Lecture 7: Maps into a Lie group

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In this lecture we study solutions $f\colon X \to G$ of a first order partial differential equation (PDE) for a map f from a smooth manifold X to a Lie group G. The PDE (or system of PDEs) is specified by a Lie algebra-valued 1-form $\omega \in \Omega^1_X(\mathfrak{g})$; the PDE is the equation $\omega = f^*(\theta)$, where $\theta \in \Omega^1_G(\mathfrak{g})$ is the Maurer-Cartan form. There is an integrability condition—a consequence of "mixed partials commute"—which is the Maurer-Cartan equation for ω. Certainly this is a necessary condition for a solution f, since the Maurer-Cartan equation holds for θ. The existence and uniqueness result is Theorem 7.8.

We give a first application of this theorem internal to the theory of Lie groups. Namely, we prove in Theorem 7.27 that a homomorphism of Lie algebras of Lie groups integrates to a homomorphism of Lie groups, assuming the domain is simply connected. More geometric applications will follow shortly, but many replace the codomain Lie group by a torsor, so in the second part of this lecture we resume the study of torsors that we touched on in Lecture 1.

Main theorem

(7.1) *The differential.* Let G be a Lie group and let $\theta \in \Omega^1_G(\mathfrak{g})$ be its parallel (left-invariant) Maurer-Cartan form (Definition 4.60). Recall that θ satisfies the Maurer-Cartan equation

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0. \tag{7.2}$$

Now suppose X is a smooth manifold and $f\colon X \to G$ is a smooth map.

Lemma 7.3. *The differential df may be identified with $f^*(\theta) \in \Omega^1_X(\mathfrak{g})$.*

Proof. The global parallelism on G is implemented by the Maurer-Cartan form, so the composition identifies the differential

$$T_xX \xrightarrow{df_x} T_{f(x)}G \xrightarrow{\theta_{f(x)}} \mathfrak{g}, \qquad x \in X, \tag{7.4}$$

Differential Geometry (Math230a, Fall '24), Dan Freed, September 25, 2024 as a $\mathfrak{g}$-valued 1-form at x, which we claim is $f^*(\theta)_x$. Namely, if $\xi_x \in T_xX$, then

$$f^*(\theta)_x(\xi_x) = \theta_{f(x)}\bigl(df_x(\xi_x)\bigr). \tag{7.5}$$ $\square$

The Maurer-Cartan equation (7.2) implies that $\omega = f^*(\theta)$ satisfies

$$0 = f^*\left(d\theta + \frac{1}{2}[\theta\wedge\theta]\right) = d\omega + \frac{1}{2}[\omega\wedge\omega]. \tag{7.6}$$

Equation (7.6) is a necessary condition on ω for a solution f to the equation $\omega = f^*(\theta)$ to exist.

(7.7) *Existence and uniqueness.* The main theorem tells that we can solve for f with specified differential ω as long as ω satisfies the integrability condition (7.6).

Theorem 7.8. *Let X be a smooth manifold, let G be a Lie group, and let $\theta \in \Omega^1_G(\mathfrak{g})$ be the Maurer-Cartan form on G.*

(1) *Suppose $f_1, f_2\colon X \to G$ are smooth maps, $f_1^*(\theta) = f_2^*(\theta)$, and X is connected. Then there exists $g \in G$ such that*

$$f_2 = L_g \circ f_1, \tag{7.9}$$

where $L_g\colon G \to G$ is left multiplication by g.

(2) *Suppose $\omega \in \Omega^1_X(\mathfrak{g})$ satisfies*

$$d\omega + \frac{1}{2}[\omega\wedge\omega] = 0. \tag{7.10}$$

Then if X is simply connected, there exists $f\colon X \to G$ such that

$$f^*(\theta) = \omega. \tag{7.11}$$

The uniqueness (1) holds for X connected, whereas the existence (2) is for X simply connected. Both hold locally, i.e., on "small" simply connected open subsets of X (that can play the role of 'X' in the theorem). We abbreviate (7.9) to the equation $f_2 = gf_1$.

