Math 231b Problem Set 6

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Problem 1. Eilenberg-MacLane spaces.

For any group π, let $K(\pi, n)$ be the n-th Eilenberg-MacLane space.

a. Let $N < G$ be a normal subgroup, with quotient group H. Show that there is a map $K(G, 1) \to K(H, 1)$ with homotopy fiber weakly equivalent to $K(N, 1)$.

Let $F_0G = \mathbb{Z}[G]$, and $F_1G = \ker(\mathbb{Z}[G] \to G)$ so that $0 \to F_1G \to F_0G \to G \to 0$ is a free resolution. Given some normal subgroup N, note that we have induced maps:

$$\begin{tikzcd}
0 \arrow[r] & F_1(G) \arrow[r] \arrow[d, "F_1(\pi)"] & F_0(G) \arrow[r] \arrow[d, "F_0(\pi)"] & G \arrow[r] \arrow[d, two heads, "\pi"] & 0 \\
0 \arrow[r] & F_1(G/N) \arrow[r] & F_0(G/N) \arrow[r] & G/N \arrow[r] & 0
\end{tikzcd}$$

Recall that to construct a Moore space we can look at the cofibers of the rows of the following induced diagram:

$$\begin{tikzcd}
 & & F(h) \arrow[d] \\
\bigvee_{\alpha \in F_1(G)} S^n \arrow[r] \arrow[d] & \bigvee_{\alpha \in F_0(G)} S^n \arrow[r] \arrow[d] & M(G, n) \arrow[d, two heads, "h"] \\
\bigvee_{\alpha \in F_1(G/N)} S^n \arrow[r] & \bigvee_{\alpha \in F_0(G/N)} S^n \arrow[r] & M(G/N, n)
\end{tikzcd}$$

Then this vertical map is the induced map between cofibers, as shown in a previous problem set. Now applying the functor $\tau_{\leq n}$ to this vertical arrow gives us a fiber sequence $\tau_{\leq n}F(h) \to K(G, n) \to K(G/N, n)$. Note that the long exact sequence of a fiber sequence gives us an exact:

$$0 \longrightarrow \pi_n(\tau_{\leq n}F(h)) \longrightarrow \pi_n(K(G, n)) \longrightarrow \pi_n(K(G/N, n)) \longrightarrow \pi_{n-1}(\tau_{\leq n}F(h)) \longrightarrow 0$$

By the way we constructed h, it's clear that $\pi_n(h) : \pi_n(K(G, n)) \to \pi_n(K(G/N, n))$ is simply the map $G \to G/N$. This implies that $\pi_{n-1}(\tau_{\leq n}F(h)) = 0$ and $\pi_n(\tau_{\leq n}F(h)) = N$. All of the lower groups are zero by the exact sequence, and the higher groups are zero by the $\tau_{\leq n}$ functor. Thus, $\tau_{\leq n}F(h) \simeq K(N, n)$.

b. Suppose that G is abelian. The same argument gives us a map $K(G, n) \to K(H, n)$ with homotopy fiber $K(N, n)$. But show also that there is a map $K(N, n) \to K(G, n)$ with homotopy fiber $K(H, n-1)$ and a map $K(H, n) \to K(N, n+1)$ with homotopy fiber $K(G, n)$. For example, what is the homotopy fiber of the map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty$ represented by twice a generator of $H^2(\mathbb{CP}^\infty)$?

In (a), we used free resolutions of the projection map $G \to G/N$ was used to construct a map $K(G,n) \to K(G/N,n)$ which induces the original projection map when taking homotopy. We can do a similar thing here, first we take the inclusion $N \to G$, which induces a map $K(N,n) \to K(G,n)$. Then by the same argument as in the previous part we get some cofiber F which satisfies the exact sequence:

$$0 \longrightarrow \pi_n(F) \longrightarrow \pi_n(K(N,n)) \longrightarrow \pi_n(K(G,n)) \longrightarrow \pi_{n-1}(F) \longrightarrow 0$$

Then we get $\pi_{n-1}(F) \cong G/N$, and this is the only nontrivial homotopy group, thus $F \simeq K(G/N, n-1)$. For the last sequence, we simply extend the homotopy fiber sequence $K(N,n) \to K(G,n) \to K(G/N,n)$ to the extra term $\Omega K(N,n) \simeq K(N,n+1)$. This will have cofiber $K(G,n)$ by the fibration sequence.

