Math 132 Problem Set 11

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Problem 1. This problem is more or less worked out in your book, at the end of §4 of Chapter 4.

See if you can do it on your own, looking at the book if you need a hint.

a. In calculus courses, line integrals in $\mathbb{R}^3$ are usually written in the form

$$\oint_C \vec{F} \cdot d\vec{r}.$$

Given a vector function $\vec{F}$, show how to construct a 1-form θ with the property that for every oriented curve C,

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C \theta.$$

Let $\theta \in \Omega^1(\mathbb{R}^3)$ be the 1-form given by $\theta = \vec{F}_x\, dx + \vec{F}_y\, dy + \vec{F}_z\, dz$, where $\vec{F}_*$ are the component functions of the vector field. Then given a parametrization $r : I \to C$ of the curve C, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(r(t)) \cdot r'(t)\, dt.$$

On the other hand, we have

$$\begin{aligned} \int_C \theta = \int_I r^*\theta &= \int_I \vec{F}_x r^* dx + \vec{F}_y r^* dy + \vec{F}_z r^* dz \\ &= \int_I \vec{F}_x r'(t) dt + \vec{F}_y r'(t) dt + \vec{F}_z r'(t) dt = \int_0^1 \vec{F}(r(t)) \cdot r'(t)\, dt. \end{aligned}$$

So these two integrals are equivalent.

b. In multivariable calculus, surface integrals are written in the form

$$\iint_S \vec{F} \cdot \mathbf{n}\, dS,$$

and computed when S is the set of points $(u, v, \mathbf{X}(u, v))$, with $(u, v) \in D$ using

$$\mathbf{n}\, dS = \mathbf{X}_u \times \mathbf{X}_v\, du\, dv$$

and

$$\iint_S \vec{F} \cdot \mathbf{n}\, dS = \iint_D \vec{F} \cdot \mathbf{X}_u \times \mathbf{X}_v\, du\, dv.$$

Reconcile this with our formulation of integration on manifolds. More specifically, show how to associate a 2-form $\omega \in \Omega^2(\mathbb{R}^3)$ to the vector function $\vec{F} = (f_1, f_2, f_3)$ with the property that for every surface S,

$$\iint_S \vec{F} \cdot \mathbf{n}\, dS = \int_S \omega.$$

Let $\omega \in \Omega^2(\mathbb{R}^3)$ be the 2-form given by

$$\omega = \vec{F}_x\, dy \wedge dz + \vec{F}_y\, dz \wedge dx + \vec{F}_z\, dx \wedge dy.$$

Now let $h : D \to S$ be the parametrization given by $h(u, v) = (u, v, \mathbf{X}(u, v))$. Then

$$\int_S \omega = \int_D h^* \omega = \int_D \vec{F} \cdot \left(-\frac{\partial \mathbf{X}}{\partial u}, -\frac{\partial \mathbf{X}}{\partial v}, 1\right) du \wedge dv = \int_D \vec{F} \cdot \vec{n}\, du \wedge dv$$

where $\vec{n} = \left(-\frac{\partial \mathbf{X}}{\partial u}, -\frac{\partial \mathbf{X}}{\partial v}, 1\right)$. Clearly, $\vec{n}(x) \perp T_x(S)$, so this is a normal vector. Letting $\mathbf{n} = \vec{n}/\|\vec{n}\|$, we can then let $dS = |\vec{n}|\, du \wedge dv$ so that

$$\int_S \omega = \int_D \vec{F} \cdot \mathbf{n}\, dS.$$

Since there is a unique unit normal vector, this formulation is the same as with the definition which was provided: $\mathbf{n}\, dS = \mathbf{X}_u \times \mathbf{X}_v\, du\, dv$.

c. In multivariable calculus courses, Stokes Theorem is usually stated in the form

$$\iint_S \mathbf{curl}\, \vec{F} \cdot \mathbf{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}.$$

Reconcile this with our formulation of Stokes Theorem.

