Math 231a Problem Set 1

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Problem 1.

(a) Let $[n]$ denote the totally ordered set $\{0, 1, \cdots, n\}$. Let $\phi : [m] \to [n]$ be an order preserving function. Identifying $[n]$ with the vertices of Δ^n, we can extend ϕ to an affine map $\Delta^m \to \Delta^n$. Write out this map in barycentric coordinates.

(b) Write $d^j : [n-1] \to [n]$ for the order preserving injection that omits j. Show that an order preserving injection $\phi : [n-k] \to [n]$ is uniquely a composition of the form $d^{j_k} d^{j_{k-1}} \cdots d^{j_1}$ with $0 \le j_1 < j_2 < \cdots < j_k \le n$. Do this by describing integers $j_1, \ldots, j_k$ directly in terms of ϕ, and then verify the straightening rule

$$d^i d^j = d^{j+1} d^i \quad \text{for} \quad i \le j.$$

(c) Show that any order preserving map $\phi : [m] \to [n]$ factors uniquely as the composition of an order preserving surjection followed by an order preserving injection.

(d) Write $s^i : [m+1] \to [m]$ for the order preserving surjection which repeats the value of i. Show that any order preserving surjection $\phi : [m] \to [n]$ has a unique expression $(s^n)^{i_n} (s^{n-1})^{i_{n-1}} \cdots (s^0)^{i_0}$.

(e) Establish a straightening rule of the form $s^i d^j$.

(f) Write down the identities satisfied by these operators.

(g) Prove that $\partial^2 = 0$.

(a) Given an order preserving map $\phi : [m] \to [n]$, define $\phi(s_0, \cdots, s_m) = (t_0, \cdots, t_n)$ where

$$t_i = \sum_{j \in \phi^{-1}(i)} s_j.$$

It is clear that $\sum_{i=0}^n t_i = 1$, and letting e_j denote the n-th vertex of Δ^m, it follows that $\phi(e_j)$ is mapped to the $\phi(j)^{\text{th}}$ vertex of Δ^n, so this is the map we need. This map is affine since it is linear in s_i.

(b) Let $\zeta : [k] \to [n]$ be the unique order preserving map with $\mathrm{Im}(\zeta) \cap \mathrm{Im}(\phi) = \varnothing$. Define $j_i = \zeta(i)$. It follows by elementary induction that $d^{j_k} d^{j_{k-1}} \cdots d^{j_1} = \phi$. Uniqueness follows because such a decomposition is uniquely determined by ζ, i.e. the elements in the image which ϕ omits, and the actual decomposition can be constructed using induction. Checking the straightening rule is pretty easy, clearly $d^i d^j(x) = d^{j+1} d^i(x)$ for $x < i$ and for $x > j+1$, and a simple casework shows that it is true in the case when $i \le x \le j+1$.

(c) We'll construct surjective $f : [m] \to [k]$ and injection $g : [k] \to [n]$. Let $k = |\phi([m])| - 1$ and consider the recursive function given by $f(0) = 0$ and

$$f(k) = \begin{cases} f(k-1) & \text{if } \phi(k) = \phi(k-1) \\ f(k-1) + 1 & \text{otherwise} \end{cases}.$$

Clearly f is an order preserving surjection. Letting $g = \phi \circ f^{-1}$, we have our order preserving injection as well. These maps give a unique factoring directly from the universal property of surjective and injective maps.

(d) Firstly, we have the straightening rule $s^i s^j = s^{j-1} s^i$ for all $i < j$. Since $(s^i)^n$ represents the surjective map which send the n integers following i inclusive to i. We can then inductively build a decomposition $(s^n)^{i_n}(s^{n-1})^{i_{n-1}} \cdots (s^0)^{i_0}$. This is unique by the straightening rule.

(e) A straightforward checking gives us:

$$s^i d^j = \begin{cases} d^{j-1} s^i & \text{if } i < j-1 \\ \mathrm{Id} & \text{if } i = j \text{ or } j-1 \\ d^j s^{i-1} & \text{if } i > j \end{cases}$$

(f) Since $(fg)^* = g^* f^*$, we have the following identities:

$$d_j d_i = d_i d_{j+1} \text{ if } i \leq j, \quad s_j s_i = s_i s_{j-1} \text{ if } i < j, \quad \text{and} \quad d_j s_i = \begin{cases} s_i d_{j-1} & \text{if } i < j-1 \\ \mathrm{Id} & \text{if } i = j \text{ or } j-1 \\ s_{i-1} d_j & \text{if } i > j \end{cases}.$$

(g) Using all of the above identities:

$$\partial^2 = \sum_{i=0}^{n-1} (-1)^i d_i \left(\sum_{j=0}^{n} (-1)^j d_j \right) = \sum_{0 \leq j < i \leq n-1} (-1)^{i+j} d_i d_j + \sum_{0 \leq i < j \leq n} (-1)^{i+j} d_{j-1} d_i =$$
$$\sum_{0 \leq j < i \leq n-1} (-1)^{i+j} d_i d_j + \sum_{0 \leq i < j \leq n-1} (-1)^{i+j} d_i d_j = 0.$$

Problem 2. Construct an isomorphism:

$$H_n(X) \oplus H_n(Y) \to H_n(X \sqcup Y).$$

Consider the map $\phi : S_n(X) \oplus S_n(Y) \to S_n(X \sqcup Y)$ given by $\phi(\sigma, \omega) = i_X(\sigma) + i_Y(\omega)$, where i_X, i_Y are the natural inclusion maps. This restricts to a map $\phi : Z_n(X) \oplus Z_n(Y) \to Z_n(X \sqcup Y)$ since if $\partial\sigma = 0$ and $\partial\omega = 0$ then $\partial(i_X(\sigma) + i_Y(\omega)) = i_X(\partial\sigma) + i_Y(\partial\omega) = 0$. Furthermore, this map is surjective, because every chain $\zeta : \Delta^n \to X \sqcup Y$ can be expressed as a $i_X(\sigma) + i_Y(\omega)$ since Δ^n is connected. Next, we compose with the surjective quotient map $Z_n(X \sqcup Y) \to H_n(X \sqcup Y)$ to get a map $\Phi : Z_n(X) \oplus Z_n(Y) \to H_n(X \sqcup Y)$.