Now letting $\iota_2 \in H^2(\mathbb{CP}^\infty)$ be a generator, the corresponding map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induces the inclusion map $2\mathbb{Z} \to \mathbb{Z}$. Thus, the fiber is $K(\mathbb{Z}/2, 2)$, which is homotopy equivalent to $\Omega\mathbb{RP}^\infty$.

Problem 2. Let Y be a simply-connected space such that $H_n(Y)$ is finitely generated for all n. Let β_n be the n-th Betti number and let n-th torsion number. Then there is a CW complex with $(\beta_n + \tau_n + \tau_{n-1})$ n-cells for each n that admits a weak equivalence to Y.

We'll build up this cell structure by induction. Starting at $n = 1$, since Y is simply connected, it follows that Y_1 is just a basepoint. Now inductively, suppose we have an n-homology equivalence $f_n : Y_n \to Y$, and Y_n has the given minimal cell structure. (induced isomorphisms $H_k(f_n)$ for $k < n$ and $H_k(f_n)$ surjective for $k = n$) Taking the homotopy cofiber, we get $H_k(C(f_n), Y_n) = 0$ for $k \leq n$ so by the Hurewicz isomorphism, we get $H_{n+1}(C(f_n), Y_n) \cong \pi_{n+1}(C(f_n), Y_n)$. We then have two exact sequences:

$$\begin{tikzcd}
H_{n+1}(C(f)) \arrow[r] & H_{n+1}(C(f_n), Y_n) \arrow[r] & H_n(Y_n) \arrow[r] & H_n(C(f_n)) \arrow[r] & 0 \\
H_{n+1}(Y_{n+1}) \arrow[r] \arrow[u] & H_{n+1}(Y_{n+1}, Y_n) \arrow[r] \arrow[u] & H_n(Y_n) \arrow[r] \arrow[u, dashed, no head] & H_n(Y_{n+1}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Since elements of $H_{n+1}(C(f_n), Y_n)$ are mapped to attachment maps of D^{n+1} to Y_n, we can get our desired generaters attached to Y_n to form Y_{n+1}.

Problem 3. Let M denote a simply-connected, compact 3-manifold. Prove that $M \simeq S^3$.

First we claim that M must be oriented.

Claim. Any simply-connected manifold is orientable.

Proof. Suppose for the sake of contradiction that M is a simply-connected, non-orientable manifold, and let $\widetilde{M}$ be its orientable double cover. This is a (connected) two-sheeted covering, which is a contradiction, since M is it's own universal cover. $\square$

Now let's compute the (reduced) homology of M. The first few groups are easy; $\widetilde{H}_0(M) = 0$ since M is connected, and $\widetilde{H}_1(M) \cong \pi_1(M) = 0$ by the Hurewicz isomorphism. Next, we have $\widetilde{H}_3(M) = \mathbb{Z}$ since M is an orientable, compact, connected manifold. All of the other homology groups $\widetilde{H}_k(M)$ must be trivial for $k > 3$ by duality. Now finally, we want to compute $\widetilde{H}_2(M)$. By duality, $H_2(M) \cong H^1(M)$, and by the universal coefficients theorem we get a short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_0(M), \mathbb{Z}) \longrightarrow H^1(M) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_1(M), \mathbb{Z}) \longrightarrow 0$$

Since both $H_0(M)$ and $H_1(M)$ are trivial, we conclude that $H^1(M)$ is trivial, and so $\widetilde{H}_2(M) = 0$. Thus, M is a $M(\mathbb{Z}, 3)$ Moore space.

By the Hurewicz theorem, we notice that $\pi_3(M) \cong H_3(M) = \mathbb{Z}$. Let $\sigma : S^3 \to M$ be some generator of $\pi_3(M)$. The map σ clearly induces an isomorphism $H_*(\sigma) : H_*(S^3) \to H_*(M)$ so it is a homotopy equivalence. This concludes the proof.

Problem 4. (Co)homological characterization of $\mathbb{CP}^n$.

