Problem Set # 1

Math 262a: Quantum theory from a geometric viewpoint I

Due: September 20

Problems

1. Let N be a smooth manifold of dimension $n = 2m$, and suppose $\omega \in \Omega^2_N$.

   (a) Prove that ω is nondegenerate (at each point of N) iff $\omega^{\wedge m} \in \Omega^{2m}_N$ is everywhere nonzero.

   (b) Now assume that ω is a symplectic form. Prove that the subspace $\mathfrak{X}^\omega_N \subset \mathfrak{X}_N$ of symplectic vector fields is closed under Lie bracket of vector fields.

   (c) Define a Lie algebra structure on Ω^0_N by transport from the Lie bracket on $\mathfrak{X}^\omega_N$ via the symplectic gradient map $f \mapsto \xi_f$ on functions. Prove the formula

   $$\{f, g\} = \xi_f \cdot g = \omega(\xi_f, \xi_g), \qquad f, g \in \Omega^0_N.$$

   (d) A *Hamiltonian vector field* is a symplectic gradient of a function. Is the subspace of Hamiltonian vector fields invariant under Lie bracket?

2. Let V be a real symplectic vector space, and suppose A is an affine space over V. Feel free to work with the model spaces.

   (a) Define the space of affine polynomial functions on A and prove that it is closed under the Poisson bracket on smooth functions.

   (b) Prove that the space of affine linear functions is also invariant. Do you recognize the resulting Lie algebra?

   (c) Prove that the space of affine quadratic functions is also invariant. Do you recognize the resulting Lie algebra?

   (d) Does the pattern continue?

3. (a) Prove that an almost symplectic manifold admits an almost complex structure. (An almost complex structure on a smooth manifold N is a section I of $\mathrm{End}(TN) \to TN$ such that $I^2 = -\mathrm{id}$ holds at every point.)

   (b) Prove that S^4 does not admit an almost symplectic structure. (Hint: Use characteristic classes)

4. Let (M, g) be a Riemannian manifold, and suppose $V\colon M \to \mathbb{R}$ is a smooth (potential energy) function. Fix $m > 0$, which we take to be the mass of a particle moving on M. Let $\mathcal{F}$ be the space of smooth maps $x\colon \mathbb{R} \to M$. Treat $\mathcal{F}$ as an infinite dimensional manifold, at least formally, assuming differential calculus carries over. Consider the Lagrangian

$$L = \left\{ \frac{m}{2}\langle \dot{x}, \dot{x} \rangle - V(x) \right\} |dt|,$$

where $x\colon \mathcal{F} \times \mathbb{R} \to M$ is the evaluation map. (You may replace '$|dt|$' with dt.) Compute the Euler-Lagrange equations, the symplectic form, and the Hamiltonian. (You will need some familiarity with Riemannian geometry.)

5. Define the notion of a symmetry of a general mechanical system. Does a symmetry have to preserve the direction (arrow) of time, or can a symmetry reverse time flow?