Lecture 6: Foliations and the global Frobenius theorem

Contents

For a vector field on a manifold there is a local existence and uniqueness theorem for integral curves (with given initial condition), as well as a global existence and uniqueness theorem for maximal integral curves. For a distribution we write the theorems for all initial conditions at once. Theorem 5.41 is the local version: a local normal form for a distribution. In this lecture we turn to the global version, which is a *foliation* on a smooth manifold.

We begin with the definition of a foliation: a partition of a smooth manifold into a union of images of injective immersions. The union is not arbitrary; there is a local model. Hence the study of foliations is a study of global phenomena. We give some examples of foliations, including the Reeb foliation of the solid torus and of the 3-sphere. We encourage the reader to have fun thinking through the geometry of these examples.

The main result (Theorem 6.22) is that an integrable distribution integrates to a foliation. The proof begins with the local Theorem 5.41 and then uses a global equivalence relation to define the global partition of the manifold into leaves. A useful corollary (Theorem 6.25) asserts that a smooth map into a foliated manifold whose image is contained in a single leaf is a smooth map into that leaf.

Foliations

(6.1) *Definition of a foliation.* Let X be a smooth manifold

Definition 6.2. A k-dimensional *foliation* $\mathcal{F}$ on X is a set $f_\alpha\colon Y_\alpha \hookrightarrow X$, $\alpha \in A$, of injective immersions such that (i) $\dim Y_\alpha = k$; (ii) Y_α is connected; (iii) the images $\mathcal{F}_\alpha := f_\alpha(Y_\alpha)$ partition X:

$$X = \bigsqcup_{\alpha\in A} \mathcal{F}_\alpha; \tag{6.3}$$

and (iv) about each $p \in X$ there exists a chart $(U; x^1, \dots, x^n)$ such that $\mathcal{F}_\alpha \cap U$ is a union of slices $\{x^i = c^i\}_{i=k+1,\dots,n}$ for some $c^{k+1}, \dots, c^n \in \mathbb{R}$. Each $\mathcal{F}_\alpha$ is called a *leaf* of the foliation, and a chart in (iv) is an *$\mathcal{F}$-coordinate system*.

Since Y_α is second countable, so too is $f_\alpha^{-1}(U) \subset Y_\alpha$. It follows that $f_\alpha^{-1}(U)$ has countably many components, hence the intersection in (iv) is contained in a union of at most *countably* many slices. An $\mathcal{F}$-coordinate system is depicted in Figure 19.

Differential Geometry (Math230a, Fall '24), Dan Freed, September 23, 2024

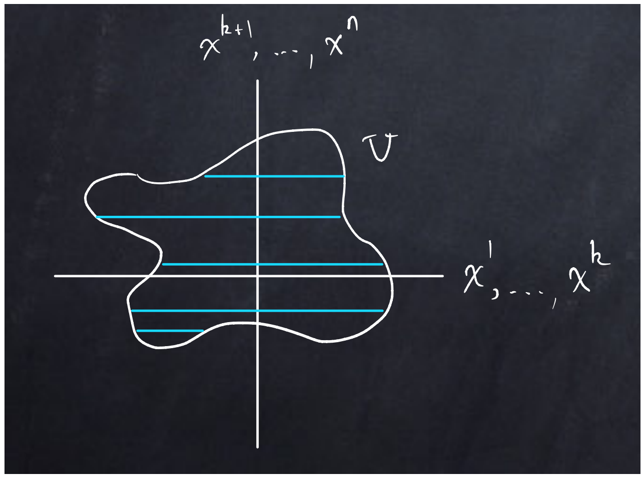

FIGURE 19. An $\mathcal{F}$-coordinate system and slices of a leaf

Remark 6.4. It is convenient to assume that an $\mathcal{F}$-coordinate system $(U; x^1, \dots, x^n)$ is *cubic* in the sense that the image $x(U) \subset \mathbb{A}^n$ is an open cube whose sides are aligned with the axes. In that case each slice is a *connected* subset of U.

(6.5) *Examples.* We give several examples to illustrate some possible behaviors.

