# Math 114 Homework 5

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1. Let $\varphi$ and $E$ be as in Rudin exercise 4. Show that there exists a measure space $X$ and measurable function $f$ such that $E = [1, 2)$.

Let $X = (0, \infty)$ and define $f(x)$ to be $x^{-1/2}$ on $(0, 1]$, 0 on $(1, 2)$, and $1/(x \log^2 x)$ on $[2, \infty)$. Let $p > 0$. Then clearly $\varphi(p) < \infty$ if and only if

$$A(p) = \int_0^1 \frac{1}{x^{p/2}} dx < \infty \quad \text{and} \quad B(p) = \int_2^\infty \left( \frac{1}{x \log^2 x} \right)^p dx < \infty.$$

It is a standard fact that $A(p) < \infty$ if and only if $p/2 < 1$, i.e. $p < 2$. For $B(p)$, we first note that $B(1) < \infty$. In fact, by substituting $u = \log x$, we have

$$\int_2^\infty \frac{1}{x \log^2 x} dx = \int_{\log 2}^\infty \frac{1}{u^2} du < \infty.$$

For $p > 1$, note that $f(x) < 1$ for $x > e$, so $f(x)^p$ is bounded above by $f(x)$ when $x > e$. Moreover, since $1/(x \log^2 x)$ is continuous (and hence integrable) on $[2, e]$, it follows that $B(p) < \infty$ when $p > 1$. It now suffices to show that $B(p) = \infty$ for $p < 1$. We first prove the following lemma.

*Lemma.* Let $0 < \alpha < 1/e$. Then $\log(1/\alpha)x^\alpha \geq \log x$ for $x \geq (1/\alpha)^{1/\alpha}$.

*Proof.* Note that we have equality when $x = (1/\alpha)^{1/\alpha}$. Now consider the derivative of $\log(1/\alpha)x^\alpha - \log x$, which is equal to

$$\alpha \log(1/\alpha)x^{\alpha-1} - \frac{1}{x} = \frac{\alpha \log(1/\alpha)x^\alpha - 1}{x}.$$

The derivative is positive if $x \geq (1/\log(1/\alpha)\alpha)^{1/\alpha}$. Since $0 < \alpha < 1/e$, $\log(1/\alpha) > 1$, so in particular, $\log(1/\alpha)x^\alpha - \log x$ is increasing when $x \geq (1/\alpha)^{1/\alpha}$. The desired result follows immediately. □

Now let $p < 1$. Pick $0 < \alpha < 1/e$ small enough so that $p(2\alpha + 1) \leq 1$. By the lemma,

$$(x \log^2 x)^p \leq \log(1/\alpha)^{2p} x^{p(2\alpha+1)}$$

for $x \geq (1/\alpha)^{1/\alpha}$. Therefore if $c = \max(2, (1/\alpha)^{1/\alpha})$, then

$$B(p) \geq \int_c^\infty f(x)^p dx \geq \frac{1}{\log(1/\alpha)^{2p}} \int_c^\infty \frac{1}{x^{p(2\alpha+1)}} dx = \infty.$$

2. Assume, in addition to the hypotheses of Exercise 4, that $\mu(X) = 1$.

(a) Prove that $\|f\|_r \le \|f\|_s$ if $0 < r < s \le \infty$.

First suppose that $s = \infty$. Then $|f| \le \|f\|_\infty$ a.e., so

$$\|f\|_r = \left(\int_X |f|^r d\mu\right)^{1/r} \le \left(\int_X \|f\|_\infty d\mu\right)^{1/r} = \|f\|_\infty.$$

The above inequality holds trivially if $\|f\|_\infty = \infty$. Now let $s < \infty$. Then $s/r > 1$, so applying Hölder's inequality to $|f|^r$ and 1 with $p = s/r$, we get

$$\int_X |f|^r d\mu \le \left(\int_X |f|^s d\mu\right)^{r/s} \left(\int_X 1 d\mu\right)^{(s-r)/s} = \left(\int_X |f|^s d\mu\right)^{r/s}.$$

Raising both sides to the power of $1/r$ yields $\|f\|_r \le \|f\|_s$.

(b) Under what conditions does it happen that $0 < r < s \le \infty$ and $\|f\|_r = \|f\|_s < \infty$.

Referring back to our use of Hölder's inequality, assuming $\|f\|_r, \|f\|_s < \infty$, we have $\|f\|_r = \|f\|_s$ if and only if $\alpha|f|^s = \beta$ a.e. for some constants $\alpha$ and $\beta$ not both 0. This occurs if and only if $|f|$ is constant a.e. Note that if $|f| = c$ a.e., we also have $\|f\|_p = c < \infty$ for all $p > 0$.

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

Let $0 < r < s$ and suppose that $f \in L^s(\mu)$. Then by (a), $\|f\|_r \le \|f\|_s < \infty$, so $f \in L^r(\mu)$. We claim that these spaces contain the same functions if and only if there there does not exist a sequence of disjoint measurable sets $E_n \subseteq X$ each of positive measure.

