Math 55b Problem Set 6

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I collaborated with AJ LaMotta for this problem set.

Problem 1. Suppose $X = U \cup V$ where U, V are open subsets of X. Suppose that $U \cap V$ is path connected and that $x_0 \in U \cap V$. Let i, j be the inclusion mappings of U and V respectively into X. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$ by Theorem 59.1.

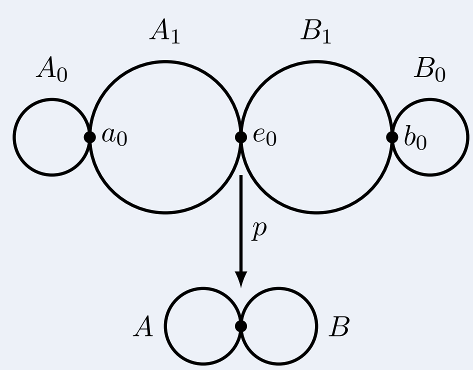
(a) What can you say about the fundamental group of X if j_* is the trivial homomorphism? If both i_* and j_* are trivial?

(b) Give an example where i_* and j_* are trivial but neither U nor V have trivial fundamental groups.

(a) If j_* is trivial, then $\pi_1(X, x_0)$ is generated by $i_*(\pi_1(U, x_0))$ so $\pi_1(X, x_0)$ is isomorphic to a quotient of the group $\pi_1(U, x_0)$. If both i_* and j_* are trivial, then $\pi_1(X, x_0)$ is trivial so x_0 is in a simply connected path component.

(b) Let $X = \mathbb{R}^2$ and define $U = \mathbb{R}^2 - \{(1, 0)\}$ and $V = \mathbb{R}^2 - \{(-1, 0)\}$. Clearly $U \cap V$ is path connected. Letting $x_0 \in U \cap V$, clearly $\pi_1(U, x_0) \cong \pi_1(V, x_0) \cong \mathbb{Z}$, since U and V have deformation retractions onto circles. Yet $\pi_1(\mathbb{R}^2, x_0)$ is trivial, so the induced homomorphisms i_* and j_* must also be trivial homomorphisms.

Problem 2. Consider the covering map indicated in the figure:



Here p wraps A_1 around A twice and wraps B_1 around B twice; p maps A_0 and B_0 homeomorphically onto A and B, respectively. Use this covering space to show that the fundamental group of the figure eight is not abelian.

Similarly to the proof in Munkres, let α be the loop going counterclockwise around A and let β be the loop going counterclockwise around B. Then lifting $\alpha * \beta$ to a path starting at e_0, we get a path going along the bottom of A_1 to a_0 and then going counterclockwise around A_0. Similarly, lifting $\beta * \alpha$ gives a path going along the top of B_1 to b_0 and then going counterclockwise around B_0 back to b_0. Since these lifts have different endpoints, $\alpha * \beta \neq \beta * \alpha$ so the fundamental group of the figure eight is not abelian.

Problem 3. Show that if $n > 1$, every continuous map $f : S^n \to S^1$ is nullhomotopic.

Recall the covering map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$. Now assume without loss of generality that $f^{-1}(1)$ is nonempty (otherwise rotate the map), let $x_0 \in f^{-1}(1)$. Then $f_*(\pi_1(S^n, x_0)) = \{0\}$ since S^n is simply connected for $n > 1$. Then by the general lifting lemma, there must be a map $\widetilde{f} : S^n \to \mathbb{R}$ such that $\widetilde{f}(x_0) = 0$ and $p \circ \widetilde{f} = f$. We claim that $f \simeq c_1$, the constant map which takes every element to 1. First note that $\widetilde{f} \simeq c_0$, since $\mathbb{R}$ is contractible. Let $H : I \times S^n \to \mathbb{R}$ be some homotopy with $H(0, x) = \widetilde{f}(x)$ and $H(1, x) = 0$. Then consider $p \circ H : I \times S^n \to S^1$. This is a homotopy with $H(0, x) = p(\widetilde{f}(x)) = f(x)$ and $H(1, x) = 1$, so $f \simeq c_1$ as desired.

Problem 4. Let $T = S^1 \times S^1$ be the torus. There is an isomorphism of $\pi_1(T, b_0 \times b_0)$ with $\mathbb{Z} \times \mathbb{Z}$ induced by projections of T onto its two factors.

(a) Find a covering space of T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $m \times 0$, where m is a positive integer.

(b) Find a covering space of T corresponding to the trivial subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(c) Find a covering space of T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $m \times 0$ and $n \times 0$ where m, n are positive integers.

