Math 212 Problem Set 3

Lev Kruglyak

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Collaborators: *AJ LaMotta*

Problem 1. In the previous problem set, you were asked (in part) to show that if g is any $L^2(\mathbb{R}^n)$ function, then there exists a unique solution f in L^2 to the equation

$$(1-\Delta)f = g. \tag{1}$$

(a). Prove that the Fourier transform maps $L^2_2(\mathbb{R}^n)$ isometrically to the completion X of the space $C^\infty_c(\mathbb{R}^n)$ using the norm

$$\|u\|^2_X = \int_{\mathbb{R}^n} (|k|^2+1)^2 |\widehat{u}(k)|^2 \, dk_1 \cdots dk_n$$

Let's first identify X as a subspace of $L^2(\mathbb{R}^n)$, namely as the set of functions with $\|u\|_X < \infty$. By the work on the previous problem set, it follows that X is complete and $C^\infty_c(\mathbb{R}^n)$ is dense in X. Note that we can rewrite the norm as

$$\|u\|^2_X = \int_{\mathbb{R}^n} |\mathcal{F}(Lu)|^2 \, dk = \|\mathcal{F}(Lu)\|^2_{L^2} = \|Lu\|^2_{L^2}.$$

where $L = (1-\Delta)$, and the last equality follows because the Fourier transform is an L^2 isometry. However, we have

$$\begin{aligned}\|u\|^2_{2,2} = \|\nabla^2 u\|^2_2 + 2\|\nabla u\|^2_2 + \|u\|^2_2 &= \|\widehat{\nabla^2 u}\|^2_2 + 2\|\widehat{\nabla^2 u}\|^2_2 + 2\|\widehat{\nabla u}\|^2_2 + \|\widehat{u}\|^2_2 \\ &= \int_{\mathbb{R}^n} (|k|^2 + (k_i)^2(k^j)^2) + 1)|\widehat{u}(k)|^2 dk = \|u\|^2_X.\end{aligned}$$

If follows that the Fourier transform $C^\infty_c(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometric linear bounded map with respect to the L^2_2 norm on C^∞_c and norm on X. By the density of $C^\infty_c(\mathbb{R}^n)$ in $L^2_2(\mathbb{R}^n)$ and completeness of X, the Fourier transform extends to a linear isometry $L^2_2 \to X$.

(b). Prove that the assignment of $g \in L^2(\mathbb{R}^n)$ to the solution f of (1) defines an isometry from $L^2(\mathbb{R}^n)$ to $L^2_2(\mathbb{R}^n)$.

This is the inverse map L^{-1} given by $\mathcal{F}^{-1} \circ M \circ \mathcal{F}$ where M is the multiplication by $1/|k|^2+1$ map. The Fourier transform maps $L^2(\mathbb{R}^n)$ isometrically onto $L^2(\mathbb{R}^n)$, the operator M maps $L^2(\mathbb{R}^n)$ isometrically onto X by the previous part, and the inverse Fourier transform maps X isometrically onto $L^2_2(\mathbb{R}^n)$. Thus, it follows that L^{-1} is an isometric isomorphism of $L^2(\mathbb{R}^n)$ and $L^2_2(\mathbb{R}^n)$.

(c). Let $q(x)$ be at least a quadratic polynomial function, let $h_{\alpha\beta}$ be a 2×2 matrix of at least linear polynomial functions, and let k be a at least a quadratic polynomial function in 2 variables. Consider the equation

$$(1-\Delta)f + h_{\alpha\beta}(f)\partial^\alpha\partial^\beta f + k(\partial^1 f, \partial^2 f) + q(f) = g.$$

Given the polynomial function q, $h_{\alpha\beta}$ and k, there exists an $\varepsilon > 0$ and $C > 0$ such that the preceding equation has a unique solution in the L_2^2 Banach space with L_2^2 norm at most $C\varepsilon$ if the L^2 norm of g is less than ε.

