Math 114 Problem Set 5

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Problem 1. Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p \leq \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$, assuming $\|f\|_\infty > 0$. Find a function f such that $E = [1, 2)$.

We'll find such a function in two steps. First, consider the function $f_1 = 1/\sqrt{x}$ on $X = [2, \infty)$. Then assuming $p \neq 2$,

$$\varphi_{f_1}(p) = \int_2^\infty \frac{1}{\sqrt{x^p}} \, dx \leq \lim_{x\to\infty} \frac{x^{1-p/2}}{1-p/2} = \begin{cases} \infty & p > 2, \\ 0 & p < 2. \end{cases}$$

For $p = 2$, we have $\varphi_{f_1}(p) = \int_2^\infty \frac{1}{x} \, dx = \infty$. So $E_{f_1} = (0, 2)$. Next, consider the function

$$f_2(x) = \frac{1}{x \log^2 x}, \qquad \text{with} \qquad \varphi_{f_2}(1) = \int_2^\infty \frac{1}{x \log^2 x} \, dx < \infty.$$

Then for any $p > 1$, we have

$$\varphi_{f_2}(p) = \int_2^\infty \frac{1}{x^p \log^{2p} x} \, dx < \infty \qquad \text{since} \qquad \frac{1}{x^p \log^{2p} x} < \frac{1}{x \log^2 x}.$$

However for any $p < 1$, Wolfram Alpha tells us the integral diverges, so $E_{f_2} = [1, \infty)$. Finally, let's add these two functions to get $f = f_1 + f_2$. We claim that $E_f = E_{f_1} \cap E_{f_2} = [1, 2)$. To first show that $E_{f_1} \cap E_{f_2} \subset E_f$, suppose $p \in E_{f_1} \cap E_{f_2}$. Then

$$\|f\|_p = \|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p < \infty.$$

Similarly if $p \in E_f$, then

$$\|f_1\|_p, \|f_2\|_p \leq \|f_1 + f_2\|_p = \|f\|_p < \infty$$

so $E_f \subset E_{f_1} \cap E_{f_2}$ and so $E_f = [1, 2)$ as desired.

Problem 2. Assume, in addition to the hypotheses of the previous exercise that

$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \le \|f\|_s$ if $1 \le r < s \le \infty$.

(b) Under what condition does it happen that $1 \le r < s \le \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $1 \le r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ fo some $r > 0$, and prove that

$$\lim_{p\to 0} \|f\|_p = \exp \int_X \log|f|\, d\mu.$$

Assume WLOG that all functions are nonnegative. We'll begin by proving a generalization of Hölder's inequality.

Claim. Let $a, b, c \ge 1$ satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Then for any functions f, g, we have $\|fg\|_c \le \|f\|_a \|f\|_b$.

Proof. Let $f' = f^c$ and $g' = g^c$. Since $\frac{1}{a/c} + \frac{1}{b/c} = 1$, we can apply Hölder's inequality to get

$$\|f'g'\|_1 \le \|f'\|_{a/c} \|g'\|_{b/c}.$$

However recall that $\|f\|_{ab} = \|f^a\|_b^{1/a}$. We can use this identity to get

$$\begin{aligned} \|f'g'\|_1 = \|(f'g')^{1/c}\|_c^c = \|fg\|_c^c \le \|f'\|_{a/c}\|g'\|_{b/c} = \|f\|_a^c \|g\|_b^c \\ \implies \|fg\|_c \le \|f\|_a \|g\|_b, \end{aligned}$$

which is what we were trying to prove. Furthermore, equality only happens when $(f')^{a/c} = f^a$ and $(g')^{b/c} = g^b$ are linearly dependent almost everywhere. $\square$

(a) Since $r < s$, let $q \ge 1$ be such that $\frac{1}{r} = \frac{1}{s} + \frac{1}{q}$. Letting $g(x) = 1$, it then follows from the generalized Hölder inequality that

$$\|f\|_r = \|f \cdot 1\|_r \le \|f\|_s \|1\|_q = \|f\|_s \cdot \mu(X)^{1/q} = \|f\|_s.$$

(b) Suppose $\|f\|_s = \|f\|_r$. Then the generalized Hölder inequality and the proof of the previous part tell us that $\|f\|^s$ and 1 are linearly dependent. This means that f is a constant function almost everywhere.