Let $p_n : S^1 \to S^1$ given by $p_n(z) = z^n$ be an n-sheeted covering of the circle, and let $q : \mathbb{R} \to S^1$ given by $q(t) = e^{2\pi i t}$ be the universal covering space of the circle.

(a) Consider the covering space $p_m \times q : S^1 \times \mathbb{R} \to S^1 \times S^1 = T$. Then by basic properties of the induced homomorphism, $(p_m \times q)_* = (p_m)_* \times q_*$ so $\mathrm{Im}((p_m \times q)_*) = m\mathbb{Z} \times 0$, which is the desired subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(b) This is just the universal cover of T, i.e. a simply connected covering space of T. Consider the covering space $q \times q : \mathbb{R}^2 \to S^1 \times S^1 = T$. Since $\mathbb{R}^2$ is simply connected, it clearly corresponds to the trivial subgroup of $\mathbb{Z} \times \mathbb{Z}$.

(c) Consider the covering space $p_m \times p_n : S^1 \times S^1 \to S^1 \times S^1 = T$. As in (a), we have $\mathrm{Im}((p_m \times p_n)_*) = \mathrm{Im}((p_m)_*) \times \mathrm{Im}((p_n)_*) = m\mathbb{Z} \times n\mathbb{Z}$, which is the desired subgroup of $\mathbb{Z} \times \mathbb{Z}$.

Problem 5. Show that there do not exist any covering maps:

(a) from $\mathbb{RP}^2$ to T (torus),

(b) from T to $\mathbb{RP}^2$,

(c) from $\mathbb{R}^2$ to $\mathbb{RP}^2$.

In lecture, we proved that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$, and $\pi_1(\mathbb{R}^2) = \{0\}$. Furthermore, for any covering map $p : E \to B$, the induced homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, p(e_0))$ is an injection. This shows that there is no covering map from $\mathbb{RP}^2$ to T or T to $\mathbb{RP}^2$ to T, since there can be no injective homorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Now suppose for the sake of contradiction that there were a covering map $p : \mathbb{R}^2 \to \mathbb{RP}^2$, and consider the natural 2-sheeted covering map $q : S^2 \to \mathbb{RP}^2$. Then letting $e_0 \in p^{-1}(b_0)$ and $e_0' \in q^{-1}(b_0)$ for some $b_0 \in \mathbb{RP}^2$, we know that $\pi_1(\mathbb{R}^2, e_0) = \pi_1(S^2, e_0') = \{0\}$ so by Theorem 79.2, there is an equivalence $h : S^2 \to \mathbb{R}^2$. However S^2 is compact while $\mathbb{R}^2$ is not, so we have a contradiction since every equivalence is a homeomorphism. Thus there is no covering map $p : \mathbb{R}^2 \to \mathbb{RP}^2$.

Problem 6. Let $p : E \to B$ and $p' : E' \to B$ be maps of topological spaces. The *fiber product* of E and E' over B, denoted $E \times_B E'$, is the set

$$E \times_B E' \;=\; \{(e, e') \in E \times E' \mid p(e) = p'(e')\},$$

with the subspace topology from the inclusion in $E \times E'$. Note that $E \times_B E'$ also has a natural map q to B, sending (e, e') to $p(e) = p'(e')$.

(a) Show that if $p : E \to B$ and $p' : E' \to B$ are covering maps, then $q : E \times_B E' \to B$ is a covering map as well.

(b) Suppose $B = S^1$, $E = S^1$ and $E' = S^1$, with the maps

$$p = p' : z \;\mapsto\; z^2.$$

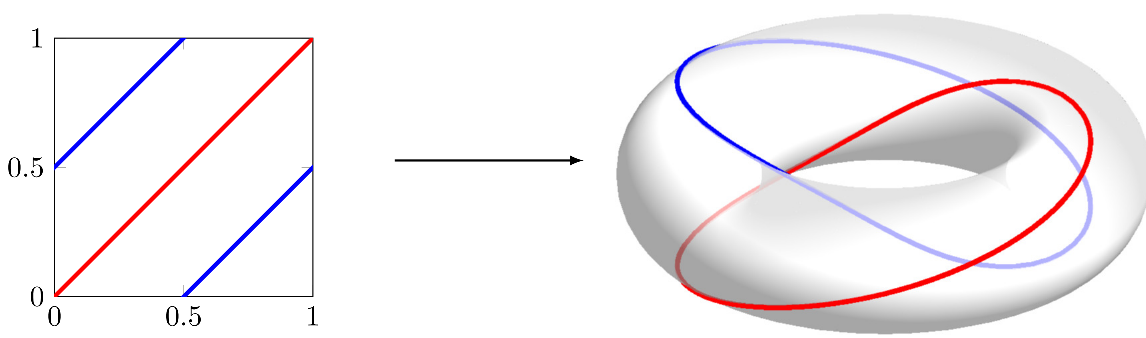
(here we think of S^1 as $\{z \in \mathbb{C} \,|\, |z| = 1\}$). Describe the covering space $q : E \times_B E' \to B$ in this case.

