Math 231b Problem Set 7

Lev Kruglyak

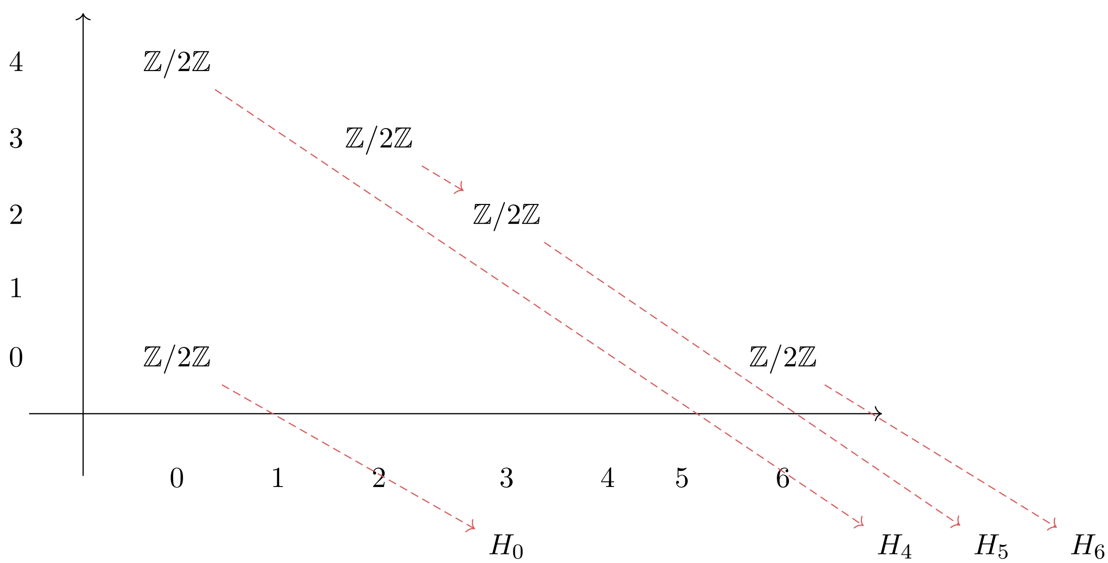
Due: April 4, 2023

Problem 1. Suppose that $F_\bullet C$ is a filtered complex of abelian groups which is first-quadrant.

Assume that the associated spectral sequence $(E^r_{*,*}, d^r)$ has E^2-term given by $E^2_{s,t} = \mathbb{Z}/2\mathbb{Z}$ if $(s,t) = (0,0), (0,4), (2,3), (3,2), (6,0)$ and $E^2_{s,t} = 0$ otherwise.

a. Determine all possible values of $H_*(C)$.

Observe that the E^2-page of the associated spectral sequence has no nontrivial boundary maps so the E^∞-page is simply equal to the E^2-page.



This immediately tells us that $H_0(C) = \mathbb{Z}/2\mathbb{Z}$, $H_4(C) = \mathbb{Z}/2\mathbb{Z}$, and $H_6(C) = \mathbb{Z}/2\mathbb{Z}$. For H_5, recall that we have a filtration $F_k H_5(C) = \mathrm{Im}(H_5(F_k C) \to H_5(C))$, and this line in the spectral sequence tells us that

$$\mathrm{gr}_2 H_5(C) = \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{gr}_3 H_5(C) = \mathbb{Z}/2\mathbb{Z}$$

with respect to this filtration. We thus have the situation where there are three abelian groups $X \subset Y \subset Z$ with $Y/X = Z/Y = \mathbb{Z}/2\mathbb{Z}$. By the first quadrant condition, it follows that A must be zero, and the group $Y = H_5(C)$. The only options here are either $\mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^{\times 2}$. Thus,

$$H_k(C) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k = 0, 4, 6, \\ \mathbb{Z}/4\mathbb{Z} \text{ or } (\mathbb{Z}/2\mathbb{Z})^{\times 2} & \text{if } k = 5, \\ 0 & \text{otherwise.} \end{cases}$$

b. Assume further that $F_\bullet C$ is a filtered complex of $\mathbb{F}_2$ vector spaces. How does this restrict the possible values of $H_*(C)$?

