Math 137 Problem Set 9

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I collaborated with Ignasi Segura Vicente on this problem set.

Throughout, K is assumed to be an algebraically closed field.

Problem 1 (bonus)**.** Let $n \geq 2$ and let $F_d \cong K^{\binom{n+d}{n}}$ be the vector space of polynomials $f \in K[X_1, \ldots, X_n]$ of degree $\leq d$.

a) If $d > 2n-3$, show that there is a nonempty Zariski open subset $U \subseteq F_d$ such that the set $\mathcal{V}(f) \subseteq K^n$ doesn't contain a straight line for any $f \in U$.

b) If $d < 2n-3$, show that for every $f \in F_d$, if $\mathcal{V}(f) \subseteq K^n$ contains a straight line, then it contains infinitely many.

c) (too difficult for a bonus problem and totally unfair) If $d \leq 2n-3$, show that there is a nonempty Zariski open subset $U \subseteq F_d$ such that the set $\mathcal{V}(f)$ contains at least one straight line for all $f \in U$.

Hint: Look at the proof of Theorem 13.5.1. What is the dimension of "the space of straight lines" in K^n? What is the dimension of the space of $f \in F_d$ such that $\mathcal{V}(f)$ contains a particular straight line?

Problem 2. Show that a polynomial $f \in K[X_1, \ldots, X_n]$ vanishes on the entire line spanned by a nonzero vector $x \in K^n$ if and only if all of its homogeneous parts f_d vanish at x.

Let $f \in K[X_1, \ldots, X_n]$ be an arbitrary polynomial with homogenization $f = \sum_{d\geq 0} f_d$. It's trivial to see that if all f_d vanish at λx for all $\lambda \in K$, so does f, since it is a sum of the f_d. Next suppose in the forward direction that f vanishes at all λx. Let D be the maximal integer such that $f_d \neq 0$, so $f = f_0 + \lambda f_1 + \cdots \lambda$. Note that for any nonzero vector $x \in K^n$, we have $f(\lambda x) = \sum_{d\geq 0}^{D} \lambda^d f_d(x)$, since each f_d is homogeneous of degree d. Say we choose some real numbers $\lambda_1, \ldots, \lambda_D$. Then we have the matrix relation

$$\begin{bmatrix} \lambda_1^0 & \lambda_1^1 & \cdots & \lambda_1^D \\ \lambda_2^0 & \lambda_2^1 & \cdots & \lambda_2^D \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_D^0 & \lambda_D^1 & \cdots & \lambda_D^D \end{bmatrix} \begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_D(x) \end{bmatrix} = 0.$$

Matrices of this form are called *Vandermonde matrices*, and we can choose for example $\lambda_1 = 1, \lambda_2 = 2, \ldots$ to get an invertible matrix in characteristic zero, and some other distinct set of values if K is an infinite field of characteristic p. This matrix being invertible means that $f_d(x) = 0$ for all d, so we are done.

Problem 3. Let $A = \mathcal{V}(I)$ for an ideal I of $K[X_1, \ldots, X_n]$. Let $S \subseteq K[X_0, \ldots, X_n]$ be the set of homogenizations of elements of I at X_0. Show that $\mathcal{V}_{\mathbb{P}^n_K}(S)$ is the Zariski closure of the image of A under the 0-th standard affine chart map φ_0.

First we'll show that $\varphi_0(A) \subset \mathcal{V}_{\mathbb{P}^n_K}(S)$, this will imply that $\overline{\varphi_0(A)} \subset \mathcal{V}_{\mathbb{P}^n_K}(S)$ since the later is an algebraic set. Let $(x_1, \dots, x_n) \in A$, so $\varphi_0(x_1, \dots, x_n) = [1 : x_1 : \cdots : x_n]$. Then $[1 : x_1 : \cdots : x_n] \in \mathcal{V}_{\mathbb{P}^n_K}(S)$ because

$$^h f(1 : x_1 : \cdots : x_n) = 1^{\deg f} f\left(\frac{x_1}{1}, \cdots, \frac{x_n}{1}\right) = 0,$$

where $^h f$ denotes the homogenization of f. Next, we'll show that

$$\mathcal{V}_{\mathbb{P}^n_K}(S) \subset \bigcap_{\varphi_0(A) \subset H \text{ algebraic}} H = \overline{\varphi_0(A)}.$$

This is equivalent to checking that every homogeneous polynomial $g \in K[x_0, \dots, x_n]$ such that $g(1, x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ vanishes everywhere in A must also satisfy $g(0, x_1, \dots, x_n)$ vanishing everywhere on $B = \bigcap_{f \in I} \mathcal{V}_{K^n}(^h f(0, x_1, \dots, x_n)) \subset K^n$. By Problem 2 however, since $g(1, x_1, \dots, x_n)$ vanishes everywhere on A, the homogeneous parts also do. This implies that $g(0, x_1, \dots, x_n)^k \in I$ since $\mathcal{I}(A) = \sqrt{I}$. Thus by definition $g(0, x_1, \dots, x_n)^k$ vanishes everywhere on B and so $g(0, x_1, \dots, x_n)$ vanishes everywhere in B as well, completing the proof.

