Math 132 Problem Set 7

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Problem 1. Let $z \in \mathbb{R}^n - X$. Prove that if x is any point of X and U any neighborhood of x in $\mathbb{R}^n$, then there exists a point of U that may be joined to z by a curve not intersecting X.

Let's set X' to be the subset of points $x \in X$ such that for any open neighborhood $U \ni x$, there is some point of U that can be joined to z by a curve that doesn't intersect X. To prove the statement, we want to show that $X' = X$. We do this in several steps. First of all, to show that X' is nonempty, we use compactness of X to find the point $x \in X$ with minimum distance from z. Taking the line segment L from x to z must clearly not intersect X, since this would violate the minimum distance condition. Then for any neighborhood U of x, the must be some $y \in U \cap (L - \{x\})$ since x is a limit point of L. This proves that X' is nonempty.

Next, we'll show that X' is open, so let $x \in X'$. Applying the local immersion theorem to the inclusion map $X \to \mathbb{R}^n$, let U be some neighborhood of x, and let $\psi : U \to \mathbb{R}^n$ be a diffeomorphism such that $\psi(X \cap U) = \mathbb{R}^{n-1} \times \{0\}$. We just need to show $X \cap U \subset X'$, so suppose we had some $a \in X \cap U$ and let V be a neighborhood of $a \in \mathbb{R}^n$. This follows by the preceding paragraph.

We'll now also show that X' is closed. Suppose x is some limit point of X', and let U be a neighborhood of $x \in \mathbb{R}^n$. The must be a point $x' \in X' \cap U$, so by definition of X', there is some point of U which is connected to z by a curve not intersecting X, so $x \in X'$ and X' is closed. Since X' is a nonempty clopen subset of a connected X, $X = X'$ so this property is true for all points in X.

Problem 2. Show that $\mathbb{R}^n - X$ has, at most, two connected components.

Suppose we had three points z_1, z_2, and z_3 in $\mathbb{R}^n - X$. Letting $\psi : U \to \mathbb{R}^n$ be a diffeomorphism from a neighborhood U of some $x \in X$. By the preceding problem, we get points $q_i \in U - X$ and paths $\sigma : q_i \to z_i$ that don't intersect X. Notice that $\sigma(q_i)$ are points in $\psi(U - X) = \mathbb{R}^{n-1} \times \mathbb{R}^\times$, which has two path components, and so there must be a path between (WLOG) $\psi(q_1)$ and $\psi(q_2)$. Pulling back by the inverse ψ^{-1} thus gives us a path between q_1 and q_2. Thus, there can be at most two path components, and hence connected components.

Problem 3. Show that if z_0 and z_1 belong to the same connected component of $\mathbb{R}^n - X$, then $W_2(X, z_0) = W_2(X, z_1)$.

Since $\mathbb{R}^n - X$ is open, it is locally path connected and so z_0 and z_1 belong to the same path component. This means that we have a smooth path $\sigma : I \to \mathbb{R}^n - X$ which takes $z_0 \mapsto z_1$. Now let $u_i : X \to S^{n-1}$ be the "direction to z_i"-map, i.e. $x \mapsto (x - z_i)/|x - z_i|$. Recall that $W_2(X, z_i) = \deg_2(u_i)$, so it suffices to show that $u_0 \simeq u_1$. Consider the homotopy $h : X \times I \to X$ given by:

$$h(x, t) = \frac{x - \sigma(t)}{|x - \sigma(t)|}.$$

This completes the proof.

Problem 4. Given a point $z \in \mathbb{R}^n - X$ and a direction vector $v \in S^{n-1}$, consider the ray r emanating from z in the direction of v,

$$r = \{z + tv \mid t \geq 0\}.$$

Check that the ray r is transversal to X if and only if v is a regular value of the direction map $u : X \to S^{n-1}$. In particular, almost every ray from z intersects X transversally.

Consider the map $u' : \mathbb{R}^n - z \to S^{n-1}$ given by the direction map $u'(y) = (y-z)/|y-z|$. If we let $i : X \to \mathbb{R}^n - z$ be the inclusion, then $u = u' \circ i$. Notice that for any direction $v \in S^{n-1}$, we get $(u')^{-1}(v) = \{z + tv : t > 0\}$. For any $y \in (u')^{-1}(v)$, then $y = x + |y-z|v \in r - z$, yet on the other hand for any $t > 0$, we get $(u')(z+tv) = v$ so $(u')^{-1}(v) = r - z$.

Let's now see that that u' is a submersion, which follows from the claim that $\dim \ker du'_u = 1$ for $y \in \mathbb{R}^n - \{0\}$. Then, for all vectors $v \in \mathbb{R}^n$ we get

$$\begin{aligned} du'_y(v) &= \lim_{t\to 0} \frac{1}{t}\left(\frac{y+tv}{|y+tv|} - \frac{y}{|y|}\right) = \frac{1}{|y|^2}\lim_{t\to 0}\left(|y|v + \frac{|y|-|y+tv|}{t}y\right) = \frac{v}{|y|} - \left(\lim_{t\to 0}\frac{|y+tv|-|y|}{t}\right)\frac{y}{|y|^2} \\ &= \frac{1}{|y|}\left(v - \frac{y\cdot v}{|y|^2}y\right) = \frac{1}{|y|}(v - \mathrm{proj}_y v). \end{aligned}$$

Thus it follows that $d(u')_y(v) = 0$ if and only if $v = \mathrm{proj}_y v$, which is true if and only if $v \in \mathrm{span}(y)$. Therefore $\ker df_y = \mathrm{span}(y)$, which is 1-dimensional as desired so f is a submersion and in particular $f \pitchfork \{v\}$.

