# CONFIGURATION SPACES IN ALGEBRAIC TOPOLOGY: LECTURE 6

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Our goal in this lecture is to understand the cohomology ring $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ with integer coefficients. Since the cases $n \in \{0, 1\}$ are exceptional and easy, we assume throughout that $n \geq 0$.

**Definition.** Let $V$ be a degreewise finitely generated non-negatively graded Abelian group $V$. The *Poincaré polynomial* of $V$ is the polynomial

$$P(V) = \sum_{i \geq 0} \mathrm{rk}(V_i) t^i.$$

If $X$ is a space of finite type, the *Poincaré polynomial* of $X$ is the $P(X) := P(H_*(X))$.

The Poincaré polynomial for graded Abelian groups is additive under direct sum and multiplicative under tensor product, so the Poincaré polynomial for spaces is additive under disjoint union and, with with appropriate torsion-freeness assumptions in place, multiplicative under Cartesian product.

**Theorem** (Leray-Hirsch)**.** *Suppose that the diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ \mathrm{pt} \arrow[r] & B \end{tikzcd}$$

*is homotopy Cartesian and that*

1. *$F$ and $B$ are path connected,*
2. *$H^*(F)$ is free Abelian,*
3. *$H^*(F)$ or $H^*(B)$ is of finite type, and*
4. *$H^*(E) \to H^*(F)$ is surjective.*

*There is an isomorphism $H^*(E) \cong H^*(B) \otimes H^*(F)$ of $H^*(B)$-modules. In particular, we have the equation*

$$P(E) = P(B)P(F).$$

The proof, which is premised on a few basic properties of the Serre spectral sequence, is deferred to a later point in the course, at which we will discuss this tool in some detail.

In order to apply the Leray-Hirsch theorem, we must verify point (3). In doing so, we employ the *Gauss maps*

$$\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\gamma_{ab}} S^{n-1}$$
$$(x_1, \ldots, x_k) \mapsto \frac{x_b - x_a}{\|x_b - x_a\|}.$$

**Lemma.** *The natural map $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \to H^*(\mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\})$ is surjective.*

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*Proof.* We first construct a collection of maps $\varphi_a : S^{n-1} \to \mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\}$ inducing a homotopy equivalence from the bouquet $\bigvee_{k-1} S^{n-1}$. We will then show that each composite

$$S^{n-1} \xrightarrow{\varphi_a} \mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\} \subseteq \mathrm{Conf}_k(\mathbb{R}^n)) \xrightarrow{\gamma_{ka}} S^{n-1}$$

is the identity, implying that $H^*(\mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\})$ is generated by classes of the form $\gamma_{ka}^*(\alpha)$ for $\alpha \in H^*(S^{n-1})$.

For the construction, we set $x_a = (3^a, 0, \ldots, 0)$ for $1 \leq a \leq k-1$ and define $\varphi_a : S^{n-1} \to \mathbb{R}^n \setminus \{x_1, \ldots, x_k\}$ by

$$\varphi_a(v)_r = (3^a, 0, \ldots, 0) + 3^a v$$

where $v \in S^{n-1}$ is regarded as a unit vector in $\mathbb{R}^n$. Then $\varphi_a(-1, \ldots, 0) = (0, \ldots, 0)$ for $1 \leq a \leq k-1$, and the induced map from $\bigvee_{k-1} S^{n-1}$ is clearly a homotopy equivalence. Finally, we have

$$\gamma_{ka} \circ \varphi_a(v) = \frac{(3^a, 0, \ldots, 0) + 3^a v - (3^a, 0, \ldots, 0)}{\|(3^a, 0, \ldots, 0) + 3^a v - (3^a, 0, \ldots, 0)\|} = \frac{3^a v}{\|3^a v\|} = v.$$

□

Write $\alpha_{ab} \in H^{n-1}(\mathrm{Conf}_k(\mathbb{R}^n))$ for the pullback along $\gamma_{ab}$ of the standard volume form on $S^{n-1}$. We are now able to give a complete additive desription of the cohomology ring in terms of these generators.

