Math 230a Problem Set 2

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Problem 1. Let V be a finite dimensional real vector space and $B : V \times V \to \mathbb{R}$ a non-degenerate bilinear form. Define:

$$\mathrm{Aut}_B(V) = \{B \in \mathrm{Aut}(V) : B(P\xi_1, P\xi_2) = B(\xi_1, \xi_2) \quad \text{for all} \quad \xi_1, \xi_2 \in V\},$$
$$\mathrm{End}_B(V) = \{A \in \mathrm{End}(V) : B(A\xi_1, \xi_2) + B(\xi_1, A\xi_2) = 0 \quad \text{for all} \quad \xi_1, \xi_2 \in V\}.$$

(a). Prove that $\mathrm{Aut}_B(V)$ is a Lie group with Lie algebra $\mathrm{End}_B(V)$.

(b). Let $V = \mathbb{R}^n$ for some $n \in \mathbb{Z}^{>0}$. Suppose B is the standard symmetric inner product. Identify $\mathrm{Aut}_B(\mathbb{R}^n)$ with the group O_n of orthogonal matrices.

(b). Let $V = \mathbb{R}^{2m}$ for some $m \in \mathbb{Z}^{>0}$. Suppose B is a non-degenerate skew-symmetric form: For the standard basis $e_1, \dots, e_{2m}$ of $\mathbb{R}^{2m}$, set

$$B(e_i, e_j) = \begin{cases} 1 & 0 < j - i \leq m, \\ -1 & 0 < i - j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Identify the group $\mathrm{Aut}_B(\mathbb{R}^{2m})$ explicitly in terms of block 2×2 matrices in which the blocks have size $m \times m$. This is the *symplectic group* Sp_2.

Problem 2.

Problem 5. Let G be a Lie group.

(a). Let V be a finite dimensional real vector space. Define a real line $|\operatorname{Det} V|$ such that an ordered n-tuple $\xi_1, \ldots, \xi_n \in V$ defines an element $|\xi_1 \wedge \cdots \wedge \xi_n| \in |\operatorname{Det} V|$ which transforms by the absolute value of the determinant of a change of basis matrix. Identify $|\operatorname{Det} V^*|$ as a certain space of functions $V^n \to \mathbb{R}$. Show that positive functions determine an orientation of $|\operatorname{Det} V^*|$. Interpret a positive function as a notion of volume for n-dimensional parallelepipeds in V. Does this induce a notion of volume for lower dimensional parallelipipeds? Identify positive elements as translationally invariant positive measures on V. Construct such a positive element from an inner product on V.

There is a natural right action of $\operatorname{Aut}(V)$ on V^n which acts on each component independently. Let's define the line by:

$$|\operatorname{Det} V| = \left\{ \epsilon : \mathscr{B}(V) \to \mathbb{R} : \epsilon(b \cdot g) = \frac{\epsilon(b)}{|\det(g)|} \quad \text{for all} \quad b \in V^n, g \in \operatorname{Aut}(V) \right\}.$$

First of all, it's clear that this space is closed under addition and scalar multiplication, so it is a real vector space. To see that it's a one dimensional space, consider the map $\mathrm{ev}_1 : |\operatorname{Det} V|$.

(b). Apply to the tangent bundle of a smooth manifold. Define the notion of a smooth positive measure on a smooth manifold. Do they always exist?

(c). The real line $|\operatorname{Det} \mathfrak{g}^*|$ consists of left-invariant measures on G. Define an action of G on this line. Compute the action in case G is compact. Compute it for $G = \mathrm{GL}_n$ and $G = \mathrm{SL}_n$.

(d). A *Haar measure* on G is a bi-invariant positive smooth measure on G. Prove that a Haar measure exists if G is compact. Normalize it so the total volume of G is 1.

(e). Write a formula for the Haar measure on the circle group $\mathbb{T} \subset \mathbb{C}$; the formula should be in terms of $\lambda \in \mathbb{T}$. What about on the multiplicative group $\mathbb{R}^\times$. What about on the additive group $\mathbb{R}$? What about on the orthogonal group O_2?

Problem 6. Suppose G is a connected compact Lie group.

(a). Let $\Omega^\bullet_{\mathrm{linv}}(G) \subset \Omega^\bullet(G)$ denote the vector subspace of left-invariant differential forms. Show that $\Omega^\bullet_{\mathrm{linv}}(G)$ is in fact a sub-differential graded algebra, i.e. it is closed under multiplication and the differential d.

(b). Construct an isomorphism

$$\textstyle\bigwedge^\bullet \mathfrak{g}^* \longrightarrow \Omega^\bullet_{\mathrm{linv}}(G).$$

Transfer the differential on $\Omega^\bullet_{\mathrm{linv}}(G)$ to $\bigwedge^\bullet \mathfrak{g}^*$ and write a formula for it. In this way you obtain a differential graded complex defined directly from the Lie algebra $\mathfrak{g}$. Observe that this definition of *any* Lie algebra.

