Problem Set # 1

Math230a: Differential Geometry

Due: September 11

Please form working groups of 3 ± 1.5 (round up) people to go over lectures and tackle problem sets. The problem sets are a crucial part of the class, and I strongly suggest that you engage with them. You can also use Discord to discuss problems, but please do not spoil others' joy by giving solutions; hints are better. The problems are as much about the journey as the destination. Those getting a grade in the course (this includes all undergraduates) should regularly hand in solutions. Do take advantage of office hours and problem sessions.

I tend to give *many* problems. They are at different levels and some are open-ended. Do as many as you can do well. I'd rather have you do fewer problems better than have you do more problems worse. Don't expect to turn in solutions to every problem every week. Again, better to turn in fewer well-written solutions than many poorly written ones.

The notes on Multivariable Analysis, Lectures 1 and 2, have material about affine space that you may want to read through. If you need reviews of exterior algebra and differential forms, they appear later in those notes and also in the notes on Differential Topology.

It is also a good idea to look over the material on ODE in Lectures 17 and 18 of the Multivariable Analysis notes. There is a global version in the middle of Lecture 14 of the Differential Topology notes. That material is good preparation for Lie derivatives, which we cover in the second lecture of the course.

For Problem 4 you may want to review Proposition 19.16 in the Differential Topology notes.

This problem set is a bit heavy on foundational problems, which is natural at the beginning to give you some review of techniques we will use right away in the course. As an antidote, a bit of classical geometry is here too. Enjoy!

1. Prove the following theorem of Menelaus. Let A be an affine space and p_0, p_1, p_2 independent points, i.e. points whose affine span is 2-dimensional. Let q_0 be a point on the line spanned by p_1, p_2; q_1 be a point on the line spanned by p_2, p_0; and q_2 be a point on the line spanned by p_0, p_1. Then q_0, q_1, q_2 are collinear if and only if
$$\frac{q_2p_0}{q_2p_1}\,\frac{q_0p_1}{q_0p_2}\,\frac{q_1p_2}{q_1p_0} = 1.$$
(Each ratio is a real number: what is the definition? No Euclidean structure is necessary, but if one is present can you define it in terms of distances? Can the ratio be 0 or ∞? Is the statement true if so?)

2. Let A, B be affine spaces over vector spaces V, W, respectively, and let $f\colon A \to B$ be a function. Do not assume finite dimensionality.

   (a) Proof or counterexample: f is affine iff for all $p_0, p_1 \in A$ and $\lambda^0, \lambda^1 \in \mathbb{R}$ with $\lambda^0 + \lambda^1 = 1$,
$$f(\lambda^i p_i) = \lambda^i f(p_i).$$

Here and always we use the summation convention: sum over indices which appear once as a subscript and once as a superscript (in an expression, i.e., on one side of an equation).

(b) Suppose that for every finite subset $S \subset A$ the *center of mass* (define!) of $f(S)$ is the image of the center of mass of S. Does it follow that f is affine?

(c) Define what it means for f to be (affine) quadratic.

(d) Define precisely an equivalence relation on quadratic functions $\mathbb{A}^2 \to \mathbb{R}$ by, roughly, considering two to be equivalent if they are related by a "change of coordinates". How many equivalence classes are there? For each equivalence class of quadratics $q\colon \mathbb{A}^2 \to \mathbb{R}$ describe the shape of $q^{-1}(0)$.

3. This problem gives practice with the index notation and summation convention that I use and strongly recommend to you. Note carefully the placement (superscript vs. subscript) of the indices in what follows. The actual name of the index (i or j or α) is arbitrary, though as always a judicious choice of notation helps you and your readers.

   Let V be an n dimensional (real) vector space. Suppose $\{e_j\}$ and $\{f_i\}$ are two bases for V which are related by the equation

$$e_j = P^i_j f_i,$$

   where P is an invertible matrix. In the matrix $P = \left(P^i_j\right)$, i is the row index and j the column index.

   (a) Suppose $\xi \in V$ is a vector. Then we can find real numbers ξ^j and $\tilde{\xi}^i$ such that $\xi = \xi^j e_j = \tilde{\xi}^i f_i$. Express $\tilde{\xi}^i$ in terms of the ξ^j.

   (b) Suppose $T\colon V \to V$ is a linear transformation. Relative to the basis $\{e_j\}$ it is expressed as the matrix A defined by $Te_j = A^i_j e_i$, and relative to the basis $\{f_i\}$ it is expressed as the matrix B defined by $Tf_i = B^j_i f_j$. What is the relationship between A and B?

   (c) The *dual space* V^* is the vector space of all linear functionals $V \to \mathbb{R}$; it is also n dimensional. Every basis of V gives rise to a *dual basis* of V^*. For the basis $\{e_j\}$ of V the dual basis $\{e^i\}$ of V^* is defined by the equation

$$e^i(e_j) = \delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

   (This equation defines the symbol δ^i_j.) The dual basis $\{f^j\}$ is defined similarly. Express f^j in terms of the e^i.

   (d) Suppose $\omega \in V^*$. Then we define its components relative to the basis $\{e^i\}$ by the equation $\omega = \omega_i e^i$ and its components relative to the basis $\{f^j\}$ by the equation $\omega = \tilde{\omega}_j f^j$. Express the ω_i in terms of the $\tilde{\omega}_j$.

   (e) The tensor and exterior powers of V have natural induced bases. For example, what is the basis for $\otimes^2 V^* = V^* \otimes V^*$, the space of bilinear forms on V? What about $\bigwedge^2 V^*$, the space of alternating bilinear forms? What about $\mathrm{Sym}^2 V^*$, the space of symmetric bilinear forms?

