Math 132 Problem Set 2

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Due: February 10, 2023

Problem 1. Using the preimage theorem, show that the Stiefel manifold from the previous problem set is a smooth manifold of dimension $2n-3$. Can you generalize your argument to the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$.

As before, we'll work with the full generality of the previous problem. Let $\mathbb{R}^{nk}$ be the space of all k-frames in $\mathbb{R}^n$, and consider the map $\phi : \mathbb{R}^{nk} \to \mathbb{R}^{\binom{k+1}{2}}$ be the map given by $\phi_q(v_1, \dots, v_k) = \langle v_q, v_q \rangle$ for $1 \le q \le k$ and when $q > k$, $\phi_q(v_1, \dots, v_k) = \langle v_{a(q)}, v_{b(q)} \rangle$ for some parametrization $1 \le a(q) < b(q) \le k$. Now by construction, the Stiefel manifold $S_{n,k}$ is exactly the preimage of the point $y = (1, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{\binom{k+1}{2}}$. So if this is a regular value, the preimage theorem would state that $S_{n,k}$ is a smooth manifold of dimension $nk - \binom{k+1}{2}$ as desired.

To show that this point is a regular value, we must show that for any $x \in \phi^{-1}(y)$ the tangent space map $d\phi_x : T_x\mathbb{R}^{nk} \to T_y\mathbb{R}^{\binom{k+1}{2}}$ is surjective. So let $(v_1, \dots, v_k) \in \phi^{-1}(y)$ be some orthonormal k-frame in $\mathbb{R}^n$. Then we have

$$\frac{\partial \phi_q}{\partial v_i} = \begin{cases} 2v_i & 1 \le q \le k, q = i, \\ v_{b(q)} & k < q, a(q) = i. \end{cases}$$

Note that is somewhat of an abuse of notation, since v_i is a k-vector in $\mathbb{R}^{nk}$, however v_i could be replaced by $v_{i,j}$ to denote the j-th component of v_i and the formula would work the same. Since these vectors are orthonormal, it follows that the rows of the $d\phi_{v_1,\dots,v_k}$ matrix are linearly independent and so the map $d\phi_{v_1,\dots,v_k}$ is surjective as desired.

Problem 2. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix, and define $f_A : \mathbb{R}^n \to \mathbb{R}$ by

$$f_A(v) = v^T \cdot A \cdot v = \sum a_{ij} v_i v_j$$

in which we are interpreting a vector $v \in \mathbb{R}^n$ as a column vector, with i^{th} coordinate v_i. Set

$$S_A = \{v \in \mathbb{R}^n : f_A(v) = 1\}.$$

Show that if $\det A \neq 0$, then S is a smooth manifold of dimension $(n-1)$. Describe the tangent space to S at a point v.

Calculating the derivative of this function:

$$f_A(v) = \sum_{0 \le i,j < n} a_{ij} v_i v_j \implies \frac{\partial f_A}{\partial v_k} = \sum_{0 \le i,j < n} \frac{\partial}{\partial v_k} a_{ij} v_i v_j = \sum_{0 \le j < n} (a_{kj} v_j + a_{jk} v_j)$$
$$= \sum_{0 \le j < n} 2a_{kj} v_j = 2v^T A$$

This means that our derivative map between tangent spaces can be expressed as:

$$\begin{aligned}(df_A)_v : T_v\mathbb{R}^n &\to T_{f_A(v)}\mathbb{R}, \\ x &\mapsto 2v^T A \cdot x.\end{aligned}$$

This is a submersion for any $v \in f_A^{-1}(1)$, since $2v^T A$ must be nonzero, so by the preimage theorem $S_A = f_A^{-1}(1)$ is a manifold of dimension $n-1$. The tangent space at a point $v \in S_A$ is kernel of $(df_A)_v$ and so consists of the vectors x such that $2v^T Ax = \langle 2v^T A, x\rangle = 0$. This is exactly the orthogonal complement of $2v^T A$ so we have

$$T_v S_A = (2v^T A)^\perp \subset T_v\mathbb{R}^n.$$

Problem 3. GP, Problem 18 of Chapter 1, Section 1

We will be making quite a lot of use of this.

a. An extremely useful function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Prove that f is smooth.

Let's first work with a modified f:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

It follows from the chain rule and power rule that on $\mathbb{R} - \{0\}$, the derivative $\partial^n f(x)/\partial x^n$ exists and takes the form $q(x)f(x)$ for some polynomial $q(x) \in \mathbb{Z}[x, x^{-1}]$. Since such polynomials are smooth everywhere except possibly 0, it follows that $f(x)$ is smooth everywhere except possibly at 0. To show that $f(x)$ is smooth at 0, we'll inductively show that $\lim_{x_0 \to 0} \left(\partial^n f(x)/\partial x^n\right)(x_0) = 0$.