Firstly, recall that there is a natural "extension by left-translation" injective map $\mathfrak{g}^* \to \mathfrak{X}^*(G)$ where $\mathfrak{X}^*(G) = \Gamma(T^*G)$ is the space of covector fields. More precisely, given some $\omega \in \mathfrak{g}^*$, the corresponding covector field ξ on G is defined by:

$$\widetilde{\omega}(\xi) = \omega \circ dL_{g^{-1}}(\xi) \quad \text{for} \quad \xi \in T_g G.$$

Such a covector field is exactly a differential 1-form, so we've exhibited a map $\mathfrak{g}^* \to \Omega^1(G)$. The form $\widetilde{\omega}$ is left-invariant because for any $h \in G$, we have

$$(L_h^* \widetilde{\omega})(\xi) = \omega \circ dL_{(hg)^{-1}} \circ dL_h(\xi) = \omega \circ dL_{g^{-1}}(\xi) = \widetilde{\omega}(\xi) \quad \text{for all} \quad \xi \in T_g G.$$

This means that we actually have a linear map $\mathfrak{g}^* \to \Omega^1_{\text{linv}}(G)$. Since the inverse is given by $\omega = \widetilde{\omega}_e$, we have an isomorphism. There is also an isomorphism $\mathbb{R} \to \Omega^0_{\text{linv}}(G)$ which sends a constant to the constant function on G. This is an isomorphism since the only left-invariant functions are the constant functions. This pair of isomorphisms uniquely extends to a graded algebra isomorphism $\bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet_{\text{linv}}(G)$.

To express the differential d as a coboundary map in $\bigwedge^\bullet \mathfrak{g}^*$, first note that $df = 0$ for any 0-form $f \in \Omega^0_{\text{linv}}(G)$ since left-invariant 0-forms are constant. To derive an expression for 1-forms, let $\omega \in \mathfrak{g}^*$ be a covector. Given vector fields $\xi_1, \xi_2 \in \mathfrak{X}(G)$, a corollary of Cartan's formula tells us that:

$$d\widetilde{\omega}(\xi_1, \xi_2) = \xi_1(\widetilde{\omega}(\xi_2)) - \xi_2(\widetilde{\omega}(\xi_1)) - \widetilde{\omega}([\xi_1, \xi_2]) = -\widetilde{\omega}([\xi_1, \xi_2]).$$

Here the terms $\xi_1(\widetilde{\omega}(\xi_2))$ and $\xi_2(\widetilde{\omega}(\xi_1))$ vanish since $\widetilde{\omega}$ is left-invariant. Let $\{\xi_i\}$ be a basis for $\mathfrak{g}$ and define structure coefficients $c^k_{i,j}$ by $[\xi_i, \xi_j] = c^k_{i,j}\xi_k$. Let $\{\theta^i\}$ be the corresponding dual basis for $\mathfrak{g}^*$. Note that:

$$d\theta^k(\xi_i, \xi_j) = -\widetilde{\theta^k}([\xi_i, \xi_j]) = -\widetilde{\theta^k}(c^q_{i,j}\xi_q) = -c^q_{i,j}\widetilde{\theta^k}(\xi_q) = -c^k_{i,j},$$

where the last equality follows since $\theta^k(\xi_q) = \delta^k_q$. Writing this this form in terms of $\bigwedge^\bullet \mathfrak{g}^*$, we get the expression

$$d\theta^k = -c^k_{i,j}\theta^i \wedge \theta^j.$$

Along with the observation that $df = 0$ for any 0-form f, using the Leibniz rule this coboundary operator extends over the entire graded algebra so that the isomorphism $\bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet_{\text{linv}}(G)$ is an isomorphism of

(c). Prove that the inclusion in (a) induces an isomorphism on cohomology. A map of cochain complexes with this property is called a *quasi-isomorphism.*

To show that inclusion (we'll call it ι) is a quasi-isomorphism, we'll prove that $\Omega^\bullet_{\text{linv}}(G)$ is a deformation retract of $\Omega^\bullet(G)$. To do this, we'll have to construct two operators, or cochain maps:

$$A : \Omega^\bullet(G) \to \Omega^\bullet_{\text{linv}}(G) \quad \text{and} \quad H : \Omega^\bullet(G) \to \Omega^{\bullet-1}(G).$$

Here, A is a cochain map satisfying $A \circ \iota = \text{id}$ and $A \circ (\iota \circ A) = A$, and H is a linear map satisfying $dH + Hd = A - 1$. Put together, these maps would prove that ι induces an isomorphism on cohomology.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{k-1}_{\text{linv}}(G) \arrow[r, "d"] \arrow[d, hook, shift left, "\iota"] & \Omega^{k}_{\text{linv}}(G) \arrow[r, "d"] \arrow[d, hook, shift left, "\iota"] & \Omega^{k+1}_{\text{linv}}(G) \arrow[r] \arrow[d, hook, shift left, "\iota"] & \cdots \\
\cdots \arrow[r] & \Omega^{k-1}(G) \arrow[r, shift left, "d"] \arrow[u, shift left, "A"] & \Omega^{k}(G) \arrow[r, shift left, "d"] \arrow[l, dashed, shift left, "H"] \arrow[u, shift left, "A"] & \Omega^{k+1}(G) \arrow[r] \arrow[l, dashed, shift left, "H"] \arrow[u, shift left, "A"] & \cdots
\end{tikzcd}$$

