Math 231a Problem Set 5

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Problem 1.

(a) Prove that complex projective space $\mathbb{CP}^n$ admits a CW structure in which $\mathrm{Sk}_{2k}\mathbb{CP}^n = \mathbb{CP}^k$ for any $0 \leq k \leq n$. Use this to compute the homology of $\mathbb{CP}^n$.

(b) Endow $\mathrm{Gr}_2(\mathbb{C}^4)$ with a CW structure and use this to compute its homology.

(a) Recall that $\mathbb{CP}^n$ is defined as

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - \{0\}\right)/\{z \sim \lambda z : \lambda \neq 0\}.$$

But then letting $U(1) \subset \mathbb{C}$ be the unitary group and considering S^{2k+1} as a subset of $\mathbb{C}^{k+1}$, we get isomorphisms

$$\begin{aligned}\mathbb{CP}^n \cong S^{2n+1}/U(1) &\cong \left\{\left(z, \sqrt{1-|z|^2}\right) \in \mathbb{C}^{n+1} : \|z\| \leq 1\right\}/\{(z,0) \sim \lambda(z,0) : \|z\| = 1, \lambda \neq 0\} \\ &\cong D^{2n}/\{z \sim \lambda z : z \in \partial D^{2n}, \lambda \neq 0\}.\end{aligned}$$

However since $\partial D^{2n}/U(1) = S^{2n-1}/U(1) \cong \mathbb{CP}^{n-1}$, it follows that $\mathbb{CP}^n$ is created from $\mathbb{CP}^{n-1}$ by attaching a $2n$-cell S^{2n-1} by the attachment map $\alpha : S^{2n-1} = \partial D^{2n} \to \partial D^{2n}/U(1) = \mathbb{CP}^{n-1}$.

To summarize this CW structure, we begin with a 0-cell, so $\mathbb{CP}^0 = *$. Then set $\mathrm{Sk}_{2k+1}\mathbb{CP}^n = \mathrm{Sk}_{2k}\mathbb{CP}^n$ for all $0 \leq k \leq n$, and $\mathrm{Sk}_{2k+2}\mathbb{CP}^n$ is the adjunction of $\mathrm{Sk}_{2k}\mathbb{CP}^n$ with $D^2 n$ by the previously mentioned attachment map $\alpha : \partial D^{2n} \to \mathrm{Sk}_{2k}\mathbb{CP}^n$.

Now the cellular chain complex of this CW structure is

$$0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0.$$

It follows that all of these maps must be trivial, so the homology groups are:

$$\boxed{H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & 0 \leq k \leq n \text{ and } k \text{ even}, \\ 0 & 0 \leq k \leq n \text{ and } k \text{ odd}, \\ 0 & k > n. \end{cases}}$$

(b) For the sake of sanity, we'll use the standard CW decomposition on the Grassmanian. Recall that a *Schubert symbol* is a sequence $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n$. Each of these symbols corresponds to a $d(\sigma)$-cell in a CW decomposition of $\mathrm{Gr}_k(\mathbb{C}^n)$ where $d(\sigma) = 2\sum_{i=1}^n (\sigma_i - i)$. It's easy to see that there are only 6 Schubert symbols in the case of $\mathrm{Gr}_2(\mathbb{C}^4)$, each of even degree so we have the homology:

$$\boxed{H_n(\mathrm{Gr}_2(\mathbb{C}^4)) = \begin{cases} \mathbb{Z} & n = 0, 2, 6, 8, \\ \mathbb{Z} \oplus \mathbb{Z} & n = 4, \\ 0 & \text{otherwise.} \end{cases}}$$

Problem 2. Describe a functor $|\cdot| : \mathbf{ssSet} \to \mathbf{Top}$ as follows. Given a semisimplicial set $X_\bullet$, we set:

$$|X_\bullet| = \frac{\coprod_n X_n \times \Delta^n}{(d_i x, y) \sim (x, d^i y)}.$$

The space $|X_\bullet|$ is called the *geometric realization* of $X_\bullet$.

(a) Given a semisimplicial set $X_\bullet$ and a nonnegative integer $n \geq 0$, define a new semisimplicial set

$$\mathrm{Sk}_n X_\bullet = \begin{cases} X_k & k \leq n, \\ \varnothing & k > n, \end{cases}.$$

where the face maps d_i are induced by those of $X_\bullet$.

Prove that $|X_\bullet|$ has a CW structure with $\mathrm{Sk}_n|X_\bullet| = |\mathrm{Sk}_n X_\bullet|$ and whose n-cells are induced by X_n.

(b) Let $S_*(X_\bullet)$ denote the semisimplicial chain complex of $X_\bullet$, and let $S_*(|X_\bullet|)$ denote the singular chain complex of $|X_\bullet|$. Define a natural injection of chain complexes $S_*(X_\bullet) \hookrightarrow S_*(|X_\bullet|)$ and prove that it induces an isomorphism on homology.

