Lecture 4: More on Lie groups

Contents

We continue our introduction to Lie groups, and we also develop a few general aspects of the calculus of differential forms.

We begin with a discussion of Lie subgroups of a Lie group. A Lie subgroup need not be closed in the topology of the Lie group. A powerful general theorem asserts that a closed subgroup of a Lie group is itself a Lie group. We state, but do not prove that theorem.

The next topic is the adjoint action of a Lie group on itself and on its Lie algebra. This is essentially conjugation in various forms; it measures the deviation of a Lie group from abelianness. For matrix groups the action is precisely conjugation.

We then make a digression to conclude a topic left over from Lecture 2, namely Lie derivatives of differential forms. In this section and in the remainder of the lecture, we introduce a few formulas (4.39), (4.48), (4.49), (4.69), (4.83) that are so fundamental in Differential Geometry that I recommend you memorize them. The section on Lie derivatives has six commutator formulas (4.30), (4.35), (4.39), (4.42), (4.43), (4.44) that are basic to the calculus of differential forms.

With this in hand, we introduce a tautological vector-valued 1-form on a Lie group: the *Maurer-Cartan form*. It satisfies the *Maurer-Cartan equation* (4.69), the basic structure equation on a Lie group. Variations of this equation will be central in our geometric discussions, for example of curvature. The form (4.81), (4.83) this equation takes on a matrix group will also recur later.

Lie subgroups

(4.1) *One-parameter subgroups on a torus.* A one-parameter subgroup of the vector group $\mathbb{R}^2$ is a homomorphism $\tilde{\gamma}\colon \mathbb{R}\to\mathbb{R}^2$, i.e., a linear map $\tilde{\gamma}(t)=(at,bt)$ for some $a,b\in\mathbb{R}$. Let $G=\mathbb{R}^2/\mathbb{Z}^2$ be the quotient by the subgroup of vectors with integer coefficients. Then G is an abelian Lie group whose underlying manifold is diffeomorphic to a 2-torus. Let $\pi\colon\mathbb{R}^2\to G$ be the quotient map. Then $\gamma=\pi\circ\tilde{\gamma}$ is a one-parameter subgroup of G. If b/a is rational, then γ is not injective, but factors through an injective Lie group homomorphism $\mathbb{R}/L\mathbb{Z}\to G$ for some $L\in\mathbb{Z}$. The image in G is a closed submanifold diffeomorphic to S^1, and it is a subgroup as well; we summarize this combination as a *closed subgroup*. If b/a is irrational, then γ is an injective immersion with dense image in G; it is not an embedding. See Figure 10 for a crude picture. The image of γ is a subgroup of G, but it is not a submanifold.

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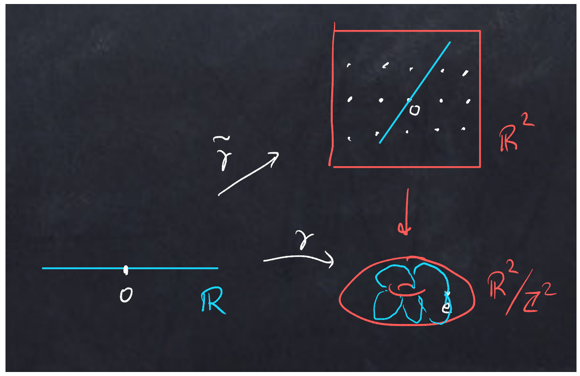

FIGURE 10. The skew line on the torus

(4.2) *Definitions and basic theorems.* The examples in (4.1) motivate the following.

Definition 4.3. Let G be a Lie group. A *Lie subgroup* $i\colon G' \to G$ is an injective homomorphism of Lie groups. It is a *closed subgroup* if $i(G') \subset G$ is closed.

The differential of an injective Lie group homomorphism is an injective homomorphism of Lie algebras, so in particular an injective Lie group homomorphism is an injective immersion. We state the following without proof.

Theorem 4.4.

