Lecture 1: Parallelism and geometric structures

Contents

We begin with the basic geometry of affine space. Affine geometry is the geometry of parallelism, which is expressed by the simply transitive action of a vector space: the action of the vector space on the affine space is called *translation*. We say that the affine space is a *torsor* over the vector space. Smooth manifolds generally do not admit a global parallelism, but rather one can introduce a structure of parallel transport along curves, as we will learn later in the course.

The second topic in this lecture is symmetry groups of vector and affine spaces. A vector space has *linear* symmetries, whereas an affine space has *affine* symmetries. An affine map between affine spaces is characterized by the property that its differential is constant (under translation), and this leads to the definition of an affine symmetry of a fixed affine space. We introduce bases of vector spaces and frames in an affine space (Definition 1.26). The spaces of bases and frames are torsors over the linear and affine symmetry groups.

The final topic is *symmetry types* (Definition 1.37). This is one of the key ideas in the first part of the course. A symmetry type is a Lie group over a general linear group. (We introduce Lie groups systematically soon.) Felix Klein, in his *Erlangen program* (**1.34**), told that a particular type of geometry is characterized by its symmetry group. A symmetry type encodes this idea. A symmetry types leads to both a type of affine geometry—as we explain in this lecture—and a type of geometry on smooth manifolds—which we explain in due course. Thus the orthogonal group as a symmetry type determines both the Euclidean geometry of affine space and Riemannian geometry of smooth manifolds. There are many, many species of geometric structures, some more common than others. The centrality of symmetry groups in geometry—as emphasized by Klein and subsequently by Cartan, Chern, Singer, Bott, . . . —guides our approach throughout the course.

We provide a brief appendix on group extensions for reference.

The notes on Multivariable Analysis and on Differential Topology contain more material about affine spaces. At points in this lecture I use concepts from the theory of smooth manifolds, such as fiber bundles, that you will find explained in the Differential Topology notes.

Affine geometry

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Definition 1.1. Let V be a vector space. An *affine space over* V is a set A equipped with a simply transitive action $+\colon A \times V \to A$ of V.

We call V the *tangent space* to A. We do not assume that V is finite dimensional. See Figure 1.

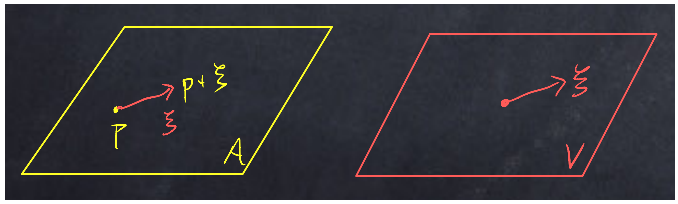

FIGURE 1. An affine space A and its tangent space V

(1.2) *The ground field.* In this class we mostly use real vector spaces, though occasionally we encounter complex vector spaces. Affine geometry may be done over any field. If V is a finite dimensional real vector space, then there is a unique topology on V for which the vector space operations are continuous, and we can transport that topology to A. Then A has a standard smooth structure and may be considered as a smooth manifold. Of course, this is backwards: a smooth manifold is locally modeled on affine space, so we need to know about affine space first.

(1.3) *Torsors.* The simple transitivity of the *translation* action of V on A means given $p, q \in A$ there exists a unique $\xi \in V$ such that $q = p + \xi$. We write $\xi = q - p$. We encounter simply transitive actions often, and we use a standard term for them.

Definition 1.4. Let G be a group. A *(left/right) torsor over* G is a set T equipped with a simply transitive (left/right) G-action. We call T a *(left/right) G-torsor.*

So an affine space is a torsor over a vector space. If G is a topological group, then a torsor over G is assumed to be a topological space and the action is assumed continuous. Similarly, if G is a Lie group, then a torsor over G is assumed to be a smooth manifold and the action is assumed smooth. The group G with left/right multiplication is the "trivial" left/right G-torsor.

