Lecture 2: Flows and Lie derivatives

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The material here relies on the fundamental theorem in the theory of ordinary differential equations; see Lectures 17 and 18 in the Multivariable Analysis notes. What is not proved there is smooth dependence of the solution on parameters. You can find a proof in several texts, such as in Arnold's books on ordinary differential equations or in *Introduction to Smooth Manifolds* by John Lee. The smooth dependence on initial conditions is used below in the construction of the flow of a vector field. There are further readings posted on Canvas that contain examples and pictures relevant to this lecture.

This is the first of several lectures that develop fundamental background material for our study of geometric structures on smooth manifolds. The topics covered are flows of vector fields and Lie derivatives, introduction to Lie groups, distributions and foliations, and the Frobenius theorem. In this lecture we already encounter several important themes of the course.

A vector field on a smooth manifold gives rise to motion. One has single motions—*integral curves*—for any initial point, and can assemble them together into a *flow* that moves sets of points. *Local* flows always exist; the existence of a *global* flow depends on completeness properties of the manifold and boundedness of the vector field. (We do not delve into these global questions here.) The dichotomy *local vs. global* is one of several important themes in the course.

Another important dichotomy is *intrinsic vs. extrinsic*. Here we use flows to transport intrinsic linear quantities—*tensors*—and then to differentiate them. This *Lie derivative* is an important ingredient in differential calculus on smooth manifolds. We work out the Lie derivative of vector fields, which leads to a *Lie algebra* structure on vector fields. In a subsequent lecture we work out the Lie derivative of differential forms.

Finally, we interpret the Lie derivative of vector fields as a measure of the failure of commutativity of the associated flows. This particular *infinitesimal* manifestation of a *local* phenomenon is another important theme of the course: we will see *curvature* as a geometric manifestation of this general link between commutativity of flows and Lie derivative. (Of course, the passage between local and infinitesimal is the basic idea of differential calculus.)

Notation: The vector space of vector fields on a smooth manifold X (i.e., the space of smooth sections of the tangent bundle $TX \to X$) is denoted $\mathfrak{X}(X)$. For a nonnegative integer $q \in \mathbb{Z}^{\geq 0}$, the vector space of smooth differential q-forms on X is denoted Ω^q_X. In particular, the algebra of

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smooth functions is denoted Ω^0_X. We often use 'n' as the dimension of a smooth manifold without a proper introduction.

Integral curves and flows

(2.1) *Integral curves.* The basic ordinary differential equation on a manifold is for a motion with prescribed velocity.

Definition 2.2. Let X be a smooth manifold, let ξ be a vector field on X, and suppose $(a,b) \subset \mathbb{R}$ is an open interval. Then a motion $\gamma\colon (a,b) \to X$ is an *integral curve* of ξ if

$$\dot\gamma(t) = \xi_{\gamma(t)}, \qquad t \in (a,b). \tag{2.3}$$

Here and always we use the notation $\dot\gamma = d\gamma/dt$.

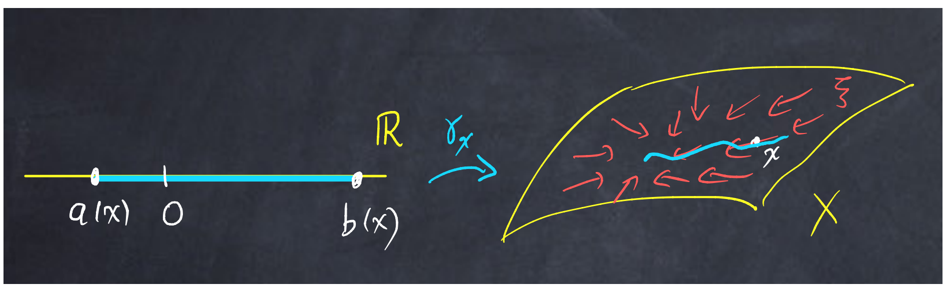

FIGURE 6. An integral curve of a vector field

A globalized form of the fundamental theorem of ODEs is the existence and uniqueness of a maximal integral curve with given initial value; see Figure 6.

