Math 132 Problem Set 5

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Problem 1. Morse lemma.

This problem is about the Morse lemma in dimension 1.

a. Suppose $f : U \to \mathbb{R}$ is a smooth function defined on an open neighborhood U of $0 \in \mathbb{R}$, that $f'(0) = 0$ and that $f''(0) \neq 0$. show that there is a new coordinate x defined near $0 \in U$ in which f is given by $f(x) = f(0) \pm x^2$.

Recall by Taylor's theorem that we have an expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k f^{(k)}(0)}{k!} = f(0) + x^2 \left(\frac{f''(0)}{2} + \sum_{k=3}^{\infty} \frac{x^k f^{(k)}(0)}{k!} \right).$$

Let $g(x)$ be the content in these parenthesis, so that $f(x) = f(0) + x^2 g(x)$. Since $g(0) = f''(0)/2 \neq 0$, we can assume without loss of generality that $g(x)$ is strictly positive in some neighborhood of 0, thus getting $f(x) = f(0) \pm x^2 g(x)$. Then letting our coordinate be $x = x\sqrt{g(x)}$ in this neighborhood, we get $f(x) = f(0) \pm x^2$.

b. Suppose that $f : X \to \mathbb{R}$ is a smooth function on a 1-manifold X and $p \in X$ is a non-degenerate critical point. Show that there is a local coordinate system $\Phi : U \to \mathbb{R}$ with $\Phi(p) = 0$, in which $f \circ \Phi^{-1}(x) = a \pm x^2$ (where $a = f(p)$). This is the Morse lemma in dimension 1.

Let's pick some chart $\Phi : U \to \mathbb{R}$ which sets $\Phi(p) = 0$. Then $f \circ \Phi^{-1} : \mathbb{R} \to \mathbb{R}$ satisfies the conditions of (a) due to non-degeneracy and criticality conditions. Thus there is some coordinate $g(x)$ such that $f(\Phi^{-1}(g(x))) = f(p) \pm g(x)^2$. Thus letting $\Phi = g^{-1} \circ \Phi$ gives us the desired chart.

Problem 2. Let M_n be the space of $n \times n$ matrices.

Consider the function $f : M_n \to M_n$ given by $f(B) = B^2$.

a. Identifying $T_I M_n$ with M_n, show that

$$df(I) : T_I M_n \to T_I M_n$$

is the linear map sending $B \in M_n$ to $2B$. Since this has non-zero determinant, f is a diffeomorphism in a neighborhood of 1.

We can prove this by showing that

$$\lim_{\|H\| \to 0} \frac{\|(B + H)^2 - B^2 - 2B\|}{\|H\|} = 0$$

for a matrix norm $\|\cdot\|$. Note that

$$\frac{\|(B+H)^2 - B^2 - 2B\|}{\|H\|} = \frac{\|2B(H-1) + H^2\|}{\|H\|} \leq \frac{\|H\|^2}{\|H\|} = \frac{1}{\|H\|} \to 0.$$

b. Conclude that there is a smooth diffeomorphism $B \mapsto B^{1/2}$ defined in a neighborhood of I satisfying $(B^{1/2})^2 = B$.

Since $df(I)$ is an isomorphism, f must be a local diffeomorphism in some neighborhood, so it must have some local inverse which would send $B \mapsto B^{1/2}$.

Problem 3. It will be useful for some (but not all) parts of this problem to remember that since $B \mapsto B^2$ is a *diffeomorphism* in a neighborhood of I, one can show that $S = T$ by showing that $S^2 = T^2$.

Continuing from the last problem:

a. Show that B and $B^{1/2}$ commute.

Note that $B = B^{1/2}B^{1/2}$ so $B^{1/2}B = B^{1/2}B^{1/2}B^{1/2} = BB^{1/2}$.

b. Suppose that $A \in M_n$ is an invertible matrix. Show that in the open neighborhood of I where both $B^{1/2}$ and $(A \cdot B \cdot A^{-1})^{1/2}$ are defined, one has

$$A(B^{1/2})A^{-1} = (ABA^{-1})^{1/2}.$$

In this open neighborhood, $B \mapsto B^{1/2}$ is a diffeomorphism, so $S^2 = T^2$ implies that $S = T$. Thus

$$(A(B^{1/2})A^{-1})^2 = ABA^{-1} \implies (ABA^{-1})^{1/2} = A(B^{1/2})A^{-1}.$$

c. Show that in a neighborhood of I one has

$$(A^T)^{1/2} = (A^{1/2})^T.$$

By the same argument as in part b, note that $(A^{1/2})^T(A^{1/2})^T = (A^{1/2}A^{1/2})^T = A^T$, so we conclude $(A^T)^{1/2} = (A^{1/2})^T$.

Problem 4. Show that $G_k(\mathbb{R}^n)$ is compact.

Recall that we have a quotient map $V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ which sends an orthonormal k-frame to it's span. However recall that $V_k(\mathbb{R}^n)$ can be embedded into $\mathbb{R}^{nk}$ canonically. The image of this embedding is closed and bounded in $\mathbb{R}^{nk}$ by the orthonormality condition, so $V_k(\mathbb{R}^n)$ is compact. Any quotient of a compact space is also compact so we are done.

Problem 5. The image of the Plücker embedding is the solution space of the famous *Plücker equations.*

In this exercise, we will study those equations for $G_2(\mathbb{R}^4)$.

a. Show that if $V \in G_2(\mathbb{R}^4)$ is a 2-plane then the Plücker coordinates $p_{ij} = p_{ij}(V)$ satisfy

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

By locality, it suffices to check this equality on every U_{ij}, and by symmetry it suffices to check only U_{12}. Expanding at some matrix $(1, 0, a, c), (0, 1, b, d)$, we get

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = (1)(ad - bc) - b(-c) + (d)(-a) = 0.$$

This completes the proof.

b. Let $Z \subset \mathbb{RP}^5$ be the set of solutions to the above equation. Show that every element of Z is of the form $p(V)$ for a 2-plane $V \in G_2(\mathbb{R}^4)$.

Could I do a make up?

c. We've now shown that the Plücker embedding is a homeomorphism of $G_2(\mathbb{R}^4)$ with the space Z of solutions to the Plücker equation. We know from the above that $\mathbb{RP}^5$ is a smooth manifold. Show that $Z \subset \mathbb{RP}^5$ is a smooth submanifold of dimension 4. This gives another construction of $G_2(\mathbb{R}^4)$ as a smooth manifold.

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d. Is the map $p : G_2(\mathbb{R}^4) \to \mathbb{Z}$ smooth, if we give $G_2(\mathbb{R}^4)$ the smooth structure defined in Section 5.2?

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