Suppose T is a right G-torsor. (This discussion works for left G-torsors as well.) A point $t_0 \in T$ induces an isomorphism of right G-torsors

$$\begin{aligned} \theta_{t_0}\colon G &\longrightarrow T \\ g &\longmapsto t_0 g \end{aligned} \tag{1.5}$$

If $t_1 \subset T$ and $t_1 = t_0 h$ for the uniquely determined $h \in G$, then $\theta_{t_1} \circ \theta_{t_0}^{-1}\colon G \to G$ is *left* multiplication by h^{-1}, which is a self-map of the right G-torsor G. One thinks of a choice of $t_0 \in G$ as a trivialization of T, and we have computed the change of trivialization in the diagram

$$\begin{tikzcd} & T & \\ G \arrow[ur, "\theta_{t_0}"] \arrow[rr, dashed] & & G \arrow[ul, "\theta_{t_1}"'] \end{tikzcd} \tag{1.6}$$

(1.7) *Model spaces.* Model geometries play an important role. Fix a dimension $n \in \mathbb{Z}^{\geq 0}$.

Definition 1.8.

(1) The standard n-dimensional vector space is

$$\mathbb{R}^n = \{(\xi^1, \dots, \xi^n) : \xi^i \in \mathbb{R}\} \tag{1.9}$$

equipped with its standard zero vector, vector addition, and scalar multiplication.

(2) The standard n-dimensional affine space is

$$\mathbb{A}^n = \{(x^1, \dots, x^n) : x^i \in \mathbb{R}\} \tag{1.10}$$

equipped with the standard translation action of $\mathbb{R}^n$.

(1.11) *Weighted averages.* Let A be an affine space, $p_0, \dots, p_k \in A$ and $\lambda^0, \dots, \lambda^k \in \mathbb{R}$. Then if $\lambda^0 + \cdots + \lambda^k = 1$ the point

$$\lambda^i p_i = \lambda^0 p_0 + \dots \lambda^k p_k \in A \tag{1.12}$$

is well-defined. (Note the summation convention in (1.12).) A *convex* weighted average is one in which $0 \leq \lambda^i \leq 1$ for all i. A *convex subset* of an affine space is a subset that contains all of its convex weighted averages.

(1.13) *Affine maps and affine subspaces.*

Definition 1.14. Let V, W be vector spaces and A, B affine spaces over V, W, respectively. A map $f\colon A \to B$ is *affine* if there exists a linear map $T\colon V \to W$ such that

$$f(p + \xi) = f(p) + T\xi, \qquad p \in A, \quad \xi \in V. \tag{1.15}$$

In this case we write $T = df$. Alternatively, if V, W are finite dimensional (or equipped with Banach space structures) we can say that a smooth map $f\colon A \to B$ is affine iff its differential $df_p\colon V \to W$ is independent of p. (Recall that in general the differential df of a differentiable map $f\colon A \to B$ is a continuous map $df\colon A \to \operatorname{Hom}(V, W)$.)

Definition 1.16. Let A be an affine space with tangent space V. A subset $A' \subset A$ is an *affine subspace* if there exists a linear subspace $V' \subset V$ such that $q' - p' \in V'$ for all $p', q' \in V'$. We call V' the *tangent space* to A'. Affine subspaces $A', A'' \subset A$ are *parallel* if they have equal tangent spaces. (If one is willing to say that affine subspaces of different dimensions are parallel, then one should require that one tangent space is a subspace of the other.)

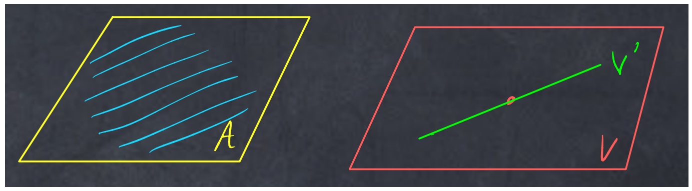

FIGURE 2. The foliation of affine subspaces with tangent space $V' \subset V$

The subspace A' is affine over V'. Translation maps affine subspaces to affine subspaces: for all $\xi \in V$, the subset $A' + \xi \subset A$ is an affine subspace with the same tangent space $V' \subset V$. We say that A' and $A' + \xi$ are *parallel*. A linear subspace $V' \subset V$ defines a *foliation* of A by affine subspaces with tangent space V'; see Figure 2.

Example 1.17 (geodesic motion)**.** Parallelism leads to the notion of a geodesic motion. Recall that a *motion* in an affine space A over V is a smooth map $\gamma\colon I \to A$ for an open interval $I \subset \mathbb{R}$. The *velocity* $\dot\gamma\colon I \to A$ is constant if and only if γ is the restriction of an affine map $\mathbb{R} \to A$. This holds iff the *acceleration* $\ddot\gamma\colon I \to A$ vanishes. Motions with zero acceleration are *geodesic* motions (or *constant velocity* motions). The range of the motion is a subset of an affine line. We will see later that on a smooth manifold we only need parallelism along curves to define geodesic motion.