(a) For $n = 0$ we have $\mathrm{Sk}_0 X_\bullet = X_0$, and for $n \geq 1$ we have

$$\mathrm{Sk}_n X_\bullet = \frac{\coprod_{k=0}^n X_k \times \Delta^k}{(d_i x, y) \sim (x, d^i y)} = \frac{\mathrm{Sk}_{n-1} \sqcup X_n \times \Delta^n}{(d_i x, y) \sim (x, d^i y)} = \mathrm{Sk}_{n-1} \cup_{\alpha_n} X_n \times \Delta^n$$

where $\alpha_n : X_n \times \Delta^n \to \mathrm{Sk}_{n-1}$ is the canonical attachment map induced by the face maps d_i. Since X_n is a discrete space, $\Delta^n \cong D^n$, and $d^i \Delta^n \subset \partial \Delta^n$, we have a CW structure.

(b) Define the map $\alpha : S_*(X_\bullet) \hookrightarrow S_*(|X_\bullet|)$ by $\alpha_n(x) = [\{x\} \times \Delta^n]$ for any $x \in X_n$ and extending linearly. This is clearly a chain map because it commutes with the face maps, and hence the boundary operator as well. To prove that it induces an isomorphism on homology, let $\sigma : \Delta^n \to |X_\bullet|$ be a cycle. This map induces isomorphisms on homology by the cellular boundary formula.

Problem 3. Let $p, q \in \mathbb{Z}$, and let $X_{p,q}$ be the 2-dimensional CW complex obtained by attaching two 2-cells to S^1 using maps of degree p and q. Compute $\pi_1(X_{p,q})$ and $H_*(X_{p,q})$.

Let's begin by calculating the fundamental group using the Seifert van-Kampen theorem. First we'll compute $\pi_1(X_p)$ where X_p is the space obtained by attaching a single 1-cell to S^1 using a map of degree p. In other words, this space is the quotient of D^2 by some equivalence relation $\sim_p$ on ∂D^2. Let $A = A_\epsilon \subset D^2$ be some subset of radius ϵ, and let $B = D^2 - A_{\epsilon/2}$. We can naturally identify these as subsets of X_p. Then applying the Siefert van-Kampen theorem and picking a suitable basepoint $x \in A \cap B$ (omitted for clarity), we get a diagram

$$\begin{tikzcd}
 & \pi_1(B) \arrow[d, "f_B"] \arrow[dr] & \\
\pi_1(A \cap B) \arrow[ur, "i_B"] \arrow[dr, "i_A"'] & \pi_1(A) *_{\pi_1(A\cap B)} \pi_1(B) \arrow[r, dashed] & \pi_1(X_{p,q}) \\
 & \pi_1(A) \arrow[u, "f_A"'] \arrow[ur] &
\end{tikzcd}$$

Since A is contractible, it follows that $\pi_1(A, x) *_{\pi_1(A \cap B, x)} \pi_1(B, x) \cong \pi_1(B, x)/i_B(\pi_1(A \cap B, x))$. First we'll show that $\pi_1(B, x) \cong \mathbb{Z}$. First observe that B is canonically homotopy equivalent to S^1 by the map which lets $\epsilon/2 \to 1$, then applies α_p, the attachment map of degree p. Similarly, $A \cap B$ is canonically homotopy equivalent to S^1. Then i_B sends a generator $[\iota]$ of $\pi_1(A\cap B, x)$ to $[\alpha(\iota)] = p[\iota]$ in $\pi_1(B, x)$ so we get $\pi_1(X_{p,q}, x) \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p$.

Now we can consider $X_{p,q}$ as the attachment of X_p to X_q along the image of S^1 in X_p, X_q, i.e. $X_{p,q} = (X_p \sqcup X_q)/(\alpha_p \sqcup \alpha_q)$. Let $U_p, U_q \subset X_{p,q}$ be open sets which ϵ expand X_p and X_q respectively. The Seifert van-Kampen theorem then gives us the pushout

$$\begin{tikzcd} & \mathbb{Z}/p \arrow[d] \arrow[dr] & \\ \mathbb{Z} \arrow[ur, two heads] \arrow[dr, two heads] & \mathbb{Z}/p *_{\mathbb{Z}} \mathbb{Z}/q \arrow[r, dashed] & \pi_1(X) \\ & \mathbb{Z}/q \arrow[u] \arrow[ur] & \end{tikzcd}$$

The group presentation of $\pi_1(X)$ then becomes $\langle x, y \mid x^p = y^q = 1, x = y\rangle$ which is exactly the group $\mathbb{Z}/\gcd(p,q)$. Here we take the convention that $\mathbb{Z}/0 = \{1\}$ so that $\gcd(x, 0) = x$ and $\gcd(0, y) = y$. So

$$\boxed{\pi_1(X_{p,q}) \cong \mathbb{Z}/\gcd(p,q).}$$

Next let's compute the homology groups. Recall that our CW structure is: X_0 consists of a single point v_0, X_1 adds an edge e_0 looped at v_0, and X_2 adds two faces f_0, f_1 by maps e_0^p and e_0^q respectively. Recall that $C_n(X_{p,q}) = \mathbb{Z}I_n$ where I_n is the set of n-cells. Thus our chain complex is

$$0 \longleftarrow \mathbb{Z}v_0 \xleftarrow{\beta} \mathbb{Z}e_0 \xleftarrow{\alpha} \mathbb{Z}f_0 \oplus \mathbb{Z}f_1 \longleftarrow 0$$