Theorem 2.4. *Let X be a smooth manifold, let ξ be a vector field on X, and fix $x \in X$. Then there exists a unique maximal integral curve*

$$\gamma_x\colon \bigl(a(x), b(x)\bigr) \longrightarrow X \tag{2.5}$$

such that $\gamma_x(0) = x$. Furthermore, $-\infty \le a(x) < 0 < b(x) \le +\infty$.

The maximality defines the functions a, b. Maximality and uniqueness mean that if $\delta\colon (c,d) \to X$ is any integral curve of ξ satisfying $\delta(0) = x$, then $a(x) \le c < 0 < d \le b(x)$ and $\delta = \gamma_x\big|_{(c,d)}$.

Definition 2.6. The vector field ξ is *complete* if $a(x) = -\infty$ and $b(x) = +\infty$ for all $x \in X$.

In other words, all integral curves of X exist for all time.

Remark 2.7.

(1) If X is compact, then every vector field on X is complete.
(2) The vector field $x^2\, \partial/\partial x$ on $\mathbb{R}$ is not complete. Nor is the vector field $\partial/\partial x$ on $\mathbb{R}^{\neq 0}$.

(2.8) *Global and local flows.* Fix a vector field $\xi \in \mathfrak{X}(X)$. We consider simultaneously all maximal integral curves of ξ and show (well, state) that they assemble into a *flow*.

Definition 2.9. A *global flow* $\varphi\colon \mathbb{R}\times X \to X$ on a smooth manifold X is a smooth function that is also a homomorphism $\mathbb{R} \to \mathrm{Diff}(X)$.

Here $\mathrm{Diff}(X)$ is the group of diffeomorphisms $X \to X$. The diffeomorphism at $t \in \mathbb{R}$ is denoted φ_t, i.e., for $t \in \mathbb{R}$ and $x \in X$ we write $\varphi_t(x) = \varphi(t,x)$. A complete vector field determines a global flow, but a general vector field determines a flow defined on an open subset of $\mathbb{R}\times X$. Set

$$\mathcal{D}_t = \big\{ x \in X : t \in \big(a(x), b(x)\big)\big\}, \qquad t \in \mathbb{R}, \tag{2.10}$$

and define

$$\varphi(t,x) = \varphi_t(x) = \gamma_x(t), \qquad \text{if } x \in \mathcal{D}_t. \tag{2.11}$$

The following theorem, which is essentially Theorem 1.48 in Warner's *Foundations of Differentiable Manifolds and Lie Groups*, gives the main properties of $\mathcal{D}_t$ and φ. I defer to that reference for the proof, assuming the fundamental theorem of ODE.

Theorem 2.12.

(1) *$\mathcal{D}_t \subset X$ is open and $\bigcup_{t>0} \mathcal{D}_t = X$.*
(2) *$\varphi_t\colon \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse φ_{-t}.*
(3) *For $t_1, t_2 \in \mathbb{R}$, we have $\varphi_{t_1}^{-1}(\mathcal{D}_{t_2}) \cap \mathcal{D}_{t_1} \subset \mathcal{D}_{t_1+t_2}$ and $\varphi_{t_2}\circ\varphi_{t_1} = \varphi_{t_1+t_2}$ on their common domain. (The two domains are equal if t_1, t_2 have the same sign.)*
(4) *If $U \subset X$ is an open subset and $x \in U$, then there exist $\epsilon \in \mathbb{R}^{>0}$ and an open subset $V \subset U$ with $x \in V$ such that $\varphi\big((-\epsilon,\epsilon)\times V\big) \subset U$.*
(5) *There exists an open subset $\mathcal{U} \subset \mathbb{R}\times X$ and a smooth function $\widehat{\varphi}\colon \mathcal{U} \to X$ such that $\{0\}\times X \subset \mathcal{U}$ and if $(t,x) \in \mathcal{U}$ then $x \in \mathcal{D}_t$ and $\widehat{\varphi}(t,x) = \varphi_t(x)$.*
(6) *If ξ is complete, then $\mathcal{D}_t = X$ for all $t \in \mathbb{R}$ and a global flow exists.*

We use 'φ' in place of '$\widehat{\varphi}$'. A *local flow* is the restriction of φ to a subset $(-\epsilon,\epsilon)\times V$, as in (4). It is the local flow that we use below to define the Lie derivative. The local flow is unique in the sense that any two local flows agree on their common domain. We say that ξ is the *generator* of the flow φ, or that ξ *generates* φ.