Global parallelism

(1.18) *Parallelism in affine space.* As stated earlier, an affine space A has global parallelism. This manifests in various ways. The structure of affine space—the simply transitive (translation) action by a vector space V—is an expression of parallelism. Thus, for example if $A' \in A$ is an affine subspace, and $p \in A$, then there is a unique affine subspace through p parallel to A'. This, of course, is the famous parallel postulate of Euclid.

FIGURE 3. A constant (parallel) vector field on A

The translation action also leads to the notion of a constant or translationally invariant or parallel vector field on an affine space A. In general, a vector field is a function $A \to V$, and a constant vector field is a constant function. So given a vector $\xi \in V$ there is a constant vector field with value ξ. (See Figure 3.) Also, a vector at any point of A extends uniquely to a constant vector field.

(1.19) *Global parallelism on smooth manifolds.*

Definition 1.20. Let X be a smooth manifold. A *(global) parallelism* on X is a vector space V and an isomorphism

$$\begin{tikzcd} X \times V \arrow[rr, "\cong"] \arrow[dr] & & TX \arrow[dl] \\ & X & \end{tikzcd} \tag{1.21}$$

of vector bundles over X.

Thus a parallelism is a structure which identifies each tangent space T_pX with a fixed vector space V. An affine space carries a canonical parallelism in this sense. A smooth manifold is *parallelizable* if a global parallelism exists. Typically global parallelisms do not exist. For example, the Hairy Ball Theorem implies that S^2 is not parallelizable. A classical theorem settles the question of which spheres are parallelizable.

Theorem 1.22 (Kervaire, Bott-Milnor 1958)**.** *Suppose the sphere S^n is parallelizable. Then its dimension satisfies $n \in \{0, 1, 3, 7\}$.*

A *Lie group* is the marriage of a group and a smooth manifold, a ceremony we will perform soon in an upcoming lecture. A Lie group G carries two canonical parallelisms via left and right translation; if G is abelian, then these agree. A torsor over a Lie group also carries a canonical parallelism; the global parallelism of affine spaces is a special case. The spheres S^0, S^1, S^3 admit Lie group structures, but S^7 does not.

Remark 1.23. Any smooth manifold admits a structure called a *linear connection* or *covariant derivative* which allows us to define parallelism along curves. There are new phenomena of *torsion*, *curvature*, and *holonomy* which do not appear in the global parallelism of affine space. This *intrinsic* parallelism on smooth manifolds has an *extrinsic* generalization: *connections on principal bundles*. Connections are a central focus of this course.

Linear and affine symmetries

Definition 1.24.

(1) Let V be a vector space. A *linear symmetry* of V is an invertible linear map $T\colon V \to V$. The group of linear symmetries of V is denoted $\mathrm{Aut}(V)$.
(2) Let A be an affine space over V. An *affine symmetry* of A is an invertible affine map $f\colon A \to A$. The group of affine symmetries of A is denoted $\mathrm{Aut}(A)$.

If V is finite dimensional (over $\mathbb{R}$), then we can endow $\mathrm{Aut}(V)$ and $\mathrm{Aut}(A)$ with the structure of a Lie group.

The differential d is a group homomorphism that fits into the *group extension*

$$1 \longrightarrow V \longrightarrow \mathrm{Aut}(A) \xrightarrow{d} \mathrm{Aut}(V) \longrightarrow 1 \tag{1.25}$$

(Recall that the differential of an affine map is constant.) The kernel of d is the normal subgroup of translations. The group extension (1.25) is split by a choice of point $p_0 \in A$; the splitting $\sigma_{p_0}\colon \operatorname{Aut}(V) \to \operatorname{Aut}(A)$ maps a linear symmetry T to the unique affine symmetry that fixes p_0 and has differential T.

Bases and frames

The word choice is a bit fraught: for a vector space we might use 'basis' and 'frame' synonymously.

Definition 1.26.

(1) Let V be an n-dimensional real vector space. A *basis* of V is a linear isomorphism

$$b\colon \mathbb{R}^n \longrightarrow V. \tag{1.27}$$

The set of all bases of V is denoted $\mathcal{B}(V)$.

(2) Let A be an n-dimensional real affine space. A *frame* in A is an affine isomorphism

$$f\colon \mathbb{A}^n \longrightarrow A. \tag{1.28}$$

The set of all frames in A is denoted $\mathcal{B}(A)$.

Remark 1.29.

