Math 55b Problem Set 4

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I collaborated with AJ LaMotta for this problem set.

Problem 1. In this problem, we'll see how to construct the *completion* of a metric space; that is, given a metric space X, we'll construct a complete metric space X^* (i.e., every Cauchy sequence in X^* has a limit) in which X sits as a dense subset. To start, let (X, d) be any metric space, and let $\mathcal{C}(X)$ denote the set of all Cauchy sequences $\{p_n\} = p_1, p_2, p_3 \dots$ in X. We define an equivalence relation $\sim$ on $\mathcal{C}(X)$ by

$$\{p_n\} \sim \{q_n\} \quad \text{iff} \quad d(p_n, q_n) \to 0, \quad \text{i.e.:} \quad \forall \epsilon > 0, \exists N : \forall n \geq N, \ d(p_n, q_n) < \epsilon.$$

We then define the set X^* to be the quotient $\mathcal{C}(X)/\sim$, that is, a point $P \in X^*$ is an equivalence class of Cauchy sequences in X. Finally, we define a distance function D on X^* by

$$D(\{p_n\}, \{q_n\}) = \lim_{n\to\infty} d(p_n, q_n)$$

We will take for granted the fact that $\mathbb{R}$ (with its usual distance) is complete.

(a) Show that $\sim$ is indeed an equivalence relation on $\mathcal{C}(X)$.

(b) Show that D is well defined and gives a metric on X^*.

(c) Show that the metric space (X^*, D) is complete.

(d) Show that the map $\iota : X \to X^*$ defined by $p \mapsto \{p, p, p, \dots\}$ is injective, and that for any $p, q \in X$ we have $D(\iota(p), \iota(q)) = d(p, q)$ (that is, ι is an *isometry*).

(e) Finally, show that the image $\iota(X) \subset X^*$ is dense.

Note that applying this construction to the metric space $\mathbb{Q}$ (with the usual distance) gives $\mathbb{R}$.

(a) We'll check the three axioms of an equivalence relation.

- **Reflexivity:** If $\{p_n\} \in \mathcal{C}(X)$, then $d(p_n, p_n) = 0$ so $\{p_n\} \sim \{p_n\}$.
- **Symmetry:** If $\{p_n\}, \{q_n\} \in \mathcal{C}(X)$ with $\{p_n\} \sim \{q_n\}$, then $d(p_n, q_n) \to 0$ so by symmetry of the distance function, $d(q_n, p_n) \to 0$.
- **Transitivity:** If $\{p_n\}, \{q_n\}, \{z_n\} \in \mathcal{C}(X)$ with $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{z_n\}$, then $d(p_n, q_n) \to 0$ and $d(q_n, z_n) \to 0$. Note that by the triangle inequality we have $d(p_n, q_n) + d(q_n, z_n) \geq d(p_n, z_n)$ so $d(p_n, z_n) \to 0$ and thus $\{p_n\} \sim \{z_n\}$.

(b) First we'll check that $D(\{p_n\}, \{q_n\})$ exists for any two Cauchy sequences $\{p_n\}, \{q_n\} \in \mathcal{C}(X)$. Consider the sequence $\{d(p_n, q_n)\}$. Let $\epsilon > 0$ and $N > 0$ such that for all $n, m \geq N$, $d(p_m, p_n) < \frac{\epsilon}{2}$ and $d(q_m, q_n) < \frac{\epsilon}{2}$. Then

$$\begin{aligned}|d(p_m, q_m) - d(p_n, q_n)| &\leq |d(p_m, q_m) - d(p_m, q_n)| + |d(p_m, q_n) - d(p_n, q_n)| \\ &\leq d(q_m, q_n) + d(p_m, p_n) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}$, however since $\mathbb{R}$ is complete, it must converge. So $D(\{p_n\}, \{q_n\})$ exists. Next, to check well-definedness, let $\{p_n\} \sim \{p'_n\}$ and $\{q_n\} \sim \{q'_n\}$. We claim that $D(\{p_n\}, \{q_n\}) = D(\{p'_n\}, \{q'_n\})$. Since $D(\{p_n\}, \{q_n\}) = \lim_{n\to\infty} d(p_n, q_n)$, it suffices to show that $\lim_{n\to\infty} |d(p_n, q_n) - d(p'_n, q'_n)| = 0$. Let $\epsilon > 0$ be arbitrary. Since $\{p_n\} \sim \{p'_n\}$ and $\{q_n\} \sim \{q'_n\}$, there must be some $N > 0$ such that for any $n \geq N$, we have $d(p_n, p'_n) < \frac{\epsilon}{2}$ and $d(q_n, q'_n) < \frac{\epsilon}{2}$. However,

