Problem Set # 4

Math230a: Differential Geometry

Due: October 2

1. Let X be a smooth manifold and $f\colon Y \hookrightarrow X$ an injective immersion. Suppose Z is a smooth manifold, $g\colon Z \to X$ is a smooth map, and assume that g factors through f: there exists a map (of sets) $h\colon Z \to Y$ such that $g = f \circ h$. Prove that h is smooth iff h is continuous.

2. Let $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ denote the division algebra of quaternions. (Recall $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ and real numbers commute with i, j, k.) Let $G \subset \mathbb{H}$ denote the subset of quaternions $g = a + bi + cj + dk$ such that $a^2 + b^2 + c^2 + d^2 = 1$. Topologize $\mathbb{H}$ using the topology of real numbers and give G the induced topology.

   (a) Prove that G is a Lie group.

   (b) If $H \subset G$ is a finite subgroup, show that the quotient map $p\colon G \to G/H$ is a covering map.

   (c) Identify the Lie algebra $\mathfrak{g}$ with the set of imaginary quaternions $\operatorname{Im}\mathbb{H} = \{q = bi + cj + dk : b, c, d \in \mathbb{R}\}$. Identify the map
   $$\begin{aligned} \varphi\colon G &\longrightarrow \operatorname{Aut}\bigl(\operatorname{Im}\mathbb{H}\bigr) \\ g &\longmapsto (q \mapsto gqg^{-1}) \end{aligned}$$
   with the adjoint representation of G. Show that φ maps onto the group SO_3 of rotations in $\mathbb{R}^3$ and the kernel of φ is cyclic of order two.

   (d) Construct an isomorphism of G with the Lie group SU_2 of 2×2 unitary matrices of unit determinant.

   (e) A 3×3 real matrix always has a fixed line, and if the matrix is orthogonal there is a fixed line that is *pointwise* fixed: the eigenvalue is 1. Show that this is so. Except for the identity matrix I, this line is unique. Show that the map $f\colon \mathrm{SO}_3 \setminus \{I\} \to \mathbb{RP}^2$ so defined is a surjective submersion. What is the inverse image of a point? Is f a fiber bundle?

   (f) Do the conjugation classes of SO_3 form a foliation? Identify the diffeomorphism types of the conjugacy classes.

   (g) If $\overline{H} \subset \mathrm{SO}_3$ is a finite subgroup, then $H \subset \varphi^{-1}(\overline{H})$ is a finite subgroup of G of twice the order of $\overline{H}$. Let $P \subset \mathbb{R}^3$ be a finite set of n vectors whose "tips" lie on a regular polygon in $\mathbb{R}^2 \subset \mathbb{R}^3$. What is the finite subgroup of rotations which preserve P? What is its lift to G?

   (h) More interesting is to let I be the set of 12 vectors whose tips lie at the vertices of a regular icosahedron. In this case the quotient space G/H is called the *Poincaré sphere*. Can you compute the order of H?

3. Let $\omega \in \Omega^1_{\mathbb{R}}$ be a 1-form on the real line, and suppose G is a Lie group.

   (a) Prove that locally about any $t \in \mathbb{R}$ there exists a map into G such that the pullback of the Maurer-Cartan form equals ω. Write the first order ordinary differential equation this map satisfies.

   (b) Do global solutions always exist?

   (c) Construct an integrable distribution on $\mathbb{R} \times G$ that is transverse to the "vertical" fibers $\{t\} \times G$ and such that leaves of the resulting foliation of $\mathbb{R} \times G$ do *not* project surjectively onto $\mathbb{R}$.

4. Let G be a Lie group with Maurer-Cartan form θ, and suppose P is a right G-torsor.

   (a) Compute the pullback of θ under multiplication $m\colon G \times G \to G$ and under inversion $i\colon G \to G$.

   (b) Use your formulas to carry out the computation (7.14) in the notes for an arbitrary Lie group G.

   (c) Suppose $\xi \in \mathfrak{g}$ is a parallel (left-invariant) vector field. Compute $(R_g)_*\xi$. Is it parallel?

   (d) Let θ_P be the transport of the Maurer-Cartan form to a $\mathfrak{g}$-valued 1-form on P. Compute $R_g^*(\theta_P)$, $g \in G$. Compute $\varphi^*(\theta_P)$ for $\varphi \in \operatorname{Aut} P$.

   (e) An element $\xi \in \mathfrak{g}$ determines a vector field $\hat{\xi} \in \mathfrak{X}(P)$. Review the definition. Compute $(R_g)_*(\hat{\xi})$. Compute $\varphi_*(\hat{\xi})$.

5. Let G be a Lie group. Recall that a left G-torsor is a smooth manifold X equipped with a simply transitive left G action. Now we relax to a *homogeneous manifold* X, which is equipped with a transitive left G-action that is not necessarily simple. For $x \in X$ the stabilizer subgroup $H \subset G$ is a closed Lie subgroup. It is true—and you may want to prove—that G/H has a canonical smooth manifold structure such that the quotient map $G \to G/H$ is a principal H-bundle, i.e., a fiber bundle whose fibers are H-torsors. Accepting this (if you don't prove it), construct a diffeomorphisms $G/H \to X$. The diffeomorphism depends on the basepoint x. If you choose a different basepoint x' can you still construct a diffeomorphism $G/H \to X$, now one that maps the basepoint of G/H (the coset H) to x'? Is it canonical? Explore the fate of parallel vector fields and the Maurer-Cartan form, both on G/H and on X. Are there canonical vector fields? Canonical 1-forms?

6. The symplectic group Sp_1 consists of $n \times n$ quaternionic matrices P such that $P\,{}^t\overline{P}$ is the identity matrix.

   (a) Identify Sp_1 with the group G in the previous problem.

   (b) Prove that Sp_n is a compact Lie group.

   (c) Describe the Lie algebra $\mathfrak{sp}_n$ as a sub Lie algebra of $\mathfrak{gl}_n\mathbb{R}$.

   (d) Compute $\dim \mathrm{Sp}_n$.

   (e) Can you recognize the homogeneous manifolds $\mathrm{Sp}_1 / \mathrm{U}_1$ and $\mathrm{Sp}_2 / \mathrm{Sp}_1$?