Math 132 Problem Set 8

Lev Kruglyak

Due: April 7, 2023

Problem 1. Suppose that $f_0 : M \to X$ and $f_1 : M \to X$ are homotopic maps. Show that f_0 and f_1 are cobordant when regarded as manifolds over X.

A homotopy between f_0 and f_1 is some map $H : M \times I \to X$ with $H(-,0) = f_0$ and $H(-,1) = f_1$. Notice that $M \times I$ is an $(n+1)$-manifold over X, with boundary $\partial M \times I = M \sqcup M$. Clearly $H|_{\partial(M\times I)}$ is a cobordism between f_0 and f_1 so we are done.

Problem 2. Suppose that $g : X \to Y$ is a continuous map of topological spaces. Show that the map sending $f : M \to X$ to $g \circ f : M \to Y$ induces a linear transformation $f_* : MO_n(X) \to MO_n(Y)$. Show that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are two composable maps then $(g \circ f)_* = g_* \circ f_*$, or in other words that the diagram

$$\begin{tikzcd} MO_d(X) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & MO_d(Y) \arrow[d, "g_*"] \\ & MO_d(Z) \end{tikzcd}$$

commutes.

This problem essentially amounts to proving that the pushforward $MO_d(-) : \mathbf{Top}^{co} \to \mathbf{Vect}_{\mathbb{F}_2}$ is a well-defined functor, where $\mathbf{Top}^{co}$ is the category of cobordism classes of topological manifolds. First, let's show that for any $f : X \to Y$ $MO_d(f)$ is well-defined on cobordism classes. Suppose $f_0 : A \to X$ and $f_1 : B \to X$ are cobordant d-manifolds over X, so there is some map $h : C \to X$ where C is a $(d+1)$-manifold, and there is a diffeomorphism $\iota/X : f_0 \sqcup f_1 \to \partial h$. Note that $\partial(f \circ h) = f \circ \partial h$ and $f \circ (f_0 \sqcup f_1) = (f \circ f_0) \sqcup (f \circ f_1)$, so we have a commutative diagram:

$$\begin{tikzcd} & \partial C \arrow[d, "\partial h"] \arrow[ddr, bend left, "\partial(f\circ h)"] & \\ A \sqcup B \arrow[ur, "\iota"] \arrow[r, "f_0 \sqcup f_1"'] \arrow[drr, bend right, "(f\circ f_0)\sqcup(f\circ f_1)"'] & X \arrow[dr, "f"] & \\ & & Y \end{tikzcd}$$

This proves that $(f \circ f_0)$ is cobordant to $(f \circ f_1)$ and so f_* is well defined. The above diagram also proves that $MO_d(f)$ is a linear map, since $MO_d(f)(f_0) + MO_d(f)(f_1) = MO_d(f)(f_0 \sqcup f_1) = MO_d(f)(f_0 + f_1)$, and $MO_d(-)$ is an $\mathbb{F}_2$-vector space. The preservation of identities is obvious, and composition follows since $MO_d(f)$ is a pushforward.

Problem 3. Suppose that M is a manifold. Show that every continuous function $f : \partial M \to \mathbb{R}$ extends to a continuous function $g : M \to \mathbb{R}$. Using this for every k, the map

$$MO_k(\mathbb{R}^n) \to MO_k(\mathrm{pt}) = MO_k$$

is an isomorphism. (Hint: Use a collar neighborhood.)

Suppose we are given $f : \partial M \to \mathbb{R}$. Let U be some collar neighborhood of ∂M in M, this comes with a diffeomorphism $\psi : U \to \partial M \times [0, 1)$, with $\psi \circ (1_{\partial M} \times 0)$ a the identity map. Consider the projections $\psi_{\partial M} = \pi_{\partial M} \circ \psi$ and $\psi_{[0,1)} = \pi_{[0,1)} \circ \psi$. Consider now the continuous $\alpha : U \to \mathbb{R}$ given by

$$g(x) = \max(0, 1 - 2\psi_{[0,1)}(x))(f \circ \psi_{\partial M})(x).$$

Let V be the complement of $\psi^{-1}(\partial M \times [0, 1/2])$ in M. Then let $\beta : V \to \mathbb{R}$ to be the zero function on V. Since $\alpha|_{\partial M} = f$, $U \cup V = M$, and $\alpha|_{U\cap V} = \beta|_{U\cap V} = 0$, it follows that we can extend f to some $g : M \to \mathbb{R}$. This also works for extending $f : \partial M \to \mathbb{R}^n$ to maps $f : M \to \mathbb{R}^n$ by the universal property of a product.

Let's now prove that $MO_k(\mathbb{R}^n) \to MO_k$ is an isomorphism. Recall that MO_k is a functor. We're interested in the induced map $\mathbb{R}^n \to \mathrm{pt}$. First of all, note that it has a left inverse given by the zero map $\mathrm{pt} \to \mathbb{R}^n$, and this gives a left inverse to $MO_k(\mathbb{R}^n) \to MO_k$. The right inverse follows because there is a cobordism $f : M \to \mathbb{R}^n$ with $0 : M \to \mathbb{R}^n$. Indeed, consider $f \sqcup 0 : \partial(M \times I) = M \sqcup M \to \mathbb{R}^n$. By the first part of the problem, let's extend this to some $g : M \times I \to \mathbb{R}^n$, which is a cobordism between f and 0. Thus $MO_k(\mathbb{R}^n) \to MO_k$.

Problem 4. Suppose that V_1, V_2, and V_3 are vector spaces over a field k. A sequence of linear transformations

$$V_1 \xrightarrow{p} V_2 \xrightarrow{q} V_3$$

is *exact* if the image of p is equal to the kernel of q. Note that this implies that the composition $q \circ p$ is zero. Show that in this situation one has $\dim V_2 \leq \dim V_1 + \dim V_3$.

By the rank-nullity theorem, we have $\mathrm{rank}\ p \leq \dim V_1$ and $\mathrm{rank}\ q \leq \dim V_3$, By exactness, we get $\mathrm{Im}\ p = \ker q$ so $\dim \ker q = \mathrm{rank}\ p$ so again by the rank-nullity theorem we get $\dim V_2 = \mathrm{rank}\ p + \mathrm{rank}\ q \leq \dim V_1 + \dim V_2$.