(c) Same as the preceding question, but now take

$$p : z \;\mapsto\; z^2 \;\text{ and }\; p' : z \;\mapsto\; z^3.$$

(a) Note that there is a natural covering map $p \times p' : E \times E' \to B \times B$ given by $(p \times p')(e, e') = (p(e), p'(e'))$. Consider the set $B_0 = \{(b, b) \mid b \in B\}$ and $E_0 = (p \times p')^{-1}(B_0)$. Then the restriction map $(p \times p')\big|_{E_0}$ is a covering map. Clearly $E_0 = E \times_B E'$ since $E \times_B E'$ has the subspace topology, and B_0 is homeomorphic to B by the map $h : B_0 \to B$ given by $(b, b) \mapsto b$. So $h \circ (p \times p')\big|_{E_0} : E \times_B E' \to B$ is a covering map and equal to q so we are done.

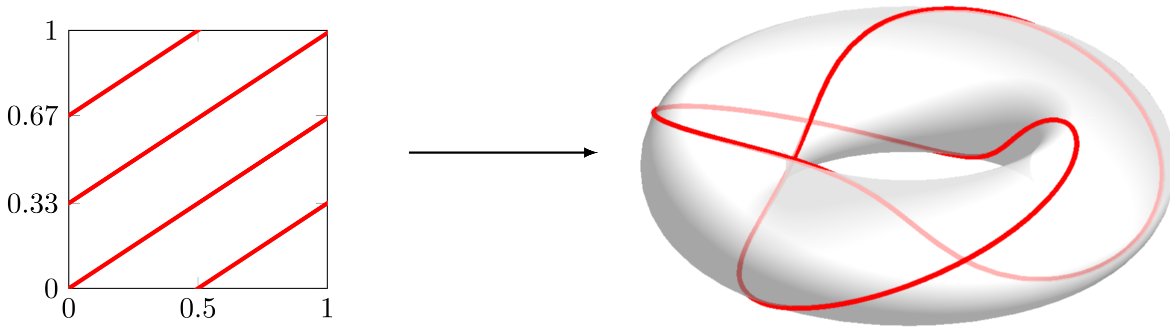
(b) The space $E \times_B E'$ is the subset of the torus $S^1 \times S^1$ defined by $\{(z_1, z_2) \in T \mid z_1^2 = z_2^2\}$. If we let $z_1 = e^{2\pi i \theta_1}$ and $z_2 = e^{2\pi i \theta_2}$, we can rewrite the condition $z_1^2 = z_2^2$ as $2\theta_1 \equiv 2\theta_2 \mod 1$. The set of solutions to this congruence looks like:[1]



So as a topological space, $E \times_B E'$ looks like a disjoint union of two circles. As a covering space with the induced map $q : E \times_B E' \to B$, each of these circles wraps twice around B, so this is a two sheeted covering of S^1, equivalent to the disjoint union of two $z \mapsto z^2$ coverings of the circle.

(c) Here we can do what we did in (b), but the modulo relation becomes $2\theta_1 \equiv 3\theta_2 \mod 1$, which looks like:

[1] Yes, this took hours but I would sooner fall upon my sword than submit a handwritten pset.



This is a connected curve known as a $(2,3)$-torus knot, but as a topological space, $E \times_B E'$ is a circle. The covering map wraps this circle around the base circle 6 times, so this covering is equivalent to the 6 sheeted covering $z \mapsto z^6$.

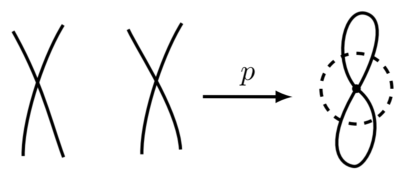
Problem 7. Let X be the figure eight space, with base point x_0 where the two circles are glued. Let a and b be the two free generators of $\pi_1(X, x_0)$ corresponding to the loops around each circle.

(a) What are the connected degree 2 coverings of X up to equivalence? Draw them! Make sure to label and orient the edges (as in Figure 60.3).