(2.13) *The flow in local coordinates.* We continue with a vector field ξ and the flow φ it generates. Let $(U; x^1, \dots, x^n)$ be a chart, and write $\xi = \xi^i \frac{\partial}{\partial x^i}$ for the vector field in the local coordinates of the chart. As in Theorem 2.12(4) choose $\epsilon > 0$ and $V \subset U$ such that if $t \in (-\epsilon,\epsilon)$ and $x \in V$, then $\varphi(t,x) \in U$. In the formulæ below the arguments t, x are implicitly restricted to $(-\epsilon,\epsilon)\times V$. Write $(t;x) = (t; x^1, \dots, x^n)$ and

$$\varphi(t,x) = \big(\varphi^1(t;x), \dots, \varphi^n(t;x)\big). \tag{2.14}$$

The elementary properties of the flow imply

$$\begin{aligned} \varphi^i(0,x) &= x^i \\ \dot\varphi^i(t;x) &= \xi^i_{\varphi(t;x)} \\ \frac{\partial\varphi^i}{\partial x^j}\bigg|_{t=0} &= \delta^i_j \\ \frac{\partial^2\varphi^i}{\partial x^k\partial x^j}\bigg|_{t=0} &= 0. \end{aligned} \tag{2.15}$$

We abbreviate the second equation: $\dot\varphi^i = \xi^i$.

Lie derivative

Continue with a smooth manifold X, a vector field $\xi \in \mathfrak{X}(X)$, and a local flow $\varphi\colon \mathcal{U} \to X$ generated by ξ. We use the flow to transport "objects" on X and so define a derivative at $t = 0$ of a curve of transported objects. Without further data, any object intrinsically associated to the tangent bundle so transports.

(2.16) *Tensor fields.* A formal definition: A tensor field is a section of a vector bundle associated to the frame bundle of X. But since we have not yet constructed the frame bundle, we content ourselves with an informal definition: A tensor field is a section of a vector bundle with is a tensor product of tensor powers of the tangent and cotangent bundles. If only the tangent bundle is involved, then the tensor field is *covariant*; if only the cotangent bundle is involved, then the tensor field is *contravariant*. A vector field is a covariant tensor field; a differential form is a contravariant tensor field. If ω is a contravariant tensor field on X, and $\psi\colon X' \to X$ is *any* smooth map, then the pullback $\psi^*\omega$ is a tensor field on X'. If η is a covariant tensor field on X, and $\psi\colon X \to X''$ is a *diffeomorphism*, then the pushforward $\psi_*\eta$ is a tensor field on X''. A tensor field that is mixed—neither covariant nor contravariant—can be pulled back or pushed forward by a diffeomorphism.

(2.17) *Differentiation along a flow.* A modification of the following works for mixed tensor fields.

Definition 2.18. Let X be a smooth manifold, $\xi \in \mathfrak{X}(X)$ a vector field on X, and $\varphi\colon \mathcal{U} \to X$ a local flow it generates. Let T be a tensor field on X that is covariant or contravariant. Then the *Lie derivative* of T with respect to ξ is

$$\mathcal{L}_\xi T = \begin{cases} \dfrac{d}{dt}\bigg|_{t=0} (\varphi_{-t})_*T, & T \text{ is covariant;} \\ \dfrac{d}{dt}\bigg|_{t=0} \varphi_t^*T, & T \text{ is contravariant.} \end{cases} \tag{2.19}$$

Intuitively, to compute $\mathcal{L}_\xi T$ at $x \in X$ we stand at X where as time t marches from $-\epsilon$ to ϵ we see a motion in a vector space at x, the fiber of the appropriate vector bundle associated to the frame bundle. It begins with upstream values of T, transported forward to x by the flow, and ends with downstream values of T, transported backward to x. Figure 7 illustrates this in case $T = \eta$ is a vector field. Note that the transport of η by the flow generated by ξ cannot be computed simply from the integral curve γ_x. (Visualize η at a point of X as a germ of a motion, and then transport the motion germs using the flow φ.)

