Math 231a Problem Set 10

Lev Kruglyak

December 7, 2022

Problem 1. Cohomology of projective space.

The goal of this problem is to compute $H^\bullet(\mathbb{RP}^n;\mathbb{F}_2)$ and $H^\bullet(\mathbb{RP}^n;\mathbb{Z})$ as rings.

a. In this problem you will prove that $H^\bullet(\mathbb{RP}^n;\mathbb{F}_2)\cong\mathbb{F}_2[x]/(x^{n+1})$, where $|x|=1$. Using induction on n, prove that this will follow if the cup product map

$$H^i(\mathbb{RP}^n;\mathbb{F}_2)\times H^j(\mathbb{RP}^n;\mathbb{F}_2)\xrightarrow{\smile} H^n(\mathbb{RP}^n;\mathbb{F}_2)$$

is nonzero for all $i,j\geq 0$ satisfying $i+j=n$.

Recall that $H^i(\mathbb{RP}^n;\mathbb{F}_2)\cong\mathbb{F}_2$ for all $0\leq i\leq n$. For all such i, let $\sigma_i\in H^i(\mathbb{RP}^n;\mathbb{F}_2)$ be the generator. To construct an isomorphism $\zeta: H^\bullet(\mathbb{RP}^n;\mathbb{F}_2)\to\mathbb{F}_2[x]/(x^{n+1})$, let $\zeta(\sigma_i)=x^i$ and extending linearly. Since $\smile$ is unital, this definition makes sense at σ_0 since $\zeta(\sigma_i\smile\sigma_0)=\zeta(\sigma_i)=\zeta(\sigma_i)\cdot\zeta(\sigma_0)$. Now since the cup product map is nonzero as described, we know that $\sigma_1\smile\sigma_{k-1}=\sigma_k$. This is actually true for all $k\leq n$ by induction and the canonical inclusion $\mathbb{RP}^{n-1}\subset\mathbb{RP}^n$. Since all the higher cohomology groups are zero, we get the aforementioned ring structure since $\sigma_i\smile\sigma_j=\sigma_{i+j \mod n+1}$.

b. Given $i,j\geq 0$ such that $i+j=n$, regard $\mathbb{RP}^i\subset\mathbb{RP}^n$ as the $[x_0:\cdots:x_n]\in\mathbb{RP}^n$ with $x_{i+1}=\cdots=x_n=0$. Similarly, regard $\mathbb{RP}^j\subset\mathbb{RP}^n$ as the $[x_0:\cdots:x_n]\in\mathbb{RP}^n$ with $x_0=\cdots=x_{i-1}=0$. Then $\mathbb{RP}^i\cap\mathbb{RP}^j=\{p\}$. Finally, regard $\mathbb{R}^n\subset\mathbb{RP}^n$ as elements of the form $[x_0:\cdots:x_{i-1}:1:x_{i+1}:\cdots:x_n]$.

Then there is a diagram of the form:

$$\begin{tikzcd}
H^i(\mathbb{RP}^n;\mathbb{F}_2)\times H^j(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\smile"] & H^n(\mathbb{RP}^n;\mathbb{F}_2) \\
H^i(\mathbb{RP}^n,\mathbb{RP}^n-\mathbb{RP}^j;\mathbb{F}_2)\times H^j(\mathbb{RP}^n,\mathbb{RP}^n-\mathbb{RP}^i;\mathbb{F}_2) \arrow[u] \arrow[d] \arrow[r, "\smile"] & H^n(\mathbb{RP}^n,\mathbb{RP}^n-\{p\};\mathbb{F}_2) \arrow[u] \arrow[d] \\
H^i(\mathbb{R}^n,\mathbb{R}^n-\mathbb{R}^j;\mathbb{F}_2)\times H^j(\mathbb{R}^n,\mathbb{R}^n-\mathbb{R}^i;\mathbb{F}_2) \arrow[r, "\smile"] & H^n(\mathbb{R}^n,\mathbb{R}^n-\{p\};\mathbb{F}_2)
\end{tikzcd}$$

Prove that the vertical maps are isomorphisms.

First let's prove a relevant lemma:

Claim. For any $i\leq n$, there is a deformation retraction $\mathbb{RP}^n-\mathbb{RP}^i$ to $\mathbb{RP}^{n-i-1}$ in $\mathbb{RP}^n$.

Proof. Consider the map $r:\mathbb{RP}^n-\mathbb{RP}^i\to\mathbb{RP}^{n-i-1}$ given by sending $[x_0:\cdots:x_{i-1}:\cdots:x_n]$ to $[x_{i-1}:\cdots:x_n]$. This is clearly well defined and continuous since these coordinates can't all be zero or else the input would be in $\mathbb{RP}^i$. The homotopy inverse can be given by the simple inclusion i sending $[x_{i-1} : \cdots : x_n]$ to $[0 : \cdots : 0 : x_{i-1} : \cdots : x_n]$.