By the cellular boundary formula, we get $\beta(e_0) = 0$, $\alpha(f_0) = pe_0$, and $\alpha(f_1) = qe_0$.Finally we can calculate the homology groups. $H_0(X_{p,q}) = \mathbb{Z}/\mathrm{Im}(\beta) = \mathbb{Z}$, $H_1(X_{p,q}) = \ker(\beta)/\mathrm{Im}(\alpha) = \mathbb{Z}/\gcd(p,q)$, and $H_2(X_{p,q}) = \ker(\alpha) = \mathbb{Z}$ if p, q are not both nonzero and $\mathbb{Z}\oplus\mathbb{Z}$ otherwise. To summarize,

$$\boxed{H_n(X_{p,q}) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}/\gcd(p,q) & n = 1, \\ \mathbb{Z} & n = 2 \text{ and } (p,q) \neq (0,0), \\ \mathbb{Z}\oplus\mathbb{Z} & n = 2 \text{ and } (p,q) = (0,0), \\ 0 & n \geq 3. \end{cases}}$$

Problem 4. Compute the homology groups of the following 2-dimensional CW complexes:

(a) The quotient of S^2 obtained by identifying the north and south poles to a point.

(b) The space obtained from S^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving the clockwise orientations of these circles.

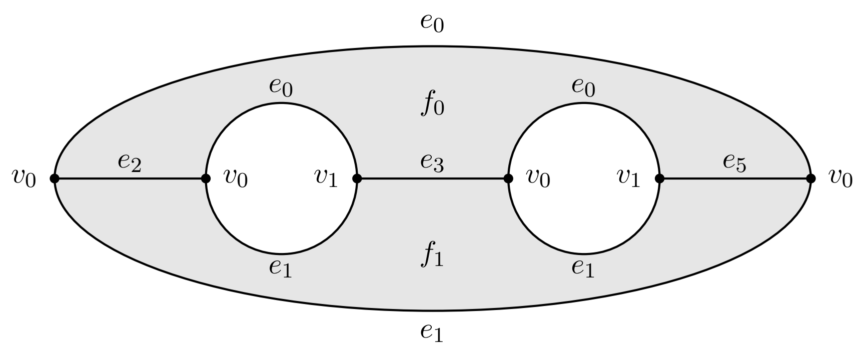
(a) Let X be the space. First of all, X is homotopy equivalent to the torus with a disk glued into the central hole. This can be given the following CW structure: X_0 consists of a single point v_0. X_1 adds two circles e_0, e_1 at v_0, and X_2 adds two disks f_0, f_1, with f_0 glued to e_0 and f_1 glued to $e_0e_1e_0^{-1}e_1^{-1}$. (Here we use the notation $e_0e_1e_0^{-1}e_1^{-1}$ to represent the attachment map which goes around e_0, then e_1, then e_0 in the other direction, then e_1 in the other direction). We thus get the following cellular chain complex:

$$0 \longleftarrow \mathbb{Z}v_0 \xleftarrow{\beta} \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \xleftarrow{\alpha} \mathbb{Z}f_0 \oplus \mathbb{Z}f_1 \longleftarrow 0$$

Here we consider these v_i, e_i, f_i as generators of $H_n(S^n)$ for the appropriate n. The cellular boundary formula gives $\beta(e_0) = v_0 - v_0 = 0$ and $\beta(e_1) = v_0 - v_0 = 0$. Similarly $\alpha(f_0) = e_0$ and $\alpha(f_1) = e_0 + e_1 - e_0 - e_1 = 0$. Then $H_0(X) = \mathbb{Z}/\mathrm{Im}(\beta) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1/\mathbb{Z}e_0 = \mathbb{Z}$, and $H_2(X) = \mathbb{Z}e_1/0 = \mathbb{Z}$. Thus

$$\boxed{H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 1, 2, \\ 0 & n \geq 3. \end{cases}}$$

(b) Let X be the space. Let's give the following CW structure to X:



So X_0 consists of two 0-cells, X_1 adds five 1-cells, and X_2 adds two 2-cells. This gives us the chain complex

$$0 \longleftarrow \mathbb{Z}^2 \xleftarrow{\beta} \mathbb{Z}^5 \xleftarrow{\alpha} \mathbb{Z}^2 \longleftarrow 0$$

with boundary maps α, β given by:

$$\begin{aligned} \beta(e_0) &= v_1 - v_0 & \qquad \alpha(f_0) &= e_0 + e_2 + e_3 + e_4 \\ \beta(e_1) &= v_1 - v_0 & \alpha(f_1) &= e_1 + e_2 + e_3 + e_4 \\ \beta(e_2) &= 0 \\ \beta(e_3) &= v_1 - v_0 \\ \beta(e_4) &= 0 \end{aligned}$$

Then it's fairly easy to see by calculating kernels and images of these maps that

$$\boxed{H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1, \\ 0 & n \geq 2. \end{cases}}$$