Math 55b Problem Set 7

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I collaborated with AJ LaMotta on this problem set.

Problem 1. Suppose that $\sum a_n$ is a *divergent* series of real positive numbers $a_n > 0$, and denote by $s_n = a_1 + \cdots + a_n$ its partial sums.

(a) Show that $\displaystyle\sum \frac{a_n}{1+a_n}$ diverges.

(b) What can you say about $\displaystyle\sum \frac{a_n}{1+na_n}$ and $\displaystyle\sum \frac{a_n}{1+n^2a_n}$?

(c) Prove that $\displaystyle\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$, and deduce that $\displaystyle\sum \frac{a_n}{s_n}$ diverges.

(d) Prove that $\displaystyle\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$, and deduce that $\displaystyle\sum \frac{a_n}{s_n^2}$ converges.

(a) We have two cases to consider. First, consider the case when $a_n \to 0$. Note that

$$\lim_{n\to\infty} \left(\frac{a_n}{1+a_n}\right)/a_n = \lim_{n\to\infty} \frac{1}{1+a_n} = 1$$

so by the ratio test $\sum \frac{a_n}{1+a_n}$ diverges. Next if $a_n \not\to 0$, there must be an infinite number of $a_n > \epsilon$ for some $\epsilon > 0$. So an infinite number of a_n satisfy $\frac{a_n}{1+a_n} > \frac{\epsilon}{1+\epsilon} > 0$ which necessarily means that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Let's start with $\sum \frac{a_n}{1+na_n}$. First note that if $a_n = \frac{1}{n}$, we have

$$\sum \frac{a_n}{1+na_n} = \sum \frac{1}{2n} = 2\sum \frac{1}{n} \to \infty$$

so in some cases the sum can diverge. On the other hand if we let $a_n = 1$ whenever n is square and $a_n = 0$ otherwise, the sum $\sum a_n$ clearly diverges because there are an infinite number of squares, yet

$$\sum \frac{a_n}{1+na_n} = \sum_{n \text{ square}} \frac{1}{1+n} = \sum \frac{1}{1+n^2} \leq \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

so the series converges. For the second series, we have

$$\sum \frac{a_n}{1+n^2a_n} \leq \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

so the series will always converge, irrespective of the choice of a_n.

(c) Notice that for any $N, k \geq 1$ we have

$$\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{S_{N+k} - S_N}{S_{N+k}} = 1 - \frac{S_N}{S_{N+k}}.$$

We claim that $\sum \frac{a_n}{s_n}$ diverges. Suppose for the sake of contradiction that $\sum \frac{a_n}{s_n} = L$ for some $L > 0$. Then for any $N \geq 0$, we have

$$\lim_{k\to\infty} \sum_{n=1}^{k} \frac{a_n}{s_n} = s_N + \lim_{k\to\infty} \sum_{n=N+1}^{k} \frac{a_n}{s_n} \geq s_N + 1 - \lim_{k\to\infty} \sum_{n=N+1}^{k} \frac{s_N}{s_{N+k}} = s_N + 1.$$

Here $\frac{s_N}{s_{N+k}} \to 0$ because $s_n \to \infty$ since $\sum a_n$ diverges. However as $N \to \infty$, $s_N + 1$ also goes to infinity and since this is a lower bound for $\sum \frac{a_n}{s_n}$, this series must also diverge.

(d) Note that for any $n > 1$, we have

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{a_n}{s_{n-1} s_n} \geq \frac{a_n}{s_n^2}.$$

So we have the upper bound

$$\sum \frac{a_n}{s_n^2} \leq \frac{a_1}{s_1^2} + \sum_{n=2} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = \frac{a_1}{s_1^2} + \frac{1}{s_1}.$$

This is of course a finite quantity so the series converges.

Problem 2.

(a) For what values of x does the series $f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2}$ converge? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly?

(b) Is f continuous wherever the series converges? Is f bounded?