$$\begin{aligned}|d(p_n, q_n) - d(p'_n, q'_n)| &\leq |d(p_n, q_n) - d(p_n, q'_n)| + |d(p_n, q'_n) - d(p'_n, q'_n)| \\ &\leq d(p_n, q_n) + d(p'_n, q'_n) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So $\lim_{n\to\infty} |d(p_n, q_n) - d(p'_n, q'_n)| = 0$ and thus $D(\{p_n\}, \{q_n\}) = D(\{p'_n\}, \{q'_n\})$. Lastly, we must check that D satisfies the axioms of a metric:

1. **Identity:** Suppose $D(\{p_n\}, \{q_n\}) = 0$. This means that $\lim_{n\to\infty} d(p_n, q_n) = 0$, so by definition $\{p_n\} \sim \{q_n\}$. Conversely, $D(\{p_n\}, \{p_n\}) = \lim_{n\to\infty} d(p_n, p_n) = 0$.

2. **Symmetry:** Clearly $D(\{p_n\}, \{q_n\}) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = D(\{q_n\}, \{p_n\})$.

3. **Triangle Inequality:** Note that $D(\{p_n\}, \{q_n\}) + D(\{q_n\}, \{z_n\}) = \lim_{n\to\infty} (d(p_n, q_n) + d(q_n, z_n)) \leq \lim_{n\to\infty} d(p_n, z_n) = D(\{p_n\}, \{z_n\})$.

(c) Suppose $\{p_{1,n}\}, \{p_{2,n}\}, \ldots$ in (X^*, D) is a Cauchy sequence. Since $\{p_{w,n}\}$ is Cauchy, let $q_w = p_{w,\kappa(w)}$ where $\kappa(w)$ is some number such that $d(q_w, p_{w,n}) < \frac{1}{w}$. First, we show that $\{q_w\}$ is a Cauchy sequence. Let $\epsilon > 0$. We claim that for any $w_1, w_2 \geq \frac{4}{\epsilon}$, we have $d(q_{w_1}, q_{w_2}) < \epsilon$. Firstly, we know by definition that $d(q_{w_1}, p_{w_1,n}) < \frac{1}{w_1}$ and $d(q_{w_2}, p_{w_2,n}) < \frac{1}{w_2}$ for $n > \max\{\kappa(w_1), \kappa(w_2)\}$. Lastly, we'll increase n until $d(p_{w_1,n}, p_{w_2,n}) < \frac{\epsilon}{2}$. (We can do this because $\{p_{w,n}\}$ is a Cauchy sequence) Then by the triangle inequality,

$$\begin{aligned}d(q_{w_1}, q_{w_2}) &\leq d(q_{w_1}, p_{w_1,n}) + d(p_{w_1,n}, p_{w_2,n}) + d(p_{w_2,n}, q_{w_2}) \\ &= \frac{1}{w_1} + \frac{\epsilon}{2} + \frac{1}{w_2} \leq \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon.\end{aligned}$$

So $\{q_w\} \in \mathcal{C}(X)$ is in the completion.

Let $\epsilon > 0$. Since $\{q_w\}$ is a Cauchy sequence, there is some N such that for any $n_1, n_2 > N$ we have $d(q_{n_1}, q_{n_2}) < \frac{\epsilon}{2}$. Let $N' = \max\{\frac{2}{\epsilon}, N\}$. For any $n \geq N$ and $m \geq \max\{\kappa(n), n\}$, we have

$$\begin{aligned}d(p_{n,m}, q_m) &= d(p_{n,m}, p_{m,\kappa(n)}) \\ &\leq d(p_{n,m}, p_{n,\kappa(n)}) + d(p_{n,\kappa(n)}, p_{m,\kappa(m)}) \\ &\leq \frac{1}{n} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This means that for all $j \geq N$ we have $D(\{p_{j,n}\}, \{q_n\}) \leq \epsilon$ so $\{p_n\}$ converges to $\{q_n\}$.

