MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY
PROJECT 3: COHOMOLOGY OF CONFIGURATION SPACES

In this project, you will compute the integral cohomology ring of the *ordered configuration spaces*

$$\mathrm{Conf}_k(\mathbb{R}^n) = \{(x_1, \dots, x_k) \mid x_i \in \mathbb{R}^n, x_i \neq x_j \text{ if } i \neq j\}.$$

The homology groups of these spaces parametrize k-ary operations on the homology of n-fold loop spaces, via the homotopy equivalences $\mathrm{Conf}_k(\mathbb{R}^n) \simeq C_n(k)$ and the maps $C_n(k) \times (\Omega^n X)^k \to \Omega^n X$.

(1) Prove that the map $p_k : \mathrm{Conf}_k(\mathbb{R}^n) \to \mathrm{Conf}_{k-1}(\mathbb{R}^n)$ which sends $(x_1, \dots, x_k)$ to $(x_1, \dots, x_{k-1})$ is a fiber bundle whose fiber over $(x_1, \dots, x_{k-1}) \in \mathrm{Conf}_{k-1}(\mathbb{R}^n)$ is $\mathbb{R}^n - \{x_1, \dots, x_{k-1}\}$.

(2) Given $1 \leq a \neq b \leq k$, define the *Gauss maps*

$$\gamma_{ab} : \mathrm{Conf}_k(\mathbb{R}^n) \to S^{n-1}$$

by

$$(x_1, \dots, x_k) \mapsto \frac{x_b - x_a}{\|x_b - x_a\|}.$$

Fix a generator ι_{n-1} of $H^{n-1}(S^{n-1})$ and let $\alpha_{ab} = \gamma_{ab}^*(\iota_{n-1}) \in H^{n-1}(\mathrm{Conf}_k(\mathbb{R}^n))$. Using the classes α_{ab}, prove that the map $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) \to H^*(\mathbb{R}^n - \{x_1, \dots, x_{k-1}\})$ is surjective.

(3) Using the Leray–Hirsch theorem, prove inductively on k that $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ is free with basis

$$S_k = \{\alpha_{a_1 b_1} \alpha_{a_2 b_2} \cdots \alpha_{a_m b_m} \mid m \geq 0, 1 \leq b_1 < \cdots < b_m \leq k, a_\ell < b_\ell\}.$$

(4) Prove the *Arnold relation*: for $1 \leq a < b < c \leq k$,

$$\alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab} = 0.$$

(Hint: Reduce to the case of $\mathrm{Conf}_3(\mathbb{R}^n)$, show that some relation must exist, then use the action of the symmetric group on three letters Σ_3 to show that the Arnold relation must hold.)

(5) Give a complete description of $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ as a graded commutative algebra.