(a) For any $k \geq 1$ let $f_j(x)$ be the partial sum function, i.e.

$$f_n(x) = \sum_{n=1}^{k} \frac{1}{1 + n^2 x^2}.$$

Now let $\epsilon > 0$. We have

$$|f(x) - f_k(x)| = \left| \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2} - \sum_{n=1}^{k} \frac{1}{1 + n^2 x^2} \right| = \left| \sum_{n=k+1}^{\infty} \frac{1}{1 + x^2 n^2} \right| \leq \frac{1}{x^2} \sum_{n=k+1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, for any $\epsilon > 0$ there is some $K(\epsilon)$ such that whenever $k \geq K(\epsilon)$, we have $\sum_{n=k+1}^{\infty} \frac{1}{n^2} < \epsilon$. Thus for any $x \neq 0$, we know that $|f(x) - f_k(x)| < \epsilon$ as long as $k \geq K(\epsilon/x^2)$. Thus the interval of convergence for the series $f(x)$ is $\mathbb{R} - \{0\}$. To see where this function is uniformly convergent, we simply need to find all intervals $I \subset \mathbb{R} - \{0\}$ for which $\frac{1}{x^2}$ is bounded above, i.e. intervals like $[a, \infty)$ and $(-\infty, -a]$ for $a > 0$. On every other interval, $f(x)$ does not uniformly converge.

(b) Yes, it is continuous on the points where it is defined. To see this, let $x \neq 0$ and $\epsilon > 0$. Then for any $y \neq 0$, we have

$$|f(x) - f(y)| = \left| \sum_{n=1}^{\infty} \left(\frac{1}{1+n^2x^2} + \frac{1}{1+n^2y^2} \right) \right| \leq \left| \left(\frac{1}{x^2} - \frac{1}{y^2} \right) \sum_{n=1}^{\infty} \frac{1}{n^2} \right| = \frac{\pi^2}{6} \left| \frac{1}{x^2} - \frac{1}{y^2} \right|.$$

Since $g(x) = \frac{1}{x}$ is continuous everywhere except for $x = 0$, we can find some $\delta > 0$ such that whenever $|x - y| < \delta$ we have $\left| \frac{1}{x^2} - \frac{1}{y^2} \right| < \epsilon \cdot \frac{6}{\pi^2}$. This would imply that $|f(x) - f(y)| < \epsilon$, completing the proof that f is continuous on $\mathbb{R} - \{0\}$.

Next, we claim that $f(x)$ is unbounded. Note that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x^2} \geq \sum_{n=1}^{\infty} \frac{1}{2n^2x^2} = \frac{1}{2x^2} \sum_{n=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{12x^2}.$$

Since $\frac{\pi^2}{12x^2}$ is unbounded, and smaller than $f(x)$, it follows that $f(x)$ is unbounded as well.

Problem 3. Let $f(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_0$ be a polynomial of degree d with *integer* coefficients, and suppose that $x \in \mathbb{R}$ satisfies $f(x) = 0$. Prove that there exists a constant $c > 0$ such that for any rational number $p/q \in \mathbb{Q}$ with $p/q \neq \alpha$,

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^d}.$$

Use this to show that $\alpha = \sum_{n=0}^{\infty} 10^{-n!}$ is transcendental (i.e. not the solution of any polynomial equation with integer coefficients).

First assume that $\left| x - \frac{p}{q} \right| \leq 1$, we'll address the other case subsequently. Since $f(t)$ is a polynomial, it is analytic infinitely differentiable and so we can apply Taylor's theorem to get an expansion at x:

$$f(t) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x)(x-t)^k = \sum_{k=1}^{d} \frac{1}{k!} f^{(k)}(x)(x-t)^k.$$

Plugging in $\frac{p}{q}$, we get

$$\begin{aligned}
\left| f\left(\frac{p}{q}\right) \right| &= \left| \sum_{k=1}^{d} \frac{1}{k!} f^{(k)}(x) \left(x - \frac{p}{q} \right)^k \right| \\
&\leq \left| x - \frac{p}{q} \right| \cdot \sum_{k=1}^{d} \left| \frac{1}{k!} f^{(k)}(x) \right| \cdot \left| x - \frac{p}{q} \right|^{k-1} \quad \leq \left| x - \frac{p}{q} \right| \cdot \sum_{k=1}^{d} \left| \frac{1}{k!} f^{(k)}(x) \right| \cdot 1^{k-1} \\
&\leq \left| x - \frac{p}{q} \right| \cdot d \cdot \max_{1 \leq k \leq d} \left| \frac{1}{k!} f^{(k)}(x) \right|.
\end{aligned}$$

Note that

$$f\left(\frac{p}{q}\right) = a_d \left(\frac{p}{q}\right)^d + a_{d-1}\left(\frac{p}{q}\right) + \cdots + a_0 = \frac{a_d p^d + a_{d-1} p^{d-1} q + \cdots + a_0 q^d}{q^d}.$$