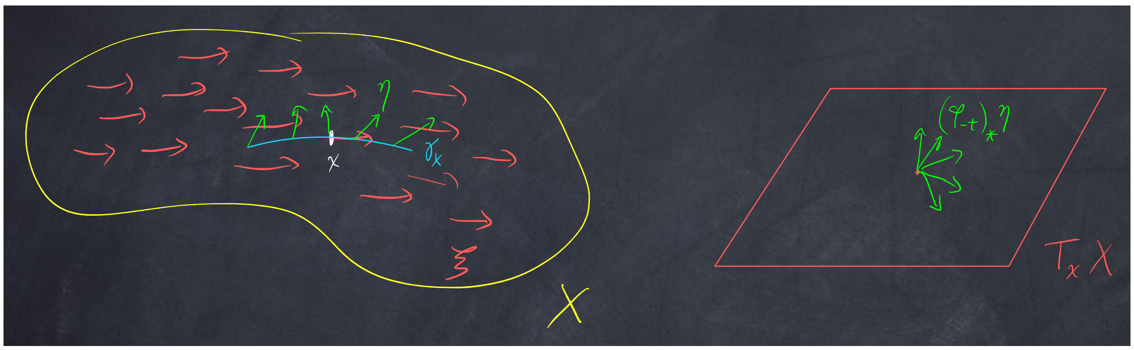

Figure 7. The Lie derivative of a vector field

(2.20) *Lie derivative of a function.* A function $f \in \Omega^0_X$ is a contravariant tensor field. (We use the notation 'Ω^q_X' for the vector space of smooth q-forms on the smooth manifold X.) Hence by (2.19), for $x \in X$ we compute

$$\begin{aligned}\mathcal{L}_\xi f(x) &= \frac{d}{dt}\Big|_{t=0} \varphi_t^* f(x) \\ &= \frac{d}{dt}\Big|_{t=0} f\big(\varphi_t(x)\big) \\ &= \frac{d}{dt}\Big|_{t=0} f\big(\gamma_x(t)\big) \\ &= (\xi f)(x) \\ &= df_x(\xi)\end{aligned} \tag{2.21}$$

Therefore, the Lie derivative of a function is the directional derivative: $\mathcal{L}_\xi f = \xi f$.

In a future lecture we compute a formula for the Lie derivative of an arbitrary differential form.

(2.22) *Lie derivative of a vector field.* Compute in a local chart, as in **(2.13)**.

Proposition 2.23. *The Lie derivative of a vector field $\eta = \eta^i \dfrac{\partial}{\partial x^i}$ with respect to $\xi = \xi^i \dfrac{\partial}{\partial x^i}$ is*

$$\mathcal{L}_\xi \eta = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}. \tag{2.24}$$

Proof. Apply (2.19). First,

$$(\varphi_{-t})_*\eta = \bigl(\varphi_t^*\eta^i\bigr)\left((\varphi_{-t})_*\frac{\partial}{\partial x^i}\right). \tag{2.25}$$

Evaluate the first factor in (2.25):

$$\varphi_t^*\eta^i(x) = \eta^i\bigl(\varphi(t;x)\bigr). \tag{2.26}$$

For the second factor in (2.25), flow by φ_{-t} the motion germ whose velocity is $\partial/\partial x^i$ at the point $\varphi(t;x)$:

$$\begin{aligned}\left((\varphi_{-t})_*\frac{\partial}{\partial x^i}\right)(x) &= \frac{d}{ds}\bigg|_{s=0}\varphi\bigl(-t;\varphi^1(t;x),\dots,\varphi^i(t;x)+s,\dots,\varphi^n(t;x)\bigr)\\ &= \frac{d}{ds}\bigg|_{s=0}\Bigl(\varphi^j\bigl(-t;\varphi^1(t;x),\dots,\varphi^i(t;x)+s,\dots,\varphi^n(t;x)\bigr)\Bigr)_j\\ &= \frac{\partial\varphi^j}{\partial x^i}\bigl(-t;\varphi(t;x)\bigr)\frac{\partial}{\partial x^j}.\end{aligned} \tag{2.27}$$

