Math 114 Problem Set 7

Lev Kruglyak

December 1, 2022

Problem 1. Let f be a function on the circle. For each $N \geq 1$ the discrete Fourier coefficients of f are defined by

$$a_N(n) = \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})e^{-2\pi ikn/N}, \quad \text{for } n \in \mathbb{Z}.$$

Let

$$a(n) = \int_0^1 f(e^{2\pi ix})e^{-2\pi inx}dx$$

denote the ordinary Fourier coefficients of f.

a. Show that $a_N(n) = a_N(n+N)$.

This follows since for any k we have $e^{-2\pi ik(n+N)/N} = e^{-2\pi iknN}$.

b. Prove that if f is continuous, then $a_N(n) \to a(n)$ as $N \to \infty$.

This follows since $a_N(n)$ is a partial Reimann sum of a continuous function, thus we have $\lim_{N\to\infty} a_N(n) = a(n)$.

Problem 2. If f is a C^1 function on the circle, prove that $|a_N(n)| \leq c/|n|$ when $0 < |n| \leq N/2$.

For any $\ell \in \mathbb{Z}$, we have

$$\begin{aligned} a_N(n)e^{2\pi i\ell n/N} &= \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})e^{-2\pi i(k-\ell)n/N} = \frac{1}{N}\sum_{k=1-\ell}^{N-\ell} f(e^{2\pi i(k+\ell)/N})e^{-2\pi ikn/N} \\ &= \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi i(k+\ell)/N})e^{-2\pi ikn/N}. \end{aligned}$$

The identity given in the hint follows immediately. Since $f(e^{2\pi ix})$ is periodic, its derivative must have a maximum M, and thus $|f(e^{2\pi ix}) - f(e^{2\pi iy})| \leq M|x-y|$ for all x, y combining this with out identity, we get

$$\begin{aligned} |a_N(n)||1 - e^{2\pi i\ell n/N}| &\leq \frac{1}{N}\sum_{k=1}^{N} |f(e^{2\pi ik/N}) - f(e^{2\pi i(k+\ell)/N})||e^{-2\pi ikn/N}| \\ &\leq \frac{1}{N}\sum_{k=1}^{N} M\Big|\frac{k}{N} - \frac{k+\ell}{N}\Big| = \frac{M|\ell|}{N} \end{aligned}$$

for all integers ℓ. Now setting ℓ such that $|\ell - N/2n| \leq 1/2$ gives $1/4 \leq \ell n/N \leq 3/4$. Thus,

$$\sqrt{2}|a_N(n)| \leq |a_N(n)||1 - e^{2\pi i \ell n/N}| \leq \frac{3M}{4|n|}.$$

Letting $c = 3M/(4\sqrt{2})$, we get $|a_N(n)| \leq c/|n|$ for $0 < |n| \leq N/2$.

Problem 3. By a similar method, show that if f is a C^2 function on the circle, then

$$|a_N(n)| \leq c/|n|^2, \quad \text{whenever } 0 < |n| \leq N/2.$$

Let $g(x) = f(e^{2\pi i x/N})$. Applying Taylor's theorem, we get get

$$g(k+\ell) = g(k) + g'(k)\ell + \frac{g''(C_\ell)}{2}\ell^2, \quad C_\ell \in [k, k+\ell].$$

This implies that

$$g(k+\ell) + g(k-\ell) - 2g(k) = \frac{g''(C_\ell) + g''(C_{-\ell})}{2}\ell^2.$$

Just like the previous problem, we use the fact that functions on compact spaces achieve their extrema, so let $M = \max(\sup_{x\in\mathbb{R}} |f'(e^{2\pi i x})|, \sup_{x\in\mathbb{R}} |f''(e^{2\pi i x})|)$. Applying the chain rule to g, we get

$$g''(x) = -\frac{4\pi^2}{N^2} e^{2\pi i x/N}\left(f'(e^{2\pi i x/N}) + e^{2\pi i x/N} f''(e^{2\pi i x/N})\right) \implies |g''(x)| \leq \frac{8M\pi^2}{N^2}.$$

This in turn implies that

$$|g(k+\ell) + g(k-\ell) - 2g(k)| \leq \frac{8M\pi^2\ell^2}{N^2}$$

Using the expression for $a_N(n)e^{2\pi i \ell n/N}$ from the previous problem, we have

$$a_N(n)(e^{2\pi i \ell n/N} + e^{-2\pi i \ell n/N} - 2) = \frac{1}{N}\sum_{k=1}^{N}[g(k+\ell) + g(k-\ell) - 2g(k)]e^{2\pi i k n/N}$$
$$\implies |a_N(n)||e^{2\pi i \ell n/N} + e^{-2\pi i \ell n/N} - 2| \leq \frac{8M\pi^2\ell^2}{N^2}.$$

Let $\ell \in \mathbb{Z}$ be such that $1/4 \leq \ell n/N \leq 3/4$ so that we have $\ell^2/N^2 \leq 9/(16n^2)$. Then

$$|e^{2\pi i \ell n/N} + e^{-2\pi i \ell n/N} - 2| = |(e^{\pi i \ell n/N} - e^{-\pi i \ell n/N})^2| = 4\sin^2(\pi \ell n/N) \geq 2$$
$$\implies 2|a_N(n)| \leq |a_N(n)||e^{2\pi i \ell n/N} + e^{-2\pi i \ell n/N} - 2| \leq \frac{9M\pi^2}{2n^2}.$$

So if we let $c = 9M\pi^2/4$, we have $|a_N(n)| \leq c/n^2$ whenever $0 < |n| \leq N/2$.

Inversion formula. As a result, prove the inversion formula for $f \in C^2$,

$$f(e^{2\pi i x}) = \sum_{m=-\infty}^{\infty} a(n)e^{2\pi i n x}$$

from its finite version.

Let N be odd and. Then for any $1 \le k \le N$ we have

$$\begin{aligned}\sum_{|n|<N/2} a_N(n)e^{2\pi ikn/N} &= \frac{1}{N}\sum_{|n|<N/2}\sum_{j=1}^{N} f(e^{2\pi ij/N})e^{2\pi i(k-j)n/N}\\ &= \frac{1}{N}\sum_{j=1}^{N} f(e^{2\pi ij/N})\sum_{|n|<N/2} e^{2\pi i(k-j)n/N}\\ &= f(e^{2piik/N}).\end{aligned}$$

Now let $x \in [0,1]$. There exists a sequence of integers $1 \le k_N \le N$ such that $k_N/N \to x$ as $N \to \infty$ so by the continuity of $f(e^{2\pi ix})$, we get

$$f(e^{2\pi ix}) = \lim_{N\to\infty} f(e^{2\pi ik_N/N}) = \lim_{N\to\infty, N \text{ odd}} \sum_{|n|<N/2} a_N(n)e^{2\pi ik_N n/N},$$

Since $|a_N(n)e^{2\pi ik_N n/N}| = |a_N(n)| \le c/n^2$ for all $0 < |n| \le N/2$, and $\sum_{n\in\mathbb{Z}-\{0\}} c/n^2 < \infty$, dominated convergence implies that

$$f(e^{2\pi ix}) = \lim_{N\to\infty} a_N(0) + \sum_{n\in\mathbb{Z}-\{0\}} \lim_{N\to\infty} a_N(n)e^{2\pi ik_N n/N} = \sum_{n=-\infty}^{\infty} a(n)e^{2\pi inx}$$

as desired.

Problem 4. Suppose w is a measurable function on $\mathbb{R}^d$ with $0 < w(x) < \infty$ for a.e. x, and K is a measurable function on $\mathbb{R}^{2d}$ that satisfies:

(i) $\int_{\mathbb{R}^d} |K(x,y)|w(y)dy \le Aw(x)$ for almost every $x \in \mathbb{R}^d$, and

(ii) $\int_{\mathbb{R}^d} |K(x,y)|w(x)dx \le Aw(y)$ for almost every $y \in \mathbb{R}^d$.

Prove that the integral operator defined by

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \quad x \in \mathbb{R}^d$$

is bounded on $L^2(\mathbb{R}^d)$ with $\|T\| \le A$.
Note as a special case that if $\int |K(x,y)|dy \le A$ for all x, and $\int |K(x,y)|dx \le A$ for all y, then $\|T\| \le A$.

We can rewrite $|K(x,y)||f(y)|$ as

$$|K(x,y)||f(y)| = \left(|K(x,y)|^{1/2}w(y)^{1/2}\right)\cdot\left(|K(x,y)|^{1/2}w(y)^{-1/2}|f(y)|\right).$$

Applying Cauchy-Schwarz to $\int_{\mathbb{R}^d} |K(x,y)|w(y)\,dy \le Aw(x)$ a.e, we get

$$\int |K(x,y)||f(y)|dy \le A^{1/2}w(x)^{1/2}\left[\int |K(x,y)||f(y)|^2 w(y)^{-1}dy\right]^{1/2}.$$

Integrating both sides by Tonelli's theorem we get

$$\int\left(\int |K(x,y)||f(y)|dy\right)^2 dx \le A\int |f(y)|^2 w(y)^{-1}\left(\int |K(x,y)|w(x)dx\right)dy.$$

Now let's apply $\int_{\mathbb{R}^d} |K(x,y)|w(x)\,dx \le Aw(y)$ a.e to get:

$$\int\left(\int |K(x,y)||f(y)|dy\right)^2 dx \le A^2\int |f(y)|^2 dy < \infty$$

since $f \in L^2(\mathbb{R}^d)$. This implies $K(x,y)f(y) \in L^1(\mathbb{R}^d)$ for almost all x. This also implies $\|T(f)\|_2 \leq A\|f\|_2$, so T is a bounded linear operator with $\|T\| \leq A$.

Problem 5. Consider the operator $T : L^2([0,1]) \to L^2([0,1])$ defined by

$$T(f)(t) = tf(t).$$

Compact operators:

a. Prove that T is a bounded linear operator with $T = T^*$, but that T is not compact.

Since $\{t \in [0,1]\}^2 = [0,1]$, we have $\|Tf\|_2 = t\|f\|_2$. Thus T is a bounded linear operator. To prove it isn't compact, note that for any $f, g \in L^2([0,1])$ we have

$$\langle Tf, g\rangle = \int_0^1 tf(t)\overline{g(t)}\, dt = \int_0^1 f(t)\overline{tg(t)}\, dt = \langle f, Tg\rangle \implies T = T^*.$$

Thus if T were compact, it would be a bounded linear Hermitian operator so it would have eigenvalues.

Note that because $0 \leq t^2 \leq 1$ for all $t \in [0,1]$,

$$\|Tf\|_2 = \left(\int_0^1 |tf(t)|^2 dt\right)^{1/2} \leq \|f\|_2.$$

It follows that T is well-defined and bounded (in particular, $\|T\| \leq 1$). Linearity is trivial. Now let $f, g \in L^2([0,1])$. Then

$$\langle Tf, g\rangle = \int_0^1 tf(t)\overline{g(t)}dt = \int_0^1 f(t)\overline{tg(t)}dt = \langle f, Tg\rangle.$$

Thus, $T = T^*$. The fact that T is not compact now follows from (b) because if T were compact, then by virtue of being a bounded linear and symmetric/Hermitian operator, $\|T\|$ or $-\|T\|$ would be an eigenvalue.

b. However, show that T has no eigenvectors.

Suppose for the sake of contradiction that $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector some $f \in L^2([0,1])$ which isn't zero almost everywhere. Then we have $\{tf = \lambda f\} = \{t = \lambda\} \cup \{f = 0\}$. Since $m(\{t = \lambda\}) = 0$ and $m(\{f = 0\}) < 1$, we have a contradiction since $Tf = \lambda f$ a.e.