(7.12) *A vector group.* The simplest case is the Lie group $G = \mathbb{R}$ under addition; the Lie algebra is $\mathfrak{g} = \mathbb{R}$ with zero Lie bracket. The Maurer-Cartan form on $\mathbb{R}_x$ is the parallel 1-form dx. If $f\colon X \to \mathbb{R}$ is a smooth function, then $f^*(dx) = df \in \Omega^1_X$. The uniqueness (1) asserts that if $df_1 = df_2$, then $f_2 = f_1 + c$ for some $c \in \mathbb{R}$, which follows since $d(f_2 - f_1) = 0$, hence is locally constant, hence by connectivity of X is constant. The proof of (1) below is a generalization of this argument. The existence (2) reduces to the claim that if $\omega \in \Omega^1_X$ is a closed 1-form ($d\omega = 0$), and if X is simply connected, then ω is exact. This is a familiar local statement: the Poincaré lemma. The global statement requires more argument, which we give more generally in the proof below.

Use a basis to generalize to $G = V$ for V a finite dimensional real vector space.

(7.13) *Computations with the Maurer-Cartan form.* Recall from (4.77) that for a matrix group G we have $\theta = g^{-1}dg$, where g is the matrix-valued function on G that identifies G as a matrix group. For an arbitrary Lie group we may compute as if we are in a matrix group and then rewrite the result without that assumption. For example, to prove the uniqueness statement Theorem 7.8(1) set $F = f_2 f_1^{-1}\colon X \to G$ and compute

$$\begin{aligned} F^*(\theta) = F^{-1}dF &= (f_1 f_2^{-1})\, d(f_2 f_1^{-1}) \\ &= (f_1 f_2^{-1})(df_2 f_1^{-1} \;-\; f_2 f_1^{-1} df_1 f_1^{-1}) \\ &= f_1 f_2^*(\theta) f_1^{-1} \;-\; f_1 f_1^*(\theta) f_1^{-1} \\ &= \operatorname{Ad}_{f_1}\bigl(f_2^*(\theta) - f_1^*(\theta)\bigr). \end{aligned} \tag{7.14}$$

The equation takes place in $\Omega^1_X(\mathfrak{g})$. In the last expression, the algebraic operator Ad_{f_1} operates pointwise on $\bigl(f_2^*(\theta) - f_1^*(\theta)\bigr) \in \Omega^1_X(\mathfrak{g})$. This last expression makes sense on any Lie group G; it needn't be a matrix group. We claim that the equality of the first and last expressions in (7.14) holds for any Lie group. The manipulations that lead to this equality can be justified in a few ways. If G embeds in a matrix group $\widetilde{G}$, then we interpret (7.14) in the ambient group $\widetilde{G}$. This works for all compact Lie groups, though the proof that G embeds in a matrix group is not easy (Peter-Weyl theorem). A universally applicable argument breaks down the computation into the pullback of θ under multiplication and inversion. Namely, for

$$\begin{aligned} m\colon G \times G &\longrightarrow G \\ i\colon G &\longrightarrow G \end{aligned} \tag{7.15}$$

we have

$$\begin{aligned} (m^*\theta)_{(g_1,g_2)} &= \operatorname{Ad}_{g_2^{-1}} \pi_1^*\theta + \pi_2^*\theta \\ (i^*\theta)_g &= -\operatorname{Ad}_g \theta \end{aligned} \tag{7.16}$$

where $\pi_i\colon G\times G \to G$, $i = 1, 2$, are the projections onto the factors. I leave the verification of (7.16) and its application to (7.14) to a homework problem.

(7.17) *The proof.* We return to the main theorem.

Proof of Theorem 7.8. For the uniqueness statement (1) set $F = f_2 f_1^{-1}\colon X \to G$. Then $F^*(\theta) = 0$ by (7.14) and the hypothesis $f_1^*(\theta) = f_2^*(\theta)$. It follows from Lemma 7.3 that $dF = 0$, hence F is locally constant. Since X is connected, F is constant.