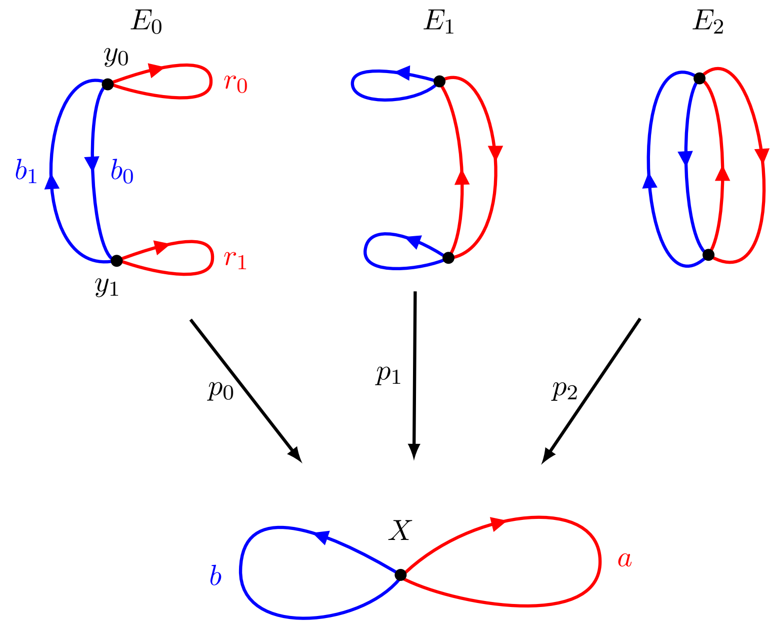
(b) For each covering you drew in (a), give a list of generators for the corresponding subgroup of $\pi_1(X, x_0)$.

(Remark: what does this say about index 2 subgroups of the free group on two generators?)

(a) To classify connected degree 2 coverings, we start by taking some open neighborhood of the gluing point and looking at its preimage in the covering space. Since the covering is degree 2, geometrically the preimage looks like two Xs.



Then to determine all of the (connected) degree 2 coverings, it suffices to find all ways to "connect" the endpoints in a way which yields a degree 2 covering. There are exactly three ways:



The spaces E_0, E_1, E_2 have natural covering maps p_0, p_1, p_2 as shown in the diagram. For example the map $p_0 : E_0 \to X$ maps the red loops in E_0 homeomorphically onto the red loop in X, and maps each blue path in E_0 onto the blue loop in X. It's clear that these are all degree 2 coverings of X.

(b) We'll calculate the corresponding subgroup of F_2 for E_0 explicitly, the rest follow similarly. Applying the Seifert Van Kampen theorem to E_0, spitting it along some small open neighborhood or y_0, we get that $\pi_1(E_0, y_0)$ is generated by the loops r_0, $b_0 * r_1 * b_1$, and $b_0 * b_1$. Letting a be the loop in X going around the red loop clockwise and b be the loop in X going counterclockwise around the blue loop. These generate $\pi_1(X, x_0) \cong F_2$. The induced subgroup of F_2 by E_0 is then generated by the images of the generators of $\pi_1(E_0, y_0)$ under the covering map p_0. Note that

$$p_0 \circ r_0 = a$$
$$p_0 \circ (b_0 * r_1 * b_1) = bab$$
$$p_0 \circ (b_0 * b_1) = b^2$$

so $(p_0)_*(\pi_1(E_0, y_0)) = \langle a, bab, b^2 \rangle$. Similarly we have $(p_1)_*(\pi_1(E_1, y_0)) = \langle a^2, aba, b \rangle$ and $(p_2)_*(\pi_1(E_2, y_0)) = \langle a^2, ab, ab^{-1} \rangle$. By the classification of covering spaces, these are all of the index two subgroups of F_2.

Problem 8. What is the fundamental group of a torus with one point removed? With two points removed? Can you come up with an example of a degree 2 covering map between these two spaces?

Recall from the previous problem set that we can find a deformation retraction of $T - \{p\}$ onto a wedge of two circles $S^1 \vee S^1$ and so $\pi_1(T - \{p\})$ is isomorphic to F_2, the free group on two generators. Similarly, we can find a deformation retraction of $T - \{p, q\}$ onto a wedge of three circles $S^1 \vee S^1 \vee S^1$ and so $\pi_1(T - \{p, q\})$ is isomorphic to F_3, the free group on three generators. In light of the counterintuitive group embedding $F_3 \to F_2$ from 55a, we'll try to construct a two sheeted covering map from $T - \{p, q\}$ to $T - \{p\}$.

Consider the two sheeted covering $f : S^1 \to S^1$ given by $z \mapsto z^2$ where $z \in S^1$ is considered as a complex number. This defines a two sheeted covering $g = f \times \mathrm{id}_{S^1} : T \to T$. Then if we pick some point $p \in T$, and let $\{q_1, q_2\} = g^{-1}(p)$ we have an induced two sheeted covering map $g|_{T - \{q_1, q_2\}} : T - \{q_1, q_2\} \to T - \{p\}$. This induces an injective group homomorphism $g_* : F_3 \to F_2$.