To prove that this is a deformation retraction, we must show that $r \circ i$ is homotopic to the identity $\mathrm{id}_{\mathbb{RP}^n - \mathbb{RP}^i}$. This is easily done by the homotopy $H : (\mathbb{RP}^n - \mathbb{RP}^i) \times I \to \mathbb{RP}^n - \mathbb{RP}^i$ with $H(x,t) = [tx_0 : \cdots : tx_{i-1} : x_i : \cdots : x_n]$. $\square$

Now lets begin with the rightmost column. Recall by the lemma that $\mathbb{RP}^n - \{p\}$ is homotopy equivalent to $\mathbb{RP}^{n-1}$ so $H^n(\mathbb{RP}^n, \mathbb{RP}^n - \{p\}; \mathbb{F}_2) \cong H^n(\mathbb{RP}^n, \mathbb{RP}^{n-1}; \mathbb{F}_2)$ which in turn is isomorphic to $H^n(\mathbb{RP}^n; \mathbb{F}_2)$ by cellular homology. The bottom map is an isomorphism by the excision theorem on pairs.

For the leftmost column, notice that by the lemma we have a diagram

$$\begin{array}{ccccccc}
H^i(\mathbb{RP}^n) & \longleftarrow & H^i(\mathbb{RP}^n, \mathbb{RP}^{i-1}) & \longleftarrow & H^i(\mathbb{RP}^n, \mathbb{RP}^n - \mathbb{RP}^j) & \longleftarrow & H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^i(\mathbb{RP}^i) & \longleftarrow & H^i(\mathbb{RP}^i, \mathbb{RP}^{i-1}) & \longleftarrow & H^i(\mathbb{RP}^i, \mathbb{RP}^i - \{p\}) & \longleftarrow & H^i(\mathbb{R}^i, \mathbb{R}^i - \{p\})
\end{array}$$

where the $\mathbb{F}_2$ coefficients are omitted for brevity. All commutative squares involved clearly consist of isomorphisms, either by cellular homology or the lemma, or excision. Since we can do the same thing swapping i and j, this proves that the leftmost vertical maps are isomorphisms.

c. Prove that the bottom product is nonzero using the relative Künneth formula and universal coefficients theorem. Conclude that $H^\bullet(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$.

By Theorem 3.18 in Hatcher (equivalent to relative Künneth formula from the previous pset), we see that there is a ring isomorphism

$$H^\bullet(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j; \mathbb{F}_2) \otimes_R H^j(\mathbb{R}^n; \mathbb{R}^n - \mathbb{R}^i; \mathbb{F}_2) \to H^\bullet(\mathbb{R}^{2n}, (\mathbb{R}^n - \mathbb{R}^j) \times \mathbb{R}^i \cup \mathbb{R}^j \times (\mathbb{R}^n - \mathbb{R}^i); \mathbb{F}_2).$$

However it follows by elementary topology that the pair $(\mathbb{R}^{2n}, (\mathbb{R}^n - \mathbb{R}^j) \times \mathbb{R}^i \cup \mathbb{R}^j \times (\mathbb{R}^n - \mathbb{R}^i))$ is homotopy equivalent to $(\mathbb{R}^n, \mathbb{R}^n - \{p\})$. This proves part a of this problem.

To see why the ring structure is $\mathbb{F}_2[x]/(x^{n+1})$, notice that there is a single nonzero element in each $H^i(\mathbb{RP}^n; \mathbb{F}_2)$ for $i \leq n$, and $\sigma_1^{\smile i} = \sigma_1^{\smile i \bmod n+1}$ if we borrow notation from a. Thus the entire ring is generated over $\mathbb{F}_2$ by a single element σ_1 of order $n+1$, which we can call x.

d. Using the map $H^\bullet(\mathbb{RP}^n; \mathbb{Z}) \to H^\bullet(\mathbb{RP}^n; \mathbb{F}_2)$ induced by mod 2 reduction $\mathbb{Z} \to \mathbb{F}_2$, compute $H^\bullet(\mathbb{RP}^n; \mathbb{Z})$ as a ring.

Recall that the integral cohomology of real projective space is given by:

$$H^i(\mathbb{RP}^{2k+1}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n, \\ \mathbb{Z}/2\mathbb{Z} & i \text{ even}, i \leq 2k, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad H^i(\mathbb{RP}^{2k}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/2\mathbb{Z} & i \text{ even}, i \leq 2k, \\ 0 & \text{otherwise}. \end{cases}$$

We can see that there is an induced map $H^i(\mathbb{RP}^n; \mathbb{Z}) \to H^i(\mathbb{RP}^n; \mathbb{F}_2)$, and by naturality of the cup product, we get the cohomology rings

$$\begin{aligned}
H^\bullet(\mathbb{RP}^{2k}; \mathbb{Z}) &\cong \mathbb{Z}[x]/(2x, x^{k+1}), & |x| &= 2, \\
H^\bullet(\mathbb{RP}^{2k+1}; \mathbb{Z}) &\cong \mathbb{Z}[x,y]/(2x, x^{k+1}, y^2, xy), & |x| = 2, |y| &= 2k+1.
\end{aligned}$$

In the odd case, this extra generator comes from the nontrivial cochain in $H^{2k+1}(\mathbb{RP}^{2k+1}; \mathbb{Z})$.

