with the understanding that $\hat{f}(0) = 2$ and $\hat{g}(0) = 1$.

3. The following exercise illustrates the principle that the decay of $\hat{f}$ is related to the continuity properties of f.

(a) Suppose that f is a function of moderate decrease on $\mathbb{R}$ whose Fourier transform $\hat{f}$ is continuous and satisfies

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right) \quad \text{as } |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hölder condition of order α, that is, that

$$|f(x+h) - f(x)| \leq M|h|^{\alpha} \quad \text{for some } M > 0 \text{ and all } x, h \in \mathbb{R}.$$

(b) Let f be a continuous function on $\mathbb{R}$ which vanishes for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of the origin. Prove that $\hat{f}$ is not of moderate decrease. In fact, there is no $\epsilon > 0$ so that $\hat{f}(\xi) = O(1/|\xi|^{1+\epsilon})$ as $|\xi| \to \infty$.

[Hint: For part (a), use the Fourier inversion formula to express $f(x+h) - f(x)$ as an integral involving $\hat{f}$, and estimate this integral separately for ξ in the two ranges $|\xi| \leq 1/|h|$ and $|\xi| \geq 1/|h|$.]

4. Bump functions. Examples of compactly supported functions in $\mathcal{S}(\mathbb{R})$ are very handy in many applications in analysis. Some examples are:

(a) Suppose $a < b$, and f is the function such that $f(x) = 0$ if $x \leq a$ or $x \geq b$ and

$$f(x) = e^{-1/(x-a)}e^{-1/(b-x)} \quad \text{if } a < x < b.$$

Show that f is indefinitely differentiable on $\mathbb{R}$.

(b) Prove that there exists an indefinitely differentiable function F on $\mathbb{R}$ such that $F(x) = 0$ if $x \leq a$, $F(x) = 1$ if $x \geq b$, and F is strictly increasing on $[a, b]$.

(c) Let $\delta > 0$ be so small that $a + \delta < b - \delta$. Show that there exists an indefinitely differentiable function g such that g is 0 if $x \leq a$ or $x \geq b$, g is 1 on $[a+\delta, b-\delta]$, and g is strictly monotonic on $[a, a+\delta]$ and $[b-\delta, b]$.

[Hint: For (b) consider $F(x) = c\int_{-\infty}^{x} f(t)\,dt$ where c is an appropriate constant.]

5. Suppose f is continuous and of moderate decrease.

(a) Prove that $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

(b) Show that if $\hat{f}(\xi) = 0$ for all ξ, then f is identically 0.

[Hint: For part (a), show that $\hat{f}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}[f(x) - f(x - 1/(2\xi))]e^{-2\pi i x\xi}\,dx$. For part (b), verify that the multiplication formula $\int f(x)\hat{g}(x)\,dx = \int \hat{f}(y)g(y)\,dy$ still holds whenever $g \in \mathcal{S}(\mathbb{R})$.]

6. The function $e^{-\pi x^2}$ is its own Fourier transform. Generate other functions that (up to a constant multiple) are their own Fourier transforms. What must the constant multiples be? To decide this, prove that $\mathcal{F}^4 = I$. Here $\mathcal{F}(f) = \hat{f}$ is the Fourier transform, $\mathcal{F}^4 = \mathcal{F}\circ\mathcal{F}\circ\mathcal{F}\circ\mathcal{F}$, and I is the identity operator $(If)(x) = f(x)$ (see also Problem 7).

7. Prove that the convolution of two functions of moderate decrease is a function of moderate decrease.

[Hint: Write

$$\int f(x-y)g(y)\,dy = \int_{|y|\leq|x|/2} + \int_{|y|\geq|x|/2}.$$

In the first integral $f(x-y) = O(1/(1+x^2))$ while in the second integral $g(y) = O(1/(1+x^2))$.]

8. Prove that f is continuous, of moderate decrease, and $\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all $x \in \mathbb{R}$, then $f = 0$.

[Hint: Consider $f * e^{-x^2}$.]

9. If f is of moderate decrease, then

$$\int_{-R}^{R}\left(1 - \frac{|\xi|}{R}\right)\hat{f}(\xi)e^{2\pi i x\xi}\,d\xi = (f * \mathcal{F}_R)(x), \tag{14}$$

where the Fejér kernel on the real line is defined by

$$\mathcal{F}_R(t) = \begin{cases} R\left(\dfrac{\sin \pi t R}{\pi t R}\right)^2 & \text{if } t \neq 0, \\ R & \text{if } t = 0. \end{cases}$$

Show that $\{\mathcal{F}_R\}$ is a family of good kernels as $R \to \infty$, and therefore (14) tends uniformly to $f(x)$ as $R \to \infty$. This is the analogue of Fejér's theorem for Fourier series in the context of the Fourier transform.

10. Below is an outline of a different proof of the Weierstrass approximation theorem.