Our formulation of the (Generalized) Stokes Theorem was that for any $\omega \in \Omega^{k-1}(X)$, we have

$$\int_X d\omega = \int_{\partial X} \omega.$$

For our case, we have $X = \mathbb{R}^3$, and S some surface. Then given a vector field $\vec{F}$, we have a differential form $\omega = \vec{F}_x\, dx + \vec{F}_y\, dy + \vec{F}_z\, dz$, with

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_{\partial S} \omega$$

by the first part of this problem. By Stokes theorem, this should be equal to $\int_S d\omega$. By an earlier book calculation, we have:

$$d\omega = \mathbf{curl}\, \vec{F} \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy) = \mathbf{curl}\, \vec{F} \cdot \mathbf{n}\, dS$$

Thus $\int_S \mathbf{curl}\, \vec{F} \cdot \mathbf{n}\, dS = \int_{\partial S} \omega$.

d. When the surface S is the boundary of a 3-dimensional region $D \subset \mathbb{R}^3$, the *divergence theorem* states that

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \mathbf{n}\, dS.$$

Show that this is also a special case of our formulation of Stokes theorem.

Here we apply Stokes Theorem to get

$$\iint_S \vec{F} \cdot \mathbf{n}\, dS = \int_{\partial D} \mathbf{curl}\, \vec{F} \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy) = \int_D d\left(\mathbf{curl}\, \vec{F} \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy)\right).$$

Some very annoying computations involving properties of exterior products and exterior derivatives then show that this latter differential form is equivalent to $\nabla \cdot \vec{F}\, dx \wedge dy \wedge dz$.

Problem 2. Conservative vector fields.

Here are some familiar facts about *conservative vector fields* stated in the language of 1-forms.

a. Suppose that ω is a 1-form on a connected manifold X, having the property that $\oint_\gamma \omega = 0$ for all closed curves γ. Show that if $p, q \in X$ are two points. Choose a smooth path $c : [0,1] \to X$ with $c(0) = p$ and $c(1) = q$. Show that $\int_0^1 c^*\omega$ is independent of the choice of c.

Suppose we have two curves c_1 and c_2. Using bump functions and the results of Exercise 4, we can reparametrize the curves c_1 and c_2 so that they are constant in neighborhoods of 0 and 1. More specifically, letting

$$\psi(x) = \begin{cases} \exp\left(\frac{-x^2}{1-x^2}\right) & x \in (0,1), \\ 1 & x \leq 0, \\ 0 & x \geq 1. \end{cases}$$

be a step function, we can use the reparametrizations $f_1(t) = 1 - f(4t-2)$ and $f_2(t) = 1 - f(4(1-t)-2)$. By Exercise 4, this doesn't affect $\int_0^1 c_i^*\omega$, but allows us to continuously stich the functions together, where the second one runs backwards. Let's call this combined close curve c. Then

$$0 = \int_0^1 c^*\omega = \int_0^1 c_1^*\omega - \int_0^1 c_2^*\omega,$$

so the two integrals are the same.

b. Prove that any 1-form ω on X with the property that $\oint_\gamma \omega = 0$ for all closed curves γ is *exact* in the sense that there is a smooth function f with $\omega = df$. A curve γ is *closed* if $\gamma(0) = \gamma(1)$.

Without loss of generality, we can assume that X is (path) connected. If it wasn't, we could run this program on each connected component, and define a global function as a disjoint union of the component functions. Now for some arbitrary point $p \in X$, let's define $f(x) = \int_p^x \omega = \int_0^1 c^*\omega$, for some path $c : p \mapsto x$. We claim that $df = \omega$. For any point $x \in X$, let (U, φ) be some coordinate chart about x, with coordinates x_i. We simply need to show that

$$\frac{\partial f}{\partial x_i}(x) = \omega_i(x) \quad \Longrightarrow \quad df = \omega.$$

Notice that we can work entirely in the chart U by picking some $p' \in U$. Then we get

$$f(x) = f(p') + \int_{p'}^{x} \omega.$$

The function $\widetilde{f}(x) = \int_{p'}^{x} \omega$ differs from f by a constant, so it suffices to prove that its derivative has the desired form. However in this chart we can simply use the fundamental theorem of calculus in $\mathbb{R}^n$, or Stokes theorem by composing with the chart diffeomorphism φ, so it's straightforward to show that $\partial f/\partial x_i(x) = \omega_i(x)$, which proves the claim.