Example 6.6 (Foliations of the 2-torus)**.** Example 5.8 illustrates parallel 1-dimensional foliations of the 2-torus $\mathbb{A}^2/\mathbb{Z}^2$. For rational $a \in \mathbb{Q}$ in (5.9) the corresponding foliation is by submanifolds diffeomorphic to a circle. In this case the $\mathcal{F}$-coordinate systems in Definition 6.2(iv) may be chosen so that each leaf cuts through the chart in zero or one slices. On the other hand, for irrational $a \in \mathbb{R} \setminus \mathbb{Q}$ the foliation is by images of injective immersions of affine lines whose images are not submanifolds. Each leaf cuts through any $\mathcal{F}$-coordinate system in countably many slices.

Example 6.7 (Fiber bundles)**.** Let $\pi\colon X \to S$ be a fiber bundle (Definition 8.24 of the Differential Topology notes). Then X is foliated by the fibers:

$$\mathcal{F} = \bigsqcup_{s\in S} X_s, \qquad X_s = \pi^{-1}(s). \tag{6.8}$$

Furthermore, each leaf (fiber) is a submanifold of X. Note in this case the *leaf space* S is a smooth manifold. The foliation in Example 6.6 has this form for rational a: the leaf space is diffeomorphic to a circle, and the torus appears as the total space of a circle bundle over the circle. For irrational a the leaf space is not a smooth manifold. (It is not even a Hausdorff topological space.)

Example 6.9 (Surjective submersions)**.** A surjective submersion $\pi\colon X \to S$ need not be a fiber bundle, and still it defines a foliation of X by submanifolds: the fibers of π. An example is

$$\begin{aligned} \pi\colon \mathbb{A}^2_{x,y} &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto (1-x^2)e^y \end{aligned} \tag{6.10}$$

Some fibers are illustrated in Figure 20.

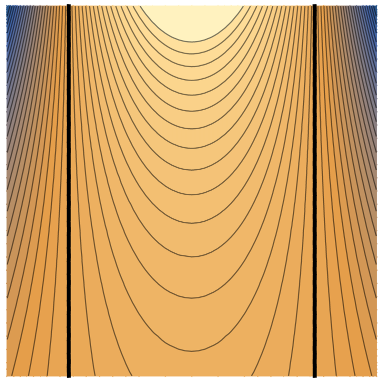

FIGURE 20. Foliation of a surjective submersion

Example 6.11 (Reeb foliation)**.** Consider Figure 21, which illustrates a foliation of a closed strip in an affine plane. It is the portion of Figure 20 with $-1 \leq x \leq 1$. The black lines are $x = \pm 1$. Imagine this x, y-plane as sitting in 3-space and revolve about the vertical blue line $x = 0$. You obtain a foliation of a solid cylinder in which the boundary cylinder is a leaf and each other leaf is diffeomorphic to an affine plane. Each leaf is a submanifold. Now observe that the foliations of the strip and of the solid cylinder are invariant under translation $y \mapsto y + c$ for any $c \in \mathbb{R}$. Restrict to $c \in \mathbb{Z}$ and take the quotient. The quotient of this $\mathbb{Z}$-action on the strip is an annulus, and now the foliation has leaves the two boundary circles together with images of affine lines under injective immersions that are not embeddings. The quotient of the $\mathbb{Z}$-action on the solid cylinder is a solid torus, and now the leaves of the foliation are the boundary torus and the images of affine planes under injective immersions that are not embeddings. This is the *Reeb foliation* of the solid torus.

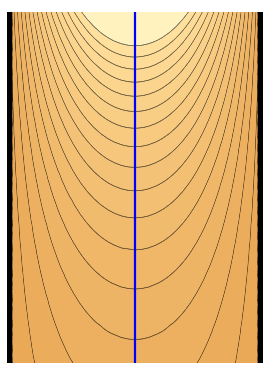

FIGURE 21. The proto-Reeb foliation

We can parlay this into a foliation of S^3 as follows. Present S^3 as the union of two solid tori along a 2-torus by writing

$$S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}. \tag{6.12}$$

The solid tori are $\{\, |z^1| \leq |z^2| \,\}$ and $\{\, |z^1| \geq |z^2| \,\}$ which intersect in the 2-torus $\{\, |z^1| = |z^2| = 1/\sqrt{2} \,\}$. Now transport the Reeb foliation of the solid torus above to these two solid tori. The foliations glue to a foliation of S^3, which can be smoothed. The result is also known as the *Reeb foliation*.