For the existence statement (2), we construct the function f, identified with its graph[1] $\Gamma(f) \subset X \times G$, as the maximal integral manifold of a distribution $E \subset T(X \times G)$ that encodes the first order differential equation (7.11). For $(x, g) \in X \times G$ identify

$$T_{(x,g)}(X \times G) \cong T_xX \oplus T_gG \cong T_xX \oplus \mathfrak{g}. \tag{7.18}$$

[1]Recall that a function $X \to G$ is a relation on $X \times G$ that satisfies an extra condition. In other words, a function *is* its graph.

Now define the distribution $E\subset T(X\times G)$ by

$$E_{(x,g)}=\Gamma(\omega_x),\qquad (x,g)\in X\times G, \tag{7.19}$$

where $\Gamma(\omega_x)\subset T_xX\oplus\mathfrak{g}$ is the graph of the linear function ω_x. Notice that E is invariant under left multiplication in G: the right hand side of (7.19) does not depend on $g\in G$, and the last identification in (7.18) is left-invariant. Alternatively, define $\Theta\in\Omega^1_{X\times G}(\mathfrak{g})$ as

$$\Theta=\pi_1^*\omega-\pi_2^*\theta, \tag{7.20}$$

where π_1,π_2 are the projections

$$\begin{tikzcd} X\times G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G \\ X \end{tikzcd} \tag{7.21}$$

Then E is the kernel of Θ. Note

$$\begin{aligned} L_g^*\Theta&=L_g^*\pi_1^*\omega-L_g^*\pi_2^*\theta\\ &=\pi_1^*\omega-\pi_2^*L_g^*\theta\\ &=\pi_1^*\omega-\pi_2^*\theta\\ &=\Theta \end{aligned} \tag{7.22}$$

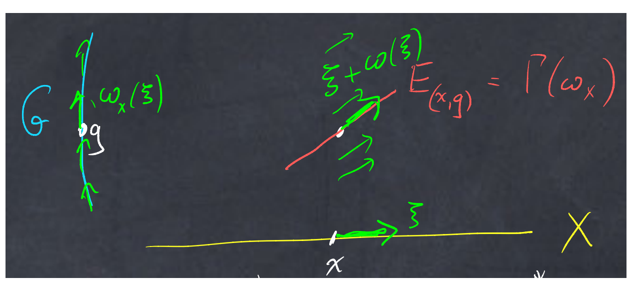

FIGURE 25. A vector field $\xi+\omega(\xi)$ in the distribution E

We claim that E is integrable. A vector field $\xi\in\mathfrak{X}(X)$ gives rise to a vector field $\xi+\omega(\xi)$ on $X\times G$ that belongs to the distribution E, as in Figure 25. This vector field is invariant under left multiplication on G. If $\xi,\eta\in\mathfrak{X}(X)$ we claim that on $X\times G$ we have

$$[\xi,\omega(\eta)]=\xi\omega(\eta). \tag{7.23}$$

The left hand side is the Lie bracket on $X\times G$ of the horizontal vector field ξ and the vertical vector field $\omega(\eta)$. The right hand side is the vertical vector field whose value on $\{x\}\times G$ is the parallel vector field on G with value the directional derivative of the function $\omega(\eta)\colon X \to \mathfrak{g}$ in the direction ξ_x. To prove (7.23), let φ_t be a local flow on X generated by ξ, and as well the induced flow on $X \times G$ that is the identity on the G factor. Then

$$[\xi, \omega(\eta)] = \frac{d}{dt}\bigg|_{t=0} (\varphi_{-t})_*\omega(\eta). \tag{7.24}$$

