Problem Set # 2

Math230a: Differential Geometry

Due: September 18

Throughout we use '$\mathfrak{g}$' to denote the Lie algebra of a Lie group G.

1. Let V be a finite dimensional real vector space and $B\colon V \times V \to \mathbb{R}$ a nondegenerate bilinear form. Define
$$\mathrm{Aut}_B(V) = \{P \in \mathrm{Aut}(V) : B(P\xi_1, P\xi_2) = B(\xi_1, \xi_2) \text{ for all } \xi_1, \xi_2 \in V\}$$
and
$$\mathrm{End}_B(V) = \{A \in \mathrm{End}(V) : B(A\xi_1, \xi_2) + B(\xi_1, A\xi_2) = 0 \text{ for all } \xi_1, \xi_2 \in V\}.$$
   (a) Prove that $\mathrm{Aut}_B(V)$ is a Lie group with Lie algebra $\mathrm{End}_B(V)$.

   (b) Let $V = \mathbb{R}^n$ for some $n \in \mathbb{Z}^{>0}$. Suppose B is the standard (symmetric) inner product. Identify $\mathrm{Aut}_B(\mathbb{R}^n)$ with the group O_n of orthogonal matrices.

   (c) Let $V = \mathbb{R}^{2m}$ for some $m \in \mathbb{Z}^{>0}$. Suppose B is a nondegenerate skew-symmetric form: for the standard basis $e_1, \dots, e_{2m}$ of $\mathbb{R}^{2m}$, set
$$B(e_i, e_{i+m}) = -B(e_{i+m}, e_i) = 1, \qquad i \in \{1, \dots, m\},$$
and set $B(e_i, e_j) = 0$ otherwise. Identify the group $\mathrm{Aut}_B(\mathbb{R}^{2m})$ explicitly in terms of block 2×2 matrices in which the blocks have size $m \times m$. This is the *symplectic group* $\mathrm{Sp}_{2m}\mathbb{R}$.

2. Let $\alpha\colon G' \to G$ be a homomorphism of Lie groups.

   (a) Prove that the differential α_* induces a linear map from left invariant vector fields on G' to left invariant vector fields on G. Furthermore, prove that this linear map is a Lie algebra homomorphism, which we denote $\dot{\alpha}\colon \mathfrak{g}' \to \mathfrak{g}$.

   (b) Prove that the diagram
$$\begin{tikzcd} \mathfrak{g}' \arrow[r, "\dot{\alpha}"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G' \arrow[r, "\alpha"] & G \end{tikzcd}$$
commutes.

3. Let G be a Lie group, let X be a smooth manifold, and suppose G acts on X. There is an induced linear map $\mathfrak{g} \to \mathfrak{X}(X)$ which maps $\xi \in \mathfrak{g}$ to the vector field $\hat{\xi}$ on X.

   (a) If G acts on the *right*, show that for all $\xi_1, \xi_2 \in \mathfrak{g}$,

   $$[\hat{\xi}_1, \hat{\xi}_2] = \widehat{[\xi_1, \xi_2]}.$$

   (b) If G acts on the *left*, show that for all $\xi_1, \xi_2 \in \mathfrak{g}$,

   $$[\hat{\xi}_1, \hat{\xi}_2] = -\widehat{[\xi_1, \xi_2]}.$$

4. Answer for each of the following Lie groups G:

$$\text{A finite dimensional real vector space } V$$
$$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$
$$\mathrm{SU}_2 = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ AA^* = I,\ \det A = 1\right\}$$
$$\mathrm{SL}_2\mathbb{R}$$
$$\mathrm{GL}_n\mathbb{R}$$
$$\mathrm{O}_n$$

   (a) Is G abelian?

   (b) Is G compact?

   (c) Is G connected?

   (d) Is G simply connected?

   (e) What is the Lie algebra of G? (Identify in concrete terms, for example as a Lie algebra of matrices.)

5. Let G be a Lie group.

   (a) As a preliminary, let V be a finite dimensional real vector space. Define a real line $|\operatorname{Det} V|$ such that an ordered n-tuple $\xi_1, \dots \xi_n \in V$ define an element $|\xi_1 \wedge \cdots \wedge \xi_n| \in |\operatorname{Det} V|$, and elements for two ordered bases are related by the absolute value of the determinant of the change of basis matrix. Identify $|\operatorname{Det} V^*|$ as a certain space of functions $V^{\times n} \to \mathbb{R}$. Show that positive functions determine an orientation of $|\operatorname{Det} V^*|$. Interpret a positive function as a notion of volume for n-dimensional parallelepipeds in V. Does this induce a notion of volume for lower dimensional parallelepipeds? Identify positive elements as translationally invariant positive measures on V. Construct such a positive element from an inner product on V.

   (b) Apply to the tangent bundle of a smooth manifold. Define the notion of a smooth positive measure on a smooth manifold. Do they always exist?

(c) The real line $|\operatorname{Det}\mathfrak{g}^*|$ consists of left-invariant measures on G. Define an action of G on this line. Compute the action in case G is compact. Compute it for $G = \mathrm{GL}_n\mathbb{R}$ and $G = \mathrm{SL}_n\mathbb{R}$.

(d) A *Haar measure* on G is a bi-invariant positive smooth measure on G. Prove that a Haar measure exists if G is compact. Normalize it so the total volume of G is 1.

(e) Write a formula for Haar measure on the circle group $\mathbb{T} \subset \mathbb{C}$; the formula should be in terms of $\lambda \in \mathbb{T}$. On the multiplicative group $\mathbb{R}^\times$ of nonzero real numbers. On the additive group $\mathbb{R}$ of real numbers. On the orthogonal group O_2.

6. Suppose G is a connected compact Lie group.

(a) Let $\Omega^\bullet_{\mathrm{linv}}(G) \subset \Omega^\bullet(G)$ denote the vector subspace of left-invariant differential forms. Show that $\Omega^\bullet_{\mathrm{linv}}(G)$ is in fact a sub-differential graded algebra, i.e., it is closed under multiplication and the differential d.

(b) Construct an isomorphism
$$\textstyle\bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet_{\mathrm{linv}}(G).$$
Transfer the differential on $\Omega^\bullet_{\mathrm{linv}}(G)$ to $\bigwedge^\bullet \mathfrak{g}^*$ and write a formula for it. In this way you obtain a differential graded complex defined directly from the Lie algebra $\mathfrak{g}$. Observe that your definition works for *any* Lie algebra (it needn't be the Lie algebra of a compact Lie group).

(c) Prove that the inclusion in part (a) induces an isomorphism on cohomology. A map of cochain complexes with this property is called a *quasi-isomorphism*. So you can compute the de Rham cohomology of G from this Lie algebra complex. (Hint: Average over G using Haar measure to construct a left-invariant form from an arbitrary form.)

(d) Use the inverse map $g \mapsto g^{-1}$ to show that the differential of a *bi-invariant* differential form vanishes. Show that the de Rham cohomology of G is isomorphic to the algebra of bi-invariant forms.

(e) Use these ideas to compute $H^\bullet_{\mathrm{dR}}(SU_2)$.

(f) (for those who know some Riemannian geometry) Endow G with a bi-invariant metric. Is there a relationship between harmonic forms and bi-invariant forms?