Math 137 Final Exam

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I affirm my awareness of the standards of the Harvard College Honor Code.

Problem 1. Let $A \subseteq K^a$, $B \subseteq K^b$, $C \subseteq K^c$ be algebraic subsets and let $\varphi : A \to B$ and $\psi : B \to C$ be morphisms. Assume that the morphism $\psi \circ \varphi : A \to C$ is finite.

(a) **(3 points)** Show that $\varphi : A \to B$ is finite.

(b) **(4 points)** Show that if φ is dominant, then $\psi : B \to C$ is also finite.

(c) **(3 points)** Show that (b) can fail without the assumption that φ is dominant.

(a) Since $\psi\circ\varphi$ is a finite morphism, by definition $\Gamma(A)$ is a finite $(\psi\circ\varphi)^*(\Gamma(C))$-module. Since $(\psi\circ\varphi)^* = \varphi^*\circ\psi^*$, this is the same as a $\varphi^*(\psi^*(\Gamma(C)))$-module. Since $\varphi^*(\psi^*(\Gamma(C))) \subset \varphi^*(\Gamma(B))$, any generating set for $\Gamma(A)$ as a $\varphi^*(\psi^*(\Gamma(C)))$ will also be a generating set for $\Gamma(A)$ as a $\varphi^*(\Gamma(B))$-module. This proves that φ is a finite morphism.

(b) Since $\psi \circ \varphi$ is a finite morphism, $\Gamma(A)$ is integral over $(\psi \circ \varphi)^*(\Gamma(C))$. This means that for any $f \in \Gamma(A)$, there is some monic polynomial $F_f \in (\psi \circ \varphi)^*(\Gamma(C))[X]$ such that $F_f(f) = 0$. Since $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$, this is actually a polynomial $F_f \in \varphi^*(\psi^*(\Gamma(C)))[X]$. Now suppose $g \in \Gamma(B)$. Then $\varphi^*(g)$ is a polynomial in $\Gamma(A)$, so there is some $F_{\varphi^*(g)}$ which annihilates $\varphi^*(g)$. Since φ^* is injective, (by dominance) there is a polynomial $H \in \psi^*(\Gamma(C))[X]$ such that $\varphi^*(H) = F_{\varphi^*(g)}$. Then $\varphi^*(H)(\varphi^*(g)) = H(g) = 0$ and we are done since g was arbitrary.

(c) Consider the morphisms $K \to K^2 \to K$ given by $t \mapsto (t, 0) \mapsto t$. The composition of these morphisms is the identity morphism which is clearly finite, and the first is finite as expected, yet the second projection morphism $K^2 \to K$ is clearly not finite since $K \to K^2$ isn't dominant.

Problem 2. **(10 points)** Show that all isomorphisms $\varphi : K \to K$ (of algebraic sets) are of the form $x \mapsto ax+b$ for $a \in K^\times$ and $b \in K$.

Recall from lecture that there is a bijective correspondence between isomorphisms of algebraic sets and K-algebra isomorphisms of their coordinate rings. So let $\varphi : K \to K$ be an isomorphism. Then $\varphi^* : K[t] \to K[t]$ is a K-algebra homomorphism. Such a map must preserve degrees and is entirely determined by $\varphi^*(t)$, so $\varphi^*(t) = at+b$ for some $a \in K^\times$ and $b \in K$. Since $\varphi^*(\lambda(x)) = \lambda(\varphi(x)) = \lambda(ax+b)$ for any $\lambda \in \Gamma(K)$, it follows that $\varphi(x) = ax + b$. It is easy to see that the converse holds as well; i.e. given any $a \in K^\times$ and $b \in K$, the map $\varphi(x) = ax + b$ is an isomorphism of algebraic sets.

Problem 3. **(10 points)** Let $K = \mathbb{C}$. Show that there is an algebraic subset V of K^n (for some n) and a surjective morphism $\varphi : K^2 \to V$ whose fibers $\varphi^{-1}(P)$ for $P \in V$ are exactly the sets of the form $\{(x, y), (-x, -y)\}$ with $(x, y) \in K^2$.

Let $n = 3$ and let $V = \mathcal{V}(y^2 - xz) \subset K^3$. Consider the morphism $\varphi : K^2 \to V$ given by $\varphi(a, b) = (a^2, ab, b^2)$. This is well defined because $(ab)^2 = a^2b^2$. Next, we'll prove that for any $(x, y, z) \in V$ (satisfying $y^2 = xz$) the preimage $\varphi^{-1}(x, y, z) = \{(a, b), (-a, -b)\}$. This will also prove surjectivity. Suppose $\varphi(a, b) = (x, y, z)$. This means that $a^2 = x$ and $b^2 = z$ so $a = \pm\sqrt{x}$ and $b = \pm\sqrt{y}$. Thus $\varphi^{-1}(x, y, z) \subset \{(\pm\sqrt{x}, \pm\sqrt{z})\}$. Checking these manually, note that $\varphi(\sqrt{x}, \sqrt{y}) = \varphi(-\sqrt{x}, -\sqrt{y}) = (x, y, z)$ yet $\varphi(-\sqrt{x}, \sqrt{y}) = \varphi(\sqrt{x}, -\sqrt{y}) = (x, -y, z)$. So only two of these are in the preimage and we are done.

Problem 4. Let $K = \mathbb{C}$. Let

$$H = \{(a, b, c, d) \in K^4 \mid a = c = 0\},$$
$$V = \{(a, b, c, d) \in K^4 \mid ab = cd\},$$
$$\Delta = \{(P, P) \in K^4 \times K^4 \mid P \in V\}.$$

(a) **(2 points)** What is the dimension of Δ?

