Math 231a Problem Set 6

Lev Kruglyak

October 25, 2022

Problem 1. Let p and q be relatively prime positive integers. Define a space $L(p,q)$ as the quotient of S^3, the unit sphere in $\mathbb{C}^2$ by the action of the group μ_p of p-th roots of unity given by

$$\zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta^q z_2).$$

Impose on $L(p,q)$ the structure of a finite cell complex with one cell in each dimension between 0 and 3. The cell complex structure is just the filtration, but you should specify the characteristic maps as well. Then compute the homology of $L(p,q)$.

We'll begin by giving a certain cell structure to S^3 which is invariant under the action of μ_p. Observe that S^3 can be identified with pairs $(r_1 e^{2\pi i\theta_1}, r_2 e^{2\pi i\theta_2})$ with $r_1^2 + r_2^2 = 1$. Consider the following subsets of S^3:

$$\begin{aligned}
e_k^0 &= \{(e^{2\pi i\theta_1}, 0) : \theta_1 = k/p\}, \\
e_k^1 &= \{(e^{2\pi i\theta_1}, 0) : \theta_1 \in [k/p, (k+1)/p]\}, \\
e_k^2 &= \{(r_1 e^{2\pi i\theta_1}, r_2 e^{2\pi i\theta_2}) : \theta_2 = k/p\}, \\
e_k^3 &= \{(r_1 e^{2\pi i\theta_1}, r_2 e^{2\pi i\theta_2}) : \theta_2 \in [k/p, (k+1)/p]\}.
\end{aligned}$$

We claim that these subsets give us a cell structure on S^3 by setting $\mathrm{Sk}_n S^3 = \coprod_{k=0}^{p-1} e_k^n$. To prove this, we'll construct attachment maps $f_k^n : D^n \to \mathrm{Sk}_n S^3$ with $f_k^n(\partial D^n) \subset \mathrm{Sk}_{n-1} S^3$ and $f_k^n\big|_{\mathrm{Int}(D^n)}$ a homeomorphism.

For $n = 0$, the maps are quite simple, with f_k^0 taking D^0 to $(e^{2\pi ik/p}, 0)$. For $n = 1$, we can identify D^1 with $[0,1]$ so our map becomes $f_k^1(t) = (e^{2\pi i(k+t)/p}, 0)$. It's clear that $f_k^1(\partial I) \in \mathrm{Sk}_0 S^3$. For $n = 2$, using the standard parametrization of D^2 in polar coordinates, consider the map

$$f_k^2(r, \theta) = \left(re^{2\pi i\theta}, \sqrt{1-r^2}e^{2\pi ik/p}\right).$$

This works because $f_k^2(1, \theta) = (e^{2\pi i\theta}, 0) \in \mathrm{Sk}_1 S^3$. Finally for $n = 3$, consider the parameterization of D^3 by "suspension" coordinates, i.e. $SD^2 = D^2 \times I/\sim$, so (r, θ, t), where $\theta \in [0,1]$, $t \in [0,1]$. Then we have the map

$$f_k^3(r, \theta, t) = \left(re^{2\pi i\theta}, \sqrt{1-r^2}e^{2\pi i(k+t)/p}\right).$$

To check the boundary, we observe that $f_k^3(r, \theta, 0) = (re^{2pii\theta}, \sqrt{1-r^2}e^{2\pi ik/p})$ and on the lower hemisphere; $f_k^3(r, \theta, 1) = (re^{2pii\theta}, \sqrt{1-r^2}e^{2\pi i(k+1)/p})$.

Next, notice that this cell decomposition is invariant under the action of μ_p, which acts as

$$\zeta \cdot e_k^0 = e_{k+1}^0, \zeta \cdot e_k^1 = e_{k+1}^1, \zeta \cdot e_k^2 = e_{k+q}^2, \text{ and } \zeta \cdot e_k^3 = e_{k+q}^3,$$

where $\zeta = e^{2\pi i/p}$. Since p, q are relatively prime, the cell decomposition on S^3 induces a cell structure on $L(p,q)$, with $\mathrm{Sk}_n L(p,q) = e^n$, where e^n is the image of e_k^n under the action of μ_p. We thus have one cell in each dimension up to 3, so the cellular homology becomes

