

Math 230a Problem Set 3

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Problem 1. Distributions of rank 2 on 3-manifolds.

a. Let M be a 3-manifold and α a non-zero 1-form. Prove that the 2-dimensional distribution determined by α is integrable if and only if $\alpha \wedge d\alpha = 0$.

Suppose α is a nowhere zero 1-form on M . We can consider this a section $\alpha \in \Gamma(T^*M)$. Since α is nowhere zero, the kernel bundle $\ker \alpha$ is a 2-dimensional subbundle of TM . Let's call this distribution E_α . Clearly, a vector field $\xi \in \Gamma(TM)$ belongs to E_α if and only if $\alpha(\xi) = 0$. The distribution is integrable if and only if for all vector fields ξ_1, ξ_2 belonging to E_α , their commutator $[\xi_1, \xi_2]$ also belongs to E_α . This means that we can we must show that:

$$\{\alpha \wedge d\alpha = 0\} \iff \{\alpha([\xi_1, \xi_2]) = 0 \text{ whenever } \alpha(\xi_1) = \alpha(\xi_2) = 0\}.$$

By the standard commutator relations between d, ι, \mathcal{L} , for any vector fields ξ_1, ξ_2 belonging to E_α , we have

$$\begin{aligned} \alpha([\xi_1, \xi_2]) &= \iota_{[\xi_1, \xi_2]}\alpha \\ &= \mathcal{L}_{\xi_1}\iota_{\xi_2}\alpha + \iota_{\xi_2}\mathcal{L}_{\xi_1}\alpha \\ &= \iota_{\xi_2}\mathcal{L}_{\xi_1}\alpha \\ &= \iota_{\xi_2}(d\iota_{\xi_1} + \iota_{\xi_1}d)\alpha \\ &= \iota_{\xi_2}d\iota_{\xi_1}\alpha + \iota_{\xi_1}d\alpha \\ &= \iota_{\xi_2}\iota_{\xi_1}d\alpha \\ &= d\alpha(\xi_2, \xi_1) \end{aligned}$$

Next, suppose ξ_1, ξ_2, ξ_3 are any vector fields, not necessarily in E_α . Then, we have

$$\begin{aligned} (\alpha \wedge d\alpha)(\xi_1, \xi_2, \xi_3) &= \iota_{\xi_1}\iota_{\xi_2}\iota_{\xi_3}(\alpha \wedge d\alpha) \\ &= \iota_{\xi_1}\iota_{\xi_2}(\iota_{\xi_3}\alpha \wedge d\alpha - \alpha \wedge \iota_{\xi_3}d\alpha) \\ &= \iota_{\xi_1}(-\iota_{\xi_3}\alpha \wedge \iota_{\xi_2}d\alpha - \iota_{\xi_2}\alpha \wedge \iota_{\xi_3}d\alpha + \alpha \wedge \iota_{\xi_2}\iota_{\xi_3}d\alpha) \\ &= \iota_{\xi_3}\alpha \wedge \iota_{\xi_1}\iota_{\xi_2}d\alpha + \iota_{\xi_2}\alpha \wedge \iota_{\xi_1}\iota_{\xi_3}d\alpha + \iota_{\xi_1}\alpha \wedge \iota_{\xi_2}\iota_{\xi_3}d\alpha \\ &= \alpha(\xi_3) \cdot d\alpha(\xi_1, \xi_2) + \alpha(\xi_2) \cdot d\alpha(\xi_1, \xi_3) + \alpha(\xi_1) \cdot d\alpha(\xi_2, \xi_3). \end{aligned}$$

Here, we make use of the identity $\iota_\xi(\alpha \wedge \beta) = \iota_\xi\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota_\xi\beta$, and cancel terms such as $\iota_{\xi_i}\iota_{\xi_j}\alpha$ and $\iota_{\xi_i}\iota_{\xi_j}\iota_{\xi_k}d\alpha$ since all negative degree forms are zero. With the two identities we derived, we can now get our result.

First, suppose that $\alpha \wedge d\alpha = 0$. Whenever we have vector fields ξ_1, ξ_2 belonging to E_α , and a vector field η not necessarily belonging to E_α , note that

$$\begin{aligned} 0 &= (\alpha \wedge d\alpha)(\xi_1, \xi_2, \eta) = \alpha(\eta) \cdot d\alpha(\xi_1, \xi_2) + \alpha(\xi_2) \cdot d\alpha(\xi_1, \eta) + \alpha(\xi_1) \cdot d\alpha(\xi_2, \eta) \\ &= \alpha(\eta) \cdot \alpha([\xi_2, \xi_1]). \end{aligned}$$

Since α is nonzero, we can find some vector field η such that $\alpha(\eta)$ is nonzero. This means that $\alpha([\xi_2, \xi_1]) = 0$, and so the distribution is integrable.

