Math 55a Problem Set 11

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- How long did this assignment take you? – 10 hours
- How hard was it? – Tough
- What resources did you use and how much help did you need? – Collaborated with AJ LaMotta
- Did you have any prior experience with this material? – No

Problem 1. For any finite abelian group G, we define the dual of G to be $\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^*)$. Show that there is a natural map from G to its double dual $\widehat{\widehat{G}}$, and that this map is an isomorphism.

Let G be some finite abelian group, and let $G' = \widehat{\widehat{G}}$ be its double dual. Consider the map $\phi : G \to G'$ sending g to the evaluation map ev_g which sends $f \in \operatorname{Hom}(G, \mathbb{C}^*)$ to the evaluation $f(g)$. This is a homomorphism because for any $g, h \in G$ we have $\mathrm{ev}_{gh} = \mathrm{ev}_g \cdot \mathrm{ev}_h$. To prove injectivity, let g be some element for which $\mathrm{ev}_g = 1$. This clearly implies that g must be the identity, since for every other element there must be some nontrivial element in the dual. To construct such a nontrivial element explicitly, decompose G as $G = \mathbb{Z}/n_1 \times \cdots \mathbb{Z}/n_k$ for some $n_1, \dots, n_k$. Then there is an element sending each generator of order n_k to a primitive root of unity of order n_k. So this shows that the kernel is trivial and that the map is injective. Surjectivity follows because $G \cong \widehat{G}$ hence they are isomorphic.

Problem 2. Let G be a group and V and W representations of G, and let $\mathrm{Hom}(V, W) = V^* \otimes W$ be given the structure of a representation as seen in class: g maps $\varphi \in \mathrm{Hom}(V, W)$ to $g \circ \varphi \circ g^{-1}$ (where g is acting on W and g^{-1} on V).

(a) Show that the invariant subspace $\mathrm{Hom}(V, W)^G = \{\varphi \in \mathrm{Hom}(V, W) \mid g(\varphi) = \varphi \ \forall g \in G\}$ is the vector space of homomorphisms $\varphi : V \to W$ of representations, that is, G-equivariant linear maps. (This is often denoted $\mathrm{Hom}_G(V, W)$).

(b) Suppose that the irreducible representations of G are $U_1, U_2, \ldots, U_c$, with $\dim(U_i) = d_i$, and suppose that $V = U_1^{\oplus m_1} \oplus \cdots \oplus U_c^{\oplus m_c}$ and $W = U_1^{\oplus n_1} \oplus \cdots \oplus U_c^{\oplus n_c}$. What is the dimension of $\mathrm{Hom}_G(V, W)$?

(a) First let $\varphi \in \mathrm{Hom}(V, W)^G$ be a G-invariant linear map, i.e. $g(\varphi) = g \circ \varphi \circ g^{-1} = \varphi$ so $g \circ \varphi = \varphi \circ g$. Then $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all $x \in V$. Conversely suppose $\varphi(g \cdot x) = g \cdot \varphi(x)$ so that $g \circ \varphi = \varphi \circ g$. Then $\varphi = g \circ \varphi \circ g^{-1} = g(\varphi)$. Thus

$$\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W).$$

(b) By a basic property of direct sums, we have the decomposition

$$\mathrm{Hom}_G(V, W) = \bigoplus_{1 \le i,j \le c} \mathrm{Hom}_G(U_i^{\oplus m_i}, U_j^{\oplus n_j}) = \bigoplus_{1 \le i,j \le c} \bigoplus_{k=1}^{m_i n_j} \mathrm{Hom}_G(U_i, U_j).$$

By Schur's lemma, and the fact that our representations are assumed to be complex, we have

$$\dim \mathrm{Hom}_G(U_i, U_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

So combining the two formulas, we get

$$\dim \mathrm{Hom}_G(V, W) = \sum_{k=1}^{c} m_k n_k.$$

Problem 3. Let U, V and W be vector spaces (not necessarily finite-dimensional).

(a) Construct a canonical isomorphism $\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$.

(b) Now suppose that U, V and W are representations of a group G. Show that the isomorphism of part (a) is in fact an isomorphism of representations.