By Exercise 7 from GP Chapter 1, Section 5, we have that $u \pitchfork \{v\}$ if and only if $X \pitchfork (u')^{-1}(v)$ in $\mathbb{R}^n - z$. But we also know that $u \pitchfork \{v\}$ if and only if v is a regular value of u so since $(u')^{-1}(v) = r - z$ and due to the fact that $\mathbb{R}^n - z$ is an open subset of $\mathbb{R}^n$, the condition $X \pitchfork (r-z)$ in $\mathbb{R}^n - z$ is the same as $X \pitchfork (r-z)$ in $\mathbb{R}^n$. This is in turn equivalent to saying that $r \pitchfork X$ since $z \notin X$. Thus, v is a regular value of u if and only if $r \pitchfork X$. Notice that by Sard's Theorem, almost every $v \in S^{n-1}$ is a regular value of u and so almost every ray r from z intersects X transversally. This is what we wanted to show.

Problem 5. Suppose that r is a ray emanating from z_0 that intersects X transversally in a nonempty (necessarily finite) set. Suppose that z_1 is any other point on r (but not on X), and let ℓ be the number of times r intersects X between z_0 and z_1. Verify that

$$W_2(X, z_0) \equiv W_2(X, z_1) + \ell \pmod 2.$$

Let $L(z_0, z_1)$ be the line segment from z_0 to z_1, and let $L_r(z_1)$ be the subray of r starting at z_1. Let $\overline{r} = r/|r|$. Let the direction maps u_i be defined as in the previous problems. Note that $L_r(z_1) \pitchfork X$, so by the previous problem it follows that $\overline{r}$ is a regular value of u_0, u_1. Thus we get $W_i(X, z_i) \equiv \deg_2(u_i) \equiv I_2(u_i, \{\overline{r}\}) \equiv |u_i^{-1}(\overline{r})|$ mod 2. Now note that $|u_0^{-1}(\overline{r})| = |r \cap X|$ and $|u_1^{-1}(\overline{r})| = |L_r(z_1) \cap X|$. Then since $r \cap X$ is the disjoint union of $L(z_0, z_1) \cap X$ and $L_r(z_1) \cap X$, it follows that $|r \cap X| = |L_r(z_1) \cap X| + \ell$, for some remainder ℓ. Thus we have

$$W_2(X, z_0) \equiv |u_0^{-1}(\overline{r})| \equiv |u_1^{-1}(\overline{r})| + \ell \equiv W_2(X, z_1) + \ell \mod 2.$$

Problem 6. Conclude that $\mathbb{R}^n - X$ has precisely two components,

$$D_0 = \{z \mid W_2(X, z) = 0\} \quad \text{and} \quad D_1 = \{z \mid W_2(X, z) = 1\}.$$

Since $\mathbb{R}^n - X$ has at most 2 connected components, and D_0 and D_1 are clearly disjoint, and in separate components by Problem 3, it suffices to show that D_0 and D_1 are nonempty. Suppose without loss of generality that D_0 is non-empty. Let's suppose D_i is nonempty for some $i \in \{0, 1\}$, so pick a point $z \in D_i$. By Problem 4, we can find some ray r which intersects X transversally. Then $|r \cap X| < \infty$, so let z' be a point on $X - r$ such that the segment of r between z and z' contains only one point of X. Then by Problem 5, $z_1 \in D_{i'}$ where i' is the other element of $\{0, 1\}$ so D_0, D_1 are both nonempty.

Problem 7. Show that if z is very large, then $W_2(X, z) = 0$.

Note that since X is compact, it must be bounded, so there is some $B > 0$ such that $|x| \leq B$ for all $x \in X$. Let $z \in \mathbb{R}^n$ be outside this bounding sphere. We claim that $z/|z|$ is not in the image of the direction map $u : X \to S^{n-1}$. Suppose conversely that there is some $x \in X$ with $z/|z| = (x - z)/|x - z|$. Manipulating this gives a contradiction with regards to the bound, so $z/|z|$ is not in the image. Then $z/|z|$ is a regular value of u so $W_2(X, z) \equiv |u^{-1}(z/|z|)| \equiv 0 \mod 2$.

Problem 8. Combine these observations to prove **The Jordan-Brouwer Separation Theorem**: The complement of the compact, connected hypersurface X in $\mathbb{R}^n$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, $\overline{D}_1$ is a compact manifold with boundary $\partial\overline{D}_1 = X$.

Problem 6 lets us split the complement of X into two connected open components D_0 and D_1, and D_1 is bounded and relatively compact. It thus makes sense to call D_1 the "inside" and D_0 the "outside" by Problem 7. Then $\overline{D_1}$ is a compact n-dimensional submanifold of $\mathbb{R}^n$. Furthermore, by the previous problems, it follows that $\partial(\overline{D_1}) = X$ since we can construct a small chart $\psi : \mathbb{H}^n \to \overline{D_1}$ at any $x \in X$.

Problem 9. Given $z \in \mathbb{R}^n - X$, let r be any ray emanating from z that is transversal to X. Show that z is inside X if and only if r intersects X in an odd number of points.

Suppose v is any vector in the direction of r, which by Problem 4 is a regular value of $u : X \to S^{n-1}$ with $u^{-1}(v) = r \cap X$. Then $W_2(X, z) \equiv |u^{-1}(v)| \equiv |r \cap X| \mod 2$. By the previous problem, z is inside of X if and only if $W_2(X, z) = 1$ which by te above argument is true if and only if $|r \cap X|$ is odd, which is what we want.