Math 231a Problem Set 8

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Problem 1. Let X denote a space. Using the long exact sequence associated to the short exact sequence of chain complexes
$$0 \longrightarrow S_*(X;\mathbb{Z}) \xrightarrow{n} S_*(X;\mathbb{Z}) \longrightarrow S_*(X;\mathbb{Z}/n) \longrightarrow 0,$$
prove that there are short exact sequences
$$0 \longrightarrow H_k(X;\mathbb{Z})/n \longrightarrow H_k(X;\mathbb{Z}/n) \longrightarrow \mathrm{tors}_n(H_{k-1}(X;\mathbb{Z})) \longrightarrow 0,$$
where, given an abelian group A, $\mathrm{tors}_n(A)$ denotes the subgroup of n-torsion elements of A.

Use this to recompute $H_*(\mathbb{RP}^n;\mathbb{F}_2)$ from $H_*(\mathbb{RP}^n;\mathbb{Z})$.

For any k, consider the slice of the exact sequence given by:
$$H_k(X;\mathbb{Z}) \xrightarrow{n} H_k(X;\mathbb{Z}) \longrightarrow H_k(X;\mathbb{Z}/n) \xrightarrow{\partial} H_{k-1}(X;\mathbb{Z}) \xrightarrow{n} H_{k-1}(X;\mathbb{Z})$$
Observe that the image of the first n map is exactly $nH_k(X;\mathbb{Z}) \subset H_k(X;\mathbb{Z})$, and the kernel of the last n map is exactly $\mathrm{tors}_n(H_{k-1}(X;\mathbb{Z}))$. Thus we have an induced exact sequence
$$0 \longrightarrow H_k(X;\mathbb{Z})/n \longrightarrow H_k(X;\mathbb{Z}/n) \longrightarrow \mathrm{tors}_n(H_{k-1}(X;\mathbb{Z})) \longrightarrow 0$$
for every k.

Now to calculate $H_*(\mathbb{RP}^n;\mathbb{F}_2)$, note that we have the exact sequences:
$$0 \longrightarrow H_k(\mathbb{RP}^n;\mathbb{Z})/2 \longrightarrow H_k(\mathbb{RP}^n;\mathbb{Z}/2) \longrightarrow \mathrm{tors}_2(H_{k-1}(\mathbb{RP}^n;\mathbb{Z})) \longrightarrow 0$$
By the previous computation of $H_*(\mathbb{RP}^n;\mathbb{Z})$, for every $0 \le k \le n$, either we have$H_k(\mathbb{RP}^n;\mathbb{Z})/2 = \mathbb{Z}/2$ or $\mathrm{tors}_2(H_{k-1}(\mathbb{RP}^n;\mathbb{Z})) = \mathbb{Z}/2$, so in either case we have an isomorphism $H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2$. This gives the homology:
$$\boxed{H_k(\mathbb{RP}^n;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}}$$

Problem 2. *Define sequential colimits.* As an example, regard $X_0 \subset X_1 \subset \cdots X$ as a sequence of spaces. Then $\varinjlim_n X_n = \bigcup_{n=0}^\infty X_n$.

Fix a commutative ring R.

a. Given a sequence M_* of R-modules, prove that

$$\varinjlim_n M_n \cong \frac{\bigoplus_{n\in\mathbb{N}} M_n}{(f_n(x_n) - x_n \text{ for } n \in \mathbb{N}, x_n \in M_n)}.$$

In particular, $\varinjlim_n M_n$ exists.

Let's call the above quotient space $\widetilde{M}$, and the quotient map $p : \bigoplus_{n\in\mathbb{N}} M_n \to \widetilde{M}$. For each M_n, we have the canonical inclusion $i_n : M_n \to \bigoplus_{n\in\mathbb{N}}$ so composing with the quotient gives us maps $p \circ i_n = g_n : M_n \to \widetilde{M}$. It's easy to check that $g_{n+1} \circ f_n = g_n$.

Now suppose there were some R-module P with maps $h_n : M_n \to P$ such that $h_{n+1} \circ f_n \circ h_n$. By the definition of coproduct, we have a unique map $h : \bigoplus_{n\in\mathbb{N}} M_n \to P$ which respects these inclusions h_n. However since they also satisfy the compatibility condition, we can pass this to a unique map in the quotient $\widetilde{h} : \widetilde{M} \to P$.

b. Suppose we have sequences $M_\bullet, N_\bullet$, and $P_\bullet$ of R-modules and an exact sequence $M_\bullet \to N_\bullet \to P_\bullet$. Prove that $\varinjlim_n M_n \to \varinjlim_n N_n \to \varinjlim_n P_n$ is an exact sequence.

We'll begin by proving the hint.

Claim. Suppose that the image of $x \in M_n$ in $\varinjlim_n M_n$ is zero. Then the image of x in M_k is zero for some $k \geq n$.

