MATH 231A: ALGEBRAIC TOPOLOGY
HOMEWORK 10
DUE: WEDNESDAY, NOVEMBER 16 AT 10:00PM ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Cohomology of projective space (25 points)

The goal of this problem is to compute $H^*(\mathbb{RP}^n; \mathbb{F}_2)$ and $H^*(\mathbb{RP}^n; \mathbb{Z})$ as rings.

(i) In this problem, you will prove that $H^n(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$, where $|x| = 1$. Using induction on n, prove that this will follow if the cup product map

$$H^i(\mathbb{RP}^n; \mathbb{F}_2) \times H^j(\mathbb{RP}^n; \mathbb{F}_2) \xrightarrow{\smile} H^n(\mathbb{RP}^n; \mathbb{F}_2)$$

is nonzero for all $i, j \geq 0$ satisfying $i + j = n$. (In fact, it is enough to prove this for $i = 1$ and $j = n - 1$.)

Given $i, j \geq 0$ such that $i + j = n$, regard $\mathbb{RP}^i \subset \mathbb{RP}^n$ as those $[x_0 : \cdots : x_n] \in \mathbb{RP}^n$ with $x_{i+1} = \cdots = x_n = 0$. Similarly, regard $\mathbb{RP}^j \subset \mathbb{RP}^n$ as those $[x_0 : \cdots : x_n] \in \mathbb{RP}^n$ with $x_0 = \cdots = x_{i-1} = 0$. Then $\mathbb{RP}^i \cap \mathbb{RP}^j = \{p\}$, where $p = [0, \ldots, 0, 1, 0, \ldots 0]$ and 1 is in the position of x_i. Finally, regard $\mathbb{R}^n \subset \mathbb{RP}^n$ as those elements of the form $[x_0 : \ldots x_{i-1} : 1 : x_{i+1} : \cdots : x_n]$.

Then there is a diagram of the form

$$\begin{tikzcd}
H^i(\mathbb{RP}^n; \mathbb{F}_2) \times H^j(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r, "\smile"] & H^n(\mathbb{RP}^n; \mathbb{F}_2) \\
H^i(\mathbb{RP}^n, \mathbb{RP}^n - \mathbb{RP}^j; \mathbb{F}_2) \times H^j(\mathbb{RP}^n, \mathbb{RP}^n - \mathbb{RP}^i; \mathbb{F}_2) \arrow[r, "\smile"] \arrow[u] \arrow[d] & H^n(\mathbb{RP}^n, \mathbb{RP}^n - \{p\}; \mathbb{F}_2) \arrow[u] \arrow[d] \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j; \mathbb{F}_2) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i; \mathbb{F}_2) \arrow[r, "\smile"] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{p\}; \mathbb{F}_2).
\end{tikzcd}$$

(ii) Prove that the vertical maps are isomorphisms. (Hint: you may find it useful to prove that $\mathbb{RP}^n - \mathbb{RP}^j$ deformation retracts onto $\mathbb{RP}^{i-1}$ in $\mathbb{RP}^n$.)

(iii) Prove that the bottom product is nonzero using the relative Künneth formula and universal coefficients theorem. Conclude that $H^n(\mathbb{RP}^n; \mathbb{F}_2 \cong \mathbb{F}_2[x]/(x^{n+1})$.

(iv) Using the map $H^*(\mathbb{RP}^n; \mathbb{Z}) \to H^*(\mathbb{RP}^n; \mathbb{F}_2)$ induced by mod 2 redution $\mathbb{Z} \to \mathbb{F}_2$, compute $H^*(\mathbb{RP}^n; \mathbb{Z})$ as a ring.

Remark: The same arguments prove that $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$ with $|x| = 2$.

2. Problem 2: Division algebras (15 points)

An *algebra* structure on $\mathbb{R}^n$ is an $\mathbb{R}$-bilinear product map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which we denote $(a, b) \mapsto ab$. It is a *division algebra* structure if, for any fixed $a, b \in \mathbb{R}^n$, the maps $x \mapsto ax$ and $x \mapsto xb$ are bijections. Note that we do not assume that the product is commutative, unital or even associative.

In this problem, you will use problem 1 to prove the following theorem of Hopf: if $\mathbb{R}^n$ admits the structure of a division algebra, then n must be a power of 2.

(i) Prove that if $\mathbb{R}^n$ is equipped with the structure of a division algebra, then the product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ induces a map $\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$.

(ii) Prove that the induced map $H^*(\mathbb{RP}^{n-1}; \mathbb{F}_2) \to H^*(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{F}_2)$ may be identified with the ring map $\mathbb{F}_2[x]/(x^n) \to \mathbb{F}_2[x_1, x_2]/(x_1^n, x_2^n)$ given by $x \mapsto x_1 + x_2$.

(iii) Prove that such a ring homomorphism can only exist when n is a power of 2.

3. Problem 3: Covers and cup products (10 points)

Let X denote a space. Prove that if $X = U_1 \cup \cdots \cup U_n$ for contractible open sets $U_i \subset X$, then the cup product of n positive dimensional classes in $H^*(X; R)$ is zero for any ring of coefficients R. As an example, conclude that the cup product of any two positive dimensional classes in $H^*(SY; R)$ is zero, where SY is the suspension in the sense of Exercise 10.10 of LAT of a space Y.