Math 55b Problem Set 2

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February 8, 2022

I collaborated with AJ LaMotta for this problem set, and it took me about 10 hours.

Problem 1. The *Zariski topology* on $\mathbb{R}^2$ is the topology generated by the basis

$$U_f = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \neq 0\}$$

where f ranges over all polynomials in $\mathbb{R}[x, y]$.

(a) Show that the subsets $U_f \subset \mathbb{R}^2$ are indeed a basis for a topology.

(b) Show that this topology is not Hausdorff.

(c) What is the closure $\overline{I}$ of the line segment $I = \{(x, 0) \mid x \in [0, 1]\} \subset \mathbb{R}^2$ in the Zariski topology?

(d) If $\mathbb{R} \subset \mathbb{R}^2$ is the x-axis, show that the subspace topology on $\mathbb{R}$ induced by the Zariski topology is the finite complement topology.

(a) To prove that U_f form a basis set for a topology, we need to show that the basis elements cover the whole space, and that the intersection of two basis elements contains a third basis element. Clearly, the basis elements cover the whole space, just consider $U_1 = \mathbb{R}^2$. Next, suppose $f_1, f_2 \in \mathbb{R}[x, y]$. Then if $U_{f_1} \cap U_{f_2} \neq \varnothing$, it must contain some point (x_0, y_0). We are done since $(x, y) = U_{(x-x_0)^2+(y-y_0)^2}$.

(b) If the topology were Hausdorff, then any subspace of it would also be Hausdorff. However by (d), $\mathbb{R} \subset \mathbb{R}^2$ has the finite complement topology, which isn't Hausdorff because any two open sets have nontrivial intersection.

(c) By (d), $\overline{I} \cap \mathbb{R}$ is equal to the closure of $I \cap \mathbb{R}$ in $\mathbb{R}$ with the finite complement topology. However in the finite complement topology, the only closed sets are collections of finite sets or the space itself. Thus $\overline{I} \cap \mathbb{R}$ must be $\mathbb{R}$ since it has an infinite number of points. However $\mathbb{R}$ is a closed subset of $\mathbb{R}^2$ because $\mathbb{R} = \mathbb{R}^2 - U_y$. So $\overline{I} = \mathbb{R} = \{(x, 0) \mid x \in R\}$.

(d) Clearly for any $f \in \mathbb{R}[x, y]$, $U_f \cap \mathbb{R} = \{x \in \mathbb{R} \mid f(x, 0) \neq 0\}$. $f(x, 0)$ is a polynomial $\mathbb{R}[x]$, so it either vanishes completely or has a finite number of roots. This is exactly an open set in the finite complement topology. Conversely, if $U = \mathbb{R} - \{x_1, x_2, \ldots, x_n\}$ is an open subset in the finite complement topology, note that $U_{(x-x_1)\cdots(x-x_n)} \cap \mathbb{R} = U$.

Problem 2. Let $x_1, x_2, x_3, \ldots$ be a sequence of points in a product space $\prod_\alpha X_\alpha$. Show that this sequence converges to a point x if and only if the sequence $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges to $\pi_\alpha(x)$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Suppose $x_1, x_2, x_3, \ldots$ is a sequence of points in $X = \prod_\alpha X_\alpha$ converging to some $x \in X$. Then since the projection maps π_α are continuous in both product and box topologies, so $\pi_\alpha(x_i)$ converge to $\pi_\alpha(x)$ in X_α.

Conversely, suppose $\pi_\alpha(x_i)$ converges to $\pi_\alpha(x)$ for all α. Let U be some open neighborhood of x, which contains some open set $\prod_\alpha U_\alpha$. By definition of the box topology, all but finitely many of the U_α are not equal to X_α, say $U_{\alpha_1}, \ldots, U_{\alpha_n}$. Since $\pi_{\alpha_j}(x_i)$ converges to $\pi_{\alpha_j}(x)$, there exist $N_{\alpha_1}, \ldots, N_{\alpha_n}$ such that $m \geq N_{\alpha_j}$ implies $\pi_{\alpha_j}(x_m) \in U_{\alpha_j}$. So letting $N = \max_j\{N_{\alpha_1}, \ldots, N_{\alpha_n}\}$, it follows that $x_m \in \prod_\alpha U_\alpha$ for all $m \geq N$.

