Answer the following questions either prove the assertions or give counter examples. If you can't prove them, try to give reasons for your answers. The deadline is 9 am Dec 10. Please latex your solution.

1. (15 pts) For a positive function f, consider two quantities

$$A := \int dy \left[\int f(x,y)^p dx \right]^{1/p}$$

$$B := \left[\int dx \left(\int f(x,y) dy \right)^p \right]^{1/p}$$

For $1 \le p < \infty$. Assume all quantities are integrable and finite. Do we know that $A \ge B$ or $A \le B$ for all functions f ? Prove your assertion. Since this is a known result, you cannot just cite a theorem. Try to prove it directly by using the fact (which you can assume) that

$$\|h\|_p = \sup_{\|g\|_q \le 1} \int hg, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{+}$$

Hint:

$$A := \int dy \left[\int f(x,y)^p dx \right]^{1/p} = \int dy \sup_{\|g(x,y)\|_{L_q(x)} \le 1} \int f(x,y) g(x,y) dx$$

Notice that g depends on x, y. Now can you rewrite B by using again (+)? Now compare the two formulas and find out why one of them will be always bigger or equal to the other one.

2. (20 points) Suppose $f_n \to f$ a.e. for all $x \in X = (0,1)$ and $\sup_n \|f_n\|_{L^2(X)} \le M$ for some M fixed. Do we know that $\lim_{n\to\infty} \|f_n - f\|_{L^2(X)} = 0$? If, in addition, $\lim_{n\to\infty} \|f_n\|_{L^2(X)} = \|f\|_{L^2(X)} < \infty$. Do we know that $\lim_{n\to\infty} \|f_n - f\|_{L^2(X)} = 0$? Hint: Parallelogram law states that

$$\|f_n - f\|^2 = 2\|f_n\|^2 + 2\|f\|^2 - \|f_n + f\|^2$$

Here all norms are L_2. Taking the limit on both sides and see what you get. You will need to use some convergence theorem to finish the problem.

3. (15 pts) Suppose μ is a probability measure (on a set, say, $\mathbb{R}^n$) and $f \ge 0$ with $\int f d\mu = 1$. Prove that

$$S(f) := \lim_{\varepsilon \downarrow 0} \int f \log(f + \varepsilon) d\mu$$

exists, $0 \le S(f)$ (the limit $S(f)$ can be infinity). Prove that, for any bounded real function h,

$$\int hf d\mu - S(f) = \int [h - (\log f)] f d\mu \le \log \left[\int e^h d\mu \right]$$

Notice that you will need to use the condition $\int f d\mu = 1$. (Hint: The last inequality can be proved by a convexity inequality. Notice that $\log x$ is a concave function while e^x is convex.) The previous steps use that $S(f) = \int f \log f d\mu$ formally, please use the correct definition $S(f) := \lim_{\varepsilon \downarrow 0} \int f \log(f + \varepsilon) d\mu$ and carry out the limit $\varepsilon \to 0$.

If you have difficulties proving the last inequality with limiting procedure, then you just try to solve the case with the assumptions that $f > 0$ and $\int f \log f d\mu < \infty$.

4. (20 pts) Let $X_1, X_2, \dots X_n$ be identically independent random variables with $\mathbb{E}X_j = 0$, $\mathbb{E}X_j^2 = 1$ and $\mathbb{E}|X_j|^3 \leq M < \infty$. This problem gives a proof of CLT with an error bound. Let ϕ be a real function such that the first three derivatives are bounded, i.e., $\sum_{j=0}^{3} \|\phi^{(j)}\|_\infty \leq M < \infty$. Let Y_j be i.i.d. normal distribution with mean zero and variance one (Y_j and X_i are independent). Rewrite

$$
\begin{aligned}
&\mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}\sum_j X_j\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}\sum_j Y_j\Big) \\
&= \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[X_1 + X_2 + \ldots + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + X_2 + \ldots + X_N]\Big) \\
&+ \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + X_2 + X_3 \ldots + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + Y_2 + X_3 \ldots + X_N]\Big) \\
&+ \ldots + \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + \ldots + Y_{N-1} + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + Y_2 + \ldots + Y_N]\Big).
\end{aligned}
$$

Prove by using Taylor theorem that

$$\Big|\mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[X_1 + X_2 + \ldots + X_N]\Big) - \mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + X_2 + \ldots + X_N]\Big)\Big| \leq C_M N^{-3/2}$$

Hint: Try to prove that

$$\Big|\mathbb{E}_{X_1}\phi\Big(\frac{1}{\sqrt{N}}[X_1 + X_2 + \ldots + X_N]\Big) - \mathbb{E}_{Y_1}\phi\Big(\frac{1}{\sqrt{N}}[Y_1 + X_2 + \ldots + X_N]\Big)\Big| \leq C_M N^{-3/2}$$

where $\mathbb{E}_{X_1}$ means taking expectation of X_1.