**Corollary.** *For any $k \geq 0$, $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ is free with basis*

$$S_k = \{\alpha_{a_1 b_1} \alpha_{a_2 b_2} \cdots \alpha_{a_m b_m} : m \geq 0,\ 1 \leq b_1 < \cdots < b_m \leq k,\ a_\ell < b_\ell\}.$$

*In particular, the Poincaré polynomial is given by*

$$P(\mathrm{Conf}_k(\mathbb{R}^n)) = \prod_{j=1}^{k-1} (1 + j t^{n-1}).$$

*Proof.* Since $\mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\}$ and $\mathrm{Conf}_{k-1}(\mathbb{R}^n)$ are path connected and the cohomology of the former is free, Fadell-Neuwirth and the previous lemma allow us to apply the Leray-Hirsch theorem. The first and third claim now follow by induction and the observation that $H^*(\mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\})$ is free with Poincaré polynomial $1 + (k-1)t^{n-1}$. For the third claim, we note that, by induction, the Leray-Hirsch theorem gives the additive isomorphism

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \cong \mathbb{Z}\langle S_{k-1} \rangle \otimes \mathbb{Z}\langle 1,\ \alpha_{ak},\ 1 \leq a \leq k-1 \rangle,$$

and it is easy to check that the map

$$S_{k-1} \times \{1,\ \alpha_{ak},\ 1 \leq a \leq k-1\} \to S_k$$

given by concatenation on the right is a bijection. □

In order to obtain a multiplicative description, we require information about the relations among the various $\alpha_{ab}$. We begin with a few easy but useful observations. Recall that $\Sigma_k$ has a right action on $\mathrm{Conf}_k(\mathbb{R}^n)$ given by

$$(x_1, \ldots, x_k) \cdot \sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(k)}).$$

This action induces a left action on cohomology.

**Proposition.** *The following relations hold in $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ for $1 \leq a \neq b \leq k$.*

(1) $\alpha_{ab} = (-1)^n \alpha_{ba}$
(2) $\alpha_{ab}^2 = 0$
(3) $\sigma^* \alpha_{ab} = \alpha_{\sigma(a)\sigma(b)}$ *for $\sigma \in \Sigma_k$.*

*Proof.* The first claim follows from the observation that the diagram

$$\begin{tikzcd} & \mathrm{Conf}_2(\mathbb{R}^n) \arrow[dl, "\gamma_{ab}"'] \arrow[dr, "\gamma_{ba}"] & \\ S^{n-1} \arrow[rr, "x\mapsto -x"] & & S^{n-1} \end{tikzcd}$$

commutes, and that the degree of the antipodal map on $S^{n-1}$ is $(-1)^n$. The second follows form the fact that the volume form on $S^{n-1}$ squares to zero. For the third claim, we have the commuting diagram

$$\begin{tikzcd} \mathrm{Conf}_k(\mathbb{R}^n) \arrow[rr, "\sigma"] \arrow[dr, "\gamma_{\sigma(a)\sigma(b)}"'] & & \mathrm{Conf}_k(\mathbb{R}^n) \arrow[dl, "\gamma_{ab}"] \\ & S^{n-1}. & \end{tikzcd}$$

□

We come now to the fundamental relation.

**Proposition** (Arnold relation)**.** *For $1 \leq a < b < c \leq k$,*

$$\alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab} = 0.$$

*Remark.* The Arnold relation holds without the assumption $a < b < c$. Indeed, this form follows from the form we have proven and the antipodal relation $\alpha_{ab} = (-1)^n\alpha_{ba}$.

We will discuss three proofs of this relation. For the time being, we concentrate on exploiting it. We write $\mathcal{A}$ for the quotient of the free graded commutative algebra on generators $\{\alpha_{ab}\}_{1\leq a\neq b\leq k}$ by the ideal generated by $\{\alpha_{ab} + (-1)^{n+1}\alpha_{ba},\, \alpha_{ab}^2,\, \alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab}\}$.

**Theorem** (Arnold, Cohen)**.** *The natural map of graded commutative algebras*

$$\mathcal{A} \to H^*(\mathrm{Conf}_k(\mathbb{R}^n))$$

*is an isomorphism.*

*Proof.* The map is surjective, since its image contains a generating set, so it will suffice to show that the basis $S_k$ exhibited above, thought of as lying in $\mathcal{A}$, spans. Indeed, it follows that $S_k$ is a basis for $\mathcal{A}$, since any relation would map to a relation in $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$, and this fact implies the claim.

