Math 55b Final Exam

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May 10, 2022

I affirm my awareness of the standards of the Harvard College Honor Code. While completing this exam, I have not consulted any external sources other than class notes and the textbooks. I have not discussed the problems or solutions of this exam with anyone, and will not discuss them until after the due date.

Signed: Lev Kruglyak

Problem 1. Let (X, d) be a compact metric space, and let $U \subset X \times X$ be an open subset of $X \times X$ (with the product topology) which contains the diagonal $\Delta = \{(p, p) \mid p \in X\}$.

(a) Show that there exists $\epsilon > 0$ such that U contains $V_\epsilon = \{(p, q) \in X \times X \mid d(p, q) < \epsilon\}$.

(b) Show that this conclusion may be false if we drop the assumption that X is compact.

(a) For each $x \in X$, let $\epsilon_x > 0$ be some radius such that $B_{\epsilon_x}(x, x) \subset U$. Such $\epsilon > 0$ is guaranteed to exist because $\Delta \subset U$ and using the definition of an open set. Thus we have an open cover of $X \times X$:

$$X \times X = (X \times X \setminus \Delta) \cup \bigcup_{x \in X} B_{\epsilon_x}(x, x).$$

(Δ is clearly closed so $X \times X \setminus \Delta$ is open.) Since $X \times X$ is compact, there must be some finite subcover which still covers the whole space. So we actually have some finite set of $x_1, \ldots, x_n \in X$ such that

$$X \times X = (X \times X \setminus \Delta) \cup \bigcup_{1 \leq i \leq n} B_{\epsilon_{x_n}}(x_n, x_n).$$

Notice that $\Delta \subset \bigcup_{1 \leq i \leq n} B_{\epsilon_{x_n}}(x_n, x_n) \subset U$. Let $\delta > 0$ be a Lebesgue number for this cover. Then by definition of a Lebesgue number, for any $x \in X$, $B_\delta(x, x)$ must lie inside one of the sets in the cover. So we have

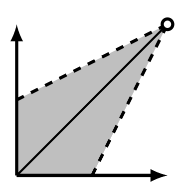
$$\Delta \subset \bigcup_{x \in X} B_\delta(x, x) \subset \bigcup_{1 \leq i \leq n} B_{\epsilon_{x_n}}(x_n, x_n) \subset U.$$

Finally we notice that for any point $(x, y) \in X \times X$, if $d(x, y) < \delta$, then $d((x, x), (x, y)) \leq \|(0, \delta)\| = \delta$. So $V_\delta \subset \bigcup_{x \in X} B_\delta(x, x) \subset U$ and we are done.

(b) Let $X = [0, 1)$ be a (non-compact) subset of the unit interval with the usual Euclidean metric. Consider the open subset $U \subset X \times X$ defined by

$$U = \{(x, y) \in X \times X \mid 2x - 1 < y < x/2 + 1/2\}.$$

Since $2x - 1 < x < x/2 + 1/2$ whenever $x \in [0, 1)$, it's clear that $\Delta \subset U$ so U satisfies the conditions of the problem. Geometrically, this set looks like the shaded gray region:



We claim that there does not exist an $\epsilon > 0$ such that $V_\epsilon \subset U$. To prove this, suppose for the sake of contradiction that there is some $\epsilon > 0$ such that $V_\epsilon \subset U$. Let $x = 1 - \epsilon$. Then

$$d\left(x, \frac{x}{2} + \frac{1}{2}\right) = \frac{x}{2} + \frac{1}{2} - x = -\frac{x}{2} + \frac{1}{2} \leq \frac{\epsilon}{2} < \epsilon$$

so $(x, x/2 + \frac{1}{2}) \in U$, which is a contradiction since $x/2 + 1/2 \not< x/2 + 1/2$.

Problem 2. Recall that a *retraction* of a topological space X onto a subspace $A \subset X$ is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. We consider the Möbius strip

$$X = (I \times I)/\sim \quad \text{where} \quad (0, y) \sim (1, 1-y) \quad \forall y \in I,$$

where $I = [0, 1]$, and its "boundary" $A = (I \times \{0, 1\})/\sim$ (the image in X of the two edges of the square that don't get identified with each other by the quotient map). Show that there does not exist a retraction of the Möbius strip X onto its boundary A.