Using the assumption that G is compact, let μ be a left-invariant Haar measure on G. Since G is compact, we scale μ by a factor of $1/\mu(G)$ so that $\mu(G) = 1$. First, let's use this measure to construct A. A succinct form for A is:

$$A = \int_G L_h^* \, d\mu(h) \quad \Longrightarrow \quad A(\omega)_g(\xi_1, \ldots, \xi_k) = \int_G (L_h^*\omega)_g(\xi_1, \ldots, \xi_k) \, d\mu(h)$$

for all $g \in G$, $\omega \in \Omega^k(G)$, and $\xi_1, \ldots, \xi_k \in T_gG$. Clearly, if ω is already left-invariant, then $A(\omega) = \omega$ since $L^*h\omega = \omega$. For any $g' \in G$, we can act on A to get:

$$L^*_{g'}A = \int_G L^*_{hg'}\, d\mu(h) = A(\omega)$$

since the transformation $h \mapsto hg'$ is a bijection and left multiplication preserves the measure μ. This shows that $A \circ \iota$ is the identity on $\Omega^\bullet_{\mathrm{linv}}(G)$ as well as $A \circ (\iota \circ A) = A$. A is a cochain map because differentials commute with integration, i.e. we have

$$A \circ d = \int_G L^*_h \circ d\, d\mu(h) = \int_G d \circ L^*_h\, d\mu(h) = d \circ A.$$

This proves A is a retract the cochain complexes. To show that A is a deformation retract, we must construct the cochain homotopy operator $H : \Omega^\bullet(G) \to \Omega^{\bullet-1}(G)$ which satisfies $dH + Hd = A - 1$. For any vector $\xi \in \mathfrak{g}$, define the operator

$$H_\xi = \int_0^1 L^*_{\exp(t\xi)} \iota_{R_\xi}\, d\mu(h).$$

where R_ξ is the right-invariant vector field generated by ξ. Computing $dH_\xi + H_\xi d$, we get:

$$\begin{aligned}
dH_\xi + H_\xi d &= d\int_0^1 L^*_{\exp(t\xi)} \iota_{R_\xi}\, d\mu(t) + \int_0^1 L^*_{\exp(t\xi)} \iota_{R_\xi} d\, d\mu(t) \\
&= \int_0^1 L^*_{\exp(t\xi)} (d\iota_{R_\xi} + \iota_{R_\xi} d)\, d\mu(t) \\
&= \int_0^1 L^*_{\exp(t\xi)} \mathcal{L}_{R_\xi}\, d\mu(t) \\
&= \int_0^1 L^*_{\exp(t\xi)} \frac{d}{ds}\bigg|_{s=0} L^*_{\exp(s\xi)}\, d\mu(t) \\
&= \int_0^1 \frac{d}{ds}\bigg|_{s=t} L^*_{\exp(s\xi)}\, d\mu(t) \\
&= L^*_{\exp(\xi)} - 1.
\end{aligned}$$

Now, let $\{U_\alpha\}$ be a locally finite open cover of G such that there are diffeomorphisms $\log_\alpha : U_\alpha \to V_\alpha \subset \mathfrak{g}$ with $\exp(\log_\alpha(g)) = g$ for all $g \in U_\alpha$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to this open cover.

Consider the operator:

$$H = \sum_\alpha \int_{U_\alpha} \psi_\alpha(h) \cdot H_{\log_\alpha(h)}\, d\mu(h).$$

Using our previous expression for $dH_\xi + H_\xi d$, we get:

$$\begin{aligned}
dH + Hd &= \sum_\alpha \int_{U_\alpha} \psi_\alpha(h) \cdot (dH_{\log_\alpha(h)} + H_{\log_\alpha(h)} d)\, d\mu(h) \\
&= \sum_\alpha \int_{U_\alpha} \psi_\alpha(h) \cdot (L^*_h - 1)\, d\mu(h) \\
&= \int_G L^*_h\, d\mu(h) - 1 \\
&= A - 1.
\end{aligned}$$

This completes the proof.

(d). Use the inverse map $g \mapsto g^{-1}$ to show that the differential of a *bi-invariant* differential form vanishes. Show that the de Rham cohomology of G is isomorphic to the algebra of bi-invariant forms.

(e). Use these ideas to compute $\mathrm{H}^\bullet_{\mathrm{dR}}(\mathrm{SU}_2)$.