B_{W^\perp}$ be the restriction of $B$ to $W^\perp$. Show that the map

$$(W, B_W) \oplus (W^\perp, B_{W^\perp}) \to (V, B)$$

sending $(w, v)$ to $w + v$ is an isometry.

**3.12.** Suppose that $\mathbf{B}_1$ and $\mathbf{B}_2$ are symmetric bilinear forms over $F = \mathbb{F}_2$. Show that $\mathbf{B}_1 \oplus \mathbf{B}_2$ is even if an only if both of $\mathbf{B}_1$ and $\mathbf{B}_2$ are even.

**3.13.** Suppose that $(V, B)$ is a non-degenerate symmetric bilinear form over $F$. A subspace $W \subset V$ is $B$-*isotropic* (or just *isotropic* if $B$ is understood) if $B(x, y) = 0$ for all $x, y \in W$. Show that if $W$ is an isotropic subspace of $V$ then $\dim W \leq \frac{1}{2} \dim V$. Show that equality can only hold if $(V, B)$ is even and that when $(V, B)$ is even there is an isotropic subspace of dimension $\frac{1}{2} \dim V$. (Hint: What is the relationship between $W$ and $W^\perp$ when $W$ is isotropic?)

**3.14.** Show that the three cases listed in Theorem 3.33 are distinct in the sense that there is no isometry between any distinct two of the three cases.

CHAPTER IV

# The Definition of a manifold

Differential topology is the study of a class of spaces on which one can do calculus.

## 1. Topological manifolds

**Definition 1.1.** A *topological manifold of dimension $d$* is a Hausdorff topological space $X$, having a countable basis for the topology, with the property that each point $x \in X$ has a neighborhood $U$ which is homeomorphic to an open subset of $\mathbb{R}^d$.

The last condition is described by saying that $X$ is locally homeomorphic to $\mathbb{R}^d$.

Circles squares and triangles are all examples of topological manifolds of dimension 1. Spheres and cubes are topological manifolds of dimension 2. The Hausdorff condition is included in order to exclude spaces like the "line with the origin doubled," obtained from the disjoint union of two copies of $\mathbb{R}^1$ by identifying the complement of the origin in one copy with the complement of the origin in the other.

Topological manifolds are easy to define, but it takes quite a bit of sophisticated topology to deal with them.

## 2. Smooth manifolds

There are several ways of defining a smooth manifolds. We start with the definition used by Milnor [**8, 6**] and adopted by Guillemin and Pollack [**1**].

Suppose that $X \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$.

**Definition 2.1.** A function $f : X \to \mathbb{R}$ is *smooth* if for each $x \in X$ there is an open set $U \subset \mathbb{R}^n$ containing $x$, and a smooth function $g : U \to \mathbb{R}$ having the property that for every $y \in U \cap X$, $g(y) = f(x)$.

**Definition 2.2.** Suppose that $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are subsets. A function $f : X \to Y$ is *smooth* if for each $i = 1, \dots, m$

$$x_i \circ f : X \to \mathbb{R}$$

is smooth, where $x_i : \mathbb{R}^m \to \mathbb{R}$ is the $i^{\text{th}}$ coordinate function.

**Definition 2.3.** Suppose $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are subsets. A map $f : X \to Y$ is a *diffeomorphism* if it is a homeomorphism and both $f$ and $f^{-1}$ are smooth.

**Definition 2.4.** A *smooth manifold of dimension* $d$ is a subset $X \subset \mathbb{R}^n$ (for some $n$) having the property that for each $x \in X$ there is a neighborhood $x \in\subset U \subset X$ and a diffeomorphism of $U$ with an open subset of $\mathbb{R}^d$.

We can say this more succinctly by saying that a smooth manifold of dimension $d$ is a subset $X \subset \mathbb{R}^n$ which is *locally diffeomorphic* to $\mathbb{R}^d$.

For the purposes of comparing this with other definitions we need to set up a bit more terminology.

Let $X$ be a topological manifold of dimension $d$.

**Definition 2.5.** A *smooth structure* on $X$ is a homeomorphism of $f : X \to Y \subset \mathbb{R}^n$ in which $Y$ is a smooth manifold. Two smooth structures

$$f_1 : X \to Y_1 \subset \mathbb{R}^{n_1}$$
$$f_2 : X \to Y_2 \subset \mathbb{R}^{n_2}$$

are *equivalent* if there is a diffeomorphism $g : Y_1 \to Y_2$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, "f_2"'] & Y_1 \arrow[d, "g"] \\ & Y_2 \end{tikzcd}$$

One of the fundamental questions in differential topology is to enumerate the set of smooth structures on a given topological manifold. For spheres the answer is pretty well understood in all dimensions except 4, 125, and 126. In dimension 2 every topological manifold has a unique smooth structure. Milnor [7] constructed 14 different smooth structures on the 7-dimensional sphere $S^7$. Later Kervaire and Milnor [4] devised a method for determining the number of smooth structures on $S^n$ in terms of number theoretic functions and the homotopy groups of spheres. Their work left open dimension $2^n - 2$ and $2^n - 3$. All but two of which were settled in [2]. At this point the answer is considered known for all dimensions except 4, 125 and 126.

## 3. Charts and atlases

One objection to the definition above is that a manifold is required to be a subset of $\mathbb{R}^n$ for some $n$. This makes it a little difficult to prove that certain things are manifolds. There are two other, more abstract approaches.

Suppose that $X$ is a topological manifold of dimension $d$.

**Definition 3.1.** A *chart* on $X$ is a pair $(U, f)$ consisting of an open subset $U \subset X$ and a continuous function $f : U \to \mathbb{R}^d$ which is a homeomorphism of $U$ with an open subset of $\mathbb{R}^d$. An *atlas* on $X$ is a covering of $X$ by charts, i.e., a collection $\{(U_\alpha, f_\alpha)\}$ of charts, having the property that $\{U_\alpha\}$ is a covering of $X$. A *smooth atlas* is an atlas having the property that for each $\alpha, \beta$ the map

$$f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^d$$

is a smooth map on $f_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^d$.

Note that interchanging the roles of $\alpha$ and $\beta$ in the definition of smooth atlas implies that $f_\alpha^{-1} \circ f_\beta$ is also smooth.

The whole point of giving a smooth structure is to be able to say which functions are smooth. Suppose that $\big(X, \{U_\alpha, f_\alpha\}\big)$ is a topological manifold equipped with a smooth atlas, and $V \subset X$ is an open subset.

**Definition 3.2.** A function $g : V \to \mathbb{R}$ is *smooth* if for each $\alpha$, the function

$$g \circ f_\alpha^{-1} : f_\alpha(V \cap U_\alpha) \to \mathbb{R}$$

is smooth.

**Definition 3.3.** Two smooth atlases $\{U_\alpha, f_\alpha\}$ and $\{V_\beta, g_\beta\}$ on $X$ are *equivalent* if they determine the same set of smooth functions on each open $V \subset X$.

**Definition 3.4.** Suppose that $X$ is a topological manifold. A *smooth structure* on $X$ is an equivalence class of smooth atlases. A *smooth manifold* is a pair $\big(X, \{(U_\alpha, f_\alpha)\}\big)\big)$ consisting of a topological manifold $X$ and a smooth structure on $X$.

Rather than work with equivalence classes of atlases one can work with a single, maximal atlas.

**Definition 3.5.** Suppose that $A = \{(U_\alpha, f_\alpha)\}\big)\big)$ is a smooth atlas of dimension $d$ on $X$. A *smooth chart* on $X$ is chart $g : V \to \mathbb{R}^d$ on $X$ with the property that for each $\alpha$, the map

$$g : f_\alpha(U_\alpha \cap V) \to g(U_\alpha \cap V)$$

and its inverse are both smooth.

A chart which is part of the original atlas is smooth. But there could also be smooth charts which are not part of the original atlas. For example, $\mathbb{R}^k$ has an atlas given by the single chart which is the identity map. The function

$$x \mapsto \frac{x}{1 - |x|^2} : B_1 \to \mathbb{R}^k$$

is a smooth chart on the open disk $B_1 = \{x \in \mathbb{R}^k \mid |x| < 1\}$. This chart was not part of the original atlas.

A smooth atlas on $X$ determines a maximal atlas consisting of all charts which are smooth with respect to the given smooth structure. One can easily check that two atlases determine the same smooth structure if they have the same maximal atlas. So a smooth structure could be defined to be a choice of maximal smooth atlas.

## 4. Sheaves

If the whole point of specifying a smooth structure is to be able to say which functions are smooth, then there should be a way of defining smooth manifolds entirely in terms of smooth functions. There is. It is done by specifying what is called a *sheaf* of functions. You can read about that notion elsewhere if you're interested. We don't require the general notion of sheaf in order to give the "sheafy" definition of smooth manifold.

For a topological space $Z$ write $C(Z)$ for the set of continuous functions $f : Z \to \mathbb{R}$.

Suppose that $X$ is a topological manifold. A *smooth structure on $X$* consists of a subset $C^\infty(U) \subset C(U)$ for every open $U \subset X$ having the properties

i) If $\{U_i\}$ is an open covering of $U$ then $f \in C(U)$ is in $C^\infty(U)$ if and only if the restriction of $f$ to each $U_i$ is in $C^\infty(U_i)$.

ii) Each point $x \in X$ there is a neighborhood $U$ of $x$ and a homeomorphism $h : U \to \mathbb{R}^d$ having the property that $f \in C^\infty(U)$ if and only if $f \circ h^{-1} : \mathbb{R}^d \to R$ is smooth.

**Definition 4.1.** A *smooth manifold* is a topological manifold equipped with a smooth structure.

## 5. Examples of manifolds

### 5.1. Real projective space.

**Definition 5.1.** The *real projective space* $\mathbf{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$.

We need to make $\mathbf{RP}^n$ into a topological space and then give this space the structure of a smooth manifold. To do so define a map

$$\mathbb{R}^{n+1} \setminus \{0\} \to \mathbf{RP}^n \tag{5.2}$$

by sending a non-zero vector $v$ to the set of all scalar multiples of $v$. Every line through the origin consists of the scalar multiples of any one of its non-zero vectors. So the map (5.2) is surjective and two vectors $v$ and $w$ determine the same line if and only if there is a non-zero $\lambda \in \mathbb{R}$ for wchich $w = \lambda v$. Thus we may identify $\mathbf{RP}^n$ as the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$. It is customary to write the equivalence class of $v = (x_0, \ldots, x_n)$ as

$$[x_0, \ldots, x_n],$$

so that the $\mathbf{RP}^n$ may be identified with the set of symbols

$$[x_0, \ldots, x_n] \neq [0, \ldots, 0]$$

with

$$[\lambda x_0, \ldots, \lambda x_n] = [x_0, \ldots, x_0]$$

for $0 \neq \lambda \in \mathbb{R}$.

This gives $\mathbf{RP}^n$ the structure of a topological space. Why is is a manifold? For $i = 0, \ldots, n$ let $U_i$ be the set of all $[x_0, \ldots, x_n] \in \mathbf{RP}^n$ with $x_i \neq 0$. Let

$$\phi_i : U_i \to \mathbb{R}^n$$

be the map given by

$$\phi_i[x_0, \ldots, x_n] = (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots x_n/x_i).$$

The map $\phi_i$ is a homeomorphism with inverse

$$\psi_i(t_1, \ldots, t_n) = [t_1, \ldots, 1, \ldots, t_n]$$

with the 1 inserted in the $i^{\text{th}}$ position. I claim that the collect $\{U_i, \phi_i\}$ forms a smooth atlas for $\mathbf{RP}^n$. This is easy and I'll leave it to you to check.

Is there a way to embed $\mathbf{RP}^n$ in $\mathbb{R}^N$ for some $N$? You had a homework problem some time ago which used bump functions and a finite smooth atlas to construct and embedding. There is another way. Note that every line through the origin in $\mathbb{R}^{n+1}$ meets the unit sphere in a pair of antipodal points. So $\mathbf{RP}^n$ can be identified with the quotient

$$S^n / v \sim -v.$$

Incidentally this shows that $\mathbf{RP}^n$ is compact. The map

$$S^n \to \mathbb{R}^{(n+1)+n(n+1)/2} = \mathbb{R}^{(n+1)(n+2)/2}$$

sending $(x_0, \dots, x_n)$ to the sequence

$$(x_0^1, \dots, x_n^2, x_0x_1, x_0x_2, \dots, x_{n-1}x_n)$$

of all degree two mononials in the variables $x_0, \dots, x_n$ turns out to be an embedding. To check this you have to check that the list of degree two monomials above separates points and tangent vectors.

### 5.2. Grassmannians.

**Definition 5.3.** For non negative integers $k \leq n$ the *Grassmannian* (of $k$-planes in $n$-space), $G_k(\mathbb{R}^n)$, is the set of $k$-dimensional linear subspaces of $\mathbb{R}^n$.

We are going to equip Grassmannian $G_k(\mathbb{R}^n)$ with the structure of a smooth manifold. Note that $G_1(\mathbb{R}^{n+1}) = \mathbf{RP}^n$, so we have already done one case of this. The general case is very similar.

A point of $G_k(\mathbb{R}^n)$ is a $k$-dimensional vector subspace $V \subset \mathbb{R}^n$. By choosing a basis of $V$ we can represent $V$ as the column space of a $n \times k$ matrix $\mathbf{v}$. Conversely, the column space of an $n \times k$ matrix $\mathbf{v}$ defines a $k$-plane as long as the columns are linearly independent.

Motivated by this, let $\tilde{V}_k(\mathbb{R}^n)$ be the set of ordered $k$-tuples $[v_1, \dots, v_k]$ of linearly independent vectors in $\mathbb{R}^n$. Writing the $v_i$ as a column vector, we may think of such a $k$-tuple as an $n \times k$ matrix of rank $k$. So $\tilde{V}_k(\mathbb{R}^n)$ is the space of $n \times k$ matrices of rank $k$. It is a subspace of the Euclidean space $M_{n,k} \approx \mathbb{R}^{nk}$ of all $n \times k$ matrices. This makes it into a topological space. From the above discussion, the map

$$p : \tilde{V}_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n),$$

sending a matrix to its column space is surjective. We make $G_k(\mathbb{R}^n)$ into a topological space by giving it the quotient space topology. In other words, we define a subset $U \subset G_k(\mathbb{R}^n)$ to be *open* if $P^{-1}(U) \subset \tilde{V}_k(\mathbb{R}^n)$ is open.

Let $\mathrm{Gl}_k(\mathbb{R})$ be the set of $k \times k$ invertible matrices of real numbers. Multiplication of matrices makes $\mathrm{Gl}_k(\mathbb{R})$ into a group. The group $\mathrm{Gl}_k(\mathbb{R})$ acts on $\tilde{V}_k(\mathbb{R}^n)$ on the right: if $\mathbf{v} \in \tilde{V}_k(\mathbb{R}^n)$ and $T \in \mathrm{Gl}_k(\mathbb{R})$ then the matrix product $\mathbf{v} \cdot T$ is also in $\tilde{V}_k(\mathbb{R}^n)$. By Exercise 5.4 if $p(\mathbf{v}) = p(\mathbf{w})$ then there exists a $T \in \mathrm{Gl}_k(\mathbb{R})$ with $\mathbf{v} \cdot T = \mathbf{w}$. This means that $p$ gives us an identification of $G_k(\mathbb{R}^n)$ with the quotient space of $\tilde{V}_k(\mathbb{R}^n)$ by the equivalence relation

$$\mathbf{v} \sim \mathbf{v} \cdot T \qquad T \in \mathrm{Gl}_k(\mathbb{R}).$$

By analogy with the case of real projective space, we represent a point $V \in Gr_2(\mathbb{R}^k)$ as an equivalence. Imitating the situation with real projective space, we can denote point in $Gr_2(\mathbb{R}^k)$ as an equivalence class $[\mathbf{v}]$ in which $v$ is an $n \times k$ matrix with linearly independent columns. Analogous to the identity

$$[x] = [\lambda x] \qquad \lambda \in \mathbb{R} - \{0\} = \mathrm{Gl}_1(\mathbb{R})$$

for points in projective space, we have

$$[\mathbf{v}] = [\mathbf{v} \cdot T] \qquad T \in \mathrm{Gl}_k(\mathbb{R}).$$

In fact

$$[\mathbf{v}] = [\mathbf{w}]$$

if and only if

$$\mathbf{w} = \mathbf{v} \cdot T$$

for some $T \in \mathrm{Gl}_k(\mathbb{R})$.

We can now give the Grassmannian the structure of a smooth manifold. For an ordered $k$-tuple $\lambda = \{i_1, \dots, i_k\}$ and $\mathbf{v} \in \tilde{V}_k(\mathbb{R}^n)$ let

$$M_\lambda(\mathbf{v})$$

be the $k \times k$ minor of $\mathbf{v}$ obtained by selecting the rows $i_1, \dots, i_k$, and let

$$\begin{aligned} \tilde{U}_\lambda &= \{\mathbf{v} \in \tilde{V}(\mathbb{R}^n) \mid \det M_\lambda(\mathbf{v}) \neq 0\} \\ U_\lambda &= P(\tilde{U}_\lambda) \subset G_k(\mathbb{R}^n). \end{aligned}$$

Using the fact that the determinant of a product is the product of the determinants one can easily check that $P^{-1}(U_\lambda) = \tilde{U}_\lambda$ and so $U_\lambda \subset G_k(\mathbb{R}^n)$ is open.

By definition, if $\mathbf{v} \in \tilde{U}_\lambda$ then the matrix $M_\lambda(\mathbf{v})$ is invertible, and so

$$\mathbf{w} = \mathbf{v} \cdot M_\lambda(\mathbf{v})^{-1}$$

has the property that $M_\lambda(\mathbf{w})$ is the identity matrix. Let $U'_\lambda \subset \tilde{U}_\lambda$ be the subset of matrices $\mathbf{w}$ for which $M_\lambda(\mathbf{w})$ is the identity matrix. Using the above one can easily check that the restriction of $P$ to $U'_\lambda$ is a homeomorphism of $U'_\lambda$ with $U_\lambda$. Now the elements of $U'_\lambda$ are arbitray $n \times k$ matrices $\mathbf{w}$ for which $k^2$ entries have been specified (the entries of $M_\lambda(\mathbf{w})$). The remaining entries are arbitrary, and so this provides a homeomorphism

$$\Psi_\lambda : U_\lambda \to \mathbb{R}^{k(n-k)}.$$

One can check that these $\Psi_\lambda$ give a smooth atlas for $G_k(\mathbb{R}^n)$.

Let's look at $G_2(\mathbb{R}^4)$ to make things more concrete. Let $V = [\mathbf{v}] \in G_2(\mathbb{R}^4)$ be a point, with

$$\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \tag{5.4}$$

Let $\lambda = \{2, 4\}$. Then the open subset $U_\lambda$ consists of all the matrices as above for which the minor

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{bmatrix}$$

has a non-zero determinant. Multiplying by the inverse of this matrix on the right, one can easily check that every equivalence class of matrices in $U_\lambda$ has a unique representative of the form

$$\begin{bmatrix} b_{11} & b_{12} \\ 1 & 0 \\ b_{31} & b_{32} \\ 0 & 1 \end{bmatrix},$$

in which the $b_{ij}$ are arbitrary. The $b_{ij}$ give a homeomorphism of $U_\lambda$ with $\mathbb{R}^4$.

It's possible to do all this in a more geometric way, without resorting to equivalence classes of matrices. For $\lambda = \{i_1, \dots, i_k\}$ let $\lambda' = \{j_1, \dots, j_{n-k}\}$ be the

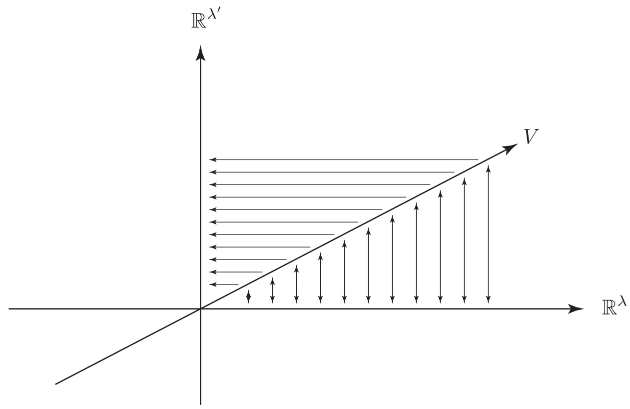



FIGURE 1. The chart $U_\lambda$

complement of $\lambda$, and write

$$\mathbb{R}^\lambda = \mathbb{R}^{\{i_1,\cdots,i_k\}} \approx \mathbb{R}^k$$
$$\mathbb{R}^{\lambda'} = \mathbb{R}^{\{j_1,\cdots,j_{n-k}\}} \approx \mathbb{R}^{n-k}$$
$$\mathbb{R}^n = \mathbb{R}^\lambda \times \mathbb{R}^{\lambda'}.$$

Then a $k$-plane $V \subset \mathbb{R}^n$ is in $U_\lambda$ if and only if the projection mapping $V \to \mathbb{R}^\lambda$ is an isomorphism. In this case $V$ may be regarded as the graph of a linear transformation $\mathbb{R}^\lambda \to \mathbb{R}^{\lambda'}$ (see Figure Figure 1). This gives an homeomorphism

$$\Psi_\lambda : U_\lambda \approx \hom(\mathbb{R}^\lambda, \mathbb{R}^{\lambda'}) \approx \mathbb{R}^{k(n-k)}.$$

These functions form a smooth atlas.

**Remark 5.5.** One of the advantages of this more "geometric" approach is that it can actually be made totally coordinate free. If you think it through you will find that you get a canonical identification of the tangent space to $G_k(\mathbb{R}^n)$ at a $k$-plane $V$ with $\hom(V, \mathbb{R}^n/V)$.

**5.3. The Plücker embedding.** We now turn to the famous Plücker embedding of $G_k(\mathbb{R}^n)$ in $\mathbf{RP}^{\binom{n}{k}-1}$. At the end of Section 5.1 we described an embedding of real projective space in a Euclidean space. Combining these gives an explicit embedding of the Grassmannian into Euclidean space.

The number $\binom{n}{k}$ comes up as the order of the set

$$S = S_k(n) = \{\lambda \subset \{1, \ldots, k\} \mid |\lambda| = k\}.$$

In fact it is easier to deal with the Eucliden space $\mathbb{R}^{S_k(n)}$, whose points are tuples $(x_\lambda)$ with $\lambda \in S_k(n)$ and $x_\lambda \in \mathbb{R}$. If you want to identify this with $\mathbb{R}^{\binom{n}{k}}$ you can choose an ordering of $S_k(n)$.

For $\lambda \in S_k(n)$ let

$$p_\lambda : \tilde{V}_k(\mathbb{R}^n) \to \mathbb{R}$$

be the function sending $\mathbf{v} \in \tilde{V}_k(\mathbb{R}^n)$ to the determinant

$$\det(\mathbf{v}_{\lambda,k})$$

of the $\lambda \times k$ minor of $\mathbf{v}$. For example if

$$\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \in \tilde{V}_2(\mathbb{R}^4)$$

and $\lambda = \{2, 4\}$ then

$$p_\lambda(\mathbf{v}) = \det \begin{pmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{pmatrix} = a_{21}\, a_{42} - a_{22}\, a_{41}.$$

Now define

$$p : \tilde{V}_k(\mathbb{R}^n) \to \mathbb{R}^{S_k(n)} - \{0\}$$

by

$$p(\mathbf{v}) = (p_\lambda(v)).$$

Since $\mathbf{v}$ has rank $k$, at least one of the $p_\lambda(\mathbf{v})$ must be non-zero.

The map $p$ gives the Plücker embedding. To spell it out, it will also help to have another notation for real projective spaces. For a real vector space $V$ let $P(V)$ be the quotient of $V - \{0\}$ by the equivalence ralation

$$v \sim \lambda v \qquad 0 \neq \lambda \in \mathbb{R}.$$

From the definition we have

$$P(\mathbb{R}^n) = \mathbf{RP}^{n-1}$$

so this gives a coordinate free way of talking about $\mathbf{RP}^n$. For $v \in V - \{0\}$ we will let $[v]$ denote the equivalence class of $v$ in $P(V)$.

Since the determinant of a product is the product of the determinants, we have

$$p_\lambda(\mathbf{v} \cdot T) = p_\lambda(\mathbf{v}) \det T$$

so that

$$\mathbf{v} \mapsto [p_\lambda(\mathbf{v})]$$

is a well defined map

$$p : G_k(\mathbb{R}^n) \to P(\mathbb{R}^{S_k(n)}).$$

One can check (Exercise 5.5) that this map is an embedding. It is called the *Plücker embedding*. The numbers $p_\lambda(\mathbf{v})$ are called the *Plücker coordinates* of $V$.

**Remark 5.6.** It might seem funny to call the numbers $x_i$ in the representative $[x_0, \ldots, x_n]$ of a point in $\mathbf{RP}^n$ *coordinates*, since they are only defined up to scale. They are, however defined *collectively* up to scale. This means that things like the set of all $[x_0, \cdots, x_n]$ for which

$$x_1^2 = x_2^2 + x_3 x_4$$

make sense, if the $x_i$ are scaled by $\lambda$, the whole expression scales by $\lambda^2$. You will meet this in 5.6.