(b) **(4 points)** Show that there is no polynomial $f \in K[A, B, C, D]$ such that $H = \mathcal{V}_V(f)$.

(c) **(4 points)** Show that there are no polynomials

$$g_1, g_2, g_3 \in K[A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2]$$

such that $\Delta = \mathcal{V}_{V\times V}(g_1, g_2, g_3)$.

(a) Since $V = \mathcal{V}_{K^4}(ab - cd)$ is the vanishing locus of a single polynomial (Krull's principal ideal theorem), $\operatorname{codim}(V, K^4) = 1$ so $\dim V = 3$. Then there is a dominant morphism $V \to \Delta$ which sends P to (P, P) so $\dim \Delta = 3$.

(b) Suppose for the sake of contradiction that there is some polynomial $f \in K[A, B, C, D]$ such that $H = \mathcal{V}_V(f) = \mathcal{V}(f) \cap V$. Restricting to the surface parametrized by $(x, y) \mapsto (x, y, y, x) \in V$, note that $f(x, y, y, x) = 0$ if and only if $x = y = 0$. But $f(x, y, y, x)$ is a single polynomial in $K[x, y]$ with vanishing locus a single point. This is a contradiction to Krull's principal ideal theorem, since its vanishing locus should be one dimensional.

(c) :(

Problem 5. **(10 points)** Let $n \geq 1$ and $d \geq 0$. Let $F = F_d$ be the vector space of polynomials $f \in K[X_1, \dots, X_n]$ of total degree at most d. Let A be the set of tuples $(f_1, \dots, f_{n+1}) \in F \times \cdots \times F$ such that $\mathcal{V}_{K^n}(f_1, \dots, f_{n+1}) \neq \varnothing$. Show that $\overline{A} \neq F \times \cdots \times F$.

Consider the space $K^n \times F^{n+1}$ equipped with natural maps $\pi_1 : K^n \times F^{n+1} \to K^n$ and $\pi_2 : K^n \times F^{n+1} \to F^{n+1}$. We also have a natural morphism $\varphi : K^n \times F^{n+1} \to K^{n+1}$ which sends a point $x \in K^n$ and tuple of polynomials $(f_1, \dots, f_{n+1})$ to the tuple $(f_1(x), \dots, f_{n+1}(x))$. We're interested in the preimage $B = \varphi^{-1}(0)$, which is algebraic due to Zariski continuity and relevant because $\pi_2(B) = A$.

Recall that for any given point $P \in K^n$, we have an evaluation map $\mathrm{ev}_P : F_d \to K$ which sends f to $f(P)$. This is clearly a morphism. We observe that $\mathrm{ev}_P^{-1}(0)$ is a nonempty algebraic subset of F_d of dimension $\dim F_d - 1$ because it is the vanishing locus of a single nonconstant, nonzero polynomial in $\Gamma(F_d)$. (Theorem 10.7) Then since $\varphi|_{\pi_1^{-1}(P)} = \mathrm{ev}_P \times \cdots \times \mathrm{ev}_P$, it follows that $\dim(\pi_1^{-1}(P) \cap B) = (n+1)(\dim F_d - 1)$ for any $P \in K^n$. Since K^n is irreducible, we apply Theorem 13.4.1 to the components of B which gives us $\dim(B) \leq \dim(\pi_1^{-1}(P) \cap B) + \dim(K^n) = (n+1)(\dim F_d - 1) + n = (n+1)\dim F_d - 1 < \dim F_d^{n+1}$. So $\dim \overline{A} \leq \dim \overline{\pi_2(B)} \leq \dim B < \dim F_d^{n+1}$ and thus $\dim \overline{A} < \dim F_d^{n+1}$. Since the dimension of $\overline{A}$ is strictly smaller than the dimension of F_d^{n+1}, we finally have $\overline{A} \neq F_d^{n+1}$ as desired.

Problem 6. **(10 points)** Which of the following statements are true? Which are false? (You don't need to give a proof or counterexample.) Any correct answer for a statement gives two points. Any incorrect answer gives zero points. If you don't answer, you get one point.

(a) **(2 points)** Any two birational irreducible algebraic subsets $V \subseteq K^n$ and $W \subseteq K^m$ have the same dimension.

(b) **(2 points)** If $V \subseteq K^n$ and $W \subseteq K^m$ are algebraic sets and $\varphi : V \to W$ is a bijective finite morphism, then φ is an isomorphism.

(c) **(2 points)** If $V \subseteq K^n$ and $W \subseteq K^m$ are algebraic sets, $\varphi : V \to W$ is a morphism, and $P \in \varphi(V)$, then $\dim(V) \le \dim(W) + \dim(\varphi^{-1}(P))$.

(d) **(2 points)** For every monomial order on the monomials in $X_1, \ldots, X_n$, for all monomials $M < N$, there exists a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $M(a_1, \ldots, a_n) < N(a_1, \ldots, a_n)$.

(e) **(2 points)** Three planes H_1, H_2, H_3 in $\mathbb{P}^3_K$ (with $H_i \neq H_j$ for all $i \neq j$) always intersect in exactly one point.

Providing explanations for myself as a sanity-check.

(a) True. Since they are birational, $K(V) \cong K(W)$ as K-algebras, so these algebras must have the same transcendence degree over K, and so the dimensions of V and W must be the same.

(b) False. Consider the morphism $\varphi : K \to \mathcal{V}(y^2 - x^3)$ given by $\varphi(t) = (t^2, t^3)$.

(c) False. Can construct a counterexample if V and W are not irreducible.

(d) True.

(e) False. Three planes intersect at a line.