(a) We'll construct a canonical isomorphism

$$\Psi : \mathrm{Hom}(U \otimes V, W) \to \mathrm{Hom}(U, \mathrm{Hom}(V, W)).$$

Let $f \in \mathrm{Hom}(U \otimes V, W)$ be arbitrary. Define $\Psi_f \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$ as

$$\Psi_f(u)(v) = f(u \otimes v).$$

This is clearly a homomorphism by basic properties of Hom and $\otimes$. Now we'll show it's injective by computing the kernel of Ψ. Suppose $\Psi_f = 1$, i.e. $\Psi_f(u)(v) = 0$ for all $u \in U, v \in V$. Then $f(u \otimes v) = 0$ so f is the zero element in $\mathrm{Hom}(U \otimes V, W)$. To prove surjectivity, let $g \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$. Then let $f \in \mathrm{Hom}(U \otimes V, W)$ be the map defined by $f(u \otimes v) = g(u)(v)$. Clearly $\Psi_f = g$ so the map is surjective. Nowhere do we use a basis so this isomorphism is canonical.

(b) Let $g \in G$ be some element, and let $f \in \mathrm{Hom}(U \otimes V, W)$ be some map. Then $\Psi_{g \cdot f}(u)(v) = (g \cdot f)(u \otimes v) = g \circ f \circ g^{-1}(u \otimes v) = g \circ f(g^{-1}u \otimes g^{-1}v)$ whereas $g \cdot \Psi_f(u)(v) = (g \circ \Psi_f \circ g^{-1})(u)(v) = g \circ f(g^{-1}u \otimes g^{-1})$. So $\Psi_{g \cdot f} = g \cdot \Psi_f$ and so Ψ is a homomorphism of representations.

Problem 4. Let V be any (finite-dimensional) representation of a group G.

(a) Show that V is irreducible if and only if V^* is.

(b) If W is any 1-dimensional representation, show that V is irreducible if and only if $V \otimes W$ is.

(a) We'll prove the contrapositive, that V is reducible if and only if V^* is reducible. Suppose V is reducible with subrepresentation $W \subset V$. Consider $\text{Ann}(W) \subset V^*$. We claim that this is a G-invariant subspace of V^*. Let $g \in G$ and $\ell \in \text{Ann}(W)$. Then $g \cdot \ell = \ell \circ g^{-1}$. Since for every $w \in W$, $(\ell \circ g^{-1})(w) = \ell(g^{-1}(w))$. Since $g^{-1}(w) \in W$, $\ell(g^{-1}(w)) = 0$ so $g \cdot \ell \in \text{Ann}(W)$. The converse direction works the same way, using the natural isomorphism of representations $V \cong V^{**}$.

(b) Again we'll prove the contrapositive, so V is reducible if and only if $V \otimes W$ is. Suppose V is reducible with subrepresentation $U \subset V$. Consider $U \otimes W \subset V \otimes W$ be the span of elements of the form $u \otimes w$ for $u \in U$ and $w \in W$. Then for any $g \in G$, $g \cdot (u \otimes w) = (g \cdot u) \otimes (g \cdot w) \in U \otimes W$. So $U \otimes W$ is a subrepresentation of $V \otimes W$. Suppose instead that $U \subset V \otimes W$ is a subrepresentation. Since W is one-dimensional, every subspace of $V \otimes W$ is of the form $U \otimes W'$ where $U \subset V$ and $W' = \{1\}$ or W. Then U is a subrepresentation of V so V is reducible.

Problem 5. Let V be the standard (2-dimensional) representation of S_3.

(a) Identify $\mathrm{Sym}^4 V$ as a direct sum of irreducible representations of S_3.

(b) Identify $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ as a direct sum of irreducible representations of S_3.

(c) On a previous homework (HW7), you constructed for any 2-dimensional vector space V a natural map $\phi : \mathrm{Sym}^2(\mathrm{Sym}^2 V) \to \mathrm{Sym}^4 V$. Show that this is a homomorphism of representations and identify the kernel of ϕ as a representation of S_3.

Let $\mathbb{C}_2$ be the standard 2-dimensional representation of S_3, $\mathbb{C}_+$ be the trivial representation, and $\mathbb{C}_-$ be the sign representation.

