Astron 140 Homework 1

Due Sept 15 at 11:59pm

1. Special relativity leads to some counter-intuitive phenomena such as the "slowing down" of a clock and "contraction of a ruler" in the moving frame. There are different ways to derive these results. In the lecture, we derived them using some thought experiments. In this exercise, let us derive them by directly using the Lorentz transformation.

   Let us consider two frames, unprimed (with coordinates labeled as t, x, y, z) and primed (t', x', y', z'). The primed frame is moving with a velocity v (in the positive-x direction) relative to the unprimed one. All events we will consider have $y = y' = 0$ and $z = z' = 0$, so you only need to consider the values of t (t') and x (x').

   First, consider two events in the unprimed frame. Event 1: $t_1 = 0$, $x_1 = 0$. Event 2: $t_2 = \Delta t$, $x_2 = 0$. In the unprimed frame, these two events simply describe a clock sitting statically at the origin for a time dilation Δt. According to the primed frame, this clock is moving with a velocity v. Using the Lorentz transformation, write down the coordinates t'_1 and x'_1 for event 1, and t'_2 and x'_2 for event 2. Interpret your results in terms of the primed frame. What is the time dilation between the same two events, now according the primed frame? Compare it with Δt and see how the two frames disagree upon the time dilation between the two events. (10 points)

2. Now, consider a ruler in the unprimed frame, lying still on the x-axis with its left end on the origin. This time, let us consider the values of the coordinates in the primed frame first. In the point of view of the primed frame, this ruler has a length L' and moves in the negative x'-direction with velocity v. As the left end of the ruler passes through the origin of the primed frame, we record it as event 1: $t'_1 = 0$, $x'_1 = 0$. As the other end passes through the origin, we record it as event 2. Convince yourself that the event 2 has the following coordinates: $t'_2 = L'/v$, $x'_2 = 0$.

   Now use the Lorentz transformation reversely. (Namely, from x' and t' to x and t, reversing those given in the lecture. Hint: you may mathematically reverse the relations in the lecture. You will find that the final expressions you got are very similar to the original ones, with simply a change from v to $-v$ and the exchange between the primed and unprimed coordinates. Think about why?). Write down the values of coordinates of these two events in the unprimed frame. Interpret the results in the unprimed frame. What is the length of this ruler according to the unprimed frame? Compare it with L'. (10 points)

   (In both problems above, please feel free if you prefer to use different methods as long as your methods use the Lorentz transformation.)

3. Two spaceships traveling in opposite directions pass one another at a relative speed of $1.0 \times 10^8 m/s$. The clock on one spaceship records a time duration of $9.0 \times 10^{-8} s$ for it to pass from the front end to the tail end of the other ship. What is the length of the second ship as measured in its own rest frame? (10 points)

4. Show the relativistic velocity composition law

$$u' = \frac{u - v}{1 - uv} \ . \tag{1}$$

All the variables are as defined in the lecture. The convention $c \equiv 1$ is used here and the problems below. Also show that, if the relative velocity between the two frames is much less than the speed of light, this composition law reduces to the one from the Galilean transformation. (10 points)

5. Show that under the Galilean transformation, the invariant intervals are Δt and $\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ (where spatial coordinates at two ends of the length interval are measured at the same time); but under the Lorentz transformation, the invariant interval is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$. (10 points)

6. * Show that if two events are timelike separated, i.e. $\Delta s^2 < 0$, there is a Lorentz frame in which they occur at the same point. Show that if two events are spacelike separated, i.e. $\Delta s^2 > 0$, there is a Lorentz frame in which they are simultaneous. (10 points)

7. ** *Twin paradox.* Alice decides to make an interstellar trip while her twin sister Barbara stays at Earth (assuming Earth is a good inertial frame). Assume that the Alice's spaceship has a constant velocity v relative to Barbara. She reaches her destination which is at a distance D away from the Earth *according to Barbara.* As soon as Alice reaches the destination, she immediately turns around the spaceship and travels back to Earth with the same velocity.

   Naively thinking, since either of them is moving relative to the other throughout the trip, according to Alice, Barbara's biological clock slows down; and according to Barbara, Alice's biological clock slows down as well. When they are reunited, naively each one thinks that the other is younger. But this cannot happen because there can only be one answer. (E.g. when they meet, they can just compare their clocks.)

   Explain *qualitatively* in simple words what is wrong in the above thinking.

   Show *quantitatively*, from both Alice's point of view and Barbara's point of view, that, at the reunion when they can compare their notes, the two views agree with each other on the conclusion about the ages of both persons. Who is the younger one? (20 points)

   Hint: The key is that the Alice's frame is not always an inertial frame. In order to return to Earth, at the destination, she has to accelerate and change the direction of her velocity. So the Lorentz transformation laws are different before and after the turnaround.

   Before, it is

$$\begin{cases} t = \gamma(t' + vx') \ , \\ x = \gamma(vt' + x') \ . \end{cases}$$

   After, it is

$$\begin{cases} t - t_0 = \gamma(t' - vx') \ , \\ x - x_0 = \gamma(-vt' + x') \ . \end{cases}$$

   Notice that the difference between them is more than the sign switch in front of velocity (because the velocity of spaceship turns around), there are also two parameters t_0 and x_0. These two parameters are zero only if Alice never changes the inertial frame. To figure out these two parameters, we use the following condition. Because the turnaround at the destination is an instant process, immediately before and after the turnaround, Alice's coordinates are the same, both in terms of her own primed frame and in terms of Barbara's unprimed frame.

   You may of course use a different method.