### Exercises

**5.1.** This problem concerns the embedding of $\mathbf{RP}^n$ in Eucliden space, described at the end of Section 5.1.

(a) Prove that each point $[x_0, \ldots, x_n] \in \mathbf{RP}^n$ has a representative having the property that

$$x_0^2 + \cdots + x_n^2 = 1.$$

(b) Show that if

$$[x_0, \ldots, x_n] = [y_0, \ldots, y_n] \in \mathbf{RP}^n$$

and

$$x_0^2 + \cdots + x_n^2 = y_0^2 + \cdots + y_n^2$$

then

$$(x_0, \ldots, x_n) = \pm(y_0, \ldots, y_n).$$

(c) Prove that the map

$$\mathbf{RP}^n \to \mathbb{R}$$

sending point

$$[x_0, \ldots, x_n]$$

to the product

$$\frac{1}{x_0^2 + \cdots + x_n^2}\,(x_0^2, x_0 x_1, \cdots, x_0 x_n, x_1^2, \cdots, x_{n^2})$$

of $1/|x|^2$ with the sequence of all degree 2 monomials in the $x_i$, is an embedding of $\mathbf{RP}^n$ in $\mathbb{R}^{(n+1)(n+2)/2}$.

**5.2.** Show that $\tilde{V}_k(\mathbb{R}^n)$ is in fact an open subset of $M_{n,k}$. (Hint: Show that

$$\tilde{U}_\lambda = \{\mathbf{v} \mid \det M_\lambda(\mathbf{v}) \neq 0\}$$

is open.)

**5.3.** Show that $G_k(\mathbb{R}^n)$ is compact. (Hint: Ever heard of Stiefel manifolds?)

**5.4.** Show that if $\mathbf{v}, \mathbf{w} \in \tilde{V}_k(\mathbb{R}^n)$ have the property that $P(\mathbf{v}) = P(\mathbf{w})$ then there exists a unique $T \in \mathrm{Gl}_k(\mathbb{R})$ with $\mathbf{v} \cdot T = \mathbf{w}$.

**5.5.** In this problem we consider the Plücker embedding

$$p : G_k(\mathbb{R}^n) \to P(\mathbb{R}^{S_k(n)}).$$

To keep things simple we will restrict to $G_2(\mathbb{R}^4)$.

(a) Suppose that $V_1, V_2 \in G_2(\mathbb{R}^4)$ are two 2-planes, and that $p(V) = p(W)$. Suppose that $p_{12}(V) \neq 0$ so that $V$ is in $U_{12}$. How that $W$ is also in $U_{12}$

(b) Given the above, $V$ and $W$ are the column spaces of unique matrices of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v_{31} & v_{31} \\ v_{41} & v_{42} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ w_{31} & w_{31} \\ w_{41} & w_{42} \end{bmatrix}$$

respectively. Show that the assumption that $p(V) = P(W)$ implies that the above two matrices must be equal, and so $V = W$.

**5.6.** The image of the Plücker embedding is the solution space of the famous *Plücker equations*. In this exercise we will study those equations for $G_2(\mathbb{R}^4)$.

(a) Show that if $V \in G_2(\mathbb{R}^4)$ is a 2-plane then the Plücker coordinates $p_{ij} = p_{ij}(V)$ satisfy

$$p_{11}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

(Hint: It suffices to check this in each $U_{ij}$. Check that by symmetry it is enough to check it in $U_{12}$, and do that case explicitly.)

(b) Now let $Z \subset \mathbf{RP}^5$ be the set of solutions to the above equation. Show that every element of $Z$ is of the form $p(V)$ for a 2-plane $V \in G_2(\mathbb{R}^4)$. (Hint: Since the $p_{ij}$ can't all be zero, you can work in the subspace of $\mathbf{RP}^5$ in which some $p_{ij} \neq 0$. Show that by symmetry you might as well take this to be $p_{11}$ and check it there. )

(c) We've now shown that the Plücker embedding is a homeomorphism of $G_2(\mathbb{R}^4)$ with the space $Z$ of solutions to the Plücker equation. We know from the above the $\mathbf{RP}^5$ is a smooth manifold. Show that $Z \subset \mathbf{RP}^5$ is a smooth submanifold of dimension 4. (Hint: The question is local, so it suffices to check in in $U_{ij}$.) This gives another construction of $G_2(\mathbb{R}^4)$ as a smooth manifold.

(d) Is the map $p : G_2(\mathbb{R}^4) \to \mathbb{Z}$ smooth, if we give $G_2(\mathbb{R}^4)$ the smooth structure defined in Section 5.2?

CHAPTER V

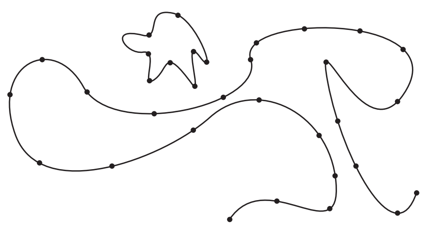
# The classification of 1-manifolds

Under construction

The main result is the following.

**Theorem 0.1.** *If $M$ is a compact smooth 1-manifold, then $M$ is diffeomorphic to a disjoint union of closed intervals and circles.*

The result is at least intuitive, and an idea of a proof springs immediately to mind. Around each point of $M$ we can find a closed neighborhood diffeomorphic to an interval. The manifold $M$ is constructed by gluing these intervals together end to end, and there aren't really many ways to do that. One might hope to achieve a classification by induction on the number of intervals.



This is more or less the idea of the proof. What one needs is a way of getting at the "size" of $M$ so that once one has taken out an interval, what is left is definitely smaller. One can do this with "arclength" as in [**8**, Appendix], or by using a Morse function as in [**1**, Appendix 2]. We will make use of a Morse function here, but use it to effect an induction as described above.

The proof will make use of some notions from point set topology. These could be threaded into the proof on the fly, and probably simplified, but these results are used quite a bit outside of differential topology, so I think it is better to take a little time with them.

**0.1. Covering spaces.** We're first going to discuss some "lifting problems." These take the form of a diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Z \arrow[ur, dashed, "g"] \arrow[r, "i"'] & Y \end{tikzcd}$$

in which the solid arrows $g$ and $i$ are given, and one wants to find maps $g$ making the diagram commute. In terms of equations, one is trying to find a function

$g : Z \to X$ satisfying $f(g(z)) = i(z)$ for all $z \in Z$. We call the map $g$ a *lift of $i$ through $f$*, or just a *lift* in the diagram. We will explore how various properties of $f$ give properties of lifts.

**Lemma 0.2.** *Suppose that $f : X \to Y$ is local homeomorphism and that $X$ is Hausdorff. If $i : Z \to Y$ is any continuous map and $g_1, g_2 : Z \to X$ are two lifts of $i$, then the set*

$$J = \{z \in Z \mid g_1(z) = g_2(z)\}$$

*is both open and closed in $Z$.*

**Remark 0.3.** In particular when $Z$ is connected, if a lift exists it is unique.

*Proof:* Let $\Delta = \{(x, x) \in X \times X\}$ be the diagonal subspace. Set

$$g = (g_1, g_2) : Z \to X \times X.$$

The map $g$ is continuous and one has $J = g^{-1}(\Delta)$. Since $X$ is Hausdorff, $\Delta$ is closed, hence so is $J$. Now suppose that $z \in J$, and write $x = g_1(z) = g_2(z)$. Let $U \subset X$ be a neighborhood of $x$ on which $f$ is a homeomorphism. Then $U \times U$ is open in $X \times X$, and so $g^{-1}(U \times U)$ is open in $Z$. I claim that $g^{-1}(U \times U) \subset J$. To see this suppose that $p$ is a point of $Z$ with $g_1(p) \in U$ and $g_2(p) \in U$. Since $f : U \to Y$ is a monomorphism, to show that $g_1(p) = g_2(p)$ it suffices to show that $f(g_1(p)) = f(g_2(p))$. But

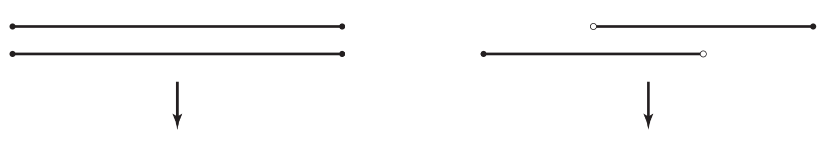
$$f(g_1(p)) = i(p) = f(g_2(p)).$$

□

**Definition 0.4.** A map $f : X \to Y$ is a *covering space* if each $y \in Y$ has a neighborhood $U$ with the property that $f^{-1}(U)$ is homeomorphic to $U \times S$ for some discrete set $S$.

(A *discrete set* $S$ is a set $S$ regarded as a topological space with the discrete topology).

The covering space condition on $f$ is stronger than the condition that $f$ be a local homeomorphism. In the diagram below, the map on the right is a local homeomorphism but not a covering space. The map on the left is a covering space.



**Remark 0.5.** When $f$ is a smooth map between smooth manifolds, being a covering space is the same thing as being a smooth fiber bundle with fiber a discrete set $S$.

**Example 0.6.** If $f : X \to Y$ is a submersion of compact manifolds (possibly with boundary) of the same dimension then $f$ is a covering space. In the closed manifold case that is the "stack of records theorem" from Problem 7, Ch. 4, §1 of [1].

**Theorem 0.7** (Path Lifting)**.** *Suppose that $f : X \to Y$ is a covering space with $X$ Hausdorff, and $\gamma : [0,1] \to Y$ is a path*

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & Y \end{tikzcd}.$$

*For each $x \in X$ there is a unique path $\tilde{\gamma} : [0,1] \to X$ lifting $\gamma$ and satisfying $\tilde{\gamma}(0) = x$.*

*Proof:* The uniqueness follows from Lemma 0.2. The lift $\tilde{\gamma}$ obviously exists if the image $\gamma([0,1])$ is contained in a set $U$ for which there is a homeomorhism $f^{-1}(U) \approx U \times S$. To handle the general case, choose, for each $t \in [0,1]$, a neighborhood $U_t$ of $\gamma(t)$ and a homeomorphism $p^{-1}(U_t) \approx U_t \times S$. The collection $\{\gamma^{-1}(U_t)\}$ is then a cover of $[0,1]$. Choose points $0 = t_1 < t_1 < \cdots < t_k < t_{k+1} = 1$ with the property that for each $i$, $[t_i, t_{i+1}]$ is contained in some $\gamma^{-1}(U_t)$ (for instance by taking $|t_{i+1} - t_i|$ to be smaller than the Lebesgue number of the covering $\{\gamma^{-1}(U_t)\}$). By the easy case of the theorem, the lifting problem for the restriction of $\gamma$ to $[t_i, t_{i+1}]$ has a unique solution once a lift has been specified for $\gamma(t_i)$. The lift $\tilde{\gamma}$ can then be constructed by starting with the restriction of $\gamma$ to $[t_0, t_1]$ and inductively constructing the unique lift along each $[t_i, t_{i+1}]$. □

**0.2. Smoothing.** Before going into the main result of this section let's agree on some terminology. Suppose that $M$ is a smooth manifold of dimension $k$, and $p \in M$ is a point. By a *coordinate neighborhood* of $p$ we will mean a pair $(\Phi, U)$ consisting of an open neighborhood $U$ of $p$ and a smooth function $\Phi : U \to \mathbb{R}^k$ satisfying $\Phi(p) = 0$, and which is a diffeomorphism of $U$ with an open neighborhood $\hat{U}$ of 0 in $\mathbb{R}^k$ (or $H^k$ if $p$ is a boundary point.) Some folks emphasize the condition that $\Phi(x) = 0$ by saying that the coordinated neighborhood is *centered at $x$*. If $f : M \to \mathbb{R}$ is a smooth function we will write $f_\Phi$ for the function

$$f_\Phi = f \circ \Phi^{-1} : \hat{U} \to \mathbb{R}$$

and call $f_\Phi$ the expression of $f$ in terms of the local coordinate system $\Phi$.

**Proposition 0.8.** *Suppose that $M$ and $N$ are smooth 1-manifolds, $p$ is an interior point of $M$ and $f : M \to N$ is a homeomorphism which restricts to a diffeomorphism of $M - \{p\}$ with $N - \{f(p)\}$. Given a neighborhood $U$ of $p$ there is a diffeomorphism $\tilde{f} : M \to N$ which agrees with $f$ outside the closure $\bar{U}$ of $U$.*

*Proof:* By working in a coordinate neighborhood of $p$ which is small enough to map to a coordinate neighborhood of $f(p)$ we may reduce to the case in which $M = \mathbb{R}$, $N = \mathbb{R}$, $x = 0$, $f(x) = 0$. Since $f$ is a homeomorphism taking 0 to 0, $f(t)$ must either have the same sign as $t$ for all $t$ or else the opposite. Replacing $f$ by $-f$ if necessary we may assume that sign $f(t) = \text{sign}(t)$ for $t \neq 0$. This now forces $f'(t) > 0$ for all $t \neq 0$. For example if for some $t > 0$ one had $f'(t) < 0$ then since $f'$ is never zero, one would have $f'(t) < 0$ for all $t > 0$ and $f$ would be decreasing on $[0, \infty)$. Since $f(0) = 0$ this contradicts the fact that $f(t) > 0$ for $t > 0$. A similar argument handles the case $t < 0$. Now choose $a, b \in \mathbb{R}$ with $a < 0 < b$ and

$[a,b] \subset U$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a bump function which is zero for $t \notin [a,b]$, identically 1 in a neighborhood of 0, and satisfies

$$\int_a^b \rho \, dt = 1.$$

Now consider

$$\tilde{f}(x) = f(a) + \int_a^x \big(c\,\rho(t) + f'(t)(1-\rho(t))\big)\, dt$$

with

$$c = f(b) - f(a) - \int_a^b f'(t)(1-\rho(t))\, dt$$

chosen so that $\tilde{f}(b) = f(b)$. Note that for $x > b$

$$\begin{aligned}\tilde{f}(x) &= f(a) + \int_a^b \big(c\,\rho(t) + f'(t)(1-\rho(t))\big)\, dt + \int_b^x \big(c\,\rho(t) + f'(t)(1-\rho(t))\big)\, dt \\ &= f(b) + \int_b^x f'(t)\, dt \\ &= f(x),\end{aligned}$$

while for $x < a$

$$\begin{aligned}\tilde{f}(x) &= f(a) + \int_a^x \big(c\,\rho(t) + f'(t)(1-\rho(t))\big)\, dt \\ &= f(a) + \int_a^x f'(t)\, dt \\ &= f(x).\end{aligned}$$

Note also that near $x = 0$ where $\rho(t) = 1$ we have

$$f'(x) = c$$

so that $\tilde{f}$ is smooth everywhere and satisfies $\tilde{f}'(t) > 0$ for all $t$. The function $\tilde{f}$ is a local diffeomorphism by the inverse function theorem. It is also one to one since it is strictly increasing. Its image is a connected subset of $\mathbb{R}$ which is unbounded, since the image of $f$ is. It follows that $f$ is a diffeomorphism. $\square$

Proposition 0.8 can be used in many forms. We will make use of using the following.

**Corollary 0.9.** *Suppose that $M$ and $N$ are 1-manifolds. If there is a homeomorphism $f : M \to N$ which restricts to a diffeomorphism $M - S \to N - S$ outside of a finite set of points $S \subset M$ then there is a diffeomorphism $\tilde{f} : M \to N$.*

*Proof:* Let's do this by induction on the number of points $|S|$ in $S$. If $S = \emptyset$ there is nothing to prove. Suppose $|S|$ has $\ell$ points and we have proved the result for the case $|S| < \ell$. Choose a point $p \in S$ and an open neighborhoods $U \subset V$ of $p$ with $\bar{U} \subset V$ and $V \cap S = \{p\}$. Write

$$g : V \to f(V),$$

for the restriction of $f$ to $V$. We can apply Proposition 0.8 to get a diffeomorphism $\tilde{g} : V \to f(V)$ which agrees with $f$ outside of $\bar{U}$. The functions $\tilde{g}$ on $V$ and $f$ on $M - \bar{U}$ patch together to give homeomorphism $M \to N$ which is a diffeomorphism outside of $S - \{p\}$. The induction hypothesis gives the further desired modification. $\square$

We will also need the following "retraction" theorem. A similar result holds in all dimensions.

**Theorem 0.10.** *Suppose that $M$ is a 1-manifold and $U$ is a neighborhood of $\partial M$. There is a neighborhood $B$ of $\partial M$ a smooth map $f : M \to M - B$ which is a diffeomorphism with its image, and is the identity outside of $U$.*

*Proof:* If $\partial M = \emptyset$ we may take $B = \emptyset$ and $f$ to be the identity map. By working in a coordinate neighborhood of each boundary point it suffices to consider the case $M = [0, \infty)$ and $U = [0, a)$. For this let $\rho : \mathbb{R} \to [0, 1] \subset \mathbb{R}$ be a smooth bump function which is identically 1 in a neighborhood of 0 and 0 outside of, say $(-a, a)$. Note that $|\rho'(x)|$ must be bounded, since $\rho'(x)$ is zero outside of a compact set. Consider the function $f : M \to M$ given by

$$f(x) = x + \epsilon\rho(x),$$

where $\epsilon$ is chosen so that $1 + \epsilon\rho'(x) > 0$ for all $x$. Then $f$ is increasing, $f(0) = \epsilon$, and $f(x) = x$ for $x \geq a$. This $f$ will do. $\square$

**0.3. The classification.** Let $f : M \to \mathbb{R}$ be a Morse function, and $S \subset M$ the set of critical points. Since the critical points of $f$ are non-degenerate, they are isolated, and therefore finite in number. Our proof will be by induction on the number of critical points. We begin with

**Proposition 0.11.** *Suppose that $M$ is a compact connected 1-manifold. If $f : M \to \mathbb{R}$ is a Morse function with no critical points, then $f$ is a diffeomorphism of $M$ with a closed interval in $\mathbb{R}$.*

*Proof:* The image of $f$ is a compact connected subset of $\mathbb{R}$, and so must be a closed interval $[a, b]$. By assumption the map $f$ is a submersion. Since $\dim M = 1$ the map $f : M \to [a, b]$ is a covering space (Example 0.6). By Theorem 0.7 (path lifting) there is a unique continuous section $g : [a, b] \to M$. Since $f$ is a local diffeomorphism, $g$ is smooth. Since $g$ is a section, we know that $f \circ g$ is the identity map of $[a, b]$. It remains to show that $g \circ f$ is the identity map of $M$. For this, note that both the identity map and $g \circ f$ are solutions to the lifting problem

$$\begin{tikzcd} & M \arrow[d, "f"] \\ M \arrow[ur, dashed] \arrow[r, "f"'] & {[a,b]} \end{tikzcd} .$$

Since $M$ is connected, they must be the same (Lemma 0.2). $\square$

Now suppose that $f$ has $k$ critical points, with $k \geq 1$. If one of the critical points is a boundary point $p$, we can compose $f$ with a retraction mapping (Theorem 0.10) which is the identity outside of a neighborhood of $p$, small enough not to contain any other critical points. This new function is a Morse function with $(k-1)$ critical points. We may therefore assume that none of the critical points is a boundary point of $M$. We may also assume that $M$ is connected, since the induction hypothesis will apply to any connected components of $M$ with fewer that $k$ critical points. Now let $p$ be an interior critical point of $f$ and $\Phi : V \to \mathbb{R}$ a coordinate neighborhood of $p$ not containing any other critical points. By our conventions on coordinate neighborhoods, $f(p) = 0$. Set

$$\begin{aligned} I &= \Phi^{-1}\big([-1,1]\big) \subset M \\ a &= \Phi^{-1}(-1) \in M \\ b &= \Phi^{-1}(1) \in M \\ I^0 &= I - \{a,b\} \\ J &= M - I^0, \end{aligned}$$

and let $J_1 \subset J$ and $J_2 \subset J$ be the connected components containing $a$ and $b$ respectively (see Figure 1). It might be the case that $J_1 = J_2$. I claim that $J = J_1 \cup J_2$ and so $M = I \cup J_1 \cup J_2$. To see this let $K = J - \big(J_1 \cup J_2\big)$. Then $K$ is closed in $J$ hence closed in $M$. On the other hand $K$ is open in $J$ and does not contain $a$ or $b$ so it is open in $J - \{a,b\} = M - I$ hence open in $M$. Since $M$ is connected $K$ is either empty or all of $M$. But $K$ does not contain $a$ or $b$ so cannot be all of $M$.

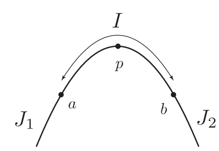



FIGURE 1. Nearby the point $p$

Now both $J_1$ and $J_2$ are connected 1-manifolds with non-empty boundary admitting a Morse function with $(k-1)$ critical points. Our induction hypothesis implies that they are diffeomorphic to closed intervals. Suppose $J_1 \neq J_2$. Choose a diffeomorphism $J_1 \to [-2,-1]$ sending $a$ to $-1$ and $J_2 \to [1,2]$ sending $b$ to 1. These diffeomorphisms fit together with the diffeomorphism

$$\Phi : I \to [-1,1]$$

to give a homeomorphism

$$M \to [-2,2]$$

which is in fact a diffeomorphism away from $\{a,b\}$. Proposition 0.8 shows that this homeomorphism may be modified to produce a diffeomorphism of $M$ with $[-2,2]$.

For the case $J_1 = J_2 = J$ choose a diffeomorphism

$$J \to \{(x,y) \in S^1 \mid y \leq 0\}$$

sending $a$ to $(-1,0)$ and $b$ to $(0,1)$, and a diffemorphism

$$I \to \{(x,y) \in S^1 \mid y \geq 0\}$$

also sending $a$ to $(-1,0)$ and $b$ to $(0,1)$ (see Figure 3). Together these combine to give a homeomorphism of $M$ with $S^1$ which is in fact a diffeomorphism away

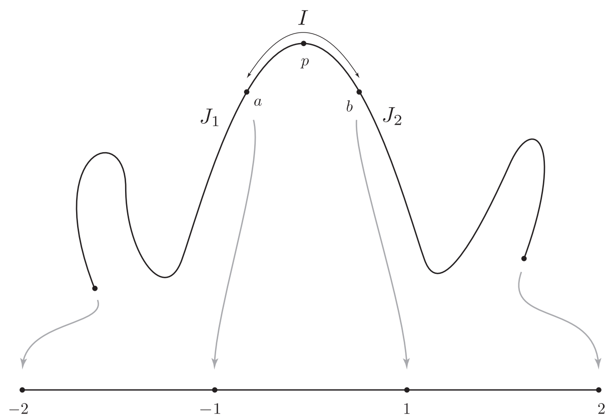



FIGURE 2. The case $J_1 \neq J_2$

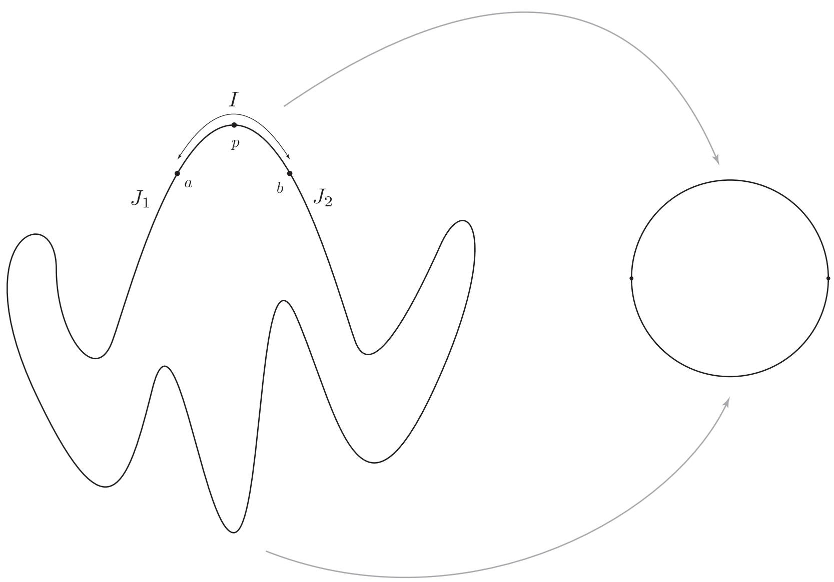



FIGURE 3. The case $J_1 = J_2$

from $\{a, b\}$. Once again, Proposition 0.8 shows that this homeomorphism may be modified to produce a diffeomorphism of $M$ with $S^1$.

CHAPTER VI

# Partitions of Unity

Under construction

## 1. basics

The goal of this section is to establish the existence of partitions of unity for subspaces of manifolds. We're going to prove this using one of the definitions of manifold that does not involve an a priori embedding in $\mathbb{R}^m$. We begin with some simple topological facts about these manifolds

**1.1. Topology of abstract smooth manifolds.** Recall that a manifold of dimension $d$ is a Hausdorff topological space $X$, having a countable basis for the topology, with the property that each point $x \in X$ has a neighborhood $U$ which is homeomorphic to an open subset of $\mathbb{R}^d$. Among other things this implies that every open cover of a manifold has a countable sub-cover.