First, suppose such a sequence $E_n$ exists. Then by disjointness, $\sum \mu(E_n) \le 1$. Hence $\mu(E_n) \to 0$ as $n \to \infty$, so we can choose a subsequence $E_{n_k}$ such that $\mu(E_{n_k}) \le 2^{-k}$ for all $k \ge 1$. Let $0 < r < s < \infty$ and define

$$f = \sum_{k=1}^\infty \mu(E_{n_k})^{-1/s} \chi_{E_{n_k}}.$$

By monotone convergence, $\int |f|^s d\mu = \sum_{k=1}^\infty 1 = \infty$, so $f \notin L^s(\mu)$. However, since $0 < r/s < 1$, we have $0 < 1 - r/s < 1$, and so by monotone convergence and a fundamental result about geometric series,

$$\int_X |f|^r d\mu = \sum_{k=1}^\infty \mu(E_{n_k})^{1-r/s} \le \sum_{k=1}^\infty \left(\frac{1}{2^{1-r/s}}\right)^k < \infty.$$

Thus $f \in L^r(\mu)$. If $s = \infty$, consider instead $f = \sum_{k=1}^\infty k\chi_{E_{n_k}}$. Since each $E_{n_k}$ has positive measure, clearly $\|f\|_\infty = \infty$. However, for any $0 < r < \infty$,

$$\int_X |f|^r d\mu = \sum_{k=1}^\infty k^r \mu(E_{n_k}) \leq \sum_{k=1}^\infty \frac{k^r}{2^k} < \infty.$$

For the other direction, suppose there does not exist a sequence of disjoint measurable sets $E_n$ with positive measure, and let $f$ be *any* complex measurable function on $X$. Consider the disjoint measurable sets $E_n = \{n - 1 \leq |f| < n\}$ for $n \geq 1$. Then only finitely many of these sets can have positive measure, so if $n$ is the largest integer such that $\mu(E_n) > 0$, we must have $|f| < n$ a.e. Therefore $f \in L^r(\mu)$ for every $r > 0$.

(d) Assume that $\|f\|_r < \infty$ for some $r < \infty$, and prove that

$$\lim_{p\to 0} \|f\|_p = \exp\left(\int_X \log |f| d\mu\right)$$

if $\exp(-\infty)$ is defined to be 0.

In addition to $\exp(-\infty) := 0$, we'll also take $\log 0 := -\infty$ as a convention. Note that $\|f\|_p$ is decreasing as $p \to 0$, and $0 \leq \|f\|_p \leq \|f\|_r < \infty$ for all $0 < p < r$. Therefore, the limit $\lim_{p\to 0} \|f\|_p$ exists. Before proceeding, we state and prove the following useful lemma:

*Lemma.* For all $x \geq 0$ and $p > 0$, $\log x \leq (x^p - 1)/p$.

*Proof.* It suffices to prove that $\log x \leq x - 1$, for then it follows that

$$p \log x = \log x^p \leq x^p - 1 \implies \log x \leq \frac{x^p - 1}{p}$$

for all $p > 0$. But note that $\log x$ is concave on $(0, \infty)$, and $x - 1$ is the tangent line to the graph of $\log x$ at $(1, 0)$. Thus $\log x \leq x - 1$ for all $x > 0$. If $x = 0$, then obviously $-\infty \leq -1$. $\square$

In light of the lemma, $\log |f| \leq (|f|^r - 1)/r$, so the integral $\int \log |f|$ is bounded above by $(\|f\|_r^r - 1)/r$. In particular, either $\int \log |f|$ is finite or $-\infty$. Consider for all $p > 0$ the function $(x^p - 1)/p$, which is strictly increasing on $(0, \infty)$ because its derivative $x^{1-1/p}$ is positive. If we let $p < \min(1, r)$, then

$$g_p(x) := |f(x)| - 1 - \frac{|f(x)|^p - 1}{p}$$

is a positive measurable function, and $g_p$ increases pointwise as $p \to 0^+$. Note that by L'Hospital's rule, $(|f|^p - 1)/p \to \log |f|$ as $p \to 0^+$. This is true even if $|f(x)| = 0$, as the limit is indeed $-\infty$ in that case. Now we apply monotone convergence to $g_p(x)$ to obtain

$$\lim_{p\to 0^+} \int_X g_p d\mu = \int_X |f| - 1 - \log|f| d\mu.$$

Subtracting $\int |f| - 1$ from both sides of the above equation, multiplying by $-1$, and then exponentiating (which preserves limits by continuity) yields

$$\lim_{p\to 0^+} \exp\left(\int_X \frac{|f|^p - 1}{p}\right) = \exp\left(\int_X \log|f| d\mu\right),$$

even when $\int \log|f| = -\infty$. Applying the lemma once more (specifically the case $\log x \leq x - 1$) allows us to conclude

$$\|f\|_p = \exp\left(\frac{1}{p}\log\left(\int_X |f|^p d\mu)\right)\right) \leq \exp\left(\int_X \frac{|f|^p - 1}{p} d\mu\right).$$

Taking the limit as $p \to 0^+$, we get $\lim_{p\to 0} \|f\|_p \leq \exp(\int \log|f|)$. The reverse inequality is easier. If $\int \log|f| = -\infty$, then the result follows because $\|f\|_p \geq 0$ for all $p > 0$. Otherwise, $\int \log|f|$ is finite. Then $f$ cannot be 0 on a set of positive measure, so we can assume WLOG that $|f| > 0$. Now applying Jensen's inequality to $|f|^p$ and the convex function $-\log x$ on $(0, \infty)$, we get

$$-\log \int_X |f|^p d\mu \leq -\int_X \log|f|^p d\mu \implies \log \int_X |f|^p d\mu \geq \int_X \log|f|^p d\mu.$$

Exponentiating both sides of the above inequality and then raising everything to the power of $1/p$ yields

$$\|f\|_p \geq \exp\left(\frac{1}{p}\int_X \log|f|^p d\mu\right) = \exp\left(\int_X \log|f| d\mu\right).$$

Taking the limit as $p \to 0$ gives $\lim_{p\to\infty} \|f\|_p \geq \exp(\int \log|f|)$.