Answer the following questions either prove the assertions or give counter examples. If you can't prove them, try to give reasons for your answers. The deadline is 11:59 pm Dec 8. Please latex your solution.

1. (15 pts) For a positive function f, consider two quantities

$$A := \int dy \left[\int f(x,y)^p dx\right]^{1/p}$$

$$B := \left[\int dx \left(\int f(x,y) dy\right)^p\right]^{1/p}$$

For $1 \le p < \infty$. Assume all quantities are integrable and finite. Do we know that $A \ge B$ or $A \le B$ for all functions f ? Prove your assertion. Since this is a known result, you cannot just cite a theorem. Try to prove it directly by using the fact that

$$\|h\|_p = \sup_{\|g\|_q \le 1} \int hg, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

2. (20 points) Suppose $f_n \to f$ a.e. for all $x \in X = (0,1)$ and $\sup_n \|f_n\|_{L^2(X)} \le M$ for some M fixed. Do we know that $\lim_{n\to\infty} \|f_n - f\|_{L^2(X)} = 0$? If, in addition, $\lim_{n\to\infty} \|f_n\|_{L^2(X)} = \|f\|_{L^2(X)} < \infty$. Do we know that $\lim_{n\to\infty} \|f_n - f\|_{L^2(X)} = 0$?

3. (15 pts) Suppose μ is a probability measure (on a set, say, $\mathbb{R}^n$) and $f \ge 0$ with $\int f d\mu = 1$. Assuming that the following integral is finite, prove that

$$S(f) := \int f \log f d\mu \ge 0.$$

Prove that, for any bounded real function h,

$$\int hf d\mu - S(f) = \int [h - (\log f)] f d\mu \le \log \Big[\int e^h d\mu\Big]$$

Notice that you will need to use the condition $\int f d\mu = 1$.

4. (20 pts) Let $X_1, X_2, \ldots X_n$ be identically independent random variables with $\mathbb{E}X_j = 0$, $\mathbb{E}X_j^2 = 1$ and $\mathbb{E}|X_j|^3 \le M < \infty$. This problem gives a proof of CLT with an error bound. Let ϕ be a real function such that the first three derivatives are bounded, i.e., $\sum_{j=0}^3 \|\phi^{(j)}\|_\infty \le M < \infty$. Let Y_j be i.i.d. normal distribution with mean zero and variance one. Rewrite

$$\begin{aligned}
&\mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}\sum_j X_j\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}\sum_j Y_j\Big) \\
&= \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[X_1 + X_2 + \ldots + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + X_2 + \ldots + X_N]\Big) \\
&+ \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + X_2 + X_3 \ldots + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + Y_2 + X_3 \ldots + X_N]\Big) \\
&+ \ldots + \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + \ldots + Y_{N-1} + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + Y_2 + \ldots + Y_N]\Big).
\end{aligned}$$

Prove by using Taylor theorem that

$$\mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[X_1 + X_2 + \ldots + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + X_2 + \ldots + X_N]\Big) \leq C_M N^{-3/2}$$

Similar bound clearly holds for the differences in the telescoping sum. From here, prove that for any ϕ satisfying $\sum_{j=0}^{3} \|\phi^{(j)}\|_\infty \leq M < \infty$,

$$\mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}\sum_j X_j\Big) - \mathbb{E}\phi\Big(\xi\Big) \leq C_M N^{-1/2},$$

where ξ is a normal distribution with mean zero and variance one.

5. (30 points) Let $X_1, X_2, \ldots X_n$ be identically independent random variables with $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = \sigma^2$. Let $S_n = X_1 + \ldots + X_n$. The weak law of large numbers states that Then for any $\varepsilon > 0$

$$\mathbb{P}\Big(|\frac{S_n}{n}| > \delta\Big) \leq \frac{\sigma^2}{n\delta^2} \tag{+}$$

Suppose that, instead of $\mathbb{E}X_j^2 = \sigma^2$, we only know that $(\mathbb{E}|X_j|^p)^{1/p} = M < \infty$ for some $1 < p < 2$. As in the class, let $\hat{X}_j = X_j \mathbf{1}(|X_j| \leq c)$, $\hat{Y}_j = X_j \mathbf{1}(|X_j| > c)$, $a_c = \mathbb{E}\hat{X}_j$ and $b_c = \mathbb{E}\hat{Y}_j$. Clearly, $X_j = \hat{X}_j + \hat{Y}_j$. Then we have

$$\begin{aligned}
\mathbb{E}\Big|\sum_j (\hat{X}_j + \hat{Y}_j)\Big| &\leq \mathbb{E}\Big|\sum_j (\hat{X}_j - a_c)\Big| + \mathbb{E}\Big|\sum_j \hat{Y}_j - b_c\Big| \\
&\leq \Big[\mathbb{E}(\sum_j \hat{X}_j - a_c)^2\Big]^{1/2} + 2n\mathbb{E}|\hat{Y}_j| \\
&= \sqrt{n}\Big[\mathbb{E}(\hat{X}_1 - a_c)^2\Big]^{1/2} + 2n\mathbb{E}|\hat{Y}_1|
\end{aligned}$$

Prove that

$$\mathbb{E}|\hat{Y}_1| \leq c^{1-p} M^p$$

$$\mathbb{E}(\hat{X}_1 - a_c)^2 \leq 2\mathbb{E}\hat{X}_1^2 + 2a_c^2 \leq 4c^{2-p}M^p$$

$$P\Big(\Big|\sum_j (\hat{X}_j + \hat{Y}_j)\Big| \geq n\delta) \leq 4\delta^{-1} \inf_{c>0} [c^{1-p/2} M^{p/2} n^{-1/2} + c^{1-p} M^p]$$

To carry out the inf, prove by calculus that, for any $\alpha, \beta > 0$, there is a constant K such that

$$\inf_{x>0} Ax^\alpha + Bx^{-\beta} = KA^{\frac{1}{1+\gamma}} B^{\frac{\gamma}{1+\gamma}}, \quad \gamma = \frac{\alpha}{\beta}$$

(Hint: Using $y = x^\alpha$ will make calculation much simpler.) Now take $\alpha = 1 - p/2, \beta = p - 1$, you can carry out the inf to finally get a bound on $\mathbb{P}\Big(|\frac{S_n}{n}| > \delta\Big)$. Besides unimportant constants, what is your final answer?