To verify that $S_k$ spans, it suffices to show that the element $\alpha_{a_1b_1}\cdots\alpha_{a_mb_m}$ may be written as a linear combination of elements in $S_k$. By the antipodal relation and graded commutativity, we may assume that $a_\ell < b_\ell$ for $1 \leq \ell \leq m$ and that $1 \leq b_1 \leq \cdots \leq b_m \leq k$. We proceed by downward induction on the largest value of $\ell$ such that $b_\ell = b_{\ell-1} =: b$. By the Arnold and antipodal relations, we have

$$\begin{aligned}\alpha_{a_1b_1}\cdots\alpha_{a_{\ell-1}b}\alpha_{a_\ell b}\cdots\alpha_{a_mb_m} &= (-1)^{n+1}\alpha_{a_1b_1}\cdots(\alpha_{a_\ell a_{\ell-1}}\alpha_{a_{\ell-1}b} + \alpha_{ba_\ell}\alpha_{a_\ell a_{\ell-1}})\cdots\alpha_{a_mb_m} \\ &= (-1)^{n+1}\alpha_{a_1b_1}\cdots\alpha_{a_\ell a_{\ell-1}}\alpha_{a_{\ell-1}b}\cdots\alpha_{a_mb_m} \\ &\quad + (-1)^{2n+1+(n-1)^2}\alpha_{a_1b_1}\cdots\alpha_{a_\ell a_{\ell-1}}\alpha_{a_\ell b}\cdots\alpha_{a_mb_m}\end{aligned}$$

After a second induction on the number of values of $\ell$ such that $b_\ell = b$, we may assume that $b_{l-2} < b$. Then, using our assumption that $a_\ell < b$ and $a_{\ell-1} < b$, these two monomials lie in the span of $S_k$ by the first induction. □

We now describe several approaches to proving the Arnold relation. The first reduction in all three cases is the observation that, using the projection $\mathrm{Conf}_k(\mathbb{R}^n) \to \mathrm{Conf}_3(\mathbb{R}^n)$ sending $(x_1, \ldots, x_k)$ to $(x_a, x_b, x_c)$, it suffices to verify the relation $\alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12} = 0$ in $H^*(\mathrm{Conf}_3(\mathbb{R}^n))$.

The first argument, due to Cohen [CLM76], is the most elementary, and we will be able to give a complete account, since it uses only techniques that we have already encountered.

*Cohen's proof of the Arnold relation.* We have already seen that $H^{2n-2}(\mathrm{Conf}_3(\mathbb{R}^n))$ is free of rank 2 with basis $\{\alpha_{12}\alpha_{23}, \alpha_{31}\alpha_{12}\}$, where we have used the antipodal relation and graded commutativity to rearrange indices in the second case. Note that another basis for this module is $\{\alpha_{12}\alpha_{23}, \alpha_{23}\alpha_{31}\}$, since the permutation $\binom{123}{321}$ interchanges this set with our known basis up to sign.

We conclude the existence of a relation

$$x\alpha_{12}\alpha_{23} + y\alpha_{23}\alpha_{31} + z\alpha_{31}\alpha_{12} = 0.$$

Applying $\tau_{12}$ to this relation, we obtain the relation

$$\begin{aligned} 0 &= x\alpha_{21}\alpha_{13} + y\alpha_{13}\alpha_{32} + z\alpha_{32}\alpha_{21} \\ &= (-1)^{(n-1)^2+2n}(x\alpha_{31}\alpha_{12} + y\alpha_{23}\alpha_{31} + z\alpha_{12}\alpha_{23}). \end{aligned}$$

Canceling the sign and subtracting the result from the known relation yields

$$(x-z)\alpha_{12}\alpha_{23} + (z-x)\alpha_{31}\alpha_{12} = 0,$$

whence $x = z$ by linear independence. Repeating the same process with $\tau_{23}$ shows that

$$(x-y)\alpha_{12}\alpha_{23} + (y-x)\alpha_{23}\alpha_{31} = 0,$$

whence $x = y$ by linear independence. We conclude that the expression in question is $x$-torsion and therefore zero, since $H^*(\mathrm{Conf}_3(\mathbb{R}^n))$ is torsion-free. □

The original proof, due to Arnold [Arn69], is of a very different flavor, but is only valid in its original form in dimension 2.

