Math 114 Review Homework

Lev Kruglyak

December 3, 2022

Problem 1. Suppose $a > 0, b > 0$ and $1 < p \leq 2$. Consider the quantities

$$U = (a+b)^p, \quad V = a^p + b^p.$$

Is it true that $U \geq V$? Prove your answer or give a counter example.

Note that $f(x) = x^p$ has second derivative $f''(x) = p(p-1)x^{p-2}$. For $x > 0$ and $p > 1$, this second derivative is positive so the function f is convex. Thus we can apply Jensen's inequality to get $f(a) + f(b) \geq f(a+b)$. Thus $U \geq V$.

Problem 2. Suppose $f_n \to f$ a.e. for all $x \in X = (0,1)$ and $\sup_n \|f_n\|_{L^2(X)} \leq M$ for some fixed M.

(a) Do we know $\|f\|_{L^2(X)} < \infty$?

(b) Do we know $\lim_{n\to\infty} \|f_n - f\|_{L^2(X)} = 0$?

(c) Do we know that $\lim_{n\to\infty} \|f_n - f\|_{L^p(X)} = 0$ for $1 < p < 2$?

(a) This is true. By Fatou's lemma, we get

$$\|f\|^2_{L^2(X)} = \int_X |f|^2 = \int_X \liminf_n |f_n|^2 \leq \liminf_n \int_X |f_n|^2 \leq \sup_n \int_X |f_n|^2 = M^2.$$

Thus $\|f\|_{L^2(X)} \leq M < \infty$.

(b) This is false. Consider the functions $f_n = n^{1/2}\chi_{[0,1/n]}$. Then clearly $f_n \to 0$ on $(0,1)$, and $\|f_n\|_{L^2(X)} = n \cdot 1/n = 1$, yet $\lim_{n\to\infty} \|f_n - 0\|_{L^2(X)} = 1$.

(c) This is true. We'll begin by proving a generalization of Hölder's inequality.

Claim. Let $a, b, c \geq 1$ satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Then for any functions f, g, we have $\|fg\|_c \leq \|f\|_a \|f\|_b$.

Proof. Let $f' = f^c$ and $g' = g^c$. Since $\frac{1}{a/c} + \frac{1}{b/c} = 1$, we can apply Hölder's inequality to get $\|f'g'\|_1 \leq \|f'\|_{a/c}\|g'\|_{b/c}$. However recall that $\|f\|_{ab} = \|f^a\|_b^{1/a}$. We can use this identity to get

$$\begin{aligned} \|f'g'\|_1 = \|(f'g')^{1/c}\|_c^c = \|fg\|_c^c \leq \|f'\|_{a/c}\|g'\|_{b/c} = \|f\|_a^c \|g\|_b^c \\ \implies \|fg\|_c \leq \|f\|_a \|g\|_b, \end{aligned}$$

which is what we were trying to prove. Furthermore, equality only happens when $(f')^{a/c} = f^a$ and $(g')^{b/c} = g^b$ are linearly dependent almost everywhere. $\square$

First of all, let's use Egorov's theorem to construct closed sets $A_\epsilon \subset X$ with $\mu(X \setminus A_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on A_ϵ. This means that for all $\delta > 0$, there is some N such that for all $n \geq N$ we have $\sup_{A_\epsilon} |f - f_n| < \delta$. Then $\lim_{n\to\infty} \|f - f_n\|_{L^p(A_\epsilon)} = 0$.

Next, notice that $\|f - f_n\|_{L^p(X)} = \|f - f_n\|_{L^p(A_\epsilon)} + \|f - f_n\|_{L^p(X\setminus A_\epsilon)}$. By Minkowski's inequality, we have

$$\|f - f_n\|_{L^p(X\setminus A_\epsilon)} \leq \|f\|_{L^p(X\setminus A_\epsilon)} + \|f_n\|_{L^p(X\setminus A_\epsilon)}.$$

Since $p < 2$, by generalized Hölder's inequality we have

$$\|f\|_{L^p(X\setminus A_\epsilon)} = \|f \cdot \chi_{X\setminus A_\epsilon}\|_{L^p(X\setminus A_\epsilon)} \leq \|f\|_{L^2(X\setminus A_\epsilon)} \|f\|_{L^{2p/p-1}(X\setminus A_\epsilon)} = \|f\|_{L^2(X\setminus A_\epsilon)} \mu(X \setminus A_\epsilon) \leq M\epsilon.$$

Similarly, we get $\|f_n\|_{L^p(X\setminus A_\epsilon)} \leq M\epsilon$. Putting this together, we get

$$\lim_{n\to\infty} \|f - f_n\|_{L^p(X)} = \lim_{n\to\infty} \|f - f_n\|_{L^p(A_\epsilon)} + \lim_{n\to\infty} \|f - f_n\|_{L^p(X\setminus A_\epsilon)} \leq 2M\epsilon.$$

Since ϵ was arbitrary, we get $\lim_{n\to\infty} \|f - f_n\|_{L^p(X)} = 0$.

