Problem Set # 3

Math230a: Differential Geometry

Due: September 25

1. (a) Let M be a 3-manifold and α a nonzero 1-form. Prove that the 2-dimensional distribution determined by α is integrable if and only if $\alpha \wedge d\alpha = 0$.

   (b) The Hopf fibration $\pi\colon S^3 \to S^2$ may be constructed by identifying S^3 as the unit sphere in $\mathbb{C}^2$ and S^2 as $\mathbb{CP}^1$; then the map is $\pi(z^1, z^2) = [z^1, z^2]$, where $(z^1)^2 + (z^2)^2 = 1$ and $[z^1, z^2]$ is the equivalence class in the projective line. The kernel of the differential π_* is an (integrable) one-dimensional distribution on S^3. Let $E \subset TS^3$ be the 2-dimensional distribution whose fiber at $p \in S^3$ is the orthogonal complement of $\ker \pi_*$ relative to the standard round metric on S^3. Is E integrable? Find a nonzero 1-form α which generates the ideal $\mathcal{I}(E)$ associated to E. Compute $d\alpha$ and $\alpha \wedge d\alpha$.

2. Suppose M is a smooth manifold and $E \subset TM$ a distribution. Define

$$\mathcal{I}(E) = \{\omega \in \Omega^\bullet_M : \omega\big|_\Delta = 0\}.$$

   (a) Prove that $\mathcal{I}(E) \subset \Omega^\bullet_M$ is an ideal.

   (b) Prove that if E has corank r—that is, if $\dim E_m + r = \dim_m M$ for all $m \in M$—then E is locally generated by r independent 1-forms.

   (c) Prove that $\mathcal{I}(E)$ is closed under d if and only if E is integrable.

   (d) Consider the distribution E on $\mathbb{A}^3_{x,y,z}$ spanned by the vector fields $\partial/\partial x$ and $x\,\partial/\partial y + \partial/\partial z$. Show that E is not integrable. Show that any point $(x, y, z) \in \mathbb{A}^3$ may be joined to $(0, 0, 0)$ by a piecewise smooth curve whose tangent line belongs to E.

3. Example or proof of nonexistence: A codimension 1 foliation on the sphere S^4.

4. (a) Let $P, Q\colon \mathbb{A}^2 \to \mathbb{R}$ be smooth functions. Define the 2-dimensional distribution E on $\mathbb{A}^2_{x,y} \times \mathbb{R}_z$ with

$$E_{(x,y,z)} = \operatorname{span}\left\{ \frac{\partial}{\partial x} + P\frac{\partial}{\partial z},\ \frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} \right\}.$$

   Compute the Frobenius tensor of E.

(b) Suppose X is a manifold and G a Lie group. Let θ^i, $i = 1, \dots, n$ be a basis of left-invariant 1-forms on G and suppose

$$d\theta^i + \frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k = 0$$

for constants c^i_{jk}. Let θ^i_X, $i = 1, \dots, n$ be 1-forms on X. Consider the ideal of differential forms on $X \times G$ generated by $\pi_2^* \theta^i - \pi_1^* \theta^i_X$, where $\pi_1 \colon X \times G \to X$ and $\pi_2 \colon X \times G \to G$ are projections. Prove that this ideal is closed under d if and only if

$$d\theta^i_X + \frac{1}{2} c^i_{jk} \theta^j_X \wedge \theta^k_X = 0$$

(c) Compute the Frobenius tensor of the distribution in (b) defined as the simultaneous kernel of the 1-forms $\pi_2^* \theta^i - \pi_1^* \theta^i_X$.

5. This is a collection of exercises on the Maurer-Cartan form.

(a) Let G be a Lie group with Maurer-Cartan form θ. Compute $R_g^* \theta$ for $g \in G$. Do this first for a matrix group, where you can write $\theta = g^{-1} dg$ for $g \colon G \to M_n \mathbb{R}$ the natural matrix-valued function on a matrix group. ($M_n \mathbb{R}$ is a vector space, so the differential of the function g is defined as a $M_n \mathbb{R}$-valued 1-form.)

(b) Let G be a Lie group and suppose T is a *right* G-torsor. Show that the Maurer-Cartan form on G transports to a canonical element of $\Omega^1_T(\mathfrak{g})$. Can you give a prose definition of this Maurer-Cartan 1-form on the torsor: "its value at a point $t \in T$ on the vector $\zeta \in T_t T$ is…"? What is the Maurer-Cartan equation? What is the pullback of the Maurer-Cartan 1-form by an element of G acting on T?

(c) Let V be an n-dimensional real vector space and $\mathcal{B}(V)$ the right $\mathrm{GL}_n(\mathbb{R})$-torsor of bases. (Recall that a basis is an isomorphism $b \colon \mathbb{R}^n \to V$.) Let Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$. Suppose $b(t)$ is a smooth curve in $\mathcal{B}(V)$. Write the basis $b(t)$ as $\{e_1(t), \dots, e_n(t)\}$ and the dual basis as $\{e^1(t), \dots, e^n(t)\}$. Prove that

$$\Theta^i_j(\dot{b}) = \langle e^i(0), \dot{e}_j(0) \rangle.$$

Heuristically, then, Θ^i_j measures the instantaneous motion of e_j in the direction of e_i, where 'direction' is determined by the entire basis $e_1, \dots, e_n$. This interpretation is important!

(d) Let A be an n-dimensional real affine space and $\mathcal{B}(A)$ the right $\mathrm{Aff}_n(\mathbb{R})$-torsor of bases of the underlying vector space at all points of A. So a point of $\mathcal{B}(A)$ is an affine isomorphism $\mathbb{A}^n \to A$. Let θ^i, Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$. (Define this basis: the single index is for infinitesimal translations, and the double index for infinitesimal linear transformations, as in (c).) Suppose $b(t)$ is a smooth curve in $\mathcal{B}(A)$ which projects to the curve $x(t)$ in A, and write the underlying basis of V as in (c). Prove that

$$\theta^i(\dot{b}) = \langle e^i(0), \dot{x}(0) \rangle.$$

Interpret this in prose terms.