4. Let X be a smooth manifold. Recall that a *vector field* is a section of the tangent bundle $TX \to X$.
   (a) Suppose $\xi\colon C^\infty(X) \to C^\infty(X)$ is a *derivation* on the algebra of smooth functions on X, i.e., ξ is a linear map and
   $$\xi(fg) = (\xi f)g + f(\xi g), \qquad f, g \in C^\infty(X).$$
   Prove that ξ defines a unique vector field whose directional derivative on functions is the given derivation. (Define these terms carefully. Henceforth we identify vector fields and the associated derivation.)
   (b) Define the *Lie bracket* $[\xi_1, \xi_2]$ of vector fields ξ_1, ξ_2 on X as the commutator of derivations:
   $$[\xi_1, \xi_2]f = \xi_1\xi_2 f - \xi_2\xi_1 f.$$
   Prove that $[\xi_1, \xi_2]$ is a derivation, hence is a vector field.
   (c) The vector space of vector fields on X is often denoted $\mathfrak{X}(X)$. For $\xi_1, \xi_2, \xi_3 \in \mathfrak{X}(X)$ prove
   $$\begin{aligned} [\xi_1, \xi_2] + [\xi_2, \xi_1] &= 0 \\ [\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] &= 0 \end{aligned}$$
   These equations express that $\big(\mathfrak{X}(X), [-,-]\big)$ is a *Lie algebra*.

5. Let X be a smooth manifold and $\Omega^\bullet(X)$ the $\mathbb{Z}$-graded commutative algebra of differential forms. Let ξ, ξ_1, ξ_2 be vector fields on X. In differential calculus we have three basic (graded) derivations on $\Omega^\bullet(X)$: $d, \mathcal{L}_\xi, \iota_\xi$, where d is the *exterior derivative* of degree $+1$, $\mathcal{L}_\xi$ is the *Lie derivative* of degree 0, and ι_ξ is the *contraction* of degree -1. We define the Lie derivative $\mathcal{L}_\xi$ in Lecture 2. In case I don't get to it, I indicate the definition of contraction in part (a). Note that d and $\mathcal{L}_\xi$ are first-order differential operators, whereas ι_ξ is algebraic. In this problem you will compute the commutators of these operations. These are the basic equations of differential calculus.
   (a) Let V be a vector space. For $\xi \in V$ let $\iota_\xi\colon V^* \to \mathbb{R}$ be the linear map which evaluates a functional $\alpha \in V^*$ on the vector ξ. Prove that there is a unique extension $\iota_\xi\colon \bigwedge^\bullet V^* \to \bigwedge^\bullet V^*$ of degree -1 which satisfies the derivation (Leibniz) formula
   $$\iota_\xi(\omega_1 \wedge \omega_2) = \iota_\xi\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \iota_\xi\omega_2, \qquad \omega_1 \in {\textstyle\bigwedge}^k V^*, \quad \omega_2 \in {\textstyle\bigwedge}^\bullet V^*.$$
   (What are the corresponding derivation formulæ for d and $\mathcal{L}_\xi$?)
   (b) Verify $[d, \iota_\xi] = \mathcal{L}_\xi$. This is the Cartan formula for the Lie derivative on differential forms.
   (c) Show $[d, \mathcal{L}_\xi] = 0$. What does this formula express about the operator d?
   (d) Note $[d, d] = 0$ is one of the defining properties of d. What is $[\iota_{\xi_1}, \iota_{\xi_2}]$?
   (e) Check $[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1, \xi_2]}$.
   (f) Compute the remaining commutator $[\mathcal{L}_{\xi_1}, \iota_{\xi_2}] = \iota_{[\xi_1, \xi_2]}$.
   (g) Let α be a 1-form. Expand $d\alpha(\xi_1, \xi_2) = \iota_{\xi_1}\iota_{\xi_2} d\alpha$ using the above formulas. The only differentiation in the final answer should be directional derivatives of functions and Lie brackets of vector fields. Try the next case: $d\beta$ for a 2-form β.

6. (a) Prove the statement that follows Definition 1.52 in the notes: For the symmetry type $\mathrm{O}_n \hookrightarrow \mathrm{GL}_n\mathbb{R}$ and a vector space V, the data $(\mathcal{B}_G(V), \theta)$ in the definition is equivalent to a choice of positive definite inner product on V.

   (b) What is the symmetry type of a complex structure on an $n = 2m$ dimensional vector space? Show that a vector space with this structure, in the sense of Definition 1.52, is a complex vector space. State and prove a converse as well.

   (c) What is the symmetry type of an n-dimensional vector space equipped with a codimension one subspace? What about the same for vector spaces equipped with an inner product?

   (d) Consider the following two 2-dimensional symmetry types. The first is $\mathrm{SO}_2 \times \mu\!\!\mu_2 \to \mathrm{GL}_2\mathbb{R}$, where $\mu\!\!\mu_2 = \{\pm 1\}$ is the cyclic group of order two and the map is projection onto the first factor followed by inclusion. The second is $\mathrm{Spin}_2 \to \mathrm{GL}_2\mathbb{R}$, which is the composition of the double cover map $\mathrm{Spin}_2 \to \mathrm{SO}_2$ followed by the inclusion $\mathrm{SO}_2 \hookrightarrow \mathrm{GL}_2\mathbb{R}$. (Both Spin_2 and SO_2 are isomorphic to the circle group $\mathbb{T}$ of unit norm complex numbers; the double cover is the map $\lambda \mapsto \lambda^2$, $\lambda \in \mathbb{T}$.) The internal symmetry group in each case is $\mu\!\!\mu_2$. Describe as best you can the geometry of a 2-dimensional affine space equipped with each geometric structure.