Math 137 Problem Set 6

Lev Kruglyak

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Throughout, K is assumed to be an algebraically closed field.

Problem 1. Show that any algebraic subset of K^n is compact with respect to the Zariski topology. (Every open cover has a finite subcover.)

It suffices to prove that K^n is compact, because algebraic subsets are closed subset of K^n, this would imply that any algebraic subset is also compact. Let $\mathcal{U}$ be some open cover of K^n and let $\mathcal{A}$ be the set of finite unions of open sets in $\mathcal{U}$. Recall that K^n is a Noetherian space because it is finite dimensional, so any chain of open sets in K^n must eventually be constant. We know by the Noetherian chain condition and Zorn's lemma that $\mathcal{A}$ contains some maximal element X for $U_i \in \mathcal{U}$. Suppose for the sake of contradiction that $X \neq K^n$, so there is some $x \in K^n - X$. Then letting $U \in \mathcal{U}$ be some open set containing x, the cover $U \cup X$ is a larger element of $\mathcal{A}$, violating maximality of X. So $K^n = X \in \mathcal{A}$ and so K^n can be expressed as a finite union of open sets in the cover.

Problem 2. Let V be an irreducible algebraic set. Show that $\mathcal{O}_V(V) = \Gamma(V)$. (In other words: if a rational function $f \in K(V)$ is defined at every point in V, then $f \in \Gamma(V)$.)

Clearly $\Gamma(V) \subset \mathcal{O}_V(V)$. To prove the other direction, we'll use a clever construction from *Algebraic Curves*. For any $f \in K(V)$ consider the set J_f defined

$$J_f = \{g \in K[X_1, \dots, X_n] \mid gf \in \Gamma(V)\}.$$

Note that J_f is an ideal in $K[X_1, \dots, X_n]$ containing $I(V)$ (If $g \in I(V)$ then $gf = 0$ in $\Gamma(V)$) Also $\mathcal{V}(J_f)$ is exactly the set of points at which f is not defined. Then if $f \in \mathcal{O}_V(V)$, it is defined everywhere so $\mathcal{V}(J_f) = \varnothing$. Thus by the nullstellensatz, $1 \in J_f$ so $1 \cdot f \in \Gamma(V)$ and so $\mathcal{O}_V(V) \subset \Gamma(V)$.

Problem 3. Let V be an irreducible algebraic set, W be any algebraic set, and let $\varphi : V \dashrightarrow W$ be a rational map. We denote by U_φ the set of points $P \in V$ at which φ is defined. Abusing notation, we write $\overline{\varphi(V)} := \overline{\varphi(U_\varphi)}$. We say φ is dominant if $\overline{\varphi(V)} = W$.

a) Show that the map $\varphi : U_f \to W$ is continuous (with respect to the subspace topologies on U_f and W).

b) Show that $\overline{\varphi(V)}$ is irreducible.

c) Let $U \subseteq U_\varphi$ be a nonempty open subset. Show that $\overline{\varphi(U)} = \overline{\varphi(V)}$.

d) Show that φ is dominant if and only if the map $\Gamma(W) \to K(V)$ sending $f \in \Gamma(W)$ to $f \circ \varphi$ is injective.

(a) Let $\{f_i, U_i\}$ be the equivalence class representing of $f \in K(V)$. Then by definition U_i form an open cover of U_f. Thus it suffices to check that f_i are continuous for all i, since $f_i = f\big|_{U_i}$. Recall that $f_i = a/b$ for some $a, b \in \Gamma(V)$ and $\mathcal{V}(b) \cap U_i = \varnothing$. Now let C be a closed subset of W, cut out by say $g_1, \dots, g_k \in \Gamma(W)$. Then

$f_i^{-1}(C) = \{x \in U \mid g_j(a(x)/b(x)) = 0\ \forall j\}$. Note that $g_j(a/b)$ is a rational function, so we can write it as p_j/q_j for some $p_j, q_j \in \Gamma(V)$ with $\mathcal{V}(q_j) \cap U_i = \varnothing$. Then $f_i^{-1}(C) = \mathcal{V}(g_1(a/b), \ldots, g_k(a/b)) = \mathcal{V}(p_1/q_1, \ldots, p_k/q_k) = \mathcal{V}(p_1, \ldots, p_k)$. So $f_i^{-1}(C)$ is an algebraic set and hence closed, so f_i is continuous as desired.

(b) This follows from (a) and due to the fact that the closure of a continuous image of a closed irreducible set is irreducible.

(c) Since U is a nonempty open subset of U_φ, U is dense in U_φ. Since φ is continuous we have $\overline{\varphi(V)} = \overline{\varphi(\overline{U})} \subset \overline{\varphi(U)}$. This completes the proof.