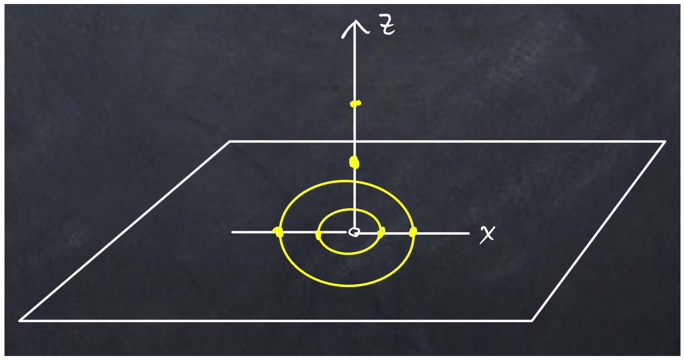

FIGURE 22. The solid torus as a quotient of upper half space minus the origin

Example 6.13 (Another construction of the Reeb foliation)**.** Define

$$\widetilde{X} = \{(x,y,z) \in \mathbb{A}^3 : z \ge 0\} \setminus \{(0,0,0)\}. \tag{6.14}$$

Note that $\widetilde{X}$ is a 3-manifold with boundary. Fix $q \in (0,1)$. Set $X = \widetilde{X}/q^{\mathbb{Z}}$ to be the quotient of $\widetilde{X}$ by the action of the infinite cyclic group of homotheties of $\mathbb{A}^3$ centered at $(0,0,0)$ with scale a power of q; see Figure 22. Consider first $\partial\widetilde{X} \approx \mathbb{A}^2 \setminus \{(0,0)\}$. A fundamental domain of the action is the annulus whose boundary circles have radii 1 and q, and under the action those boundary circles are identified. Hence the quotient is a 2-torus. Now the entire quotient $X = \widetilde{X}/q^{\mathbb{Z}}$ is a 3-manifold with boundary a 2-torus, and I leave as a visualization exercise proving that X is diffeomorphic to a solid 3-dimensional torus.[1] Now let $\widetilde{\mathcal{F}}$ be the foliation of $\widetilde{X}$ by horizontal planes $z = c$, $c \in \mathbb{R}^{\ge 0}$. (We remove the origin.) There is a quotient foliation $\mathcal{F}$ of X. The leaf $\{z = 0\}$ of $\widetilde{\mathcal{F}}$ descends to the leaf ∂X of $\mathcal{F}$ diffeomorphic to a 2-torus. The homotheties acts freely on the leaf space of $\widetilde{\mathcal{F}}$ minus the leaf $\{z = 0\}$, so each other leaf of $\mathcal{F}$ is diffeomorphic to an affine plane. The claim is that $\mathcal{F}$ is diffeomorphic to the Reeb foliation of Example 6.11.

(6.15) *Codimension one foliations.* If a connected closed manifold X admits a codimension one foliation $\mathcal{F}$, then it is not difficult to see that the Euler number $\mathrm{Euler}(X)$ must vanish. (It is useful to use the tangent bundle $T\mathcal{F}$, defined shortly.) Thus, for example, even dimensional spheres do not admit a codimensional one foliation. Thurston proved the converse statement.

Theorem 6.16 (Thurston)**.** *A connected closed smooth manifold X admits a codimension one foliation iff* $\mathrm{Euler}(X) = 0$.

The global Frobenius theorem

(6.17) *The tangent bundle to a foliation.* Let X be a smooth manifold and suppose $\mathcal{F}$ is a foliation on X.

---

[1] You might first consider the x,z-plane (restricting to $z \ge 0$). You can also consider the rays emanating from the origin in that plane, and then revolution about the nonnegative z-axis. In addition, you can consider the foliation of $\widetilde{X}$ by rays emanating from the origin, and see what happens to that foliation in the quotient.

Definition 6.18. The *tangent bundle* $T\mathcal{F} \to X$ to $\mathcal{F}$ is

$$\bigsqcup_{p\in X} T_p\mathcal{F}_{\alpha(p)} \to X, \qquad p \in \mathcal{F}_{\alpha(p)}. \tag{6.19}$$

In (6.19) we write A for the index set of the foliation (the leaf set) and $\alpha\colon X \to A$ for the function that assigns to each $p \in X$ its leaf.

Lemma 6.20. *$T\mathcal{F} \to X$ is a vector bundle. Furthermore, the distribution $T\mathcal{F}$ is integrable.*

Proof. Local triviality and integrability follow from the existence of local $\mathcal{F}$-coordinate systems (Definition 6.2(iv)). ☐

(6.21) *Global Frobenius theorem.* The theorem asserts that every integrable distribution is the tangent bundle to a foliation.

