Math 132
Problem Set 3
Spring, 2023

This problem set is due on Friday, Feb 17th. Please make your answers as complete and clear as possible. You are allowed to discuss these problems with others in the class, but your writing should be your own.

1. Show using the preimage theorem that the tangent space to the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^n$ at a point $[v_1, v_2]$, is the vector space of vectors $(u, w) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\begin{aligned} v_1 \cdot u &= 0 \\ v_2 \cdot w &= 0 \\ v_1 \cdot w + v_2 \cdot u &= 0. \end{aligned}$$

2. (GP §5, Problem 1)

(a) Suppose that $A : \mathbb{R}^k \to \mathbb{R}^n$ is a linear map and V is a vector subspace of $\mathbb{R}^n$. Check that $A \pitchfork V$ means just $A(\mathbb{R}^k) + V = \mathbb{R}^n$.

(b) If V and W are linear subspaces of $\mathbb{R}^n$, then $V \pitchfork W$ means just $V + W = R^n$.

3. (GP, §5, Problem 2) Which of the following linear spaces itersect transversally?

(a) The xy plane and the z-axis in $\mathbb{R}^3$

(b) The xy plane and the plane spanned by $\{(3, 2, 0), (0, 4, -1)\}$ in $\mathbb{R}^3$.

(c) The plane spanned by $\{(1, 0, 0), (2, 1, 0)\}$ and the y axis in $\mathbb{R}^3$.

(d) $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^\ell$ in $\mathbb{R}^n$. (Depends on k, ℓ, n).

(e) $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^\ell \times \{0\}$ in $\mathbb{R}^n$. (Depends on k, ℓ, n).

(f) $V \times \{0\}$ and the diagonal in $V \times V$.

(g) The symmetric ($A^T = A$) and skew symmetric ($A^T = -A$) matrices in $M(n)$.

(You can just submit "yes" and "no" answers.)

4. (GP, §5, Problem 9). Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A : V \to V$, consider the graph $W = \{(v, Av)\})$. Show that $W \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of A.

5. (GP, §5, Problem 10). Let $f : X \to X$ be a map with fixed point x; that is, $f(x) = x$. If $+1$ is not an eigenvalue of $df : T_x(X) \to T_x(X)$ then x is called a *Lefschetz fixed point* of f. A map f is called a *Lefschetz* map if all of its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

6. Prove Assertion 3.3 in the section *Colloquialisms in differential topology* of the course lecture notes.

7. (This problem is from the section *Colloquialisms in differential topology* in the lecture notes.) Write expanded versions of the following assertions. (You don't have to prove anything.)

(a) Locally every immersion looks like the standard immersion $\mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^\ell$ which sends x to $(x, 0)$.

(b) Locally every submersion looks like the standard submersion $\mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^k$ sending (x, y) to x.

(c) Every transverse pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\pitchfork"] & Y \arrow[d, "\subset"] \\ X \arrow[r, "\subset"'] & M \end{tikzcd}$$

in which X and Y submanifolds of M, looks, near every $w \in W$, like

$$\begin{tikzcd} \mathbb{R}^\ell \arrow[rr, "x\mapsto(x,0)"] \arrow[d, "x\mapsto(0,x)"'] & & \mathbb{R}^\ell \times \mathbb{R}^m \arrow[d, "(x,y)\mapsto(0,x,y)"] \\ \mathbb{R}^k \times \mathbb{R}^\ell \arrow[rr, "(a,b)\mapsto(a,b,0)"'] & & \mathbb{R}^k \times \mathbb{R}^\ell \times \mathbb{R}^m . \end{tikzcd}$$

8. Suppose that M is a smooth manifold of dimension 2, that X and Y are submanifolds of M of dimension 1 intersecting transversally and that x is a point of $X \cap Y$. Show that there is a coordinate neighborhood $\Phi : U \to \mathbb{R}^2$ centered at $x \in M$ under which $\Phi(X \cap U)$ is the x-axis and $\Phi(Y \cap U)$ is the y-axis.

(Hint: *Near x one can find a coordinate neighborhood $\Phi_1 : U_1 \to \mathbb{R}^2$ under which the inclusion $X \subset M$ corresponds to the x-axis. Composing Φ_1 with projection to the y-axis gives a function $\rho_1 : U_1 \to \mathbb{R}$ defined in a neighborhood of $x \in M$, for which 0 is a regular value, and $\Phi_1^{-1}(0) = U_1 \cap X$. There is a similar function $\rho_2 : U_2 \to \mathbb{R}$ doing the same for Y. Now consider the function $(\rho_1, \rho_2) : U_1 \cap U_2 \to \mathbb{R}^2$, and go from there*).