Let's write $N(p,q) = a_d p^d + a_{d-1}p^{d-1}q + \cdots + a_0 q^d$ so that $f\left(\frac{p}{q}\right) = \frac{N(p,q)}{q^d}$. Similarly let $M(x) = \max_{1\le k\le d}\left|\frac{1}{k!}f^{(k)}(x)\right|$. Note that $M(x) > 0$ since any polynomial has at least one nonvanishing derivative at any given point. Then

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{N(p,q)}{q^d}\right| \le \left|x - \frac{p}{q}\right| \cdot d \cdot M(x).$$

Rearranging terms around in the inequality, we get

$$\left|x - \frac{p}{q}\right| \ge \frac{|N(p,q)/dM(x)|}{q^d} \ge \frac{1/dM(x)}{q^d}.$$

So letting $c = 1/dM(x)$ works. Note that $c > 0$. Now if $\left|x - \frac{p}{q}\right| > 1$, write $\frac{p}{q} = n + \frac{p'}{q}$ for some $p', n \in \mathbb{Z}$ such that $\left|x - \frac{p'}{q}\right| \le 1$ with $x - \frac{p'}{q}$ and n having the same sign. Then

$$\left|x - \frac{p}{q}\right| = \left|x - \frac{p'}{q} - n\right| = \left|x - \frac{p'}{q}\right| + |n| \ge \left|\frac{c}{q^d}\right|.$$

Now let $\alpha = \sum_{n=0}^{\infty} 10^{-n!}$ as in the problem. Suppose for the sake of contradiction that α is not transcendental, say that it is the root of an integral polynomial. For every $k \ge 0$, let $\alpha_k = \sum_{n=0}^{k} 10^{-n!}$. Note that α_k can be expressed as a rational number with denominator $10^{k!}$. Then by the preceding argument, there is some constant c with

$$\frac{c}{10^{d\cdot k!}} \le |\alpha - \alpha_k| = \left|\sum_{n=0}^{\infty} 10^{-n!} - \sum_{n=0}^{k} 10^{-n!}\right| = \left|\sum_{n=k+1}^{\infty} 10^{-n!}\right| \le 10^{-(k+1)!+1}.$$

This implies that $c \le 10^{d\cdot k! - (k+1)!+1}$ for all $k \ge 0$. This of course implies that $c = 0$ since $10^{d\cdot k!-(k+1)!+1} \to 0$ as $k \to \infty$. But this is a contradiction since $c > 0$ so α must be transcendental.

Problem 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $|f''(x)| \le 1$ for all $x \in \mathbb{R}$, and $\lim_{x\to+\infty} f(x) = 0$ (i.e., $\forall \epsilon > 0\ \exists M$ such that $x > M \Rightarrow |f(x)| < \epsilon$). Show that $\lim_{x\to+\infty} f'(x) = 0$.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\begin{cases} f(p/q) = 1/q & \text{if } p/q \in \mathbb{Q} \text{ is an irreducible fraction, } p, q \in \mathbb{Z},\ q > 0 \\ f(x) = 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) At which points of $\mathbb{R}$ is f continuous?

(b) At which points of $\mathbb{R}$ is f differentiable?

(c) Show that f is the pointwise limit of a sequence of continuous functions.

(d) Show that f is Riemann integrable, and that $\int_0^1 f(x)\,dx = 0$.

(e) (optional, extra credit) Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined similarly but with $g(p/q) = 1/q^3$ (and still $g(x) = 0$ for $x \notin \mathbb{Q}$). Find examples of points at which g is differentiable, and of points at which g is continuous but not differentiable. (Hint: use Problem 3.)

(a) We claim that f is continuous precisely at the irrational numbers. Firstly, let $\frac{p}{q} \in \mathbb{Q}$ be rational, and suppose for the sake of contradiction that f is rational at $\frac{p}{q}$. This means that for every $\epsilon > 0$ there exists some $\delta > 0$ such that $|x - \frac{p}{q}| < \delta$ implies that $|f(x) - f(\frac{p}{q})| < \epsilon$. However contained in any interval $(\frac{p}{q} - \delta, \frac{p}{q} + \delta)$ are an infinite number of irrational numbers x, so $|f(x) - f(\frac{p}{q})| = \frac{1}{q}$ which may not be less than ϵ.

