Cohomology Rings of Configuration Spaces

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In this paper, we prove the following theorem.

Definition 1. *Recall that a* configuration space *of* $\mathbb{R}^n$ *is defined as:*

$$\mathrm{Conf}_k(\mathbb{R}^n) = \{(x_1, \dots, x_k) : x_i \in \mathbb{R}^n, x_i \neq x_j \text{ for } i \neq j\}.$$

Theorem 1. *We have the following isomorphism of graded commutative algebras, where* $1 \leq a < b < c \leq k$*:*

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \cong \left\langle \alpha_{ab} \mid \alpha_{ab}^2,\ \alpha_{ab} + (-1)^{n+1}\alpha_{ba},\ \alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab} \right\rangle.$$

To prove this theorem, we'll place $\mathrm{Conf}_k(\mathbb{R}^n)$ into a fiber sequence, then apply the Leray-Hisrch theorem.

Theorem 2 (Leray-Hirsch)**.** *Let* $\pi : E \to B$ *be a fiber bundle with fiber* F*. Suppose that for each* t*, the abelian group* $H^t(F)$ *is free of finite rank. Assume that the restriction* $H^*(E) \to H^*(F)$ *is surjective. Because* $H^t(F)$ *is a free abelian group for each* t*, the surjection* $H^*(E) \to H^*(F)$ *admits a splitting; pick one, say* $s : H^*(F) \to H^*(E)$*. The projection map renders* $H^*(E)$ *a module over* $H^*(B)$*. The* $H^*(B)$*-linear extension of* s*,*

$$\overline{s} : H^*(B) \otimes H^*(F) \to H^*(E)$$

is then an isomorphism of $H^*(B)$*-modules.*

To apply this, we first need to build a fiber sequence around $\mathrm{Conf}_k(\mathbb{R}^n)$.

Lemma 1. *Let* $p_k : \mathrm{Conf}_k(\mathbb{R}^n) \to \mathrm{Conf}_{k-1}(\mathbb{R}^n)$ *be the map which sends* $(x_1, \dots, x_k)$ *to* $(x_1, \dots, x_{k-1})$*. Then* p_k *is a fiber bundle, and the fiber at a point* $(x_1, \dots, x_{k-1})$ *is the space* $\mathbb{R}^n - \{x_1, \dots, x_{k-1}\}$*.*

Proof. We just need to prove the local triviality condition, i.e. given some point $(x_1, \dots, x_{k-1})$, we need an open neighborhood $\mathcal{U} \subset \mathrm{Conf}_{\|-\infty}(\mathbb{R}^{\setminus})$, and an isomorphism $\varphi_{\mathcal{U}}$ such that the following diagram commutes:

$$\begin{tikzcd}
p_k^{-1}(\mathcal{U}) \arrow[r, "\varphi_{\mathcal{U}}"] \arrow[d, "p_k"'] & \mathcal{U} \times (\mathbb{R}^n - \{x_1, \dots, x_{k-1}\}) \arrow[dl] \\
\mathcal{U} &
\end{tikzcd}$$

Let $\epsilon > 0$ be some real value such that $x_j \neq B_\epsilon(x_i)$ for $i \neq j$. Then, let $\mathcal{U} = \bigcup_i B_\epsilon(x_i)$, which is a subset of $\mathrm{Conf}_{k-1}(\mathbb{R}^n)$. Note that $p_k^{-1}(\mathcal{U})$ is the set:

$$p_k^{-1}(\mathcal{U}) = \{(x_1, \dots, x_k) : \|x_i\| < \epsilon \text{ for } i \leq k-1\}.$$

Now we can set $\varphi_{\mathcal{U}}$ to be the identity homeomorphism. This proves the local triviality condition. $\square$

Another condition which must be met to apply the Leray-Hirsch theorem is that $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \to H^*(\mathbb{R}^n - \{x_1, \dots, x_{k-1}\})$ is surjective. To do this, we establish some notation.

Definition 2. *Let* $1 \leq a < b \leq k$. *The* Gauss map $\gamma_{ab} : \mathrm{Conf}_k(\mathbb{R}^n) \to S^{n-1}$ *is given by*

$$\gamma_{ab}(x_1, \dots, x_k) = \frac{x_b - x_a}{\|x_b - x_a\|}.$$

Letting $\iota_{n-1} \in H^{n-1}(S^{n-1})$ *be any choice of generator, set* $\alpha_{ab} = \gamma_{ab}^*(\iota_{n-1})$ *for some* $1 \leq a < b \leq k$.

