Math 55b Midterm

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"I affirm my awareness of the standards of the Harvard College Honor Code. While completing this exam, I have not consulted any external sources other than class notes and the textbook (Munkres). I have not discussed the problems or solutions of this exam with anyone, and will not discuss them until after the due date."

Signed: **Lev Kruglyak**

Problem 1 (14 points)**.** Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) If $A_i \subset X_i$ are closed subsets for all $i \in I$, then $\prod_{i \in I} A_i$ is a closed subset of $\prod_{i \in I} X_i$ with the product topology.

(b) If $x_1, x_2, \dots \in X$ are limit points of a subset $A \subset X$, and if the sequence x_n converges to a limit $x \in X$, then x is a limit point of A.

(c) If a subspace A of a topological space X is connected, then its closure $\overline{A} \subset X$ is connected.

(d) If X is Hausdorff, and $A \subset X$ is compact, then its boundary $\partial A = \overline{A} - \operatorname{int}(A)$ is compact.

(e) $[0,1] \subset \mathbb{R}_\ell$ with the lower limit topology (generated by the basis $\{[a,b),\ a < b\}$) is compact.

(f) The addition map $f : \mathbb{R}_\ell \times \mathbb{R}_\ell \to \mathbb{R}_\ell$ defined by $f(x,y) = x + y$ is continuous (equipping $\mathbb{R}_\ell$ with the lower limit topology and $\mathbb{R}_\ell \times \mathbb{R}_\ell$ with the product topology).

(g) The set of all uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ (i.e., such that $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\forall p, q \in \mathbb{R}$, $|p - q| < \delta \Rightarrow |f(p) - f(q)| < \epsilon$) is a closed subset of the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the uniform topology.

(a) This is true. For brevity, let $X = \prod_{i \in I} X_i$ and for any $x \in X$, let $x_i \in X_i$ be the projection of x to its i-th component. Note that

$$\prod_{i \in I} A_i = \bigcap_{i \in I} \{x \in X \mid x_i \in A_i\} = \bigcap_{i \in I} \pi_i^{-1}(A_i)$$

where $\pi_i : X \to X_i$ is the projection map onto the i-th component. However the product topology is the coarsest topology for which the projection maps are continuous, so $\pi_i^{-1}(A_i)$ is closed in X. Thus $\bigcap_{i \in I} \pi_i^{-1}(A_i)$ is closed in X, since it is an intersection of infinitely many closed sets.

(b) This is true. Suppose $U \ni x$ is an open neighborhood. Since x is a limit of the sequence $x_1, x_2, \dots$, U must contain some x_n. Yet since x_n is a limit point of A, U must intersect A nontrivially. This implies that x is a limit point of A, because U was an arbitrary open neighborhood.

(c) This is true. To make life easier for us, we'll use an equivalent formulation of connectedness:

Claim. A space X is connected if and only if every continuous function $f : X \to \{0,1\}$ is constant, where $\{0,1\}$ has the discrete topology.

Proof. If X is connected then $f(X)$ is connected in $\{0,1\}$ so $f(X) = \{0\}$ or $\{1\}$. Conversely, if every continuous function $f : X \to \{0,1\}$ is constant then X must be connected, because otherwise we could construct a continuous map which maps different connected components of X to 0 and 1. $\square$

Now since $A \subset X$ is connected, an easy extension of the above claim implies every continuous map $f : X \to \{0,1\}$ must be constant when restricted to A. Say without loss of generality that $f(A) = 0$. Then $A \subset f^{-1}(\{0\})$ and thus $f^{-1}(\{0\})$ is a closed set containing A. This means that $\overline{A} \subset f^{-1}(\{0\})$ so $f(\overline{A}) = 0$. Since f was an arbitrary continuous function, it follows that $\overline{A}$ is connected.

(d) This is true. Since X is Hausdorff and A is a compact subset, A is closed so $\overline{A} = A$. Now suppose $\mathcal{U}$ is an open cover of ∂A. Then $\mathcal{U} \cup \{\text{int}(A)\}$ is an open cover of A so it must contain a finite subcover $\mathcal{V}$. Removing $\text{int}(A)$ from $\mathcal{V}$ if needed, we get an open subcover of ∂A.

(e) This is false. Consider the open cover

$$[0,1] \subset \left[\frac{1}{2}, 2\right) \cup \bigcup_{0<x<\frac{1}{2}} [0,x).$$

This cannot have a finite subcover because every element in the cover contains a point which isn't in any of the other elements.