(d) Clearly ι is an isometry, because if $p, q \in X$ then $D(\iota(p), \iota(q)) = \lim_{n\to\infty} d(p, q) = d(p, q)$. This automatically implies that ι is an injection because distinct points $p, q \in X$ have images $\iota(p), \iota(q)$ with $D(\iota(p), \iota(q)) = d(p, q) > 0$ so $\iota(p) \neq \iota(q)$ by the identity axiom of distances.

(e) Let $\{p_n\} \in X^*$ and $\epsilon > 0$. We'll find a $q \in X$ with $D(\iota(q), \{p_n\}) < \epsilon$. Since $\{p_n\}$ is a Cauchy sequence, there must be some $N > 0$ such that whenever $n \geq N$ we have $d(p_n, p_N) < \epsilon$. So letting $q = p_N$, note that $D(\iota(q), \{p_n\}) = \lim_{n\to\infty} d(p_N, p_n) < \epsilon$. This completes the proof.

Problem 2.

(a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

(b) If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

(a) First of all, since p has a right inverse, it must be a surjective map of sets. Next, suppose $V \subset Y$ is an open set. Then $p^{-1}(V)$ is open by continuity. Conversely if $V \subset Y$ is an arbitrary subset with $p^{-1}(V)$ open, then $f^{-1}(p^{-1}(V)) = V$ is open. Since p is surjective and $V \subset Y$ is open if and only if $p^{-1}(V)$ is open, p is a quotient map.

(b) Letting $\iota_A : A \to X$ be the inclusion map, note that $r \circ \iota_A$ is the identity map on A. Thus by (a), r is a quotient map.

Problem 3. Let $X = \mathbb{R} \times \{1, 2\}$, where $\{1, 2\}$ is equipped with the discrete topology, and consider the equivalence relation given by $(x, 1) \sim (x, 2)$ for all $x \neq 0$ (but $(0, 1) \not\sim (0, 2)$). Show that the quotient topology on $X/\sim$ is not Hausdorff.

Let x_1 and x_2 be the images of $(0, 1)$ and $(0, 2)$ under the quotient map. Suppose for the sake of contradiction that $U \ni x_1$ and $V \ni x_2$ are disjoint open sets in $X/\sim$. Letting $\pi : X \to X/\sim$ be the quotient map, $\pi^{-1}(U) \ni (0, 1)$ and $\pi^{-1}(V) \ni (0, 2)$ are disjoint open sets. Let $B_{\epsilon_1}(0, 1) \subset U$ and $B_{\epsilon_2}(0, 2) \subset V$ for some $\epsilon_1, \epsilon_2 > 0$. Assume without loss of generality that $\epsilon_1 \leq \epsilon_2$. Then $\pi(B_{\epsilon_1}(0, 1)) \subset U \cap V$ so U and V are not disjoint. Thus $X/\sim$ is not Hausdorff.

Problem 4. Let $X = \mathbb{R}^{n+1} - \{0\}$. We define an equivalence relation on X by $x \sim y \Leftrightarrow x = \alpha y$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The quotient $X/\sim$ is called the n-dimensional real projective space, $\mathbb{RP}^n$.

(a) By considering the image under the quotient map of the open subset $X_0 = \{(x_1, \ldots, x_{n+1}) \in X \mid x_{n+1} \neq 0\}$, show that $\mathbb{RP}^n$ contains an open subset U_0 which is homeomorphic to $\mathbb{R}^n$ and whose complement is homeomorphic to $\mathbb{RP}^{n-1}$.

(b) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient space $S^n/\sim$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$ and $a \sim b \Leftrightarrow a = \pm b$ (i.e., we identify *antipodal* points on the sphere).

(c) Show that the quotient map $p : S^n \to \mathbb{RP}^n$ is a two-to-one *covering map*, i.e. that every point of $\mathbb{RP}^n$ has a neighborhood U such that $p^{-1}(U)$ is the disjoint union of two open subsets $U_1, U_2 \subset S^n$, such that the restriction of p to $U_i \to U$ is a homeomorphism for each $i = 1, 2$.

(d) Show that $\mathbb{RP}^1$ is homeomorphic to S^1. (Note: the analogue for $n \geq 2$ is false).

