Math 212 Problem Set 1

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Problem 1. Find an example of a continuous function on $\mathbb{R}$ which goes to zero at infinity and which isn't the Fourier transform of a function in $L^1(\mathbb{R})$.

Recall that the Fourier transform $\mathscr{F}$ maps $L^1(\mathbb{R})$ functions to $C_0^0(\mathbb{R})$ functions.

Problem 2. The Schwarz space $\mathcal{S}(\mathbb{R}^d)$ is defined as

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d, \mathbb{R}) \;\middle|\; \lim_{|x|\to\infty} |(\mathcal{D}f)(x)| = 0 \quad \forall \mathcal{D} \in \mathrm{Diff}(\mathbb{R}[x_1, \ldots, x_d]) \right\}$$

where $\mathrm{Diff}(\mathbb{R}[x_1, \ldots, x_d])$ denotes the space of polynomial differential operators in d variables. Prove that the Fourier transform maps the vector space $\mathcal{S}$ to itself.

Suppose $f \in \mathcal{S}(\mathbb{R}^d)$ is a Schwarz function. First, we claim that $f \in L^p(\mathbb{R}^d)$. Note that

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f|^p \right)^{1/p} =$$