(1) *Let $i\colon G' \to G$ be a Lie subgroup. Then $i(G') \subset G$ is closed iff i is an embedding.*
(2) *A closed subgroup of a Lie group is a Lie subgroup.*

The second assertion states that if a subset $G' \subset G$ is a subgroup in the group structure and is closed in the topology, then G' is a submanifold and the group operations on G' are smooth. The proofs of these assertions are not immediate; see Warner.

Remark 4.5. Theorem 4.4(2) is powerful, as pointed out in Example 3.27 for instance.

The adjoint action

(4.6) *The nonlinear action of G by conjugation.* Define the left action (Definition 3.8) of G on G by conjugation:

$$\mathrm{AD}_g(x) = gxg^{-1}, \qquad g, x \in G. \tag{4.7}$$

The *orbit* of $x \in G$ under conjugation is the *conjugacy class* of x, which is a closed submanifold of G. The *stabilizer* of $x \in G$ is the *centralizer* $Z(x) \subset G$, which is a closed subgroup and hence, by Theorem 4.4(2), is a Lie subgroup.

(4.8) *The linear action of G on its Lie algebra.* Let $\xi \in \mathfrak{g}$ be a parallel vector field on G. Then we check that the differential of AD_g maps ξ to a parallel vector field using the canonical motion (germ) with velocity ξ:

$$\mathrm{AD}_g(xe^{t\xi}) = gxe^{t\xi}g^{-1} = (gxg^{-1})(ge^{t\xi}g^{-1}). \tag{4.9}$$

This linear action of G on its Lie algebra of parallel vector fields is denoted

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{Aut}(\mathfrak{g}). \tag{4.10}$$

It is a canonical *linear representation* of G, the *adjoint representation.* Note that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group (Example 3.24), and (4.10) is a Lie group homomorphism. Under the identification (3.18) of $\mathfrak{g}$ with T_eG we have $\mathrm{Ad}_g = d(\mathrm{AD}_g)_e$. Furthermore, linear transformations in the image of the adjoint representation preserve the Lie bracket:

$$\mathrm{Ad}_g[\xi,\eta] = [\mathrm{Ad}_g\,\xi, \mathrm{Ad}_g\,\eta], \qquad g \in G, \quad \xi,\eta \in \mathfrak{g}. \tag{4.11}$$

In other words, we can read '$\mathrm{Aut}(\mathfrak{g})$' not just as linear automorphisms, but as automorphisms which preserve the Lie algebra structure. This follows from a computation with the flows generated by ξ,η (see Proposition 2.58 and Remark 3.55), which we evaluate at $e \in G$:

$$\left.\frac{\partial^2}{\partial t \partial s}\right|_{\substack{t=0\\s=0}} \mathrm{AD}_g(e^{t\xi}e^{s\eta}e^{-t\xi}e^{-s\eta}) = \left.\frac{\partial^2}{\partial t \partial s}\right|_{\substack{t=0\\s=0}} (ge^{t\xi}g^{-1})(ge^{s\eta}g^{-1})(ge^{-t\xi}g^{-1})(ge^{-s\eta}g^{-1}). \tag{4.12}$$

(4.13) *The infinitesimal linear action of $\mathfrak{g}$ on $\mathfrak{g}$.* By Lemma 3.40 the Lie group homomorphism (4.10) induces a Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}). \tag{4.14}$$

This is the *infinitesimal adjoint representation* of the Lie algebra $\mathfrak{g}$. Furthermore, ad_ξ, $\xi \in \mathfrak{g}$, acts as a derivation on $\mathfrak{g}$, which follows by differentiating (4.11) in g:

$$\mathrm{ad}_\xi[\eta,\zeta] = [\mathrm{ad}_\xi(\eta),\zeta] + [\eta,\mathrm{ad}_\xi(\zeta)]. \tag{4.15}$$

In fact, we can compute the adjoint action explicitly.