$$C_n(L(p,q)) = \begin{cases} \mathbb{Z} & n \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Next we compute the boundary maps $\partial_n : C_n(L(p,q)) \to C_{n-1}(L(p,q))$. Let $\widetilde{e^n}$ be the generator in $C_n(L(p,q))$. Obviously $\partial\widetilde{e^0} = 0$, and $\partial\widetilde{e^1} = 0$ since the endpoints of e^1 are the same. For e^2, the map $f^2\big|_{S^1} \to \mathrm{Sk}_1 L(p,q) = S^1$ has degree p, so $\partial\widetilde{e^2} = p\widetilde{e^1}$. Lastly, $\partial\widetilde{e^3} = 0$ because the bounding hemispheres of e^3 map to the same 2-cell. So the homology becomes:

$$\boxed{H_n(L(p,q)) = \begin{cases} \mathbb{Z} & n = 0, 3, \\ \mathbb{Z}/p & n = 1, \\ 0 & \text{otherwise.} \end{cases}}$$

Problem 2. Show that the Euler characteristic is a "cut-and-paste" invariant, in the following sense. Let X and Y be subcomplexes of the finite CW complex $X \cup Y$. Show that

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y).$$

Recall that for a finite CW complex A, the Euler characteristic is defined as

$$\chi(A) = \sum_{k=0} e_{A,k}(-1)^k$$

where $e_{A,k}$ is the number of k-cells in A. Then for X, Y finite subcomplexes of A, we have $e_{X \cup Y,k} = e_{X,k} + e_{Y,k} + e_{X \cap Y,k}$ so $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$.

Problem 3. A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show that there exist even maps of any given even degree.

Recall that the antipodal map $\alpha : S^n \to S^n$ given by $x \mapsto -x$ has degree $(-1)^{n+1}$. For any even map $f : S^n \to S^n$, we have $f \circ \alpha = f$ so $(\deg f)(\deg \alpha) = \deg f$. Thus $\deg f = (-1)^{n+1} \deg f$. So for even dimensional spheres, we have $\deg f = -\deg f$ and so $\deg f = 0$.

For odd dimensional spheres, let $\rho : S^n \to \mathbb{RP}^n$ be the standard two sheeted covering quotient map. Since the map f is even, it passes to the quotient so we get an $\widetilde{f} : \mathbb{RP}^n \to S^n$. This similarly induces a homology triangle:

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "\rho"] & S^n \\ \mathbb{RP}^n \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} H_n(S^n) \arrow[r, "f_*"] \arrow[d, "\rho_*"] & H_n(S^n) \\ H_n(\mathbb{RP}^n) \arrow[ur, "\widetilde{f}_*"'] & \end{tikzcd}$$

All the homology groups are $\mathbb{Z}$ with $\rho_*(x) = 2x$, so since $f_* = \widetilde{f}_* \circ \rho_*$, it follows that f is even degree. To explicitly construct a map of degree $2k$ for any k, consider S^{2n-1} as a subspace of $\mathbb{C}^n$. Let

$$f_k(z_1, \ldots, z_n) = \frac{(z_1^2, \ldots, z_n^2)}{\|(z_1^2, \ldots, z_n^2)\|}.$$

This is clearly an even degree $2k$ map.

Problem 4. The goal of this problem is to prove the generalized Jordan curve theorem:

(i) For an embedding $h : D^k \hookrightarrow S^n$, we have $\widetilde{H}_*(S^n \setminus h(D^k)) \cong 0$.

(ii) For an embedding $h : S^k \hookrightarrow S^n$, we have $\widetilde{H}_*(S^n \setminus h(S^k)) \cong \mathbb{Z}[n-k-1]$.

We'll begin by proving (a) by induction on k. Let's replace D^k by the cube I^k.

(a) Prove (i) above in the case $k = 0$.

(b) Suppose that we are given an embedding $h : I^k \hookrightarrow S^n$ and assume that statement (i) is true for $k-1$. Using the Mayer-Vietoris sequence, prove that if there exists a nonzero class $\alpha \in \widetilde{H}_i(S^n \setminus h(I^k))$ then it maps to a nonzero element in $\widetilde{H}_i(S^n \setminus h([0, \frac{1}{2}] \times I^{k-1}))$ or $\widetilde{H}_i(S^n \setminus h([\frac{1}{2}, 1] \times I^{k-1}))$.

(c) Conclude by iterating (b) that there is a sequence of closed intervals $I \supset I_1 \supset I_2 \supset \cdots$ where I_ℓ has length $2^{-\ell}$ such that the image of α in $\widetilde{H}_i(S^n \setminus h(I_\ell \times I^{k-1}))$ is nonzero for all $\ell \geq 1$.