Theorem 6.22. *Let X be a smooth manifold and $E \subset TX$ an integrable distribution.*

1. *There exists a foliation $\mathcal{F}$ such that $E = T\mathcal{F}$.*
2. *Each leaf of $\mathcal{F}$ is a maximal connected integral manifold of E.*
3. *Points $p_0, p_1 \in X$ lie in the same leaf of $\mathcal{F}$ iff there exists a piecewise smooth motion in X from p_0 to p_1 whose velocity vectors lie in E.*

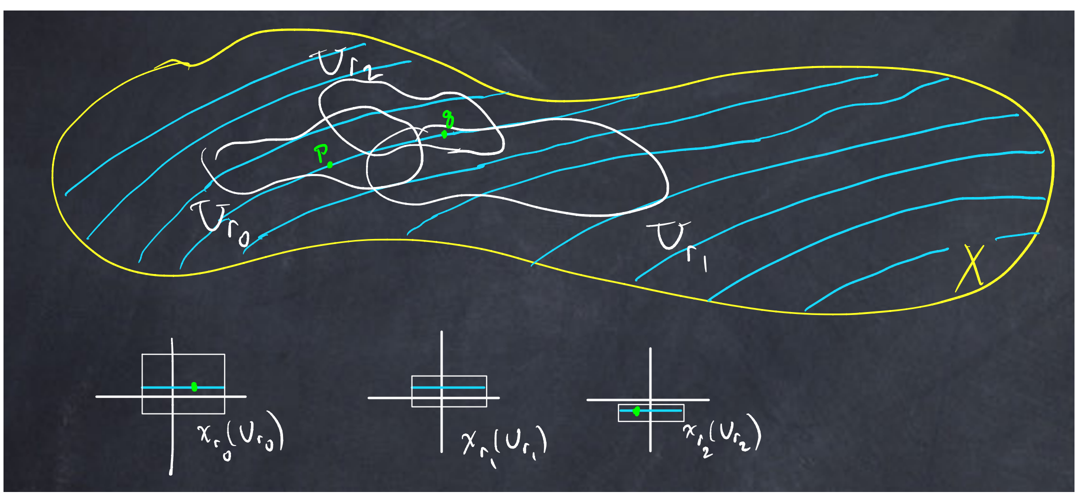

FIGURE 23. A sequence of slices in some $\mathcal{F}_\alpha$

Proof. Apply Theorem 5.41 and second countability of X to cover X by a (finite or countable) sequence $(U_r; x_r^1, \dots, x_r^n)$, $r = 1, 2, \dots$, of E-coordinate systems, which we assume are cubic in the sense of Remark 6.4. Observe that if $S \subset U_r$ is a slice, then for $t \neq r$ the intersection $S \cap U_t$ is empty or is a finite or countable union of slices of U_t. Define an equivalence relation $\sim$ on X that sets $p \in S_r \subset U_r$ equivalent to $q \in S_t \subset U_t$ iff there exist $r_0, \dots, r_\ell \in \mathbb{Z}^{>0}$ and slices $S_{r_i} \subset U_{r_i}$ such that $r_0 = r$, $r_\ell = t$, $S_{r_0} = S_r$, $S_{r_\ell} = S_t$, and $S_{r_{i-1}} \cap S_{r_i} \neq \emptyset$ for all $1 \le i \le \ell$. Then we claim that the equivalence classes

(6.23)
$$X = \bigsqcup_{\alpha \in A} \mathcal{F}_\alpha$$

of $\sim$ are the leaves of a foliation $\mathcal{F}$. Let Y be an equivalence class. Then each $p \in Y$ lies in a slice $S_r \in U_r$ in one of the chosen E-coordinate systems. If rank $E = k$, then use the first k coordinates in that chart as coordinates on $S_r \subset Y$. In this way we simultaneously topologize Y and construct an atlas. Coordinate charts have C^∞ overlaps since the E-coordinate systems do. The topology on Y is clearly locally Euclidean and path connected, hence is connected. It is also second countable: there are at most countably many sequences $U_{r_0}, \dots, U_{r_\ell}$, and if $p \in S_{r_0} \subset U_{r_0}$, then there are at most countably many sequences $S_{r_0}, \dots, S_{r_\ell}$ in which[2] $S_{r_{i-1}} \cap S_{r_i} \neq \emptyset$, hence in all there are at most countably many slices that appear in Y. The inclusion $Y \hookrightarrow X$ is smooth and is an injective immersion. The E-coordinate systems are $\mathcal{F}$-coordinate systems, and this proves that (6.23) is a foliation. The local form shows $E = T\mathcal{F}$, which completes the proof of (1).

