Math 132
Problem Set 6
Spring, 2023

This problem set is due on Friday, March 24th Please make your answers as complete and clear as possible. You are allowed to discuss these problems with others in the class, but your writing should be your own.

1. Give an example of a 1-manifold M having the property that ∂M consists of 3 points. Show that there are infinitely many such manifolds, no two of which are diffeomorphic. Why doesn't this contradict our classification result?

2. Suppose that X is a compact n-manifold, for $i = 1, \dots, k$ and there are smooth functions functions

$$\phi_i : X \to \mathbb{R}^n$$
$$\lambda_i : X \to [0, 1]$$

with the following properties

i) For all i, the image of ϕ_i contains the open ball $B_2 \subset \mathbb{R}^n$ of radius 2.
ii) Set $V_i = \phi_i^{-1}(B_2)$. The map ϕ_i restricts to a diffeomorphism $V_i \to B_2$.
iii) The open subsets $U_i = \phi_i^{-1} B_1$ cover X, in which $B_1 \subset \mathbb{R}^n$ is the open ball of radius 1.
iv) For $x \in U_i$, $\lambda_i(x) = 1$.
v) For $x \in X - V_i$, $\lambda_i(x) = 0$.

One can construct such data using some coordinate charts and a bump function. Using these, define a map

$$g : X \to \left(\mathbb{R}^n \times \mathbb{R}^1\right)^k = \mathbb{R}^{k(n+1)}$$

by

$$g(x) = \big((\phi_1, \lambda_1), \cdots, (\phi_k, \lambda_k)\big).$$

Show that the map g is an embedding. Since X is compact this amounts to showing that it is an immersion and one to one. (Hint: I think you should be able to do this on your own, but if you get stuck, look at pages 23-4 of the book of Hirsh listed on the course web page).

3. Recall the tubular neighborhood theorem: If $X \subset \mathbb{R}^n$ is a smooth manifold there is an open neighborhood $X \subset U \subset \mathbb{R}^n$ of X, and a smooth map $r : U \to X$ having the property that for all $x \in X$, $r(x) = x$. (The map r is called a *retraction*). Using the tubular neighborhood theorem, prove the following results.

(a) Suppose that $Z \subset X$ is a compact submanifold of a compact manifold X (both without boundary), there is an open neighborhood $Z \subset U \subset X$ and a smooth retraction $r : U \to Z$.

(b) If X is a compact manifold with non-empty boundary, ∂X, there is an open neighborhood $\partial X \subset U \subset X$ and a retraction $r : U \to \partial X$. (Hint: Use that fact that X comes to you as a subset of $\mathbb{R}^N$.)

4. This problem proves the existence of a *collar* neighborhood of the boundary of a manifolds. Suppose that X is a compact smooth manifold with non empty boundary ∂X.

(a) Show that there is a smooth function $f : X \to [0, \infty) \subset \mathbb{R}$ having the properties that
(1) $f^{-1}(0) = \partial X$
(2) For each $x \in \partial X$ the map $df : T_x X \to T_0\mathbb{R}$ is surjective.
(Hint: Partitions of unity are your friend.)

(b) Now let (U, r) be the neighborhood of ∂X and the retraction $r : U \to \partial X$ constructed in the previous problem. Show that the map $(r, f) : U \to \partial X \times [0, \infty)$ restricts to a diffeomorphism $U' \to \partial X \times [0, \varepsilon)$ for some neighborhood $U' \subset U$ of ∂X. Such a neighborhood is called a *collar neighborhood*.

5. This is Problem 4 of Chapter 2, §3 of GP. Let X and Y be submanifolds of $\mathbb{R}^N$. Show that for almost all $a \in \mathbb{R}^N$, the translate

$$X + a = \{x + a \mid x \in X\}$$

intersects Y transversally.