**Lemma 1.1.** *Each point of a manifold $M$ has a neighborhood with compact closure.*

*Proof:* Just take the open unit ball in a coordinate neighborhood. □

**Lemma 1.2.** *Every compact subset $K \subset M$ has a neighborhood with compact closure.*

*Proof:* For each point of $K$ choose a neighborhood in $M$ with compact closure. Since $K$ is compact it is contained in the union of finitely many of these. This finite union has compact closure. □

**Lemma 1.3.** *Every manifold is a countable union of compact sets.*

*Proof:* As mentioned earlier, the fact that there is a countable basis for the topology of $M$ implies that every open cover has a countable sub-cover. For each point of $M$ choose an open neighborhood with compact closure. By the countability assumption $M$ is the union of a countable subset of these. □

**Lemma 1.4.** *There exists a nested sequence*

$$K_1 \subset V_1 \subset K_2 \subset V_2 \subset \cdots$$

*with $M = \bigcup K_i$ in which $K_i$ is compact and $V_i$ is a neighborhood of $K_i$ whose closure is contained in $K_{i+1}$.*

*Proof:* Using Lemma 1.3 choose compact subsets $B_1, B_2, \cdots$ with $M = \bigcup B_i$. Let $K_1 = B_i$ and using Lemma 1.2 choose a neighborhood $K_1 \subset V_1$ with compact closure $\bar{V_2}$. Now set $K_2 = \bar{V_2} \cup B_2$ and let $V_2$ be a neighborhood of $K_2$ with compact closure. Continuing this way leads to the desired sequence. The fact that $\bigcup K_i = M$ follows from the fact that $B_i \subset K_i$. □

**Definition 1.5.** A collection $\{K_i\}$ of subsets of $X$ is *disconnected* if there are open neighborhoods $K_i \subset V_i$ with $V_i \cap V_j = \emptyset$ for $i \neq j$.

**Proposition 1.6.** *Each manifold $M$ can be written as $M = I_1 \cup I_2 \cup I_3$ in which each $I_i$ is the union of a countable set of disconnected compact subspaces.*

*Proof:* Let $K_1 \subset V_1 \subset K_2 \subset V_2 \subset \cdots$ be the sequence provided by Lemma 1.4. Set

$$L_i = K_i - V_{i-2} \subset V_i - K_{i-2} = W_i.$$

Note that $L_i$ is compact and $W_i$ is open. Note also that $\bigcup L_i = X$. Indeed for each $x$ we may find an $i$ with

$$x \in K_i - K_{i-1} \subset K_i - V_{i-2} = L_i.$$

Also note that $W_i \cap W_{i+3} = \emptyset$, since

$$W_{i+3} \cap W_i = (V_{i+3} - K_{i+1}) \cap (V_i - K_{i-2}) \subset (V_{i+3} - K_{i+1}) \cap V_i$$

which is empty since $V_i \subset K_{i+1}$. Now take

$$\begin{aligned} I_1 &= \coprod L_{3i+1} \\ I_2 &= \coprod L_{3i+2} \\ I_3 &= \coprod L_{3i+3}. \end{aligned}$$

□

### 1.2. Bump functions.

First some helpful terminology.

**Definition 1.7.** The *support* of a continuous real-valued function $f : X \to \mathbb{R}$ on a topological space $X$ is the closure of the set $\{x \mid f(x) \neq 0\}$.

The support of $f$ will be denoted $\operatorname{Supp} f$, and we will write

$$\operatorname{Supp}^+ f = \{x \mid f(x) > 0\}.$$

Note that $\operatorname{Supp}^+ f$ is open and $\operatorname{Supp} f$ is closed.

Now for the bump functions. In Exercise 18 of Chapter 1, §1 you constructed for $0 < a < b$ a smooth function $\rho = \rho_a^b : \mathbb{R}^k \to [0,1] \subset \mathbb{R}$ having the properties that

$$\rho(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| \ge b \end{cases}$$

and that $0 < \rho(x) < 1$ for $a < |x| < b$. In fact in the range $a < |x| < b$, $\rho$ is a strictly decreasing function of $|x|$.

**Definition 1.8.** Suppose that $M$ is a smooth manifold and $p \in M$ and $U \subset$ is a neighborhood of $p$. A *bump function at $p$ supported in $U$* is a smooth function $\rho : M \to [0,1] \subset \mathbb{R}$, with $\operatorname{Supp} \rho \subset U$ and taking the value 1 in a neighborhood of $p$.

Bump functions subordinate to arbitrary neighborhoods always exist. Indeed suppose $p \in M$ and $U$ is a neighborhood of $p$. Choose a coordinate neighborhood $\Phi : U' \to \mathbb{R}^k$ with $U' \subset U$, and numbers $0 < a < b$ having the property that the closed ball of radius $b$ is contained in $\Phi(U')$. Then the function

$$\rho_a^b \circ \Phi : U' \to \mathbb{R}$$

and the constant function 0 on $M - \Phi^{-1}\big(\operatorname{Supp} \rho_a^b\big)$ agree on the intersection

$$U' \cap \big(M - \Phi^{-1}\big(\operatorname{Supp} \rho_a^b\big)\big) = U' - \Phi^{-1}\big(\operatorname{Supp} \rho_a^b\big)$$

and so patch together to give a smooth bump function $\rho : M \to \mathbb{R}$ subordinate to $U$.

**1.3. Partitions of unity.** Suppose that $X$ is a topological space and $\mathcal{U} = \{U_\alpha\}$ is a collection of subsets whose interiors cover $X$.

**Definition 1.9.** A *partition of unitiy subordinate to* $\mathcal{U}$ is a sequence $\theta_1, \theta_2, \cdots$ of functions

$$\theta_i : X \to [0, 1] \subset \mathbb{R}$$

with the following properties

(1) For each $i$, there exists an $\alpha_i$ such that the support of $\theta_i$ is contained in $U_{\alpha_i}$.

(2) Each $x \in X$ has a neighborhood on which all but finitely many of the $\theta_i$ are zero.

(3) For each $x$ one has

$$\sum_i \theta_i(x) = 1.$$

Our interest is in establishing the existence of partitions of unity. If instead of (i) we have

(3′) For each $x$ there exists an $i$ with $\theta_i(x) \neq 0$

then the functions

$$\eta_i = \frac{\theta_i(x)}{\sum_j \theta_j(x)}.$$

form a partition of unity subordinate to $\mathcal{U}$.

**Definition 1.10.** An *positive partition subordinate to* $\mathcal{U}$ is a sequence $\theta_1, \theta_2, \cdots$ of functions

$$\theta_i : X \to [0, 1] \subset \mathbb{R}$$

satisfying (1), (2), (3′) above.

Here is the main result.

**Theorem 1.11.** *Suppose that $M$ is a manifold, $X \subset M$ is a subspace and $\mathcal{U} = \{U_\alpha\}$ is a collection of sets whose interiors cover $X$. There exists a sequence $\{\theta_i\}$ of smooth functions on $M$ whose restriction to $X$ forms a partition of unity subordinate $\{U\}$.*

**Remark 1.12.** I've sneaked in an extra assertion in the statement of Theorem 1.11. Not only does it assert that partitions of unity exist for every subspace of a smooth manifold, but it says that the functions forming one are the restrictions of smooth functions on the manifold. This extra information comes up often and is useful. It also helps streamline the proof.

We will prove Theorem 1.11 by starting with the case in which $X$ is compact, and building up to the general case from there.

**Lemma 1.13.** *The theorem is true when $X$ is compact.*

*Proof:* First coose for each $\alpha$ an open subset $\tilde{U}_\alpha \subset M$ with $U_\alpha \cap X = \operatorname{int} U_\alpha$. Next, for each point $x \in X$ choose an $\alpha$ with $x \in \operatorname{int} U_\alpha$ and a bump function $\theta_x$ at $x$ supported in $\tilde{U}_\alpha \subset M$. The open sets $\operatorname{Supp}^+ \theta_x$ cover $X$. Since $X$ is compact we can find a finite number of such functions $\theta_i$ having the property that the subsets $\operatorname{Supp}^+ \theta_i$ cover $X$. This collection of functions restricts to a positive partition subordinate to $\mathcal{U}$. $\square$

**Lemma 1.14.** *If the theorem is true for a finite collection $\{I_1, \dots, I_k\}$ of subsets of $M$ it is true for their union.*

*Proof:* If $\{\theta_i^\ell\}$ is a countable collection of smooth functions on $M$ restricting to partition of unity subordinate to $\mathcal{U}$ in $I_\ell$, then the set

$$\bigcup_\ell \{\theta_i^\ell\}$$

restricts to a positive partition subordinate to $\mathcal{U}$. $\square$

**Lemma 1.15.** *The theorem is true when $X$ is the union of a countable disconnected collection of compact subsets.*

*Proof:* Suppose $X = \bigcup K_i$ with $K_i$ compact, and there are neighborhoods $K_i \subset V_i$ with $V_i \cap V_j = \emptyset$ when $i \neq j$. Using Lemma 1.15 choose for each $k$ a set $\{\theta_i^k\}$ of smooth functions on $M$ whose restriction to $K_k$ is a partition of unity subordinate to the covering $\{V_k \cap U_\alpha\}$. $\square$

**Corollary 1.16.** *The theorem is true if $X \subset M$ is open.*

*Proof:* If $X$ is open then $X$ itself is a manifold and so admits the decomposition provided by Proposition 1.6. The result then follows from Lemmas 1.14 and 1.15 above. $\square$

*Proof of Theorem 1.11:* As in the proof of Lemma 1.13 choose for each $\alpha$ an open subset $\tilde{U}_\alpha \subset M$ with $U_\alpha \cap X = \operatorname{int} U_\alpha$. Let $M' = \bigcup_\alpha \tilde{U}_\alpha$. By Corollary 1.16 there a collection of smooth functions $\{\theta_i\}$ on $M$ whose restriction to $M'$ forms a partition of unity subordinate to the cover $\{\tilde{U}_\alpha\}$. This same set restricts to a partition of unity on $X$ subordinate to the collection $\mathcal{U}$. $\square$

**1.4. Shrinking and Strong partitions.** Often one needs something slightly stronger than what we have called a partition of unity.

**Definition 1.17.** Suppose that $X$ is a space and $\mathcal{U} = \{U_\alpha\}$ is a collection of subspaces whose interiors cover. A partition of unity $\{\theta_i\}$ subordinate to $\mathcal{U}$ is *strong* if the correspondence $i \mapsto \alpha_i$ is one to one.

**Remark 1.18.** If $\{\theta_i\}$ is a partition of unity subordinate to $\mathcal{U}$ then the subset $\{U_\alpha \mid \alpha = \alpha_i \text{ for some } i\}$ is also a collection of subsets whose interiors cover. So just by forgetting some of the $U_\alpha$ one can arrange that the map $i \mapsto \alpha_i$ is a surjection. If $\{\theta_i\}$ is a strong partition of unity then this arranges things so that the map $i \mapsto \alpha_i$ is a bijection.

**Remark 1.19.** There isn't much consistency in the terminology in this situation. Some authors use "partition of unity" to refer to what we are calling a strong partition of unity. Some authors take the property that the map $i \mapsto \alpha_i$ be a bijection to be part of the definition of "partition of unity." We will see below (Proposition 1.23) that strong partitions of unity always exist for coverings of subsets of manifolds, so it doesn't make much difference in the end.

**Remark 1.20.** If $\{\theta_i\}$ is a partition of unity and if the map

$$i \to \alpha_i$$

is finite to one then by defining

$$\theta_\alpha = \sum_{\alpha_i = \alpha} \theta_i$$

one gets a countable list of functions forming a strong partition of unity. The issue is the situation when $i \mapsto \alpha_i$ is not finite to one. In that case the functions $\theta_\alpha$ are still defined, but the support of $\theta_\alpha$ is the closure

$$\operatorname{closure}\Big(\bigcup_{\alpha_i = \alpha} \operatorname{Supp} \theta_i\Big).$$

Though the union of the supports will be contained in $U_\alpha$ the closure might not be.

**Remark 1.21.** Even if the covering $\mathcal{U}$ finite, the partition of unity $\{\theta_i\}$ provided by Theorem 1.11 might be countable, so you don't get a strong partition automatically. When $X$ is compact the proof does lead to a strong partition of unity, using Remark 1.20.

Our aim is to construct strong partitions of unity. The trick for doing this is the notion of a *shrinking*. Suppose that $\mathcal{U} = \{U_\alpha\}$ is an open covering of a topological space $X$.

**Definition 1.22.** A *shrinking* of $\mathcal{U}$ is an open covering $\mathcal{V} = \{V_\alpha\}$ having the property that for each $\alpha$, the closure of $V_\alpha$ is contained in $U_\alpha$.

**Proposition 1.23.** *If $X$ is a subspace of a smooth manifold, every open cover has a shrinking.*

*Proof:* Suppose that $\mathcal{U}$ is an open cover and let $\theta_i$ be a partition of unity subordinate to $\mathcal{U}$, so that there is a function $i \mapsto \alpha_i$ with $\operatorname{Supp} \theta_i \subset U_i$. We might as well assume that the function $i \mapsto \alpha_i$ is surjective since for $\alpha \notin \{\alpha_i\}$ we can just take $V_\alpha$ to be the empty set. For each $\alpha$ let

$$\phi_\alpha = \sum_{\alpha_i = \alpha} \theta_i.$$

Then each of the functions $\phi_\alpha$ has the property that for all $x$

$$0 \le \phi_\alpha(x) \le 1$$
$$\sum_\alpha \phi_\alpha(x) = 1$$
$$\phi_\alpha^{-1}((0,1]) \subset U_\alpha.$$

Let $V_i = \phi_i^{-1}(1/2^{n+1}, 1]$. We have

$$\bar{V}_i \subset \phi_i^{-1}[1/2, 1] \subset U_i$$

so we're done if we show that the $V_i$ cover $X$. This is equivalent to showing that for every $x \in X$ there is an $n$ with $\phi_n(x) > 1/2^{n+1}$. But if for all $n$, $\phi_n(x) \le 1/2^{n+1}$ then

$$1 = \sum_n \phi_n(x) \le \sum_n 1/2^{n+1} = 1/2$$

which is a contradiction. □

**Proposition 1.24.** *Suppose that $X$ is a subset of a manifold $M$ and $\mathcal{U}$ is collection of subsets of $X$ whose interiors cover. There is a countable collection $\{\theta_i\}$ of smooth functions on $M$ whose restriction to $X$ is a strong partition of unity subordinate to $\mathcal{U}$.*

*Proof:* Let $\mathcal{V} = \{V_\alpha\}$ be a shrinking of $\mathcal{U}$ and $\{\phi_i\}$ a partition of unity subordinate to $\mathcal{V}$. Let $\theta_\alpha = \sum_{\alpha_i = \alpha} \phi_I$. Then

$$\operatorname{Supp} \theta_\alpha \subset \bar{V}_\alpha \subset U_\alpha$$

so $\{\theta_\alpha\}$ is the desired strong partition of unity. □

# Part 2

# Surfaces

This entire part is under construction

CHAPTER VII

# Cobordism

## 1. Gluing

Let's begin with something that we have used frequently. Suppose $M = U \cup V$ is a manifold, written as a union of two open sets $U$ and $V$, and $N$ is another manifold. A function $f : M \to N$ is smooth if and only if the restrictions of $f$ to $U$ and $V$ are both smooth. This is just a consequence of the fact that smoothness is something one checks locally. Conversely, suppose we are given smooth functions

$$\begin{aligned} f_U &: U \to N \\ f_V &: V \to N \end{aligned}$$

having the property that for $x \in U \cap V$ one has $f_U(x) = f_V(x)$. Then the function

$$f : M \to N$$

given by

$$f(x) = \begin{cases} f_U(x) & x \in U \\ f_V(x) & x \in V \end{cases}$$

is well-defined and smooth. This process is referred to as "patching" or "gluing"smooth functions.

One can also "patch" or "glue" smooth manifolds.

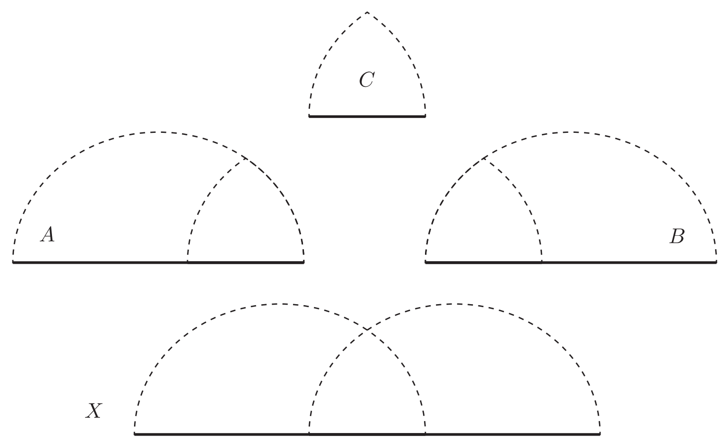
**Proposition 1.1.** *Suppose that $A$, $B$ and $C$ are smooth manifolds of dimension $n$, and that we are given a diagram of smooth maps*

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & A \\ B & \end{tikzcd}$$

*in which $i$ and $j$ are diffeomorphims with open subsets of $A$ and $B$ respectively. Let*

$$X = A \underset{C}{\cup} B = A \amalg C \amalg B / x \sim i(x) \sim j(x) \qquad (x \in C).$$

*The space $X$ has a unique structure of a smooth manifold for which the inclusion maps $A \to X$ and $B \to X$ are smooth. Moreover, $\partial X = \partial A \underset{\partial C}{\cup} \partial B$.*




Note that our assumptions imply that $A \to X$ and $B \to X$ are inclusions of open subsets of $X$, and that $A \cap B$ may be identified with $C$, via $i$ or $j$.

*Proof:* The space $X$ obviously has a countable base for the topology since $A$ and $B$ do. For $U \subset X$ define a map $f : U \to \mathbb{R}$ to be smooth if and only if the restrictions of $f$ to $U \cap A \subset A$ and $U \cap B \subset B$ are smooth. With this definition $X$ is locally diffeomorphic to $H^n$ since $A$ and $B$ are. The assertion about $\partial X$ is straightforward. $\square$

**Remark 1.2.** The manifold $X$ is characterized by the following universal property: given the commuting solid arrow diagram of smooth maps

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & X \arrow[dr, dashed, "\exists"] & \\
& & M
\end{tikzcd}$$

there exists a unique smooth map $X \to M$ making the diagram commute.

It can be useful to have a criteria for $X = A \underset{C}{\cup} B$ to be Hausdorff. The following lemma provides one.

**Lemma 1.3.** *Suppose that $X = A \cup B$ is a topological space, with $A$ and $B$ open subsets and write $C = A \cap B$. Suppose in addition that $A$, $B$, and $C$ are Hausdorff. If there is a continuous function $\phi : X \to \mathbb{R}$ with $\phi(A \setminus C) \subset [1, \infty)$, $\phi(B \setminus C) \subset (-\infty, -1]$ then $X$ is Hausdorff (see Figure 1).*

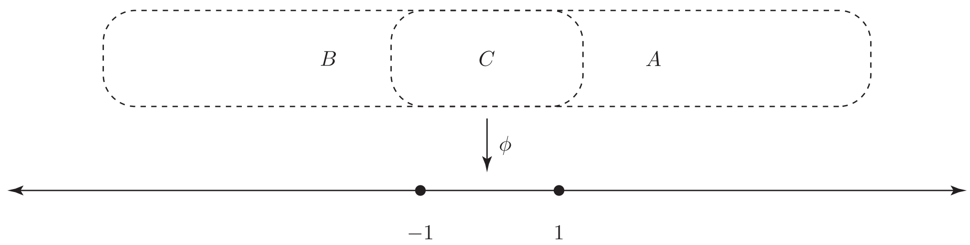



FIGURE 1. A Hausdorff criterion

**Remark 1.4.** The function $\phi$ is determined by its restriction to $A$ and $B$ so in the situation of Proposition 1.1 one can construct a function $\phi$ from a pair of functions

$$\phi_A : A \to \mathbb{R}$$
$$\phi_B : B \to \mathbb{R}$$

which agree on $C$ in the sense $\phi_A \circ i = \phi_B \circ j$, and which have the property that

$$A \setminus i(C) \subset \phi_A^{-1}([1, \infty))$$
$$B \setminus j(C) \subset \phi_B^{-1}((-\infty, -1]).$$

**Remark 1.5.** There is nothing special about the values $-1$ and 1. They could be replaced by any real numbers $s < t$.

*proof of Lemma 1.3:* Suppose that $x$ and $y$ are points in $X$. If $x$ and $y$ are both in $A$, then since $A$ is Hausdorff there are neighborhoods $x \in U_x \subset A$ and $y \in U_y \subset A$ with $U_x \cap U_y = \emptyset$. Since $A$ is open in $X$ so are the subsets $U_x$ and $U_y$. The same argument takes care of the case when $x$ and $y$ are both in $B$ or $C$. This leaves only the situation when $x \in A \setminus C$ and $y \in B \setminus C$. In that case we have $\phi(x) = 1$ and $\phi(y) = -1$ and we can take

$$U_y = \phi^{-1}(-\infty, 0)$$
$$U_x = \phi^{-1}(0, \infty)$$

$\square$

We now come to an important method for constructing smooth manifolds.

**Theorem 1.6** (Gluing). *Suppose that $N$ is a smooth manifold with boundary,*

$$M_1, M_2 \subset \partial N$$

*are unions of components of $\partial N$, which are disjoint ($M_1 \cap M_2 = \emptyset$), and $f : M_1 \to M_2$ is a diffeomorphism. Then the space*

$$N' = N/x \in M_1 \sim f(x) \in M_2$$

*can be given the structure of a smooth manifold in such a way that the maps $N \to N'$ is smooth*

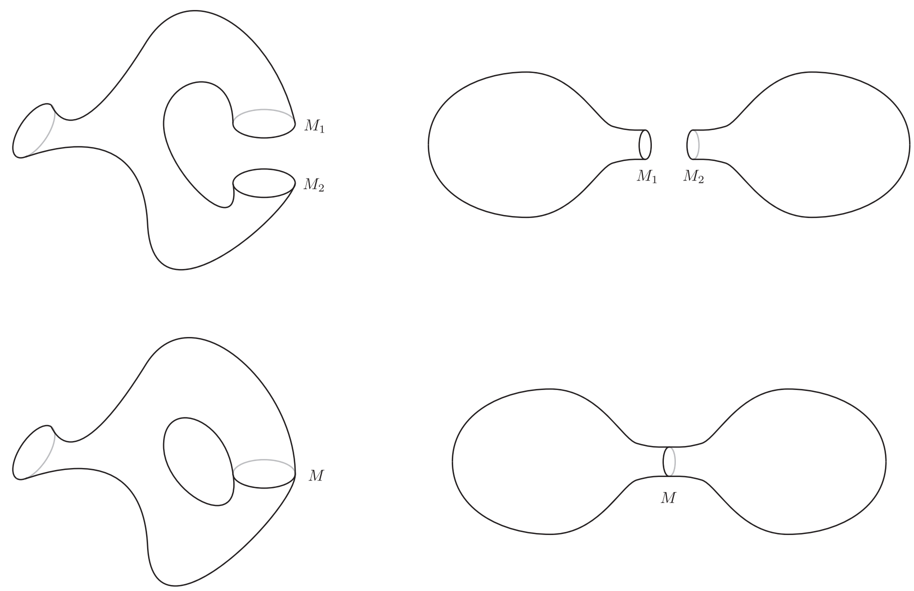



FIGURE 2. Examples of gluing

Figure 2 illustrates two examples of gluing.

*Proof:* Write $M \subset N'$ for the image of $M_1$ (which coincides with the image of $M_2$.) Let $g_1 : C_1 \to M_1 \times [0,1)$ be a collar neighborhood of $M_1$ and $g_2 : C_2 \to M_2 \times (-1,0]$ a collar neighborhood of $M_2$. Let

$$\begin{aligned} A &= C_1 \amalg C_2/x \in M_1 \sim f(x) \in M_2 \subset N' \\ B &= U = N' - M = N - (M_1 \amalg M_2) \\ C &= A \cap B. \end{aligned}$$

The functions $g_1$ and $g_2$ patch together to give a homeomorphism

$$g : A \to M \times (-1,1)$$

which we use to make $A$ into a smooth manifold. We are in a position to be able to use Proposition 1.1 to give $N'$ the structure of a smooth manifold, once we know that $N'$ is Hausdorff. We will do this by using the criterion Lemma 1.3. For this let $\rho : (-1,1) \to \mathbb{R}$ be a bump function taking value 1 at 0 and vanishing outside of $[-1/2,1/2]$. The desired function $\phi$ is constructed by patching together the composition

$$A \xrightarrow{g} M \times (-1,1) \xrightarrow{\text{pr}} (-1,1) \xrightarrow{\rho} \mathbb{R}$$

(the map labeled "pr" is projection to the second factor) and the constant function 0 on $N' - g^{-1}[-1/2,1/2]$. $\square$

The smooth structure given by Theorem 1.6 depends on the choice of collar neighborhoods, and so is not unique. It is, however, unique up to diffeomorphism. As in Theorem 1.6, suppose that $N$ is a smooth manifold with boundary,

$$M_1, M_2 \subset \partial N$$

are unions of components of $\partial N$, which are disjoint, $f : M_1 \to M_2$ is a diffeomorphism

$$N' = N/x \in M_1 \sim f(x) \in M_2.$$

Write $M \subset N'$ for the subspace $M_1 \simeq M_2$, and let $U \subset N'$ be any neighborhood of $M$. The following result is proved later as Theorem IX.4.2.

**Theorem 1.7.** *If $N_1'$ and $N_2'$ are two smooth structures on $N'$ with the properties that the maps $N \to N_1'$ and $N \to N_2'$ are smooth, there is a diffeomorphism $N_1' \to N_2'$ which the identity on $M_1$ and the identity outside of $U$.* □

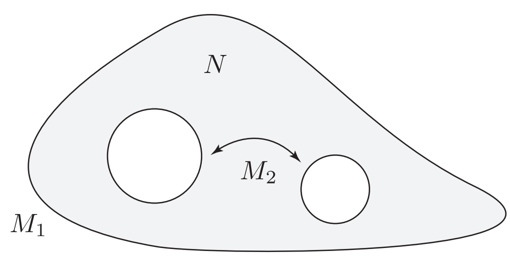
## 2. Cobordism

With the fundamentals of patching and gluing established we now turn to the main constructions of interest. Unless otherwise specified all manifolds will be assumed to be compact.