Proposition 4.16. *For $\xi,\eta \in \mathfrak{g}$ we have*

$$\mathrm{ad}_\xi(\eta) = [\xi,\eta]. \tag{4.17}$$

Proof. As in (4.12) we compute at $e \in G$ and for $g \in G$ write

$$\mathrm{Ad}_g(\eta)(e) = \frac{d}{ds}\Big|_{s=0} g e^{s\eta} g^{-1}. \tag{4.18}$$

Hence

$$\mathrm{ad}_\xi(\eta)(e) = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} e^{t\xi} e^{s\eta} e^{-t\xi}. \tag{4.19}$$

Since the flow φ_t generated by ξ is right multiplication by $e^{t\xi}$, the expression on the right hand side of (4.19) is[1] $\frac{d}{dt}\big|_{t=0} \varphi_{-t}(\eta)$, which by (2.19) is $(\mathcal{L}_\xi \eta)(e)$. Conclude with (2.54). $\square$

Remark 4.20. Under the identification (4.17), equation (4.15) is a form of the Jacobi identity (2.36) for the Lie bracket.

(4.21) *Adjoint action for matrix groups.* Suppose G is a matrix group, which means that $G \subset \mathrm{Aut}(V)$ for some finite dimensional real vector space V. In that case the Lie algebra $\mathfrak{g} \subset \mathrm{End}(V)$ is a subset of endomorphisms, after identifying it with the tangent space to G at the identity $e = \mathrm{id}_V$. Then from (4.9) with $x = e$ we deduce that

$$\mathrm{Ad}_P(A) = PAP^{-1}, \qquad P \in G, \quad A \in \mathfrak{g}. \tag{4.22}$$

In other words, the adjoint action is precisely conjugation for matrix groups.

The Lie derivative on differential forms

(4.23) *Derivations on graded algebras.* We begin with the $\mathbb{Z}$-graded variation of **(2.29)**.

Definition 4.24. Let $A = \bigoplus_{q\in\mathbb{Z}} A^q$ be a $\mathbb{Z}$-graded algebra. A homogeneous map $D\colon A \to A$ of degree k is s *derivation* if

$$D(\alpha\beta) = (D\alpha)\beta + (-1)^{kq}\alpha(D\beta), \qquad \alpha \in A^q, \quad \beta \in A. \tag{4.25}$$

If D_1, D_2 are derivations of degrees k_1, k_2, respectively, then

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1k_2} D_2 \circ D_1 \tag{4.26}$$

is a derivation of degree $k_1 + k_2$.

Remark 4.27.

[1] The value of the parallel vector field η at $\varphi_t(e) = e^{t\xi}$ is $\frac{d}{ds}\big|_{s=0} e^{t\xi}e^{s\eta}$.

(1) In (4.25) and (4.26) we use the *Koszul sign rule*, which loosely imposes a sign $(-1)^{pq}$ which commuting objects of degrees p and q in a $\mathbb{Z}$-graded algebra.

(2) The commutator (4.26) (with sign) endows the vector space $\mathrm{Der}(A)$ of derivations of A with the structure of a *$\mathbb{Z}$-graded Lie algebra*. As an exercise, write this $\mathbb{Z}$-graded variation of Definition 2.33. You'll need to apply the Koszul sign rule carefully and consistently.

Example 4.28 (Derivations on differential forms)**.** Let X be a smooth manifold and let $A = \Omega^\bullet_X$ be the $\mathbb{Z}$-graded algebra of differential forms. The Cartan differential $d\colon \Omega^\bullet_X \to \Omega^{\bullet+1}_X$ is a derivation of degree $+1$. For any vector field $\xi \in \mathfrak{X}(X)$, the Lie derivative $\mathcal{L}_\xi$ is a derivation of degree 0. To verify this, let φ_t be the flow generated by ξ, differentiate

$$\varphi_t^*(\alpha \wedge \beta) = \varphi_t^*\alpha \wedge \varphi_t^*\beta, \qquad \alpha, \beta \in \Omega^\bullet_X \tag{4.29}$$

in t, and apply (2.19). Furthermore,

$$[d, \mathcal{L}_\xi] = 0 \tag{4.30}$$

follows by differentiating

$$\varphi_t^* d\alpha = d\varphi_t^*\alpha, \qquad \alpha \in \Omega^\bullet_X, \tag{4.31}$$

in t.