Combine (2.19), (2.25), (2.26), and (2.27) to obtain

$$\begin{aligned}(\mathcal{L}_\xi\eta)(x) &= \frac{d}{dt}\bigg|_{t=0}\left\{\eta^i\bigl(\varphi(t;x)\bigr)\frac{\partial\varphi^j}{\partial x^i}\bigl(-t;\varphi(t;x)\bigr)\frac{\partial}{\partial x^j}\right\}\\ &= \frac{\partial\eta^i}{\partial x^k}\dot\varphi^k\frac{\partial\varphi^j}{\partial x^i}\frac{\partial}{\partial x^j}\bigg|_{t=0} - \eta^i\frac{\partial\dot\varphi^j}{\partial x^i}\frac{\partial}{\partial x^j}\bigg|_{t=0} + \frac{\partial^2\varphi^j}{\partial x^k\partial x^i}\dot\varphi^k\frac{\partial}{\partial x^j}\bigg|_{t=0}\\ &= \frac{\partial\eta^j}{\partial x^i}\xi^k\frac{\partial}{\partial x^i} - \eta^i\frac{\partial\xi^j}{\partial x^i}\frac{\partial}{\partial x^j},\end{aligned} \tag{2.28}$$

where we use (2.15) to pass to the last equation. Rearrange the indices to obtain (2.24). $\square$

Lie derivatives and derivations

(2.29) *Derivations.* The following may be generalized to derivations on a module over a ring.

Definition 2.30. Let A be an algebra. A linear map $D\colon A\to A$ is a *derivation* if

$$D(fg) = (Df)g + f(Dg), \qquad f,g\in A. \tag{2.31}$$

If D_1, D_2 are derivations of A, then so is the *commutator* or *Lie bracket*

$$[D_1,D_2] = D_1\circ D_2 - D_2\circ D_1. \tag{2.32}$$

A simple manipulation with (2.32) proves that the bracket endows the vector space of derivations of A with the structure of a *Lie algebra.*

Definition 2.33. A *Lie algebra* $\bigl(L,[-,-]\bigr)$ is a vector space L endowed with a bilinear operation

$$[-,-]\colon L\times L\longrightarrow L \tag{2.34}$$

which for all $D_1,D_2,D_3\in L$ satisfies

$$[D_1,D_2]+[D_2,D_1]=0 \tag{2.35}$$

$$\bigl[D_1,[D_2,D_3]\bigr]+\bigl[D_3,[D_1,D_2]\bigr]+\bigl[D_2,[D_3,D_1]\bigr]=0 \tag{2.36}$$

The *Jacobi identity* (2.36) can be written in alternative useful forms. For example, it is the statement that the operation $[D_1,-]$ of left bracketing with D_1 is a derivation:

$$\bigl[D_1,[D_2,D_3]\bigr]=\bigl[[D_1,D_2],D_3\bigr]+\bigl[D_2,[D_1,D_3]\bigr]. \tag{2.37}$$

Equivalently, right bracketing with D_1 is a derivation. Another equivalent form of the Jacobi identity is the map $D\mapsto[D,-]$ is a homomorphism of Lie algebras.

(2.38) *Vector fields as derivations.* Let $\xi\in\mathfrak{X}(X)$ be a vector field on a smooth manifold X. Then ξ defines a derivation on functions via directional derivative:

$$\begin{aligned} D_\xi\colon \Omega^0_X &\longrightarrow \Omega^0_X\\ f&\longmapsto \xi f\end{aligned} \tag{2.39}$$

Conversely, every derivation on the algebra Ω^0_X is directional derivative along a vector field.

Proposition 2.40. *Let $D\colon\Omega^0_X\to\Omega^0_X$ be a derivation. Then there exists a vector field $\xi\in\mathfrak{X}(X)$ such that $D=D_\xi$.*

First we prove the following.

Lemma 2.41. *Suppose h is a smooth real-valued function defined on a convex neighborhood U of $0\in\mathbb{A}^n$, and assume $h(0)=0$. Then there exist smooth functions $g_i\colon U\to\mathbb{R}$, $i=1,\dots,n$, such that*

$$h(x)=g_i(x)x^i. \tag{2.42}$$

Furthermore,

$$g_i(0)=\frac{\partial h}{\partial x^i}(0). \tag{2.43}$$

Proof. For any $x \in U$ write

$$h(x) = \int_0^1 dt\, \frac{d}{dt} h(tx) = \int_0^1 dt\, \frac{\partial h}{\partial x^i}(tx)\, x^i, \tag{2.44}$$