This argument fails in the box topology because the maximum of an infinite collection of integers need not exist, so if you had a sequence in an infinite product space which converges slower and slower in each coordinate, it may never converge in the box topology.

Problem 3. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

First, we claim that $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$ in the product topology, i.e. $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. Recall that any open set in $\mathbb{R}^\omega$ is of the form $U = U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \cdots$. Then, the sequence $x = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$ is in U for some $x_i \in U_i$. But $x \in \mathbb{R}^\infty$, so every open set in $\mathbb{R}^\omega$ intersects $\mathbb{R}^\infty$ nontrivially, which implies density.

In the box topology, $\mathbb{R}^\infty$ is closed, so $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. To show this suppose $x \notin \mathbb{R}^\infty$, so there are an infinite number of terms in x which are nonzero. Then

$$U = \prod_i U_i \quad \text{where } U_i = \begin{cases} (x_i - |x_i|/2, x_i + |x_i|/2) & x_n \neq 0 \\ (-1, 1) & x_n = 0 \end{cases}$$

is an open set containing x which is disjoint from $\mathbb{R}^\infty$.

Problem 4. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) - (A \times B)$$

is connected.

Suppose $F : (X \times Y) - (A \times B) \to \{0, 1\}$ is a continuous function. We claim that F must be constant, since this will imply connectedness of $(X \times Y) - (A \times B)$. (If it was disconnected, then F could take on different values for all of the connected components.)

Fix some $a \in X - A$ and $b \in Y - B$. Now consider an arbitrary point $(x, y) \in (X \times Y) - (A \times B)$. We have two cases. If $x \notin A$, the set $\{x\} \times Y$ is connected so $F(x, y) = F(x, b)$. Similarly, $X \times \{b\}$ is connected, so $F(x, b) = F(a, b)$. So $F(x, y) = F(a, b)$. Otherwise, if $x \in A$ then

$y \notin B$, so we can use a similar argument to get $F(x, y) = F(a, b)$. Since $F(a, b)$ was fixed, it follows that F is constant so $(X \times Y) - (A \times B)$ is connected.

Problem 5. Let $\mathbb{R}_\ell$ denote the real line with the *lower limit topology*, generated by the basis consisting of all intervals $[a, b)$, $a < b$.

(a) Show that $\mathbb{R}_\ell$ is *totally disconnected*, i.e. its only (nonempty) connected subspaces are subsets consisting of a single point.

(b) Say a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous from the right* (in the usual topology) if, $\forall a \in \mathbb{R}$, $\lim_{x \to a^+} f(x) = f(a)$, i.e. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $a < x < a + \delta \Rightarrow |f(x) - f(a)| < \epsilon$. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right if and only if it is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.

(c) What functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$?

(d) What can you say about functions which are continuous as maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$?

(a) Note that we have open sets for any $b \in \mathbb{R}$:

$$(-\infty, b) = \bigcup_{a \le b} [a, b), \quad [b, \infty) = \bigcup_{b \le a} [b, a).$$

Then any set X containing more than a single element, say $a < b$ can be separated by disjoint open sets of the form $a \in (-\infty, b)$ and $b \in [b, \infty)$.

(b) First suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right, and fix $a \in \mathbb{R}$. For each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $a < x < a + \delta$, we have $|f(x) - f(a)| < \epsilon$. This implies that $f([a, a + \delta)) \subset B_\epsilon(f(p))$, so $f^{-1}((a, b))$ is open in the lower limit topology. Conversely suppose $f : \mathbb{R}_\ell \to \mathbb{R}$ is continuous. Then for any $a \in \mathbb{R}$ and $\epsilon > 0$, by continuity we have $f^{-1}((a - \epsilon, a + \epsilon))$ is open in $\mathbb{R}_\ell$. Pick some basis element $[x, y) \subset f^{-1}((a - \epsilon, a + \epsilon))$. containing a. Then we can find some δ such that $[a, a + \delta) \subset [x, y)$, so for any $x \in [a, a + \delta)$, $|f(x) - f(a)| < \epsilon$ by construction. This is exactly a right continuous function.