(4.32) *Interior product (contraction).* Interior product is an algebraic operation on differential forms: it acts pointwise. There is no differentiation; by contrast, the derivations d and $\mathcal{L}_\xi$ act as first order differential operators. Hence we treat contraction algebraically.

Let V be a vector space.[2] For $\xi \in V$ we give three equivalent characterizations of the derivation

$$\iota_\xi\colon {\textstyle\bigwedge}^\bullet V^* \longrightarrow {\textstyle\bigwedge}^{\bullet-1} V^* \tag{4.33}$$

as follows:

(1) ι_ξ is dual to left exterior multiplication $\epsilon_\xi\colon \bigwedge^\bullet V \to \bigwedge^{\bullet+1} V$.

(2) ι_ξ is a derivation of degree -1 and $\iota_\xi\colon V^* \to \mathbb{R}$ is evaluation on ξ.

(3) If we identify $\alpha \in \bigwedge^q V^*$ with alternating q-linear functionals on V, then

$$(\iota_\xi\alpha)(\xi_1, \dots, \xi_{q-1}) = \alpha(\xi, \xi_1, \dots, \xi_{q-1}), \qquad \xi_1, \dots, \xi_{q-1} \in V. \tag{4.34}$$

This last equation tells that ι_ξ is the operation "plug ξ into the first slot". All three characterizations are useful in practice. It is easy to verify

$$[\iota_\xi, \iota_\eta] = 0, \qquad \xi, \eta \in V. \tag{4.35}$$

[2] We needn't assume the ground field is $\mathbb{R}$, nor that V is finite dimensional.

For example, this is dual to the (graded) commutativity of exterior multiplication using characterization (1). Take $\xi = \eta$ in (4.35) to deduce

$$(\iota_\xi)^2 = \frac{1}{2}[\iota_\xi, \iota_\xi] = 0. \tag{4.36}$$

Let X be a smooth manifold. Define the interior product $\iota_\xi \colon \Omega^\bullet_X \to \Omega^{\bullet-1}_X$ on differential forms pointwise as interior product on the exterior algebra of the cotangent space.

(4.37) *The Cartan*[3] *formula.* Now we can compute the Lie derivative on differential forms, a formula you should know so well that you could recite it in your sleep.

Theorem 4.38. *Let $\xi \in \mathfrak{X}(X)$ be a vector field on X. Then*

$$\mathcal{L}_\xi = [d, \iota_\xi] = d\iota_\xi + \iota_\xi d. \tag{4.39}$$

Proof. Both $\mathcal{L}_\xi$ and $[d, \iota_\xi]$ are degree 0 derivations of $\Omega^\bullet_X$, both commute with d, and they agree on functions Ω^0_X; see (4.26), (4.29), and (2.21). Any differential form is locally a sum of terms of the form $f_0\, df_1 \wedge \cdots \wedge df_q$, where $f_0, f_1, \dots, f_q \in \Omega^0_X$. The result follows. $\square$

Remark 4.40. The derivations of interior product, Lie derivative, and Cartan d endow

$$\mathfrak{X}(X) \oplus \mathfrak{X}(X) \oplus \mathbb{R} \tag{4.41}$$

with the structure of a $\mathbb{Z}$-graded Lie algebra, where the degrees of the summands are $-1, 0, +1$. An element $\eta \in \mathfrak{X}(X)$ in the first summand acts on $\Omega^\bullet_X$ as ι_η. An element $\xi \in \mathfrak{X}(X)$ in the second summand acts on $\Omega^\bullet_X$ as $\mathcal{L}_\xi$. A real number $a \in \mathbb{R}$ acts on $\Omega^\bullet_X$ as a times Cartan d. The bracketing relations are (4.30), (4.35), (4.39), and

$$[d, d] = 0 \tag{4.42}$$

$$[\mathcal{L}_\xi, \iota_\eta] = \iota_{[\xi,\eta]} \tag{4.43}$$

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]} \tag{4.44}$$

Remark 4.45. This $\mathbb{Z}$-graded Lie algebra encodes what is needed to compute in first order differential calculus. If one supplements with integration over the fibers of a fiber bundle, and the bracket of that operation with the operations in (4.41), then one has almost all one needs to compute many formulas in calculus. (One needs integration over fiber bundles whose fibers are manifolds with boundary... and then eventually manifolds with corners...).