(1) Automorphisms of $\mathbb{R}^n$ act on the *right* on bases, and automorphism of V act on the *left*. Both actions are simply transitive. The right and left torsor structures are indicated thus:

$$\operatorname{Aut}(V) \circlearrowright \mathcal{B}(V) \circlearrowleft \mathrm{GL}_n\mathbb{R} \tag{1.30}$$

In a similar way we have the right and left torsors

$$\operatorname{Aut}(A) \circlearrowright \mathcal{B}(A) \circlearrowleft \mathrm{Aff}_n\mathbb{R}, \tag{1.31}$$

where $\mathrm{Aff}_n\mathbb{R} = \mathrm{Aff}(\mathbb{A}^n)$ is the group of affine symmetries of $\mathbb{A}^n$.

(2) My convention is that *right* group actions are *structural* and that *left* group actions are *geometric*. Thus the left action in (1.30) is induced from the geometric action of $\operatorname{Aut}(V)$ on V, and in (1.31) from the geometric action of $\operatorname{Aut}(A)$ on A. These are the natural symmetries. The right actions are "internal"; they depend on the choice of frame or basis. You might think of the left action as "active" and the right as "passive". Example: there is a right (internal) action on bases that exchanges the first and second basis elements.

(3) Both $\mathcal{B}(V)$ and $\mathcal{B}(A)$ have natural smooth manifold structures, and the actions in (1.30) and (1.31) are Lie group actions.

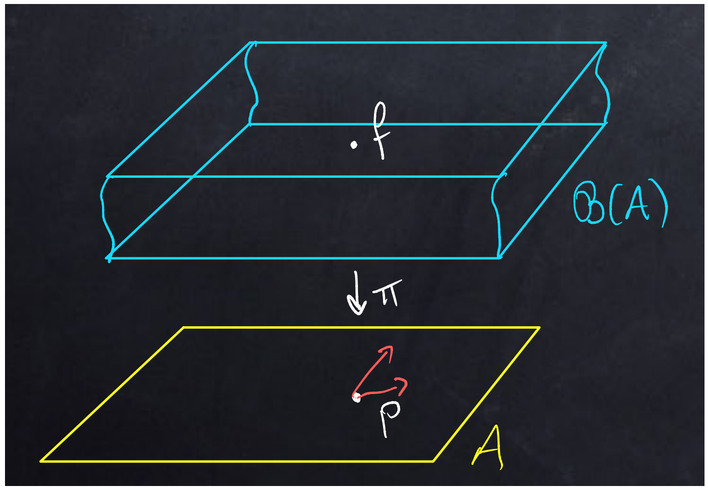

FIGURE 4. The bundle of frames of the affine space A

(4) Let $0 \in \mathbb{A}^n$ denote the point $(0, \dots, 0)$. The map

$$\begin{aligned} \mathcal{B}(A) &\longrightarrow A \times \mathcal{B}(V) \\ f &\longmapsto f(0),\ df_0 \end{aligned} \tag{1.32}$$

is a diffeomorphism. Furthermore, the diffeomorphism is equivariant for the left V-action and the right $\mathrm{GL}_n\mathbb{R}$-action, and those actions commute. (What is the right $\mathrm{GL}_n\mathbb{R}$-action on $\mathcal{B}(A)$? Its definition depends on the basepoint $0 \in \mathbb{A}^n$.)

(5) The projection

$$\begin{aligned} \pi\colon \mathcal{B}(A) &\longrightarrow A \\ f &\longmapsto f(0) \end{aligned} \tag{1.33}$$

is a fiber bundle of a special type: a *principal* $\mathrm{GL}_n\mathbb{R}$-*bundle*. The total space $\mathcal{B}(A)$ carries a right $\mathrm{GL}_n\mathbb{R}$-action which commutes with π, and the fibers of π are right $\mathrm{GL}_n\mathbb{R}$-torsors. We identify the fiber of π over $p \in A$ as the set of bases of $T_pA = V$ by sending f to its differential $df_0\colon \mathbb{R}^n \xrightarrow{\cong} V$.

We observe that π is the quotient map of the right $\mathrm{GL}_n\mathbb{R}$-action on $\mathcal{B}(A)$.

A smooth manifold carries an analog of (1.33)—the *bundle of frames* or *frame bundle*—which plays an important role in our study of differential geometry. However, on a general smooth manifold we do not have the geometric left action of a group. On affine space the left V-action on π encodes global parallelism. On a smooth manifold we will introduce a different notion of parallelism on the total space of the frame bundle.