(c) Note that any continuous map $f : \mathbb{R} \to \mathbb{R}_\ell$ must map connected sets to connected sets, so $f(\mathbb{R})$ is a connected set in $\mathbb{R}_\ell$. However the only connected sets in $\mathbb{R}_\ell$ are single points, so f is a constant function.

(d) Suppose $f : \mathbb{R}_\ell \to \mathbb{R}_\ell$ is some continuous function. By (b), since $\mathbb{R}_\ell$ is finer than $\mathbb{R}$ so any continuous function $f : \mathbb{R}_\ell \to \mathbb{R}_\ell$ is also continuous $f : \mathbb{R}_\ell \to \mathbb{R}$. So f must be right continuous by (b).

Note that if $f : \mathbb{R}_\ell \to \mathbb{R}_\ell$ is continuous, then for any $x \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for any $y \in \mathbb{R}$ satisfying $x \le y < x + \delta$ we have $f(x) \le f(y) < f(x) + \epsilon$. So f must be monotonically increasing. This is a necessary but not sufficient condition.

Problem 6.

(a) Show that no two of the spaces $(0,1)$, $(0,1]$, and $[0,1]$ are homeomorphic.

(b) Suppose there exist embeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.

(c) Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

We'll introduce a powerful invariant to help us answer these questions.

Claim. Let X be a topological space. Let

$$n(X) = \{x \in X \mid X - \{x\} \text{ is path connected } \}.$$

Then if two spaces X, Y are homeomorphic, their associated sets $n(X), n(Y)$ are bijective.

Proof. We'll show that there are injections $\iota_1 : n(X) \to n(Y)$ and $\iota : n(Y) \to n(X)$. Indeed, suppose $x \in n(X)$. Then $f(X - \{x\}) = f(X) - \{f(x)\} = Y - \{f(x)\}$ since f is a bijection. However f is also continuous so this image is path connected. Thus $f(x) \in n(Y)$. This gives an injection $n(X) \to n(Y)$ induced by f. We can do the same thing for f^{-1}, and so there must be a bijection $n(X) \xrightarrow{\sim} n(Y)$. □

(a) Observe that $n(0,1) = \varnothing$, $n(0,1] = \{1\}$, and $n[0,1] = \{0,1\}$, hence none of them can be homeomorphic because any homeorphism would induce bijections of their associated sets.

(b) Consider $(0,1) \hookrightarrow [0,1]$ embedded in the natural way, and $[0,1] \hookrightarrow (0,1)$ embedded by the map sending $[0,1] \to [1/3, 2/3] \subset (0,1)$. However these spaces aren't homeomorphic by (a).

(c) Using our invariant, note that

$$n(\mathbb{R}^n) = \begin{cases} \mathbb{R} & n = 1 \\ \varnothing & n > 1 \end{cases}$$

because any point from $\mathbb{R}^n$ clearly leaves it path connected for $n > 1$, whereas removing a point x from $\mathbb{R}$ leaves it split into two components, $(-\infty, x) \cup (x, \infty)$.

Problem 7. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show that there exists a point $x \in S^1$ such that $f(x) = f(-x)$.

Consider the function $g(x) = f(x) - f(-x)$. Now pick some $x \in S^1$. If $g(x) = 0$, we are done since that would mean that $f(x) = f(-x)$. Otherwise, assume without loss of generality that $g(x) > 0$. Then $g(-x) = f(-x) - f(x) = -g(x) < 0$.

Now reparameterizing the input of g so that it can be considered as a function $\overline{g}(t) : [0,1] \to \mathbb{R}$ with $\overline{g}(0) = g(x)$ and $\overline{g}(1) = g(-x)$. Recall that $\overline{g}(0) > 0$ and $\overline{g}(1) < 0$. Thus by the intermediate value theorem there must be a point $t \in [0,1]$ such that $\overline{g}(t) = 0$. This corresponds to a point $y \in S^1$ satisfying $f(y) = f(-y)$ so we are done.