so set

$$g_i(x) = \int_0^1 dt\, \frac{\partial h}{\partial x^i}(tx). \tag{2.45}$$

Equation (2.43) follows immediately from (2.45) upon setting $x = 0$. $\square$

Proof of Proposition 2.40. First, use the derivation property to show that $D(1) = 0$ and then D applied to any constant function vanishes. Next, suppose f_1, f_2 are functions that agree in a neighborhood U of a point $x \in X$. Then if ρ is a cutoff function—$\rho(x) = 1$ and ρ vanishes on the complement of U—we have $\rho f_1 = \rho f_2$, and apply D to conclude that $D(f_1)(x) = D(f_2)(x)$. It follows that a derivation of Ω^0_X gives rise to a derivation of Ω^0_U for all open $U \subset X$. Furthermore, a derivation of Ω^0_X is determined by the collection of induced derivations on Ω^0_U as U ranges over an open cover of X.

Cover X by charts $(U; x^1, \dots, x^n)$ such that the image of U is a convex subset of $\mathbb{A}^n$. Define the vector field on U

$$\xi_x = (Dx^i)(x) \frac{\partial}{\partial x^i}, \qquad x \in U. \tag{2.46}$$

We claim that the derivation D induces on Ω^0_U is D_ξ. Namely, fix $x_0 \in U$ and translate the coordinates so that $x^i(x_0) = 0$ for all i. Identify $U \subset X$ with its image in $\mathbb{A}^n$, and so $x_0 \in X$ with $0 \in \mathbb{A}^n$. Fix $f \in \Omega^0_U$. Then Lemma 2.41 produces functions $g_i \in \Omega^0_U$ such that

$$f(x) = f(0) + g_i(x) x^i \tag{2.47}$$

and $g_i(0) = \partial f/\partial x^i(0)$. Then

$$Df(0) = D\bigl(f - f(0)\bigr)(0) = D(g_i x^i)(0) = (Dx^i)(0) g_i(0) = Dx^i(0) \frac{\partial f}{\partial x^i}(0) = (\xi f)(0). \tag{2.48}$$

This proves that $D = D_\xi$ as derivations of Ω^0_U, from which it follows that ξ patches to a global vector field on X and $D = D_\xi$ as derivations of Ω^0_X. $\square$

(2.49) *The Lie derivative is the Lie bracket.*

Proposition 2.50. *Let X be a smooth manifold and let $\xi, \eta \in \mathfrak{X}(X)$ be vector fields on X. Then*

$$[D_\xi, D_\eta] = D_{\mathcal{L}_\xi \eta}. \tag{2.51}$$

Proof. Compute in local coordinates using Proposition 2.23. For $f \in \Omega^0_X$ we have

$$
\begin{aligned}
[D_\xi, D_\eta]f &= \xi(\eta f) - \eta(\xi f) \\
&= \left(\xi^i \frac{\partial}{\partial x^i}\right)\left(\eta^j \frac{\partial f}{\partial x^j}\right) - \left(\eta^i \frac{\partial}{\partial x^i}\right)\left(\xi^j \frac{\partial f}{\partial x^j}\right) \\
&= \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} \\
&= (\mathcal{L}_\xi \eta) f
\end{aligned} \tag{2.52}
$$

□

Transport the commutator of derivations of Ω^0_X to a Lie bracket on vector fields, using Proposition 2.40: for $\xi, \eta \in \mathfrak{X}(X)$ characterize $[\xi, \eta] \in \mathfrak{X}(X)$ by