Let $\pi : \mathbb{R}^{n+1} - \{0\} \to (\mathbb{R}^{n+1} - \{0\})/\sim$ be the quotient map.

(a) We claim that $\pi(X_0)$ is an open subset of $\mathbb{RP}^n$ homeomorphic to $\mathbb{R}^n$. First consider the map $f : X_0 \to \mathbb{R}^n$ which takes $(x_1, \ldots, x_{n+1})$ to $(x_1/x_{n+1}, \ldots, x_n/x_{n+1})$. This map is clearly a continuous surjection and well defined since $x_{n+1} \neq 0$. This map also passes to the quotient, giving us a bijective continuous map $\widetilde{f} : \pi(X_0) \to \mathbb{R}^n$. There is also a continuous inverse map given by $(x_1, \ldots, x_n) \mapsto [(x_1, \ldots, x_n, 1)]$. Next, let's look at the complement of $\pi(X_0)$. This complement is the image under the quotient map of the closed set

$X_1 = \{(x_1, \ldots, x_n, 0) \in X\}$. Consider the map $g : X_1 \to \mathbb{RP}^{n-1}$ which takes $(x_1, \ldots, x_n, 0)$ to $[(x_1, \ldots, x_n)]$. Again once we pass to the quotient $\widetilde{g} : \pi(X_1) \to \mathbb{RP}^{n-1}$ we get a homeomorphism.

(b) Consider the inclusion $i : S^n \to \mathbb{R}^{n+1} - \{0\}$. Then $\pi \circ i : S^n \to \mathbb{RP}^n$ is a continuous surjection, so passing to the quotient, we get a continuous surjection $\widetilde{\pi \circ i} : S^n/\sim \to \mathbb{RP}^n$. However $S^n/\sim$ is a compact space and $\mathbb{RP}^n$ is Hausdorff so any continuous surjection from $S^n/\sim$ to $\mathbb{RP}^n$ is a homeomorphism.

(c) Let $x \in \mathbb{RP}^n$ be a point. Assume without loss of generality that its last coordinate is nonzero, so $x_{n+1} \neq 0$. Thus $x \in U_0$, the open subset from (a). Next it's clear that $p^{-1}(U_0) = V_+ \sqcup V_-$ where $V_+ = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} > 0\}$ and $V_- = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < 0\}$. These are clearly disjoint and open. Furthermore the restriction $p\big|_{V_+}$ is clearly a continuous surjection, however proving it's injective is a bit more tricky. Suppose $(x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}) \in V_+$ with $p(x_1, \ldots, x_{n+1}) = p(y_1, \ldots, y_{n+1})$. This means that $x_i = \alpha y_i$ for some $\alpha \in \mathbb{R}$. Since $x_{n+1} = \alpha y_{n+1}$ and x_{n+1} and y_{n+1} have the same sign, it follows that $\alpha > 0$. However note that $x_1^2 + \cdots + x_{n+1}^2 = 1$ and $(\alpha x_1)^2 + \cdots + (\alpha x_{n+1})^2 = \alpha^2 = 1$, so $\alpha = 1$. This means that $x_i = y_i$ and the function is injective. So the restriction to $V_\pm$ is a homeomorphism.

(d) We know by (b) that $\mathbb{RP}^1$ is homeomorphic to $S^1/\sim$, where $x \sim y$ iff $x = \pm y$. Letting $S_1 \subset \mathbb{C}$, consider the continuous surjective map $f : S^1 \to S^1$ given by $f(z) = z^2$. Since $x \sim y$ implies that $f(x) = f(y)$, we can pass to the quotient to get a continuous surjection $\widetilde{f} : S^1/\sim \to S^1$. Again, since $S^1/\sim$ is compact and S^1 is Hausdorff, this map must be a homeomorphism. So $\mathbb{RP}^1 \cong S^1$.

Problem 5. Let X be a topological space, and consider the equivalence relation on X defined by $x \sim y$ if there exists a path in X from x to y. The equivalence classes are called *path components* of X. Define $\pi_0(X)$ to be the set of path components of X.

(a) If $f : A \to Y$ is continuous and A is path-connected, show that $f(A)$ is path-connected and thus contained in a single path component of Y.

(b) Show that if $f : X \to Y$ is a continuous function, there is an induced map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$.