Problem 8. Let $f : X \to X$ be a continuous map. Show that if $X = [0, 1]$, there is a point $x \in X$ such that $f(x) = x$. The point x is called a *fixed point* of f. What happens if $X = [0, 1)$ or $(0, 1)$?

Let $g(x) = f(x) - x$. Then $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$ so by the intermediate value theorem there must be an $t \in [0, 1]$ such that $g(t) = 0$. Then $f(t) = t$ so we are done.

Problem 9. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Let C be a compact subspace of a metric space (X, d). It is clearly closed because compact subspaces are closed in Hausdorff spaces, and every metric space is Hausdorff. To prove that is is bounded, write

$$C \subset \bigcup_{x \in C} B_\epsilon(x) \quad \text{for some } \epsilon > 0.$$

Since C is compact and this is an open cover, there must be some finite collection of points $x_1, x_2, \ldots, x_n \in C$ such that $C \subset B_\epsilon(x_1) \cup \cdots \cup B_\epsilon(x_n)$. Now let $x \in C$ be arbitrary. Then $C \subset B_r(x)$ where $r = \epsilon + \max_{1 \leq i \leq n} d(x, x_i)$ by the triangle inequality. So C is bounded by a ball of radius r.

To show an example of a metric space where every closed bounded subspace is not compact, consider any infinite set X equipped with the discrete metric, so every set is open. Note that every set is closed and bounded in this space. However any infinite subset S cannot be compact since $S = \bigcup_{s \in S} \{s\}$ is an open cover of S with no finite subcover.

Problem 10. Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

First we'll prove that for any point $a \in A$, there exists open disjoint sets $U \ni a$ and $V \supset B$. Indeed, since X is Hausdorff, we have disjoint open sets $U_b \ni a$ and $V_b \ni b$ for every $b \in B$. Considering $\{V_b\}_{b \in B}$ as an open cover of B, there must be some finite sub collection of the sets also covering B by compactness, say $V_{b_1}, V_{b_2}, \ldots, V_{b_n}$. Then $U = \bigcap_i U_{b_i}$ and $V = \bigcup_i V_{b_i}$ satisfy the required conditions.

Now we can construct disjoint open $U \supset A$ and $V \supset B$. By the first paragraph, we have disjoint open sets $U_a \ni a$ and $V_a \supset B$ for every $a \in A$. Since $\{U_a\}_{a \in A}$ is an open cover of A, there must be some finite subcover, say $U_{a_1}, U_{a_2}, \ldots, U_{a_n}$. Then $U = \bigcup_i U_{a_i}$ and $V = \bigcap_i V_{a_i}$ satisfy the conditions of the problem.

Problem 11. Let X be the union of $\mathbb{R}^n$ and one additional point called ∞. Consider the topology with basis given by open balls in $\mathbb{R}^n$ plus the sets $U_r = \{\infty\} \cup \{x \in \mathbb{R}^n \mid |x| > r\}$. Show that X is a compact Hausdorff space.

To show that the space is Hausdorff, it suffices to show that ∞ can be separated from any point in $\mathbb{R}^n$ by disjoint open sets, since any two distinct points in $\mathbb{R}^n$ can already be separated by disjoint open sets. Let $x \in \mathbb{R}^n$. Then letting $U_r(x) = \{\infty\} \cup \{x \in \mathbb{R}^n \mid |x| > r\}$ and choosing some $\epsilon > 0$ we have $\infty \in U_\epsilon(x)$ and $x \in B_\epsilon(x)$, and these sets are clearly disjoint.

To prove compactness, let $\mathcal{U}$ is some open cover of X. Pick some $V \in \mathcal{U}$ such that $\infty \in V$. We can assume without loss of generality that V is a basis element (otherwise split it into a union of basis elements), so $V = X - C$ for some compact ball C. So when restricted to $\mathbb{R}^n$ the set $\mathcal{U} - \{V\}$ is an open cover of C. (Clearly any set of the form $\mathbb{R}^n - C_1$ is open since $\mathbb{R}^n$ is a Hausdorff space so any compact C_1 is closed) Since C is compact, there is some finite subcover $\mathcal{V} \subset \mathcal{U} - \{V\}$. The $\mathcal{V} \cup \{V\}$ is an open cover of X.