Math 231b Problem Set 10

Lev Kruglyak

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Problem 1. Using the splitting principle, prove the *Wu formula* for the action of the Steenrod squares on the mod 2 reduction of the Chern classes:

$$\mathrm{Sq}^{2i}(c_j) = \sum_k \binom{j+k-i-1}{k} c_{i-k}c_{j+k}.$$

The splitting principle tells us that we only need to check this formula for Chern classes of sums of line bundles. First, let's prove a helpful lemma, which further reduces the scope of proof.

Claim. Suppose the Wu formula holds for classes $c_k^{(1)}(\zeta_1)$ and $c_k^{(n)}(\zeta_2)$ over X, where ζ_1 is a line bundle and ζ_2 is a general bundle. Then it holds for $c_k^{(p+1)}(\zeta_1 \oplus \zeta_2)$.

Proof. We know by assumption that

$$\mathrm{Sq}^{2i}(c_j(\zeta_i)) = \sum_k \binom{j+k-i-1}{k} c_{i-k}(\zeta_i) \smile c_{j+k}(\zeta_i).$$

Using the Whitney sum formula for Chern classes, Cartan formula, and properties of Steenrod squares, we can simplify

$$\begin{aligned}\mathrm{Sq}^{2i}(c_j^{(p+q)}(\zeta_1 \oplus \zeta_2)) = \mathrm{Sq}^{2i}\left(\sum_{a+b=j} c_a(\zeta_1) \smile c_b(\zeta_2)\right) &= \sum_{a+b=j} \mathrm{Sq}^{2i}\left(c_a(\zeta_1) \smile c_b(\zeta_2)\right) \\ &= \sum_{a+b=j} \sum_{c+d=2i} \mathrm{Sq}^c(c_a(\zeta_1)) \smile \mathrm{Sq}^d(c_b(\zeta_2))\end{aligned}$$

At this point I became a bit stuck and was unable to get unstuck, I'm still fairly sure that the inductive step holds. □

So now we only need to prove the Wu formula for line bundles. However this follows from the base relation

$$\mathrm{Sq}^2(c_1) = c_1 \smile c_1$$

This forms a base case for the Wu relation.

Problem 2. Let $n = 2^m(2s+1)$ denote a positive integer which is divisible by 2 exactly m times. In this problem, you will use Steenrod operations to prove that S^{n-1} does not admit 2^m vector fields which are linearly independent at every point of S^{n-1}.

Let $V_k(\mathbb{R}^n)$ denote the space of k orthonormal vectors in $\mathbb{R}^n$. Then $V_1(\mathbb{R}^n)$ may be identified with S^{n-1}.

a. Consider the map $p_{k+1} : V_{k+1}(\mathbb{R}^n) \to V_1(\mathbb{R}^n) = S^{n-1}$ which sends $(v_1, \ldots, v_{k+1})$ to v_{k+1}. Prove that S^{n-1} admits k linearly independent vector fields if and only if p_{k+1} admits a section $S^{n-1} \to V_{k+1}(\mathbb{R}^n)$.

First suppose p_{k+1} admits a section $s : S^{n-1} \to V_{k+1}(\mathbb{R}^n)$. We have projection maps $\pi_i : V_{k+1}(\mathbb{R}^n) \to \mathbb{R}^n$ which send $(v_1, \ldots, v_{k+1})$ to v_i. Then $\pi_i \circ s$ for $i \leq k$ is a set of k linearly independent vector fields on S^{n-1}. In the opposite direction, given k linearly independent vector fields v_i on S^{n-1}, we can apply the Gram-Schmidt formula

$$u_k = v_k - \sum_{j=1}^{k-1} \mathbf{proj}_{u_j}(v_k)$$

to each tuple $(v_1(x), \ldots, v_k(x), x)$. These adjusted orthonormal vector fields are thus still continuous, and can be called $\widetilde{v}_i$. We then define a map $S^{n-1} \to V_{k+1}(\mathbb{R}^n)$ which sends x to $(\widetilde{v}_1(x), \ldots, \widetilde{v}_k(x), x)$.

b. There is a map $\mathbb{RP}^{n-1} \to O(n) \cong V_n(\mathbb{R}^n)$ which sends a line in $\mathbb{R}^n$ to the reflection across its normal hyperplane. Prove that the composition of this map with the map $V_n(\mathbb{R}^n) \to V_k(\mathbb{R}^n)$ taking $(v_1, \ldots, v_n)$ to $(v_{n-k+1}, \ldots, v_n)$ factors through a map $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \to V_k(\mathbb{R}^n)$. When $k = 1$, prove that $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \cong V_1(\mathbb{R}^n)$ is a homemorphism.

The map $\mathbb{RP}^{n-1} \to O(n)$ sends a line ℓ to the matrix reflecting across it's normal hyperplane. In a basis where $\ell/|\ell|$ is the first vector, this is a diagonal matrix with first coordinate -1 and the others set to 1. In $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1}$, this would be the identity matrix, so the map factors through.

c. There is a commutative diagram

$$\begin{tikzcd}
S^{n-k} \arrow[r] \arrow[d] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \arrow[r] \arrow[d] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-k} \arrow[d] \\
S^{n-k} \arrow[r] & V_k(\mathbb{R}^n) \arrow[r] & V_{k-1}(\mathbb{R}^n)
\end{tikzcd}$$

where the top tow is a cofiber sequence and the bottom row is a fiber sequence. Using the Serre long exact sequence, prove by induction on k that $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \to V_k(\mathbb{R}^n)$ is a $(2n-2k)$-equivalence.

Not sure.

d. When $2k \leq n$, prove that S^{n-1} admits $(k-1)$ everywhere linearly independent vector fields if and only if the map $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \to \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \simeq S^{n-1}$ admits a section up to homotopy.

Not sure.

e. Using the action of the Steenrod operations on $H^*(\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1}; \mathbb{F}_2)$, prove that S^{n-1} does not admit 2^m linearly independent sections, where $n = 2^m(2s+1)$.

Not sure.