Problem 3. Suppose f is a positive function on $[0, 1]$. Which of the following is larger? Prove your claim.

$$\int_0^1 f(x) \log f(x)\, dx, \qquad \int_0^1 f(s)\, ds \int_0^1 \log f(t)\, dt.$$

We claim that the first is larger. Consider the function $g : x \to -\log(x)$. This has second derivative $g''(x) = x^{-2}$ which is positive on $(0, \infty)$. Since $\mu([0, 1]) = 1$, we can apply Jensen's inequality to get

$$-\log \int_0^1 f(x)\, dx \leq -\int_0^1 \log f(x)\, dx \implies \int_0^1 \log f(x)\, dx \leq \log \int_0^1 f(x)\, dx.$$

Similarly, the function $h : x \to x\log(x)$ has second derivative $h''(x) = 1/x$ on $(0, \infty)$ so we get

$$\int_0^1 f(x)\, dx \log \int_0^1 f(x)\, dx \leq \int_0^1 f(x) \log f(x)\, dx.$$

Combining the inequalities together, we get

$$\int_0^1 f(x)\, dx \int_0^1 \log f(x)\, dx \leq \int_0^1 f(x)\, dx \log \int_0^1 f(x)\, dx \leq \int_0^1 f(x) \log f(x)\, dx.$$

Problem 4. Let $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $1 \leq p, q, r < \infty$. For any positive functions f, g, h on $\mathbb{R}^n$, prove that

$$\int_{\mathbb{R}^n} fgh\, dx \leq \|f\|_p \|g\|_q \|h\|_r.$$

Note that $\frac{1}{p} + \frac{1}{qr/(q+r)} = 1$ so we can apply Hölder's inequality to get

$$\int_{\mathbb{R}^n} fgh\, dx = \|f \cdot gh\|_1 \leq \|f\|_p \|gh\|_{qr/(q+r)}.$$

Then by generalized Hölder inequality, (proved in Problem 2) since $\frac{1}{q} + \frac{1}{r} = \frac{1}{qr/(q+r)}$, we get

$$\|gh\|_{qr/(q+r)} \leq \|g\|_q \|h\|_r \implies \int_{\mathbb{R}^n} fgh\, dx \leq \|f\|_p \|g\|_q \|h\|_r.$$

Problem 5. Let μ be a positive measure on X. A sequence $\{f_n\}$ of complex measurable functions on X is said to *converge in measure* to the measurable function f if for every $\epsilon > 0$ there corresponds an N such that

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$$

for all $n > N$. (This notion is of importance in probability theory.) Assume $\mu(X) < \infty$ and prove the following statements:

(a) If $f_n(x) \to f(x)$ a.e., then $f_n \to f$ in measure.

(b) If $f_n \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure; here $1 \leq p \leq \infty$.

(c) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to f a.e.

(a) By Egorov's theorem, for every $\epsilon > 0$, let A_ϵ be a closed subset of X with $\mu(X \setminus A_\epsilon)$ and $f_n \to f$ uniformly on A_ϵ. Then there is some N such that for all $n \geq N$ we have $|f(x) - f_n(x)| < \epsilon$ for all $x \in A_\epsilon$. Thus

$$\mu(\{x : |f(x) - f_n(x)| > \epsilon\}) \leq \mu(X \setminus A_\epsilon) < \epsilon.$$

This means that $f_n \to f$ in measure.

(b) Let $\epsilon > 0$. Let N be such that for all $n \geq N$, $\|f - f_n\|_p < \epsilon^{\frac{p+1}{p}}$. This means

$$\int_X |f - f_n|^p < \epsilon^{p+1}.$$

By Tchebychev's inequality and since $p \geq 1$, we get

$$\mu(\{|f - f_n| > \epsilon\}) = \mu(\{|f - f_n|^p > \epsilon^p\}) \leq \frac{1}{\epsilon^p} \int_X |f - f_n|^p < \epsilon$$

for all $n \geq N$. The case when $p = \infty$ follows by definition, since $\|f_n - f\|_\infty \to 0$ means that $|f - f_n| < \epsilon$ almost everywhere.

(c) Consider the set

$$E_{n,\epsilon} = \{x : |f(x) - f_n(x)| > \epsilon\}.$$

Consider some subsequence $\{n_k\}$ such that $\mu(E_{n_k,1/2^k}) < 1/2^k$. Then since $\sum_k \mu(E_{n_k,1/2^k}) < \infty$, we can apply the Borel-Cantelli lemma to get $\mu(\limsup_k E_{n_k,1/2^k}) = 0$. This means that $f_n \to f$ almost everywhere, since $\limsup_k E_{n^k,1/2^k}$ is exactly the set where the functions do not converge.

Problem 6. What about the converse of 5a and 5b, i.e., we assume that $f_n \to f$ in measure then whether $f_n \to f$ a.e. or in L_p? There is no need to consider the case $\mu(X) = \infty$.

In the previous problem we've shown that convergence almost everywhere implies convergence in measure and that L^p convergence implies convergence in measure. If convergence in measure implied either one of these, then convergence almost everywhere and L^p convergence would be the same which is not true. To see why, consider the functions f_n on $(0, 1)$ given by

$$f_n(x) = n\chi_{[0,1/n]}.$$

Then $f_n \to f$ almost everywhere, but $\|f - f_n\|_p = n^{1/p}$. Conversely the functions $f_n = \chi_{E_n}$ where $E_n = [a2^{-k}, (a+1)2^{-k}]$ for $n = 2^k + a$ converges in L^1 but not on $X = (0, 1)$.