Suppose for the sake of contradiction that there was a retraction of the Möbius strip M onto its boundary ∂M. Let H be the half circle $\{(x, 1/2) \mid x \in [0, 1]\}/\sim$ which goes around the center of the Möbius strip. We have the sequence of maps:

$$\partial M \xrightarrow{p} H \xrightarrow{i} M$$

where $p : \partial M \to H$ sends $(x, y) \mapsto (x, 1/2)$ and i is the obvious inclusion map. Since $(1, y)$ and $(0, 1-y)$ map to the same element under p, it is fairly easy to show that p is actually a 2-sheeted covering map, and since ∂M is path connected it follows that

$$[\pi_1(H, p(x_0)) : p^*(\pi_1(\partial M, x_0)] = |p^{-1}(p(x_0))| = 2$$

for any $x_0 \in \partial M$, so $\pi_1(\partial M)$ is an index 2 subgroup of $\pi_1(H)$. Next we can show that H is actually a deformation retract of M by the homotopy $H : M \times [0, 1] \to M$ given by $H((x, y), t) = (x, (1-t)y + t/2)$. So the inclusion map i induces an isomorphism $i^* : \pi_1(H) \to \pi_1(M)$. Thus $(i \circ p)^*(\pi_1(\partial M))$ is an index 2 subgroup of $\pi_1(M)$.

Now let $r : M \to \partial M$ be the retraction. By functoriality, it follows that $r^* : \pi_1(M) \to \pi_1(\partial M)$ is injective. So we have the maps

$$\pi_1(\partial M) \xrightarrow{i^* \circ p^*} \pi_1(M) \xrightarrow{r^*} \pi_1(\partial M).$$

Note that ∂M is a circle, and M deformation retracts onto the half circle H, so the fundamental groups $\pi_1(\partial M)$ and $\pi_1(M)$ are both infinite cyclic groups, i.e. isomorphic to $\mathbb{Z}$. Let $b \in \pi_1(\partial M)$ and $m \in \pi_1(M)$ be generators. Then $(i \circ p)^*(b) = \pm 2m$ since it is an index 2 subgroup. Since $r \circ i = \mathrm{id}_{\partial M}$, we have $r^* \circ i^* = \mathrm{id}_{\partial M}$, so $r^* \circ i^* \circ p^*(b) = p^*(b) = b$ but also equals $r^*(\pm 2m) = \pm 2r^*(m)$. Since $r^*(m) \in \pi_1(\partial M)$, we must have $r^*(m) = nb$ for some n. But then $\pm 2r^*(m) = \pm 2nb = b$ which is impossible, so we have our contradiction.

Problem 3. Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$, and for $k < n$ let S^k be the unit sphere in the $(x_1, \ldots, x_{k+1})$-coordinate plane, i.e.

$$S^k = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{k+2} = \cdots = x_{n+1} = 0\}.$$

Determine the fundamental group $\pi_1(S^n \setminus S^k)$.

We'll first prove a lemma relevant to this problem:

Lemma. For any $n > 0$ and $k < n$, we have the homotopy equivalence

$$S^n \setminus S^k \simeq S^{n-k-1}$$

where S^k is embedded into S^n in the obvious way as described in the problem.

Proof. In this proof, it will be useful to make a distinction between different ways of embedding S^k into S^n. So let $\iota_1 : S^k \to S^n$ be defined in the same way as in the problem, i.e. $\iota_1(x_1, \dots, x_{k+1}) = (x_1, \dots, x_{k+1}, 0, \dots, 0) \in S^n$. Next, we will consider the embedding $\iota_2 : S^{n-k-1} \to S^n$ given by $\iota_2(x_1, \dots, x_{n-k}) = (0, \dots, 0, x_1, \dots, x_{n-k})$. (i.e. the first $k+1$ entries are zero) We claim that $\iota_2(S^{n-k-1})$ is a deformation retraction of $S^n \setminus \iota_1(S^k)$. (Note that $\iota_2(S^{n-k-1}) \cap \iota_1(S^k) = \varnothing$. Consider the homotopy $F : (S^n \setminus \iota_1(S^k)) \times [0,1] \to S^n \setminus \iota_1(S^k)$ given by

$$F((x_1, \dots, x_{n+1}), t) = \frac{((1-t)x_1, \dots, (1-t)x_{k+1}, x_{k+2}, \dots, x_{n+1})}{\|((1-t)x_1, \dots, (1-t)x_{k+1}, x_{k+2}, \dots, x_{n+1})\|}.$$