(a) Let e_1 and e_2 be basis eigenvectors of $\mathbb{C}_2$ with respect to τ as in the lecture notes, so $\tau e_1 = \lambda e_1$, $\sigma e_1 = e_2, \tau e_2 = \lambda^2 e_2, \sigma e_2 = e_1$ for $\lambda = e^{2\pi i/3}$. The basis for $\mathrm{Sym}^4 V$ is then

Name	Basis Element	Eigenvalues of τ
s_1	$e_1 \otimes e_1 \otimes e_1 \otimes e_1$	λ
s_2	$e_1 \otimes e_1 \otimes e_1 \otimes e_2$	λ^2
s_3	$e_1 \otimes e_1 \otimes e_2 \otimes e_2$	1
s_4	$e_1 \otimes e_2 \otimes e_2 \otimes e_2$	λ
s_5	$e_2 \otimes e_2 \otimes e_2 \otimes e_2$	λ^2

Suppose we decomposed $\mathrm{Sym}^4\mathbb{C}_2 = \mathbb{C}_+^{\oplus a} \oplus \mathbb{C}_-^{\oplus b} \oplus \mathbb{C}_2^{\oplus c}$. Then $\dim\ker(\tau - 1) = 1$ so $a + b = 1$. It is clear that $b = 0$ since $\mathrm{Sym}^4(V)$ has no signed component. So $a = 1$ and $c = 2$. So

$$\mathrm{Sym}^4\mathbb{C}_2 \cong \mathbb{C}_+ \oplus \mathbb{C}_2^{\oplus 2}.$$

(b) Let e_1, e_2 be the eigenbasis from (a).

Name	Basis Element	Eigenvalues of τ
s_1	$(e_1 \otimes e_1) \otimes (e_1 \otimes e_1)$	λ
s_2	$(e_1 \otimes e_1) \otimes (e_1 \otimes e_2)$	λ^2
s_3	$(e_1 \otimes e_1) \otimes (e_2 \otimes e_2)$	1
s_4	$(e_1 \otimes e_2) \otimes (e_1 \otimes e_2)$	1
s_5	$(e_1 \otimes e_2) \otimes (e_2 \otimes e_2)$	λ
s_6	$(e_2 \otimes e_2) \otimes (e_2 \otimes e_2)$	λ^2

As before, we have the rule $a + b = 2$, so since $b = 0$, it follows that $a = 2$ and so $c = 2$. Thus

$$\mathrm{Sym}^2(\mathrm{Sym}^2\mathbb{C}_2) \cong \mathbb{C}_+^{\oplus 2} \oplus \mathbb{C}_2^{\oplus 2}.$$

(c) We can check that the map $\Psi : \mathrm{Sym}^2(\mathrm{Sym}^2\mathbb{C}_2) \to \mathrm{Sym}^4\mathbb{C}_2$ is indeed a homomorphism of representations simply by checking that the action on each basis element is invariant under the action of S_3. This is straightfoward using the above computed tables. The kernel of Ψ is $(e_1 \otimes e_1) \otimes (e_2 \otimes e_2) - (e_1 \otimes e_2) \otimes (e_1 \otimes e_2)$. This is a one dimensional space on which τ acts trivially and σ acts trivially. So the kernel is $\mathbb{C}_+$.

Problem 6. Again, let V be the standard representation of S_3. Show that $\mathrm{Sym}^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^3(\mathrm{Sym}^2 V)$ as representations of S_3.

Let's build the same eigenvalue tables we computed for the previous problem. For $\mathrm{Sym}^2(\mathrm{Sym}^3\mathbb{C}_2)$ we have

