Astron 140 Problem Set 4

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Due: October 9, 2024

Problem 1. In vector decomposition into components along basis vectors, the completeness condition $e_\mu \otimes e^\mu = 1$ ensures that all vectors (not just a subset of vectors) can be decomposed using these basis vectors. In this exercise, let us rigorously prove the above statement in general.

(a). Prove that given an arbitrary vector A and the basis vectors e_μ (and the inverse basis vectors e^μ), the components of A along the basis vectors e_μ, denoted as A^μ, are given $A^\mu = A \cdot e^\mu$.

By the definition of the dot product operation in a vector space, we have:

$$A \cdot e^\mu = (A^\mu e_\mu) \cdot (e^\nu) = A^\mu (e_\mu \cdot e^\nu) = A^\mu \delta_\mu^\nu = A^\mu.$$

(b). Prove that after getting the components along the basis vectors e_μ, the vector $A^\mu e_\mu$ equals to A if the completeness condition is satisfied. Namely, these basis vectors are complete and no others are needed.

Next, by the completeness condition we can write

$$A = (e_\mu \otimes e^\mu) A = e_\mu (e^\mu \cdot A) = e_\mu (A \cdot e^\mu) = e_\mu A^\mu = A^\mu e_\mu.$$

(c). To improve your understanding, look at the following counterexample. For example, in the 3D Cartesian coordinates, if only e_x and e_y are given, check that these two basis alone do not satisfy the 3D completeness condition. As a consequence, convince yourself that the two basis vectors given above are not enough to decompose an arbitrary 3D vector. This can be done only after e_z is introduced.

The vectors e_x, e_y alone do not satisfy the completeness conditions since

$$(e_x \otimes e^x + e_y \otimes e^y) e_z = e_x (e^x \cdot e_z) + e_y (e^y \cdot e_z) = 0.$$

However if we introduce e_z, we get

$$(e_x \otimes e^x + e_y \otimes e^y + e_z \otimes e^z) e_z = e_x (e^x \cdot e_z) + e_y (e^y \cdot e_z) + e_z (e^z \cdot e_z) = e_z.$$

Problem 2. This is a simple explicit example on basis and inverse-basis vectors. The basis vectors for a 2D space are given explicitly by

$$e_1 = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = b \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

(a). Find the inverse basis vectors $\{e^i\}$ so that $e_i \cdot e^j = \delta_i^j$.

The inverse vectors must satisfy

$$e_1 \cdot e^1 = 1, \quad e_1 \cdot e^2 = 0, \quad e_2 \cdot e^1 = 0, \quad e_2 \cdot e^2 = 1.$$

Letting $e^1 = (x^1, y^1)$ and $e^2 = (x^2, y^2)$, we get a system of equations

$$\begin{cases} ax^1 = 1, \\ ax^2 = 0, \\ (b\cos\theta)x^1 + (b\sin\theta)y^1 = 0, \\ (b\cos\theta)x^2 + (b\sin\theta)y^2 = 1, \end{cases} \implies \quad e^1 = \begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} 1 \\ -\cot\theta \end{pmatrix}, \quad e^2 = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \frac{1}{b}\begin{pmatrix} 0 \\ \csc\theta \end{pmatrix}.$$

(b). Write out the metric matrices g_{ij} and g^{ij} and check their inverse relationship:

$$g_{ij}g^{jk} = \delta_i^k.$$

Using the identities $g_{ij} = e_i \cdot e_j$ and $g^{ij} = e^i \cdot e^j$, we get the matrices

$$g_{ij} = \begin{pmatrix} a^2 & ab\cos\theta \\ ab\cos\theta & b^2 \end{pmatrix} \quad \text{and} \quad g^{ij} = \begin{pmatrix} \csc^2\theta/a^2 & -\cot\theta\csc\theta/ab \\ -\cot\theta\csc\theta/ab & \csc^2\theta/b^2 \end{pmatrix}.$$

Multiplying these matrices, we get

$$\begin{aligned} &\begin{pmatrix} a^2 & ab\cos\theta \\ ab\cos\theta & b^2 \end{pmatrix} \begin{pmatrix} \csc^2\theta/a^2 & -\cot\theta\csc\theta/ab \\ -\cot\theta\csc\theta/ab & \csc^2\theta/b^2 \end{pmatrix} \\ &\qquad = \begin{pmatrix} \csc^2\theta - \cos\theta\cot\theta\csc\theta & (a/b)(-\cot\theta\csc\theta + \csc^2\theta\cos\theta) \\ (b/a)(\cos\theta\csc^2\theta - \cot\theta\csc\theta) & -\cos\theta\cot\theta\csc\theta + \csc^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

(c). Show that the completeness condition $e_i \otimes e^i = I$ is satisfied.