Similar bounds clearly hold for the differences in every differences in the telescoping sum. From here, prove that for any ϕ satisfying $\sum_{j=0}^{3} \|\phi^{(j)}\|_\infty \leq M < \infty$,

$$\Big|\mathbb{E}\phi\Big(\frac{1}{\sqrt{N}}\sum_j X_j\Big) - \mathbb{E}\phi\Big(\xi\Big)\Big| \leq C_M N^{-1/2},$$

where ξ is a normal distribution with mean zero and variance one. You can use that sum of independent normal random variables are normal so that $\frac{1}{\sqrt{N}}[Y_1 + Y_2 + \ldots + Y_N]$ is a normal distribution.

5. (30 points) Let $X_1, X_2, \dots X_n$ be identically independent random variables with $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = \sigma^2$. Let $S_n = X_1 + \ldots + X_n$. The weak law of large numbers states that Then for any $\varepsilon > 0$

$$\mathbb{P}\Big(|\frac{S_n}{n}| > \delta\Big) \leq \frac{\sigma^2}{n\delta^2} \tag{+}$$

Suppose that, instead of $\mathbb{E}X_j^2 = \sigma^2$, we only know that $(\mathbb{E}|X_j|^p)^{1/p} = M < \infty$ for some $1 < p < 2$. As in the class, let $\hat{X}_j = X_j\mathbf{1}(|X_j| \leq c)$, $\hat{Y}_j = X_j\mathbf{1}(|X_j| > c)$, $a_c = \mathbb{E}\hat{X}_j$ and $b_c = \mathbb{E}\hat{Y}_j$. Clearly,

$X_j = \hat{X}_j + \hat{Y}_j$. Then we have

$$\begin{aligned}
\mathbb{E}\Big|\sum_j (\hat{X}_j + \hat{Y}_j)\Big| &\le \mathbb{E}\Big|\sum_j (\hat{X}_j - a_c)\Big| + \mathbb{E}\Big|\sum_j \hat{Y}_j - b_c\Big| \\
&\le \Big[\mathbb{E}\Big\{\sum_j (\hat{X}_j - a_c)\Big\}^2\Big]^{1/2} + 2n\mathbb{E}|\hat{Y}_j| \\
&= \sqrt{n}\Big[\mathbb{E}(\hat{X}_1 - a_c)^2\Big]^{1/2} + 2n\mathbb{E}|\hat{Y}_1|
\end{aligned}$$

Prove that

$$\mathbb{E}|\hat{Y}_1| \le c^{1-p}\mathbb{E}|X_1|^p \le c^{1-p}M^p \tag{1}$$

$$\mathbb{E}(\hat{X}_1 - a_c)^2 \le 2\mathbb{E}\hat{X}_1^2 + 2a_c^2 \le 4c^{2-p}M^p \tag{2}$$

$$P\Big(\Big|\sum_j (\hat{X}_j + \hat{Y}_j)\Big| \ge n\delta) \le 4\delta^{-1}\inf_{c>0}[c^{1-p/2}M^{p/2}n^{-1/2} + c^{1-p}M^p] \tag{3}$$

Hint: You should try to start with a proof of (1) and assuming that X_1 is positive. To prove (2), you need to use that

$$\mathbb{E}\hat{X}_1^2 \le c^{2-p}\mathbb{E}|X_1|^p$$

Also, (3) is a simple consequence of (1) and (2) (you should think about how to link probability bound of a random variable with expectation the random variable). Even if you cannot prove (1) or (2), you can assume it and continue with (3).

Finally, to carry out the inf, prove by calculus (this part you only need freshman calculus) that, for any $\alpha, \beta > 0$, there is a constant K such that

$$\inf_{x>0} Ax^{\alpha} + Bx^{-\beta} = KA^{\frac{1}{1+\gamma}}B^{\frac{\gamma}{1+\gamma}}, \quad \gamma = \frac{\alpha}{\beta}$$

(Hint: Using $y = x^{\alpha}$ will make calculation much simpler.) Now take $\alpha = 1 - p/2, \beta = p - 1$, you can carry out the inf to finally get a bound on $\mathbb{P}\Big(|\frac{S_n}{n}| > \delta\Big)$. Besides unimportant constants, what is your final answer?