Suppose that $M_1$ and $M_2$ are two smooth manifolds of dimension $n$.

**Definition 2.1.** A *cobordism* of $M_1$ and $M_2$ is a smooth compact manifold $N$ with boundary, together with a diffeomorphism

$$\iota : M_1 \amalg M_2 \to \partial N.$$




At times will be useful to drop the assumption that $N$, $M_1$ and $M_2$ are compact, in which case we will refer to $(N, \iota)$ as a *non-compact* cobordism.

**Example 2.2.** The empty set is a manifold of dimension $n$ for every $n$. Taking $M_2 = \emptyset$ we see that if $N$ is a manifold with boundary, then $\partial N$ is cobordant to $\emptyset$.

**Example 2.3.** Let $N = M \times [0,1]$, so that $\partial N = M \amalg M$. Taking $M_1 = M_2 = M$ we see that every manifold is cobordant to itself, and in fact that diffeomorphic manifolds are cobordant. Taking $M_1 = M \amalg M$ and $M_2 = \emptyset$ we find that for every $M$, the manifold $M \amalg M$ is cobordant to $\emptyset$.

**Example 2.4.** Suppose $f : M \to W$ is a smooth map, $x, y \in W$ are two regular values, and $\gamma : [0,1] \to W$ is a path between them which is transverse to $f$. Then the transverse pullback

$$\begin{tikzcd}
f^{-1}\gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\pitchfork"] & M \arrow[d, "f"] \\
{[0,1]} \arrow[r, "\gamma"'] & W
\end{tikzcd}$$

given by

$$f^{-1}\gamma = \{(t, m) \in [0,1] \times M \mid \gamma(t) = f(m)\}$$

is a cobordism between $f^{-1}(0)$ and $f^{-1}(1)$

**Example 2.5.** Suppose that $Z \subset W$ is a closed submanifold, $f, g : M \to W$ are smooth maps which are transverse to $Z$. If $f$ and $g$ are homotopic then $f^{-1}Z$ and $g^{-1}Z$ are cobordant. In fact a smooth homotopy $h : M \times [0,1] \to W$ which is transverse to $Z$ gives a cobordism $h^{-1}(Z)$ between them.

**Theorem 2.6.** *Both cobordism and non-compact cobordism are an equivalence relations.*

*Proof:* That cobordism is symmetric is more or less immediate from the definition: given a diffeomorphism $M_1 \amalg M_2 \to \partial N$ representing a cobordism of $M_1$ with $M_2$ we can simply interchange the labels of $M_1$ and $M_2$. The fact that every $M$ is cobordant to itself is part of Example 2.3 above. For transitivity, suppose we are given

$$M_1 \xrightarrow{N_{12}} M_2 \xrightarrow{N_{23}} M_3.$$

Let $N$ be a manifold constructed using the Gluing Theorem 1.6 from $N_{12}$, $N_{23}$ and the given diffeomorphisms of $M_2$. Then $N$ is a cobordism from $M_1$ to $M_3$. $\square$

**Remark 2.7.** Unless otherwise stated we use "cobordism" to refer to compact cobordism.

We now consider the set $MO_n$ of cobordism classes of compact $n$-manifolds without boundary. There is a small set-theoretic point to make here. The collection of all $n$-manifolds does not form a set. However since every $n$-manifold is diffeomorphic to a submanifold of $\mathbb{R}^{2n+1}$ the collection of closed manifolds up to diffeomorphism does form a set. Since diffeomorphic manifolds are cobordant (Example 2.3) this means the collection of cobordism classes of closed $n$-manifolds forms a set.

The disjoint union of manifolds makes $MO_n$ into a commutative monoid with unit. The unit is the empty $n$-manifold. By Example 2.3, for every $M$ one has $M + M = 0$. This means that the monoid $MO_n$ is actually a group, and in fact a vector space over the field $\mathbb{Z}/2$.

**Example 2.8.** A compact 0-manifold is just a finite set of points, and by the classification of 1-manifolds the number of points in the boundary of any compact 1-manifold is even. It follows that $MO_0 = \mathbb{Z}/2$.

**Example 2.9.** Again by the classification of 1-manifolds every compact closed 1-manifold is diffeomorphic to a disjoint union of circles. Since the circle is the boundary of a disk, every compact closed 1-manifold is a boundary and $MO_1 = 0$.

**Example 2.10.** We will see later in this course that $MO_2 = \mathbb{Z}/2$ with generator the cobordism class of the real projective plane $\mathbf{RP}^2$.

In fact all of the vector spaces $MO_n$ have been computed. The Cartesian product of manifolds makes the sum of the groups $\bigoplus MO_n$ into a graded commutative ring, known as the *unoriented cobordism ring*. Amazingly, the structure of this ring is known [**10, 5**]. It is isomorphic to the polynomial ring $\mathbb{Z}/2[x_2, x_4, x_5, x_6, \cdots]$, with $x_n \in MO_n$ and with the subscripts ranging through all $n$ not of the form $2^k - 1$. For $n$ even, the cobordism class $x_n$ contains $\mathbf{RP}^n$. Finding manifolds in $x_n$ when $n$ is odd is a bit more tricky.

## 3. Manifolds over a space

We now turn to a straightforward but extremely useful generalization of these ideas. Let $X$ be a topological space.

**Definition 3.1.** An *$n$-manifold over $X$* is a continuous map $f : M \to X$ in which $M$ is a smooth manifold.

Note that when $X$ is the one point space then a manifold over $X$ is just a manifold (there is a unique map from any space to the one point space).

Everything we have done so far extends to the situation of manifolds over $X$. For instance the *boundary* of $f : M \to X$ is the restriction $\partial f : \partial M \to X$ of $f$ to $\partial M$. A *smooth map* $g/X : f_1 \to f_2$ from $f_1; M_1 \to X$ to $f_2 : M_2 \to X$ is a diagram

$$\begin{tikzcd} M_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & M_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

in which $g$ is a smooth map. Diffeomorphisms of manifolds over $X$ are defined in the evident way.

Suppose that $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are two $n$-manifolds over $X$.

**Definition 3.2.** A *cobordism* of $f_1$ and $f_2$ is an $(n+1)$-manifold $h : N \to X$ over $X$, together with a diffeomorphism

$$\iota/X : f_1 \amalg f_2 \to \partial h.$$

The following result is straightforward.

**Proposition 3.3.** *Cobordism is an equivalence relation on manifolds over $X$.*

**Remark 3.4.** We will frequently use the fact that homotopic maps $M \to X$ are cobordant when regarded as manifolds over $X$ (Exercise 3.1).

**Definition 3.5.** The $n^{\text{th}}$ *bordism homology group* $MO_n(X)$ of $X$ is the set equivalence classes of compact closed $n$-manifolds over $X$.

The disjoint union of manifolds over $X$ makes $MO_n(X)$ into a vector space over $\mathbb{Z}/2$.

**Notation 3.6.** A compact $k$-manifold $M$ over $X$ without boundary is denoted $(M, f)$. The equivalence class of $(M, f)$ in $MO_k(X)$ will be denoted $[M, f]$.

**Example 3.7.** Every continous map $f : S^1 \to \mathbb{R}^n$ extends to a continuous map $g : D^2 \to \mathbb{R}^n$ by setting $g(r, \theta) = rf(\theta)$. From the classification of 1-manifolds this implies that $MO_1(\mathbb{R}^n) = 0$.

To go further it will be helpful to know that when $X$ is a smooth manifold the vector space $MO_n(X)$ can be described in terms of smooth maps. We will need the following result.

**Proposition 3.8.** *Suppose $X$ and $M$ are smooth (not necessarily compact) manifolds and $f : M \to X$ is a continuous function. There is a homotopy $h : M \times [0,1] \to X$ with $h(x, 0) = f(x)$ and $h(x, 1)$ a smooth function of $X$. If $f$ is smooth on a closed subset $K \subset M$ then $h$ can be chosen so that $h(x,t) = f(x)$ for all $x \in K$ and all $0 \le t \le 1$.*

I'll give the proof of this later. It's just a variation on the theorem of Weierstrass approximating continuous functions by polynomial functions.

**Proposition 3.9.** *Suppose $X$ is a smooth manifold. Every manifold $f : M \to X$ over $X$ is cobordant to an $f' : M \to X$ with $f'$ a smooth map. If*

$$f_1 : M_1 \to X$$
$$f_2 : M_2 \to X$$

*are smooth and $h : N \to X$ is a cobordism between them, then there is a smooth map $\tilde{h} : N \to X$ which is a cobordism between them.*

*Proof:* This is more or less immediate from Proposition 3.8. □

Here is one application.

**Proposition 3.10.** *The vector space $MO_1(S^2)$ is the trivial vector space.*

*Proof:* By Proposition 3.9 any cobordism class in $MO_1(S^2)$ contains a smooth map $f : M \to S^2$. Let $x \in S^2$ be a regular value. Since $M$ has dimension 1 the point $x$ is not in the image of $f$. But then $f$ factors through $S^2 - \{x\}$ which is diffeomorphic to $\mathbb{R}^2$. The claim now follows from Example 3.7. □

We now exhibit some non-trivial elements in $MO_n(X)$.

**Proposition 3.11.** *Suppose that $X$ is a smooth manifold and $Z \subset X$ is a compact, closed submanifold. The function associating to a smooth map $f : M \to X$ the mod 2 intersection number $I(f, Z)$ defines a homomorphism*

$$I_Z : MO_n(X) \to \mathbb{Z}/2.$$

**Example 3.12.** Suppose that $X$ is a compact closed manifold of dimension $n$. The identity map of $X$ defines an element $[X] \in MO_n(X)$. Let $Z$ be a point of $X$. The function $I_Z : MO_n(X) \to \mathbb{Z}/2$ sends $[X]$ to 1. This means that $[X]$ is non-zero, even if $X$ is a boundary. In that case it shows that a smooth manifold never admits a retraction to the boundary. You can think of the fact that $[X]$ is non-zero as a generalization of that fact.

### Exercises

**3.1.** Suppose that $f_0 : M \to X$ and $f_1 : M \to X$ are homotopic maps. Show that $f_0$ and $f_1$ are cobordant when regarded as manifolds over $X$.

**3.2.** Suppose that $g : X \to Y$ is a continuous map of topological spaces. Show that the map sending $f : M \to X$ to $g \circ f : M \to Y$ induces a linear transformation $MO_n(X) \to MO_n(Y)$.

**3.3.** Suppose that $M$ is a manifold. Show that every continuous function $f : \partial M \to \mathbb{R}$ extends to a continuous function $g : M \to \mathbb{R}$. Using this generalize Example 3.7 to show that for every $k$, the map

$$MO_k(\mathbb{R}^n) \to MO_k(\text{pt}) = MO_k$$

is an isomorphism. (Hint: Use a collar neighborhood).

**3.4.** Generalize Proposition 3.10 to show that if $k < n$ then the map $MO_k(S^n) \to MO_k(\mathrm{pt})$ is an isomorphism.

**3.5.** Prove Proposition 3.11.

**3.6.** This is a slight generalization of Exercise 3.5. Suppose that $X$, $Y$, and $Z$ are smooth manifolds without boundary. For this exercise we assume that $X$ and $Z$ are compact, but $Y$ need not be. Suppose also that

$$\dim X + \dim Z = \dim Y.$$

Given smooth maps

$$\begin{aligned} f &: X \to Y \\ g &: Z \to Y \end{aligned}$$

we have defined the mod 2 intersection number $I_2(f, g)$. Using Propositions 3.8 and 3.9 show that this can be extended to a symmetric bilinear map

$$MO_{\dim X}(Y) \times MO_{\dim Z}(Y) \to \mathbb{Z}/2.$$

This is called the *mod* 2 *intersection form* of $Y$.

**3.7.** As in Exercise 3.6, suppose that $X$, $Y$, and $Z$ are smooth manifolds without boundary and that $Y$ is not necessarily compact. Given transverse maps

$$\begin{aligned} f &: X \to Y \\ g &: Z \to Y \end{aligned}$$

form the transverse pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\pitchfork"] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd},$$

so that $W = \{(x, y) \in X \times Z \mid f(x) = g(y)\}$. From the composition in the diagram we may regard $W$ as a smooth manifold over $Y$. Using Propositions 3.8 and 3.9 show that this construction induces a bilinear map

$$MO_{n-k}(Y) \times MO_{n-\ell}(Y) \to MO_{n-(k+\ell)}(Y).$$

This is called the *intersection product* on the bordism homology of $M$. How is this related to the intersection form?

CHAPTER VIII

# Surgery

## 1. Surgery

**1.1. Surgery and cobordism.** To go further we need to be able to construct some cobordisms. The technique for doing so is called *surgery*. Suppose that $M$ is a manifold and $A \subset M$ is a closed submanifold of the interior of $M$ of codimension $k$. The technique of surgery cuts out a neighborhood of a submanifold of $A \subset M$ and replaces it with a new manifold. To perform surgery requires two pieces of data: a *framing* and a *bounding manifold*.

**Definition 1.1.** Suppose that $M$ is a manifold and $A \subset M$ is a submanifold of of codimension $k$. A *framing* of $A$ in $M$ is a neighborhood $U$ of $A$ and a diffeomorphism

$$\Phi = (r, \phi) : U \to A \times \mathbb{R}^k$$

under which the inclusion of $A$ in $U$ corresponds to $a \mapsto (a, 0)$.

**Remark 1.2.** It will make the discussion simpler at times to drop $\Phi$ from the notation and think of $A \times \mathbb{R}^k$ as an open sub-manifold of $M$.

**Definition 1.3.** Suppose that $A$ is a manifold. A *bounding manifold* of $A$ is a manifold $B$ and a diffeomorphism $\partial B \approx A$.


FIGURE 1. Surgery data

With a choice of framing and bounding manifold in hand, surgery can be performed. Suppose now that $A$ is a subset of the interior of $M$. The subspace $M_0 = M - A \times \mathrm{int}(D^k)$ is a manifold with boundary, and $\partial M_0 = \partial M \amalg A \times S^{k-1}$ (this follows easily from the observation that $M_0 \cap U = \{x \in U \mid |\phi(x)| \geq 1\}$ and the fact that 1 is a regular value). On the other hand, from our bounding manifold $B$ we have a diffeomorphism

$$\partial B \times S^{k-1} \approx A \times S^{k-1},$$

so we can use gluing to create a new manifold

$$M' = M_0 \underset{A \times S^{k-1}}{\cup} B \times S^{k-1}.$$

One says that $M'$ is constructed from $M$ by *surgery*.

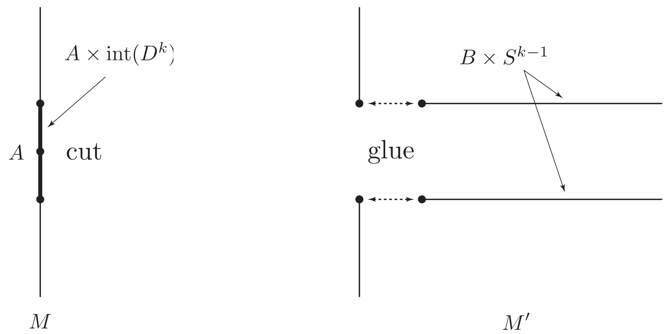



FIGURE 2. Surgery

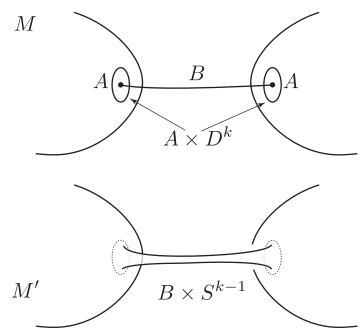



FIGURE 3. Another picture of surgery

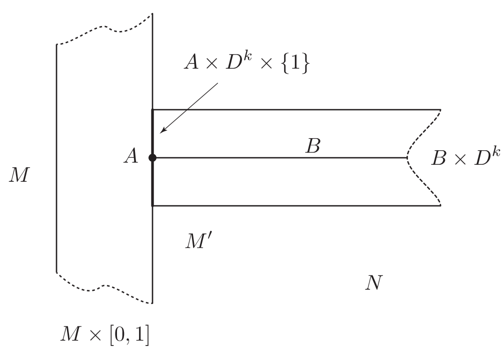



FIGURE 4. The surgery cobordism

To specify the smooth structure on $M'$, requires a choice of a collar neighborhood $A \times [0, \infty) \subset B$ and a collar neighborhood of $A \times S^{k-1}$ in $M_0$. The latter may be easily constructed from the diffeomorphism $\Phi : U \to A \times \mathbb{R}^k$ with $A \times \mathbb{R}^k$.

It's not quite obvious at first, but in fact there is a natural cobordism between $M$ and $M'$ when $M$ is closed. Let $N$ be the topological space constructed from the disjoint union of

$$M \times [0, 1] \quad \text{and} \quad B \times D^k$$

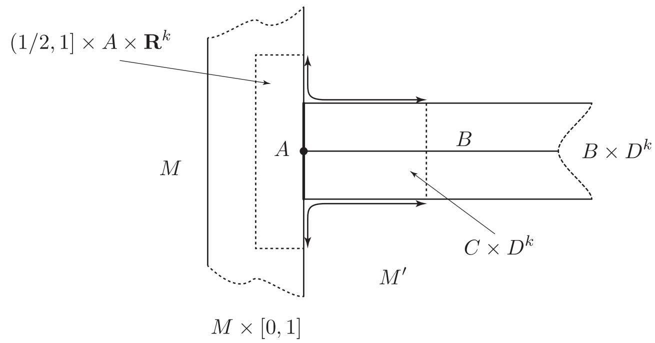



FIGURE 5. Smooth structure on the surgery cobordism

by identifying the copy of $A \times D^k$ in $B \times D^k$ with the one in $M \times \{1\}$. We will use pasting to give $N$ the structure of a smooth manifold provided with an identification

$$\partial N = M \amalg M'$$

(see Figures 4 and 5).

We can write $N$ as the union of the open sets.

$$S = N - A \times D^k \times \{1\}$$
$$T = (1/2,1] \times A \times \mathbb{R}^k \cup C \times D^k,$$

in which the open subset $C \subset B$ is a collar neighborhood of $A$, equipped with a chosen diffeomorphism $C \to A \times [0,\infty)$. The space $S$ is the disjoint union of $M \times [0,1] - D^k \times A \times \{1\}$, which is an open subset of $M \times [0,1]$, and $(B - A) \times D^k$, which is an open subset of $B \times D^k$. We give these the corresponding smooth manifold structures. In terms of the collar structure $C = A \times [0,\infty)$ we can write

$$T = (1/2,1] \times A \times \mathbb{R}^k \bigcup_{A \times D^k} A \times D^k \times [0,\infty)$$

in which $(a,v) \in A \times D^k$ is identified with $(1,a,v)$ in $[0,1] \times A \times \mathbb{R}^k$ and with $(a,v,0) \in A \times D^k \times [0,\infty)$. This space is the product $T' \times A$, where

$$T' = (1/2,1] \times \mathbb{R}^k \bigcup_{D^k} D^k \times [0,\infty)$$

which we may identify with the subspace

$$R = \{(v,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \leq 0 \text{ or } |v| \leq 1\} \subset \mathbb{R}^n \times \mathbb{R}.$$

Our gluing construction makes use of the specific homeomorphism of the boundary

$$\mathbb{R}^k - \operatorname{int}(D^k) \cup S^k \times [0,\infty)$$

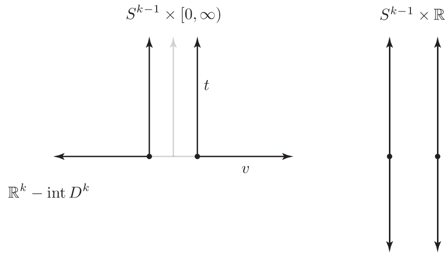
of this region with $S^k \times \mathbb{R}$ given by

$$(v,t) \mapsto \begin{cases} (v,t) & t > 0 \\ (v/|v|, -|v|) & t = 0. \end{cases}$$

Now this homeomorphism extends to a homeomorphism

$$R \to \{(v,t) \mid v \leq 1\} = D^k \times \mathbb{R}$$

which is smooth at all interior points. Crossing with the identity map of $A$ gives the desired smooth structure.




The choice of framing is important for surgery, and it can be useful to know how to construct one.

**Proposition 1.4.** *Suppose that $M$ is smooth manifold and $A \subset M$ is a closed submanifold. Suppose that $V \subset M$ is a neighborhood of $A$ and $f : V \to \mathbb{R}^k$ is a smooth function for which $0$ is a regular value and $f^{-1}(0) = A$. There is an $\epsilon < 0$, a neighborhood $U \subset V$ of $A$ and a retraction $r : U \to A$ having the property that the map*

$$U \xrightarrow{(r,f)} A \times \mathbb{R}^k$$

*is a diffeomorphism of $U$ with $A \times B_\epsilon$.*

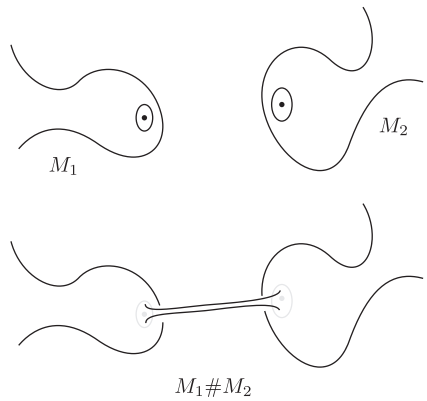
Composing with a standard diffeomorphism $B_\epsilon \to \mathbb{R}^k$ then gives a framing of $A$ in $M$.

*Proof:* By the $\varepsilon$-neighborhood theorem there is a neighborhood $U' \subset V$ of $A$ and a retraction $r : U' \to A$. The derivative of the map

$$U' \xrightarrow{(r,f)} A \times \mathbb{R}^k$$

is an isomorphism at all points of $A$, so by the inverse function theorem it is a diffeomorphism of a neighborhood $U'' \subset U'$ of $A$ with a neighborhood of $A \times \{0\}$ in $A \times \mathbb{R}^k$. Choose $\epsilon > 0$ so that $A \times B_\epsilon$ is contained in this neighborhood, and let $U \subset U''$ be the subspace corresponding to $A \times B_\epsilon$. $\square$

**Example 1.5.** Suppose that $M_1$ and $M_2$ are $k$-dimensional manifolds and $x \in M_1$ and $y \in M_2$ are points. Choose coordinate neighborhoods $\Phi_x : U_x \to \mathbb{R}^k$ and $\Phi_y : U_y \to \mathbb{R}^k$. Take $A = \{x, y\}$, and for bounding manifold $B = [0, 1]$ with $\partial B \to A$ the map sending $0$ to $x$ and $1$ to $y$. This provides the surgery data to do surgery on $M = M_1 \amalg M_2$. The resulting manifold $M'$ is the *connected sum* of $M_1$ and $M_2$ and often denoted $M_1 \# M_2$.




**Remark 1.6.** Different choices of framing can lead to different manifolds after surgery. See Exercise 1.1.

**1.2. Surgery over a space.** The construction of cobordisms via surgery extends in the evident manner to the case of manifolds over a space $X$. Suppose we are given a manifold $f : M \to X$ over $X$, a framed submanifold $A \subset M$ and a bounding manifold $g : B \to X$ of $f|_A : A \to X$. We suppose in addition that we have specified a *thickening*

$$\tilde{g} : B \times D^k \to X$$

extending the restriction of $f$ to $A \times D^k$. We then define

$$h : N \to X$$

to be the continuous map whose restriction to $M \times [0,1]$ sends $(m,t)$ to $f(m)$ and whose restriction to $B \times D^k$ is $g$. The map $h$ is a cobordism between $(M, f)$ and $(M', f')$ where $M' = L \underset{A\times S^{k-1}}{B} \times S^{k-1}$ and $f'$ is composed of the restriction of $f$ to $L = M - A \times \operatorname{int}(D^k)$ and the restriction of $g$ to $B \times S^{k-1}$.

While it is often important to be able to choose the thickening $B \times D^k \to X$ it is also useful to know that one can be constructed directly from the bounding map $B \to X$. In other words, surgery can be performed on a framed $A \subset M$ over $X$ and a choice of bounding manifold $g : B \to X$. The smooth structure on $M'$ depends on a further choice (the collar of $A$ in $B$), which does not change the map to $X$.

**Lemma 1.7.** *Suppose that $B$ is manifold with compact boundary $A$, and $k \geq 0$ is an integer. The inclusion*

$$A \times D^k \underset{A\times\{0\}}{\cup} B \to B \times D^k$$

*admits a retraction*

$$B \times D^k \to A \times D^k \underset{A\times\{0\}}{\cup} B.$$

*Proof:* Let's first consider the case in which $B = A \times [0,\infty)$. In this case $B \times D^k = A \times [0,\infty) \times D^k$ and

$$A \times D^k \underset{A\times\{0\}}{\cup} B$$

is the product of $A$ with $D^k \underset{0}{\cup} [0,\infty)$. In this case we take the retraction to be the product of the identity map of $A$ with any retraction of $r : D^k \times [0,\infty) \to D^k \cup [0,\infty)$ having the property

$$r(v,t) = t \quad \text{if } t \geq 2. \tag{1.8}$$

For instance the map given by projection from $(2,2)$ in the $|v|t$ plane:

$$r(t,v) = \begin{cases} t & t \geq 2 \\ 2 - 2\frac{(t-2)}{(|v|-2)} & |v| \leq t \\ 2 - 2\frac{(|v|-2)}{(t-2)}\, v & |v| \geq t \end{cases}$$

For the general case let $C_1 = A \times [0,\infty) \subset B$ be a collar neighborhood of $A$ in $B$ and $C_2 \subset B$ the complement in $B$ of the closed subspace corresponding to $A \times [0,2]$. On $C_1 \times D^k$ we use the retraction defined above. On $C_2 \times D^k$ we use the projection to $D^k$. These maps agree on $C_1 \cap C_2 \times D^k$ by virtue of the specification (1.8). □

By Lemma 1.7 the thickening map $B \times D^k \to X$ can be taken to be the composition of the retraction

$$B \times D^k \to A \times D^k \underset{A\times\{0\}}{\cup} B$$

with the map $g : B \to X$. We will call this canonical thickening map the *retractive thickening map.*

**Exercises**

**1.1.** Surgery depends on the framing of $A$ in $M$. Let $M = S^2$, $A$ the set consisting of the north and south poles, $B = [0,1]$ and identify $\partial B$ with $A$ by sending 0 to the north pole and 1 to the south pole. Find two different framings of $A$, one providing a surgery giving the torus and the other the Klein bottle.

