MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY
HOMEWORK 10
DUE: TUESDAY, APRIL 25 AT 12:00AM (MIDNIGHT) ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at:
`https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Wu formula (15 points)

Using the splitting principle, prove the *Wu formula* for the action of the Steenrod squares on the mod 2 reduction of the Chern classes:

$$\mathrm{Sq}^{2i}(c_j) = \sum_k \binom{j+k-i-1}{k} c_{i-k}c_{j+k}.$$

2. Problem 2: Vector fields on spheres (35 points)

Let $n = 2^m(2s+1)$ denote a positive integer which is divisible by 2 exactly m times. In this problem, you will use Steenrod operations to prove that S^{n-1} does not admit 2^m vector fields which are linearly independent at every point of S^{n-1}.[1]

Let $V_k(\mathbb{R}^n)$ denote the space of k orthonormal vectors in $\mathbb{R}^n$. Then $V_1(\mathbb{R}^n)$ may be identified with S^{n-1}.

(1) Consider the map $p_{k+1} : V_{k+1}(\mathbb{R}^n) \to V_1(\mathbb{R}^n) = S^{n-1}$ which sends $(v_1, \dots, v_{k+1})$ to v_{k+1}. Prove that S^{n-1} admits k linearly independent vector fields if and only if p_{k+1} admits a section $S^{n-1} \to V_{k+1}(\mathbb{R}^n)$. (Hint: Use Gram-Schmidt to prove that if S^{n-1} admits k linearly independent vector fields, then it admits k orthonormal vector fields.)

(2) There is a map $\mathbb{RP}^{n-1} \to O(n) \cong V_n(\mathbb{R}^n)$ which sends a line in $\mathbb{R}^n$ to the reflection across its normal hyperplane. Prove that the composition of this map with the map $V_n(\mathbb{R}^n) \to V_k(\mathbb{R}^n)$ taking $(v_1, \dots, v_n)$ to $(v_{n-k+1}, \dots, v_n)$ factors through a map $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \to V_k(\mathbb{R}^n)$. When $k = 1$, prove that $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \to V_1(\mathbb{R}^n)$ is a homeomorphism.

(3) There is a commutative diagram

$$\begin{tikzcd} S^{n-k} \arrow[r] \arrow[d] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \arrow[r] \arrow[d] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-k} \arrow[d] \\ S^{n-k} \arrow[r] & V_k(\mathbb{R}^n) \arrow[r] & V_{k-1}(\mathbb{R}^n), \end{tikzcd}$$

where the top row is a cofiber sequence and the bottom row is a fiber sequence. Using the Serre long exact sequence, prove by induction on k that $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \to V_k(\mathbb{R}^n)$ is a $(2n-2k)$-equivalence.

(4) When $2k \leq n$, prove that S^{n-1} admits $(k-1)$ everywhere linearly independent vector fields if and only if the map $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1} \to \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \simeq S^{n-1}$ admits a section up to homotopy. (Hint: for the if direction, use the fact that $V_k(\mathbb{R}^n) \to V_1(\mathbb{R}^n)$ is a fibration to show that homotopy sections can be rigidified to genuine sections.)

(5) Using the action of the Steenrod operations on $H^*(\mathbb{RP}^{n-1}/\mathbb{RP}^{n-k-1}; \mathbb{F}_2)$, prove that S^{n-1} does not admit 2^m linearly independent sections, where $n = 2^m(2s+1)$.

[1] This result is not optimal. To obtain the optimal bound, Adams followed the same general strategy, replacing cohomology with *K-theory* and Steenrod operations with *Adams operations*.