On the other hand suppose x is irrational. Then for every $\epsilon > 0$, choose the largest value of q so that $\frac{1}{q} < \epsilon$. Then let $\delta = \min_{q' \leq q, p \in \mathbb{Z}} |x - \frac{p}{q'}|$. Thus any rational number $\frac{p}{q'} \in (x - \delta, x + \delta)$ will have $q' \geq q$ so $|f(x) - f(\frac{p}{q'})| = \frac{1}{q'} < \epsilon$. We can disregard irrational numbers in this interval since $f(y) = 0$ and so $|f(x) - f(y)| = 0 < \epsilon$.

(b) Since a function can only be differentiable at points where it is continuous, it suffices to only check its differentiability at irrational numbers. We claim it is not differentiable. Let $x \in \mathbb{R} - \mathbb{Q}$ be irrational. If the derivative exists at x, it would have to be zero, since we can find another irrational number y arbitrarily close to x with $f(x) - f(y) = 0$. So let $\epsilon > 0$ and suppose there is a rational y with

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{|f(y)|}{|x - y|} < \epsilon.$$

We don't care about the case when y is irrational since then the ratio would simply be zero. Say $y = \frac{p}{q}$, then

$$\epsilon > \frac{|f(\frac{p}{q})}{|x - \frac{p}{q}|} = \frac{\frac{1}{q}}{|x - \frac{p}{q}|} \geq \frac{\frac{1}{q}}{\min_{p \in \mathbb{Z}} |x - \frac{p}{q}|} = \frac{\frac{1}{q}}{\frac{1}{2q}} = 2.$$

This is a contradiction, since ϵ need not be greater than 2. So f isn't differentiable at x.

(c) Recall that the rationals are countable, so we can order them $\{\frac{p_i}{q_i}\}_{i \geq 0}$. Now let's construct functions $f(x, i)$, with $0 \leq f(x, i) \leq 1$ for all x, i, and defined recursively as follows. First, $f(x, 0)$ is defined as a function which takes on values $f(\frac{p_0}{q_0}, 0) = \frac{1}{q_0}$, and any other values between 0 and < 1 elsewhere. Next, for any i, define $f(x, i)$ as some function with $f(\frac{p_i}{q_i}, i) = \frac{1}{q_i}$, and $f(x, i) = 1$ whenever x is in some chosen open neighborhood of $\{\frac{p_k}{q_k}\}_{k \leq i}$ which doesn't contain $\frac{p_i}{q_i}$. Then clearly since $f(x, i) \in [0, 1]$ we have a well defined infinite product

$$f(x) = \prod_{i \geq 0} f(x, i).$$

Clearly $f(\frac{p_i}{q_i}) = \frac{1}{q_i}$, and by choosing the values of $f(x, i)$ appropriately we can get $f(x) = 0$ for irrational x. (Indeed it is a product of infinitely many values < 1) All of these functions are continuous so we are done.

(d) Since $\inf_{x \in [a,b]} f(x) = 0$ for any interval $[a, b] \subset [0, 1]$, the lower Riemann integral $I_{-}(f) = \sup_{[a,b]} \inf_{x \in [a,b]} f(x) = 0$. Similarly, we claim that $\inf_{[a,b]} \sup_{x \in [a,b]} f(x) = 0$. Consider an interval $[a, b]$ where every rational number $\frac{p}{q} \in [a, b]$ has $q > Q$ for some $Q > 2$. (This can be done because there are a finite number of rational numbers to avoid) Then $\sup_{x \in [a,b] f(x) = \frac{1}{Q}}$. Since $\frac{1}{Q}$ can be made arbitrarily small, it follows that the upper Riemann integral is also zero. Thus $I_{-}(f) = I_{+}(f) = \int_0^1 f(x)\, dx = 0$.

Problem 6. Let p, q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that if f, g are integrable, $f, g \geq 0$, and $\int_a^b f^p\,dx = \int_a^b g^q\,dx = 1$, then $\int_a^b fg\,dx \leq 1$.

(b) Use this to deduce *Hölder's inequality* for integrable functions:

$$\left|\int_a^b fg\,dx\right| \leq \left(\int_a^b |f|^p\,dx\right)^{1/p}\left(\int_a^b |g|^q\,dx\right)^{1/q}.$$

(c) (optional, extra credit) The L^p norm of $f \in C^0([a,b])$ is defined to be $\|f\|_p = \left(\int_a^b |f|^p\,dx\right)^{1/p}$. The triangle inequality for the L^p norm is known as *Minkowski's inequality*:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Prove Minkowski's inequality for $p > 1$ by observing that $|f+g|^p \leq |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$, and using Hölder's inequality to bound the integral of the right hand side.

