MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY
HOMEWORK 9
DUE: TUESDAY, APRIL 18 AT 12:00AM (MIDNIGHT) ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: James Splitting of $\Sigma\Omega S^{2n+1}$ (15 points)

(a) Given two path-connected pointed spaces X and Y, prove that there is a splitting
$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$
(Hint: use the pinch map $\Sigma Z \to \Sigma Z \vee \Sigma Z$.)[1]

(b) Composition of loops $\Omega S^{2n+1} \times \Omega S^{2n+1} \to \Omega^{2n+1}$ makes $H_*(\Omega S^{2n+1})$ into a ring. Prove that $H_*(\Omega S^{2n+1}) \cong \mathbb{Z}[x_{2n}]$. (Hint: the product $H_*(\Omega S^{2n+1}) \otimes H_*(\Omega S^{2n+1}) \to H_*(\Omega S^{2n+1}$ is dual to a coassociative and counital coproduct $H^*(\Omega S^{2n+1}) \to H^*(\Omega S^{2n+1}) \otimes H^*(\Omega S^{2n+1})$ which itself is a map of rings.)

(c) Prove that $\Sigma\Omega S^{2n+1} \simeq \bigvee_{k=1}^{\infty} S^{2kn+1}$.[2]

2. Problem 2: EHP sequence (15 points)

(a) Using 1(c), construct a map $H : \Omega S^{2n+1} \to \Omega S^{4n+1}$ which induces an isomorphism in H_{4n}.

(b) Let $E : S^{2n} \to \Omega S^{2n+1}$ denote the adjoint to the identity on S^{2n+1}. Using the Serre spectral sequence, prove that
$$S^{2n} \xrightarrow{E} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{4n+1}$$
is a mod $\mathcal{C}_2$-fiber sequence, i.e. that the map $S^{2n} \to F$ induces a mod $\mathcal{C}_2$-isomorphism on homotopy groups.[3] The induced long exact sequence of mod $\mathcal{C}_2$ homotopy groups is called the *EHP sequence*.

3. Problem 3: Cohomology of $V_2(\mathbb{R}^n)$ (10 points)

Let $V_2(\mathbb{R}^n)$ denote the space of pairs (x_1, x_2) of orthonormal vectors in $\mathbb{R}^n$.

(a) Identify the map $V_2(\mathbb{R}^n) \to S^{n-1}$ which sends $(x_1, x_2) \mapsto x_1$ with the unit sphere bundle associated to the tangent bundle of S^{n-1}.

(b) Using the fact that $\langle e(TM), [M]\rangle = \chi(M)$, compute the cohomology rings $H^*(V_2(\mathbb{R}^n); \mathbb{F}_2)$ and $H^*(V_2(\mathbb{R}^n); \mathbb{Z})$.

4. Problem 4: Cohomology of G_2 (10 points)

The exceptional Lie group G_2 lies in a fiber sequence
$$S^3 \to G_2 \to V_2(\mathbb{R}^6).$$
Compute the integral and mod 2 cohomology groups of G_2 using the Serre spectral sequence. Explain why the Serre spectral sequence is unable to uniquely determine the ring structure without some additional input.

---

[1] The statement is true without the path-connected assumption, but including this assumption simplifies the proof by allowing you to use homology Whitehead.

[2] The same results are all true for ΩS^{2n}, but the proof requires a little bit more work, since the rationalization of $H^*(\Omega S^{2n+1})$ is no longer generated by a single element.

[3] Again, the same result is true for ΩS^{2n}. There is also an odd-primary version, which is a bit more complicated.