Math 230a Problem Set 8

Lev Kruglyak

Due: October 30, 2024

Collaborators: *AJ LaMotta, Ignasi Vicente*

Problem 3.

(a). Prove that the action of a compact Lie group G on a smooth manifold is proper.

We need to show that the action of G on a smooth manifold M by left multiplication is proper, i.e. the map

$$\Phi : G \times M \to M \times M, \quad \Phi(g, x) = (gx, x)$$

is proper. Suppose $K \subset M \times M$ is compact. Then the projection $\pi_M(K)$ onto M must be compact, and $\Phi^{-1}(K) \subset G \times \pi_M(K)$. This is a closed subset of a compact space and so must be compact.

(b). Let G be a Lie group and suppose H is a closed Lie subgroup. Prove that the action of H on G by left multiplication is proper. What if $H \subset G$ is not closed?

We need to show that the action of H on G by left multiplication is proper, i.e. the map

$$\Phi : H \times G \to G \times G, \quad \Phi(h, g) = (hg, g)$$

This map Φ is a composition of the inclusion $H \times G \to G \times G$ with the homeomorphism $G \times G \to G \times G$ given by $(g, g') \mapsto (gg', g')$. The latter map is proper because it is a homeomorphism, and the former map is proper because the inclusion of a closed subspace is proper. Thus the action is proper. is proper.

Problem 4. Principal bundles and homotopy theory.

(a). Let $Q \to [0, 1] \times M$ be a principal bundle. Choose a connection on Q. Use parallel transport to construct an isomorphism $Q|_{\{0\}\times M} \to Q|_{\{1\}\times M}$.

For each point $p \in M$, the parallel transport along the vertical path $\gamma(t) = (t, p)$ in $[0, 1] \times M$ which yields a map from the fiber of Q over $(0, m)$ to that over $(1, m)$. This gives us the desired isomorphism $Q|_{\{0\}\times M} \to Q|_{\{1\}\times M}$.

(b). Let $f_t : M \to N$ be a smooth homotopy and $P \to N$ a principal bundle. Prove that $f_0^*P \cong f_1^*P$.

Such a homotopy is a smooth map $f : [0, 1] \times M \to N$. Pulling back the principal bundle $P \to N$ by f then gives us a principal bundle $f^*P \to [0, 1] \times M$. We can apply the results of the previous problem to get an isomorphism $f^*P|_{\{0\}\times M} \cong f^*P|_{\{1\}\times M}$. However, note that $f^*P|_{\{t\}\times M} = f_t^*P$ so we have our desired isomorphism $f_0^*P \cong f_1^*P$.

(c). Prove that a principal bundle over a contractible manifold is trivializable.

Let M be a contractible manifold, say by some map $f : [0,1] \times M \to M$ with $f_0 = \mathrm{id}_M$ and $f_1 = c_p$ for some chosen point $p \in M$. For any principal bundle $P \to M$, we thus have $P \cong \mathrm{id}_M^* P \cong c_p^* P$ by the previous problem. However, the pullback of any bundle by a constant map is trivial, so this isomorphism gives a trivialization of P.

(d). Classify up to isomorphism principal U_1-bundles on S^n for all n.

First, let's construct the classifying space for U_1. Recall that the data of a principal U_1-bundle $P \to X$ is equivalent to the data of a complex line bundle $f : \overline{P} \to X$. This means that it suffices to classify complex line bundles on S^n. Let's pick some embedding of X into affine complex space $\mathbb{A}^k_{\mathbb{C}}$ and simultaneous linear embedding of $\overline{P}$ into the tangent space $T\mathbb{A}^k_{\mathbb{C}} = \mathbb{A}^k_{\mathbb{C}} \times \mathbb{C}^k$ for k large enough. This gives us a map $B(f) : X \to \mathbb{CP}^{k-1}$ which sends the fiber at a point $x \in X$ to the complex line $\overline{P}_x$ embedded in $T_x\mathbb{A}^k_{\mathbb{C}} = \mathbb{C}^k$.