[3] This is Henri Cartan, son of Élie.

(4.46) *Two basic formulas.* The following formulas should also inhabit your dreams.

Proposition 4.47. *Let X be a smooth manifold, let $\alpha, \beta \in \Omega^1_X$ be 1-forms, and $\xi, \eta \in \mathfrak{X}(X)$ be vector fields. Then*

$$d\alpha(\xi,\eta) = \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi,\eta]) \tag{4.48}$$

$$(\alpha\wedge\beta)(\xi,\eta) = \alpha(\xi)\beta(\eta) - \alpha(\eta)\beta(\xi) \tag{4.49}$$

Proof. For (4.48), compute using the commutation relations of (4.41):

$$\begin{aligned} d\alpha(\xi,\eta) &= \iota_\eta\iota_\xi d\alpha \\ &= \iota_\eta(\mathcal{L}_\xi\alpha - d\iota_\xi\alpha) \\ &= \mathcal{L}_\xi\iota_\eta\alpha - \iota_{[\xi,\eta]}\alpha - \iota_\eta d\iota_\xi\alpha \\ &= \xi\alpha(\eta) - \alpha([\xi,\eta]) - \eta\alpha(\xi). \end{aligned} \tag{4.50}$$

The proof of (4.49) is similar:

$$\begin{aligned} (\alpha\wedge\beta)(\xi,\eta) &= \iota_\eta\iota_\xi(\alpha\wedge\beta) \\ &= \iota_\eta(\iota_\xi\alpha\wedge\beta - \alpha\wedge\iota_\xi\beta) \\ &= (\iota_\xi\alpha)(\iota_\eta\beta) - (\iota_\eta\alpha)(\iota_\xi\beta) \\ &= \alpha(\xi)\beta(\eta) - \alpha(\eta)\beta(\xi). \end{aligned} \tag{4.51}$$

$\square$

Differential forms on Lie groups

(4.52) *Parallel forms and Lie algebra cohomology.* Let G be a Lie group. The global parallelism by left translation determines a subalgebra $\Omega^\bullet_{\parallel}(G) \subset \Omega^\bullet(G)$ of parallel differential forms.

Lemma 4.53. *$\Omega^\bullet_{\parallel}(G)$ is closed under d.*

Recall that this is not true for a general global parallelism on a smooth manifold.

Proof. A differential form $\alpha \in \Omega^\bullet_\bullet G$ lies in $\Omega^\bullet_{\parallel}(G)$ iff $L_g^*\alpha = \alpha$ for all $g \in G$. The lemma follows since $L_g^* \circ d = d \circ L_g^*$. $\square$

Evaluation at the identity gives an isomorphism of $\mathbb{Z}$-graded algebras

$$\Omega^\bullet_{\parallel}(G) \xrightarrow{\mathrm{ev}_e} {\textstyle\bigwedge}^\bullet\mathfrak{g}^*. \tag{4.54}$$

Use it to transfer d to a differential on $\bigwedge^\bullet\mathfrak{g}^*$. The resulting differential graded algebra, the *Chevalley-Eilenberg complex*, defines the Lie algebra cohomology of $\mathfrak{g}$ and can be generalized to Lie algebras which do not arise as Lie algebras of a Lie group, as well as to Lie algebras over more general fields.

(4.55) *Vector-valued differential forms.* Let X be a smooth manifold and W a real vector space.

Definition 4.56. The $\mathbb{Z}$-graded vector space $\Omega^\bullet_X(W) = \bigwedge^\bullet(X;W)$ of *W-valued differential forms* is the space of smooth sections of the vector bundle over X whose fiber at $x \in X$ is $\bigwedge^\bullet T^*_xX \otimes W$.

