Math 222 Problem Set 6

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Due: March 12, 2025

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Problem 1. A torus has a generator. We prove this in a sequence of steps.

Problem 2. True or false. Provide a proof or example.

(a). There is a non-trivial homomorphism $\mathrm{SO}_3 \to \mathrm{Sp}_1$.

This is false. Since SO_3 is a simple (abstract) group, any non-trivial homomorphism would have to be injective and not surjective. But since Sp_1 is homeomorphic to S^3, by removing a point not in the image of the homomorphism would give an embedding of SO_3 in $\mathbb{R}^3$. However, SO_3 is homeomorphic to $\mathbb{RP}^3$ which is a closed 3-manifold and thus cannot be embedded in 3-dimensional Euclidean space.

(b). There is a non-trivial homomorphism $\mathrm{Sp}_1 \to \mathrm{SO}_3$.

Embed $\mathbb{R}^3 \subset \mathbb{H}$ by sending a basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ to $\{i, j, k\}$ in $\mathbb{H}$. Then conjugation by a unit quaternion $q \in \mathrm{Sp}_1$ leaves this hyperplane invariant, since $\Re(qxq^{-1}) = \Re(x)$. Furthermore, conjugation by a unit norm quaternion is an isometry on $\mathbb{H}$, and so is an isometry on $\mathbb{R}^3 \subset \mathbb{H}$. We thus get a homomorphism

$$\begin{array}{rcl} \mathrm{Sp}_1 & \longrightarrow & \mathrm{SO}_3 \\ q & \longmapsto & (x \mapsto qxq^{-1}). \end{array}$$

(c). There is a non-trivial homomorphism $\mathrm{SO}_5 \to \mathrm{Sp}_2$.

(d). There is a non-trivial homomorphism $\mathrm{Sp}_2 \to \mathrm{SO}_5$.