Now, there is a tautological complex line bundle $\xi_k : E_k \to \mathbb{CP}^{k-1}$ where

$$E_k = \{(x, \ell) \in \mathbb{C}^k \times \mathbb{CP}^{k-1} : x \in \ell\}$$

with the map ξ_k given by obvious projection onto the $\mathbb{CP}^{k-1}$ component. For any map $F : X \to \mathbb{CP}^{k-1}$ we can pull back this tautological complex line bundle ξ_k to get a complex line bundle on X. Conversely, given a line bundle $f : E \to X$, it can be shown that $B(f)^*\xi_k \cong f$. To avoid dependence on k, we can pass to the limit and consider maps $X \to \mathbb{CP}^\infty$. Assuming X is a finite CW complex, any map $X \to \mathbb{CP}^\infty$ is homotopic to a map $X \to \mathbb{CP}^k$ for large enough k so this works both ways. We know that homotopic maps correspond to isomorphic bundles, and we can also show that isomorphic bundles correspond to homotopic maps.

We thus get an isomorphism:

$$\mathrm{Bun}_{\mathrm{U}_1}(X) \xrightarrow{\sim} [X, \mathbb{CP}^\infty]$$

For the case of spheres, classifying principal U_1-bundles over S^n up to isomorphism thus becomes equivalent to computing $[S^n, \mathbb{CP}^\infty]$. Recall that $\mathbb{CP}^\infty$ is the Eilenberg-Maclane space $K(\mathbb{Z}, 2)$, and so represents the cohomology theory $\mathrm{H}^2(-;\mathbb{Z})$. Thus, we have

$$\mathrm{Bun}_{\mathrm{U}_1}(X) \cong \mathrm{H}^2(S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

We should give explicit U_1-bundles on S^2 corresponding to every integer $n \in \mathbb{Z}$, since this is the only non-trivial case. Let $\mathbb{S}(TS^2)$ be the sphere bundle of the tangent bundle TS^2 – this corresponds to $1 \in \mathbb{Z}$. Then, any integer n can be obtained by pulling back this bundle by a map $S^2 \to S^2$ of degree n. For example, the trivial bundle is obtained by pulling back the constant map.

Problem 6.

(a). Let P be a Riemannian manifold equipped with a free action of a Lie group H by isometries. Assume that the quotient map $\pi : P \to X$ is a principal H-bundle. Use the metric to construct a connection on π.

Let's consider the distribution

$$DH = (\ker d\pi)^\perp$$

where $d\pi$ is the differential of the bundle, and the orthogonal complement is taken with the Riemannian structure on P. This distribution must be horizontal since at any point, we have $T_pP = (\ker d\pi)_p \oplus D_p$, and we know that π is a submersion so $d\pi_p$ must be surjective. Thus, D_p must be mapped isomorphically onto the tangent space at a basepoint, i.e. $T_{\pi(p)}X$.

To see that this horizontal distribution is H-equivariant, suppose $h \in H$ is some group element and $\rho(h) : P \to P$ is the corresponding isometry. This is a morphism of bundles, so $d\rho(h)$ preserves $\ker d\pi$. Since it is further an isometry, it must respect orthogonal complements as well and so preserves D.

Being an H-equivariant horizontal distribution, D gives rise to a connection on π.

(b). Let G be a Lie group and let H be a closed Lie subgroup. Define the notion of a *bi-invariant Riemannian metric* on G. Give examples of a Lie group and a bi-invariant metric on it. Give an example of a Lie group which does not admit a bi-invariant metric.

A bi-invariant Riemannian metric is a metric for which the left and right multiplication maps are isometries. For a simple example, take any abelian Lie group and pick a Haar measure. A counterexample would be SL_2, on which the left and right Haar measures are distinct.

