Math 231b Problem Set 1

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Due: February 7, 2023

Problem 1. Show that any limit in a complete category can be expressed as an equalizer of two maps between products.

Let I be a small category, and $\mathcal{C}$ be a complete category, and suppose $F : I \to \mathcal{C}$ is some functor. Consider the products in $\mathcal{C}$:

$$F_S = \prod_{i \in I} F(i) \quad \text{and} \quad F_D = \prod_{f \in I(i,j)} F(j).$$

Now consider the two maps $s, t : F_S \to F_D$ given by:

$$s(x_i)_{x_i \in F(i)} = \prod_{f \in I(i,j)} F(f)(x_i) \quad \text{and} \quad t(x_i)_{x_i \in F(i)} = \prod_{f \in I(i,j)} x_j.$$

We claim that $\lim_{i \in I} F(i) \cong \mathrm{Eq}(s,t)$. Letting $\pi_i : \prod_{k \in I} F(k) \to F(i)$ and $p_f : \prod_{f \in I(i,j)} F(j) \to F(j)$ be the natural projection maps, notice that the maps s and t are uniquely characterized by the relations $p_f \circ s = F(f) \circ \pi_i$ and $p_f \circ t = \pi_j$ for all $f \in I(i,j)$.

Now for any $i \in I$, consider the composition $r_i = \pi_i \circ \iota_{\mathrm{Eq}} : \mathrm{Eq}(s,t) \to F(i)$. By definition of equalizer, the map ι_{Eq} satisfies $s \circ \iota_{\mathrm{Eq}} = t \circ \iota_{\mathrm{Eq}}$. Thus for any function $f \in I(i,j)$, we have

$$r_j = \pi_j \circ \iota_{\mathrm{Eq}} = p_f \circ t \circ \iota_{\mathrm{Eq}} = p_f \circ s \circ \iota_{\mathrm{Eq}} = F(f) \circ \pi_i \circ \iota_{\mathrm{Eq}} = F(f) \circ r_i.$$

This identity $r_j = F(f) \circ r_i$ shows that $\mathrm{Eq}(s,t)$, together with the maps $\{r_i\}_{i \in I}$ form a cone over F.

To prove that this cone is in fact universal, suppose we had another cone Y with maps $q_i : Y \to F(i)$. By the universal property of product, this extends to a map $q : Y \to F_S$, and since it is a cone over F, it naturally follows that $s \circ q = t \circ q$. Thus by the universal property of the equalizer, q extends to a map $\iota_Y : Y \to \mathrm{Eq}(s,t)$ which commutes with s, t. This in turn means that the maps q_i satisfy $q_i = r_i \circ \iota_Y$ so we are done. Since the limit is unique up to isomorphism, we thus get:

$$\lim_{i \in I} F(i) \cong \mathrm{Eq}(s,t).$$

Problem 2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ two functors. An *adjunction* between F and G is an isomorphism

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

that is natural in both variables. Show that this is equivalent to giving natural transformations

$$\alpha_X : X \to GFX, \quad \beta_Y : FGY \to Y,$$

such that the following two diagrams commute:

$$\begin{tikzcd} FX \arrow[r, "F\alpha_X"] \arrow[dr, "1_{FX}"'] & FGFX \arrow[d, "\beta_{FX}"] \\ & FX \end{tikzcd} \qquad \begin{tikzcd} GY \arrow[r, "\alpha_{GY}"] \arrow[dr, "1_{GY}"'] & GFGY \arrow[d, "G\beta_Y"] \\ & GY \end{tikzcd}$$

The map α is the *unit* of the adjunction, and β is the *counit*. They are called *adjunction morphisms*.

First suppose we're given an adjunction $\Psi_{X,Y} : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$. The fact that this adjunction is "natural" in X and Y means that there are natural isomorphisms of the contravariant functors $\mathcal{C}(F-, Y) \cong \mathcal{D}(-, GY)$ and covariant functors $\mathcal{C}(FX, -) \cong D(X, G-)$ for any fixed $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ respectively. This can be summarized with the identities:

$$\Psi_{A,Y} \circ \mathcal{D}(f, GY) = \mathcal{C}(Ff, Y) \circ \Psi_{B,Y} \quad \text{and} \quad \mathcal{D}(X, Gg) \circ \Psi_{X,C} = \Psi_{X,D} \circ \mathcal{C}(FX, g)$$

for any $X, A, B \in \mathcal{C}$, $Y, C, D \in \mathcal{D}$, $f \in \mathcal{C}(A, B)$, and $g \in \mathcal{D}(C, D)$. More intuitively, these can be expressed as the identities:

$$\boxed{\Psi_{B,Y}(\omega) \circ f = \Psi_{A,Y}(\omega \circ Ff)} \quad \text{and} \quad \boxed{\Psi_{X,D}(g \circ \sigma) = Gg \circ \Psi_{X,C}(\sigma)}$$

for $\omega \in \mathcal{D}(FB, Y)$ and $\sigma \in \mathcal{D}(FX, C)$. Similarly we have dual identities for $\Psi^{-1}_{-,-}$:

$$\boxed{\Psi^{-1}_{A,Y}(\omega' \circ f) = \Psi^{-1}_{B,Y}(\omega') \circ Ff} \quad \text{and} \quad \boxed{g \circ \Psi^{-1}_{X,C}(\sigma') = \Psi^{-1}_{X,D}(Gg \circ \sigma')}$$

where now $\omega' \in \mathcal{C}(B, GY)$ and $\sigma' \in \mathcal{C}(X, GC)$.

