Math 231A: Algebraic Topology
Final exam
Due: Friday, December 9 at 12:00pm (noon) on Canvas

December 6, 2022

- The exam is out of a total of 100 points and is worth 25% of your total grade.
- When you have finished the exam, please upload your solutions to Canvas.
- The official due date for the exam is 12pm (noon) on Friday, December 9. However, I will accept late exams until 12pm (noon) on Sunday, December 11. I will not accept any exams turned in later than this unless there are extenuating circumstances.
- You are not allowed to collaborate on the exam. The only outside resources you may consult are your own notes and the texts listed in the syllabus.
- If you have any questions about the exam (e.g. about the statement of one of the questions), please feel free to ask me them over email.
- Good luck!

1. (10 points) Let $V_1, V_2 \subset \mathbb{R}^6$ denote complementary 3-planes (i.e. V_1 and V_2 span $\mathbb{R}^6$ as a vector space). Compute the integral homology groups of $\mathbb{R}^6 \setminus (V_1 \cup V_2)$.

2. (10 points) Given a space X with the property that $\bigoplus_{i=0}^{\infty} H_i(X)$ is finitely generated, define its *Euler characteristic* to be $\chi(X) = \sum_{i=0}^{\infty}(-1)^i \mathrm{rk}(H_i(X))$. Use the universal coefficients theorem to prove that $\chi(X) = \sum_{i=0}^{\infty}(-1)^i \dim_k H_i(X; k)$ for $k = \mathbb{Q}$ or $\mathbb{F}_p$, where p is a prime number.

3. (25 points) Consider the following three spaces:

$$A = \mathbb{CP}^3 \qquad B = S^2 \times S^4 \qquad \text{and} \qquad C = S^2 \vee S^4 \vee S^6$$

   (a) (10 points) Compute the integral cohomology groups of these three spaces.

   (b) (10 points) Prove that A and B are not homotopy equivalent.

   (c) (5 points) Prove that C is not homotopy equivalent to any compact manifold.

4. (30 points) Let M and N denote connected n-manifolds. A *connected sum* $M \# N$ of M and N is a new connected n-manifold defined in the following way. Choose Euclidean open balls $B_1 \subset \mathbb{R}^n \subset M$ and $B_2 \subset \mathbb{R}^n \subset N$. Then $M \setminus B_1$ and $N \setminus B_2$ contain embedded $(n-1)$-spheres

which are the boundaries of the balls B_i, and we define $M\#N$ by gluing $M \setminus B_1$ and $N \setminus B_2$ along any homeomorphism of these $(n-1)$-spheres. More concisely, we write:

$$M\#N \coloneqq M \setminus B_1 \cup_{S^{n-1}} N \setminus B_2.$$

For example, if M_g and M_h are oriented surfaces of genus g and h, then $M_g\#M_h$ is an oriented surface of genus $g+h$. [1] In the following, we let $M\#N$ denote an arbitrary choice of connected sum of M and N.

(a) (10 points) Assume that $n \geq 2$. Prove that $M\#N$ is orientable if and only if M and N are orientable.

(b) (10 points) Assume that M and N are compact. If M or N is orientable, prove that

$$H_i(M\#N) \cong H_i(M) \oplus H_i(N)$$

for $0 < i < n$.

(c) (10 points) Assume that M and N are compact. If M and N are both nonorientable, prove that

$$H_i(M\#N) \cong H_i(M) \oplus H_i(N)$$

for $0 < i < n-1$, whereas $H_{n-1}(M\#N)$ is obtained from $H_{n-1}(M)\oplus H_{n-1}(N)$ by replacing a $\mathbb{Z}/2\mathbb{Z}$ summand by a $\mathbb{Z}$ summand.

5. (25 points) Let M denote a compact connected 3-manifold. Suppose that $H_1(M) \cong \mathbb{Z}^{\oplus r} \oplus F$ for a finite abelian group F and nonnegative integer r.

   (a) (10 points) If M is orientable, prove that $H_2(M) \cong \mathbb{Z}^{\oplus r}$.

   (b) (15 points) If M is nonorientable, prove that $H_2(M) \cong \mathbb{Z}^{\oplus r-1} \oplus \mathbb{Z}/2\mathbb{Z}$. In particular, one must have $r \geq 1$ in this case. (Hint: consider what you know about other coefficients and use the UCT.)

   Remark: This implies that the homology groups of a compact connected 3-manifold are completely determined by its fundamental group and (non)orientability!

[1] It turns out that the homeomorphism class of $M\#N$ does not depend on many of the choices made above. In particular, connected sum may be made into a well-defined operation on oriented homeomorphism classes of manifolds. This is not a simple result and relies on the Annulus theorem, proved by Kirby in dimensions ≥ 5 and Quinn in dimension 4. (The version for smooth manifolds is significantly simpler.) However, this does not descend to a well-defined operation on homeomorphism classes of manifolds: if we let $\overline{\mathbb{CP}^2}$ denote $\mathbb{CP}^2$ with its orientation reversed, then $\mathbb{CP}^2\#\mathbb{CP}^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ are not even homotopy equivalent. In fact, it turns out that their intersection forms differ.