Since the flow does not move the points of G, the right hand side of (7.24) reduces to the definition of the directional derivative, which proves (7.23). Now compute

$$\begin{aligned} [\xi + \omega(\xi), \eta + \omega(\eta)] &= [\xi, \eta] + [\xi, \omega(\eta)] - [\eta, \omega(\xi)] + [\omega(\xi), \omega(\eta)] \\ &= [\xi, \eta] + \xi\omega(\eta) - \eta\omega(\xi) + [\omega(\xi), \omega(\eta)] \\ &= [\xi, \eta] + \omega\big([\xi, \eta]\big) + d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] \\ &= [\xi, \eta] + \omega\big([\xi, \eta]\big) + d\omega(\xi, \eta) + \frac{1}{2}[\omega \wedge \omega](\xi, \eta) \\ &= [\xi, \eta] + \omega\big([\xi, \eta]\big). \end{aligned} \tag{7.25}$$

In the second equation we apply (7.23), in the third we apply (4.48), in the penultimate we apply (4.49), and in the last we apply the integrability condition (7.10). This proves that sections of E are closed under Lie bracket, so E is integrable.

An integral manifold Γ_0 of E is locally the graph of the function $f\colon U \to G$ for an open set $U \subset X$, since the differential of the projection $\pi\colon X \times G \to X$ is an isomorphism restricted to E. More precisely, cutting down Γ_0 if necessary we have that $\pi_1\big|_{\Gamma_0}\colon \Gamma_0 \to X$ is invertible with inverse $s\colon U \to \Gamma_0 \hookrightarrow X \times G$ a local section of π_1. Then $f = \pi_2 \circ s$. Compute

$$f^*(\theta) = s^*\pi_2^*\theta = s^*(-\Theta + \pi_1^*\omega) = s^*\pi_1^*(\omega) = \omega \tag{7.26}$$

since the restriction of Θ to $\Gamma_0 \subset X \times G$ vanishes. (The tangent distribution E is the kernel of Θ.) In other words, integral manifolds of E satisfy (7.11).

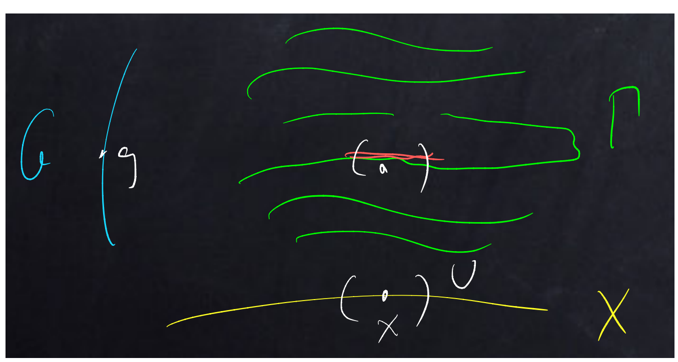

FIGURE 26. A maximal integral manifold of E

Fix $(x_0, g_0) \in X \times G$. The global Frobenius Theorem 6.22 gives a maximal integral manifold $\Gamma \subset X \times G$ through (x_0, g_0), as depicted in Example 1.44. The projection $\pi = \pi_2\big|_{\Gamma}\colon \Gamma \to X$ is a local diffeomorphism, since the differential is an isomorphism, hence for all $x \in X$ in the image $\pi(\Gamma) \subset X$ there is a local inverse $U \to \Gamma$ for some neighborhood $U \subset X$ of x. Suppose that local inverse maps x to $g \in G$. Since E is invariant under left translations in G, if $(x, g') \in \Gamma$ for some $g' \in G$, then composition with left multiplication by $g'g^{-1}$ maps a local inverse with $x \mapsto g$ to a local inverse with $x \mapsto g'$. It follows that U is evenly covered by π. Also, π is an open map since it is a local diffeomorphism, hence $\pi(\Gamma) \subset X$ is open.