Computing the sum $e_1 \otimes e^1 + e_2 \otimes e^2 = I$, we get

$$\begin{aligned} e_1 \otimes e^1 + e_2 \otimes e^2 &= \begin{pmatrix} a \\ 0 \end{pmatrix} \otimes \frac{1}{a}\begin{pmatrix} 1 \\ -\cot\theta \end{pmatrix} + \begin{pmatrix} b\cos\theta \\ b\sin\theta \end{pmatrix} \otimes \frac{1}{b}\begin{pmatrix} 0 \\ \csc\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\cot\theta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cot\theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Problem 3.

(a). Show that $\Lambda^{\mu}_{\mu'}\Lambda^{\nu'}_{\mu} = \delta^{\nu'}_{\mu'}$.

For any 4-vectors A and B, recall that $A_\mu B^\mu$ is a scalar. Since scalars don't transform under coordinate transformations:

$$A_{\mu'}B^{\mu'} = \Lambda^{\mu}_{\mu'}A_\mu \Lambda^{\mu'}_{\nu}B^\nu = (\Lambda^{\mu}_{\mu'}\Lambda^{\mu'}_{\nu})A_\mu B^\nu = A_\mu B^\mu.$$

This immediately implies that $\Lambda^{\mu}_{\mu'}\Lambda^{\nu'}_{\mu} = \delta^{\nu'}_{\mu'}$.

(b). Show $T^\mu_{\mu\rho}$ transforms as a $(0,1)$-tensor.

For a general tensor of rank (p,q), the transformation rule is given by

$$T^{\nu_1'\nu_2'\dots\nu_q'}_{\mu_1'\mu_2'\dots\mu_p'} = \Lambda^{\nu_1}_{\nu_1'}\Lambda^{\nu_2}_{\nu_2'}\cdots\Lambda^{\nu_q'}_{\nu_q}\Lambda^{\mu_1'}_{\mu_1}\Lambda^{\mu_2'}_{\mu_2}\cdots\Lambda^{\mu_p'}_{\mu_p}T^{\nu_1\nu_2\dots\nu_q}_{\mu_1\mu_2\dots\mu_p}.$$

This means that for the tensor $T^\mu_{\mu\rho}$, we have the transformation law

$$T^{\mu'}_{\mu'\rho'} = \Lambda^{\mu}_{\mu'}\Lambda^{\mu'}_{\mu}\Lambda^{\rho'}_{\rho}T^{\mu}_{\mu\rho} = \delta^{\mu'}_{\mu'}\Lambda^{\mu}_{\mu\rho} = \Lambda^{\rho'}_{\rho}\Lambda^{\mu}_{\mu\rho}.$$

This is a $(0,1)$ transformation law, so $T^\mu_{\mu\rho}$ transforms as a $(0,1)$-tensor.

(c). Show that in Minkowski spacetime, ∂_μ transforms as a covariant vector and the D'Alembertian $\Box = \partial^\mu\partial_\mu$ is a scalar.

By the chain rule, for any vector A we have the transformation law:

$$\partial_{\nu'}A^\lambda e_\lambda = \frac{\partial x^\mu}{\partial x^{\nu'}}\partial_\mu A^\mu e_\lambda \quad\implies\quad \partial_{\nu'} = \frac{\partial x^\mu}{\partial x^{\nu'}}\partial_\mu.$$

This requires the assumption that spacetime is flat, i.e. $\partial_\nu e_\mu = \delta_{\nu\mu}$ otherwise ∂_μ has higher order terms in the transformation law.

For the D'Alembertian operator, note that

$$\begin{aligned}\Box' = \partial^{\mu'}\partial_{\mu'} = g^{\mu'\nu'}\partial_{\mu'}\partial_{\nu'} &= \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}g^{\mu\nu}\right)\left(\frac{\partial x^\sigma}{\partial x^{\mu'}}\partial_\sigma\right)\left(\frac{\partial x^\rho}{\partial x^{\nu'}}\partial_\rho\right)\\ &= \delta^\sigma_\mu\delta^\rho_\nu g^{\mu\nu}\partial_\sigma\partial_\rho\\ &= g^{\mu\nu}\partial_\mu\partial_\nu = \partial^\mu\partial_\nu = \Box.\end{aligned}$$

Since there is no transformation, the operator transforms as a constant.

Problem 4.

(a). Show that Maxwell's equations can be written in the following tensor form:

$$\partial_\mu F^{\mu\nu} = -j^\nu, \quad \partial_\mu\widetilde{F}^{\mu\nu} = 0.$$

Why doe the form explicitly show that the Maxwell equations are Lorentz invariant?