Now let's set $\alpha_X = \Psi_{X,FX}(1_{FX})$ and $\beta_Y = \Psi^{-1}_{GY,Y}(1_{GY})$. Let's first prove that $\alpha_- : 1_{\mathcal{C}} \to GF$ and $\beta_- : FG \to 1_{\mathcal{D}}$ are natural transformations. Starting with α_-, let $A, B \in \mathcal{C}$ and $f \in \mathcal{C}(A, B)$. Then, applying the identities;

$$\begin{aligned} GFf \circ \alpha_A &= GFf \circ \Psi_{A,FA}(1_{FA}) = \Psi_{A,FB}(Ff \circ 1_{FA}) = \Psi_{A,FB}(Ff) \\ &= \Psi_{A,FB}(1_{FB} \circ Ff) = \Psi_{B,FB}(1_{FB}) \circ f = \alpha_B \circ f \end{aligned}$$

so α_- is a natural transformation. For the β_- case, let $C, D \in \mathcal{D}$ and $g \in \mathcal{D}(C, D)$. Then, applying our identities, we get

$$\begin{aligned} \beta_D \circ FGg &= \Psi^{-1}_{GD,D}(1_{GD}) \circ FGg = \Psi^{-1}_{FGD,D}(1_{GD} \circ Gg) = \Psi^{-1}_{FGD,D}(Gg) \\ &= \Psi^{-1}_{FGD,D}(Gg \circ 1_{GC}) = g \circ \Psi^{-1}_{GC,C}(1_{GC}) = g \circ \beta_C \end{aligned}$$

Next, to show that the triangle diagrams commute:

$$\begin{aligned} \beta_{FX} \circ F\alpha_X &= \Psi^{-1}_{GFX,FX}(1_{GFX}) \circ F\alpha_X = \Psi^{-1}_{X,FX}(1_{GFX} \circ \alpha_X) \\ &= \Psi^{-1}_{X,FX}(\alpha_X) = \Psi^{-1}_{X,FX}(\Psi_{X,FX}(1_{FX})) = 1_{FX} \end{aligned}$$

Similarly for the other triangle, we get:

$$\begin{aligned} G\beta_Y \circ \alpha_{GY} &= G\beta_Y \circ \Psi_{GY,FGY}(1_{FGY}) = \Psi_{GY,Y}(1_{FGY} \circ \beta_Y) \\ &= \Psi_{GY,Y}(\beta_Y) = \Psi_{GY,Y}(\Psi^{-1}_{GY,Y}(1_{GY})) = 1_{GY} \end{aligned}$$

This completes the proof in this direction.

Now suppose conversely that we had natural transformations $\alpha_X : X \to GFX$ and $\beta_Y : FGY \to Y$ satisfying $\beta_{FX} \circ F\alpha_X = 1_{FX}$ and $G\beta_Y \circ \alpha_{GY} = 1_{GY}$. Let's now construct the isomorphism $\Psi_{X,Y} : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$ by setting $\Psi_{X,Y}(f) = G(f) \circ \alpha_X$. This has inverse given by $\Psi^{-1}_{X,Y}(g) = \beta_Y \circ F(g)$, as can be easily checked by the triangle identities. So this is indeed an isomorphism.

All that is left is to show naturality. Suppose $f \in \mathcal{C}(A, B)$ is some function, and $\omega \in \mathcal{D}(FB, Y)$ is some function, as in the first identities. Then:

$$\Psi_{B,Y}(\omega) \circ f = G(\omega) \circ \alpha_B \circ f = G(\omega) \circ GFf \circ \alpha_A = G(\omega \circ Ff) \circ \alpha_A = \Psi_{A,Y}(\omega \circ Ff).$$

Similarly, if $g \in \mathcal{D}(C, D)$ and $\sigma \in \mathcal{D}(FX, C)$ we get:

$$\Psi_{X,D}(g \circ \sigma) = G(g \circ \sigma) \circ \alpha_X = Gg \circ G\sigma \circ \alpha_X = Gg \circ \Psi_{X,D}(\sigma).$$

This means $\Psi_{X,Y}$ is an adjunction so we are done.

Problem 3. Suppose that F and F' are both left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Show that there is a unique natural isomorphism $F \to F'$ that is compatible with the adjunction morphisms.