Let's calculate the kernel of this map. Suppose (σ, ω) are chains with $i_X(\sigma) + i_Y(\omega) = \partial\zeta$ for some $\zeta : \Delta^{n+1} \to X \sqcup Y$. Then by connectedness of Δ^{n+1}, we can write $\zeta = i_X(\sigma') + i_Y(\omega')$ for some $\sigma' \in Z_{n+1}(X)$ and $\omega' \in Z_{n+1}(Y)$. Then $i_X(\sigma) + i_Y(\omega) = \partial(i_X(\sigma') + i_Y(\omega'))$. Moving ∂ around and rearranging, we get $i_X(\sigma - \partial\sigma') = i_Y(\omega - \partial\omega')$. By connectedness, it follows that $\sigma = \partial\sigma'$ and $\omega = \partial\omega'$. So the kernel is $B_n(X) \oplus B_n(Y)$, and so by the first isomorphism theorem we get an isomorphism $\tilde{\Phi} : H_n(X) \oplus H_n(Y) \to H_n(X \sqcup Y)$.

Problem 3. Compute the homology groups of the following semisimplicial sets:

(a) The semisimplicial set T_* with underlying sets $T_0 = \{v\}, T_1 = \{a, b, c\}, T_2 = \{U, L\}$ and $T_n = \varnothing$ for $n \geq 3$, and face maps given by

$$d_0U = b,\ d_1U = c,\ d_2U = a$$
$$d_0L = a,\ d_1L = c,\ d_2L = b$$
$$d_ia = d_ib = d_ic = v \text{ for } i = 0, 1.$$

(b) The semisimplicial set K_* with underlying sets $K_0 = \{v\}, K_1 = \{a, b, c\}, K_2 = \{U, L\}$ and $T_n = \varnothing$ for $n \geq 3$, and face maps given by

$$d_0U = b,\ d_1U = c,\ d_2U = a$$
$$d_0L = a,\ d_1L = b,\ d_2L = c$$
$$d_ia = d_ib = d_ic = v \text{ for } i = 0, 1.$$

(c) Given any nonnegative integer n, the unique simplicial set $A[n]_*$ with $A[n]_k$ consisting of a single element for $k \leq n$ and $A[n]_k$ empty for $k > n$.

(a) From the definition of the semisimplicial set, it's clear that $H_n(T_*)$ is trivial for $n > 2$. We must then only check three cases:

1. $H_0(T_*)$: Here $\text{Im}(\partial_1) = \{1\}$ and $\text{Ker}(\partial_0) = \mathbb{Z}v$ so $H_0(T_*) = \mathbb{Z}$.
2. $H_1(T_*)$: In this case, $\text{Im}(\partial_2) = \mathbb{Z}(b-c+a)$ and $\text{Ker}(\partial_1) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$ so $H_1(T_*) = \langle a, b, c \mid b-c+a \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$.
3. $H_2(T_*)$: In this case, $\text{Im}(\partial_3) = \{1\}$ and $\text{Ker}(\partial_2) = \mathbb{Z}(U - L)$ so $H_2(T_*) \cong \mathbb{Z}$

Putting everything together, we thus have

$$H_n(T_*) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ \{1\} & \text{otherwise} \end{cases}.$$

(b) As before, we have $H_n(K_*)$ trivial for $n > 2$ and $\mathbb{Z}$ if $n = 0$, so we must only check two cases:

1. $H_1(K_*)$: In this case, $\text{Im}(\partial_2) = \mathbb{Z}(b-c+a) \oplus \mathbb{Z}(a-b+c) = \mathbb{Z}(a-b+c) \oplus \mathbb{Z}(2a)$ and $\text{Ker}(\partial_1) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$ so $H_1(K_*) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.
2. $H_2(K_*)$: Lastly, $\text{Im}(\partial_3) = \{1\}$ and $\text{Ker}(\partial_2) = \{1\}$, so $H_2(K_*) = \{1\}$.

To conclude,

$$H_n(K_*) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ \{1\} & \text{otherwise} \end{cases}.$$

(c) Here the boundary maps $\partial_k : \mathbb{Z}A[n]_k \to \mathbb{Z}A[n]_{k-1}$ are given by

$$\partial_k = \begin{cases} 0 & \text{if } k \text{ odd, or } k > n \\ 1 & \text{if } k \text{ even} \end{cases}.$$

Then the kernels and images are given by

$$\mathrm{Ker}(\partial_k) = \begin{cases} \mathbb{Z} & \text{if } k \text{ odd, or } k > n \\ 0 & \text{if } k \text{ even} \end{cases} \qquad \text{and} \qquad \mathrm{Im}(\partial_k) = \begin{cases} 0 & \text{if } k \text{ odd, or } k > n \\ \mathbb{Z} & \text{if } k \text{ even} \end{cases}$$

so the homology groups are

$$H_k(A[n]) = \mathrm{Im}(\partial_{k+1})/\mathrm{Ker}(\partial_k) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = n \text{ and } n \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$