## 2. A moving lemma

We now give an application of surgery. Let's consider the following situation. Suppose that $X$ is a smooth 2-manifold and $Z \subset X$ is an embedded, connected 1-manifold. For intersection theory to work we don't need to assume that $Z$ is compact, but we do need to assume that the map $Z \to X$ is proper.

**Theorem 2.1** (Moving Lemma). *Any* 1*-manifold $f : M \to X$ over $X$ is cobordant one which is transverse to $Z$ and has the property that*

$$\#\{f^{-1}(Z)\} < 2.$$

The moving lemma actually holds more generally when $X$ has dimension $n$, $Z$ is connected of dimension $k$ and $M$ has dimension $(n-k)$. In Exercise 2.1 below you will be asked to prove the special case when $Z$ consists of a single point.

*Proof:* Moving $f$ by a homotopy, we may suppose that $f$ is transverse to $Z$. Also, we may suppose that the number of points in $f^{-1}(Z)$ is at least 2, otherwise there is nothing to prove. Now consider the special case in which $X = \mathbb{R}^2$ and $Z$ is the $x$-axis. Write $f = (f_1, f_2)$. Suppose that $p \neq q$ are points of $M$ with $f(p)$ and

$f(q)$ in $\mathbb{R}$. Since $f$ is transverse to the $x$-axis the map $f_2$ is a diffeomorphism in a neighborhood of $V_p$ of $p$ and $V_q$ of $q$. Choose coordinate neighborhoods

$$\Phi_p : U_p \to \mathbb{R}$$
$$\Phi_q : U_q \to \mathbb{R}$$

with the properties that

$$\Phi_p^{-1}([-1,1]) \subset V_p \quad \text{and}$$
$$\Phi_q^{-1}([-1,1]) \subset V_q.$$

To ease the notation a bit, write

$$g_p = f_2 \circ \Phi_p^{-1} : [-1,1] \to \mathbb{R}$$
$$g_q = f_2 \circ \Phi_q^{-1} : [-1,1] \to \mathbb{R}$$

Changing the sign of $\Phi_p$ and $\Phi_q$ if necessary, we may assume that both $g_p(1)$ and $g_q(1)$ are positive. Since $g_p$ and $g_q$ are diffeomorphisms with their image, and $g_p(0) = g_q(0) = 0$ this implies that both $g_p(-1)$ and $g_p(-1)$ are negative.

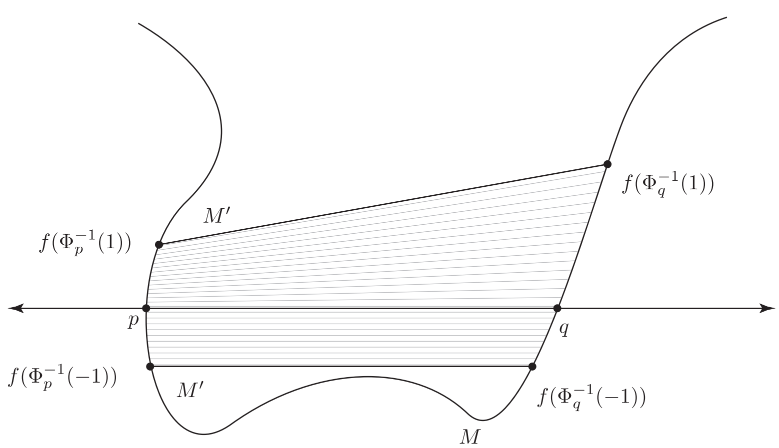
We now set up our surgery data. We take $A = \{p, q\}$ and use $\Phi_p$ and $\Phi_q$ to provide a framing. We take $B = [0,1]$ with bounding map $g : B \to \mathbb{R}^2$ given by $g(s) = (1-s)f(p) + sf(q)$. We define the thickening

$$h : B \times [-1,1] \to \mathbb{R}^2$$

by

$$h(s,t) = (1-s)\,\Phi_p^{-1}(t) + s\,\Phi_q^{-1}(t).$$

By construction, the restriction of $h$ to $B \times \{-1,1\}$ does not intersect the $x$-axis. Doing surgery on this data then produces a map $f' : M' \to \mathbb{R}^2$ meeting the $x$-axis in two fewer points.



Now we reduce the general case to this one. Since $Z \cap f(M)$ is compact we may find, by Proposition 2.3 below, an open neighborhood $U$ of $Z \cap f(M)$ and a diffeomorphism $U \to \mathbb{R}^2$ under which $Z \cap U$ corresponds to the $x$-axis. Now set up the surgery data for $f^{-1}(U) \to U$ as before, and use this data to perform surgery on $f : M \to X$. $\square$

**Remark 2.2.** There's a useful principle embedded in the above proof. While cobordism involves the entire manifold, the surgery data is local. This often lets one reduce surgery to something in Euclidian space.

We have used.

**Proposition 2.3.** *Suppose that $X$ is a manifold of dimension $2$, $Z \subset X$ is a smooth connected $1$-manifold, and $S \subset Z$ a proper compact subset. There is a neighborhood $U$ of $S$ in $X$ and a diffeomorphism $\Phi : U \to \mathbb{R}^2$ under which $Z \cap U$ corresponds to the $x$-axis.*

The proof of Proposition 2.3 requires a little foundation.

**Lemma 2.4.** *Suppose that $Z$ is a connected $1$-manifold and $S \subset Z$ is a compact proper subset. There exists a $1$-dimension compact connected submanifold $K \subset Z$ containing $S$ in its interior and which is diffeomorphic to $[0, 1]$.*

*Proof:* If $Z$ is diffeomorphic to $[0, 1]$ there is nothing to prove. Since $S$ is compact, and not all of $Z$ we can find a small open interval in $Z$ disjoint from $S$. If $Z$ is diffeomorphic to $S^1$ we can take the submanifold $K$ to be the complement of this interval. This takes care of the cases in which $Z$ is compact.

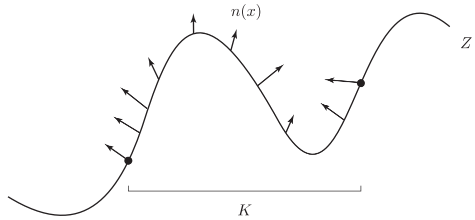
Since $Z$ is connected we may find paths in $Z$ joining all of the components of $S$. The union of the images of these paths is a connected compact subset $W \subset Z$ containing $S$. Since $Z$ is not compact, $W$ is a proper subset of $Z$. This means we might as well suppose that $S$ is connected. Now choose a proper smooth function $f : Z \to \mathbb{R}$. Since $S$ is compact, so is $f(S)$. Choose regular values $a < b$ with $f(S) \subset [a, b]$. Then $f^{-1}([a, b])$ is a submanifold of $Z$ containing $S$. Replacing $Z$ with the connected component of $f^{-1}([a, b])$ containing $S$ reduces us to the compact case. $\square$

**Lemma 2.5.** *Suppose that $X$ is a $2$-manifold, $Z \subset X$ a submanifold of dimension $1$, and $K \subset Z$ a submanifold diffeomorphic to $[0, 1]$. There exists and open neighborhood $U$ of $K$, and a smooth function $f : U \to \mathbb{R}$ having $0$ as a regular value and with $K \subset f^{-1}(0)$.*

*Proof:* Let $\gamma : [0, 1] \to K$ be a diffeomorphism. We first construct a lift

$$\begin{tikzcd} & TX \arrow[d] \\ K \arrow[ur, "n"] \arrow[r] & X \end{tikzcd}$$

having the property that for each $x \in K$, the vector $n(x)$ spans $T_xX/T_xZ$.




To do this choose a finite collection of points $t_0 = 0 < t_1 < \cdots < t_n = 1$ with the property that for each $i$, $\gamma([t_i, t_{i+1}])$ is contained in a neighborhood $U_i$ equipped with a diffeomorphism $\Phi_i : U_i \to \mathbb{R}^2$ under which $Z \cap U$ corresponds the $x$-axis. Suppose we have defined $n(t)$ for $t \in [t_{i-1}, t_i]$ and write

$$d\Phi_i(n(p)) = (a, b) \in \mathbb{R}^2 = T_{\Phi(p)}\mathbb{R}^2$$
$$p = \eta(\gamma(t_i)).$$

Note that $b$ is non-zero since $\eta(\gamma(t_i))$ is non-zero in $TX/TZ$. We extend the definition of $\eta(t)$ to $t \in [t_i, t_{i+1}]$ by setting $\eta(s)$ to be the unique vector with $d\Phi_{i+1}(\eta(\gamma(s))) = d\Phi_{i+1}(a, b)$. this needs to refer to $d\Phi_{i+1}$

Having defined $\eta$ we now let $\phi_i : U_i \to \mathbb{R}$ be the composition

$$U_i \xrightarrow{\Phi_i} \mathbb{R}^2 \xrightarrow{\pm y} \mathbb{R}$$

where the sign is chosen so that for all $t \in [t_i, t_{i+1}]$ $d\phi_i(\eta(\gamma(t)))$ is positive (the sign is constant since $[t_i, t_{i+1}]$ is connected, $\eta$ is continuous, and $d\phi_i(\eta(t))$ is never 0). Let $\{\theta_i\}$ be a partition of unity subordinate to the cover $\{U_i\}$. Then we may take

$$f = \sum \theta_i \phi(i).$$

$\square$

*Proof of Proposition 2.3:* By Lemma 2.4 We may find a compact connected $K \subset Z$ containing $S$ in its interior and a diffeomorphism of $K$ with $[0, 1]$. Lemma 2.5 then provides a neighborhood $U$ of $K$ and smooth function $f : U \to \mathbb{R}$ having 0 as a regular value and with $K \subset f^{-1}(0)$. By the Neighborhood retract theorem, there is a neighborhood $W$ of $Z$ and a retraction $r : W \to Z$. Let $L = r^{-1}(K)$. Then $L$ is a 2-manifold with boundary, containing $K$ as a retract. The derivative of

$$F : L \to K \times \mathbb{R}$$
$$F(x) = (f(x), r(x))$$

is an isomorphism at all points of $K \subset L$ and so restricts to a diffeomorphism of a neighborhood of $K$ in $L$ with a neighborhood of $K \times \{0\}$ in $K \times \mathbb{R}$. Restricting to a smaller open if necessary we may assume that $F$ restricts to a diffeomorphism of a neighborhood $V$ of $K$ in $L$ with $K \times (-a, a) \subset K \times \mathbb{R}$ for some $a$. The same $F$ then restricts to a diffeomorphism of

$$U = V - r^{-1}(\partial K)$$

with $\operatorname{int} K \times (-a, a)$. Now just use a diffeomorphisms of $\operatorname{int} K$ and $(-a, a)$ with $\mathbb{R}$ to construct the desired $f$. $\square$

**Exercises**

**2.1.** Suppose that $X$ is a smooth $n$-manifold and $Z \subset X$ consists of a single point $p \in X$. Show that if $f : M \to X$ is any closed $n$-manifold over $X$ and $I(f, Z) = 0$ then $f$ is cobordant to a map $f' : M' \to X$ whose image does not contain $p$.

**2.2.** In Exercise VII.3.4 you determined $MO_k(S^n)$ in terms of $MO_k(\mathrm{pt})$ for $k < n$. We now consider the case $k = n$, for $n > 0$.

(a) Show that if $X$ is any non-empty space, the map $MO_n(X) \to MO_n(\mathrm{pt})$ is surjective.

(b) Let $Z \subset S^n$ consist of a single point $p$. Show that the map

$$I(Z, -\,) : MO_n(S^n) \to \mathbb{Z}/2$$

restricts to an isomorphism of the kernel of the "augmentation" map

$$MO_n(S^n) \to MO_n(\mathrm{pt})$$

with $\mathbb{Z}/2$. Conclude that the map

$$MO_n(S^n) \to MO_n(\mathrm{pt}) \oplus \mathbb{Z}/2$$

whose components are the two maps above is an isomorphism.

(c) Show that the kernel of the augmentation map is generated by the identity map of $S^n$. (Since $MO_1(\mathrm{pt}) = 0$ this implies that $MO_1 S^1 = \mathbb{Z}/2$ generated by the identity map of $S^1$.)

**2.3.** Suppose that $g : S^n \to S^n$ is a diffeomorphism. Show that for any $f : S^n \to X$ the cobordism classes of $f$ and of $f \circ g$ are the same. Among other things, this means that the cobordism class of an embedded circle is independent of the parameterization.

CHAPTER IX

# Computations

## 1. Computations

Armed with the moving lemma we can now make some computations. We have already shown that $MO_1(S^2) = 0$. Now let $X = S^1 \times \mathbb{R}$, set $Z = \{p\} \times \mathbb{R} \subset X$, where $p = (0,1) \in S^1$, and define $f : S^1 \to X$ by $f(t) = (t, 0)$.

**Proposition 1.1.** *The vector space $MO_1(S^1 \times \mathbb{R})$ has dimension 1 and is generated by the cobordism class of $f$.*

*Proof:* Consider the map

$$MO_1(S^1 \times \mathbb{R}) \to \mathbb{Z}/2$$

sending $g : M \to X$ to $I(g, Z)$. This map sends $f$ to 1 so it is surjective. Suppose that $I(g, Z) = 0$. By the moving lemma $g$ is cobordant to a map $\tilde{g} : \tilde{M} \to X$ whose image does not meet $Z$. This shows that $\tilde{g}$ is in the image of $MO_1(X - Z)$. But $X - Z$ is diffeomorphic to $\mathbb{R}^2$ so $\tilde{g} = 0$ since $MO_1(\mathbb{R}^2) = 0$. $\square$

Now consider $X = S^1 \times S^1$ and define

$$Z_1 = S^1 \times \{p\} \subset S^1 \times S^1$$
$$Z_2 = \{p\} \times S^1 \subset S^1 \times S^1.$$

**Proposition 1.2.** *The map $MO_1 S^1 \times S^1 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ given by*

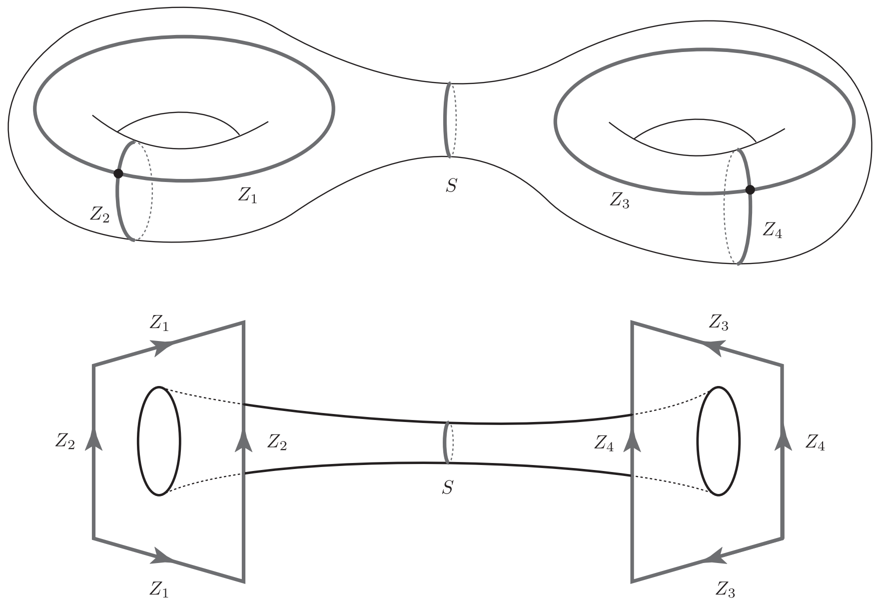
$$f \mapsto (I(f, Z_1), I(f, Z_2))$$

*is an isomorphism.*

*Proof:* Under the above map, $Z_1$ goes to $(0,1)$ and $Z_2$ goes to $(1,0)$ so the map is surjective. It suffices to show that if $f : M \to X$ is a 1-manifold over $X$ with $I(f, Z_1) = I(f, Z_2) = 0$ then $f = 0 \in MO_1(X)$. By the moving lemma we may suppose that $f(M) \cap Z_1 = \emptyset$ and that $f$ is transverse to $Z_2$. Then $f$ factors through $(S^1 - \{p\}) \times S^1$ and $I(f, Z_2 - \{p\}) = 0$. That $f = 0$ then follows from proof of Proposition 1.1. $\square$

### Exercises

**1.1.** Let $X$ be the connected sum of $S^1 \times S^1$ with itself and consider the picture below.




Show that the map

$$MO_1(X) \to (\mathbb{Z}/2)^4$$

sending $f : M \to X$ to

$$\big(I(f, Z_1), I(f, Z_2), I(f, Z_3), I(f, Z_4)\big)$$

is an isomorphism. (Hint: Show that the map is surjective. Next show that $S = 0$. Show that if $f$ is in the kernel then $f$ factors through $X - Z_1$ and from there through $X - Z_1 - Z_2$ and on through $U = X - Z_1 - Z_2 - Z_3 - Z_4$. This last manifold $U$ is a cylinder and $MO_1(U) = \mathbb{Z}/2$ generated by $S$. So $f$ is either zero or $S$. But it doesn't matter since $S = 0$ in $MO_1(X)$.)

**1.2.** Suppose that $V_1$, $V_2$, and $V_3$ are vector spaces over a field $k$. A sequence of linear transformations

$$V_1 \xrightarrow{p} V_2 \xrightarrow{q} V_3$$

is *exact* if the image of $p$ is equal to the kernel of $q$. Note that this implies that the composition $q \circ p$ is zero. Show that in this situation one has $\dim V_2 \leq \dim V_1 + \dim V_3$.

**1.3.** Suppose that $X$ is a smooth 2-manifold and $Z \subset X$ is an embedding of 1 manifold (recall that this means that the map $Z \to X$ is proper). Define maps

$$p : MO_1(X \setminus Z) \to MO_1(X)$$
$$q : MO_1(X) \to \mathbb{Z}/2$$

by

$$p([M, f]) = [M, i \circ f]$$
$$q([M, f]) = I_2(f, Z),$$

where $i : X \setminus Z \to X$ is the inclusion. Show that the sequence

$$MO_2(X \setminus Z) \xrightarrow{p} MO_1(X) \xrightarrow{q} \mathbb{Z}/2$$

is exact. Can you find an example in which $MO_1(X) \neq 0$ and the map $q$ is the zero map?

**1.4.** Suppose that $X$ is a smooth 2-manifold and that there is a chain of open set

$$\mathbb{R}^n = U_n \subset U_{n-1} \subset \cdots \subset U_1 \subset U_0 = X$$

having the property that for all $U_{k+1} = U_k \setminus Z_k$ where $Z_k \subset U_k$ is an embedding of a connected 1-manifold without boundary (so, in particular proper). Show in this case that $\dim MO_1(X) \leq n$ .

## 2. The neighborhood retract theorem

We are making frequent use of something called the $\varepsilon$-neighborhood theorem, but which I'm often calling the "neighborhood retract theorem." Just for clarity I'll describe it here.

**Definition 2.1.** Suppose $X$ is a manifold. An *embedded submanifold* is a submanifold $M \subset X$ having the property that the inclusion map is proper and transverse to $\partial X$.

**Theorem 2.2** (Neighborhood Retract Theorem)**.** *Suppose that $X$ is a manifold and $Z \subset X$ is an embedded submanifold. There exists an open neighborhood $U$ of $Z$ and a retraction $r : U \to Z$.*

A *retraction* is a mapping $r$ satisfying $r(x) = x$ for $x \in Z$.

*Proof:* First observe that if $X \subset Y$ is an embedded submanifold of a bigger manifold $Y$ and $Z$ is a retract of a neighborhood $U \subset Y$ then $Z$ is a retract of $U \cap X$. By embedding $X$ in a half space $H^n$ we can reduce to the case $X = H^n$. For this case let $NZ$ be the normal bundle of $Z$ in $H^n$ and

$$g : NZ \to H^n$$

the map sending $(x, v)$ to $x + v$. The derivative of $g$ map is an isomorphism at all points $(x, 0) \in Z \subset NX$ and so by the inverse function theorem is a diffeomorphism in a neighborhood of each point. This means that for every compact subset $K \subset M$ there is an $\epsilon > 0$ such that the restriction of $g$ to

$$N_\epsilon K = \{(x, v) \in NZ \mid x \in K, |v| < \epsilon\}$$

is one to one. Choose a proper map $f : H^n \to \mathbb{R}$, and for each $i \in \mathbb{Z}$ a number $\epsilon_i > 0$ for which the restriction of $g$ to

$$N_{\epsilon_i} f^{-1}([i, i + 1])$$

is one to one. Let $\epsilon : \mathbb{R} \to \mathbb{R}$ be any smooth function having the property that for all $i$ and all $t \in [i, i + 1]$, the inequality

$$\epsilon(t) < \epsilon_i$$

holds. Set.

$$N_\epsilon Z = \{(x, v) \in NZ \mid |v| < \epsilon(f(x))\}.$$

One easily checks that the restriction of $g$ to $N_\epsilon Z$ is a diffeomorphism with an open neighborhood $U$ of $Z$ in $X$. The retraction $r$ is then the composition of $g^{-1}$ with the map $N_\epsilon Z \to Z$ sending $(x, v)$ to $x$. □

Since $N_\epsilon X$ is diffeomorphic to $NX$ (the $\epsilon$-ball in $\mathbb{R}^n$ is diffeomorphic to all of $\mathbb{R}^n$) the proof of the above result actually gives a little more.

**Theorem 2.3** (Tubular Neighborhood Theorem). *Suppose that $X$ is a manifold and $Z \subset X$ is an embedded submanifold with normal bundle $N$. There exists an open neighborhood $U$ of $Z$ and a diffeomorphism $U \to N$ under which the inclusion $Z \to U$ corresponds to the zero section.* □

The *zero section* is the map $Z \to N$ sending $x \in Z$ to $(x, 0)$.

## 3. An orphaned subsection

**3.1. The intersection form.** Now suppose that $X$ is a 2-manifold. The mod 2-intersection number of 1-manifolds over $X$ gives a symmetric bilinear form $I$ on $MO_1X$. This is the *intersection form* of $X$. When $X$ is compact this form turns out to be finite dimensional and non-degenerate. The non-degeneracy will be established below (???).

## 4. Gluing redux

To finish cleaning up some technical loose ends we need to return to the situation of Theorem VII.1.6, and state a slightly improved gluing result. We suppose that $N$ is a smooth $n$-manifold with boundary,

$$M_1, M_2 \subset \partial N$$

are unions of boundary components which are disjoint and

$$f_i : M \to M_i$$

is a diffeomorphism. Write $N' = N/f_1(x) \sim f_2(x)$ for the quotient space. We are interested in giving $N'$ the structure of a smooth manifold for which the map $N \to N'$ is smooth. In Theorem VII.1.6 such a smooth structure was constructed. Now we are interested in the uniqueness.

**Definition 4.1.** Let $W$ be a topological space. The *support* of a homeomorphism $g : W \to W$ (denoted $\operatorname{Supp} g$) is the closure of the set of points $x \in W$ for which $g(x) \neq x$.

**Theorem 4.2.** *In the situation described above, there exists a smooth manifold $X$ and a homeomorphism*

$$N' \to X$$

*having the property that the composition $N \to N' \to X$ is smooth. Given two such homeomorphisms*

$$N' \to X_1$$
$$N' \to X_2$$

*and a neighborhood $M \subset U \subset N'$ there is a diffeomorphism $f : X_1 \to X_2$ and a homeomorphism $g : N' \to N'$, supported in $U$, making the diagram*

$$\begin{tikzcd} N' \arrow[r] \arrow[d, "g"'] & X_1 \arrow[d, "f"] \\ N' \arrow[r] & X_2 \end{tikzcd}$$

*commute.*

The proof of Theorem 4.2 requires the following parameterized version of the smoothing lemma described in the lecture on the classification of 1-manifolds.

**Proposition 4.3.** *Suppose that $M$ is a compact manifold and*

$$h = (\pi, f) : M \times (-r, r) \to M \times \mathbb{R}$$

*is a continuous map which is an open embedding and satisfies the following:*

i) *For all $x \in M$, $h(x, 0) = (x, 0)$;*

ii) *for all $-r < t < r$ the map $\pi(x, t) : M \times \{t\} \to M$ is a diffeomorphism;*

iii) *the restriction of $h$ to $M \times (0, r)$ is a smooth embedding $M \times (0, r) \to M \times (0, \infty)$;*

iv) *The restriction of $h$ to $M \times (-r, 0)$ is a smooth embedding $M \times (-r, 0) \to M \times (-\infty, 0)$;*

*Given $0 < \epsilon < r$ there is a smooth $\tilde{h} : M \times (-r, r) \to M \times \mathbb{R}$ which is a diffeomorphism with the image of $h$, and agrees with $h$ outside of $M \times (-\epsilon, \epsilon)$ and on $M \times \{0\}$.*

*Proof:* We will construct a modification $\tilde{f} : M \times (-r, r) \to \mathbb{R}$ of the map $f$, which is smooth, agrees with $f$ outside of a closed subset of $M \times (-\epsilon, \epsilon)$ and on $M \times \{0\}$ and has the property that $\partial f(x, t)/\partial t > 0$ for all $t \in (-r, r)$. Having done this we can then set $\tilde{h}(x, t) = (\pi(x, t), \tilde{f}(x, t))$. One easily checks that $\tilde{h}$ is a diffeomorphism with its image.