Lemma 2. *The restriction map* $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \to H^*(\mathbb{R}^n - \{x_1, \dots, x_{k-1}\})$ *is surjective.*

Proof. We prove this by showing that $H^*(\mathbb{R}^n - \{x_1, \dots, x_{k-1}\})$ is generated by α_{ak}. We will construct right inverse maps $\psi_i : S^{n-1} \to \mathbb{R}^n - \{x_1, \dots, x_{k-1}\}$ to γ_{ik}. Since all $\mathbb{R}^n - \{x_1, \dots, x_{k-1}\}$ are homotopic for distinct x_i, we can assume that $x_i = (f(i), 0, \dots, 0)$ for some real function with $f(0) = 1$. Then let's set $\psi_i(v) = x_i + e^i v$ for any $v \in S^{n-1}$. Clearly

$$\gamma_{ik} \circ \psi_i(v) = \frac{x_i + f(i)v - x_i}{\|x_i + f(i)v - x_i\|} = v.$$

Furthermore, note that $\psi_i : S^{n-1} \to \mathbb{R}^n - \{x_1, \dots, x_{k-1}\}$ give us a homotopy equivalence between $\bigvee_{k-1} S^{n-1}$ and $\mathbb{R}^n - \{x_1, \dots, x_{k-1}\}$. Thus, it follows that $H^*(\mathbb{R}^n - \{x_1, \dots, x_{k-1}\})$ is generated by $\gamma_{ik}^*(\alpha) = \alpha_{ik}$. Thus the map is surjective. $\square$

Lemma 3. *For some $k \geq 0$, let B_k be the set of classes:*

$$B_k = \bigsqcup_{m \geq 0} \left\{ \prod_{i \geq 1}^{m} \alpha_{a_i b_i} \,:\, 1 \leq b_1 < \cdots < b_m \leq k, \alpha_i < b_i \right\}.$$

Then as a $\mathbb{Z}$-module, $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \cong \mathbb{Z}\big\langle B_k \big\rangle$.

Proof. This is proven by induction on k, via repeated applications of the Leray-Hirsch theorem. The base case of $k = 1$ is trivial, since $\mathrm{Conf}_1(\mathbb{R}^n) = \mathbb{R}^n$. Suppose the claim is true for $k-1$. By the Leray-Hirsch theorem we get an isomorphism of $\mathbb{Z}$-modules:

$$\begin{aligned} H^*(\mathrm{Conf}_k(\mathbb{R}^n)) &\cong H^*(\mathbb{R}^n - \{x_1, \ldots, x_{k-1}\}) \otimes H^*(\mathrm{Conf}_{k-1}(\mathbb{R}^n)) \\ &\cong \mathbb{Z}\big\langle 1, \alpha_{ak} \,:\, 1 \leq a \leq k-1 \big\rangle \otimes \mathbb{Z}\big\langle B_{k-1} \big\rangle. \end{aligned}$$

This completes the proof, since appending α_{ak} to B_{k-1} gives us B_k. $\square$

Having understood the additive structure of $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$, let's now look at the multiplicative structure. We can first prove some basic relations.

Lemma 4. $\alpha_{ab} = (-1)^n \alpha_{ba}$.

Proof. Observe that we have a commutative diagram, where -1 is the antipodal map.

$$\begin{tikzcd} \mathrm{Conf}_k(\mathbb{R}^n) \arrow[d, "\gamma_{ab}"'] \arrow[dr, "\gamma_{ba}"] & \\ S^{n-1} \arrow[r, "-1"'] & S^{n-1} \end{tikzcd} \qquad \begin{tikzcd} H^*(\mathrm{Conf}_k(\mathbb{R}^n)) & \\ \mathbb{Z}[x_{n-1}] \arrow[u, "\gamma_{ab}^*"] & \mathbb{Z}[x_{n-1}] \arrow[l, "(-1)^n"] \arrow[ul, "\gamma_{ba}^*"'] \end{tikzcd}$$

The second, induced diagram follows because the antipodal map on S^{n-1} has degree $(-1)^n$. Since $\gamma_{ab}^* = (-1)^n \alpha_{ba}^*$, it follows that $\gamma_{ab}^*(\iota_{n-1}) = (-1)^n \alpha_{ba}^*(\iota_{n-1})$ so we get our relation. $\square$

Lemma 5. $\alpha_{ab}^2 = 0$.

Proof. This follows because:

$$\begin{aligned} \alpha_{ab}^2 &= \gamma_{ab}^*(\iota_{n-1}) \smile \gamma_{ab}^*(\iota_{n-1}) \\ &= \gamma_{ab}^*(\iota_{n-1} \smile \iota_{n-1}) \\ &= 0. \end{aligned}$$

The final equality is zero because $H^*(S^{n-1}) \cong \mathbb{Z}[x_n]/(x_n^2)$. $\square$

Notice that there is a right action of the symmetric group Σ_k on $\mathrm{Conf}_k(\mathbb{R}^n)$ given by $(x_1, \ldots, x_k) \cdot \sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$. This gives us the following relation.

