Math 132
Problem Set 4
Spring, 2018

This problem set is due on Friday, Feb. 24th Please make your answers as complete and clear as possible. You are allowed to discuss these problems with others in the class, but your writing should be your own.

1. Suppose that $X \subset \mathbb{R}^N$ is a smooth manifold, of dimension n, and let k be a non-negative integer. Show that the set of

$$((v_1, \dots, v_k), x) \in \mathbb{R}^{Nk} \times \mathbb{R}^N$$

with $x \in X$ and $v_i \in T_x X$ for $i = 1, \dots, k$ is a smooth manifold of dimension $n(k+1)$.

2. Suppose that $k \leq \ell$. Show that the set of linear transformations $T : \mathbb{R}^k \to \mathbb{R}^\ell$ having rank less than k has measure zero in the space of all linear transformation $\mathbb{R}^k \to \mathbb{R}^\ell$. (Hint: Consider making use of the previous problem, and the space of pairs (T, v) with $T : \mathbb{R}^k \to \mathbb{R}^\ell$ a linear transformation and $v \in \mathbb{R}^k$ a unit vector in the kernel of T.)

3. Suppose that F is a smooth manifold of dimension k and M is a smooth manifold of dimension ℓ. A *smooth fiber bundle* is a subspace $E \subset \mathbb{R}^n$ and a smooth map $p : E \to M$ with the property that for each point $x \in M$ there an open neighborhood $U \subset M$ of x and a diffeomorphism $p^{-1}(U) \to U \times F$ having the property that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "p"'] & U \times F \arrow[d, "(u,x)\mapsto u"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

commutes.

(a) Show that if $p : E \to M$ is a smooth fiber bundle, then E is in fact a smooth manifold of dimension $k + \ell$.

(b) Show that the tangent bundle of a manifold is a smooth fiber bundle.

4. In this problem we return to the Stiefel manifold $V_k(\mathbb{R}^n)$ which has made so many appearances in our problem sets. For a unit vector $v \in S^{n-1}$ let $R_v : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal transformation which sends v to $-v$ an fixes all the vectors w with $\langle v, w \rangle = 0$. This transformation is called *reflection through the hyperplane perpendicular to* v. It is given by the formula

$$R_v(x) = x - 2\langle x, v \rangle v.$$

Given two vectors $v, w \in S^{n-1}$, with $v \neq -w$ set $m = (v + w)/|v + w|$ and define $R_{v,w} = R_m \circ R_v$. The orthogonal transformation $R_{v,w}$ induces the unique rotation in the 2-plane spanned by v and w sending v to w (see Figure 1). Finally, let $p : V_k(\mathbb{R}^n) \to S^{n-1}$ be the map given by

$$p([v_1, \cdots, v_k]) = v_k.$$

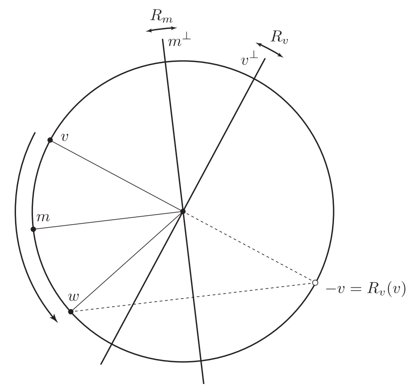

FIGURE 1. The rotation $R_{v,w}$

(a) Given $v \in S^{n-1}$ let $U = S^{n-1} - \{-v\}$, and $H = v^\perp$. Show that the map
$$g : p^{-1}(U) \to U \times V_{k-1}(H)$$
given by
$$g\big([v_1, \dots, v_{k-1}, x]\big) = \big(x, [Rv_1, \dots, Rv_{k-1}]\big),$$
where $R = R_{x,v}$, is a diffeomorphism.

(b) Show that the map p is a smooth fiber bundle with fiber $V_{k-1}(\mathbb{R}^{n-1})$. (HINT: a choice of orthonormal basis of H gives a diffeomorphism of $V_{k-1}(H)$ with $V_{k-1}(\mathbb{R}^{n-1})$)

(c) Using this give another proof that the Stiefel manifold is a smooth manifold.

5.

(a) (GP §8, Problem 4) Let $S(X)$ be the set of points $(x, v) \in T(X)$ with $|v| = 1$. Prove that $S(X)$ is a $(2k-1)$-dimensional submanifold of $T(X)$; it is called the *sphere bundle* of X. [HINT: You can do this with "bundles" as above or by considering the map $(x, v) \mapsto |v|^2$].

(b) Can you think of a relationship between the sphere bundle of S^{n-1} and a Stiefel manifold?

6. *This is a makeup problem. You may submit this if you had trouble with Problem 8 on Problem Set 3.* The point of this problem is to prove one case of the assertion of that locally, every transverse intersection looks like the intersection of coordinate planes. Suppose that X is a smooth manifold of dimension n, $W \subset X$ is a submanifold of dimension p and $Z \subset X$ is a submanifold of dimension $q = n - p$. Suppose also that Z and W are transverse. Let $x \in Z \cap W$ be a point.

(a) Show that there are smooth functions
$$f_W : U \to \mathbb{R}^q$$
$$f_Z : U \to \mathbb{R}^p$$
defined on some open neighborhood $U \subset X$ of x, having the property that 0 is a regular value, and with
$$f_W^{-1}(0) = W \cap U$$
$$f_Z^{-1}(0) = Z \cap U\ .$$

(b) Show that the maps

$$df_W : T_x W \to \mathbb{R}^q$$
$$df_Z : T_x Z \to \mathbb{R}^p$$

are zero.

(c) Using the transversality assumption, show that the maps

$$df_W : T_x Z \to \mathbb{R}^q$$
$$df_Z : T_x W \to \mathbb{R}^p$$

are isomorphisms.

(d) Using the above statements and the inverse function theorem, show that the map

$$U \xrightarrow{(f_W, f_Z)} \mathbb{R}^p \times \mathbb{R}^q$$

is a diffeomorphism of a possibly smaller neighborhood U' of x with an open neighborhood of the origin in $\mathbb{R}^p \times \mathbb{R}^q$.

(e) Show that under this diffeomorphism, $W \cap U'$ corresponds to the coordinate plane $\mathbb{R}^p \times \{0\}$ and $Z \cap U'$ corresponds to the coordinate plane $\{0\} \times \mathbb{R}^q$.