Let X denote a simple space.

a. If X has homology groups $H_*(X; \mathbb{Z}) \cong \mathbb{Z}[0] \oplus \mathbb{Z}[2] \oplus \cdots \oplus \mathbb{Z}[2n]$ and cohomology ring $H^*(X; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$ where $|x| = 2$. Prove that $X \simeq \mathbb{CP}^n$.

First of all, by cellular approximation, we can assume without loss of generality that X is a CW complex. Next by Problem 2, we can further restrict by giving X a CW structure with only a single cell in each dimension $0, 2, \ldots, 2n$ since $H_*(X; \mathbb{Z}) \cong \mathbb{Z}[0] \oplus \mathbb{Z}[2] \oplus \cdots \oplus \mathbb{Z}[2n]$. Now recall by representability of cohomology that we have a natural bijection

$$[X, K(\mathbb{Z}, 2)]_* \to H^2(X; \mathbb{Z})$$

which sends $f : X \to K(\mathbb{Z}, 2)$ to the pullback $f^*(\iota_2)$ for some fundamental $\iota_2 \in H^2(K(\mathbb{Z}, 2); \mathbb{Z})$. Since $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$, and $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$ by assumption, we will consider the preimage $\sigma : X \to \mathbb{CP}^\infty$ of a generator of $H_2(X; \mathbb{Z})$. Notice that this map $\sigma : X \to \mathbb{CP}^\infty$ induces an isomorphism $\sigma^* : H^2(\mathbb{CP}^\infty; \mathbb{Z}) \to H^2(X; \mathbb{Z})$. Recall that X has no cells of dimension greater than $2n$, so by skeletal approximation, σ can be factored through some map $\zeta : X \to \mathbb{CP}^n$. Since the inclusion $\mathbb{CP}^n \to \mathbb{CP}^\infty$ induces an isomorphism on H^2, by functoriality we get an induced isomorphism $\zeta^* : H^2(\mathbb{CP}^n; \mathbb{Z}) \to H^2(X; \mathbb{Z})$.

We claim that ζ^* is an isomorphism in every dimension. Since the odd dimensional cohomology groups, and higher cohomology groups past dimension $2n$ are all zero, we are only interested in the cohomology groups of dimension $2k$ for $k \leq n$. For any such k, the cohomology ring structure of X and $\mathbb{CP}^n$ give us "lifting" isomorphisms

$$H^2(X; \mathbb{Z}) \to H^{2k}(X; \mathbb{Z}) \quad \text{and} \quad H^2(\mathbb{CP}^n; \mathbb{Z}) \to H^{2k}(\mathbb{CP}^n; \mathbb{Z})$$

which send some ω to $\omega \smile \cdots \smile \omega$. By naturality of the cup product, and by extension this map, we get a commutative square

$$\begin{tikzcd}
H^{2k}(\mathbb{CP}^n; \mathbb{Z}) \arrow[r, "\zeta^*"] & H^{2k}(X; \mathbb{Z}) \\
H^{2k}(\mathbb{CP}^n; \mathbb{Z}) \arrow[u, "\smile"] \arrow[r, "\zeta^*"] & H^2(X; \mathbb{Z}) \arrow[u, "\smile"']
\end{tikzcd}$$

Since the bottom and side arrows are isomorphisms, it follows that the top arrow is as well. Thus it follows that ζ induces an isomorphism on all cohomology groups. Since X is simple, this implies that ζ is a homotopy equivalence, so $X \simeq \mathbb{CP}^\infty$.

b. Prove that $[\mathbb{CP}^n, \mathbb{CP}^n] \cong \mathbb{Z}$ via the map sending a map $\mathbb{CP}^n \to \mathbb{CP}^n$ to the induced homomorphism on H_2.