(c) By (a), it follows that $L^r(\mu) \supset L^s(\mu)$ since $\|f\|_r \le \|f\|_s$. We claim that they are equal if and only if X contains no sequence of disjoint sets of positive measure.

Suppose first that such a sequence exists, say A_k. We have $\sum_k \mu(A_k) \le 1$ by disjointness of A_k, and so we must have $\mu(A_k) \to 0$. Let us choose a sequence k_n such that $\mu(E_{k_n}) \le 2^{-n}$, and define

$$f = \sum_{n=1}^{\infty} \mu(E_{k_n})^{-1/s} \chi_{E_{k_n}}.$$

Monotone convergence then implies that

$$\int |f|^s\, d\mu = \sum_{n=1}^{\infty} 1 = \infty \implies f \notin L^s(\mu).$$

Yet since $0 < 1 - r/s < 1$, we get

$$\int_X |f|^r \, d\mu = \sum_{n=1}^{\infty} \mu(E_{k_n})^{1-r/s} \leq \sum_{n=1}^{\infty} \left(\frac{1}{2^{1-r/s}} \right)^n < \infty.$$

Thus $f \in L^r(\mu)$. We can do a similar argument to get the edge case $s = \infty$.

Next, assume that there does not exist such a sequence. For any measurable function f, the sequence of sets $E_\ell = \{x \in X : f(x) \in [\ell, \ell+1)\}$ for any integer $\ell \in \mathbb{Z}$ is a sequence of disjoint measurable sets. So only a finite number of these can have positive measure. But this shows that the function is bounded, and since $\mu(X) = 1$, we have $f \in L^p(\mu)$ for all p.

(d) Since $\|f\|_p \geq 0$ decreases as $p \to 0$, the limit $\lim\limits_{p \to 0} \|f\|_p$ exists. Now recall that for $x \geq 0$, we have $\log x \leq (x^p - 1)/p$. In particular $\int \log |f|$ is bounded by $(\|f\|_r^r - 1)/r$. Letting $p < \min(1, r)$ we can look at the functions

$$g_p(x) = |f(x)| - 1 - \frac{|f(x)|^p - 1}{p}$$

is positive. By monotone convergence, we get

$$\lim_{p \to 0} \int_X g_p \, d\mu = \int_X |f| - 1 - \log |f| \, d\mu \implies \lim_{p \to 0} \exp \left(\int_X \frac{|f|^p - 1}{p} \right) = \exp \left(\int_X \log |f| \right).$$

Finally, since $\log x \leq x - 1$, we can conclude

$$\|f\|_p = \exp \left(\frac{1}{p} \log \left(\int_X |f|^p \, d\mu \right) \right) \leq \exp \left(\int_X \frac{|f|^p - 1}{p} \, d\mu \right).$$

As $p \to 0$, we get $\lim\limits_{p \to 0} \|f\|_p \leq \exp(\int_X \log |f|)$.

To prove the reverse inequality, we can assume that $|f| > 0$. Applying Jensen's inequality on the convex function $-\log x$, we get

$$-\log \int_X |f|^p \, d\mu \leq - \int_X \log |f|^p \, d\mu \implies \log \int_X |f|^p \, d\mu \geq \int_X \log |f|^p \, d\mu.$$

If we exponentiate both sides and raise to the $1/p$th power gives

$$\|f\|_p \geq \exp \left(\frac{1}{p} \int_X \log |f|^p \, d\mu \right) = \exp(\int_X \log |f| \, d\mu).$$

As before, when we take the limit as $p \to 0$, we get $\lim\limits_{p \to \infty} \|f\|_p \geq \exp(\int_X \log |f| \, d\mu)$. Thus we can conclude

$$\lim_{p \to \infty} \|f\|_p = \exp \left(\int_X \log |f| \, d\mu \right).$$