Problem 4. Any invertible linear map $g : K^{n+1} \to K^{n+1}$ induces a map $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ sending the line spanned by $x \in K^{n+1}$ to the line spanned by $g(x) \in K^{n+1}$. Maps $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ of this form are called *projective transformations*.

a) Consider the projective line $\mathbb{P}^1_K = K \sqcup \{\infty\}$. Let P, Q, R be three distinct points in $\mathbb{P}^1_K$. Show that there is a projective transformation $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ sending P to 0, Q to 1, and R to ∞.

b) We say that points $P_1, \dots, P_m$ in $\mathbb{P}^n_K$ are *in general linear position* if no $d+2$ of them lie on a d-dimensional linear subspace for any $0 \le d \le \min(m-2, n-1)$.

Let the points $P_1, \dots, P_{n+2} \in \mathbb{P}^n_K$ be in general linear position and let $Q_1, \dots, Q_{n+2} \in \mathbb{P}^n_K$ be in general linear position. Show that there is a unique projective transformation $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ sending P_i to Q_i for $i = 1, \dots, n+2$.

(a) Let p, q, r be nonzero representatives of the equivalence classes $P, Q, R \in \mathbb{P}^1_K$ respectively, i.e. nonzero points on the lines P, Q, R. The lines are distinct, so p, q, r are pairwise linearly independent. Let's choose p, r. We know that these form a basis for K^2. Then we can write $q = ap + br$ for some nonzero $a, b \in K$. Now consider the linear transformation $L : K^2 \to K^2$ given by sending p to $[1/a : 0]$ and r to $[0 : 1/b]$ for some nonzero $k \in K$. Then in the induced linear transformation $\widetilde{L} : \mathbb{P}^1_K \to \mathbb{P}^1_K$ sends P to a line of slope $0/(1/a) = 0$, Q to a line of slope $(b/b)/(a/a) = 1$, and R to a line of slope $(1/b)/0 = \infty$.

(b) As in (a), let's chose representatives $p_1, \dots, p_{n+2} \in K^{n+1}$ for the lines $P_1, \dots, P_{n+2}$ and $q_1, \dots, q_{n+2}$ for $Q_1, \dots, Q_{n+2}$. Since these lines are in general linear position, $(p_1, \dots, p_{n+1})$ and $(q_1, \dots, q_{n+1})$ are bases for K^{n+1}. Then write $p_{n+2} = a_1 p_1 + \cdots + a_{n+1} p_{n+1}$ and $q_{n+2} = b_1 p_1 + \cdots + b_{n+1} p_{n+1}$. Then the linear map $L : K^{n+1} \to K^{n+1}$ which sends p_i to $q_i b_i / a_i$.

We claim that the induced linear map $\widetilde{L} : \mathbb{P}^n_K \to \mathbb{P}^n_K$ sends P_i to Q_i. Note that for all $i \le n+1$, $L(p_i) = (b_i/a_i) q_i$ so $\widetilde{L}(P_i) = Q_i$. For $i = n+2$, we have $L(p_i) = (b_1/a_1) a_1 q_1 + \cdots + (b_{n+1}/a_{n+1}) a_{n+1} q_{n+1} = b_1 q_1 + \cdots + b_{n+1} q_{n+1} = q_{n+2}$ so $\widetilde{L}(P_i) = Q_i$ and we are done.

Problem 5 (Pappus's hexagon theorem)**.** Let $g \neq h$ be lines in $\mathbb{P}^2_K$ that intersect in the point P. Let A, B, C be points on g and A', B', C' be points on h (all seven points P, A, B, C, A', B', C' distinct). Let Z be the point of intersection of the lines AB' and $A'B$. Let Y be the point of intersection of the lines AC' and $A'C$. Let X be the point of intersection of the lines BC' and $B'C$. Show that X, Y, Z are colinear. *(Hint: Apply a projective transformation to for example make* $P = [0 : 0 : 1]$, $A = [1 : 0 : 0]$, $B = [1 : 0 : 1]$, $C = [r : 0 : 1]$, $A' = [0 : 1 : 1]$, $B' = [0 : 1 : 0]$, $C' = [0 : s : 1]$. *Then compute* X, Y, Z.*)*

Since A, B, A', and B' are in general linear position, by the previous problem there is a projective transformation L which maps A to $[0 : 0 : 1]$, B to $[1 : 0 : 1]$, A' to $[0 : 1 : 1]$, and B' to $[0 : 1 : 0]$ since these are also in general linear position. This projection maps P to $[0 : 0 : 1]$ because it is the intersection of the lines AB and $A'B'$. Similar arguments show that C and C' must map to $[r : 0 : 1]$ and $[0 : s : 1]$ respectively. Now we can calculate that the intersection of $f(A)f(B')$ and $f(A')f(B')$ is $[-1 : 1 : 0]$, the intersection of $f(A')f(C)$ and $f(A)f(C')$ is $[r - rs : s : 1]$, and the intersection of $f(B)f(C')$ and $f(B')f(C)$ is $[r : s - rs : 1]$. It's easy to check that these points of intersection are colinear in the image of the projective transformation, so they must be colinear in the preimage, i.e. X, Y, Z are colinear.