To see why this function is continuous, note that the only way the fraction would be undefined/discontinuous would be if $\|((1-t)x_1, \dots, (1-t)x_{k+1}, x_{k+2}, \dots, x_{n+1})\| = 0$. If $t \neq 1$, this would only happen if $x_1 = x_2 = \dots = x_{n+1} = 0$, which is impossible since $\sum_{i=1}^{n+1} x_i^2 = 1$. If $t = 1$, then

$$\|((1-t)x_1, \dots, (1-t)x_{k+1}, x_{k+2}, \dots, x_{n+1})\| = \|(0, \dots, 0, x_{k+2}, \dots, x_{n+1})\| \neq 0$$

because not all $x_{k+2}, \dots, x_{n+1}$ can be zero. (Recall that our input space is $S^n \setminus \iota_1(S^k)$. So this is a continuous, defined map. To see why its a homotopy, note that $F((x_1, \dots, x_{n+1}, 0) = (x_1, \dots, x_{n+1})$, and $F((x_1, \dots, x_{n+1}), 1) = \iota_2(x_{k+2}, \dots, x_{n+1})$. Lastly, $F(\iota_2(x_1, \dots, x_{n-k}), t) = \iota_2(x_1, \dots, x_{n-k})$ for any $t \in [0,1]$. This completes the proof of the deformation retraction, which is a homotopy equivalence of the spaces in question. $\square$

Since homotopy equivalences preserve fundamental groups, we have the relation $\pi_1(S^n \setminus S^k) \cong \pi_1(S^{n-k-1})$. (This is a slight abuse of notation since these spaces are not path connected in the case when $n = k+1$, in this case, the space is homotopy equivalent to two points so its trivial irrespective of chosen basepoint.) Since $\pi_1(S^k) = \mathbb{Z}$ only if $k = 1$ and trivial otherwise, we thus have the final form:

$$\pi_1(S^n \setminus S^k, x_0) \cong \begin{cases} \{*\} & n \neq k+2 \\ \mathbb{Z} & n = k+2 \end{cases}.$$

where x_0 is an arbitrary basepoint.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function, i.e. f is differentiable and f' is continuous.

(a) Show that, for every $\epsilon > 0$ and $M > 0$, there exists $\delta > 0$ such that

$$|f(x+t) - (f(x) + tf'(x))| \leq \epsilon |t| \qquad \text{for all } x \in [-M, M] \text{ and all } t \in (-\delta, \delta).$$

(b) Show that there exist constants $c_n > 0$ (independent of f) such that the sequence of functions

$$g_n(x) = c_n \int_{-\pi/n}^{\pi/n} f(x+t) \sin(nt)\, dt$$

converges to $f'(x)$, uniformly on every bounded interval $[-M, M] \subset \mathbb{R}$.

(a) Since f is C^1, by the definition of differentiability we have

$$\lim_{t\to 0} \frac{f(x+t)-f(x)}{t} = f'(x),$$

and this limit converges at every point. If we restrict $x \in [-M, M]$ to a compact interval, the limit uniformly converges. So for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$t \in (-\delta, \delta) \implies \left|\frac{f(x+t)-f(x)}{t} - f'(x)\right| < \epsilon$$
$$|f(x+t) - (f(x) - tf'(x))| \le \epsilon|t| \quad \forall x \in [-M, M].$$

This is exactly what we wanted.

(b) We claim that $c_n = n^2/2\pi$ work. Since the results from (a) hold over any compact interval $[-M, M]$, we can fix some arbitrary such interval as a domain for x. We'll begin by making a few observations. First note that

$$\int_{-\pi/n}^{\pi/n} (f(x) + tf'(x))\sin(nt)\,dt = f(x)\underbrace{\int_{-\pi/n}^{\pi/n} \sin(nt)\,dt}_{0} + f'(x)\int_{-\pi/n}^{\pi/n} t\sin(nt)\,dt = f'(x)\frac{2\pi}{n^2} = \frac{f'(x)}{c_n}$$

Now for the next part let's prove a useful lemma.