Geometric structures: symmetry types

To define a geometric structure we (1) specify a symmetry type on the model vector space $\mathbb{R}^n$, (2) extend to the model affine space $\mathbb{A}^n$ imposing translation invariance, and (3) use frames to define a structure on a general n-dimensional affine space. As mentioned in the introduction to this lecture, a symmetry type also defines a notion of geometric structure on smooth manifolds, an idea we develop in due course.

(1.34) *Erlangen program.* Linear geometry is the study of invariants under linear symmetry. Such concepts as basis, spanning subset, linear subspace are all examples of linear notions. In other words, the set of all bases or of linear subspaces (of fixed dimension) in a vector space V is acted on by the group $\mathrm{Aut}(V)$. A similar statement can be made for affine geometry and affine symmetry. In a similar vein, differential topology is the study of C^∞ concepts. By contrast, Euclidean geometry is the study of invariants under the Euclidean group of symmetries, which is a subgroup of the group of affine symmetries. Whereas the notion of a triangle makes sense in both affine and Euclidean geometry, the notion of the perimeter or of the area of a triangle is only defined in Euclidean geometry. On the other hand, a triangle is not a C^∞ concept, so it does not exist in differential topology.

In 1872 Felix Klein introduced a dictum which characterizes different types of geometry by their symmetry. A "small" interpretation of Klein's *Erlangen program* in the case of affine geometry focuses on a single affine space A and tells that affine geometry on A is the study of structures/properties/quantities of/in A which are invariant under the action of $\mathrm{Aut}(A)$ on A. A "large" interpretation says that affine geometry is the study of the category of affine spaces and affine maps between them. The notion of a *symmetry type* generates both of these interpretations.

Remark 1.35. A general principal is that a smaller symmetry group has more invariants, so in this context a smaller symmetry group leads to richer geometry.

(1.36) *Symmetry types.* Following (1.34) we define a linear geometric structure on $\mathbb{R}^n$ by a group acting linearly on $\mathbb{R}^n$.

Definition 1.37. An n-dimensional (linear) *symmetry type* is a pair (G_n, λ_n) in which G_n is a Lie group and $\lambda_n\colon G_n \to \mathrm{GL}_n\mathbb{R}$ is a homomorphism of Lie groups.

Roughly, (G_n, λ_n)-geometry is the study of structures/properties/quantities in $\mathbb{R}^n$ invariant under the G_n-action. A symmetry type induces a canonical symmetry group of the model space $\mathbb{A}^n$ via pullback of group extensions:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{R}^n \arrow[r, dashed] \arrow[d, equal] & \mathcal{G}_n \arrow[r, dashed] \arrow[d, dashed, "\tilde\lambda_n"] & G_n \arrow[r] \arrow[d, "\lambda_n"] & 1 \\ 1 \arrow[r] & \mathbb{R}^n \arrow[r] & \mathrm{Aff}_n\mathbb{R} \arrow[r] & \mathrm{GL}_n\mathbb{R} \arrow[r] & 1 \end{tikzcd} \tag{1.38}$$

We call the pair $(\mathcal{G}_n, \tilde\lambda_n)$ an *affine symmetry type*. Note that $\mathbb{R}^n \subset \mathcal{G}_n$ is a *normal* subgroup. The existence of a normal subgroup of translations is the statement that the affine symmetry type includes translation invariance. Quotienting by the normal subgroup of translations passes in the other direction from an affine symmetry type to a linear symmetry type. Also, a choice of $p \in \mathbb{A}^n$ splits the group extension (1.38): sitting over G_n is the subgroup of $\mathcal{G}_n$ consisting of symmetries that fix p. We use the canonical choice $p = 0 = (0, \dots, 0)$, as in (1.32).

Remark 1.39. For some symmetry types (G_n, λ_n) there is a geometric structure on $\mathbb{R}^n$ whose group of invariants is $G_n \subset \mathrm{GL}_n\mathbb{R}$. This is true in Example 1.40 below: the orthogonal group $\mathrm{O}_n \subset \mathrm{GL}_n\mathbb{R}$ is the subgroup of matrices that preserves the standard inner product. For any symmetry type (G_n, λ_n), the invariants of $G_n \circlearrowright \mathbb{R}^n$ are important elements of the geometry. They may be tensors of various types or other kinds of geometric objects. But one needn't find a collection that *characterizes* G_n. Rather, the symmetry type defines the geometry completely. Also, we neither assume that λ_n is injective nor that λ_n is surjective. In particular, λ_n may have a nontrivial kernel $K_n \subset G_n$, a normal subgroup of *internal symmetries*. One can imagine an "internal geometric structure" whose symmetry group is K_n, and in affine space such an internal geometric structure exists at each point and is acted on by translations; see Example 1.43 below. Traditionally, Cartan's notion of "G-structure" has λ_n injective, so no internal geometric structures. Notice that this definition excludes Example 1.44.