$$D_{[\xi,\eta]} = \big[D_\xi, D_\eta\big]. \tag{2.53}$$

Then Proposition 2.50 shows that

$$\mathcal{L}_\xi \eta = [\xi, \eta]. \tag{2.54}$$

Corollary 2.55. $\big(\mathfrak{X}(X), [-,-]\big)$ *is a Lie algebra.*

A geometric interpretation of the Lie derivative

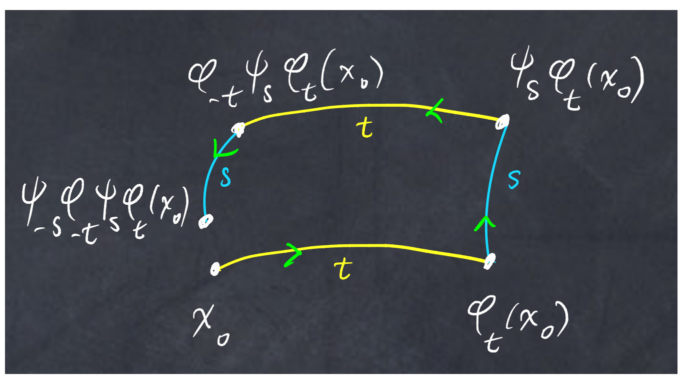

FIGURE 8. The commutator of flows

Let $\xi, \eta \in \mathfrak{X}(X)$ be vector fields on a smooth manifold X, and fix $x_0 \in X$. We compute the Lie derivative $\mathcal{L}_\xi \eta$ at x_0 in terms of the commutator of the flows that ξ, η generate. Let φ_t, ψ_s be local flows near x_0 generated by ξ, η, and fix local coordinates $x^1, \dots, x^n$ near x_0. For sufficiently small $\epsilon > 0$ and $t, s \in (-\epsilon, \epsilon)$, write the function

$$(t, s) \longmapsto \psi_{-s}\varphi_{-t}\psi_s\varphi_t(x_0) \tag{2.56}$$

as

$$(t, s) \longmapsto x^i(t, s). \tag{2.57}$$

Proposition 2.58. *We have*

$$\mathcal{L}_\xi \eta(x_0) = \frac{\partial^2 x^i}{\partial t \partial s} \frac{\partial}{\partial x^i}\bigg|_{\substack{t=0\\s=0}}. \tag{2.59}$$

Proof. From (2.56) compute

$$\frac{\partial x^i}{\partial s} \frac{\partial}{\partial x^i}\bigg|_{s=0} = -\eta + (\varphi_{-t})_*\eta, \tag{2.60}$$

where the right hand side is evaluated at x_0. Differentiate (2.60) in t at $t=0$ to derive (2.59). □

Consider the diagonal motion derived from (2.56) as

$$t \longmapsto \psi_{-t}\varphi_{-t}\psi_t\varphi_t(x_0), \tag{2.61}$$

which we write as

$$t \longmapsto \tilde{x}^i(t). \tag{2.62}$$

Corollary 2.63. *We have*

$$\mathcal{L}_\xi \eta(x_0) = \frac{1}{2}\frac{d^2\tilde{x}^i}{dt^2} \frac{\partial}{\partial x^i}\bigg|_{\substack{t=0\\s=0}}. \tag{2.64}$$

Lie brackets and smooth maps

In general, if $\varphi\colon X' \to X$ is a smooth map, and $\xi' \in \mathfrak{X}(X')$ is a vector field, then there is not a vector field defined on X. For to define a putative pushforward at $x \in X$ we would apply the differential of φ to ξ' at an inverse image point of x. But $\varphi^{-1}(x)$ may not be a singleton: if it is empty, there is no value defined, and if its cardinality is ≥ 2 there may be multiple values defined.

Definition 2.65. Suppose $\varphi\colon X' \to X$ is a surjective map, $\xi' \in \mathfrak{X}(X')$, and $\xi \in \mathfrak{X}(X)$. We say ξ', ξ are *φ-related* if $\varphi_*(\xi'_{x'}) = \xi_{\varphi(x')}$ for all $x' \in X'$.

Proposition 2.66. *If ξ'_1, ξ'_2 are φ-related to ξ_1, ξ_2, respectively, then $[\xi'_1, \xi'_2]$ is φ-related to $[\xi_1, \xi_2]$.*

Proof. ξ'_i and ξ_i are φ-related iff for all $f \in \Omega^0_X$ we have $\varphi^*(\xi_i f) = \xi'_i(\varphi^* f)$. If so, then

$$\begin{aligned} \varphi^*\big([\xi_1,\xi_2]f\big) &= \varphi^*(\xi_1\xi_2 f - \xi_2\xi_1 f) \\ &= \xi'_1\varphi^*(\xi_2 f) - \xi'_2\varphi^*(\xi_1 f) \\ &= \xi'_1\xi'_2\varphi^* f - \xi'_2\xi'_1\varphi^* f \\ &= [\xi'_1,\xi'_2]\varphi^* f. \end{aligned} \tag{2.67}$$

It follows that $[\xi'_1, \xi'_2]$ is φ-related to $[\xi_1, \xi_2]$. □