In the converse direction, suppose that for any vector fields ξ_1, ξ_2 belonging to E_α , we have $\alpha([\xi_1, \xi_2]) = 0$. Let U be an open set on which we can find such fields $\xi_1, \xi_2 \in \Gamma(E_\alpha; U)$ which are also linearly independent. Finally, suppose that there is a third field $\eta \in \Gamma(TM; U)$, which is linearly independent to ξ_1, ξ_2 so that these fields span the tangent bundle TM restricted to U . On U , we have

$$(\alpha \wedge d\alpha)(\xi_1, \xi_2, \eta) = \alpha(\eta) \cdot \alpha([\xi_2, \xi_1]) = 0.$$

However, these fields formed a local frame the tangent bundle, so it follows that $\alpha \wedge d\alpha$ vanishes on U . Since our choice of U was arbitrary, we can do this over the entire manifold to show that $\alpha \wedge d\alpha$ vanishes globally.

b. The Hopf fibration $\pi : S^3 \rightarrow S^2$ may be constructed by identifying S^3 as the unit sphere in \mathbb{C}^2 and S^2 as \mathbb{CP}^1 ; then the map is a restriction of the canonical projection $(\mathbb{C}^2)^\times \rightarrow \mathbb{CP}^1$. The kernel $E' = \ker d\pi$ is an (integrable) one-dimensional distribution on S^3 . Let $E \subset TS^3$ be the 2-dimensional distribution given by the orthogonal complement of E' with respect to the standard round metric. Is E integrable? Find a nonzero 1-form α which generates the ideal $\mathcal{I}(E)$ associated to E . Compute $d\alpha$ and $\alpha \wedge d\alpha$.

We can consider S^3 as the subset of \mathbb{C}^2 given by points (z_1, z_2) with $|z_1|^2 + |z_2|^2 = 1$. Choosing coordinates:

$$\begin{cases} z_1 = \cos(\theta) \cdot e^{i(\phi+\psi)} \\ z_2 = \sin(\theta) \cdot e^{i(\phi-\psi)} \end{cases} \quad \text{where } (\theta, \phi, \psi) \in [0, \pi/2] \times [0, \pi]^2.$$

In these coordinates, the Hopf fibration becomes $\pi(\theta, \phi, \psi) \rightarrow (\theta, \phi)$ where we use the standard spherical coordinate system on S^2 .

Next, let's see what form the round metric takes on S^3 using the coordinate system we provided. Recall that the round metric on S^3 is the pullback of the Euclidean metric on \mathbb{R}^4 from the canonical embedding $\mathbb{C}^2 \rightarrow \mathbb{R}^4$. Expanding our coordinate system in \mathbb{R}^4 , and using standard trigonometric identities, up to scaling, the metric g on S^3 has matrix form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\cos(\theta) \\ 0 & 2\cos(\theta) & 1 \end{pmatrix} \iff g = d\theta^2 + d\phi^2 + 2\cos\theta d\phi d\psi + d\psi^2.$$

Now suppose at a point $(\theta, \phi, \psi) \in S^3$, the tangent vector $v \in T_{(\theta, \phi, \psi)}S^3$ is in the kernel of $d\pi$. This means that v must have no $\partial/\partial\phi$ or $\partial/\partial\psi$ components. The space of vectors $v = (x, y, z)$ in the complement of this kernel must then satisfy, for every $t \in \mathbb{R}$,

$$\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\cos(\theta) \\ 0 & 2\cos(\theta) & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \implies (2\cos\theta)y + z = 0.$$

In other words, E is spanned by the vector fields

$$\xi_1 = \frac{\partial}{\partial\theta} \quad \text{and} \quad \xi_2 = \frac{\partial}{\partial\psi} + 2\cos(\theta) \frac{\partial}{\partial\phi}.$$

This distribution is thus **not integrable**, since a simple calculation shows that the commutator of these vector fields is $[\xi_1, \xi_2] = -2\sin(\theta)\partial/\partial\phi$, which does not belong to E .

Next, we need to find a 1-form α which generated the ideal associated to the distribution. Since we have vector fields which already span E , it's clear that the 1-form $\alpha = 2\cos(\theta)d\psi - d\phi$ vanishes on E , and clearly must be a generator for $\mathcal{I}(E)$ since it has minimal degree. Computing $d\alpha$ and $\alpha \wedge d\alpha$, we get

$$d\alpha = -2\sin(\theta)d\theta \wedge d\psi \implies \alpha \wedge d\alpha = (2\cos(\theta)d\psi - d\phi) \wedge (-2\sin(\theta)d\theta \wedge d\psi) = 2\sin(\theta)d\phi \wedge d\theta \wedge d\psi$$

Since $\alpha \wedge d\alpha$ is non-zero, this agrees with the previous part of the problem.

Problem 2. Suppose M is a smooth manifold and $E \subset TM$ is a distribution. Define:

$$\mathcal{I}(E) = \{\omega \in \Omega^\bullet(M) : \omega|_E = 0\}.$$

a. Prove that $\mathcal{I}(E) \subset \Omega_M^\bullet$ is an ideal.