Example 1.40 (Euclidean geometry)**.** Let $G_n = \mathrm{O}_n$ be the orthogonal group and $\lambda_n\colon \mathrm{O}_n \hookrightarrow \mathrm{GL}_n\mathbb{R}$ be the inclusion. This is the symmetry group of the standard positive definite inner product on $\mathbb{R}^n$, and linear orthogonal geometry is the study of linear invariants, such as the length of a vector, the angle between two vectors, the volume of parallelepipeds of all dimensions, etc. The induced standard affine symmetry group $\mathcal{G}_n$ in (1.38) is the Euclidean group, a subgroup of $\mathrm{Aff}_n\mathbb{R}$. It is the symmetry group of Euclidean geometry in the standard affine space, including the traditional rotations, reflections, and translations of Euclidean geometry.

Example 1.41 (oriented volume geometry)**.** Let $G_n = \mathrm{SL}_n\mathbb{R}$ be the group of invertible $n \times n$ matrices of determinant one, and let $\lambda_n\colon \mathrm{SL}_n\mathbb{R} \hookrightarrow \mathrm{GL}_n\mathbb{R}$ be the inclusion. The invariant in this geometry is the signed volume of an n-dimensional parallelepiped.

Example 1.42 (Lorentz geometry)**.** This is formally similar to Example 1.40, but the positive definite inner product is replaced by a standard nondegenerate symmetric bilinear form on $\mathbb{R}^n$ of signature $(1, n-1)$. In this case $G_n = \mathrm{O}_{1,n-1}$ is the Lorentz group and again λ_n is the inclusion. The induced affine symmetry group is a form of the Poincaré group. The symmetry type one uses to define a general relativistic quantum field theory uses a subgroup of index two in G_n (that preserves a time orientation).

Example 1.43 (internal symmetry)**.** Let K be any Lie group and set $G_n = \mathrm{GL}_n\mathbb{R} \times K$ with λ_n projection onto the first factor. The induced affine symmetry group is $\mathcal{G}_n = \mathrm{Aff}_n\mathbb{R} \times K$. Think of this as geometry on $\mathbb{A}^n$ equipped with some internal structure at each point which transforms under the symmetry group K. For example, if K is cyclic of order two, this is the symmetry group of the product bundle (covering) $\mathbb{A}^n \times \{0,1\} \to \mathbb{A}^n$: the internal symmetry group K permutes the fiber over each point of $\mathbb{A}^n$.

Example 1.44 (spin geometry)**.** The spin group $G_n = \mathrm{Spin}_n$ is a double cover of the identity component $\mathrm{SO}_n \subset \mathrm{O}_n$, and that double cover map followed by the inclusion defines $\lambda_n\colon \mathrm{Spin}_n \to \mathrm{GL}_n\mathbb{R}$. It is difficult to describe a structure on $\mathbb{R}^n$ whose precise symmetry group defines the spin symmetry type.

Remark 1.45 (stable symmetry types)*.* The symmetry types in the previous example can be defined in all dimensions coherently, and implicitly that is what we understand when we say 'Euclidean geometry' or 'Lorentz geometry'. We define a *stable symmetry type* as follows. First, for each $n \in \mathbb{Z}^{\geq 0}$ define an inclusion

$$\begin{aligned} \mathrm{GL}_n\mathbb{R} &\longrightarrow \mathrm{GL}_{n+1}\mathbb{R} \\ A &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \end{aligned} \tag{1.46}$$

A stable symmetry type is a collection $\{(G_n, \lambda_n)\}_{n\in\mathbb{Z}^{\geq 0}}$ of symmetry types, one in each dimension, and inclusions $i_n\colon G_n \hookrightarrow G_{n+1}$ such that the diagram of group homomorphisms

$$\begin{tikzcd} G_n \arrow[r, hook, "i_n"] \arrow[d, "\lambda_n"'] & G_{n+1} \arrow[d, "\lambda_{n+1}"] \\ \mathrm{GL}_n\mathbb{R} \arrow[r, hook] & \mathrm{GL}_{n+1}\mathbb{R} \end{tikzcd} \tag{1.47}$$

commutes and is a pullback diagram. Note that the internal symmetry group $K_n = \ker \lambda_n$ is independent of n up to an isomorphism determined by the data.