By the previous parts, the inverse operator L^{-1} is an isometric isomorphic from $L^2(\mathbb{R}^2)$ onto $L_2^2(\mathbb{R}^2)$. Now define the nonlinear operator

$$\mathcal{N}(f) = h_{\mu\nu}(f)\partial^\mu\partial^\nu f + k(\partial_1 f, \partial_2 f) + q(f)$$

so that the differential equation can be written $L(f) + \mathcal{N}(f) = g$. Next, consider the linear operator $\mathcal{T} : L_2^2(\mathbb{R}^2) \to L_2^2(\mathbb{R}^2)$ given by

$$\mathcal{T}(f) = L^{-1}(g - \mathcal{N}(f)).$$

To solve the differential equation, we need a fixed point of $\mathcal{T}$. Since $f \in L_2^2(\mathbb{R}^2)$, its second derivatives are in $L^2(\mathbb{R}^2)$ and its first derivatives belong to $W^{1,2}(\mathbb{R}^2)$. In two dimensions, the Sobolev embedding implies that these first derivatives lie in some $L^p(\mathbb{R}^2)$ spaces for some $p > 2$. Consequently, products of the derivatives can be estimated in $L^2(\mathbb{R}^2)$. Thus, there exists a constant $C_1 > 0$ such that for sufficiently small f,

$$\|\mathcal{N}(f)\|_{L^2} \le C_1 \|f\|_{L_2^2}^2.$$

Moreover, a similar argument shows that $\mathcal{N}$ is locally Lipschitz. For each $f, h \in L_2^2(\mathbb{R}^2)$ with sufficiently small norms, there exists a constant $C_2 > 0$ such that

$$\|N(f) - N(h)\|_{L^2} \le C_2\Big(\|f\|_{L_2^2} + \|h\|_{L_2^2}\Big)\|f-h\|_{L_2^2}.$$

Now, let's choose a radius $R > 0$ and consider the closed ball $B_R = \{f \in L_2^2(\mathbb{R}^2) : \|f\|_{L_2^2} \le R\}$. We now choose R small enough so that for all $f, h \in B_R$ we have

$$C_2\Big(\|f\|_{L_2^2} + \|h\|_{L_2^2}\Big) \le 2C_2 R < 1.$$

Also, choose some $\varepsilon > 0$ so that if $\|g\|_{L^2} < \varepsilon$, then $\|L^{-1}(g)\|_{L_2^2} = \|g\|_{L^2} < R$. For any $f \in B_R$, we have

$$\|\mathcal{T}(f)\|_{L_2^2} = \|L^{-1}(g - \mathcal{N}(f))\|_{L_2^2} = \|g - \mathcal{N}(f)\|_{L^2}.$$

Since $\|\mathcal{N}(f)\|_{L^2} \le C_1\|f\|_{L_2^2}^2 \le C_1 R^2$, we obtain $\|\mathcal{T}(f)\|_{L_2^2} \le \|g\|_{L^2} + C_1 R^2 < R$ provided the norm $\|g\|_{L^2}$ is sufficiently small. Then, for any $f, h \in B_R$, we have

$$\|\mathcal{T}(f) - \mathcal{T}(h)\|_{L_2^2} = \|L^{-1}(\mathcal{N}(h) - \mathcal{N}(f))\|_{L_2^2} = \|\mathcal{N}(f) - \mathcal{N}(h)\|_{L^2}.$$

Using the local Lipschitz property of $\mathcal{N}$, we get

$$\|\mathcal{T}(f) - \mathcal{T}(h)\|_{L_2^2} \le C_2\Big(\|f\|_{L_2^2} + \|h\|_{L_2^2}\Big)\|f-h\|_{L_2^2} \le 2C_2 R\,\|f-h\|_{L_2^2}.$$

Since $2C_2 R < 1$, the mapping $\mathcal{T}$ is a contraction on B_R. This completes the proof.