Proof. We'll use the isomorphism given in (a) to simplify this proof. Thus, the condition in the claim can be rephrased as

$$x_n = \sum_{k=1}^{N} (y_k - f_k(y_k))$$

for some $y_i \in M_i$, where x_n is the image of x in M_n. A simple inductive argument then shows that $x_{N+1} = 0$. □

Note that the universal property of sequential limits give us maps $\Phi : \varinjlim_n M_n \to \varinjlim_n N_n$ and $\Psi : \varinjlim_n N_n \to \varinjlim_n P_n$. The rest is a fairly straightforward diagram chase using the claim.

c. Given a sequence of R-modules $M_\bullet$ and an R-module N, prove that there is a natural isomorphism

$$(\varinjlim_n M_n) \otimes_R N \cong \varinjlim_n (M_n \otimes_R N).$$

Suppose P is an R-module. Since there is a bijection

$$\operatorname{Hom}\left((\varinjlim_n M_n) \otimes_R N, P\right) \cong \operatorname{Hom}\left(\varinjlim_n M_n, \operatorname{Hom}(N, P)\right),$$

a map from $(\varinjlim_n M_n) \otimes_R N$ to P is the same as a linear map $\varinjlim_n M_n \to \operatorname{Hom}(N, P)$, which in turn is a family of maps $g_n : M_n \to \operatorname{Hom}(N, P)$

Problem 3. Flatness of $\mathbb{Q}$.

Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module.

a. Prove that $\mathbb{Q}$ is isomorphic to the sequential colimit of the following diagram:

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \cdots$$

Firstly note that we have a canonical diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "2"] \arrow[dr, bend right, "1"'] & \mathbb{Z} \arrow[r, "3"] \arrow[d, "1/2"'] & \mathbb{Z} \arrow[r, "4"] \arrow[dl, "2/3"] & \mathbb{Z} \arrow[dll, bend left, "3/4"] \\
 & \mathbb{Q} & &
\end{tikzcd}$$

where each map is the multiplication by $n/(n+1)$ map. Any other $\mathbb{Z}$-module M which such maps must contain all of the rational numbers (i.e. multiplicative inverses for $\mathbb{Z}^{\times}$), so there will be a unique map $\mathbb{Q} \to M$.

b. Using part (a) and Problem 2, prove that $\mathbb{Q}$ is a *flat* $\mathbb{Z}$-module, i.e. that the functor $- \otimes \mathbb{Q} : \mathrm{Ab} \to \mathrm{Vect}_{\mathbb{Q}}$ is exact.

Suppose $0 \to A \to B \to C \to 0$ is an exact sequence. This gives us the commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
A \otimes \mathbb{Z} \arrow[r, "\otimes 2"] \arrow[d] & A \otimes \mathbb{Z} \arrow[r, "\otimes 3"] \arrow[d] & A \otimes \mathbb{Z} \arrow[r, "\otimes 4"] \arrow[d] & \cdots \\
B \otimes \mathbb{Z} \arrow[r, "\otimes 2"] \arrow[d] & B \otimes \mathbb{Z} \arrow[r, "\otimes 3"] \arrow[d] & B \otimes \mathbb{Z} \arrow[r, "\otimes 4"] \arrow[d] & \cdots \\
C \otimes \mathbb{Z} \arrow[r, "\otimes 2"] \arrow[d] & C \otimes \mathbb{Z} \arrow[r, "\otimes 3"] \arrow[d] & C \otimes \mathbb{Z} \arrow[r, "\otimes 4"] \arrow[d] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}$$

By Problem 2b, we get a short exact sequence:

$$0 \longrightarrow \varinjlim_n (A \otimes \mathbb{Z}) \longrightarrow \varinjlim_n (B \otimes \mathbb{Z}) \longrightarrow \varinjlim_n (C \otimes \mathbb{Z}) \longrightarrow 0$$

Then by Problem 2c and Part a, we get a short exact sequence:

$$0 \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow B \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow C \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0$$

Thus $- \otimes_{\mathbb{Z}} \mathbb{Q}$ is exact.

c. Given a finitely generated abelian group $A \cong \mathbb{Z}^{\oplus r} \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$, prove that $A \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus r}$.