(c) Show that π_0 is a functor from the category of topological spaces (with continuous functions) to the category of sets; i.e. verify that (i) for the identity map $\mathrm{id}_X : X \to X$, $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$, and (ii) given $f : X \to Y$ and $g : Y \to Z$, $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.

(a) Let $x, y \in f(A)$ be distinct points. Pick any $a \in f^{-1}(x)$ and $b \in f^{-1}(y)$ and since A is path connected we can find a path $\gamma : I \to A$ with $\gamma(0) = a$ and $\gamma(1) = b$. Then $f \circ \gamma : I \to Y$ is a path with $(f \circ \gamma)(0) = x$ and $(f \circ \gamma)(1) = y$. So $f(A)$ is path connected.

(b) Write $X = \bigsqcup_{i \in \pi_0(X)} X_i$ where X_i are the path connected components of X, and write $Y = \bigsqcup_{j \in \pi_0(Y)} Y_j$ where Y_i are the path connected components of Y. For every $i \in \pi_0(X)$, note that by (a), $f(X_i)$ is path connected, so it must be contained in some Y_j. So define the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ by taking $i \in \pi_0(X)$ to the $j \in \pi_0(Y)$ such that $f(X_i) \subset Y_j$.

(c) For the identity map id_X note that $f(X_i) = X_i$ so $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$. Next, note that if $f(X_i) \subset Y_{\pi_0(f)(i)}$ and $g(Y_{\pi_0(f)(i)}) \subset Z_{(\pi_0(g) \circ \pi_0(f))(i)}$ then clearly $(g \circ f)(X_i) \subset Z_{(\pi_0(g) \circ \pi_0(f))(i)}$. Yet $(g \circ f)(X_i) \subset Z_{\pi_0(g \circ f)(i)}$. So $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.

Problem 6. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Let $H_h : I \times X \to Y$ be a homotopy between h and h', so $H_h(0, x) = h(x)$ and $H_h(1, x) = h'(x)$. Similarly, let $H_k : [0, 1] \times Y \to Z$ be a homotopy between k and k'. Now consider the homotopy $H_{k \circ h} : [0, 1] \times X \to Z$ given by $H_{k \circ h}(t, x) = H_k(t, H_h(t, x))$. Note that $H_{k \circ h}(0, x) = H_k(0, H_h(0, x)) = (k \circ h)(x)$ and $H_{k \circ h}(1, x) =$

$H_k(1, H_h(1,x)) = (k' \circ h')(x)$. This is clearly continuous because it is a composition of continuous functions. It's thus a homotopy between $k \circ h$ and $k' \circ h'$.

Problem 7. Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

(a) Let $f : X \to I$ be a continuous function. Consider the map $H : I \times X \to I$ given by $H(t, x) = (1-t) \cdot f(x)$. This is clearly continuous, so it is a homotopy between f and the constant map $c_0 : X \to I$ which maps every element to zero. Since every function in $[X, I]$ is homotopic to a constant map, there is only one equivalence class in $[X, I]$.

(b) Let $f : I \to Y$ be a continuous function. First note that the map $H : I \times I \to Y$ given by $H(t, x) = f((1-t) \cdot x + t)$. Then $H(0, x) = f(x)$ but $H(1, x) = f(1)$. So f is homotopic to the constant map $c_{f(1)}$. However since Y is path connected, every constant map is homotopic, i.e. given $a, b \in Y$, there is a path $\gamma : I \to Y$ connecting a, b. Then $H : I \times I \to Y$ given by $H(t, x) = \gamma(t)$ is a homotopy between c_a and c_b. So $[I, Y]$ contains a single point corresponding to the homotopy class of constant functions.

Problem 8. A space X is said to be contractible if the identity map $\mathrm{id}_X : X \to X$ is nulhomotopic.

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

(a) For both spaces, consider the constant map $c_0 : X \to X$ which takes x and maps it to $0 \in X$. Next, look at the homotopy $H : I \times X \to X$ given by $H(t, x) = (1-t) \cdot x$. Then $H(0, x) = \mathrm{id}_X(x)$ and $H(1, x) = c_0(x)$. This clearly a continuous homotopy between id_X and a constant map, so I and $\mathbb{R}$ are both contractible.

