Math 137 Problem Set 8

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I collaborated with AJ LaMotta and Ellen Fitzsimons for this problem set.

Problem 1. We call $P \in K^n$ a *point of symmetry* for a subset $S \subseteq K^n$ if the reflection $2P - Q$ across P of any point $Q \in S$ lies in S. Assuming that K has characteristic zero, show that any nonempty algebraic subset $S \subseteq K^n$ that doesn't contain a straight line has at most one point of symmetry.

Suppose $S \subset K^n$ is an algebraic subset with no straight lines in it. Further suppose that S has two points of symmetry, say $P_1, P_2 \in S$. Then by the conditions of symmetry and since we're in characteristic zero, we have a countably infinite set of points $k(P_1 - P_2) + P_1$ which must be in S. Now consider the line L traced out by $t(P_1 - P_2) + P_1 \subset K^n$ for $t \in \mathbb{R}$. Then $L \cap S$ should be an algebraic set. It can't be zero dimensional because zero dimensional sets are finite, yet $L \cap S$ contains an infinite number of points. So it must be one dimensional, and hence $L \cap S = L$. This means that $L \subset S$, a contradiction. So S has at most one point of symmetry.

Problem 2. Let $\varphi : V \to W$ be a dominant morphism between irreducible algebraic sets. Assume that there is a nonempty Zariski open subset U of W such that $|\varphi^{-1}(w)| < \infty$ for all $w \in U$. Show that $\dim(V) = \dim(W)$.

Recall Theorem 13.4.2 from the lectures:

Theorem 13.4.2. Let $\varphi : V \to W$ be a dominant morphism between irreducible algebraic sets. If $B \subset W$ is irreducible and A is an irreducible component of $\varphi^{-1}(B)$ with $\overline{\varphi(A)} = B$, then

$$\operatorname{codim}(A, V) \leq \operatorname{codim}(B, W).$$

Applying this theorem, we let $B = \{w\}$ for some point $w \in W$ with $|\varphi^{-1}(w)| > 0$. Such a point must exist because φ is dominant and a map between irreducible algebraic sets. Then $A = \varphi^{-1}(w)$ clearly satisfies the conditions of the theorem, so we have

$$\operatorname{codim}(\varphi^{-1}(w), V) \leq \operatorname{codim}(\{w\}, W) \implies \dim(V) \leq \dim(W).$$

Conversely, we have $\dim(V) \geq \dim(W)$ because φ is dominant, so we have equality $\dim(V) = \dim(W)$.

Problem 3. For $r \leq n$, consider the set $V_r \subseteq M_n(K)$ of $n \times n$-matrices of rank at most r. You've shown on problem set 4 that V_r is an algebraic subset of $M_n(K) = K^{n\times n}$. Show that its dimension is $2nr - r^2$.

In a rank at most r matrix, there are no more than r linearly independent rows in the matrix, with the rest being linear combinations of the r rows. This motivates the following construction: For any subset $S \subset \{1, \dots, n\}$ of size r, let $\varphi_S : K^{rn+(n-r)r} \to V_r$ be the map which sends the first r vectors of size n to the S rows in the matrix, and writing the remaining $n - r$ rows as a linear combination of the first r vectors using the $(n - r)r$

remaining coefficients. Then

$$V_r = \bigcup_{S \in \mathcal{P}(\{1,\ldots,n\})} \overline{\varphi_S(K^{nr+(n-r)r})}.$$

Notice that all of these maps are dominant finite, so we get that the dimension of V_r is the dimension of $K^{nr+(n-r)r}$, which is $2nr - r^2$.

Problem 4. Let $V \subseteq K^n$ be an irreducible algebraic set and let $P \in K^n$ be a point not contained in V. Show that the Zariski closure of the join of V and $\{P\}$ has dimension $\dim(V) + 1$.

First, we'll prove a lemma about joins:

Lemma. Let V, P be as in the problem. Then $\overline{J(V, \{P\})}$ is irreducible.

Proof. Consider the morphism $\varphi : V \times K \to K^n$ given by $\varphi(x,t) = tx + (1-t)P$. Then $J(V,\{P\})$ is the image of this morphism, so since $V \times K$ is irreducible, it follows that $\overline{J(V,\{P\})}$ must be as well. $\square$

The morphism from the lemma gives restricts to a dominant morphism $V \times K \to \overline{J(V,\{P\})}$ so since both are irreducible, $\dim(\overline{J(V,\{P\})}) = \dim(V) + 1 = n + 1$.

Problem 5 (bonus)**.** Let $V_1, \ldots, V_m$ be any irreducible algebraic subsets of K^n of codimension at least 2. Show that there is an irreducible algebraic subset $W \subsetneq K^n$ containing $V_1 \cup \cdots \cup V_m$.
Hint: What is the dimension of the space of polynomials of degree at most d vanishing on $V_1 \cup \cdots \cup V_m$? What is the dimension of the space of polynomials that are not irreducible?

Problem 6. Let $n \geq 2$ and $d \geq 1$. Consider the vector space $F_d \cong K^{\binom{n+d}{n}}$ of polynomials f in $K[X_1, \ldots, X_n]$ of degree at most d. Show that there is a function $0 \neq r \in \Gamma(F_d)$ (a polynomial in the $\binom{n+d}{n}$ coefficients of f) such that $r(f) = 0$ for all reducible polynomials $f \in F_d$.

Let's consider the space $V \subset F_d$ which is the Zariski closure of the set of all reducible polynomials in F_d. We proved in lecture that

$$\dim V \leq \binom{a+n}{n} + \binom{b+n}{n} - 1$$

for any positive integers a, b which sum to d. Recall that for any integer $m > 0$, we have

$$\binom{m+n}{n} = \frac{(m+n)!}{m! \cdot n!} = \frac{(m+1)(m+2)\cdots(m+n)}{n!} = \frac{p(m)}{n!} + 1.$$

where $p(m) = (m+1)(m+2)\cdots(m+n) - n!$. Notice that $p(m)$ is a degree $n \geq 2$ polynomial with no constant term, and since $a, b \geq 1$ we have the inequality $a^k + b^k < (a+b)^k$ for all $k \geq 2$. This means $p(a) + p(b) < p(d)$ so

$$\binom{a+n}{n} + \binom{b+n}{n} = \frac{p(a)+p(b)}{n!} + 2 < \frac{p(d)}{n!} + 1 = \binom{d+n}{n}.$$

Since $\dim V < \dim F_d$, V is a proper subspace of F_d. Hilbert's Nullstellensatz then implies that $\mathcal{I}_{F_d}(V) \neq \varnothing$, and so there must be some polynomial $r \in \mathcal{I}_{F_d}(V)$ which vanishes on V but not on all of F_d.