(d) Let $\widetilde{\varphi} : \Gamma(W) \to K(V)$ be the "pullback" map, defined by sending f to $f \circ \varphi$. We claim that φ is dominant iff $\widetilde{\varphi}$ is injective. First suppose φ is dominant. We know from lecture that we have a K-algebra homomorphism $\varphi^* : K(W) \to K(V)$, and this is an extension of $\widetilde{\varphi}$, i.e. $\varphi^*\big|_{\Gamma(W)} = \widetilde{\varphi}$. Since $K(W)$ is a field and φ^* is a homomorphism, it follows that φ^* is injective and so $\widetilde{\varphi}$ is injective. Conversely, suppose $\widetilde{\varphi}$ is injective. We claim that $\mathcal{I}_W(\varphi(U_\varphi)) = \{0\}$. Indeed, suppose $f \in \mathcal{I}_W(\varphi(U_\varphi))$ is some function. Then for any point $p \in U_\varphi$, we have $(f \circ \varphi)(p) = 0$ so $\widetilde{\varphi}(f) = 0$ everywhere since $f \circ \varphi$ vanishes on a nonempty open set. Thus since we assumed $\widetilde{\varphi}$ was injective, we have $f = 0$. Since $\mathcal{I}_W(\varphi(U_\varphi)) = \{0\}$, Hilbert's Nullstellensatz implies that $\overline{\varphi(U_\varphi)} = \overline{\varphi(V)} = W$.

Problem 4. Are $a = X^2 \in \mathbb{C}(X)$ and $b = X^3 + X + 1 \in \mathbb{C}(X)$ algebraically independent over $\mathbb{C}$? If not, find a polynomial $f \in \mathbb{C}[S, T]$ with $f(a, b) = 0$.

They are not algebraically independent, because the trancendence degree of $\mathbb{C}(X)$ over $\mathbb{C}$ is one, so the size of any algebraically independent set is of at most size one. So we should be able to find an explicit polynomial $f \in \mathbb{C}[S, T]$ which satisfies $f(a, b) = 0$. Let $f(s, t) = t^2 - s^3 - 2s^2 - s - 2t + 1$. Then

$$f(a, b) = (X^3 + X + 1)^2 - X^6 - 2X^4 - X^2 - 2X^3 - 2X^3 - 2 + 1 = 0$$

so a and b are algebraically dependent.

Problem 5. Let I be any ideal of $K[X_1, \ldots, X_n]$ and let $V = \mathcal{V}(I)$. Let $S = K[X_1, \ldots, X_n]/I$.

a) Show that $\dim(V)$ is the maximum number of (over K) algebraically independent elements of S. *Note:* We call elements $f_1, \ldots, f_d$ of any K-algebra S *algebraically independent* if there is no polynomial $0 \neq g \in K[Y_1, \ldots, Y_d]$ such that $g(f_1, \ldots, f_d) = 0$ in S.

b) Show that $\dim(V)$ is the maximum size of a sublist of $X_1, \ldots, X_n$ consisting of algebraically independent elements of S.

(a) Recall that $\dim(V)$ is the size of a transcendence basis of $K(V)/K$, so let $X_{i_1}, \ldots, X_{i_d}$ be a transcendence basis for $K(V)/K$ where $d = \dim(V)$, and we chose the X_{i_k} from among the generators $X_1, \ldots, X_n$. Suppose there existed a nonzero polynomial $g \in K[Y_1, \ldots, Y_d]$ such that $g(X_{i_1}, \ldots, X_{i_d}) \equiv 0 \mod I$. Then since $I \subset \sqrt{I}$, we also must have $g(X_{i_1}, \ldots, X_{i_d}) \equiv 0 \mod \sqrt{I}$ which would mean that $g(X_{i_1}, \ldots, X_{i_d}) = 0$ in $K(V)$, a contradiction to the algebraic independence of the $X_{i_1}, \ldots, X_{i_d}$. This shows that there are at least $\dim(V)$ algebraically elements of S. To prove that $\dim(V)$ is in fact the maximum number of algebraically independent elements of S, let $f_1, \ldots, f_{d'}$ be some algebraically independent collection of elements with $d' > d$. This implies that they are algebraically dependent in $K(V)$, so we have some nonzero $g \in K[Y_1, \ldots, Y_{d'}]$ such that $g(f_1, \ldots, f_{d'}) = 0 \mod \sqrt{I}$. Since $g(f_1, \ldots, f_{d'}) \in \sqrt{I}$, by definition of a radical ideal there is some k such that $g^k(f_1, \ldots, f_{d'}) \in I$. Since $g^k \neq 0$, we have our algebraic dependence $\mod I$.

(b) This is proven by (a), we even constructed a sublist of $X_1, \ldots, X_{\dim V}$ which is algebraically independent in S.