Math 132 Problem Set 1

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Problem 1. Suppose that $f : X \to Y$ is a diffeomorphism. Prove that at each $x \in X$, the derivative

$$df_x : T_xX \to T_{f(x)}Y$$

is an isomorphism of tangent spaces.

Since f is a diffeomorphism, it must have some smooth inverse $g : Y \to X$. For any $x \in X$, consider the linear map $dg_{f(x)} : T_{f(x)}Y \to T_xY$. Since $g \circ f = \mathrm{id}_X$, and $d(\mathrm{id}_X)_x = \mathrm{id}_{T_xX}$, we have

$$dg_{f(x)} \circ df_x = \mathrm{id}_{T_xX}$$

by the chain rule. A similar argument shows that

$$df_x \circ dg_{f(x)} = \mathrm{id}_{T_{f(x)}Y}.$$

Since it has a linear inverse, it follows that df_x is an isomorphism.

Problem 2. Write elements of $\mathbb{R}^{2n}$ as $n \times 2$ matrices, which you should think of as pairs of column vectors $[v_1, v_2]$. With this in mind, consider the set $V \subset \mathbb{R}^{2n}$ of orthonormal pairs $[v_1, v_2]$. By definition, this means the pairs $[v_1, v_2]$ satisfying $v_1 \cdot v_2 = 0, v_1 \cdot v_1 = 1, v_2 \cdot v_2 = 1$. This turns out to be a smooth manifold known as a *Stiefel manifold*. Can you guess the dimension of this manifold?

More generally, there is a Stiefel manifold of orthonormal k-tuples $[v_1, \dots, v_k]$ of vectors $v_i \in \mathbb{R}^n$. It is naturally a subspace of $\mathbb{R}^{nk}$. Can you guess the dimension of this manifold?

We'll do the general case. Let's denote the Stiefel manifold of orthonormal k-tuples in $\mathbb{R}^n$ as $S_{n,k}$, and for now we can give this the subspace topology as a subspace of $\mathbb{R}^{nk}$. We can then consider $S_{n,k}$ as the locus of the following system of (nonlinear) equations:

$$\begin{cases} v_{i,1}^2 + \cdots + v_{i,n}^2 = 1, & 0 \le i < k, \\ v_{i,1}v_{j,1} + \cdots + v_{i,n}v_{j,n} = 0, & 0 \le i < j < k. \end{cases}$$

The first equation carves out a manifold homeomorphic to $S^{n-1} \times \mathbb{R}^{nk-n}$, notably one with codimension 1 in $\mathbb{R}^{nk}$. Similarly, the second equation carves out some more complicated space, but also of codimension one, since $v_{i,1}$ can be expressed in terms of the other variables. Since we have $\binom{k+1}{2}$ of these "independent" equations, and assuming their locuses intersect nicely, we can guess that:

$$\boxed{\dim S_{n,k} = nk - \binom{k+1}{2}} \implies \dim S_{n,2} = 2n - 3$$

Just as a sanity check, we note that this agrees with our intuition in the case when $k = 1$, since $S_{n,1}$ is homeomorphic to S^{n-1}. Similarly, we expect a symmetry $\dim S_{n,k} = \dim S_{n,n-k}$ since orthonormal k-tuples can be put into correspondence with their orthogonal complements. This also holds of the guessed formula.

Problem 3. Smooth functions.

This problem involves the definition of smooth functions.

a. Suppose that $f : M \to N$ is a function between smooth manifolds. Show that f is smooth if and only if for each smooth $g : N \to \mathbb{R}$ the composition $g \circ f$ is smooth.

The forward direction is clear, since the composition of two smooth functions is once again smooth. In the reverse direction, suppose that for each smooth $g : N \to \mathbb{R}$, the composition $g \circ f$ is smooth. To show this, we'll first prove the case when $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$.

Let's write $f = (f_1, \ldots, f_n)$. For any $0 \le i < n$, let $g_i : \mathbb{R}^n \to \mathbb{R}$ be the function that selects the ith coordinate. This is clearly a smooth function, so by assumption $f_i = g_i \circ f$ should also be a smooth function from $M \to \mathbb{R}$. This means that the partial derivatives $\partial f_i / \partial x_j$ exist for all $0 \le j < m$. Since we can do this for all $0 \le i < n$, it follows that all partial derivatives of f exist, hence it is smooth.