First we'll prove a useful inequality.

Claim. Let p, q be positive integers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for any nonnegative u, v, we have

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Proof. We can assume that $u, v \neq 0$ since the case when one or both are zero is trivial. Then

$$uv = e^{\log uv} = e^{\log u + \log v} = e^{\frac{1}{p}p\log u + \frac{1}{q}q\log v} = e^{\frac{1}{p}\log u^p + \frac{1}{q}\log v^q}.$$

However since e^x is a strictly convex function and $\frac{1}{p} + \frac{1}{q} = 1$, by Jensen's inequality we have

$$e^{\frac{1}{p}\log u^p + \frac{1}{q}\log v^q} \leq \frac{1}{p}e^{\log u^p} + \frac{1}{q}e^{\log v^q} = \frac{u^p}{p} + \frac{v^q}{q}.$$

So this proves the inequality. $\square$

(a) Applying this inequality to the given integrals, we get

$$\int_a^b fg\,dx \leq \frac{1}{p}\int_a^b |f|^p\,dx + \frac{1}{q}\int_a^b |g|^q\,dx = 1,$$

which is the desired inequality.

(b) Using the notation of the L^p norm, let $f' = |f|/\|f\|_p$ and $g' = |g|/\|g\|_q$. (We'll address the cases when $\|f\|_p$ and/or $\|g\|_q$ are zero later. Then f', g' satisfy the conditions of (a), so we get

$$\int_a^b f'g'dx = \frac{1}{\|f\|_p\|g\|_q}\int_a^b |f||g|dx \leq 1$$

$$\int_a^b |f||g|dx \leq \left(\int_a^b |f|^p dx\right)^{1/p}\left(\int_a^b |g|^q dx\right)^{1/q}.$$

which is the desired inequality. Now suppose that one of $\|f\|_p$ or $\|g\|_q$ is zero, say without loss of generality that $\|g\|_q = 0$.[1] Let $c > 0$. Then by the claim, $c|f| \leq \frac{1}{p}|f|^p + \frac{1}{q}c^q$ so

$$c\int_a^b |g|dx \leq \frac{1}{q}\int_q^b |g|^p dx + \frac{1}{q}\int_a^b c^q dx = \frac{c^q(b-a)}{q}$$
$$\int_a^b |g|dx \leq \frac{c^{q-1}(b-a)}{q}.$$

Thus, $\int_a^b |g|dx = 0$ since we can make $\frac{c^{q-1}(b-a)}{q}$ arbitrarily small. This means that $\int_a^b |f||g|dx \leq (\max_{[a,b]}|f|)\int_a^b |g|dx = 0$ so

$$\left|\int_a^b fgdx\right| \leq \int_a^b |f||g|dx = 0 = \left(\int_a^b |f|^p dx\right)^{1/p}\left(\int_a^b |g|^q dx\right)^{1/q}.$$

This completes the proof.

Problem 7. Prove that the series given by the sum of the inverses of the primes, $\sum 1/p$, is divergent. (This shows that the primes are not too sparse as a subset of the positive integers).

Let $N \in \mathbb{Z}_+$. Then by unique factorization of integers, we have

$$\sum_{k=1}^{N}\frac{1}{k} \leq \prod_{p \text{ prime}}^{N}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \text{ prime}}^{N}\frac{1}{1-\frac{1}{p}}.$$

Next, we claim that for any $x \in [0, \frac{1}{2}]$, $1 - x \geq 4^{-x}$. This is easy to see, note that $1 - x = 4^{-x}$ has only two solutions at $x = 0$ and $x = \frac{1}{2}$. Since $1 - x$ and 4^{-x} are both continuous, by the intermediate value theorem it suffices to check the inequality only at a single $x \in (0, \frac{1}{2})$. Note that $1 - \frac{1}{4} = \frac{3}{4}$ and $4^{-\frac{1}{4}} = \frac{\sqrt{2}}{2}$. Then $\frac{\sqrt{2}}{2} < \frac{3}{4}$ because $\frac{1}{2} < \frac{9}{16}$. So $1 - x \geq 4^{-x}$ on this interval. Equivalently, $\frac{1}{1-x} \leq 4^x$. Since $0 \leq \frac{1}{p} \leq \frac{1}{2}$ for any prime p, we can apply this inequality to the product, getting:

$$\prod_{p \text{ prime}}^{N}\frac{1}{1-\frac{1}{p}} \leq \prod_{p \text{ prime}}^{N} 4^{\frac{1}{p}} = 4^{\left(\sum_{p \text{ prime}}^{N}\frac{1}{p}\right)}.$$