For $W = \mathbb{R}^m$ an element of $\Omega^\bullet_X(W)$ is a column vector of differential forms on X. Pointwise exterior multiplication gives $\Omega^\bullet_X(W)$ the structure of a $\mathbb{Z}$-graded module over $\Omega^\bullet_X$. Furthermore, the differential d acts on $\Omega^\bullet_X(W)$ compatibly with the module structure:

$$d(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^q \alpha \wedge d\sigma, \qquad \alpha \in \Omega^q_X, \quad \sigma \in \Omega^\bullet_X(W). \tag{4.57}$$

In fact, (4.57) can be used to define d on $\Omega^\bullet_X(W)$: first let $\sigma = w$ be a constant function in $\Omega^0_X(W)$, in which case define $d(\alpha w) = (d\alpha)w$, and then extend to $\Omega^\bullet_X(W)$ by the Leibniz rule (4.57). We say that $\Omega^\bullet_X(W)$ is a *differential graded module* over the differential graded algebra $\Omega^\bullet_X$.

Remark 4.58. We will generalize to differential forms valued in a vector bundle $W \to X$. Then we need an additional structure—a *covariant derivative* on $W \to X$—to define a first-order differential operator of degree $+1$ on $\Omega^\bullet_X(W)$. In that case the square is not necessarily zero; it is the *curvature*.

(4.59) *The Maurer-Cartan form.* There is a tautological vector-valued 1-form on a Lie group G.

Definition 4.60. Let G be a Lie group. The *Maurer-Cartan form* $\theta \in \Omega^1_G(\mathfrak{g})$ assigns to $\xi_x \in T_xG$, $x \in G$, the parallel vector field $\xi \in \mathfrak{g}$ whose value at x is ξ_x.

It follows immediately from the definition that θ is parallel:

$$(L^*_g\theta)_x(\xi_x) = \theta_{gx}\big((L_g)_*\xi_x\big) = \xi = \theta_x(\xi_x), \qquad g, x \in G, \quad \xi_x \in T_xX, \tag{4.61}$$

since the parallel vector field ξ which extends ξ_x has value $(L_g)_*\xi_x$ at $gx \in G$. The tautological nature of θ is brought out by applying the isomorphism (4.54): the image of θ under

$$\Omega^1_{\|}(G;\mathfrak{g}) \xrightarrow{\ \mathrm{ev}_e\ } \mathfrak{g}^* \otimes \mathfrak{g} \cong \mathrm{End}(\mathfrak{g}) \tag{4.62}$$

is the identity endomorphism $\mathrm{id}_{\mathfrak{g}}$.

(4.63) *The wedge-bracket.* For any smooth manifold X, define the *wedge-bracket*

$$[-\wedge -]\colon \Omega^\bullet_X(\mathfrak{g}) \times \Omega^\bullet_X(\mathfrak{g}) \longrightarrow \Omega^\bullet_X(\mathfrak{g}) \tag{4.64}$$

on $\mathfrak{g}$-valued differential forms by simultaneously executing wedge product and Lie bracket:

$$[\alpha\xi \wedge \beta\eta] = (\alpha \wedge \beta)[\xi,\eta], \qquad \alpha,\beta \in \Omega^\bullet_X, \quad \xi,\eta \in \mathfrak{g}. \tag{4.65}$$

This suffices, since the general element of $\Omega^\bullet_X(\mathfrak{g})$ is locally a sum of forms of this type. The Koszul sign in the wedge product and the skew-symmetry of the Lie bracket combine to yield

$$[\sigma \wedge \tau] = -(-1)^{pq}\,[\tau \wedge \sigma], \qquad \sigma \in \Omega^p_X(\mathfrak{g}), \quad \tau \in \Omega^q_X(\mathfrak{g}). \tag{4.66}$$

Notice in particular that the wedge-bracket is symmetric on $\mathfrak{g}$-valued 1-forms.