Lemma. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, n a positive integer, with $|g(t)| < \epsilon|t|$ for all $t \in (-\pi/n, \pi/n)$ and some $\epsilon > 0$. Then

$$\left|\int_{-\pi/n}^{\pi/n} g(t)\sin(nt)\,dt\right| \le \frac{2\pi}{n^2}\epsilon.$$

Proof. A simple calculation shows that:

$$\left|\int_{-\pi/n}^{\pi/n} g(t)\sin(nt)\,dt\right| \le \int_{-\pi/n}^{\pi/n} |g(t)\sin(nt)|\,dt = \int_0^{\pi/n} |g(t)|\sin(nt)\,dt + \int_{-\pi/n}^{0} |g(t)|(-\sin(nt))\,dt$$
$$\le \epsilon\int_0^{\pi/n} t\sin(nt)\,dt + \epsilon\int_{-\pi/n}^{0} (-t)(-\sin(nt))\,dt = \frac{2\pi}{n^2}\epsilon.$$

This is exactly what we set out to prove. □

Recall from (a) that for any $\epsilon > 0$, there is a $\delta > 0$ with $|f(x+t) - (f(x) + tf'(x))| \le \epsilon|t|$ whenever $t \in (-\delta, \delta)$. Applying the lemma and the first observation for n such that $\pi/n < \delta$, we get

$$\left|\int_{-\pi/n}^{\pi/n} (f(x+t) - (f(x) + tf'(x)))\sin(nt)\,dt\right| \le \frac{2\pi}{n^2}\epsilon$$
$$\left|\int_{-\pi/n}^{\pi/n} f(x+t)\sin(nt)\,dt - \frac{f'(x)}{c_n}\right| \le \frac{\epsilon}{c_n}$$
$$\left|c_n\int_{-\pi/n}^{\pi/n} f(x+t)\sin(nt)\,dt - f'(x)\right| \le \epsilon.$$

So for all $n > \pi/\delta$, we know that the integral is within ϵ of $f'(x)$, and this ϵ is constant over $x \in [-M, M]$. So we have uniform convergence to $f'(x)$ over $[-M, M]$.

Problem 5. Let f be an analytic function on the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$, and suppose that the restriction of f to the real axis is an odd real-valued function, i.e. for all $t \in (-1, 1)$, $f(t) \in \mathbb{R}$ and $f(-t) = -f(t)$. Show that f takes imaginary values on the imaginary axis.

Since f is analytic over D, there must be a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where a_n are the Taylor coefficients. Notice that all $a_n \in \mathbb{R}$ because $a_n = f^{(n)}(0)/n!$ and all derivatives of f at zero must be real. (We can approach the origin from the real axis in the limit, and since the function is real valued the limit must be too.) Then since f restricted to $(-1, 1) \subset \mathbb{R}$ is odd, we have the relation

$$f(t) = -f(-t) = \sum_{n=0}^{\infty} (-1)^{n+1} a_n z^n = \sum_{n=0}^{\infty} a_n z^n \quad \forall t \in (-1, 1).$$

Since these two sums are equal over an infinite subset of D with an accumulation point, they must be equal over the entire unit disc D by the identity theorem. By a standard argument it follows that their coefficients must be equal since they are convergent power series centered at the same point. Then $a_n = (-1)^{n+1} a_n$ so $a_{2n} = 0$ for all n. This leaves only the odd terms so

$$f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}.$$

Using this form, for any $t \in (-1, 1)$ we have

$$f(it) = \sum_{n=0}^{\infty} a_{2n+1} (it)^{2n+1} = i \underbrace{\sum_{n=0}^{\infty} a_{2n+1} (-1)^n t^{2n+1}}_{\in \mathbb{R}}.$$

This last summation is real because $t \in \mathbb{R}$ and $a_{2n+1} \in \mathbb{R}$, so $f(it)$ is strictly imaginary.

Problem 6. Let $f(z) = \dfrac{\pi^3 \cos(\pi z)}{\sin^3(\pi z)}$.