To construct $\tilde{f}$ first note that our conditions imply that $\partial f(x, t)/\partial t > 0$ for all $t \neq 0$. We need to modify $\partial f(x, t)/\partial t$ so that it is smooth near $t = 0$. Choose $0 < a < \epsilon$ and let $\rho : \mathbb{R} \to \mathbb{R}$ be a bump function which is zero for $t \notin [-a, a]$, identically 1 in a neighborhood of 0, and satisfies

$$\int_{-a}^{a} \rho \, dt = 1.$$

Now let

$$\tilde{f}(x, t) = f(x, -a) + \int_{-a}^{t} \left( c(x)\, \rho(s) + \tfrac{\partial}{\partial s} f(x, s)(1 - \rho(s)) \right) ds$$

with

$$c(x) = f(x, a) - f(x, -a) - \int_{-a}^{a} \tfrac{\partial}{\partial s} f(x, s)(1 - \rho(s))\, ds$$

chosen so that $\tilde{f}(x,a) = f(x,a)$. Note that $c(x) > 0$, since,

$$\begin{aligned} c(x) &= f(x,a) - f(x,-a) - \int_{-a}^{a} \tfrac{\partial}{\partial s} f(x,s)(1-\rho(s))\, ds \\ &> f(x,a) - f(x,-a) - \int_{-a}^{a} \tfrac{\partial}{\partial s} f(x,s)\, ds = 0. \end{aligned}$$

For $t > a$ one has

$$\begin{aligned} \tilde{f}(x,t) &= f(a) + \int_{-a}^{a} \big(c(x)\,\rho(s) + \tfrac{\partial}{\partial s} f(x,s)(1-\rho(s))\big)\, ds \\ &\qquad + \int_{a}^{t} \big(c(x)\,\rho(s) + \tfrac{\partial}{\partial s} f(x,s))(1-\rho(s))\big)\, ds \\ &= f(x,a) + \int_{a}^{t} \tfrac{\partial}{\partial s} f(x,s)\, ds \\ &= f(x,t), \end{aligned}$$

while for $t < -a$

$$\begin{aligned} \tilde{f}(x,t) &= f(x,-a) + \int_{-a}^{t} \big(c\,\rho(s) + \tfrac{\partial}{\partial s} f(x,s)(1-\rho(s))\big)\, ds \\ &= f(x,-a) + \int_{-a}^{t} \tfrac{\partial}{\partial s} f(x,s)\, ds \\ &= f(x,-a) + \int_{-a}^{t} \tfrac{\partial}{\partial s} f(x,s)\, ds \\ &= f(x,s). \end{aligned}$$

Note also that near $t = 0$ where $\rho(t) = 1$ we have

$$\tfrac{\partial}{\partial t} f(x,t) = c(x) > 0$$

so that $\tilde{f}$ is smooth everywhere and satisfies $\tfrac{\partial}{\partial t}\tilde{f}(x,t) > 0$ for all $t$. $\square$

In preparation for the proof of Theorem 4.2 we isolate the data that needs to be specified in order to give $N'$ a smooth structure. Suppose that a smooth structure $j : N' \to X$ has been constructed. Identify $M$ with its image in $X$ under the composition

$$M \xrightarrow{f_1} M_1 \subset N \to N' \to X.$$

By the neighborhood retract theorem, $M$ is a smooth retract of a neighborhood $V \subset M$. As in the construction of collar neighborhoods, by making use of the inward pointing tangent vectors on $M_1$ we may also find a smooth function $f : V \to \mathbb{R}$ having $0$ as a regular value, and with $f^{-1}(0) = M$. These last two properties may be expressed in a way that does not reference $X$: the composition

$$\pi^{-1}(U) \to U \xrightarrow{f} \mathbb{R}$$

has $0$ as a regular value with $f^{-1}(0) = f_0(M) \amalg f_1(M)$, and the composition

$$\pi^{-1}(V) \to V \xrightarrow{r} M$$

is smooth. Thus a smooth structure on $N'$ gives rise a triple $(V, r, f)$ consisting of an open neighborhood $M \subset V \subset N'$, a retraction $r : V \to M$ with the property that

$$\pi^{-1}V \to V \to M$$

is smooth, and a function $f : V \to \mathbb{R}$ with the property that $\pi^{-1}(V) \to V \xrightarrow{f} \mathbb{R}$ has 0 as a regular value and cuts out $f_0^{-1}(M) \amalg f_1^{-1}(M)$. Conversely, such a triple defines a smooth manifold $X$. The underlying topological space is $N'$ (hence Hausdorff) and a map $g : Z \to N'$ is smooth if and only if the maps

$$g^{-1}(N' - M) \to N' - M - N - (M_0 \amalg M_1)$$
$$g^{-1}(V) \to V \xrightarrow{r} M$$
$$g^{-1}(V) \to V \xrightarrow{f} \mathbb{R}$$

are smooth. Another way to describe this smooth structure is to note that one can choose $\epsilon > 0$ for which the map $N \to N' \xrightarrow{f} \mathbb{R}$ has no critical values in $(-\epsilon, \epsilon)$ and for which $V_\epsilon = f^{-1}((-\epsilon, \epsilon))$ is contained in $V$. The smooth structure on $V_\epsilon$ is the one for which the homeomorphism

$$V_\epsilon \xrightarrow{(r,f)} M \times (-\epsilon, \epsilon) \tag{4.4}$$

is a diffeomorphism. The smooth structure on $N' - M$ has been specified. Since $N'$ is Hausdorff and $N' - M$ and $V_\epsilon$ form an open covering, this gives $N'$ a smooth structure by gluing.

This smooth structure is local near $M$ in the sense that if $M \subset V' \subset V$ then $(V', r, f)$ and $(V, r, f)$ define the same $X$.

*Proof of Theorem 4.2:* Since the smooth structure is determined by the one on $V$, and the diffeomorphism is extended from a diffeomorphism of $V$ by the identity on the complement of a closed subset of $V$, we can replace $N'$ by $V$. By choosing collar neighborhoods of $M_1$ and $M_2$ we are therefore reduced to the case

$$N = M \times (-\infty, 0] \amalg M \times [0, \infty), \tag{4.5}$$

with $f_1$ and $f_2$ the maps identifying $M$ with $M \times \{0\}$ in the first and second factors respectively, and $N' = M \times \mathbb{R}$. The existence is then obvious. For the uniqueness we suppose we are given a triple $(V, r, f)$ with $M \times \{0\} \subset V \subset M \times \mathbb{R}$, etc. Since we are allowed to shrink $V$, we may suppose that $V = M \times (-\delta, \delta)$ and that $r$ has the property that for each $t \in (-\delta, \delta)$ the restriction of $r$ to $M \times \{t\} \to M$ is a diffeomorphism. Reparameterizing $(-\delta, \delta)$ we might as well suppose that $V$ is all of $M \times \mathbb{R}$ and that for all $t$, the map $r$ restricts to a diffeomorphism

$$r_t : M \times \{t\} \to M.$$

Choose any $\ell > 0$. By Proposition 4.3, there is a diffeomorphism

$$\tilde{h} : M \times \mathbb{R} \to M \times \mathbb{R}$$

which is the identity outside of $M \times [-\ell, \ell]$ and has the property that $f \circ h$ is smooth. Using this diffeomorphism we may therefore suppose that $f$ is given by $f(x, t) = t$.

Now write $(M \times \mathbb{R})'$ for the topological space $M \times \mathbb{R}$ equipped with the smooth structure specified by $f(x,t) = x$ and the map $r(x,t)$. Our goal is to produce a diffeomorphism

$$M \times \mathbb{R} \to (M \times \mathbb{R})'$$

which is the identity outside, say, $M \times [-2,2]$. To do this let

$$g : M \times \mathbb{R} \to M$$

be the isotopy gotten by smoothing out the concatenation

$$g^*(x,t) = \begin{cases} r_t(x)^{-1} & -1 \le t \le 1 \\ r_{2-t}(x)^{-1} & 1 \le t \le 2 \\ r_{-2-t}(x)^{-1} & -2 \le t \le -1 \\ x & t \notin [-2,2]. \end{cases}$$

at the value $t = \pm 1$ and $\pm 2$ using a bump function, as in the proof of Lemma 7.3. The function $g$ is the smooth for all $t \neq 0$. Using $g$ form the map

$$G : M \times \mathbb{R} \to (M \times \mathbb{R})'$$
$$G(x,t) = (g(x,t), t).$$

The map $G$ is a homeomorphism, and is smooth for all $t \neq 0$. It is also smooth at $t = 0$ since $f(G(x,t)) = t$ and for $|t| < 1$, $r(G(x,t)) = x$. The map $G$ is then the desired diffeomorphism. $\square$

## 5. Special position

Our next main theorem is the following:

**Theorem 5.1.** *Suppose that $X$ is a 2-manifold. Every 1-manifold over $X$ is cobordant to an embedding $f : M \to X$.*

In fact one can go a bit further.

**Theorem 5.2.** *Suppose that $X$ is a connected 2-manifold. Every compact closed 1-manifold over $X$ is cobordant to an embedding $f : S^1 \to X$.*

In other words, when $X$ is connected, every element of $MO_1(X)$ is represented by an embedded circle.

The proof relies on a result of Whitney which we now describe.

**Definition 5.3.** A *regular immersion* is an immersion $f : M \to N$ having the property that for all $x \neq y \in M$, if $f(x) = f(y)$ then

$$df : T_x M \oplus T_y M \to T_{f(x)} N$$

is surjective.

The condition is often described by saying that $f$ has *transverse self-intersections*. This *almost* means that $f \pitchfork f$ (see Exercise 5.1). A point $y \in N$ is a *double point* if $f^{-1}(y)$ has two points. We say that $f$ *has at worst double points* if for every $y \in N$ the set $f^{-1}(y)$ has at most two points.

**Theorem 5.4.** *Every map from a $k$-manifold $M$ to a $2k$-manifold $N$ is homotopic to a regular immersion $f : M \to N$ with at worst double points*

We will give a proof of Theorem 5.4 when $k = 1$. Assuming it for the moment we turn to the proof of Theorem 5.1.

*Proof of Theorem 5.1:* First suppose that $X = \mathbb{R}_1 \coprod \mathbb{R}_2$ is the disjoint union of two copies of $\mathbb{R}$, $Y = \mathbb{R}^2$ and $f : X \to Y$ sends $a \in \mathbb{R}_1$ to $(a, 0)$ and $b \in \mathbb{R}_2$ to $(0, b)$. This map is a regular immersion and the only double point is 0. We now do surgery on this map to remove the double point. Write $t_i \in \mathbb{R}_i$ for a real number $t$ regarded as a point of $\mathbb{R}_i$. We take $A = \{0_1, 0_2\}$ to be the union of the two origins and $g : B = [0, 1] \to Y$ to be the constant map 0, with boundary identification sending 0 to $O_1$ and 1 to $O_2$. The map $A \to Y$ comes with a framing (the two lines themselves). The thickening

$$h : B \times [-1, 1] \to \mathbb{R}^2$$

requires a bit of care, as we need that the restriction of $h$ to $B \times \{-1, 1\}$ extend to an embedding of $\mathbb{R}_1 - (0, 1) \cup B \times \{-1, 1\} \cup \mathbb{R}_2 - (0, 1)$ in $\mathbb{R}^2$. To do this choose $0 < \epsilon < 1$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function having the property that

$$\phi(t) = \begin{cases} 0 & -\infty < t < -\epsilon \\ 1 & \epsilon < t < \infty \end{cases}$$

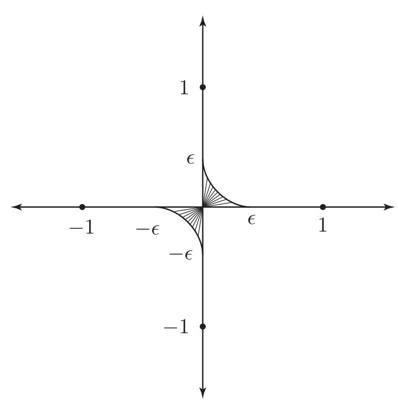
and which is strictly increasing on $(-\epsilon, \epsilon)$. Let $g(t) = t\phi(t)$ and set

$$G(t) = (g(-t), g(t))$$

As one can easily check, the function $G$ is an embedding of $\mathbb{R}$ into $\mathbb{R}^2$ which runs down the positive $x$-axis until $(\epsilon, 0)$ then turns and runs up the $y$-axis from $(0, \epsilon)$ onward. We take

$$h(s, t) = s\, G(t).$$

We leave the reader to check that after the surgery there are no double points, and the new map is an embedding.



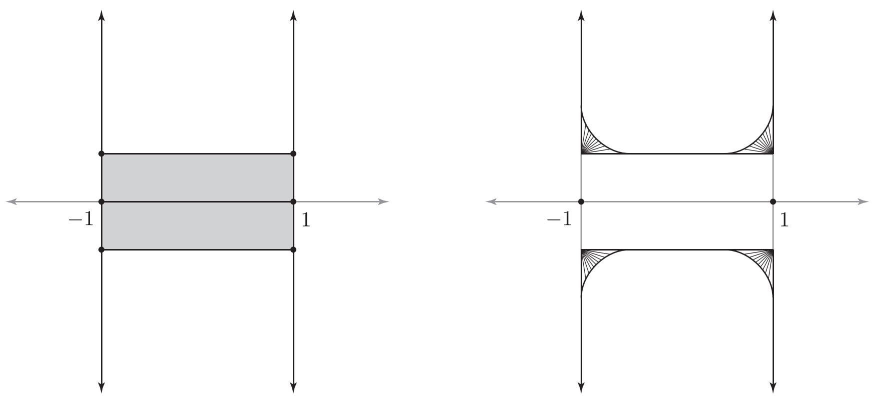

Now, as in the proof of the Moving Lemma we reduce to this case by showing that locally, the general situation looks like this. Suppose that $p, q \in M$ satisfy $f(p) = f(q) = x$. By working in coordinate neighborhood $U$ of $x$ and restricting

$f^{-1}(U) \subset M$ we might as well assume that $X$ is $\mathbb{R}^2$. Now consider the map

$$F : M \times M \to \mathbb{R}^2$$
$$(x, y) \mapsto f(x) + f(y).$$

Since $f$ has transverse self intersections, $dF_{(p,q)}$ is an isomorphism. It is therefore a diffeomorphism in a neighborhood of $(p, q)$. Now just choose coordinate neighborhoods $U_p$ and $U_q$ of $p$ and $q$ and use $F : U_p \times U_q \to \mathbb{R}^2$ as a coordinate neighborhood of 0 to show that locally any transverse intersection looks like the inclusion of the coordinate axes in the plane. □

*Proof of Theorem 5.2:* Suppose that $f : Z \to X$ is a closed 1-manifold over $X$. By Theorem 5.1 $Z$ is cobordant to an embedding. So we might as well suppose that $f : Z \to X$ is an embedding. We will show below that we can choose two path components $Z_1, Z_2$ of $Z$, an open $U \subset X$ equipped with a diffeomorphism $\Phi : U \to \mathbb{R}^2$ under which $U \cap Z_1$ corresponds to the line $x = -1$ and $U \cap Z_2$ corresonds to the line $x = 1$ and having the property that $U \cap Z = U(\cap Z_1 \cup Z_2)$. Assuming this we set up surgery data by taking $A =$




□

**Lemma 5.5.** *Suppose that $p, q \in \mathbb{R}^n$ are two distinct points. There is a diffeomorphism of $\phi : \mathbb{R}^n$ with $\phi(p) = q$ and with $\phi(x) = x$ outside of some compact $K$ containing $p$ and $q$.*

*Proof:* Let's first do the case of $\mathbb{R}^1$. Using a change of coordinates it suffices to consider the case in which $p = 0$ and $q$ is any other point $\lambda > 0$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth bump function which is 1 is an neighborhood of 0 and 0 outside a compact set. Choose $\lambda > 0$ so that $|\lambda\rho'(t)| < 1$ for all $t \in \mathbb{R}$, and consider

$$f(x) = x + \lambda\rho(x).$$

For $x$ outside the compact set where $\rho = 0$ we have $f(x) = x$. For any $x$ we have $f'(x) = 1 + \lambda\rho'(x) > 0$ so $f$ is strictly increasing and a local diffeomorphism. It follows that $f$ is a diffeomorphism. For the general case write points in $\mathbb{R}^n$ as pairs $(x, y)$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Again, by an affine linear change of coordinates it suffices to consider $p = (0, 0)$ and $q = (\lambda, 0)$ for any choice of $\lambda > 0$. The map

$$(x, y) \mapsto (x + \lambda\rho(|y|)\rho(x), y)$$

is a diffeomorphism sending $(0, 0)$ to $(\lambda, 0)$ and is the identity outside a compact set. □

**Corollary 5.6.** *Suppose that $X$ is a smooth connected $k$-manifold and $p, q \in X$ are two points. There is a diffeomorphism $\phi : X \to X$ with $\phi(p) = q$ and which is the identity outside a compact subset containing $p$ and $q$.*

For writing out the proof it will help to have some terminology.

**Definition 5.7.** Let $\phi : X \to X$ be a diffeomorphism. The *support of* $\phi$ is the closure of the set of $x$ for which $\phi(x) \neq x$. If the support of $\phi$ is compact we will say that $\phi$ is *compactly supported.*

Thus the assertion of Corollary 5.6 is that any point of a connected manifold can be taken to any other point by a compactly supported diffeomorphism. Note that the support of a composition of diffeomorphisms is contained in the union of the supports, so that the composition of compactly supported diffeomorphisms is compactly supported.

*Proof:* Fix $p$ and consider the set $T$ of all $q$ for which the claim holds. It certainly holds all $q$ in any neighborhood $U$ of $q$ for which there exists a diffeomorphism $\Phi : U \to \mathbb{R}^k$. One constructs the desired diffeomorphism by patching the diffeomorphism of $U$ sending $p$ to $p'$ provided by Lemma 5.5 with the identity map of the complement in $X$ of the support of $\phi$. From this we learn that the set of such $q$ is non-empty. It is also open since if $p \in T$ and $U$ is a neighborhood of $p$ diffeomorphic to $\mathbb{R}^k$ one can compose the compactly supported diffeomorphism sending $p$ to $q$ with the one sending $q$ to $q' \in U$. It is also closed since if one cannot send $p$ to $q$ one also can't send $p$ to any $q' \in U$. This implies that $T$ is non-empty and connected, hence all of $X$. □

**Corollary 5.8.** *Suppose that $X$ is a manifold and $C \subset \operatorname{int} X$ is a closed set whose complement is connected. Given $p, q \in X$ there is a compactly supported diffeomorphism of $X$ sending $p$ to $q$ and which is the identity on $C$.*

*Proof:* Apply Corollary 5.6 to $X - C$ to get a compactly supported diffeomorphism $\phi : X - C \to X - C$ sending $p$ to $q$. Patching $\phi$ $X - C$ with the identity map of $X - \operatorname{Supp}\phi$ gives the desired diffeomorphism. □

**Lemma 5.9.** *If $X$ is a connected manifold of dimension $k \geq 2$ and $S \subset X$ is a discrete set, then $X - S$ is connected.*

*Proof:* Suppose $p, q \in X - S$. Choose a path $\gamma : [0, 1] \to X$ from $p$ to $q$ which is transverse to $S$. Since $\dim X \geq 2$ the transversality condition is that $\phi([0, 1]) \cap S = \emptyset$. Thus $\gamma$ is a path in $X - S$ connecting $p$ and $q$. □

**Corollary 5.10.** *A diffeomorphism taking any finite set to any other finite set.*

**Corollary 5.11.** *If $X$ is a connected manifold and $S \subset X$ is a finite set, there is a single coordinate neighborhood containing $S$.*

**Corollary 5.12.** *Suppose that $X$ is a connected manifold, and $p \neq q$ are two points of $X$. There is an embedding $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$.*

*Proof:* If $\dim X = 1$ we can find a compact connected submanifold of $X$ containing $p$ and $q$. The result in that case follows from the classification of 1-manifolds. If $\dim X > 1$ we can assume that $p$ and $q$ are in a single coordinate chart and so reduce the question to the case $X = \mathbb{R}^n$ where it is obvious. □

**Proposition 5.13.** *Suppose that $Z$ is a 1-manifold embedded in $\mathbb{R}^2$. There is a line $\ell \subset \mathbb{R}^2$ which is transverse to $Z$.*

I upgraded this to a Proposition, since the proof is pretty important.

*Proof:* For a point $[a, b, c] \in \mathbf{RP}^2$, the equation

$$ax + by = c,$$

defines a line in $\mathbb{R}^2$ provided $(a, b) \neq (0, 0)$. As one easily checks, this gives a bijection between the set of lines in $\mathbb{R}^2$ and $\mathbf{RP}^2 - \{[0, 0, 1]\}$, making the set of lines in $\mathbb{R}^2$ into a smooth manifold of dimension 2. Define a map

$$\tau : Z \to \mathbf{RP}^2 - \{[0, 0, 1]\}$$

by sending a point $z \in Z$ to the line tangent to $Z$ through the point $z$. I claim this is a smooth map. The assertion is local in $Z$ so for each $z_0 \in Z$ we may choose a smooth function $f$ defined in a neighborhood $U$ of $z_0$ having 0 as a regular value and with $f^{-1}(0) = Z \cap U$. Then for $z = (s, t) \in U$, we have $\tau(z) = [a, b, c]$ with

$$\begin{aligned} a &= f_x \\ b &= f_y \\ c &= sf_x + tf_y, \end{aligned}$$

which is smooth. Now we can appeal to Sard's theorem to find a point $[a, b, c] \in \mathbf{RP}^2 - \{[0, 0, 1]\}$ which is not in the image of $\tau$. The corresponding line $\ell$ is transverse to $Z$. □

Suppose $p, q \in \mathbb{R}^2$ are two points having the property that the line $\ell_{p,q}$ through them is transverse to $\eta_1$ and $\eta_2$. Then for $v, w \in \mathbb{R}^2$ in some neighborhood of 0, the line $\ell_{p+v,q+w}$ will also be transverse to $\ell_1$ and $\ell_2$. This defines a map $L : B \times B \to U_{\eta_1,\eta_2}$, where $B$ is a small disk around the origin in $\mathbb{R}^2$. The derivative of $L$

$$dL : \mathbb{R}^2 \times \mathbb{R}^2 \to T(\eta_1 \times \eta_2)$$

is surjective (in fact a diagonal matrix), with kernel spanned by $(p - q, 0)$ and $(0, p - q)$.

**Lemma 5.14.** *Suppose that $Z \subset \mathbb{R}^2$ is a 1-manifold and $p \in \mathbb{R}^2$ is a point containing a line which is not tangent to $Z$. Then the set of lines through $p$ which are tangent to $Z$ has measure zero.*

*Proof:* Let $\eta$ be any line and consider the projection mapping $\pi_{p,\eta} : V_\eta \cap Z \to \eta$. Our assumptions imply that $V_\eta \cap Z$ is not empty. The critical points of $\pi_{p,\eta}$ are exactly the points $z \in Z$ for which the line through $p$ and $z$ is tangent to $Z$ at $z$. By Sard's theorem the set of critical values has measure zero. This implies the result. $\square$

**Lemma 5.15.** *Suppose that $Z \subset \mathbb{R}^2$ is an embedded 1-manifold with $\partial Z = \emptyset$, and $Z_1, Z_2$ are disjoint components of $Z$. There is a line $\ell$ in $\mathbb{R}^2$ which is transverse to $Z$ and has the property that $\ell \cap Z_1$ and $\ell \cap Z_2$ are non-empty.*

*Proof:* We first remark that by embedding we mean that the inclusion map is proper. This implies that if $Z_1$ is contained in a line $\ell$, then in fact $Z_1 = \ell$.