(d) Let $\{x\} = \bigcap_{\ell=1}^{\infty} I_\ell$. Prove that the image of α in $\widetilde{H}_i(S^n \setminus h(\{x\} \times I^{k-1}))$ is nonzero. Conclude that (i) holds by induction on k.

(e) Using (a) and the Mayer Vietoris sequence, prove (ii) by induction on k.

(a) In the case $k = 0$, we must show that $\widetilde{H}_*(S^n \setminus \{p\}) \cong 0$ for any $p \in S^n$. However there is a stereographic projection homeomorphism $\sigma : S^n \setminus \{p\} \to \mathbb{R}^n$ so $\widetilde{H}_*(S^n \setminus \{p\}) \cong \widetilde{H}_*(\mathbb{R}^n) \cong 0$.

(b) Let $H = S^n \setminus h(I^k)$, $H^+ = S^n \setminus h([0, 1/2] \times I^{k-1})$, and $H^- = S^n \setminus h([1/2, 1] \times I^{k-1})$. Note that the interiors of H^+ and H^- form a cover of S^n. The Mayer-Vietoris sequence gives us an exact sequence

$$\cdots \longrightarrow \widetilde{H_{i+1}}(H) \xrightarrow{\partial_*} \widetilde{H}_i(H^+ \cap H^-) \longrightarrow \widetilde{H}_i(H^+) \oplus \widetilde{H}_i(H^-) \longrightarrow \widetilde{H}_i(H) \longrightarrow \cdots$$

However $H^+ \cap H^- = S^n \setminus h(1/2 \times I^{k-1})$, so by the inductive assumption $\widetilde{H}_i(H^+ \cap H^-) = 0$. This means that the map $\widetilde{H}_i(H^+) \oplus \widetilde{H}_i(H^-) \to \widetilde{H}_i(H)$ is an isomorphism, so any nonzero class in the latter must have come from a nonzero class in the either of the formers.

(c) We construct this sequence inductively. Let I_1 be the interval $[0, 1/2]$ if α came from a nonzero element of $\widetilde{H}_i(H^+)$ and $[1/2, 1]$ otherwise. Then we apply (b) again to I_1 to get the next interval I_2, and keep doing this repeatedly.

(d) By the previous part we have a diagram

$$\begin{tikzcd}
 & & & \widetilde{H}_i(H_x) & \\
\widetilde{H}_i(H) \arrow[r] & \widetilde{H}_i(H_1) \arrow[r] \arrow[urr] & \widetilde{H}_i(H_2) \arrow[r] \arrow[ur] & \widetilde{H}_i(H_3) \arrow[r] \arrow[u] & \cdots \arrow[ul]
\end{tikzcd}$$

Since α maps to nonzero elements in all the terms in this sequence, it follows that it maps to a nonzero cycle in $H_x = S_n \setminus h(x \times I^{k-1})$. But this is trivial by the inductive hypothesis so $\widetilde{H}_i(H)$ is trivial and the induction is complete.

(e) Suppose the claim is true for $k-1$. Letting D^k_+ and D^k_- be the upper and lower hemispheres of S^k, Mayer-Vietoris gives us the sequence

$$\cdots \longrightarrow \widetilde{H_{i+1}}(S^n \setminus h(S^k)) \xrightarrow{\partial_*} \mathbb{Z}[n-k] \longrightarrow 0 \longrightarrow \widetilde{H}_i(S^n \setminus h(S^k)) \longrightarrow \cdots$$

Thus since the kernel is one dimensional, we get $\widetilde{H}_*(S^n \setminus h(S^k)) \cong \mathbb{Z}[n-k-1]$.

Problem 5. Prove that if U is an open set in $\mathbb{R}^n$ and $h : U \to \mathbb{R}^n$ is a continuous injection, then the image $h(U)$ is an open set in $\mathbb{R}^n$ and h is a homeomorphism onto $h(U)$.

It suffices to prove that for any open ball $B \subset U$, $h(B)$ is open in $\mathbb{R}^n$ and $h\big|_B$ is a homeomorphism. First, we replace $\mathbb{R}^n$ by its one point compactification, which is homeomorphic to S^n by stereographic projection. By the previous problem, $S^n \setminus h(\partial B)$ is an open set with exactly two path components X_1, X_2, which are also connected. $h(B)$ is clearly connected and $h(S^n \setminus h(\overline{B}))$ is connected by the previous problem as well, so we have $h(B) \cup (S^n \setminus h(\overline{B})) = X_1 \cup X_2$. Thus $h(B) = X_1$ without loss of generality and so $h(B)$ is open. Since $h\big|_B$ is a bijective open map, it is a homeomorphism so we are done.