Firstly, note that by the skeletal approximation theorem, we have a canonical isomorphism $[\mathbb{CP}^n, \mathbb{CP}^n] \cong [\mathbb{CP}^n, \mathbb{CP}^\infty]$ induced by the CW inclusion $\mathbb{CP}^n \to \mathbb{CP}^\infty$. Furthermore this isomorphism also clearly preserves induced homomorphisms between homology groups, so it is sufficient to investigate $[\mathbb{CP}^n, \mathbb{CP}^\infty]$. Recall that the universal coefficient theorem gives us a map $h : H^2(\mathbb{CP}^n;\mathbb{Z}) \to \mathrm{Hom}(H_2(\mathbb{CP}^n;\mathbb{Z}),\mathbb{Z})$ which sends σ to $x \mapsto \sigma(x)$. This map is also part of a short exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_1(\mathbb{CP}^n;\mathbb{Z}),\mathbb{Z}) \longrightarrow H^2(\mathbb{CP}^n;\mathbb{Z}) \xrightarrow{h} \mathrm{Hom}(H_2(\mathbb{CP}^n;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0$$

Since $\mathbb{CP}^1$ is simply connected, the Ext term vanishes, and it follows that h is an isomorphism. Now let ψ be the representation isomorphism $[\mathbb{CP}^n, \mathbb{CP}^\infty] \to H^2(\mathbb{CP}^n;\mathbb{Z})$. This map sends f to $f^*(\iota_2)$ for some universal $\iota_2 \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ so we get a diagram:

$$\begin{tikzcd}
{[\mathbb{CP}^n, \mathbb{CP}^\infty]} \arrow[r, "\psi"] \arrow[d, "H_2"'] & H^2(\mathbb{CP}^n;\mathbb{Z}) \arrow[d, "h"'] \\
\mathrm{Hom}(H_2(\mathbb{CP}^n;\mathbb{Z}), H_2(\mathbb{CP}^\infty;\mathbb{Z}) \arrow[r, "-\circ h(\iota_2)"'] & \mathrm{Hom}(H_2(\mathbb{CP}^n;\mathbb{Z}),\mathbb{Z})
\end{tikzcd}$$

Clearly this diagram commutes. Recall that ψ is an isomorphism, h is an isomorphism. Similarly, $-\circ h(\iota_2)$ is an isomorphism because $h(\iota_2)$ is as a consequence of ι_2 being a generator. Thus by commutativity, H_2 must be an isomorphism as well. This completes the proof since $\mathrm{Hom}(H_2(\mathbb{CP}^n;\mathbb{Z}), H_2(\mathbb{CP}^\infty;\mathbb{Z}))$ is isomorphic to $\mathbb{Z}$.

Problem 5. Let Y be a simple space and N an integer, and suppose that $N\pi_*(Y) = 0$. Let (X, A) be a relative CW complex and assume that $H_*(X, A; \mathbb{F}_p) = 0$ whenever the prime p divides N. Show that the restriction map $[X, Y] \to [A, Y]$ is bijective.

A consequence of the obstruction theorem implies that the restriction map $[X, Y] \to [A, Y]$ is bijective if the cohomology groups $H^{n+1}(X, A; \pi_n(Y)) = 0$ for all n, so we prove this. Note that for every prime $p|N$, we have an exact sequence

$$0 \to p \cdot \pi_n(Y) \to \pi_n(Y) \to \pi_n(Y)_p \to 0$$

where $\pi_n(Y)_p$ is the p-torsion component of $\pi_n(Y)$. Then $\pi_n(Y)_p$ naturally has the structure of an $\mathbb{F}_p$-vector space, so it splits $\pi_n(Y)_p = \bigoplus_i \mathbb{F}_p$. By the exactness of cohomology in coefficients, we get a short exact sequence

$$0 \to H^{n+1}(X, A; p \cdot \pi_n(Y) \to H^{n+1}(X, A; \pi_n(Y) \to H^{n+1}(X, A; \pi_n(Y)_p) \to 0.$$

Since $H_*(X, A; \mathbb{F}_p) = 0$, the universal coefficients theorem implies that $H^*(X, A; \mathbb{F}_p) = 0$ so $H^{n+1}(X, A; \pi_n(Y)_p) = \bigoplus_i H^{n+1}(X, A; \mathbb{F}_p) = 0$. Thus we get an isomorphism:

$$H^{n+1}(X, A; p \cdot \pi_n(Y)) \cong H^{n+1}(X, A; \pi_n(Y))$$

Now $p \cdot \pi_n(Y)$ satisfies $(N/p)p \cdot \pi_n(Y) = 0$. This means we can induct all the way down until $\pi_*(Y) = 0$, completing the proof.