(f) This is true. Let $[b,a)$ be an open set in $\mathbb{R}_\ell$. Then we claim that

$$f^{-1}([b,a)) = \bigcup_{t\in\mathbb{R}} \left[t, t + \frac{a-b}{2}\right) \times \left[b-t, b-t+\frac{a-b}{2}\right).$$

The $\supset$ direction follows because for any $t \in \mathbb{R}$ and $(x,y) \in \left[t, t+\frac{a-b}{2}\right) \times \left[b-t, b-t+\frac{a-b}{2}\right)$ and simply by adding inequalities together we get

$$\begin{cases} t \le x < t + \frac{a-b}{2} \\ b-t \le y < b-t+\frac{a-b}{2} \end{cases} \implies b \le x+y < a$$

so $(x,y) \in f^{-1}([b,a)$. Conversely for any $(x,y) \in f^{-1}([b,a))$, let $t = \frac{x-y+b}{2}$. Since $b \le x+y < a$, we have $x \ge b-y$ and $x < a-y$ so $\frac{x-y+b}{2} \le x < \frac{x-y+a}{2}$. This is the same as saying $x \in \left[t, t+\frac{a-b}{2}\right)$. Similarly, we get $y \in \left[b-t, b-t+\frac{a-b}{2}\right)$. So $(x,y) \in \left[t, t+\frac{a-b}{2}\right) \times \left[b-t, b-t+\frac{a-b}{2}\right)$. So $f^{-1}([b,a))$ is an arbitrary union of open sets in $\mathbb{R}_\ell \times \mathbb{R}_\ell$ and hence is open. This means that f is continuous.

(g) This is false. Suppose the set of uniformly continuous functions was closed. This would mean that any convergent sequence of uniformly continuous functions must converge to a function which is also uniformly continuous. This is clearly false, consider the non-uniformly continuous function $f(x) = e^x \sin(1/e^x)$. This is the limit of the sequence of uniformly continuous functions

$$f_n = \begin{cases} f(x) & \text{if } x > -n \\ f(-n) & \text{if } x \le -n \end{cases}.$$

This gives the desired contradiction.

Problem 2 (6 points)**.** Let X, Y be topological spaces. The graph of $f : X \to Y$ is the subset $G_f = \{(x, f(x)) \,|\, x \in X\}$ of $X \times Y$.

(a) Show that if Y is Hausdorff and $f : X \to Y$ is continuous then its graph G_f is a closed subset of $X \times Y$ (with the product topology).

(b) Show that if Y is compact and the graph G_f is closed in $X \times Y$ then f is continuous.

(c) Give an example showing that the result of (b) need not hold if Y is not compact.

(a) Let $(x, y) \in X \times Y - G_f$, so $y \neq f(x)$. Since Y is Hausdorff, there are open sets in Y such that $U \ni y$, $V \ni f(x)$ and $U \cap V = \varnothing$. Now consider the open set $f^{-1}(V) \times U \ni (x, y)$. We claim that $f^{-1}(V) \times U \cap G_f = \varnothing$. Indeed, if $(t, f(t)) \in f^{-1}(V) \times U$ then $f(t) \in V$ and $f(t) \in U$, which would be a contradiction since V and U are disjoint. So we have found an open neighborhood of (x, y) which does not intersect the graph. Since (x, y) was arbitrary point not on the graph, the graph is closed.

(b) Let $V \subset Y$ be an open set, and pick an arbitrary $x \in f^{-1}(V)$. Clearly $\{x\} \times (Y - V)$ does not intersect G_f so since the graph is closed, for every $(x, y) \in \{x\} \times (Y - V)$ there is some open set $U_y \ni (x, y)$ disjoint from G_f. So $\bigcup_{y \in Y - V} U_y$ is an open cover of $\{x\} \times (Y - V)$ disjoint from G_f. Note that $Y - V$ is compact since it is a closed subset of a compact space. Thus we can apply the tube lemma to find a tube of the form $U \times (Y - V) \subset \bigcup_{y \in Y - V} U_y$, where $U \subset X$ is an open neighborhood of X. Note that $U \times (Y - V)$ must be disjoint from G_f. We now claim that $U \subset f^{-1}(V)$. Indeed, for any $t \in U$, $f(t)$ must be in V because otherwise $(t, f(t)) \in U \times (Y - V)$ which would contradict the fact that $U \times (Y - V)$ is disjoint from G_f. So U is an open neighborhood of $x \in f^{-1}(V)$. Since x was arbitrary, it follows that $f^{-1}(V)$ is open and so f is continuous.

(c) Let $X, Y = \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Clearly, f is not continuous at $x = 0$, since $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = \infty$ yet $f(0) = 0$. The graph of f however is closed in $\mathbb{R}^2$ since it is a union of two curves and the origin point, which are both closed in $\mathbb{R}^2$.