(c). Assuming a bi-invariant metric exists and is chosen, use it to construct a connection on $\pi : G \to G/H$. Compute the curvature of this connection.

If G has a bi-invariant Riemannian structure, then H acts on G by isometries so we can apply the first problem to get a connection on π. Since the curvature Ω of this connection is the negative of the Frobenius tensor of the horizontal distribution. This is exactly

$$\Omega_e(\xi, \eta) = -[\xi, \eta]_{\mathfrak{h}}, \quad \Omega \in \Omega^2(G; \mathfrak{h})$$

where the subscript $\mathfrak{h}$ denotes projection to the subspace $\mathfrak{h} \subset \mathfrak{g}$.

(d). Apply to the Hopf bundle $\mathrm{U}_1 \to S^3 \to S^2$. What about the Hopf bundle $\mathrm{Sp}_1 \to S^7 \to S^4$? The connection you construct on the latter is the basic *instanton* (self-dual connection).

The first Hopf bundle can be expressed as the quotient map $\mathrm{U}_1 \to \mathrm{SU}_2 \to \mathrm{SU}_2 / \mathrm{U}_1$. Recall that the Lie algebra $\mathfrak{su}_2$ can be explicitly given as the vector space

$$\mathfrak{su}_2 = \left\{ \begin{pmatrix} ia & -\bar{z} \\ z & -ia \end{pmatrix} : a \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

The Lie algebra $\mathfrak{su}_1$ of U_1 embeds into $\mathfrak{su}_2$ by diagonal matrices. If we give a basis for $\mathfrak{su}_2$ by

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

then the commutator relations are $[u_3, u_1] = 2u_2, [u_1, u_2] = 2u_3, [u_2, u_3] = 2u_1$. The embedding $\mathfrak{su}_1 \to \mathfrak{su}_2$ then becomes the inclusion of u_3 into u_1, u_2, u_3. By the previous problem, the curvature 2-form is

$$\Omega_e(\xi, \eta) = -[\xi, \eta]_{\mathfrak{su}_1} \qquad \Longrightarrow \qquad \Omega_e = -2u_3 \, du_1 \wedge du_2.$$

Problem 7. Let G be a Lie group and let $\pi : P \to X$ be a principal G-bundle. A *gauge transformation* of π is a diffeomorphism $\varphi : P \to P$ that is G-equivariant and covers the identity map of X.

(a). Construct a function $\psi : P \to G$ that satisfies $\varphi(p) = p \cdot \psi(p)$ for all $p \in P$. How does ψ transform under the G-action on P?

For any $p \in P$, let $\psi(p)$ be the unique element of G such that $\varphi(p) = p \cdot \psi(p)$. We know that such an element exists and is unique because of the transitivity and freeness of the G-action on the fibers of P.

Now suppose $h \in G$. By G-equivariance of φ, we get

$$\begin{cases} \varphi(p \cdot h) = (p \cdot h) \cdot \psi(p \cdot h) \\ \varphi(p) \cdot h = p \cdot \psi(p) \cdot h \end{cases} \qquad \Longrightarrow \qquad \psi(p \cdot h) = h^{-1} \cdot \psi(p) \cdot h.$$

In other words, ψ transforms by conjugation.

(b). Express ψ as a section of a fiber bundle associated to π. What kind of fiber bundle is it?

(c). Do gauge transformation always exist?

(d). Are there any simplifications if G is abelian? If G is discrete?

(e). Let $\mathrm{Aut}(P)$ denote the group of G-equivariant diffeomorphisms of P, and let $\mathrm{Aut}(\pi)$ denote the group of gauge transformations. Construct an exact sequence

$$1 \longrightarrow \mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(P) \longrightarrow \mathrm{Diff}(X)$$

where $\mathrm{Diff}(X)$ is the group of diffeomorphisms of X. Is the last map surjective? Give a proof or counterexample to verify your answer.