Rearranging terms and using the order preserving properties of the logarithm, we get

$$\sum_{p \text{ prime}}^{N}\frac{1}{p} \geq \frac{1}{\log 4}\log\left(\sum_{k=1}^{N}\frac{1}{k}\right).$$

Thus since $\sum_{k=1}^{N}\frac{1}{k}$ diverges as $N \to \infty$, so does $\sum_{p \text{ prime}}^{N}\frac{1}{p}$.

[1] Credit to AJ LaMotta for providing a hint which motivates this argument.

Problem 8. This problem gives a weaker form of Stirling's formula, which asserts that

$$n! \sim n^n e^{-n}\sqrt{2\pi n}.$$

Let $\{x\} = \lfloor x \rfloor$. Define

$$f(x) = (1 - \{x\}) \log \lfloor x \rfloor + \{x\} \log(\lfloor x \rfloor + 1),$$

and

$$g(x) = \frac{x}{\lfloor x + \frac{1}{2} \rfloor} - 1 + \log \left\lfloor x + \frac{1}{2} \right\rfloor.$$

Describe or sketch the graphs of f and g, and show that $f(x) \le \log x \le g(x)$ for all $x \ge 1$. Then show that for n a positive integer,

$$\int_1^n f(x)\,dx = \log(n!) - \frac{1}{2}\log n \quad \text{and} \quad \int_1^n g(x)\,dx < \log(n!) - \frac{1}{2}\log n + \frac{1}{8}.$$

Conclude, by integrating $\log x$ over $[1, n]$, that

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n < 1$$

for all integers $n \ge 2$, and hence that

$$e^{7/8}\, n^n e^{-n}\sqrt{n} < n! < e\, n^n e^{-n}\sqrt{n}.$$

Notice that $f(x)$ is a "polygonal" approximation of $\log x$, with $f(n) = \log n$ for all $n \in \mathbb{Z}_+$, and being piecewise linear for all $x \in [n, n+1]$. Clearly $f(x) \le \log x$ because $\log x$ is a concave function. $g(x)$ similarly satisfies $g(n) = \log n$ for all $n \in \mathbb{Z}_+$, but takes on the values of the tangent line to $\log x$ at n on the interval $\left[n - \frac{1}{2}, n + \frac{1}{2}\right)$. Just as with $f(x)$, $g(x) \ge \log x$ because $\log x$ is a concave function. Now we calculate the integrals.

$$\begin{aligned}
\int_1^n f(x)dx = \sum_{k=1}^{n-1} \int_0^1 f(x+k)dx &= \sum_{k=1}^{n-1} \int_0^1 (1-x)\log k + x\log(k+1)dx \\
&= \sum_{k=1}^{n-1} \left(\log k \int_0^1 (1-x)dx + \log(k+1) \int_0^1 x\,dx \right) \\
&= \log(n!) - \frac{1}{2}\log n.
\end{aligned}$$

Similarly for $g(x)$,

$$\begin{aligned}
\int_1^n g(x)\,dx &= \left(\sum_{k=1}^{n-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{n} - 1 + \log n\,dx \right) + \int_{n-\frac{1}{2}}^{n} \frac{x}{n} - 1 + \log n\,dx \\
&= \log(n!) - \frac{1}{2}\log n + \frac{1}{8}.
\end{aligned}$$

Now integrating on both sides of the inequality,

$$\begin{gathered}
\log(n!) - \frac{1}{2}\log n \leq \int_1^n \log x \; dx \leq \log(n!) - \frac{1}{2}\log n + \frac{1}{8} \\
\log(n!) - \frac{1}{2}\log n \leq n\log n - n + 1 \leq \log(n!) - \frac{1}{2}\log n + \frac{1}{8} \\
\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n < 1.
\end{gathered}$$

Then applying e^x to both sides, we get

$$\begin{gathered}
e^{7/8} < \frac{n!e^{-n}}{n^{n+\frac{1}{2}}} < e \\
e^{7/8}n^n e^{-n}\sqrt{n} < n! < en^n e^{-n}\sqrt{n}.
\end{gathered}$$

This is the desired inequality.