*Arnold's proof of the Arnold relation ($n = 2$).* Since there is no torsion, it suffices to prove the relation holds in cohomology with coefficients in $\mathbb{C}$. Make the identification $\mathbb{R}^2 \cong \mathbb{C}$. The class $\alpha_{ab}$ is obtained by pulling back a standard generator of $H^1(S^1)$ along the composite

$$\mathrm{Conf}_k(\mathbb{C}) \xrightarrow{(z_a, z_b)} \mathrm{Conf}_2(\mathbb{C}) \xrightarrow{z_2 - z_1} \mathbb{C}^\times \xrightarrow{\frac{z}{\|z\|}} S^1.$$

A representative for this generator in $H^1(\mathbb{C}^\times)$ is given by the differential form $dz/z$, since, by the residue theorem

$$\frac{1}{2\pi i}\int_{S^1} \frac{dz}{z} = 1.$$

Therefore, we may represent $\alpha_{ab}$ by the differential form

$$\omega_{ab} = \frac{dz_b - dz_a}{z_b - z_a}.$$

The claim now follows from the easy observation that the differential forms $\omega_{ab}$ satisfy the Arnold relation. □

This line of argument actually yields the far stronger result of a quasi-isomorphism

$$H^*(\mathrm{Conf}_k(\mathbb{C})) \xrightarrow{\sim} \Omega^*(\mathrm{Conf}_k(\mathbb{C}); \mathbb{C})$$

of differential graded algebras, where the cohomology is regarded as a chain complex with zero differential; in jargon, $\mathrm{Conf}_k(\mathbb{C})$ is *formal*.

In higher dimensions, the corresponding differential forms satisfy the relation only up to a coboundary, i.e., we have the equation

$$\omega_{12}\omega_{23} + \omega_{23}\omega_{31} + \omega_{31}\omega_{12} = d\beta.$$

Roughly, the differential form $\beta$ is obtained by integrating the form $\omega_{14}\omega_{24}\omega_{34}$ along the fibers of the projection $\pi : \mathrm{Conf}_4(\mathbb{R}^n) \to \mathrm{Conf}_3(\mathbb{R}^n)$ onto the first three coordinates. To see why this might be the case, we imagine that might imagine that a fiberwise version of Stokes' theorem should give that the boundary of the fiberwise integral should be the fiberwise integral along the "boundary" of the fiber, which in turn should be a sum of four terms: the first three terms are the loci where $x_i = x_4$ for $1 \leq i \leq 3$, and the fourth lies at infinity, where $x_4$ is very far away. We might imagine that the three terms in the Arnold relation arise from these first three terms and that the term at infinity vanishes.

Of course, the fiber of this projection is non-compact, so, in order to make this kind of reasoning precise, one must replace the configuration spaces with their *Fulton-MacPherson compactifications* $\mathrm{Conf}_k[\mathbb{R}^n]$, which is defined as the closure of the image of $\mathrm{Conf}_k(\mathbb{R}^n)$ under the maps

$$\mathrm{Conf}_k(\mathbb{R}^n) \to (\mathbb{R}^n)^k \times (S^{n-1})^{\binom{n}{2}} \times [0,\infty]^{\binom{n}{3}}$$

given by the inclusion in the first factor, the Gauss maps $\gamma_{ab}$ for $1 \leq a < b \leq k$ in the second, and the relative distance functions $\delta_{abc}(x_1, \ldots, x_k) = \frac{\|x_a - x_b\|}{\|x_a - x_c\|}$ for $1 \leq a < b < c \leq k$ in the third—see the original references [FM94, AS94] or the detailed account [Sin04]. It turns out that $\mathrm{Conf}_k[\mathbb{R}^n]$ is a manifold with corners on which the integration described above can actually be carried out.

Using this compactification, Kontsevich [Kon99] was able to carry out an analogue of Arnold's program from above. The basic observation is that the construction of $\beta$ is an example of a more systematic method for generating differential forms from graphs, which, when pursued fully, yields a zig-zag of quasi-isomorphisms

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \xleftarrow{\sim} \mathrm{Gph}_{n,k} \xrightarrow{\sim} \Omega^*(\mathrm{Conf}_k[\mathbb{R}^n]).$$

Thus, in higher dimensions, too, configuration spaces are formal. See [LV13] for a detailed proof of the formality theorem.

The third proof of the Arnold relation will proceed through a geometric, intersection-theoretic analysis following [Sin06]. In order to pursue this direction, we will need to understand something about the homology of $H_*(\mathrm{Conf}_k(\mathbb{R}^n))$, a task that will occupy our attention in the next lecture.

## References


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