We claim the image $\pi(\Gamma) \subset X$ is also closed. Namely, suppose $x_0 \in X$ is a limit point of the image. Then we claim that there exists a neighborhood U of x_0 such that $\pi_1^{-1}(U) \subset X \times G$ is foliated by integral manifolds of E. Certainly there is an integral manifold through every point of $\pi_1^{-1}(x_0)$, but for general distributions we do not have uniformity to conclude that there is a neighborhood U of x_0 to which each integral manifold maps surjectively by π_1. In this case the distribution is invariant under left multiplication on G, and this gives the desired uniformity: the image of any integral manifold under left translation is another integral manifold. Now since x_0 is a limit point of $\pi(\Gamma)$ it follows that one of these integral manifolds Γ' intersects Γ nontrivially. The maximality of Γ then implies that $\Gamma' \subset \Gamma$, and therefore $x_0 \in \pi(\Gamma)$. This proves that $\pi(\Gamma) \subset X$ is closed. Since X is connected, it follows that π is surjective.

Since π evenly covers its image, it is a covering map. And since X is assumed simply connected, it now follows that π is a diffeomorphism. Now define $f = \pi_2 \circ \pi^{-1}$, where $\pi_2 \colon \Gamma \to G$ is the restriction of projection $X \times G \to G$. Then (7.26) shows that f satisfies the differential equation (7.11). This concludes the proof of existence. $\square$

Lie group homomorphisms

Our first application of Theorem 7.8 is internal to the theory of Lie groups; see Remark 3.42.

Theorem 7.27. *Let G be a Lie group and let G' be a simply connected Lie group. Suppose $\dot\psi \colon \mathfrak{g}' \to \mathfrak{g}$ is a homomorphism of their Lie algebras. Then there exists a unique Lie group homomorphism $\psi \colon G' \to G$ whose differential is $\dot\psi$.*

Proof. Let $\theta_G, \theta_{G'}$ denote the Maurer-Cartan 1-forms on G, G', respectively, and let e', e be the respective identity elements. Set $\omega = \dot\psi(\theta_{G'}) \in \Omega^1_{G'}(\mathfrak{g})$. Theorem 7.8 produces a unique smooth map $\psi \colon G' \to G$ such that

$$\begin{aligned} \psi^*(\theta_G) &= \dot\psi(\theta_{G'}) \\ \psi(e') &= e \end{aligned} \tag{7.28}$$

We claim ψ is a homomorphism, which is equivalent to

$$\psi \circ L_{g'} = L_{\psi(g')} \circ \psi, \qquad g' \in G' \tag{7.29}$$

as maps $G' \to G$ for all choices of g'. Observe that both sides Ψ of (7.29) satisfy

$$\begin{aligned} \Psi^*(\theta_G) &= \dot\psi(\theta_{G'}) \\ \Psi(e') &= \psi(g'), \end{aligned} \tag{7.30}$$

and therefore are equal by the uniqueness Theorem 7.8(1). $\square$

Right G-torsors

(7.31) *Recollection.* We repeat Definition 1.4, but now in the smooth category.

Definition 7.32. Let G be a Lie group. A *right G-torsor* is a smooth manifold P equipped with a smooth simply transitive right G-action $P \times G \to P$.

The simple transitivity is equivalent to the invertibility of the map $P \times G \to P \times P$ whose first component is projection and whose section is the action.

Example 7.33 (The trivial right G-torsor)**.** The Lie group acts on itself by right multiplication, and that action is simply transitive. Denote this canonical right G-torsor as G_G.

(7.34) *Transport of structure.* We reprise (1.5) and (1.6). Let P be a right G-torsor. Then any $p \in P$ determines an isomorphism of right G-torsors (cf. (1.5) and (1.6))

$$\begin{aligned} \psi_p \colon G_G &\longrightarrow P \\ g &\longmapsto pg \end{aligned} \tag{7.35}$$

Hence any two right G-torsors are isomorphic, though not canonically so. The overlap between two parametrizations is the composition in the diagram

$$\begin{tikzcd} & P & \\ G \arrow[ur, "\psi_p"] \arrow[rr, dashed] & & G \arrow[ul, "\psi_{ph}"'] \end{tikzcd} \tag{7.36}$$

where $h \in G$. This composition is easily computed to be the left multiplication $L_{h^{-1}} \colon G \to G$. In other words, a right G-torsor is identified with G_G up to a *left* multiplication on G. It follows that the result of transporting a (left) parallel tensor on G to P via ψ_p is independent of $p \in P$.