Recall that $\mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$ and $\mathbb{Z}/d \otimes \mathbb{Q} = 0$. Since tensor products distribute over direct sums, we get

$$A \otimes \mathbb{Z} \cong (\mathbb{Z} \otimes \mathbb{Q})^{\otimes r} \oplus (\mathbb{Z}/n_1 \otimes \mathbb{Q}) \oplus \cdots \oplus (\mathbb{Z}/n_k \otimes \mathbb{Q}) = \mathbb{Q}^{\oplus r}.$$

d. By tensoring with $\mathbb{Q}$ and using basic facts from linear algebra, prove that the rank of finitely generated abelian groups is additive in short exact sequences. That is, given a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of finitely generated abelian groups, prove that

$$\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C).$$

Since $- \otimes_{\mathbb{Z}} \mathbb{Q}$ is exact, we get an exact sequence:

$$0 \longrightarrow A \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow B \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow C \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0$$

By Part c, this becomes an exact sequence of $\mathbb{Q}$-vector spaces:

$$0 \longrightarrow \mathbb{Q}^{\oplus \mathrm{rank}(A)} \longrightarrow \mathbb{Q}^{\oplus \mathrm{rank}(B)} \longrightarrow \mathbb{Q}^{\oplus \mathrm{rank}(C)} \longrightarrow 0$$

Thus by linear algebra, we get $\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C)$.

Problem 4. Tor computations.

Let R denote a commutative ring.

a. Given a non zero-divisor $x \in R$ and an R-module M, compute $\mathrm{Tor}^R_*(R/x, M)$.

Here we have a simple free resolution of R/x given by the sequence:

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow R/x \longrightarrow 0$$

This gives rise to a chain complex:

$$0 \longrightarrow M \otimes_R R \xrightarrow{- \otimes_R \cdot x} M \otimes_R R \longrightarrow 0$$

From this we get our Tor modules:

$$\begin{aligned}
&\mathrm{Tor}^R_0(R/x, M) = M \otimes_R R/\mathrm{Im}(M \otimes_R R \to M \otimes_R R) = M \otimes_R R/M \otimes_R Rx \cong M \otimes_R R/x, \\
&\mathrm{Tor}^R_1(R/x, M) = \ker(M \otimes_R R \to M \otimes_R R) \cong \mathrm{tors}_x(M).
\end{aligned}$$

Thus the higher Tor modules are

$$\mathrm{Tor}^R_k(R/x, M) = \begin{cases} M \otimes_R R/x & k = 0, \\ \mathrm{tors}_x(M) & k = 1, \\ 0 & k \geq 2. \end{cases}$$

b. Given two ideals $I, J \subset R$, prove that $\mathrm{Tor}^R_1(R/I, R/J) = (I \cap J)/IJ$.

Since we have a short exact sequence of R-modules:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

This gives rise to a long exact sequence of Tor modules:

$$\begin{tikzcd}
 & \mathrm{Tor}^R_2(R, R/J) \arrow[r] & \mathrm{Tor}^R_2(R/I, R/J) \arrow[dll] & \\
\mathrm{Tor}^R_1(I, R/J) \arrow[r] & \mathrm{Tor}^R_1(R, R/J) \arrow[r] & \mathrm{Tor}^R_1(R/I, R/J) \arrow[dll] & \\
R/J \otimes_R I \arrow[r] & R/J \otimes_R R \arrow[r] & R/J \otimes_R R/I \arrow[r] & 0
\end{tikzcd}$$

Since R is flat as an R-module, $\mathrm{Tor}^R_k(R, R/J) = 0$ for all $k \geq 1$. Thus we have an isomorphism

$$\mathrm{Tor}^R_1(R/I, R/J) \cong \ker(R/J \otimes_R I \to R/J \otimes_R R).$$

But we also have the canonical isomorphism $R/J \otimes RI \cong I/IJ$ and $R/J \otimes_R R \cong R/J$ so this map is in fact the canonical map $I/IJ \to R/J$ which sends $x + IJ \to x + J$. This is clearly well defined, and the only elements which map to zero are those $x + IJ$ with $x \in J$. Thus this kernel is $(I \cap J)/IJ$ so

$$\mathrm{Tor}_1^R(R/I, R/J) \cong (I \cap J)/IJ.$$

c. Compute $\mathrm{Tor}_*^{R[x]/x^n}(R, R)$ for any $n \geq 2$.

In this case, we can use the free resolution approach again, but the free resolution is considerably more complex. Consider the resolution:

$$\longrightarrow R[x]/x^n \xrightarrow{\cdot x} R[x]/x^n \xrightarrow{\cdot x^{n-1}} R[x]/x^n \xrightarrow{\cdot x} R[x]/x^n \longrightarrow\!\!\!\!\!\rightarrow R \longrightarrow 0$$

For this to be an exact sequence, we need the maps to alternate back and forth between $\cdot x$ and $\cdot x^{n-1}$. This gives a chain complex:

$$\xrightarrow{\cdot x} R[x]/x^n \otimes_{R[x]/x^n} R \xrightarrow{\cdot x^{n-1}} R[x]/x^n \otimes_{R[x]/x^n} R \xrightarrow{\cdot x} R[x]/x^n \otimes_{R[x]/x^n} R \longrightarrow 0$$

Taking the homology of this chain complex, it's clear that we have

$$\mathrm{Tor}_*^{R[x]/x^n}(R, R) \cong R \otimes_{R[x]/x^n} R \cong R.$$