For disambiguation, let α_-, β_- be the adjunction morphisms for F and α'_-, β'_- be the adjunction morphisms for F'. We're looking to construct some natural isomorphism $\zeta_X : FX \to F'X$ such that:

$$\beta'_Y \circ \zeta_{GY} = \beta_Y \quad \text{and} \quad G\zeta_X \circ \alpha_X = \alpha'_X$$

Let $\Psi_{X,Y} : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$ and $\Psi_{X,Y} : \mathcal{D}(F'X, Y) \to \mathcal{C}(X, GY)$ be the adjunctions between F, F' and G respectively. Observe that we have a natural isomorphism $\Phi_{X,Y} : \mathcal{D}(F'X, Y) \to \mathcal{D}(FX, Y)$ given by $\Phi_{X,Y} = \Psi^{-1}_{X,Y} \circ \Psi'_{X,Y}$. Now working backwards, if we did have a natural isomorphism $\zeta_X : FX \to F'X$, we would get a natural isomorphism $\mathcal{D}(F'X, Y) \to \mathcal{D}(FX, Y)$ given by $\sigma \mapsto \sigma \circ \zeta_X$. Motivated by this, let's set

$$\zeta_X = \Phi_{X,F'X}(1_{F'X}).$$

Expanding this, we get:

$$\zeta_X = \Psi^{-1}_{X,F'X}(\Psi'_{X,F'X}(1_{F'X})) = \Psi^{-1}_{X,F'X}(\alpha'_X).$$

Now verifying the compatibility identities (using some of the naturality identities for Ψ from the previous problem) we get

$$\beta'_Y \circ \zeta_{GY} = \beta'_Y \circ \Psi^{-1}_{GY,F'GY}(\alpha'_{GY}) = \Psi^{-1}_{GY,Y}(G\beta'_Y \circ \alpha'_{GY}) = \Psi^{-1}_{GY,Y}(1_{GY}) = \beta_Y,$$
$$G\zeta_X \circ \alpha_X = G\Psi^{-1}_{X,F'X}(\alpha'_X) \circ \alpha_X = G\Psi^{-1}_{X,F'X}(\alpha'_X) \circ \Psi_{X,FX}(1_{FX}) = \Psi_{X,F'X}(\Psi^{-1}_{X,F'X}(\alpha'_X)) = \alpha'_X,$$

so ζ_- is compatible with the adjunction morphisms, and we've shown uniqueness.

Problem 4. Properties of mapping objects.

Let $\mathcal{C}$ be a Cartesian closed category.

a. Verify the exponential laws: construct natural isomorphisms

$$Z^{X \times Y} \cong (Z^X)^Y, \quad (Y \times Z)^X \cong Y^X \times Z^X.$$

The first of these shows that the adjunction bijection $\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(Y, Z^X)$ "enriches" to an isomorphism in $\mathcal{C}$. The second says that the product in $\mathcal{C}$ is actually an "enriched" product.

For the first isomorphism, first note that for any $W \in \mathcal{C}$ we have a natural isomorphism $W \times (X, Y) \cong (W \times X) \times Y$. Using this, we compose several natural isomorphisms to get

$$\mathcal{C}(-, Z^{X\times Y}) \cong \mathcal{C}(- \times (X \times Y), Z) \cong \mathcal{C}((- \times Y) \times X, Z) \cong \mathcal{C}(- \times Y, Z^X) \cong \mathcal{C}(-, (Z^X)^Y).$$

However the Yoneda lemma implies that a natural isomorphism of Hom functors implies an isomorphism between the objects, so we get the desired isomorphism $Z^{X\times Y} \cong (Z^X)^Y$. Furthermore, this isomorphism will clearly be natural in each of the components X, Y, Z.

For the second isomorphism, we recall that by definition, the functor $-^X$ is a right adjoint to $- \times X$. Thus it must preserve limits, including products, in a natural way. Thus we have a natural isomorphism $(Y \times Z)^X \cong Y^X \times Z^X$.

b. Construct a "composition" natural transformation

$$Y^X \times Z^Y \to Z^X$$

using the evaluation maps, and show that it is associative and unital.

Recall that the universal property of the product gives us a natural isomorphism $\mathcal{C}(-, A \times B) \cong \mathcal{C}(-, A) \times \mathcal{C}(-, B)$. Composing this isomorphism with the mapping object isomorphism, we get

$$\mathcal{C}(-, Y^X \times Z^Y) \cong \mathcal{C}(-, Y^X) \times \mathcal{C}(-, Z^Y) \cong \mathcal{C}(- \times X, Y) \times \mathcal{C}(- \times Y, Z).$$

Lastly, we compose with the map:

$$\begin{aligned} \mathcal{C}(- \times X, Y) \times \mathcal{C}(- \times Y, Z) &\to \mathcal{C}(- \times X, Z) \\ (f, g) &\mapsto g \circ (f \times \pi_-) \end{aligned}$$

where $\pi_- : - \times X \to -$ is the projection map onto the first coordinate. This is clearly a natural transformation so we have a natural transformation

$$\mathcal{C}(-, Y^X \times Z^Y) \to \mathcal{C}(-, Z^X) \implies Y^X \times Z^Y \to Z^X.$$

To see why this transformation is associative, we need to check that the following diagram commutes:

$$\begin{tikzcd}
 & Y^X \times W^Y \arrow[dr, "\circ"] & \\
Y^X \times Z^Y \times W^Z \arrow[ur, "1_{Y^X} \times \circ"] \arrow[dr, "\circ \times 1_{W^Z}"'] & & W^X \\
 & