Math 231b Problem Set 9

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Problem 1. James splitting of $\Sigma\Omega S^{2n+1}$.

First we'll prove a lemma.

a. Given two path-connected pointed spaces X and Y, prove that there is a splitting

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

Define a map $\psi : \Sigma(X \times Y) \to \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ by setting $\psi = \Sigma\pi_X + \Sigma\pi_Y + \Sigma q$, where $q : X \times Y \to X \wedge Y$ is the quotient fibration. Since all the spaces involved are simply connected (CW complexes), it suffices to prove that ψ induces an isomorphism on integral homology to prove that it is a weak, hence homotopy equivalence by the homotopy Whitehead theorem. Recall that for a field k, the Kunneth theorem gives us a natural isomorphism

$$H_n(X \times Y; k) \cong \bigoplus_{p+q=n} H_p(X; k) \otimes_k H_q(Y; k).$$

Since suspension simply acts as a homology lifting functor, we get a similar natural isomorphism

$$H_n(\Sigma(X \times Y); k) \cong \bigoplus_{p+q=n+1} H_p(X; k) \otimes_k H_q(Y; k).$$

Note that the suspended projections $\Sigma\pi_X$ and $\Sigma\pi_Y$ send all terms to zero, except for $H_{n+1}(X; k) \otimes H_0(Y; k)$ and $H_0(X; k) \otimes H_{n+1}(Y; k)$ respectively. Meanwhile, the middle terms are exactly isomorphic to $H_n(\Sigma(X \wedge Y); k)$ by the long exact sequence associated to the sequence $X \vee Y \to X \times Y \to X \wedge Y$. So ψ_* is an isomorphism for all homology with coefficients in a field. Thus, by the universal coefficients theorem, it induces an isomorphism for all integral homology. This completes the proof.

b. Composition of loops $\Omega S^{2n+1} \times \Omega S^{2n+1} \to \Omega S^{2n+1}$ makes $H_*(\Omega S^{2n+1})$ into a ring. Prove that $H_*(\Omega S^{2n+1}) \cong \mathbb{Z}[x_{2n}]$. This is because the product $H_*(\Omega S^{2n+1}) \otimes H_*(\Omega S^{2n+1}) \to H_*(\Omega S^{2n+1})$ is dual to a coassociative and counital product $H^*(\Omega S^{2n+1}) \to H^*(\Omega S^{2n+1}) \otimes H^*(\Omega S^{2n+1})$ which itself is a map of rings.

Recall that $H^*(\Omega S^{2n+1}) \cong \Gamma[x]$, where $\Gamma[x]$ is the divided power algebra. Then the natural operation on homology is dual to a coassociative and counital coproduct $H^*(\Omega S^{2n+1}) \to H^*(\Omega S^{2n+1}) \otimes H^*(\Omega S^{2n+1})$, so we have such a map $\psi : \Gamma[x] \to \Gamma[x] \otimes \Gamma[x]$. Clearly, this map endows $\Gamma[x]$ with the divided power coalgebra structure, and so is dual to a polynomial ring multiplication on $H_*(\Omega S^{2n+1}) \cong \mathbb{Z}[x_{2n}]$.

c. Prove that $\Sigma\Omega S^{2n+1} \simeq \bigvee_{k=1}^{\infty} S^{2kn+1}$.

Recall that in the previous problem, we had the composition map $\Omega S^{2n+1} \times \Omega S^{2n+1} \to \Omega S^{2n+1}$, which induces the map $\mathbb{Z}[x_{2n}] \otimes \mathbb{Z}[x_{2n}] \to \mathbb{Z}[x_{2n}]$ given by $(f, g) \mapsto f \cdot g$. Applying the suspension functor gives us a map:

$$\Sigma(\Omega S^{2n+1} \times \Omega S^{2n+1}) \to \Sigma\Omega S^{2n+1}.$$

By basic properties of the suspension, recall that

$$H_k(\Sigma\Omega S^{2n+1}) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } 2kn+1, k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is exactly the homology of $\bigvee_{k=1}^{\infty} S^{2kn+1}$. Repeatedly applying (a) to $\Sigma(\Omega S^{2n+1} \times \Omega S^{2n+1})$ then allows us to construct a map $\bigvee_{k=1}^{\infty} S^{2kn+1} \to \Sigma\Omega S^{2n+1}$ which is an isomorphism on homology.

Problem 2. EHP sequence.

We will use the previous problem.

a. Using Problem 1c, construct $H : \Omega S^{2n+1} \to \Omega S^{4n+1}$ which induces an isomorphism in H_{4n}.

Firstly, we have a canonical map $\bigvee_{k=1}^{\infty} S^{2kn+1} \to \bigvee_{k=1}^{\infty} S^{4kn+1}$ which maps S^{4n+1} to S^{4n+1}, and all other S^{2kn+1} to the basepoint. Composing with the homotopy equivalence maps from 1(c), we thus get a map $\Sigma H : \Sigma\Omega S^{2n+1} \to \Sigma\Omega S^{2n+1}$, which induces an isomorphism in H_{4n+1}. Composing with the natural projections $\Sigma X \to X$ and inclusion $X \to \Sigma X$, we thus get our map H, which still is an isomorphism in H_{4n}.

b. Let $E : S^{2n} \to \Omega S^{2n+1}$ denote the adjoint to the identity on S^{2n+1}. Using the Serre spectral sequence, prove that

$$S^{2n} \xrightarrow{E} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{4n+1}$$

is a mod $\mathcal{C}_2$-fiber sequence, i.e. that the map $S^{2n} \to F$ induces a mod $\mathcal{C}_2$-isomorphism on homotopy groups. The induced long exact sequence of mod $\mathcal{C}_2$ homotopy groups is called the *EHP sequence.*

I'm not sure how to do this.

Problem 3. Cohomology of $V_2(\mathbb{R}^n)$.

Let $V_2(\mathbb{R}^n)$ denote the space of pairs (x_1, x_2) of orthonormal vectors in $\mathbb{R}^n$.

a. Identify the map $\pi : V_2(\mathbb{R}^n) \to S^{n-1}$ which sends $(x_1, x_2) \mapsto x_1$ with the unit sphere bundle associated to the tangent bundle of S^{n-1}.

For any vector $v \in S^{n-1}$, its fiber $\pi^{-1}(v)$ is the set of pairs of vectors (v, x) with $|x| = 1$ and $v \perp x$. This is exactly the unit sphere bundle.

b. Using the fact that $\langle e(TM), [M] \rangle = \chi(M)$, compute the cohomology rings $H^*(V_2(\mathbb{R}^n); \mathbb{F}_2)$ and $H^*(V_2(\mathbb{R}^n); \mathbb{Z})$.

By the previous part, we see that we have a spherical fibration:

$$S^{n-2} \longrightarrow V_2(\mathbb{R}^n) \xrightarrow{\pi} S^{n-1}$$

We can thus apply the Gysin sequence to get the cohomology rings of the Stiefel manifolds, assuming $n \geq 3$. So for any commutative ring R, we have a long exact sequence:

$$\cdots \longrightarrow H^k(S^{n-1};R) \xrightarrow{\pi^*} H^k(V_2(\mathbb{R}^n);R) \longrightarrow H^{k-n+2}(S^{n-1};R) \xrightarrow{E} H^{k+1}(S^{n-1};R) \longrightarrow \cdots$$

Here the E map is given by $E(\zeta) = e(TS^{n-1}) \smile \zeta$. There are four special cases we must worried about. First of all, if $k = 0, n-2, n-1, 2n-3$, the terms $H^k(S^{n-1};R)$ and $H^{k-n+2}(S^{n-1};R)$ vanish, which implies that $H^k(V_2(\mathbb{R}^n))$ is trivial. We now go through these cases one by one to fill in the non-trivial degrees of cohomology.

If $k = 0$, $H^{k-n+2}(S^{n-1};R) = 0$ since $n \geq 3$, so we have an exact sequence:

$$0 \longrightarrow R \longrightarrow H^0(V_2(\mathbb{R}^n);R) \longrightarrow 0$$

Thus $H^0(V_2(\mathbb{R}^n);R) \cong R$. Next, for $k = n-2, n-1$, we get a combined exact sequence:

$$0 \longrightarrow H^{n-2}(V_2(\mathbb{R}^n);R) \longrightarrow H^0(S^{n-1};R) \xrightarrow{E} H^{n-1}(S^{n-1};R) \longrightarrow H^{n-1}(V_2(\mathbb{R}^n);R) \longrightarrow 0$$

Here we have two different cases based on R. If $R = \mathbb{Z}/2$, the fact that $\langle e(TS^{n-1}), [S^{n-1}] \rangle = \chi(S^{n-1})$ is always even implies that E is the zero map $\mathbb{Z}/2 \to \mathbb{Z}/2$. Thus $H^{n-2}(V_2(\mathbb{R}^n);\mathbb{F}_2) \cong \ker(E) = \mathbb{F}_2$ and $H^{n-1}(V_2(\mathbb{R}^n);\mathbb{F}_2) \cong \operatorname{coker}(E) = \mathbb{F}_2$. Finally, we use the Gysin sequence to see that $H^{2n-3}(V_2(\mathbb{R}^n);\mathbb{F}_2) \cong \mathbb{F}_2$, so we have

$$H^*(V_2(\mathbb{R}^n);\mathbb{F}_2) \cong \mathbb{F}_2[x_{n-2}, x_{n-1}]/(x_{n-2}^2, x_{n-1}^2).$$

This ring is commutative because $(n-2)(n-1)$ is always even. In the $\mathbb{Z}$ case, we recall that $\chi(S^{n-1}) = 0$ when n is even and 2 when n is odd. For this former case, we have the same algebra. In the latter case, the $n-1$ cohomology becomes $\mathbb{Z}/2$, and the $n-2$ cohomology vanishes so we get a different presentation:

$$H^*(V_2(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z}[x_{n-2}, x_{n-1}]/\left(x_{n-2}^2, x_{n-1}^2\right) & n \text{ even} \\ \mathbb{Z}[x_{n-1}, x_{2n-3}]/\left(x_{n-1}^2, x_{n-1}x_{2n-3}, 2x_{n-1}\right) & n \text{ odd} \end{cases}.$$

Problem 4. The exceptional Lie group G_2 lies in a fiber sequence

$$S^3 \longrightarrow G_2 \longrightarrow V_2(\mathbb{R}^6).$$

Compute the integral and mod 2 cohomology groups of G_2 using the Serre spectral sequence. Explain why the Serre spectral sequence is unable to uniquely determine the ring structure without some additional input.

By the previous problem, $H^*(V_2(\mathbb{R}^6);R) = R[x_4, x_5]/(x_4^2, x_5^2)$ for $R = \mathbb{Z}$ and $\mathbb{Z}/2$. Using the cohomology of S^3, the Serre spectral sequence gives us the E_2 page:

$$\begin{array}{c|cccccc}
t & & & & & & \\
3 & R & & R & R & & R \\
2 & & & & & & \\
1 & & & & & & \\
0 & R & & R & R & & R \\
\hline
 & 0 & \cdots & 4 & 5 & \cdots & 9 \quad s
\end{array}$$

Thus we have the following cohomology groups: (for $\mathbb{Z}$ and $\mathbb{Z}/2$)

$$H^k(G_2; R) \cong \begin{cases} R & k = 0, 3, 4, 5, 7, 8, 9, 12, \\ 0 & \text{otherwise.} \end{cases}$$

Calling the generators corresponding to each degree y_i, we see a couple of things. First of all, we know that y_4, y_5 come from x_4 and x_5, and so $y_9 = y_4 y_5$. There simply isn't any information dictating what y_3^2 is for example, so we can't understand the multiplicative structure. This would require some other fibration with known cohomology.