Math 137 Problem Set 3

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Throughout, K is assumed to be an algebraically closed field.

Problem 1. Let A be an irreducible algebraic subset of K^n and let B be an irreducible algebraic subset of K^m. Show that the cartesian product $A \times B$ is an irreducible algebraic subset of $K^n \times K^m = K^{n+m}$.

Problem 2. If S is an integral ring extension of R and T is an integral ring extension of S, prove that T is an integral ring extension of R.

Problem 3. Show that any integral ring-finite ring extension is module-finite.

Problem 4. Find a ring-finite ring extension S of a ring R for which S is a field, but which is not module-finite.

Problem 5.

(a) What is the integral closure of $K[X^2, X^3] \subseteq K[X]$ in $K(X)$?

(b) Let $R = \mathbb{Z}/4\mathbb{Z}$. What is the integral closure of R in $R[X]$?

Problem 6. Let X be a topological space. Let $A \neq \varnothing$ be a closed subset and equip it with the subspace topology. Show that the following are equivalent:

(a) We cannot write $A = A_1 \cup A_2$ with closed subsets $A_1, A_2 \subsetneq A$.

(b) Any two open subsets $\varnothing \neq U_1, U_2 \subseteq A$ (for the subspace topology) intersect.

(c) Every open subset $\varnothing \neq U \subseteq A$ (for the subspace topology) is dense in A.

Such a set A is called *irreducible*. (Note that the irreducible algebraic subsets of K^n are exactly the irreducible closed subsets for the Zariski topology.)

Problem 7. Let $f : X \to Y$ be a continuous map between topological spaces. Let $A \subseteq X$ be a closed irreducible subset. Show that $\overline{f(A)} \subseteq Y$ is irreducible.