(a) Give a formula expressing $f(z)$ as an infinite sum of rational functions. (Justify your answer).

(b) Calculate $\displaystyle\int_\gamma z^2 f(z)\, dz$, where γ is the circle of radius $\sqrt{55}$ centered at the origin.

(a) Recall from lecture that we have the infinite sum expansion:

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

This converges uniformly on any compact set not including a pole, so we can differentiate both sides to get:

$$-\frac{\pi^2 \cdot 2\pi \cos(\pi z)}{\sin^4(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{-2(z-n)}{(z-n)^4} \implies f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^3}$$

(b) Plugging in the expression from (a), we get

$$\int_\gamma z^2 f(z)\, dz = \int_\gamma z^2 \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}\, dz = \sum_{n \in \mathbb{Z}} \int_\gamma \frac{z^2}{(z-n)^2}\, dz.$$

Here we can swap the sum and integrals because the sum converges uniformly to $z^2 f(z)$ on γ. Then by the residue theorem, we have

$$\int_\gamma \frac{z^2}{(z-n)^3}\,dz = \begin{cases} 2\pi i \text{ Res}\left(\frac{z^2}{(z-n)^3}, n\right) & |n| < \sqrt{55} \\ 0 & \text{otherwise} \end{cases}.$$

We'll calculate the residue by applying the residue theorem again; so let $S^1(\epsilon, z)$ be the loop centered at $z \in \mathbb{C}$ of radius $\epsilon > 0$. Then

$$\begin{aligned}
\text{Res}\left(\frac{z^2}{(z-n)^3}\right) = \frac{1}{2\pi i}\int_{S^1(\epsilon,n)} \frac{z^2}{(z-n)^3}\,dz &= \frac{1}{2\pi i}\int_{S^1(\epsilon,0)} \frac{(z+n)^2}{z^3}\,dz \\
&= \frac{1}{2\pi i}\left(\int_{S^1(\epsilon,0)} \frac{1}{z}\,dz + 2n\int_{S^1(\epsilon,0)} \frac{1}{z^2} + n^2\int_{S^1(\epsilon,0)} \frac{1}{z^3}\right) \\
&= \text{Res}\left(\frac{1}{z}, 0\right) + 2n\text{Res}\left(\frac{1}{z^2}, 0\right) + n^2\text{Res}\left(\frac{1}{z^3}, 0\right) = 1.
\end{aligned}$$

Combining all of the results, we have

$$\int_\gamma z^2 f(z)\,dz = \sum_{n\in\mathbb{Z}} \int_\gamma \frac{z^2}{(z-n)^2}\,dz = \sum_{|n|<\sqrt{55}} 2\pi i = 30\pi i.$$

Problem 7. Assume $f(z)$ is analytic in the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$, and $|f(z)| < 1$ for all $z \in D$ (i.e., f maps D to itself). Show that if f has two fixed points (i.e., if there exist $a \neq b \in D$ with $f(a) = a$ and $f(b) = b$), then $f(z) = z$ for all $z \in D$. *(Hint: Schwarz lemma)*

Consider the function $\zeta : D \to D$ given by

$$\zeta(z) = \frac{z-a}{1-\bar{a}z}.$$

We've shown on a previous problem set that since $|a| < 1$, this is a biholomorphic map $D \to D$. Recall that

$$\zeta^{-1}(z) = \frac{z-a}{-\bar{a}z-1}.$$

Now consider $g(z) = \zeta(f(\zeta^{-1}(z)))$. Since all functions map $D \to D$, so does $g(z)$. Also $g(0) = \zeta(f(a)) = \zeta(a) = 0$. But since we have another fixed point b for f, it follows that $\zeta(b)$ is a fixed point for g since $\zeta(f(\zeta^{-1}(\zeta(b)))) = \zeta(f(b)) = \zeta(b)$. (Note that $\zeta(b) \neq 0$ because ζ is biholomorphic.) So $|g(\zeta(b))| = |\zeta(b)|$. Applying the Schwarz lemma on g, we get that $g(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$, but since $g(z)$ has a nonzero fixed point we can simply conclude that $g(z) = z$. Thus $\zeta^{-1} \circ g \circ \zeta = f$ must also be the identity, so $f(z) = z$ as desired.