First of all, for $f(x)$ itself has limit by basic calculus:

$$\begin{aligned}\lim_{x\to 0} f(x) = \lim_{x \to 0} e^{-1/x^2} &= \lim_{x \to -\infty} e^x = 0 \\ &\text{since } \lim_{x\to 0} \frac{-1}{x^2} = -\infty.\end{aligned}$$

Now more generally, for any n, the derivative $\partial^n f(x)/\partial x^n$ is bounded above (from the right) side by $e^{-1/x}$ and from below by e^{-1/x^2} so by the squeeze theorem, we get

$$\lim_{x_0 \to 0^+} \partial^n f(x)/\partial x^n(x_0) = 0.$$

We can do a similar thing on the left side by bounding it above by $e^{1/x}$ so the limit must be equal to 0 and so all the derivatives are continuous. Lastly, we check that this "limit completed" derivative is indeed the derivative, but this is easy to see since the n-th derivative at 0 is always zero by induction.

Now since the n-th derivative at zero is zero, we can replace the negative side of this function by 0 and it will still be smooth. Thus we get our smooth function:

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

b. Suppose that $a < b$ are real numbers. Show that $g(x) = f(x-a)f(b-x)$ is a smooth function, positive on (a,b) and zero elsewhere. It follows that

$$h(x) = \frac{\int_{-\infty}^{x} g(x)\,dx}{\int_{-\infty}^{\infty} g(x)\,dx} \text{ is smooth} \implies h(x) = \begin{cases} 0 & x < a, \\ 1 & x > b, \\ 0 < h(x) < 1 & x \in (a,b). \end{cases}$$

In fact h is monotone increasing on (a,b).

As a product of smooth functions, clearly $g(x) = f(x-a)f(b-x)$ is a smooth function. Furthermore, since $f(x-a) = 0$ iff $x > a$ and $f(b-x) = 0$ iff $x < b$, it follows that $g(x)$ is zero on $\mathbb{R} - (a,b)$ and positive on (a,b). There isn't really much else to show here, the rest follows as stated in the problem.

c. Now construct a smooth function on $\mathbb{R}^k$ that equals 1 on the ball of radius a, is zero outside the ball of radius b and is strictly between 0 and 1 at points x with $a < |x| < b$. Such a function is called a *bump* function, and will play a very important role in our later work.

Let $B_{a,b} : \mathbb{R}^k \to \mathbb{R}$ be the function defined as $B_{a,b}(x) = 1 - h(\|x\|)$, which clearly satisfies all the properties we want it to by the previous problem. The only thing we have to check is that it is a smooth function. Recall that $\|-\| : \mathbb{R}^n \to \mathbb{R}$ is smooth everywhere except for zero. However the function $1 - h(-)$ is constant in a neighborhood around 0, so the composition $1 - h(\|-\|)$ must be smooth at zero. This completes the proof.

Problem 4. Suppose that y is a regular value of $f : X \to Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$. Prove that there exists a neighborhood U of $y \in Y$ such that $f^{-1}(U)$ is a disjoint union $V_1 \sqcup \cdots \sqcup V_N$, where V_i is an open neighborhood of x_i, and f maps each V_i diffeomorphically onto U.

By the preimage theorem, $f^{-1}(y)$ is a submanifold of X of dimension 0. Suppose for the sake of contradiction that $f^{-1}(y)$ is an infinite set of points. Since X is compact, this means that $f^{-1}(y)$ has a limit point, which violates the manifold condition of being locally diffeomorphic to $\mathbb{R}^0$. Thus $f^{-1}(y)$ is a finite set, say composed of $\{x_1, \ldots, x_N\}$.

Now since y is a regular value, $df_x : T_xX \to T_yY$ is a submersion for any $x \in f^{-1}(X)$. Since T_xX and T_yY have the same dimensions, it must be an isomorphism. Thus by the inverse function theorem, we have local diffeomorphisms $\psi_i : \mathcal{V}_{x_i} \to \mathcal{U}_{x_i}$. We can shrink the sets $\mathcal{V}_{x_i}$ so that they are all disjoint. Now let $U = Y - f(X - \bigcup_i \mathcal{V}_{x_i})$. Since $X - \bigcup_i \mathcal{V}_{x_i}$ is compact, its image is compact and hence closed so U is open as desired. Letting $V_i = \mathcal{V}_{x_i} \cap f^{-1}(U)$ gives us the required properties.

Problem 5. Prove that the set of all 2×2 matrices of rank 1 is a three-dimensional submanifold of $\mathbb{R}^4 = M(2)$.

The rank of a 2×2 matrix can either be $0, 1$, or 2. It is only rank 0 if the matrix is the zero matrix, otherwise it is rank 1 if the determinant is 0. So consider the smooth function $\det : M(2) - \{0\} \to \mathbb{R}$. Since $\det(a,b,c,d) = ad - bc$, it's easy to check that $0 \in \mathbb{R}$ is a regular value of det, indeed every real number is a regular value since the derivative map is nonzero and hence surjective for any $(a,b,c,d) \in M(2) - \{0\}$. Thus $\det^{-1}(0)$, the set we are investigating, is a smooth manifold of dimension $4 - 1 = 3$.