From symmetry types to vector and affine geometries

A symmetry type lets us define linear and affine geometric structures of that type on abstract vector and affine spaces. The construction follows a basic theme: torsors control geometric structures. It is the geometric structures on affine spaces that we generalize later to geometric structures on smooth manifolds.

We begin with an important general maneuver on torsors.

(1.48) *The mixing construction.* Let $\lambda\colon G \to \overline{G}$ be a homomorphism of (Lie) groups, and suppose T is a right G-torsor. Define the right $\overline{G}$-torsor $\lambda(T)$ as the *mixing construction*

$$\lambda(T) = T \times_G \overline{G} = T \times \overline{G}/\sim \tag{1.49}$$

where the equivalence relation $\sim$ is

$$(t, \bar{g}) \sim (tg, \lambda(g)^{-1}\bar{g}), \qquad t \in T, \quad g \in G, \quad \bar{g} \in \overline{G}. \tag{1.50}$$

In other words, $\lambda(T)$ is the quotient of $T \times \overline{G}$ by the right G-action $(t, \bar{g}) \xrightarrow{g} (tg, \lambda(g)^{-1}\bar{g})$.

We invite the reader to visualize the mixing construction using Figure 5. We later encounter this construction fiber-by-fiber in a fiber bundle of torsors, i.e., in a principal bundle.

Think through this construction in terms of trivializations (1.5) of the torsor T. Show that a trivialization of T—a point $t_0 \in T$—identifies $\lambda(T)$ with $\overline{G}$. How does a change of trivialization of T change this identification?

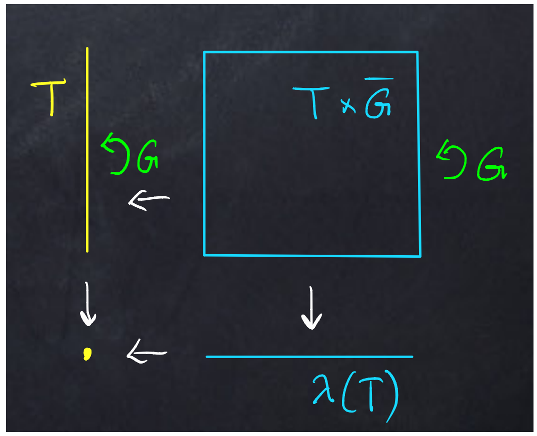

FIGURE 5. The mixing construction

(1.51) *Geometric structure on a vector space.* Given a symmetry type (G_n, λ_n), we specify the data of a geometric structure of type (G_n, λ_n) on an arbitrary n-dimensional vector space V.

Definition 1.52. Fix $n \in \mathbb{Z}^{\ge 0}$ and a symmetry type (G_n, λ_n). Let V be an n-dimensional vector space. A (G_n, λ_n)*-structure* on V is a pair $(\mathcal{B}_G(V), \phi)$ consisting of a right G_n-torsor $\mathcal{B}_G(V)$ and an isomorphism $\phi\colon \mathcal{B}(V) \xrightarrow{\cong} \lambda_n\bigl(\mathcal{B}_G(V)\bigr)$ of right $GL_n\mathbb{R}$-torsors.

In the case of Euclidean structure (Example 1.40) the data $(\mathcal{B}_G(V), \phi)$ is equivalent to a choice of positive definite inner product on V. In that case $\mathcal{B}_G(V)$ can be identified with the space of orthonormal bases.

(1.53) *Geometric structure on an affine space.* Here is the affine extension of Definition 1.52.

Definition 1.54. Fix a symmetry type (G_n, λ_n) and let $(\mathcal{G}_n, \tilde\lambda_n)$ be the induced affine symmetry type. Let A be an n-dimensional affine space. A $(\mathcal{G}_n, \tilde\lambda_n)$*-structure* on A is a pair $\bigl(\mathcal{B}_G(A), \tilde\phi\bigr)$ consisting of a right $\mathcal{G}_n$-torsor $\mathcal{B}_G(A)$ and an isomorphism $\tilde\phi\colon \mathcal{B}(A) \xrightarrow{\cong} \tilde\lambda_n\bigl(\mathcal{B}_G(A)\bigr)$ of right $\mathrm{Aff}_n\mathbb{R}$-torsors.

