Math 212 Problem Set 1

Lev Kruglyak

Due: February 10, 2025

Problem 1. Find an example of a continuous function on $\mathbb{R}$ which goes to zero at infinity and which isn't the Fourier transform of a function in $L^1(\mathbb{R})$.

Recall that the Fourier transform $\mathscr{F}$ maps $L^1(\mathbb{R})$ functions to $C_0^0(\mathbb{R})$ functions by the equation

$$\mathscr{F}(f)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} f(x)\, dx.$$

Now consider the following function (inspired by this MathOverflow question)

$$F(k) = \begin{cases} \operatorname{sgn}(k)/\log|k| & |k| \geq e, \\ k/e & |k| \leq e. \end{cases}$$

This is clearly a continuous function which vanishes at infinity but is also not in $L^1(\mathbb{R})$ since its integral is divergent, being bounded below by the harmonic series since $1/\log(x) > 1/x$ for $x > 1$.

Suppose for the sake of contradiction that $F(k) = \mathscr{F}(f)(k)$ for some $f \in L^1(\mathbb{R})$. Note, conjugation acts in the following way under Fourier transform:

$$\mathscr{F}(\overline{f})(k) = \int_{\mathbb{R}} e^{ikx} \overline{f(x)}\, dx = \overline{\int_{\mathbb{R}} e^{-ikx} f(x)\, dx} = \overline{\mathscr{F}(f)(-k)}.$$

In our case, since F is an odd function, it follows that $\mathscr{F}(\overline{f})(k) = \mathscr{F}(-f)(k)$. By injectivity of the Fourier transform, this means that $\overline{f} = -f$ and so f is purely imaginary (almost everywhere). This means that we can write

$$\mathscr{F}(f)(k) = \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(kx) g(x)\, dx$$

where $f(x) = ig(x)$. Next, we expand the inequality

$$\begin{aligned} \left| \int_0^\infty \frac{\mathscr{F}(f)(k)}{k}\, dk \right| = \left| \int_0^\infty \frac{1}{k} \int_{\mathbb{R}} \sin(kx) f(x)\, dx\, dx \right| &\leq \left| \int_0^\infty \int_{\mathbb{R}} \frac{\sin(kx) f(x)}{k}\, dx\, dx \right| \\ &= \left| \int_{\mathbb{R}} f(x) \int_0^\infty \frac{\sin(kx)}{k}\, dk\, dx \right| \\ &= \frac{\pi}{2} \cdot \|f\|_{L^1}. \end{aligned} \tag{1}$$

Here, we've used the fact that $\int_0^\infty \sin(kx)/k\, dk$ is a piecewise constant function in k with absolute value $\pi/2$. One implication of (1) is that $\int_0^\infty F(k)/k\, dx$ is finite. However, we can bound the integral of $F(k)/k$ from below by

$$\int_0^\infty \frac{F(k)}{k}\, dk \geq \int_e^\infty \frac{1}{k \log(k)}\, dk = \int_1^\infty \frac{1}{u}\, du = \infty.$$

where $u = \log(k)$. This is a direct contradiction to the assumption that f is L^1, since (1) would then imply that $\|f\|_{L^1} \geq \infty$.

Problem 2. The Schwarz space $\mathcal{S}(\mathbb{R})$ is defined as

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \lim_{|x|\to\infty} |x^\alpha f^{(\beta)}(x)| = 0, \quad \forall \alpha, \beta \in \mathbb{N} \right\}$$

Prove that the Fourier transform maps the vector space $\mathcal{S}$ to itself.

Suppose $f \in \mathcal{S}(\mathbb{R}^d)$ is a Schwarz function. We'll use the fact that $x^\alpha f^{(\beta)} \in C_0^0(\mathbb{R})$ to swap polynomials and derivatives with the integral sign. First, we note that

$$k^\alpha \frac{\partial^\beta}{\partial k^\beta} \mathscr{F}(f)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\partial^\beta}{\partial k^\beta} e^{ikx} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} k^\alpha (ix)^\beta f(x)\, dx.$$

For any function $g(x) \in \mathbb{C}_0^0(\mathbb{R})$ with $\lim_{x\to\infty} e^x g(x) = 0$, integration by parts gives us the identity

$$\int_{\mathbb{R}} e^{ikx} g(x)\, dx = \left[\frac{e^{ikx}}{ik} g(x) \right]_{-\infty}^{\infty} + \frac{1}{ik} \int_{\mathbb{R}} e^{ikx} g'(x)\, dx = \frac{1}{ik} \int_{\mathbb{R}} e^{ikx} g'(x)\, dx.$$

In particular, this rule holds for any $g \in \mathcal{S}(\mathbb{R})$. Using this integration rule with $g(x) = k^\alpha (ix)^\beta f(x)$ gives

$$\begin{aligned}
\left| k^\alpha \frac{\partial^\beta}{\partial k^\beta} \mathscr{F}(f)(k) \right| &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{|k|} \left| \int_{\mathbb{R}} e^{ikx} k^\alpha (\beta (ix)^{\beta-1} f(x) + (ix)^\beta f'(x)) \right| \\
&\leq \frac{1}{|k|\sqrt{2\pi}} \left| \int_{\mathbb{R}} e^{ikx} \beta (ix)^{\beta-1} f(x) \right| + \frac{1}{|k|\sqrt{2\pi}} \left| \int_{\mathbb{R}} e^{ikx} (ix)^\beta f'(x) \right|.
\end{aligned}$$

We can inductively do this procedure to get a factor of $1/|k|^N$ in front of each term.