MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY
HOMEWORK 8
DUE: TUESDAY, APRIL 11 AT 12:00AM (MIDNIGHT) ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Homology with local coefficients (10 points)

(a) Let X denote a path-connected and semilocally simply-connected space, and let $\widetilde{X} \to X$ denote its universal cover. Prove that $S_*(\widetilde{X}; R)$ is a complex of free $R[\pi_1(X)]$-modules, where $\pi_1(X)$ acts via deck transformations on $\widetilde{X}$. (Hint: a basis is given by a choosing a lift $\Delta^n \to \widetilde{X}$ for each $\Delta^n \to X$.)

(b) In the setting of part (a), prove that a short exact sequence of $R[\pi_1(X)]$-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ gives rise to a long exact sequence

$$\cdots \to H_{n+1}(X; M_3) \to H_n(X; M_1) \to H_n(X; M_2) \to H_n(X; M_3) \to H_{n-1}(X; M_1) \to \dots.$$

(c) Prove that $H_*(K(G,1); M) \cong \mathrm{Tor}_*^{R[G]}(R, M)$ by noting that $S_*(\widetilde{K(G,1)}; R)$ is a resolution of R by free $R[G]$-modules. This is usually called the *group homology* of G with coefficients in M and is denoted by $H_*(G; M)$.

2. Problem 2: Example of homology with local coefficients (10 points)

Let $\mathbb{Z}(-1)$ denote the $\mathbb{Z}[C_2]$-module on which the generator of C_2 acts by -1. Compute $H_*(\mathbb{RP}^n; \mathbb{Z}(-1))$. (Hint: there is a short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[C_2] \to \mathbb{Z}(-1) \to 0$.) Read Remark 62.4 of LAT.

3. Problem 3: Cohomology of $U(n)$ (10 points)

Using the fibrations $U(n-1) \to U(n) \to U(n)/U(n-1) \cong S^{2n-1}$, prove by induction on n that $H^*(U(n); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_3, \dots, x_{2n-1}]/(x_1^2, x_3^2, \dots, x_{2n-1}^2)$.

4. Problem 4: Homology of fiber of degree 2 map (10 points)

Do Exercise 62.6 of LAT.

5. Problem 5: Weak equivalence implies homology isomorphism (10 points)

Do Exercise 64.6 of LAT.