Define a map from $L : Z_1 \times Z_2 \to \mathbf{RP}^2 - \{[0, 0, 1]\}$ by sending $(z_1, z_2)$ to the unique line through $z_1$ and $z_2$. The points $z_1$ and $z_2$ are distinct since $Z_1 \cap Z_2 = \emptyset$. I claim there is a point $(z_1, z_2) \in Z_1 \times Z_2$ at which $L$ is a local diffeomorphism. Assuming this for the moment, this shows that the image of $L$ contains an open set and so cannot be contained in a set of measure zero, so there is a line $\ell$ transverse to $Z$ and contained in the image of $L$. To verify the claim, note by the description above that if $p \in Z_1$ and $q \in Z_2$ are points having the property that $\ell_{pq}$ is transverse to $Z_1$ at $p$ and $Z_2$ at $q$ then $dL$ is an isomorphism at $(p, q)$. So it suffice to exhibit a point $(p, q) \in Z_1 \times Z_2$ with this property. $\square$

We have used

**Lemma 5.16.** *Suppose that $Z_1, Z_2 \subset \mathbb{R}^2$ are disjoint 1-manifolds having the property that $Z_1 \amalg Z_2$ is not contained in a line. There are points $p \in Z_1$ and $q \in Z_2$ having the property that the line $\ell_{pq}$ between them is not tangent to $Z_1$ at $p$ and $Z_2$ at $q$.*

*Proof:* Choose $p \in Z_1$ and $q \in Z_2$. Since $Z_1 \amalg Z_2$ is not contained in a line there is a point $r$ in one of them not contained in $\ell_{pq}$. We might as well suppose $r \in Z_1$. Now the lines $\ell_{pq}$ and $\ell_{rq}$ meet at $q$ and are distinct, so they can't both be tangent to $Z_2$ at $q$. Relabeling, we can assume we've found $p \in Z_1$ and $q \in Z_2$ having the property that $\ell_{pq}$ is not tangent to $Z_2$ at $q$. Now choose another line $\eta$ not parallel to $\ell_{pq}$ and not containing $p$ (for instance the tangent line to $Z_2$ at $q$), and consider the projection mapping

$$\pi : U_{p,\eta} \cap Z_2 \to \eta.$$

By construction $\pi$ is a local diffeomorphism near $q$. This means that for $q'$ near $q$ the line $\ell_{pq'}$ is not tangent to $Z_2$ at $q'$, and is distinct from $\ell_{pq}$. Now both $\ell_{pq}$ and $\ell_{pq'}$ cannot be tangent to $Z_1$ at $p$. Relabeling we have shown that there are points $p \in Z_1$ and $q \in Z_2$ with $\ell_{pq}$ not tangent to $Z_1$ at $p$ and not tangent to $Z_2$ at $q$. $\square$

**Proposition 5.17.** *Suppose that $Z$ is a compact submanifold of $\mathbb{R}^2$ transverse to a line $\ell$. There is a coordinate neighborhood $U$ of $\ell$ under which $\ell$ corresponds to the $x$-axis and $Z \cap U$ to vertical lines.*

*Proof:* Not that the condition that $Z$ be transverse to $\ell$ means that $\partial Z \cap \ell = \emptyset$. Using an affine transformation of $\mathbb{R}^2$ we may arrange that $\ell$ *is* the $x$-axis. Label the points of intersection of $Z$ with $\ell$ by $\{p_1, p_2, \dots, p_k\}$, with $p_i = (x_i, 0)$. We will find an $\epsilon > 0$ and a retraction

$$r : U = \mathbb{R} \times (-\epsilon, \epsilon) \to \mathbb{R}$$

with the properties

i) one has

$$Z \cap U = \coprod_{i=1}^{k} Z_i$$

in which $Z_i$ is the connected component of $Z \cap U$ containing $p_i$

ii) the projection mapping

$$\pi : U \to (-\epsilon, \epsilon)$$

restricts, for each $i$, to a diffeomorphism

$$Z_i \to (-\epsilon, \epsilon).$$

iii) $r^{-1}(\{p_1, \dots, p_k\}) = Z \cap U$

Given this, the desired coordinate neighborhood can be taken to be

$$U \xrightarrow{(r, p_2)} \mathbb{R} \times (-\epsilon, \epsilon) \xrightarrow{\mathrm{Id} \times g} \mathbb{R} \times \mathbb{R} \tag{5.18}$$

where $g$ is any diffeomorphism of $(-\epsilon, \epsilon)$ with $\mathbb{R}$ taking 0 to 0.

We now turn to the construction of $\epsilon$ and $r$. Since $Z$ is transverse to $\ell$ the restriction to $Z$ of the projection mapping to the $y$-axis has 0 as a regular value. By the inverse function theorem it is a diffeomorphism of a neighborhood of each $p_i$ with $(\epsilon_i, \epsilon_i)$. We first take $\epsilon$ strictly smaller than all of the $\epsilon_i$. The complement in $\pi^{-1}[-\epsilon, \epsilon] \cap Z$ of the components containing the $x_i$ is a compact space not meeting the $x$-axis. Taking $\epsilon$ even smaller we may assume this compact space is empty. This choice of $\epsilon$ guarantees i) and ii).

To construct the retraction $r$, let

$$U_i = \{(x, y) \mid |x - x_i| < \epsilon, |y| < \epsilon\},$$

and consider the map

$$\begin{aligned} Z \times \mathbb{R} &\to \mathbb{R}^2 \\ (z, t) &\mapsto z + (0, t). \end{aligned}$$

Since $Z$ is transverse to the $x$-axis this map is a diffeomorphism in a neighborhood of each $p_i$. Shrinking $\epsilon$ further, if necessary, we may assume that for each $i$, $U_i$ is contained in this neighborhood, and that the open sets $U_i$ are disjoint. We define

$$r_i : U_i \to \mathbb{R}$$

by $r_i(g(z, t)) = t + x_i$. The map $r_i$ is a retraction of $U_i$ to the intersection of $U_i$ with the $x$-axis, and has the property that $r_i^{-1}(x_i) = Z \cap U_i$. Next let $V_i$ be the complement in $(x_i, x_{i+1}) \times \mathbb{R}$ of the points $x \in U_j$ for which $r_j(x) \notin (x_i, x_{i+1})$

Finally, set

$$U_0 = \{(x, y) \mid |y| < \epsilon\} - \{p_1, \dots, p_k\},$$

and define $r_0 : U_0 \to \mathbb{R}$ by $r_0(x, y) = x$. The collection $\mathcal{U}\{U_0, U_1, \cdots, U_k\}$ covers $\mathbb{R} \times (-\epsilon, \epsilon)$. Let $\{\theta_0, \dots, \theta_k\}$ be a partition of unity subordinate to $\mathcal{U}$, with $\operatorname{Supp} \theta_i \subset U_i$, and define $r$ by

$$r = \sum \theta_i r_i,$$

where, as usual, we have extended $\theta_i r_i$ to be a smooth function on all of $\mathbb{R} \times (-\epsilon, \epsilon)$ by taking it to be zero on the complement of $\operatorname{Supp} \theta_i$. but they are not. You need to arrange that $U_i$ contains $Z_i$ and that $U_0$ does not contain the projection of $Z_i$ to the $x$-axis. yikes

□

*Proof of Theorem 5.2* :. UNDER CONSTRUCTION. □

*Proof of Theorem 5.4 when* $k = 1$:. UNDER CONSTRUCTION. □

## Exercises

**5.1.** Why does the condition that $f$ be a regular immersion (Definition 5.3) not imply that $f \pitchfork f$? Show that $f$ being a regular immersion is equivalent to $f$ being an immersion, and the map

$$M \times M - \Delta_M \xrightarrow{f \times f} N \times N$$

being transverse to the diagonal of $N$ ($\Delta_M = \{(x, x) \in M \times M\}$ is the diagonal of $M$).

**5.2.** Generalize the argument in the proof of Theorem 5.1 to show that every regular immersion $f : M \to N$ of a manifold of dimension $k$ to a manifold of dimension $2k$ is cobordant to an embedding.

**5.3.** The Whitney theorem (Theorem 5.4) can be proved in a manner analogous to the Transversality Theorem of [**1**, Chapter 2, §3]. For an integer $n > 0$ write

$$X^n = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The key point of the proof of the Whitney theorem is to show that given a smooth map $f : X \to Y$ of manifolds, there is a manifold $S$, a point $s_0 \in S$, and a map $h : X \times S \to Y$ with $h(x, s_0) = f(x)$ and having the property that for $n = 1, \dots, k$, the map

$$h^{(n)} : X^{(n)} \times S \to Y^n$$

given by

$$h^{(n)}(x_1, \dots, x_n, s) = (h(x_1, s), \dots, h(x_n, s))$$

is a submersion. Assuming this, show that for almost all $s \in S$ the map

$$h_s(x) = h(x, s) : X^{(n)} \to Y^n$$

is transverse to the diagonal $\Delta_Y = \{(y, \dots, y) \in Y^n\}$. By considering the cases $n = 2, 3$ show that when $2 \dim X \leq \dim Y$, every smooth map $X \to Y$ is homotopic to a regular immersion with at worst double points.

**5.4.** Suppose that $p \in \mathbb{R}^2$ is a point and $\eta \subset \mathbb{R}^2$ is a line not containing $p$. Let $U \subset \mathbb{R}^2$ be the complement of the line through $p$ which is parallel to $\eta$. The *projection mapping* $\pi_{p,\eta} : U \to \eta$ is the mapping sending $x \in U$ to the intersection of the line through $p$ and $x$ with $\eta$.

(a) Show that $\pi_{p,\eta}$ is a submersion and that for $x \in U$ the kernel of $d\pi_{p,\eta}$ at $x$ is the line spanned by $\pi_{p,\eta}(x) - x$.

(b) Suppose that $Z \subset \mathbb{R}^2$ is a 1-manifold. Using the above, show that a point $y \in \eta$ is a regular value for the restriction of $\pi_{p,\eta}$ to $U \cap Z$ if and only if the line through $p$ and $y$ is not tangent to $Z$ at any point.

**5.5.** In this exercise we consider the set of all lines $\ell \subset \mathbb{R}^2$.

(a) Let $\eta_1, \eta_2$ be two distinct parallel lines in $\mathbb{R}^2$, and $U_{\eta_1,\eta_2}$ the set of lines *not* parallel to them. Define a mp

$$\Phi_{\eta_1,\eta_2} : U_{\eta_1,\eta_2} \to \eta_1 \times \eta_2$$

by sending a line $\ell$ to its points of intersection with $\eta_1$ and $\eta_2$. Show that the maps $\Phi_{\eta_1,\eta_2}$ give the set of lines in $\mathbb{R}^2$ the structure of a smooth manifold.

(b) Here is another way of getting a smooth structure on the set of lines in $\mathbb{R}^2$. A line has an equation of the form

$$ax + by + c = 0$$

with $(a, b) \neq (0, 0)$. The triples $(a, b, c)$ and $(a', b', c')$ define the same line if and only if there is a $\lambda \neq 0 \in \mathbb{R}$ for which $(a', b', c') = \lambda(a, b, c)$. This gives a map from the set of lines in $\mathbb{R}^2$ to $\mathbf{RP}^2 - p$ where $p$ is the point $[0, 0, 1]$. Show that this map is a diffeomorphism.

## 6. Separation

The following notion has come up several times, and so we give it a name.

**Definition 6.1.** Suppose that $X$ is a manifold of dimension $n$. A sub manifold $Z \subset X$ of codimension $k$ is said to *cut out* by a function $\phi : X \to \mathbb{R}^k$ if 0 is a regular value of $\phi$ and $\phi^{-1}(0) = Z$.

**Proposition 6.2.** *Suppose that $X$ is a manifold of dimension 2. A submanifold $Z \subset X$ is cut out by a function $\phi : X \to \mathbb{R}$ if and only if $I(Z, f) = 0$ for all closed one manifolds $f : M \to X$ over $X$.*

*Proof:* The if direction is easy. For the only if direction, we might as well suppose that $X$ is connected, as we can define $\phi$ separately on each component. Choose a point $p \in X - Z$ and define a function $s(x) \in \{0, 1\}$ for $x \in X - Z$ by choosing a path $\gamma : [-1, 1] \to X$ with $\gamma(-1) = s$, $\gamma(1) = x$, which is transverse to $Z$, and setting

$$s(x) = I(\gamma, Z) = \#\{\gamma^{-1}(Z)\}.$$

To see that the function $s$ is well-defined, suppose $\tilde{\gamma}$ is another such path and let $f : S^1 \to X$ be the map

$$f(x, y) = \begin{cases} \gamma(x) & y \geq 0 \\ \tilde{\gamma}(x) & y \leq 0 \end{cases}$$

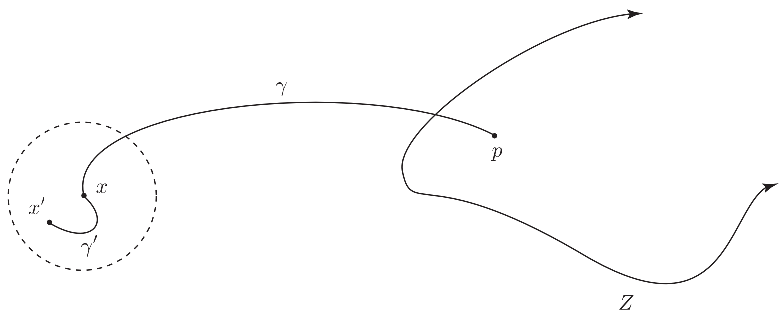
(smoothed out using near $x = \pm 1$ using Exercise 6.1). Then

$$I(\gamma, Z) + I(\tilde{\gamma}, Z) = I(f, Z) = 0$$

by our assumption on $Z$. The function $s$ is locally constant. Indeed, given $x \in M - Z$ choose a connected neighborhood $x \in V \subset M - Z$. Given a point $x' \in V$ choose a path $\gamma'$ from $x$ to $x'$. Given a path $\gamma$ from $p$ to $x$ which is transverse to $Z$, the path

$$\gamma' * \gamma(t) = \begin{cases} \gamma((2t+1) & -1 \leq t \leq 0 \\ \gamma'((2t-1) & 0 \leq t \leq 1, \end{cases}$$

reparamaterized near $t = 0$ to be smooth (see Exercise 6.1) is transverse to $Z$ and satisfies $I(\gamma' * \gamma, Z) = I(\gamma, Z)$, so that $s(x) = s(x')$




Now choose a countable collection of open sets $\{U_i\}$ covering $Z$ and diffeomorphisms $\Phi_i : U_i \to \mathbb{R}^2$ under which $Z \cap U_i$ corresponds to the $x$-axis. Reversing the $y$-coordinate if necessary, we may assume that $\Phi_i$ carries the points $x \in U_i - Z$ with $s(x) = 0$ to the points in $\mathbb{R}^2$ with positive $y$-coordinate. Let $\phi_i : U_i \to \mathbb{R}$ be the $y$-coordinate of $\Phi_i$. For $x \in Z$ choose an $i$ with $x \in U_i$ and define $T_x^+ X \subset T_x X$ to $v$ for $d\Phi_i(v) \in T_{\Phi(x)}\mathbb{R}^2 = \mathbb{R}^2$ has positive $y$ coordinate. Note that this is independent of $i$, since for every $i$, $j$, $\Phi_j \Phi_i^{-1}$ carries points with positive $y$-coordinate to points with positive $Y$-coordinate. Note that $T_x^+ X$ can also be described as the set of $v \in T_x X$ such that $d\phi_i(v) > 0$. Let $W = X - U$ and $\{\psi, \theta_i\}$ be a partition of unity subordinate to $\{W, U_i\}$ with

$$\operatorname{Supp} \psi \subset W$$
$$\operatorname{Supp} \theta_i \subset U_i.$$

Let

$$f(x) = (-1)^{s(x)} \psi(x) + \sum \theta_i(x) \phi_i(x).$$

Since $\operatorname{Supp} \psi \subset X - Z$ the function $\psi$ vanishes in a neighborhood of $Z$, so for $x \in Z$ we have both $\psi(x) = 0$ and $d\psi(x) = 0$. If $x \in Z$ then

$$f(x) = \sum \theta_i(x) \phi_i(x) = 0.$$

On the other hand, if $x \notin Z$ then $f(x)$ is strictly positive if $s(x) = 0$ and strictly negative if $s(x) = 1$. This implies that $f^{-1}(0) = Z$. To check that 0 is a regular value, suppose $x \in Z$ and let $v \in T_x^+ X$. Then

$$\begin{aligned} df_x(v) &= \sum d\theta_i(v)\, \phi_i(x) + \theta_i(x) d\phi_i(x) \\ &= \sum \theta_i(x)\, d\phi_i(v) > 0 \end{aligned}$$

since for all $i$, $\theta_i(x) \geq 0$, for some $i$, $\theta_i(x) > 0$, and for all $i$ with $x \in U_i$, $d\phi_i(v) > 0$. Thus 0 is a regular value. $\square$

**Theorem 6.3.** *If $X$ is a compact closed manifold of dimension 2, then the intersection form $(MO_1(X), I)$ is non-degenerate.*

*Proof:* We must show that if $a \in MO_1(X)$ has the property that $I(a,b) = 0$ for all $b$ then $a = 0$. By Theorem 5.1 we may represent $a$ by an embedded submanifold $Z \subset X$. Since $I(Z, b) = 0$ for all $b$, Proposition 6.2 implies that $Z$ is cut out by a smooth function $f : X \to \mathbb{R}$. But then $Z = \partial f^{-1}\big([0,\infty)\big)$ so $a = 0$. $\square$

Proposition 6.2 has many applications. To go further with it we need to know the local structure of embedded 1-manifold $Z$ in a 2-manifolds $X$. Lemma VIII.2.5 provides the answer in case $Z = [0, 1]$. The remaining case is when $Z = S^1$. To have something to compare our neighborhoods to we fix some definitions.

**Definition 6.4.** The *open cylinder* is the manifold $S^1 \times \mathbb{R}$. The *closed cylinder* is the manifold $S^1 \times [-1, 1]$.

For any $a < b$, the open cylinder is diffeomorphic to $S^1 \times (a, b)$ and the closed cylinder is diffeomorphic to $S^1 \times [a, b]$. The open cylinder is also the quotient of

$$\mathbb{R} \times \mathbb{R}$$

by the equivalence relation $(x, y) \sim (x + 1, y)$. A specific diffeomorphism is given by $(x, y) \mapsto (\cos(2\pi\, x)\ \sin(2\pi\, x), y)$.

**Definition 6.5.** The *open Möbius band* is the quotient of the manifold

$$\mathbb{R} \times \mathbb{R}$$

by the equivalence relation

$$(x, t) \sim (x + 1, -t). \tag{6.6}$$

The *closed Möbius band* is the quotient of $\mathbb{R} \times [-1, 1]$ by the equivalence relation (6.6). The *meridian circle* of the Möbius band is the subspace of points $(x, 0)$.

For any $a > 0$, the open and closed Möbius bands are diffeomorphic, respectively, to the quotient of $\mathbb{R} \times (-a, a)$ and $\mathbb{R} \times [-a, a]$ by (6.6).

**Lemma 6.7.** *Suppose that $X$ is a 2-manifold (not necessarily compact) and $Z \subset \operatorname{int} X$ is an embedded circle. If $I(Z, Z) = 0$ then there exist a neighborhood $U$ of $Z$ which is diffeomorphic to the open cylinder under which $Z$ corresponds to $S^1 \times \{0\}$. If $I(Z, Z) = 1$ there is a neighborhood of $Z$ which is diffeomorphic to the open Möbius band, under which $Z$ corresponds to the inclusion of the meridian circle.*

*Proof:* Let $N$ be the normal bundle to $Z$ in $X$. By the neighborhood retract theorem there is a neighborhood $U$ of $Z$ in $M$ which is diffeomorphic to $N$ and under which the inclusion of $Z$ corresponds to the map $x \mapsto (x, 0) \in N$. It therefore suffices to construct the appropriate diffeomorphism of $N$ with the open cylinder or the open Möbius band. We will use some material from the notes on 1-manifolds. Let $C \subset N$ the subset of pairs $(x, v)$ with $|v| = 1$. The map $C \to Z$ is a covering space. Choose a diffeomorphism $g : S^1 \to Z$ and let $p = g(0, 1) \in Z$. There are two points $(p,v) \in C$ lying over $p$. Pick one. Consider the diagram

$$\begin{tikzcd}
\{0\} \arrow[r, "{(p,v)}"] \arrow[d] & & C \arrow[d] \\
\mathbb{R} \arrow[r, "\gamma"'] \arrow[urr, "\tilde{\gamma}"] & S^1 \arrow[r, "g"'] & Z
\end{tikzcd}$$

where $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$, and the top arrow sends the point 0 to some choice of vector $v$ over the image of $(0,1)$. By the path lifting property a unique lift $\tilde{\gamma}$ exists. There are two possibilities. If $\tilde{\gamma}(1) = (p,v)$ then we can define a diffeomorphism of the open cylinder with $N$ by sending the equivalence class of $(x,t)$ to $t\tilde{\gamma}(x)$. On the other hand if $\tilde{\gamma}(1) = (p,-v)$, we get a diffeomorphism of the open Möbious band with $N$ from the same formula. The two cases are distinguished by the self intersection number of the central circle. In the case of the cylinder it is 0 and in the case of the Möbius band it is 1. $\square$

**Exercises**

**6.1.** Suppose that $\gamma : [-1,1] \to X$ is a continuous map which is smooth except at 0 then given $\epsilon$, there is a smooth map $g : [-1,1] \to [-1,1]$ with $g(x) = x$ for $|x| > \epsilon$ and with the property that $\gamma \circ g$ is smooth. (HINT: Choose an appropriate bump function $\phi$ and take $g(x) = (1 - \phi(x))g(x)$).

**6.2.** Why is compactness required in Theorem 6.3?

**6.3.** Compute the intersection form for $\mathbf{RP}^2$ and for the Klein bottle.

**6.4.** Compute the intersection form for $\mathbf{RP}^2 \# \mathbf{RP}^2$. Can you compute it for the connected sum of any number of copies of $\mathbf{RP}^2$?

**6.5.** This problem is devoted to establishing the *surgery formula.* Suppose that $X$ is a compact closed 2-manifold and $f : S^1 \to X$ is an embedding. Let $[f] \in MO_1 X$ denote the cobordism class of $f$. Suppose that $I(f,f) = 0$. By Lemma 6.7 the map $f$ extends to a diffeomorphism of $\tilde{f} : S^1 \times \mathbb{R} \to U$ where $U$ is a neighborhood of $f(S^1)$. Take $B = D^2$ and use this data to perform surgery on $X$ to produce a manifold $X'$. The surgery formula is

$$MO_1(X') = [f]^{\perp}/[f].$$

(a) Let $V = [f]^{\perp} \subset MO_1 X$. Define a map $V \to MO_1 X'$ by representing $v \in V$ by a map $g : M \to X$ which does not meet $\tilde{f}(S^1 \times [-1,1])$, and composing with the map to $X'$ (see Part (e)). Show that this map is well defined.

(b) Show that $[f]$ is in the kernel of the map you just constructed, so there is a map $V/[f] \to MO_1 X'$.

(c) Let $p \in X'$ be the point corresponding to to $(0,-1) \in B \times S^0$ and $q \in X'$ the point corresponding to $(0,1) \in B \times S^0$. Show that the map constructed in Part (b) surjective by choosing 1-manifolds over $X'$ which are transverse to $\{p,q\}$.

(d) Show that the kernel of this map is generated by $[f]$ as follows. Suppose that $g : M \to X$ has image disjoint from $S^1 \times [-1, 1]$ and that the image of $g$ in $X'$ is the boundary of $h : N \to X'$. Assume that $h$ is transverse to $\{p, q\}$. By cutting out small disks around $p$ and $q$ and removing their inverse images in $N$ produce a cobordism in $X$ to a multiple of $[f]$.

(e) Here is something I should have pointed out to make things a little clearer. From the definition of surgery, we are given a diffeomorphism $X' - B \times S^0$ with $X - A \times (-1, 1)$. As above, let $p \in X'$ be the point corresponding to to $(0, -1) \in B \times S^0$ and $q \in X'$ the point corresponding to $(0, 1) \in B \times S^0$. Show that the above diffeomorphism extends to a diffeomorphism of $X' - \{p, q\}$ with $X - f(S^1)$. In this way a manifolds $g : M \to X$ over $X$ disjoint from $f(S^1)$ may be identified with manifolds over $X'$ disjoint from $\{p, q\}$.

**6.6.** Using the surgery formula show that if $M_1$ and $M_2$ are compact 2-manifolds then $MO_1(M_1 \# M_2)$ is the orthogonal sum $MO_1(M_1) \oplus MO_1(M_2)$.

**6.7.** Show that every finite dimension symmetric bilinear form over $\mathbb{F}_2$ occurs as the intersection form of a compact, connected 2-manifold.

## 7. Isotopies and diffeomorphisms of $S^1$

The next two sections contain some further material needed for the classification of 2-manifolds.

**Theorem 7.1.** *Every diffeomorphism of $\phi : S^1 \to S^1$ extends to a diffeomorphism $\tilde{\phi} : D^2 \to D^2$.*

The proof of Theorem 7.1 requires the notion of *isotopy*.

**Definition 7.2.** Suppose that $M$ is a manifold, and $f_0, f_1 : M \to M$ are diffeomorphisms. An *isotopy* of $f_0$ with $f_1$ is a smooth homotopy $h : M \times I \to M$ from $f_0$ to $f_1$ having the property that for each $t \in [0, 1]$ the map $h_t(x) = h(t, x) : M \to M$ is a diffeomorphism. Two diffeomorphisms are *isotopic* if there is an isotopy between them.

The term "isotopy" is used more generally to refer to any homotopy preserving whatever structure you are interested in. One talks about "isotopies of embeddings," "isotopies of immersions," etc. Hirsch [**3**] uses the term "diffeotopy" to refer to "isotopies of diffeomorphisms."

We need to dispense with a technical point.

**Lemma 7.3.** *"Isotopic" is an equivalence relation.*

*Proof:* The only property requiring proof is transitivity. Suppose that $h^0$ is an isotopy of $f_0$ with $f_1$ and $h^1$ is an isotopy of $f_1$ with $f_2$. One would like to define an isotopy $h : M \times I \to M$ of $f_0$ with $f_2$ by taking

$$h(x, t) = \begin{cases} h^0(x, 2t) & 0 \le t \le 1/2 \\ h^1(x, 2t - 1) & 1/2 \le t \le 1\,. \end{cases} \tag{7.4}$$

However this map need not be smooth at the points $(x, 1/2)$. This can be remedied by "slowing down" the isotopies before concatenating them. Choose $0 < \epsilon < 1$ and a smooth function $\phi : \mathbb{R} \to [0,1] \subset \mathbb{R}$ which is identically zero for $x < \epsilon$, strictly increasing on $(\epsilon, 1-\epsilon)$ and strictly 1 for $x > \epsilon$. Now define an isotopy $h$ by

$$h(x,t) = \begin{cases} h^0(x, \phi(2t)) & 0 \le t \le 1/2 \\ h^1(x, \phi(2t-1)) & 1/2 \le t \le 1\,. \end{cases} \tag{7.5}$$

One easily checks that this $h$ is smooth. □

We will call the concatenation of the isotopies used in the above proof the *smooth concatenation*.