By the same logic as in the previous problem, in this case we get

$$H_k(C) = \begin{cases} \mathbb{F}_2 & \text{if } k = 0, 4, 6, \\ \mathbb{F}_2^{\oplus 2} & \text{if } k = 5, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2. Let R be any ring and C_* a chain complex of projective (or even just flat) R-modules, and let M be an R-module. Construct a "universal coefficient spectral sequence"

$$E^2_{s,t} = \mathrm{Tor}^R_s(H_t(C_*), M) \implies H_{s+t}(C_* \otimes_R M)$$

in the following manner. Let $M \leftarrow P_*$ be a projective resolution of M as an R-module. Form the double complex $C_* \otimes_R P_*$; and study the associated pair of spectral sequences. Observe that this returns a short exact sequence as in the Universal Coefficient Theorem if R is a PID.

We'll use the provided hint, so let P_* be a projective resolution of M, considered as a chain complex. Now let $A = C_* \otimes_R P_*$ be the tensor product, here considered a double complex with the natural grading; $d_h(c \otimes p) = dc \otimes p$ and $d_v(c \otimes p) = c \otimes dp$. The total complex $\overline{A}_*$ is given by

$$\overline{A}_n = \bigoplus_{s+t=n} C_s \otimes_R P_t \quad \text{with} \quad da = d_h a + (-1)^s d_v a.$$

We claim that the associated spectral sequence is exactly the desired one. Recall tha to get the associated spectral sequence, we take the natural filtration

$$F_p(\overline{A}_n) = \bigoplus_{\substack{s+t=n,\\ s \leq p}} C_s \otimes_R P_t \subset \overline{A}_n.$$

The associated spectral sequence with respect to this filtration has E^0-page:

$$\begin{array}{ccc} C_2 \otimes_R P_0 & C_2 \otimes_R P_1 & C_2 \otimes_R P_2 \\ \downarrow & \downarrow & \downarrow \\ C_1 \otimes_R P_0 & C_1 \otimes_R P_1 & C_1 \otimes_R P_2 \\ \downarrow & \downarrow & \downarrow \\ C_0 \otimes_R P_0 & C_0 \otimes_R P_1 & C_0 \otimes_R P_2 \end{array}$$

where the vertical arrows are the d_C differentials. Here, for each s, the vertical sequence $(E^0_{s,*}, d^0)$ is the chain complex $C_* \otimes_R P_s$. Since P_s is a projective module, it follows that $H_t(E^0_{s,*}, d^0) \cong H_t(C) \otimes_R P_s$. This naturally respects all of the differential maps, so the E^1-page is:

$$\begin{array}{ccccc}
H_2(C) \otimes_R P_0 & \longleftarrow & H_2(C) \otimes_R P_1 & \longleftarrow & H_2(C) \otimes_R P_2 \\
H_1(C) \otimes_R P_0 & \longleftarrow & H_1(C) \otimes_R P_1 & \longleftarrow & H_1(C) \otimes_R P_2 \\
H_0(C) \otimes_R P_0 & \longleftarrow & H_0(C) \otimes_R P_1 & \longleftarrow & H_0(C) \otimes_R P_2
\end{array}$$

Similarly to before, each horizontal sequence $(E^1_{*,t}, d^1)$ is the chain complex $H_t(C) \otimes_R P_*$. Thus, we have the E^2-page:

$$E^2_{s,t} = H_s(H_t(C) \otimes_R P_*) = \mathrm{Tor}^R_s(H_t(C), M),$$

where the second equality follows because P_* is a projective resolution of M. By the discussion in Miller's notes, we have a convergence $E^2_{s,t} \implies H_{s+t}(C_* \otimes_R P_*)$. But by homological algebra, $H_{s+t}(C_* \otimes_R P_*) \cong H_{s+t}(C_* \otimes_R M)$ since P_* is a projective resolution, which is what we wanted.

Now to recover the UCT short exact sequence from this generalized spectral sequence, recall that we have a short exact sequence:

$$0 \longrightarrow (\mathrm{coker}\ d^2)_{n-1} \longrightarrow H_n(C_* \otimes_R M) \longrightarrow (\ker d^2)_n \longrightarrow 0$$

Firstly, note that $(\ker d^2)_n = \ker\left(E^2_{1,n-1} \to E^2_{-1,n-1}\right) = \mathrm{Tor}^R_1(H_{n-1}(C_*), M)$. By a similar note, it follows that $(\mathrm{coker}\ d^2)_{n-1} = H_n(C_*) \otimes_R M$, so we get our desired UCT sequence.