Lemma 6. *For any $\sigma \in \Sigma_k$, we have $\sigma^* \alpha_{ab} = \alpha_{\sigma(a)\sigma(b)}$.*

Proof. For this case, we get a similar diagram:

$$\begin{tikzcd} \mathrm{Conf}_k(\mathbb{R}^n) \arrow[r, "\sigma"] \arrow[d, "\gamma_{\sigma(a)\sigma(b)}"'] & \mathrm{Conf}_k(\mathbb{R}^n) \arrow[dl, "\gamma_{ab}"] \\ S^{n-1} & \end{tikzcd} \qquad \begin{tikzcd} H^*(\mathrm{Conf}_k(\mathbb{R}^n)) & H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \arrow[l, "\sigma^*"'] \\ \mathbb{Z}[x_{n-1}] \arrow[u, "\gamma^*_{\sigma(a)\sigma(b)}"] \arrow[ur, "\gamma^*_{ab}"'] & \end{tikzcd}$$

Thus we get the desired relation. ☐

Next we prove the Arnold relation.

Lemma 7. *For $1 \leq a < b < c \leq k$, we have $\alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab} = 0$.*

Proof. Recall that $H^{2n-2}(\mathrm{Conf}_3(\mathbb{R}^n))$ is free, with basis $\{\alpha_{12}\alpha_{23}, \alpha_{23}\alpha_{31}\}$. Applying the relations, we also have a basis $\{\alpha_{12}\alpha_{23}, \alpha_{31}\alpha_{12}\}$. Thus, there must be some linear dependence

$$x \cdot \alpha_{12}\alpha_{23} + y \cdot \alpha_{23}\alpha_{31} + z \cdot \alpha_{31}\alpha_{12} = 0.$$

To solve for these coefficients, we apply the previous lemma. Let $\tau_{12} \in \Sigma_3$ be the transposition switching 1 and 2. Applying it to the linear dependence, and using the commutativity rules gives us:

$$\begin{aligned} x \cdot \alpha_{21}\alpha_{13} + y \cdot \alpha_{13}\alpha_{32} + z \cdot \alpha_{32}\alpha_{21} &= 0 \\ (-1)^n x \cdot \alpha_{12}\alpha_{13} + (-1)^n y \cdot \alpha_{13}\alpha_{23} + (-1)^{2n} z \cdot \alpha_{12}\alpha_{23} &= 0 \end{aligned}$$

We do a similar thing with τ_{23}, to get another dependence. It is straightforward to check that solving the resulting system of equations gives us $x = y = z$, so by additive freeness of $H^*(\mathrm{Conf}_3(\mathbb{R}^n))$ we can cancel x and get our desired relation in $H^*(\mathrm{Conf}_3(\mathbb{R}^n))$:

$$\alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab} = 0.$$

Now in the general case $k > 3$, given some a, b, c, we have a map $\varphi_{abc} : \mathrm{Conf}_k(\mathbb{R}^n) \to \mathrm{Conf}_3(\mathbb{R}^n)$ which sends $(x_1, \ldots, x_k)$ to (x_a, x_b, x_c). Notice that φ_{abc} induces an injective map of cohomology rings $H^*(\mathrm{Conf}_3(\mathbb{R}^n)) \to H^*(\mathrm{Conf}_k(\mathbb{R}^n))$. Thus, the Arnold relation lifts to the cases $k > 3$. ☐

Finally, we can prove the desired theorem: the commutativity relation, the nilpotency relation, and the Arnold relation completely describe the multiplicative structure of $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$. Specifically, the quotient of the free graded commutative algebra on the generators $\{\alpha_{ab}\}_{1\leq a\neq b\leq k}$ by the relations. Given some element of this free graded commutative algebra, using the nilpotency and commutativity relation, we can rewrite such an element in the form $\alpha_{a_1b_1}\cdots\alpha_{a_mb_m}$, where $a_i < b_i$, and $1 \leq b_1 \leq \cdots b_m \leq k$. We can then repeatedly use the Arnold relation to force $b_1 < \cdots < b_m$. Specifically, if $b_\ell = b_{\ell-1}$, we have

$$\begin{aligned} &\alpha_{a_1b_1}\cdots\alpha_{a_{\ell-1}b_{\ell-1}}\alpha_{a_\ell b_\ell}\cdots\alpha_{a_mb_m} \\ = (-1)^*\alpha_{a_1b_1}\cdots&\left(\alpha_{a_\ell a_{\ell-1}}\alpha_{a_{\ell-1}b_{\ell-1}+\alpha_{b_\ell a_\ell}\alpha_{a_\ell a_{\ell-1}}}\right)\cdots\alpha_{a_mb_m}\end{aligned}$$

By repeated induction, we can thus force this element into a linear combination of elements in B_k, the additive spanning basis for $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$.

Recall that the additive basis for $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ was given by B_k, and consists of elements of the form $\alpha_{a_1b_1}\cdots\alpha_{a_mb_m}$.