(b) Let X be a contractible space, say $H : I \times X \to X$ is a homotopy of id_X and some constant map c_x for a fixed $x \in X$. Now let $a, b \in X$ be distinct points. Then $\gamma_a : I \to X$ given by $\gamma_a(t) = H(t, a)$ is a path from a to x. Similarly, we get a path γ_b from b to x. Thus we can construct a path γ by

$$\gamma(t) = \begin{cases} \gamma_a(2t) & 0 \le t < \frac{1}{2} \\ \gamma_b(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

This is clearly continuous by the gluing lemma, and $\gamma(0) = a$ and $\gamma(1) = b$ so it is a path from a to b. Thus X is path connected.

(c) Let $f : X \to Y$ be a continuous function. Since Y is contractible, there is a homotopy $H : I \times Y \to Y$ with $H(0, y) = y$ and $H(1, y) = p$ for some fixed $p \in Y$. Consider the homotopy $H_f : I \times X \to Y$ given by $H_f(t, x) = H(t, f(x))$. This is clearly continuous, and $H_f(0, x) = H(0, f(x)) = f(x)$ and $H_f(1, x) = H(1, f(x)) = p$. So any f is homotopic to the constant path c_p. Thus $[X, Y]$ consists of a single element.

(d) Let $f : X \to Y$ be a continuous function. X is contractible, so there is a homotopy $H : I \times X \to X$ with $H(0, x) = x$ and $H(1, x) = p$ for some fixed $p \in X$. Next, consider the homotopy $H_f : I \times X \to Y$ given by $H_f(t, x) = f(H(0, x))$. Then $H_f(0, x) = f(x)$ and $H_f(1, x) = f(p)$. So every function f is homotopic to a constant path $c_{f(p)}$. Since Y is path connected, for any two functions $f_1, f_2 : X \to Y$ we can find a path $\gamma : I \to Y$ such that $\gamma(0) = f_1(p)$ and $\gamma(1) = f_2(p)$. By the previous part of the argument, $f_1 \sim c_{f_1(p)}$ and $f_2 \sim c_{f_2(p)}$. However $c_{f_1(p)} \sim c_{f_2(p)}$ because there is a homotopy $H_{f_1,f_2} : I \times X \to Y$ given by $H(t, x) = \gamma(t)$. So by the transitive property of homotopy, $f_1 \sim f_2$ and so $[X, Y]$ has one element.

Problem 9 (optional, extra credit).

(a) Show that the collection $\mathcal{B} = \{[a,b) \mid a < b,\ a,b \in \mathbb{Q}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$ which is strictly finer than the standard topology and strictly coarser than the lower limit topology.

(b) Show that $\mathcal{T}$ is regular (T3) and second-countable, hence metrizable by Urysohn's theorem.

(c) Show that $(\mathbb{R}, \mathcal{T})$ is homeomorphic to a subspace of $\mathbb{R}$ with its usual topology (this shows more directly that $\mathcal{T}$ is metrizable).

(a) Clearly $\mathcal{B}$ is a basis, since it covers the whole space, and whenever $[a,b)$ intersects $[c,d)$ nontrivially, we can find $[e,f) \subset [a,b) \cap [c,d)$. It's strictly finer than the standard topology because $[0,1)$ is not open in the standard topology, since every open ball around 0 is not fully contained in $[0,1)$. It is also strictly coarser that the lower limit topology because $[\sqrt{2}, 2)$ is not in $\mathcal{B}$, since every open $[a,b)$ containing $\sqrt{2}$ must partly lie outside $[\sqrt{2}, 2)$.

(b) Clearly $\mathcal{B}$ is second countable, because it is a subset of the countable set $\mathbb{Q}^2$. To prove it is a regular space, let $(-\infty, a) \cup [b, \infty) = C \subset \mathbb{R}$ be closed and let p be in $\mathbb{R} - C$. If $p = a$, then $(-\infty, a) \cup [b - (b-a)/3, \infty) \supset C$ and $[p, p + (b-a)/3] \supset \{p\}$. Otherwise, let $e = \min\{p - a, b - p\}$. Then $(-\infty, a + e/2) \cup [b + e/2, \infty) \supset C$ and $[p - \epsilon, p + \epsilon] \supset \{p\}$. This proves that $(\mathbb{R}, \mathcal{B})$ is regular.