Remark 1.55. Recall Remark 1.29(5) in which we construct the principal bundle of frames of affine space. Now, to a $(\mathcal{G}_n, \tilde\lambda_n)$-structure $\bigl(\mathcal{B}_G(A), \tilde\phi\bigr)$ on an affine space A, let

$$\pi_G\colon \mathcal{B}_G(A) \longrightarrow A \tag{1.56}$$

be the quotient map of the right G_n-action on $\mathcal{B}_G(A)$. (This action uses the splitting described immediately preceding Remark 1.39.) The map $\tilde\phi$ associates to each point $f_G \in \mathcal{B}_G(A)$ a frame based at the point $\pi_G(f_G) \in A$. The map π_G is a principal G_n-bundle.

Example 1.57 (Euclidean structure)**.** For the symmetry type in Example 1.40 we say an affine space equipped with an O_n-structure is a *Euclidean space*. (We sometimes abbreviate '(G_n, λ_n)-structure' by 'G_n-structure' if λ_n is unmistakable.) In this case a point of $\mathcal{B}_O(A)$ in the fiber over $p \in A$ is an orthonormal frame of the tangent space $T_pA = V$, and (1.56) is called the *bundle of orthonormal frames*.

Appendix: group extensions

We include this appendix for reference. We do not comment further on topology, but say once and for all that for topological groups all group homomorphisms are assumed continuous, and for Lie groups all group homomorphisms are assumed smooth.

Definition 1.58. A *group extension* is a sequence of group homomorphisms

$$1 \longrightarrow G' \xrightarrow{\;i\;} G \xrightarrow{\;\pi\;} G'' \longrightarrow 1 \tag{1.59}$$

that is exact in the sense that the kernel of any homomorphism equals the image of the preceding homomorphism. We call G' the *kernel* and G'' the *quotient*.

Exactness at G' implies that i is injective. The inclusion i identifies G' with its image, which is a subgroup of G. At G the exactness implies that π factors through an injective map of the quotient group G/G' into G'', and exactness at G'' implies that this is an isomorphism. We use it to identify G'' as this quotient.

There is a category of group extensions with fixed kernel and quotient.

Definition 1.60. Let G', G'' be groups and G^{τ_1}, G^{τ_2} group extensions with kernel G' and quotient G''. A *morphism of group extensions* G^{τ_1}, G^{τ_2} is a group homomorphism $\varphi\colon G^{\tau_1} \to G^{\tau_2}$ which fits into the commutative diagram

$$\begin{tikzcd} & & G^{\tau_1} \arrow[dd, "\varphi"] \arrow[dr] & & \\ 1 \arrow[r] & G' \arrow[ur] \arrow[dr] & & G'' \arrow[r] & 1 \\ & & G^{\tau_2} \arrow[ur] & & \end{tikzcd} \tag{1.61}$$

As usual, φ is an *isomorphism* if there exists a homomorphism $\psi\colon G^{\tau_2} \to G^{\tau_1}$ of group extensions so that the compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps, which is simply equivalent to the condition that φ be an isomorphism of groups..

There is a notion of a trivialization of a group extension.

Definition 1.62. A *splitting* of the group extension (1.59) is a homomorphism $j\colon G'' \to G$ such that $\pi \circ j = \mathrm{id}_{G''}$.

Not every group extension is split: for example, the cyclic group of order 4 is a nonsplit extension of the cyclic group of order 2 by the cyclic group of order 2:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{1.63}$$

Definition 1.64. Suppose (1.59) is a group extension and $\rho''\colon \widetilde{G}'' \to G''$ a group homomorphism. Then there is a *pullback* group extension with kernel G' and quotient $\widetilde{G}''$ which fits into the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G' \arrow[r] \arrow[d, equal] & \widetilde{G} \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho"] & \widetilde{G}'' \arrow[r] \arrow[d, "\rho''"] & 1 \\ 1 \arrow[r] & G' \arrow[r] & G \arrow[r, "\pi"] & G'' \arrow[r] & 1 \end{tikzcd} \tag{1.65}$$

It is defined by setting

$$\widetilde{G} = \big\{(g, \tilde{\gamma}'') \in G \times \widetilde{G}'' : \pi(g) = \rho''(\tilde{\gamma}'')\big\}. \tag{1.66}$$