Problem 2. An *algebra* structure on $\mathbb{R}^n$ is an $\mathbb{R}$-bilinear product map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which we denote $(a, b) \mapsto ab$. It is a *division algebra* structure if, for any fixed $a, b \in \mathbb{R}^n$, the maps $x \mapsto ax$ and $x \mapsto bx$ are bijections. Note that we do not assume that the product is commutative, unital, or even associative.

In this problem, you will use Problem 1 to prove the following theorem of Hopf: if $\mathbb{R}^n$ admits the structure of a division algebra, then n must be a power of 2.

a. Prove that if $\mathbb{R}^n$ is equipped with the structure of a division algebra, then the product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ induces a map $\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$.

For any nonzero $v \in \mathbb{R}^n$, let $[v] \in \mathbb{RP}^{n-1}$ be the projection. Let $\cdot$ be the product map on the division algebra structure on $\mathbb{R}^n$. Finally, consider the map $\widetilde{\times} : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$ given by $([v_1], [v_2]) \mapsto [v_1 \cdot v_2]$. This is clearly the canonical map obtained by passing to the quotients, and it's well defined because none of the v_i are zero.

b. Prove that the induced map $H^\bullet(\mathbb{RP}^{n-1}; \mathbb{F}_2) \to H^\bullet(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2)$ may be identified with the ring map $\mathbb{F}_2[x]/(x^n) \to \mathbb{F}_2[x_1, x_2]/(x_1^n, x_2^n)$ given by $x \mapsto x_1 + x_2$.

Recall that the Künneth theorem gives us an isomorphism

$$H^\bullet(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2) \cong H^\bullet(\mathbb{RP}^{n-1}; \mathbb{F}_2) \otimes H^\bullet(\mathbb{RP}^{n-1}; \mathbb{F}_2)$$

Using the identification of $H^\bullet(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2)$ with $\mathbb{F}_2[x]/(x^n)$, we get a canonical identification of $H^\bullet(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2)$ with $\mathbb{F}_2[x_1, x_2]/(x_1^n, x_2^n)$. Now let $\widetilde{\times}^\bullet$ be the induced ring homomorphism from $H^\bullet(\mathbb{RP}^{n-1}; \mathbb{F}_2) \to H^\bullet(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2)$. Such a homomorphism is determined by where x maps to, so we only need to see what happens at the first degree cohomology level.

Recall that $H^1(\mathbb{RP}^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2$ and $H^1(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. By the nondegeneracy conditions of the bilinear product, the induced map of $\widetilde{\times}$ on H^1 must be $1 \mapsto (1, 1)$, i.e. it can't be zero in either coordinate. This is because we can consider the induced inverse map in each coordinate. Thus the ring map is given by $x \mapsto x_1 + x_2$.

c. Prove that such a ring homomorphism can only exist when n is a power of 2.

This map is a homomorphism if and only if $(x_1 + x_2)^n \equiv x_1 + x_2 \mod 2$, which in turn is equivalent to $2 \mid \binom{n}{k}$ for all $0 < k < n$. By Lucas' theorem, $\nu_2\left(\binom{n}{k}\right) = s(k) + s(n-k) - s(n)$ where $s(\ell)$ is the sum of digits in the binary expansion of ℓ. Thus our condition can be rephrased as:

$$s(k) + s(n-k) \geq 1 + s(n) \quad \forall 0 < k < n.$$

This can be proven using induction to only hold when n is a power of 2.

Problem 3. Let X denote a space. Prove that if $X = U_1 \cup \cdots \cup U_n$ for contractible open sets $U_i \subset X$, then the cup product of n positive dimensional classes in $H^\bullet(X; R)$ is zero for any ring of coefficients R. As an example, conclude that the cup product of any two positive dimensional classes in $H^\bullet(SY; R)$ is zero, where SY is the suspension.

Recall that the relative cup product gives us a commutative diagram:

$$\begin{tikzcd} H^{p_1}(X, U_1; R) \times \cdots \times H^{p_n}(X, U_n; R) \arrow[r, "\smile"] \arrow[d] & H^p(X, U_1 \cup \cdots \cup U_n; R) \arrow[d] \\ H^{p_1}(X; R) \times \cdots \times H^{p_n}(X; R) \arrow[r, "\smile"] & H^p(X; R) \end{tikzcd}$$

where $p = p_1 + \cdots + p_n$. Notice that since U_i are contractible, we have isomorphisms $H^{p_i}(X, U_i; R) \cong H^{p_i}(X; R)$ by excision. Similarly, $U_1 \cup \cdots U_n = X$, so $H^p(X, U_1 \cup \cdots \cup U_n; R) \cong 0$. Thus the bottom map in the diagram is zero, meaning the n-fold cup product of any cochains is zero.

In the case of a suspension SX, we can chose contractible open sets C_+ and C_- to be the images of $X \times [0, 0.5 + \epsilon]/\sim$ and $X \times [0.5 - \epsilon, 1]/\sim$ for some small ϵ. Then by the problem, it follows that the cup product of any two positive dimensional classes is zero.