Name	Basis Element	Eigenvalues of τ
s_1	$(e_1 \otimes e_1 \otimes e_1) \otimes (e_1 \otimes e_1 \otimes e_1)$	1
s_2	$(e_1 \otimes e_1 \otimes e_1) \otimes (e_1 \otimes e_1 \otimes e_2)$	λ
s_3	$(e_1 \otimes e_1 \otimes e_1) \otimes (e_1 \otimes e_2 \otimes e_2)$	λ^2
s_4	$(e_1 \otimes e_1 \otimes e_1) \otimes (e_2 \otimes e_2 \otimes e_2)$	1
s_5	$(e_1 \otimes e_1 \otimes e_2) \otimes (e_1 \otimes e_1 \otimes e_2)$	λ^2
s_6	$(e_1 \otimes e_1 \otimes e_2) \otimes (e_1 \otimes e_2 \otimes e_2)$	1
s_7	$(e_1 \otimes e_1 \otimes e_2) \otimes (e_2 \otimes e_2 \otimes e_2)$	λ
s_8	$(e_1 \otimes e_2 \otimes e_2) \otimes (e_1 \otimes e_2 \otimes e_2)$	λ
s_9	$(e_1 \otimes e_2 \otimes e_2) \otimes (e_2 \otimes e_2 \otimes e_2)$	λ^2
s_{10}	$(e_2 \otimes e_2 \otimes e_2) \otimes (e_2 \otimes e_2 \otimes e_2)$	1

So by the same argument as in the previous problem, we obtain the decomposition

$$\mathrm{Sym}^2(\mathrm{Sym}^3\mathbb{C}_2) \cong \mathbb{C}_- \oplus \mathbb{C}_+^{\oplus 3} \oplus \mathbb{C}_2^{\oplus 3}.$$

Next, we compute the same table for $\mathrm{Sym}^3(\mathrm{Sym}^2\mathbb{C}_2)$.

Name	Basis Element	Eigenvalues of τ
s_1	$(e_1 \otimes e_1) \otimes (e_1 \otimes e_1) \otimes (e_1 \otimes e_1)$	1
s_2	$(e_1 \otimes e_1) \otimes (e_1 \otimes e_1) \otimes (e_1 \otimes e_2)$	λ
s_3	$(e_1 \otimes e_1) \otimes (e_1 \otimes e_1) \otimes (e_2 \otimes e_2)$	λ^2
s_4	$(e_1 \otimes e_1) \otimes (e_1 \otimes e_2) \otimes (e_1 \otimes e_2)$	λ^2
s_5	$(e_1 \otimes e_1) \otimes (e_1 \otimes e_2) \otimes (e_2 \otimes e_2)$	1
s_6	$(e_1 \otimes e_1) \otimes (e_2 \otimes e_2) \otimes (e_2 \otimes e_2)$	λ
s_7	$(e_1 \otimes e_2) \otimes (e_1 \otimes e_2) \otimes (e_1 \otimes e_2)$	1
s_8	$(e_1 \otimes e_2) \otimes (e_1 \otimes e_2) \otimes (e_2 \otimes e_2)$	λ
s_9	$(e_1 \otimes e_2) \otimes (e_2 \otimes e_2) \otimes (e_2 \otimes e_2)$	λ^2
s_{10}	$(e_2 \otimes e_2) \otimes (e_2 \otimes e_2) \otimes (e_2 \otimes e_2)$	1

So $\mathrm{Sym}^3(\mathrm{Sym}^2\mathbb{C})$ has the same decomposition,

$$\mathrm{Sym}^3(\mathrm{Sym}^2\mathbb{C}) \cong \mathbb{C}_- \oplus \mathbb{C}_+^{\oplus 3} \oplus \mathbb{C}_2^{\oplus 3}.$$

Hence the two representations are isomorphic.

Problem 7. Show by example that, over fields of characteristic $p > 0$, complete reducibility may fail. In other words, find an example of a finite group G, a finite-dimensional vector space over $\mathbb{F}_p$, an action of G on V (that is, a homomorphism $\rho : G \to \mathrm{GL}(V)$) and a subspace $W \subset V$ invariant under G such that no complementary invariant subspace of V exists.

Consider the 2 dimensional $\mathbb{F}_2$-representation of $\mathbb{Z}/2$ defined where $k \in \mathbb{Z}/2$ acts by the matrix

$$\rho(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Let's look at the subrepresentation $V \subset \mathbb{F}_2^2$ given by

$$V = \left\{ \begin{bmatrix} k \\ 0 \end{bmatrix} \mid k \in \mathbb{Z}/2 \right\}.$$

If V had some complement V', this would imply that $\rho(k)$ is a diagonalizable matrix, which it is not over $\mathbb{F}_2$.