Problem 3. There are many variations on bordism homology. An important one is *oriented bordism*. One defines an *oriented k-manfifold* over a space X to be a pair (M, f) consisting of an oriented k-manifold M and a continuous map $f : M \to X$. The notion of *cobordism* is a little more subtle in this case. A *cobordism* of (M_0, f_0) and (M_1, f_1) consists of an oriented $(k+1)$-manifold N, a continuous map $h : N \to X$, and an oriented diffeomorphism $M_1 \amalg (-M_0) \approx \partial N$ having the property that for $i = 0, 1$, the composition $M_i \to N \xrightarrow{h} X$ is f_i.

As a "reminder" when M is an oriented manifold, the symbol $(-M)$ refers to M with the opposite orientation. The *oriented bordism homology group* $MSO_k(X)$ is defined to be the set of equivalence classes of oriented closed k-manifolds over X modulo the equivalence relation of oriented cobordism. The set $MSO_k(X)$ is a commutative monoid under disjoint union.

a. Suppose that $f : M \to X$ is an oriented k-manifold over X. Show that

$$(M, f) + (-M, f) = 0 \in MSO_k(X).$$

(Thus $MSO_k(X)$ is an abelian group.)

Since $M \times I$ is an oriented $(k+1)$-manifold with boundary $M \amalg (-M)$, we can use the projection $M \times I$ composed with f to get a cobordism between $(M \amalg (-M), f \amalg f)$ and $\varnothing$.

b. Now suppose that X is a smooth manifold and $\omega \in \Omega^k(X)$ is a k-form which is *closed* in the sense that $d\omega = 0$. The proof that $MO_k(X)$ can be computed by smooth maps and smooth cobordism applies to $MSO_k(X)$. Show that sending $f : M \to X$ to $\int_M f^*\omega$ gives a well-defined homomorphism

$$MSO_k(X) \to \mathbb{R}.$$

We need to show that this map is well-defined, and a homomorphism. For the well defined part, let (N, h) be a cobordism between (M_0, f_0) and (M_1, f_1). By Stokes theorem, we have:

$$\begin{aligned}\int_{M_1} f_1^*\omega - \int_{M_0} f_0^*\omega = \int_{M_1 \amalg (-M_0)} (f_1 \amalg f_0)^*\omega &= \int_N d(h^*\omega) \\ &= \int_N h^*(d\omega) = 0.\end{aligned}$$

It thus follows that the map is well defined. To prove that it is a homomorphism, suppose we had manifolds (M_0, f_0) and (M_1, f_1). Then we have

$$\int_{M_0 \amalg M_1} (f_0 \amalg f_1)^*\omega = \int_{M_0} f_0^*\omega + \int_{M_1} f_1^*\omega$$

so the map is a well-defined homomorphism.

c. A k-form ω is *exact* if there is a $(k-1)$-form η with $d\eta = \omega$. Show that if ω is exact then the above homomorphism is zero. Thus there is a map

$$H^k_{\mathrm{DR}}(X) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\} \to \mathrm{Hom}(MSO_k(X), \mathbb{R}).$$

Let (M, f) be a manifold over X. By Stokes theorem, we get

$$\int_M f^*\omega = \int_M f^* d\eta = \int_M df^*\eta = 0.$$

So the homomorphism extends to a mapping of De Rham cohomology.

Problem 5. Prove that for the $(n-1)$ sphere of radius r in $\mathbb{R}^n$, the Gaussian curvature is everywhere $1/r^{n-1}$.

Let $S^{n-1}_r(0)$ be the unit sphere of radius r. Recall that the Gauss map $g : S^{n-1}_r(0) \to S^{n-1}$ sends x to $\vec{n}(x)$. Then the Gaussian curvature is the Jacobian: $J_g(x) = \kappa(x)$. In this case, $g(x) = x/r$, so the Jacobian matrix is an $(n-1) \times (n-1)$ diagonal with entries equal to $1/r$, and so the Jacobian is thus a constant function with value $\frac{1}{r^{n-1}}$. Thus $\kappa(x) = \frac{1}{r^{n-1}}$.