Clearly $\mathcal{I}(E)$ is additively closed. For any $\omega, \eta \in \Omega^\bullet(M)$ with $\omega \in \mathcal{I}(E)$ and $\xi_i \in \Gamma(E)$, recall that

$$(\omega \wedge \eta)(\xi_1, \dots, \xi_n) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \cdot \omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(k)}) \cdot \eta(\xi_{\sigma(k+1)}, \dots, \xi_{\sigma(n)}),$$

where $|\omega| = k$ and $n = |\omega| + |\eta|$.

Clearly, every term in this sum will vanish since $\omega|_E = 0$ and all vector fields belong to E .

b. Prove that if E has corank r – that is, if $\dim E_x + r = \dim_x M$ for all $x \in M$ – then E is locally generated by r independent 1-forms.

It suffices to consider connected manifolds when the dimension is constant, since the result is local anyways.

c. Prove that $\mathcal{I}(E)$ is closed under d if and only if E is integrable.

For any k -form $\omega \in \Omega^\bullet(M)$ and vector fields $\xi_1, \dots, \xi_{k+1} \in \Gamma(TM)$, we have

$$\begin{aligned} d\omega(\xi_1, \dots, \xi_{k+1}) &= \sum_{1 \leq i \leq k+1} (-1)^{i+1} \xi_i \omega(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{k+1}) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j+1} \omega([\xi_i, \xi_j], \xi_1, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_{k+1}), \end{aligned}$$

where $\widehat{\xi_i}$ denotes omission of the i -th entry. This identity can be proved inductively using the standard commutator relations between ι, d , and \mathcal{L} .

Now, recall that E is integrable if and only if for any vector fields $\xi_1, \xi_2 \in \Gamma(E)$ belonging to E , their commutator $[\xi_1, \xi_2]$ belongs to E as well. Supposing E is integrable and that ω vanishes on E , it's clear that $d\omega$ vanishes on E since all the vector fields in the terms of the sum are in E , and ω vanishes on vector fields in E . Conversely, if $\omega \in \mathcal{I}(E)$ and $d\omega$ vanishes, then we know that for all vector fields $\xi_1, \xi_2 \in \Gamma(E)$ and $\xi_3, \dots, \xi_{k+1} \in \Gamma(TM)$ we have

$$\begin{aligned} 0 &= d\omega(\xi_1, \dots, \xi_{k+1}) = \sum_{1 \leq i \leq k+1} (-1)^{i+1} \xi_i \omega(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{k+1}) \\ &\quad + \sum_{0 \leq i < j \leq k+1} (-1)^{i+j+1} \omega([\xi_i, \xi_j], \xi_1, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_{k+1}) \\ &= \omega([\xi_1, \xi_2], \xi_3, \dots, \xi_{k+1}) \end{aligned}$$

d. Consider the distribution E on $\mathbb{A}_{x,y,z}^3$ spanned by the vector fields $\partial/\partial x$ and $x\partial/\partial y + \partial/\partial z$. Show that E is not integrable. Show that any point $(x, y, z) \in \mathbb{A}$ may be joined to the origin by a piecewise smooth curve whose tangent line belongs to E .

Problem 3. Example or proof of nonexistence: A codimension 1 foliation on the sphere S^4 .

Problem 4. The Frobenius tensor.

- a.** Let $P, Q : \mathbb{A}^2 \rightarrow \mathbb{R}$ be smooth functions. Define the 2-dimensional distribution E on $\mathbb{A}_{x,y}^2 \times \mathbb{R}_z$ with

$$E_{(x,y,z)} = \text{span} \left\{ \frac{\partial}{\partial x} + P \frac{\partial}{\partial z}, \frac{\partial}{\partial y} + Q \frac{\partial}{\partial z} \right\}.$$

Compute the Frobenius tensor of E .

- b.** Suppose X is a manifold and G a Lie group. Let $\{\theta^i\}$ be a basis of left-invariant 1-forms on G and suppose

$$d\theta^i + \frac{1}{2}c_{jk}^i \theta^j \wedge \theta^k = 0$$

for constants c_{jk}^i . Let $\{\theta_X^i\}$ be 1-forms on X . Consider the ideal of differential forms on $X \times G$ generated by $\pi_G^* \theta^i - \pi_X^* \theta_X^i$, where $\pi_X : X \times G \rightarrow X$ and $\pi_G : X \times G \rightarrow G$ are projections. Prove that this ideal is closed under d if and only if

$$d\theta_X^i + \frac{1}{2}c_{jk}^i \theta_X^j \wedge \theta_X^k = 0.$$

- c.** Compute the Frobenius tensor of the distribution in (b) defined as the simultaneous kernel of the 1-forms $\pi_G^* \theta^i - \pi_X^* \theta_X^i$.