Math 230a Problem Set 6

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Due: October 16, 2024

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Problem 1. Let A be a 3-dimensional Euclidean space and $\Sigma \subset A$ a cooriented surface. A point $p \in \Sigma$ is *umbilic* if the second fundamental form at p is a multiple of the metric at p. Suppose e_1, e_2, e_3 is a moving frame on an open subset of Σ and $\theta^1, \theta^2, \Theta^2_1, \Theta^3_1, \Theta^3_2$ the induced 1-forms from $\mathcal{B}_{\mathrm{O}}(A)$. Express the umbilic condition in terms of these forms.

A point $p \in \Sigma$ is umbilic if and only if for all $\xi, \eta \in T_p\Sigma$ we have

$$-\langle D_\xi \nu, \eta\rangle_p = \lambda \langle \xi, \eta\rangle_p$$

for some constant $\lambda \in \mathbb{R}$. If we plug in the basis vectors e_i, this amounts to the condition

$$\langle D_{e_i}\nu, e_j\rangle_p = -\lambda\delta_{ij}.$$

Rewriting this in terms of the Θ forms, we get

$$(\Theta^3_1)_p(e_1) = (\Theta^3_2)_p(e_2), \quad (\Theta^3_1)_p(e_2) = 0, \quad \text{and} \quad (\Theta^3_2)_p(e_1) = 0.$$

In particular, we have $h_{ij} = \lambda\delta_{ij}$.

Suppose now that *every* point of Σ is umbilic. Then there is a function $\lambda : \Sigma \to \mathbb{R}$ such that the second fundamental form is λ times the first fundamental form. Prove that λ is locally constant.

This follows from the Codazzi-Mainardi equations.

Problem 2. Let M be a smooth manifold of dimension n.

Before getting started with the constructions, let's consider the product tangent bundle $T(M^{\times n}) = TM^{\times n}$, and the diagonal $\Delta_M \subset M^{\times n}$. This diagonal is of course a submanifold of $M^{\times n}$ canonically diffeomorphic to M by a map

$$\begin{array}{rcl} \delta : M & \longrightarrow & \Delta_M \subset M^{\times n} \\ x & \longmapsto & (x, \dots, x). \end{array}$$

We can also take the restriction of the tangent bundle to this submanifold, let's denote this $\overline{T}\Delta_M \subset TM^{\times n}$. At this point, the pullback bundle $\delta_*\overline{T}\Delta_M$ is the fiber bundle of ordered n-tuples of tangent vectors of M. Let's denote this canonical bundle $\overline{\mathcal{B}}(M)$.

(a). Construct the principal GL_n-bundle of frames $\mathcal{B}(M) \to M$.

Now, $TM^{\times n}$ is a $2n^2$-manifold, and can thus be embedded into some affine space $\mathbb{A}^k$ for k large enough – let's call this embedding $\iota : TM^{\times n} \to \mathbb{A}^k$. By composing the diagonal map with the zero section and the embedding, we get an embedding $\iota_0 : M \to \mathbb{A}^k$, where $\iota_0 = \iota \circ s_0 \circ \delta$. These embeddings then give us a smooth map

$$\begin{aligned} \det \; : \; \overline{T}\Delta_M &\longrightarrow \mathbb{R} \\ (x; v_1, \ldots, v_n) &\longmapsto \det(\iota(v_1) - \iota_0(x), \ldots, \iota(v_n) - \iota_0(x)). \end{aligned}$$

Note that $\det^{-1}(\mathbb{R}^\times)$ is an open subset of $\overline{T}\Delta_M$ and so must be a manifold. Let's finally define the total space frame bundle to be

$$\mathcal{B}(M) = \delta_* \det^{-1}(\mathbb{R}^\times) = \left\{ (x; v_1, \ldots, v_n) \in \overline{\mathcal{B}}(M) : \operatorname{span}\{v_1, \ldots, v_n\} = T_x M \right\}$$

(b). Construct the principal O_n-bundle of orthonormal frames $\mathcal{B}_{\mathrm{O}}(M) \to M$ given a Riemannian metric.

A Riemannian metric gives us an inner product on each tangent space. Let's consider the set

$$\mathcal{B}_{\mathrm{O}}(M) = \left\{ (x; v_1, \ldots, v_n) \in \mathcal{B}(M) : \langle v_i, v_j \rangle = \delta_{ij} \right\}.$$

This set of equations has no singularities and so forms a manifold. It's clear that the fibers are diffeomorphic to O_n, and we have local trivializations by embedding in affine space as before.

Problem 3. Does there exist an example of a Riemannian 2-manifold Σ and an embedded oriented circle $C \subset \Sigma$ such that parallel transport around C is a reflection (rather than a rotation)? If so, what does this say about the lift of C to the orientation double cover, which is identified with $\mathcal{B}_{\mathrm{O}}(\Sigma)/\mathrm{SO}_2 \to \Sigma$? Does there exist an example in which Σ is a submanifold of a Euclidean 3-space?

Consider the Möbius strip M, viewed as a nontrivial line bundle π over S^1. Picking any orientation of S^1 and including it into M by the zero section gives us an oriented subcircle $C \subset M$. Since M is a line bundle, note that the tangent space of the Möbius strip at any point $p \in M$ is given by $T_pM = T_{\pi(p)}S^1 \oplus M_{\pi(p)}$. The parallel transport of any vector in $T_{\pi(p)}S^1$ around C is just given by the parallel transport in S^1, and parallel transports of vectors in $M_{\pi(p)}$ are given by locally constant sections of M. It's clear that a frame (v_1, v_2) with v_1 a vector in the former space and v_2 a vector in the latter space is sent to $(v_1, -v_2)$ after a full parallel transport around C. This is a reflection, not a rotation.

This implies that the orientation double cover is connected, since any frame can be reflected by a parallel transport. In particular, it implies that the lift of C to $\mathcal{B}_{\mathrm{O}}(\Sigma)/\mathrm{SO}_2$ can be identified with the double cover $S^1 \to S^1$. We see that any surface satisfying the condition must be nonorientable. If we let surfaces be non-compact, then our example of the Möbius bundle works. Otherwise, if we require compact closed manifolds, the smallest example can be embedded in Euclidean 4-space.