This immediately implies the claim when M is an open subset of $\mathbb{R}^m$ since smoothness is preserved by restriction. (i.e. a local property) To complete the proof, we can now work in the full generality and let M and N be k and ℓ dimensional smooth manifolds in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. To show that $f : M \to N$ is smooth, let $x \in M$ and let $U \subset \mathbb{R}^m$ be some open neighborhood of x in $\mathbb{R}^m$. Yet $f\big|_U : U \to \mathbb{R}^n$ must be smooth by the earlier argument since the "composition with g" condition is preserved by function restriction so we are done.

b. Suppose that M is a smooth manifold, $\{U_i\}$ is a covering of M by open subsets. Show that a function $f : M \to \mathbb{R}$ is smooth if and only if the restriction of f to each U_i is smooth.

The forward direction is clear since smoothness is preserved by restriction. Now conversely suppose that the restrictions $f\big|_{U_i}$ are smooth for all U_i. Then for any point $x \in M$, we can choose some U_i containing x, and since $f\big|_{U_i}$ is smooth, the function f is smooth at x so we are done.

c. The above result is not true if the condition that the U_i be open is dropped. Can you find a counterexample?

Let $M = \mathbb{R}$ and consider the covering of it by $(-\infty, 0), \{0\}, (0, \infty)$. Now let $f : M \to \mathbb{R}$ be given by $f(x) = x^{1/3}$. This is clearly smooth on $(0, \infty)$ and $(-\infty, 0)$. At $\{0\}$, it is trivially smooth, since we can take any smooth function passing through 0 to extend f locally. However as a whole, f isn't smooth because it has undefined first derivative at $x = 0$.

Problem 4. GP, Section 2, Problem 11.

Tangent spaces.

a. Suppose that $f : X \to Y$ is a smooth map, and let $F : X \to X \times Y$ be $F(x) = (x, f(x))$. Show that

$$dF_x(v) = (v, df_x(v)).$$

Clearly F is a smooth map because it is a product of smooth maps. (S1P14) By S2P9d, we have for any smooth $f : X \to A$ and $g : X \to B$ the relation

$$d(f \times g)_x = df_x \times dg_x.$$

Thus we have $dF_x = d(\mathrm{id})_x \times df_x$ and so $dF_x(v) = (v, df_x(v))$.

b. Prove that the tangent space to graph of f at the point $(x, f(x))$ is the graph of $df_x : T_x(X) \to T_{f(x)}(Y)$.

By S2P9a, we have an natural equality $T_{(x,y)}X \times Y = T_xX \times T_yY$ with similarly induced maps. We have an induced map of tangent spaces

$$dF_x : T_xX \to T_{(x,f(x))}X \times Y = T_xX \times T_{f(x)}Y.$$

Since this map is trivial on it's first component, and the tangent space at the graph of f at the point $(x, f(x))$ is exactly the image of this map, we have the desired result.

Problem 5. A *curve* in a manifold X is a smooth map $t \to c(t)$ of an interval of $\mathbb{R}$ into X. The *velocity vector* of the curve c at time t_0 (denoted $dc/dt(t_0)$) is defined to be the vector $dc_{t_0}(1) \in T_{x0(X)}$, where $x_0 = c(t_0)$, and $dc_{t_0} : \mathbb{R} \to T_{x_0}X$. In the case when $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t_0), \ldots, c_k'(t_0)).$$

Prove that every vector in T_xX is the velocity vector of some curve in X, and conversely.

In the first part, we have (without loss of generality) a smooth map $c : [0, 1] \to \mathbb{R}^k$. Thus by definition of derivative we get $dc/dt(t) = dc_t(1) = (\partial c_1/\partial t(t), \ldots, \partial c_k/\partial t(t))$ which is exactly what we want. For the second part, let's fix some point x, and chart $\psi_x : \mathbb{R}^k \to U_x \subset X$ around x, where ψ_x is a diffeomorphism and U_x is an open neighborhood of x in X.

Recall that the tangent space T_xX is defined as the image of $d(\iota \circ \psi_x)_x$ where $\iota : M \to \mathbb{R}^n$ is the canonical inclusion. Now for any vector $v \in T_xX$, let $v' \in \mathbb{R}^k$ be the preimage under this map, so that $d(\iota \circ \psi_x)_x(v') = v$. Now consider the curve $c_v : [-1, 1] \to M$ given by $c_v(t) = \psi_x(tv')$. Then $d(c_v)_0(1) = v$ by construction.