Recall that the Maxwell equations (up to some scaling to get rid of constants) can be written in the form

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \rho, &\quad \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t},\\ \nabla\cdot\mathbf{B} &= 0, &\quad \nabla\times\mathbf{B} &= \mathbf{J} + \frac{\partial\mathbf{E}}{\partial t}.\end{aligned}$$

Expanding these equations into components, we get the system of eight equations

$$\begin{aligned}-\frac{\partial B_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, &\quad \frac{\partial E_x}{\partial t} + J_x &= \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z},\\ -\frac{\partial B_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, &\quad \frac{\partial E_y}{\partial t} + J_y &= \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x},\\ -\frac{\partial B_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, &\quad \frac{\partial E_z}{\partial t} + J_z &= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y},\end{aligned}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho, \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.$$

Let's define the 4-vector j^μ by

$$j^\mu = \left(\rho, \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}\right).$$

Similarly, we define the electromagnetic field strength tensor by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \implies \begin{aligned} \partial_\mu F^{\mu 0} &= -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} = -\rho \\ \partial_\mu F^{\mu 1} &= -\frac{\partial E_x}{\partial t} - \frac{\partial B_y}{\partial z} + \frac{\partial B_z}{\partial y} = -J_x \\ \partial_\mu F^{\mu 2} &= -\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} = -J_y \\ \partial_\mu F^{\mu 3} &= -\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} = -J_z \end{aligned}$$

Therefore, if we set $j^\nu = (\rho, \mathbf{J})$, we get the equation $\partial_\mu F^{\mu\nu} = -j^\nu$. To get the other half of the equations, we can construct the dual electromagnetic field strength tensor as

$$\widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} \implies \begin{aligned} \partial_\mu \widetilde{F}^{\mu 0} &= -\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} - \frac{\partial B_z}{\partial z} = 0 \\ \partial_\mu \widetilde{F}^{\mu 1} &= -\frac{\partial B_x}{\partial t} - \frac{\partial E_y}{\partial z} + \frac{\partial E_z}{\partial y} = 0 \\ \partial_\mu \widetilde{F}^{\mu 2} &= -\frac{\partial B_y}{\partial t} + \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0 \\ \partial_\mu \widetilde{F}^{\mu 3} &= -\frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} = 0 \end{aligned}$$

This gives the equation $\partial_\mu \widetilde{F}^{\mu\nu} = 0$, so Maxwell's equations can be written concisely as

$$\partial_\mu F^{\mu\nu} = -j^\nu \quad \text{and} \quad \partial_\mu \widetilde{F}^{\mu\nu} = 0.$$

We can also express $\widetilde{F}^{\mu\nu}$ as:

$$\widetilde{F}^{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}$$

where $\epsilon_{\mu\nu\rho\lambda} = 1$ is antisymmetric.

Since all sides of this tensor equation are tensors, they are manifestly Lorentz invariant since they transform covariantly under the Lorentz transformation.

(b). From the first of Maxwell's equations show that the electromagnetic current 4-vector is divergenceless:

$$\partial_\mu j^\mu = 0.$$

Demonstrate in terms of j^μ's components that this is just the electric charge conservation law.

Note that

$$\partial_\mu j^\mu = -\partial_\mu \partial_\nu F^{\nu\mu}.$$

In other words, $\partial_\mu j^\mu$ is the contraction of a symmetric tensor $\partial_\mu \partial_\nu$ and an antisymmetric tensor $F^{\nu\mu}$. Thus, by swapping indices we get

$$\partial_\mu j^\mu = -\partial_\mu \partial_\nu j^{\nu\mu} = \partial_\nu \partial_\mu j^{\mu\nu} = -\partial_\mu j^\mu.$$

Thus, $\partial_\mu j^\mu = 0$. In components, this gives us equations

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \frac{\partial \nabla \cdot \mathbf{E}}{\partial t} + \nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

since $\nabla \cdot \mathbf{E}$ is constant. This is exactly the charge conservation law.

Problem 5. Demonstrate the energy-momentum conservation law:

$$\partial_\mu T^{\mu\nu} = 0.$$

Recall that in Minkowski spacetime, the energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\Delta p^\mu}{\Delta s^\nu} \quad \text{where} \quad \Delta s^\mu = \{\Delta x^1 \Delta x^2 \Delta x^3,\ \Delta t \Delta x^2 \Delta x^3,\ \Delta t \Delta x^3 \Delta x^1,\ \Delta t \Delta x^1 \Delta x^2\}.$$

Now let's work in some infinitesimal volume V. The net change of momentum Δp^μ in this volume is due to flux of momentum across its boundary surfaces. For each pair of faces perpendicular to the x^ν axis, the net momentum flux is

$$\Delta p^\mu_\nu = T^{\mu\nu}(x^\nu)\Delta s^\nu - T^{\mu\nu}(x^\nu + \Delta x^\nu)\Delta s^\nu.$$

However, using a Taylor approximation, (V is supposed to be infinitesimal) we get

$$T^{\mu\nu}(x^\nu + \Delta x^\nu) - T^{\mu\nu}(x^\nu) \approx \partial_\nu T^{\mu\nu} \Delta x^\nu.$$

This means that overall we have

$$\Delta p^\mu = -\sum_\nu \partial_\nu T^{\mu\nu} \Delta x^\nu \Delta s^\nu = -(\partial_\nu T^{\mu\nu})\Delta V,$$

since $\Delta x^\nu \Delta s^\nu$ is exactly the volume element. The conservation of relativistic 4-momentum states that $\Delta p^\mu = 0$ and so $\partial_\nu T^{\mu\nu} = 0$.