Remark 7.37. Recall that a smooth structure on a topological manifold is an atlas of C^∞-compatible charts. The C^∞-compatibility means that C^∞ objects and concepts in affine space transport unambiguously to smooth manifolds. The principle here is the same.

(7.38) *Vector fields on P.* Let $\zeta \in \mathfrak{g}$ be a parallel (left-invariant) vector field on G. By transport of structure **(7.34)** it determines a vector field $\widehat{\zeta}$ on P. Using the identification of the Lie algebra $\mathfrak{g}$ as the set of one-parameter subgroups of G (Theorem 3.46), we give a useful formula for the vector field $\widehat{\zeta}$:

$$\widehat{\zeta}_p = \frac{d}{dt}\bigg|_{t=0} pe^{t\zeta}, \qquad p \in P. \tag{7.39}$$

Remark 7.40. Since left and right multiplication on G commute, the parallelism defined by right multiplication is left invariant, so transports to a parallelism on a right G-torsor P. More simply, a simply transitive group action on a manifold gives rise to a parallelism, and the right action of G on P is simply transitive. The vector field $\widehat{\zeta}$ is *not* right parallel on P. Namely, for $p \in P$, $h \in G$,

$$\widehat{\zeta}_{ph} = \frac{d}{dt}\Big|_{t=0} phe^{t\zeta} = \frac{d}{dt}\Big|_{t=0} p\left(he^{t\zeta}h^{-1}\right)h = (R_h)_*(\widehat{\mathrm{Ad}_h\,\zeta})_p, \tag{7.41}$$

where $R_h\colon P \to P$ is the action of h and Ad_h is the adjoint action of $h \in G$ on the Lie algebra $\mathfrak{g}$.

Remark 7.42. A right G-torsor is a *left* torsor for the action of its automorphism group $\mathrm{Aut}(P)$, which we define in the next lecture, and that action induces a (left) parallelism on P. The vector field $\widehat{\zeta}$ *is* parallel with respect to that parallelism.

(7.43) *The Maurer-Cartan form on P.* The Maurer-Cartan form $\theta \in \Omega^1_G(\mathfrak{g})$ is (left) parallel, so transports to a 1-form $\theta_P \in \Omega^1_P(\mathfrak{g})$ that we call the Maurer-Cartan form on P. The Maurer-Cartan equation transports to the equation

$$d\theta_P + \frac{1}{2}[\theta_P \wedge \theta_P] = 0 \qquad \text{in } \Omega^2_P(\mathfrak{g}). \tag{7.44}$$

Lemma 7.45. *Fix $\zeta \in \mathfrak{g}$ and let $\widehat{\zeta} \in \mathfrak{X}(P)$ denote the corresponding vector field on P. Then*

$$\theta_P(\widehat{\zeta}) = \zeta. \tag{7.46}$$

The right hand side of (7.46) is a constant $\mathfrak{g}$-valued function on P.

Proof. For $p \in P$ we see from (7.35) and (7.39) that $d(\psi_p)(\zeta) = \widehat{\zeta}$. Then $(\theta_P)(\widehat{\zeta}) = \theta(\zeta) = \zeta$. ☐

Examples of right G-torsors

We have encountered some of these examples earlier.

Example 7.47 (Affine space)**.** Let V be a finite dimensional real vector space, which we regard as a Lie group under vector addition. Then a right V-torsor is an affine space over V (Definition 1.1).