An isotopy $h : M \times I \to M$ gives a diffeomorphism

$$\begin{aligned} \hat{h} : M \times I &\to M \times I \\ (m,t) &\mapsto (h(m,t),t). \end{aligned}$$

Conversely, if

$$g = (g_1, g_2) : M \times I \to M \times I$$

is a diffeomorphism for which $g_2$ is the projection mapping to $I$, then $g_1$ is an isotopy.

**Proposition 7.6.** *Every diffeomorphism of $S^1$ is isotopic to either the identity or to the map $(x,y) \mapsto (x,-y)$.*

*Proof:* Suppose that $\phi : S^1 \to S^1$ is a diffeomorphism. Since rotations of the circle are isotopic to the identity we may assume, by transitivity, that $\phi(0,1) = (0,1)$. Now consider the diagram

$$\begin{tikzcd} \{0\} \arrow[rr] \arrow[d] & & \mathbb{R} \arrow[d, "e"] \\ \mathbb{R} \arrow[r, "e"'] \arrow[urr, dashed, "f"] & S^1 \arrow[r, "\phi"'] & S^1 \end{tikzcd} \tag{7.7}$$

in which $e(t) = (\cos(t), (\sin(t)))$. Since $e : \mathbb{R} \to S^1$ is a covering space, there is an unique lift $f$ making the diagram commute. In the form of an equation this means $\phi$ is given by

$$\phi(\cos(\theta), \sin(\theta)) = (\cos(f(\theta), \sin(f(\theta)))).$$

Since for all $x$, $f'(x) \neq 0$ we either have $f'(x) > 0$ for all $x$ or $f'(x) < 0$ for all $x$. Suppose $f'(x) > 0$ for all $x$. For $0 \le s \le 0$ define

$$f_s(x) = (1-s)f(x) + s\,x.$$

One checks

$$f'_x(x) = (1-s)f'(x) + s > 0$$

so that $f_s$ is strictly increasing. In fact $f_s$ is a diffeomorphism (see Exercise 7.3). The map

$$\begin{gathered} h : S^1 \times I \to S^1 \\ h((\cos(\theta), \sin(\theta)), s) = (\cos(f_s(\theta)), \sin(f_s(\theta))) \end{gathered}$$

is then an isotopy of $\phi$ with the identity. A similar argument using

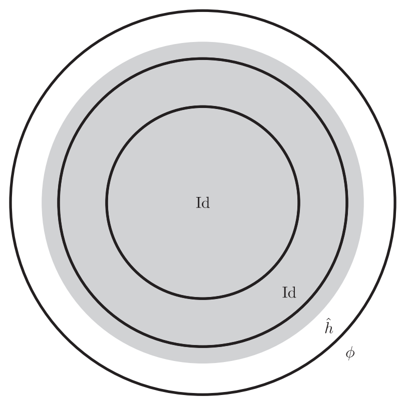
$$f_s(x) = (1-s)f(x) - s\,x$$

in case $f'(x) < 0$ gives an isotopy of $\phi$ with the map $(x,y) \mapsto (x,-y)$. □

*Proof of Theorem 7.1:* Suppose that $\phi : S^1 \to S^1$ is a diffeomorphism. Since the map $(x, y) \mapsto (x, -y)$ extends to a diffeomorphism of the disk, it suffice to consider the case in which there is an isotopy $h$ from $\phi$ to the identity map. Let $g : S^1 \times [0, 1] \to S^1$ be the smooth concatenation of $h$ with the identity map. Identifying $S^1 \times [0, 1]$ with the annulus

$$\{v \in \mathbb{R}^2 \mid 1/2 \leq |v| \leq 1\}$$

we extend $\hat{g}$ to a diffeomorphism of the disk by patching it with the identity map on the disk of radius 1/2.




□

**Exercises**

**7.1.** Given an example of isotopies $h^0$ and $h^1$ of diffeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ for which (7.4) is not smooth.

**7.2.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function with the property that $f'(x) \neq 0$ and $f(x + 2\pi) = f(x) + 2\pi$. Show that $f$ is a diffeomorphism.

**7.3.** Consider the function $f$ in the proof of Proposition 7.6. Show that for all $x \in \mathbb{R}$ one has

$$f(x + 2\pi) = \begin{cases} f(x) + 2\pi & \text{if } f'(0) > 0 \\ f(x) - 2\pi & \text{if } f'(0) < 0. \end{cases}$$

Conclude that $f$ is a diffeomorphism. Show the function $f_s$ is a diffeomorphism.

## 8. More on connected sums

**Theorem 8.1.** *Suppose that $X$ is a smooth connected manifold of dimension $n$ and $\phi : U \to \mathbb{R}^n$ is a diffeomorphism of an open subset of $X$ with $\mathbb{R}^n$. Given an* *embedding* $f : D^n \to \operatorname{int} X$*, there is a diffeomorphism* $g : X \to X$ *making the diagram*

$$\begin{tikzcd} D^n \arrow[r, "i"] \arrow[d, "f"'] & \mathbb{R}^n \arrow[r, "\Phi^{-1}"] & U \arrow[d] \\ M \arrow[rr, "g"'] & & M \end{tikzcd}$$

*commute, in which* $i$ *is one of the maps*

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$$
$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, -x_n).$$

In other words, up to diffeomorphism there are at most two possible embeddings of an $n$-disk into the interior of an $n$-manifold.

I'm not sure if the notion of *embedding* of a manifold with boundary into another has been defined. But it is supposed to mean a proper map which is a diffeomorphism with its image. This means that the derivative is an isomorphism at every point. This in turn implies that any embedding of a disk extends to a diffeomorphism of a slightly thicker open disk with an open nieghborhood of the image. Write $D^n_r$ for the closed disk of radius $r$. By choosing a diffeomorphism $\phi : (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$ which is the identity on a neighborhood of $[-1, 1]$ we can arrange that any embedding of $D^n \to X$ extends to a diffeomorphism $\mathbb{R}^n \to U \subset X$ of all of $\mathbb{R}^n$ with an open subset $U$ of $X$. That is, any embedding $D^n \to X$

*Proof:*

□

CHAPTER X

# Applications

## 1. Classification of 2-manifolds

## 2. Immersion of surfaces I

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## 3. Quadratic forms in characteristic 2

**3.1. Abstract quadratic forms.** A quadratic form is a homogeneous polynomial of degree 2 in $n$ variables. Examples are

$$q(x) = \lambda x^2$$
$$q(x, y) = x^2 + xy$$
$$q(x, y, z) = x^2 + xz + z^2 + yz.$$

In a standard course in linear algebra you would learn how to classify quadratic forms over $\mathbb{R}$ and to understand the shape of the "quadrics" defined by $q(v) = 1$. Here I want to focus on the case of $F = \mathbb{F}_2$. But to begin we will work over an arbitrary field.

For convenience I want to work with a more abstract "coordinate free" definition of a quadratic form.

**Definition 3.1.** Suppose $V$ is a vector space over a field $F$. A *quadratic function* on $V$ is a function

$$q : V \to F$$

having the property that

$$B(x, y) = q(x + y) - q(x) - q(y)$$

is a bilinear form on $V$. A *quadratic form* on $V$ is a quadratic function having the additional property that

$$q(\lambda v) = \lambda^2 q(v)$$

for all $\lambda \in F$ and all $v \in V$.

The bilinear form $B(x, y) = q(x + y) - q(x) - q(y)$ is called the *underlying bilinear form* of $q$ (or sometimes the *associated bilinear form.*) Note that bilinear form underlying a quadratic function is always symmetric.

**Exercise 3.1.** Show that if $q$ and $q'$ are two quadratic functions on $V$ with the same underlying bilinear form then $q - q'$ is linear. Show that if $q$ and $q'$ are actually quadratic *forms* and the characteristic of $F$ is not 2 then in fact $q - q' = 0$.

**Exercise 3.2.** Suppose that $V = F^n$ and that we are given elements $a_{ij} \in F$ for $i, j = 1, \ldots, n$. Define $f : V \to F$ by

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

Show that $f$ is a quadratic form.

**3.2. Quadratic forms and quadratic polynomials.** We now describe how a choice of basis lets one describe any quadratic form as a homogeneous polynomial of degree 2.

Suppose that $(V, q)$ is a quadratic form over a field $F$. We now make no restrictions on the characteristic of $F$. Suppose that $\alpha = \{v_1, \ldots, v_n\}$ is an ordered basis of $V$. Given a quadratic form $q : V \to F$ define elements $a_{ij} \in F$ by

$$a_{ij} = \begin{cases} q(v_i + v_j) & i \neq j \\ q(v_i) & i = j, \end{cases}$$

and set

$$q_\alpha(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

This is the quadratic polynomial associated to $q$ by the ordered basis $\alpha$.

$$x_\alpha = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

**Proposition 3.2.** *If* $v = x_1 v_1 + \cdots + x_n v_n$ *then*

$$q(v) = q_\alpha(x_1, \ldots, x_n).$$

One way to prove this is to repeatedly use the properties of $q$ to expand $q(x_1 v_1 + \cdots + x_n v_n)$. That's actually not so bad. I want to outline a slightly more elegant approach that involves some useful other ideas.

Stepping back a bit we have two quadratic forms on $V$ and we wish to show they are the same. One of them is $q$. The other is the form $q'(v)$ computed by writing $v = x_1 v_1 + \cdots + x_n v_n$ and setting $q'(v) = q_\alpha(x_1, \ldots, x_n)$.

**Exercise 3.3.** Show that $q$ and $q'$ have the same underlying bilinear forms, ie that

$$q(x + y) - q(x) - q(y) = q'(x + y) - q'(x) - q'(y).$$

(Hint: Since both sides are bilinear it suffices to do this for $x = v_i$ and $y = v_j$).

**Exercise 3.4.** Using Exercise 3.1 show that if $q_1$ and $q_2$ are two quadratic functions with the same underlying bilinear form, $\{v_1, \ldots, v_n\}$ generate $V$, and $q_1(v_i) = q_2(v_i)$ for $i = 1, \ldots, n$ then $q_1 = q_2$.

**Exercise 3.5.** Using the above exercises prove Proposition 3.2.

**3.3. Isometries and orthogonal sums.** Note that the bilinear form $B$ associated to a quadratic form $q$ satisfies

$$B(x,x) = q(2x) - 2q(x) = 4q(x) - 2q(x) = 2q(x), \tag{3.3}$$

or in other words that

$$q(x) = \frac{1}{2}B(x,x).$$

This means that if the characteristic of $F$ is not 2, the theory of quadratic forms is equivalent to the theory of symmetric bilinear forms. In characteristic 2 there is an interesting difference. In either case there is a close correspondence, and there is some terminology that reflects that.

**Definition 3.4.** Suppose that $F$ is a field. A *quadratic form over* $F$ is a pair $(V, q)$ consisting of a finite dimensional vector space over $F$ and a quadratic form $q : F \to V$.

**Definition 3.5.** A quadratic form $(V, q)$ over $F$ is *non-degenerate* if the associated symmetric bilinear form $B(x, y) = q(x + y) - q(x) - q(y)$ is non-dengnerate.

**Definition 3.6.** The *orthogonal sum* of two quadratic forms $(V_1, q_1)$ and $(V_2, q_2)$ is the form $(V_1 \oplus V_2, q_1 \oplus q_2)$ in which

$$(q_1 \oplus q_2)(v_1, v_2) = q_1(v_1) + q_2(v_2).$$

It is easy to check that the bilinear form associated to the orthogonal sum of two quadratic forms is the orthogonal sum of the two associated bilinear forms.

**Definition 3.7.** Suppose that $(V_1, q_1)$ and $(V_2, q_2)$ are quadratic forms over a field $F$. An *isometry*

$$T : (V_1, q_1) \to (V_2, q_2)$$

between two quadratic forms over a field $F$ is an invertible linear transformation $T : V_1 \to V_2$ satisfying

$$q_2(T(v)) = q_1(v)$$

for all $v \in V$.

Theorem III.3.13 and the relation between quadratic forms and symmetric bilinear forms described above gives

**Theorem 3.8.** *Suppose that $(V, q)$ is a quadratic form over a field $F$ whose characteristic is not 2. There is a basis $v_1, \dots, v_n$ of $V$ and elements $\lambda_1, \dots, \lambda_n \in F$ such that*

$$q(x_1v_1 + \cdots + x_nv_n) = \lambda_1x_1^2 + \cdots + \lambda_nx_n^2.$$

This means that over fields of characteristic not equal to 2 one can find a change of variables for every homogeneous polynomial of degree 2 which makes it a sum of multiples of squares. The situation in characteristic 2 is a little more involved.

**3.4. Quadratic forms over $\mathbb{F}_2$.** For this section we assume the field $F$ is the field $\mathbb{F}_2$. Note that if $(V, q)$ is a quadratic form over $F$ then the identity (3.3) implies that the associated symmetric bilinear form is even. Corollary III.3.34 then tells us that this symmetric bilinear form must be isometric to $\mathbf{H}^n \oplus \mathbf{0}^m$. Our aim is to classify all of the quadratic forms. We will restrict our attention to those that are non-degenerate. In that case the associated symmetric bilinear form is isometric to $\mathbf{H}^n$ for some $n$. In particular this means that the dimension of $V$ must be even.

Let's try and work out all the non-degenerate quadratic forms on $F^2$. By Proposition 3.2 there are exactly 8 possiblities for $q$;

$$\begin{aligned}
q(x,y) &= 0\\
q(x,y) &= x^2\\
q(x,y) &= xy\\
q(x,y) &= x^2+xy\\
q(x,y) &= y^2\\
q(x,y) &= x^2+y^2\\
q(x,y) &= xy+y^2\\
q(x,y) &= x^2+xy+y^2.
\end{aligned}$$

Of these the only ones that are non-degenerate are

$$\begin{aligned}
q(x,y) &= xy\\
q(x,y) &= x^2+xy\\
q(x,y) &= xy+y^2\\
q(x,y) &= x^2+xy+y^2.
\end{aligned}$$

As one can easily check, the underlying symmetric bilinear form in all four cases is just **H**. To go further let's make a table of values:

| | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
|---:|---|---|---|---|
| $xy$ | 0 | 0 | 0 | 1 |
| $x^2+xy$ | 0 | 1 | 0 | 0 |
| $xy+y^2$ | 0 | 0 | 1 | 0 |
| $x^2+xy+y^2$ | 0 | 1 | 1 | 1 |

Note that $q$ either takes the value 0 three times and 1 one time, or it takes the value 0 once and the value 1 three times.

**Exercise 3.6.** Show that the quadratic forms in the above table taking the value 0 three times are all isometric.

Now that you've done the above exercise you know that, up to isometry, there are exactly two non-degenerate quadratic forms over $\mathbb{F}_2$ in dimension 2. We will see that this holds in general. To do this we need to name some quadratic forms. Set

$$\begin{aligned}
\mathbf{H}_+ &= (F^2, xy)\\
\mathbf{H}_- &= (F^2, x^2+xy+y^2).
\end{aligned}$$

**Theorem 3.9.** *Suppose that $(V,q)$ is a non-degenerate quadratic form over $\mathbb{F}_2$. Then $(V,q)$ is isometric to one of*

$$\mathbf{H}_+^{\oplus n} \quad \textit{or} \quad \mathbf{H}_- \oplus \mathbf{H}_+^{\oplus n-1}.$$

We will prove Theorem 3.9 by induction on $\dim V$. You checked the case $\dim V = 2$ in Exercise 3.6.

**Lemma 3.10.** *Suppose that $(V, q)$ is a quadratic form over $F = \mathbb{F}_2$ and that $\dim V \geq 4$. There exists a vector $v \in V$ with $q(v) = 0$.*

*Proof:* Write $B(x, y) = q(x + y) - q(x) - q(y)$ for the underlying symmetric bilinear form. Since $B$ is even and non-degenerate we know from Theorem III.3.33 that $(V, B)$ is isometric to $\mathbf{H}^n$ for some $n \geq 1$. We may therefore suppose that $V = F^{2n}$ and that $B$ is $H^n$. Let $v = e_1 + e_2 + e_3 + e_4$. We compute

$$\begin{aligned} q(v) = q(e_1 + e_2 + e_3 + e_4) &= q(e_1 + e_2) + q(e_3 + e_4) + B(e_1 + e_2, e_3 + e_4) \\ &= q(e_1 + e_2) + q(e_3 + e_4) \\ &= q(e_1) + q(e_2) + B(e_1, e_2) + q(e_3) + q(e_4) + B(e_3, e_4) \\ &= 1 + 1 + 1 + 1 + 1 + 1 = 0. \end{aligned}$$

□

We now return to the proof of Theorem 3.9 and suppose we are given a non-degenerate quadratic form $(V, q)$ of dimension greater than 2. Since the dimension of $V$ must be even we know that the dimension of $V$ is at least 4. By Lemma 3.10 there is a vector $v \in V$ with $q(v) = 0$. Write $B(x, y) = q(x + y) - q(x) - q(y)$ for the underlying symmetric bilinear form (I could have written + instead of − since we are in characteristic 2). Since $B$ is non-degenerate there is a vector $w \in V$ with $B(v, w) = 1$. It follows that

$$q(v + w) + q(w) = q(v + w) + q(v) + q(w) = 1.$$

This means that one of $q(w)$ or $q(v + w)$ is equal to 1. Since

$$B(v, v + w) = B(v, v) + B(v, w) = B(v, w)$$

there is no loss of generality if we assume that $q(w) = 0$. Let $W \subset V$ be the subspace of $V$ spanned by $v$ and $w$. As in the proof of Theorem III.3.33 let

$$V' = \{x \in V \mid B(v, x) = B(w, x) = 0\}.$$

Write $q_W$ for the restriction of $q$ to $W$ and $q'$ for the restriction of $q$ to $V'$. I claim that

$$(V, q) \approx (W, q_W) \oplus (V', q').$$

We observed in the proof of Theorem 3.9 that this is true as vector spaces. So to verify the claim we have to check that for $x \in W$ and $y \in V'$ one has

$$q(x + y) = q(x) + q(y).$$

But $q(x + y) - q(x) - q(y) = B(x, y) = 0$, so this does indeed hold.

Finally, note that $(W, q_W)$ is isometric to $\mathbf{H}_+$:

$$\begin{aligned} q(xv + yw) &= q(xv) + q(yw) + B(xv, yw) \\ &= x^2 q(v) + xyB(v, w) + y^2 q(w) \\ &= xy. \end{aligned}$$

We have shown that if the dimension of $V$ is greater than 2, there is an isometry of $(V, q)$ with $\mathbf{H}_+ \oplus (V', q')$. Theorem 3.9 then follows by induction.

**3.5. The Arf invariant.** We'd like to know that the two forms in Theorem 3.9 are not in fact isometric to each other. The situation is a little like that of Theorem III.3.33, for which we were able to use the notion of "even" and "odd" forms to distinguish some of the cases, and the dimension of a maximal isotropic subspace to distinguish the others. There is an analogue of this notion for quadratic forms called the "Arf invariant."

Suppose that $(V, q)$ is a non-degenerate quadratic form over $\mathbb{F}_2$. One way of getting a measure of $q$ is to compare the number of solutions to the equation $q(v) = 0$ with the number of solutions to $q(v) = 1$. This suggest looking at the number

$$\#\{v \in V \mid q(v) = 0\} - \#\{v \in V \mid q(v) = 1\}.$$

It turns out that there is a clever choice of normalizing factor for this difference.

**Definition 3.11.** Suppose that $(V, q)$ is a non-degenerate quadratic form over $\mathbb{F}_2$. The *Arf invariant* of $(V, q)$ is the number

$$\begin{aligned} \operatorname{Arf}(V, q) &= \frac{1}{2^{\dim V/2}} \sum_{v \in V} (-1)^{q(v)} \\ &= \frac{1}{2^{\dim V/2}} \big(\#\{v \in V \mid q(v) = 0\} - \#\{v \in V \mid q(v) = 1\}\big). \end{aligned}$$

**Exercise 3.7.** Show that $\operatorname{Arf}(\mathbf{H}_+) = 1$ and $\operatorname{Arf}(\mathbf{H}_-) = -1$.

**Proposition 3.12.** *Suppose that* $\mathbf{q_1} = (V_1, q_1)$ *and* $\mathbf{q_2} = (V_2, q_2)$ *are non-degenerate symmetric bilinear forms over* $\mathbb{F}_2$. *Then*

$$\operatorname{Arf}(\mathbf{q_1} \oplus \mathbf{q_2}) = \operatorname{Arf}(\mathbf{q_1}) * \operatorname{Arf}(\mathbf{q_2}).$$

**Exercise 3.8.** Prove Proposition 3.12 by expanding out the right hand side.

Clearly isometric quadratic forms have the same Arf invariant. In fact

**Theorem 3.13.** *Two non-degenerate quadratic forms over* $\mathbb{F}_2$ *are isometric if and only if they have the same dimension and the same Arf invariant.*

**Exercise 3.9.** Show that the forms $\mathbf{H}_+^{\oplus n}$ and $\mathbf{H}_- \oplus \mathbf{H}_+^{\oplus n-1}$ are not isometric by computing their Arf invariants.

**Exercise 3.10.** Using Theorem 3.9 and the previous exercise prove Theorem 3.13.

**Exercise 3.11.** The quadratic forms $\mathbf{H}_+^2$ and $\mathbf{H}_-^2$ both have Arf invariant 1. Can you find an isometry between them?

The important thing about the normalizing factor in the definition of the Arf in variant is the following fact which is immediate from Exercise 3.7, Theorem 3.9 and Proposition 3.12.

**Corollary 3.14.** *If* $\mathbf{q}$ *is a non-degenerate quadratic form over* $\mathbb{F}_2$ *then*

$$\operatorname{Arf}(\mathbf{q}) = \pm 1.$$

□

Corollary 3.14 has a pretty cool consequence. Suppose that $(V, q)$ is a non-degenerate quadratic form over $\mathbb{F}_2$. Let $a$ and $b$ be the numbers

$$\begin{aligned} a &= \#\{v \in V \mid q(v) = 0\} \\ b &= \#\{v \in V \mid q(v) = 1\}. \end{aligned}$$

Write $\dim V = 2m$. Then we have

$$a + b = 2^n$$
$$a - b = \pm 1.$$

It follows that either

$$a = 2^{m-1}(2^m - 1)$$
$$b = 2^{m-1}(2^m + 1)$$

or

$$a = 2^{m-1}(2^m + 1)$$
$$b = 2^{m-1}(2^m - 1).$$

So we actually know the number of times $q$ takes on the value 0 or 1. You can think of this as the analogue for $\mathbb{F}_2$ of knowing the shape of $q(v) = 1$. Just to highlight it, I'll state it as a result.

**Corollary 3.15.** *Suppose that* $\mathbf{q} = (V, q)$ *is a non-degenerate quadratic form over* $\mathbb{F}_2$*, with* $\dim V = 2m$*. Then*

$$\#\{v \in V \mid q(v) = 1\} = \begin{cases} 2^{m-1}(2^m - 1) & \operatorname{Arf}(\mathbf{q}) = 1 \\ 2^{m-1}(2^m + 1) & \operatorname{Arf}(\mathbf{q}) = -1. \end{cases}$$

## 4. March 28th

Classification of 2-manifolds.

## 5. april 5th

loose ends (being written)

## 6. april 7

Immersions of orientable (even intersection form) surfaces.

(1) immersions up to isotopy

(2) immersions of cylinders

(3) form a group under surgery

(4) is generated by "one full twist"

(5) is Z/2

(6) linking number definition (umm... why does this require the bounding manifold to be immerssed? yikes)

(7) ended with the quadratic function to Z/2.

**6.1. suggested exercises.** Show that if an embedded cylinder can be the collar of any immersed surface it must have no full twists.
hot wheels move?
$M^{(2)} \times S \to N^2$ transverse to the diagonal? A submersion? double point? triple point?

## 7. Mayer-Vietoris

(1) maps of bordism groups

(2) the connecting homomorphism

(3) the long exact sequence.

## 8. Immersions

Maybe you need to define a "ribbon" to be a curve on an immersed surface, two being equivalent if they are cobordant on a surface. germs. surgery. might be cleaner. get $\mathbb{Z}/4$?

## 9. Exercises

(1) the basic pushout thingy

(2) to get collars, etc

(3) equivalence of two definitions of framing

(4) $MO_n\mathbb{R}^n \to MO_n(\mathrm{pt})$ is an isomorphism. (map of bo)

(5) show that if $M_1$ and $M_2$ can be embedded in $\mathbb{R}^n$ then so can their connected sum.

(6) homotopy invariance of $MO_n$?

(7) (well-definedness of connected sum) Show that if $M$ is a connected manifold and $x$ and $y$ are points of $M$ there is an embedding $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. (fix $x$ and show that the set of all $y$ which can be joined is both open and closed.) Show that there is a diffeomorphism taking any point to any other. Show that there is a diffeomorphism taking a shrinking of any framing to a shrinking of any other. (there is also a diffeomorphism which is isotopic to the identity)

(8) (well-definedness of surgery) Isotopic embeddings?

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