(4.67) *The Maurer-Cartan equation.* This is also known as the *structure equation* of a Lie group.[4]

Theorem 4.68. *The Maurer-Cartan form $\theta \in \Omega^1_G(\mathfrak{g})$ satisfies*

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0. \tag{4.69}$$

Proof. Apply Proposition 4.47. Fix parallel vector fields $\xi, \eta \in \mathfrak{g}$. Then (4.48) yields

$$\begin{aligned} d\theta(\xi,\eta) &= \xi\theta(\eta) - \eta\theta(\xi) - \theta([\xi,\eta]) \\ &= -\theta([\xi,\eta]) \\ &= -[\xi,\eta] \end{aligned} \tag{4.70}$$

because the pairing of a parallel 1-form with a parallel vector field is a parallel function, i.e., a constant function. Also, by (4.49) we have

$$\begin{aligned} \frac{1}{2}[\theta \wedge \theta](\xi,\eta) &= \frac{1}{2}\big([\theta(\xi),\theta(\eta)] - [\theta(\eta),\theta(\xi)]\big) \\ &= [\xi,\eta]. \end{aligned} \tag{4.71}$$

Combine (4.70) and (4.71) to obtain (4.69). ∎

(4.72) *Maurer-Cartan in a basis.* Let $\xi_1, \dots, \xi_m$ be a basis of $\mathfrak{g} = \mathfrak{X}_{\|}(G)$, and let $\theta^1, \dots, \theta^m$ be the dual basis of $\mathfrak{g}^* = \Omega^{\bullet}_{\|}(X)$. The *structure constants* $c^i_{jk} \in \mathbb{R}$ of $\mathfrak{g}$ relative to this basis encode the Lie bracket:

$$[\xi_j, \xi_k] = c^i_{jk}\xi^i, \qquad i \in \{1, \dots, m\}. \tag{4.73}$$

The Maurer-Cartan form (Definition 4.60) is

$$\theta = \theta^i \xi_i, \tag{4.74}$$

and we compute

$$[\theta \wedge \theta] = [\theta^j\xi_j \wedge \theta^k\xi_k] = (\theta^j \wedge \theta^k)[\xi_j, \xi_k] = c^i_{jk}\theta^j \wedge \theta^k \xi_i. \tag{4.75}$$

The Maurer-Cartan equation (4.69) in this basis is a system of m equations:

$$d\theta^i + \frac{1}{2}c^i_{jk}\theta^j \wedge \theta^k = 0, \qquad i \in \{1, \dots, m\}. \tag{4.76}$$

[4]This is the fourth formula of this lecture that is worthy of special memorization.

(4.77) *Maurer-Cartan on a matrix group.* Consider the matrix group $G = \mathrm{GL}_n\mathbb{R}$. There is a tautological matrix-valued function

$$g\colon\ \mathrm{GL}_n\mathbb{R} \longrightarrow M_n\mathbb{R}, \tag{4.78}$$

which is simply the inclusion of $\mathrm{GL}_n\mathbb{R}$ into the vector space of $n\times n$ real matrices. It is precisely this function (4.78) which justifies the moniker 'matrix group'. It follows directly from Definition 4.60 and (3.25) that the Maurer-Cartan form on $\mathrm{GL}_n\mathbb{R}$ is

$$\Theta = g^{-1}dg, \tag{4.79}$$

which is an $n \times n$ matrix of 1-forms on $\mathrm{GL}_n\mathbb{R}$. (The capitalized notation 'Θ' is intentional.) The right hand side uses matrix multiplication. The Maurer-Cartan equation in matrix form follows directly by differentiation:

$$d\Theta = -g^{-1}dg\, g^{-1} \wedge dg = -\Theta \wedge \Theta, \tag{4.80}$$

or in the form we will use it

$$d\Theta + \Theta \wedge \Theta = 0. \tag{4.81}$$

Write the matrix Θ of 1-forms in terms of scalar 1-forms

$$\Theta = \left(\Theta^i_j\right), \qquad \Theta^i_j \in \Omega^1_{\mathrm{GL}_n\mathbb{R}}. \tag{4.82}$$

Then (4.81) expands to n^2 equations, another entry for your rapidly expanding mathematical memory bank:

$$d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j = 0, \qquad i, j \in \{1, \dots, n\}. \tag{4.83}$$

We will use these structure equations often.