Example 7.48 (Torsors in homogeneous manifolds)**.** Let $\widetilde{G}$ be a Lie group and suppose $\widetilde{G}$ acts transitively on a smooth manifold X via a left action. Fix $x \in X$ and let $G = \mathrm{Stab}_x \subset \widetilde{G}$ be the closed Lie subgroup of elements that fix x. Then for any $x' \in X$ define the right G-torsor

$$P_{x'} = \{\tilde{\gamma} \in \widetilde{G} : \tilde{\gamma}x = x'\}. \tag{7.49}$$

Note that $P_{x'}$ is a left G-coset in $\widetilde{G}$.

As a particular example, let $\widetilde{G} = \mathrm{O}_2$ be the orthogonal group and $X = \{x, x'\}$ a set of cardinality 2. Let $g \in \mathrm{O}_2$ act as the id_X if $\det(g) = +1$ and as the nontrivial permutation if $\det(g) = -1$.

Then $G = \mathrm{Stab}_x = \mathrm{SO}_2 \subset \mathrm{O}_2$, and $P_{x'} = \mathrm{O}_2 \setminus \mathrm{SO}_2$. Hence reflections are a right torsor over rotations.

As another example, let $\widetilde{G} = \mathrm{SO}_3$ act by rotations on the unit $S^2 \subset \mathbb{R}^3$, and let $x = (1,0,0)$ be the north pole. Then $G = \mathrm{Stab}_x \cong \mathrm{SO}_2$ is the group of rotations about the $(0,0,*)$-axis, which can be identified with rotations in the $(*,*,0)$-plane. The rotations that move x to a fixed $x' \in S^2$ are a torsor over SO_2.

Example 7.50 (Bases in a vector space)**.** We conclude with a key example for us. Let V be a finite dimensional real vector space. Recall from Definition 1.26 the right $\mathrm{GL}_n\mathbb{R}$-torsor of bases

(7.51) $$\mathcal{B}(V) = \{b\colon \mathbb{R}^n \longrightarrow V : b \text{ is an isomorphism}\}.$$

We use the notation in **(4.77)** to write the (transported) Maurer-Cartan form on $\mathcal{B}(V)$ as a matrix of scalar 1-forms $\Theta^i_j \in \Omega^1_{\mathcal{B}(V)}$ that satisfy the Maurer-Cartan equation (4.83):

(7.52) $$d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j = 0.$$

(7.53) *Geometric interpretation of Θ^i_j on $\mathcal{B}(V)$.* Let $b \in \mathcal{B}(V)$ be a basis, which we identify with the image $e_1, \dots, e_n$ in V of the standard basis of $\mathbb{R}^n$. Let $e^1, \dots, e^n$ be the dual basis of V^*. A tangent vector $\xi \in T_b\mathcal{B}(V)$ is represented as a motion germ $e_1(t), \dots, e_n(t)$ with initial value the given basis $e_1, \dots, e_n$. Let $\dot{e}_j = e'_j(0)$ be the initial velocity of the j^{th} basis element.

Proposition 7.54. $(\Theta^i_j)_b(\xi) = \langle e^i, \dot{e}_j \rangle$.

The brackets on the right hand side denote the duality pairing of V^* and V. In words, Θ^i_j measures the instantaneous rate at which e_j is moving towards e_i, relative to the given basis. This geometric interpretation is important in the sequel.

Proof. Choose $A \in \mathfrak{gl}_n\mathbb{R}$ so that $\xi = \widehat{A}$. Then by Lemma 7.45 we have $(\Theta^i_j)_b(\xi) = A^i_j$. Choose the motion germ with $e_j(t) = b\left[(e^{tA})_j\right]$ is the image of the j^{th} column, regarded as a vector in $\mathbb{R}^n$, and so $\dot{e}_j = b(A_j)$. Hence the basis b maps the pairing $\langle \hat{e}^i, A_j \rangle = A^i_j$ of A_j with the standard dual basis element $\hat{e}^i \in (\mathbb{R}^n)^*$ to the desired pairing $\langle e^i, \dot{e}_j \rangle$. $\square$