Since $T\mathcal{F} = E$, each leaf $\mathcal{F}_\alpha$ is an integral manifold of E. If $f\colon Y \hookrightarrow X$ is any (connected) integral manifold, then by Theorem 5.41(2) locally its image lies in a slice of an E-coordinate system $(U_r; x_r^1, \dots, x_r^n)$. Any $p, q \in f(Y)$ can be joined by a piecewise smooth path, and along that path we can find a finite sequence of slices $S_{r_0}, \dots, S_{r_\ell}$ as in the definition of $\sim$. In other words, $p \sim q$. This proves $f(Y) \subset \mathcal{F}_\alpha$ for some α, which is the maximality claimed in (2).

If $p_0, p_1 \in X$ lie in the same leaf of $\mathcal{F}$, then from the definition of $\sim$ we construct a smooth motion from p_0 to p_1 which lies in the slices that connect the points. The velocity vectors are in E. Conversely, given such a motion the velocity vectors lie in $E = T\mathcal{F}$, which means that if $p_0 \in \mathcal{F}_\alpha$, then locally the motion lies in $\mathcal{F}_\alpha$, from which it globally lies in $\mathcal{F}_\alpha$. $\square$

(6.24) *Maps into a leaf.* We sometimes encounter a smooth map whose codomain has a foliation and the map factors through the inclusion of a leaf. In that situation we would like to know that the factored map is smooth. This is not true in general for a map that factors through an injective immersion—see Figure 24—but it is true if that injective immersion is the leaf of a foliation.

Theorem 6.25. *Let X be a smooth manifold equipped with a foliation $\mathcal{F}$, and suppose Z is a smooth manifold and $\varphi\colon Z \to X$ is a smooth map. Assume there is a leaf of $\mathcal{F}$, the image $\mathcal{F}_\alpha$ of an injective immersion $f_\alpha\colon Y_\alpha \hookrightarrow X$, and a map $\varphi'\colon Z \to Y_\alpha$ such that $\varphi = f_\alpha \circ \varphi'$:*

(6.26)
$$\begin{tikzcd} Z \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi'"'] & X \\ & Y_\alpha \arrow[u, hook, "f_\alpha"'] \end{tikzcd}$$

Then φ' is smooth.

[2] A slice in one chart U_r intersects at most countably many slices in any U_s by the argument following Definition 6.2.

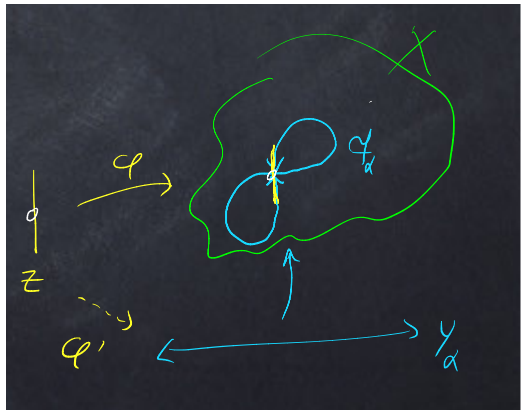

FIGURE 24. A map that factors through an injective immersion but the factored map is not continuous

Proof. Fix $z \in Z$ and let $p = \varphi(z)$. Choose an E-coordinate system $(U; x^1, \dots, x^n)$ about p from the sequence in the proof of Theorem 6.22, and choose the slice $S \subset U$ such that $p \in S$. The inverse image $\varphi^{-1}(U) \subset Z$ is open; let $V \subset Z$ be the component of $\varphi^{-1}(U)$ that contains z. Then $\varphi(V) \subset U$ is connected, $p \in \varphi(V)$, hence by the factorization (6.26) we conclude that $\varphi(V)$ lies in the component of $\mathcal{F}_\alpha \cap U$ that contains p. But from the first part of the proof of Theorem 6.22, that component is S. (The intersection $\mathcal{F}_\alpha \cap U$ is a countable union of slices.) Now $x^1, \dots, x^k$ are local coordinates on S (assuming $\operatorname{rank} E = k$), and since φ is smooth, so too is the local representation $(x^1, \dots, x^k) \circ \varphi$ of φ'. $\square$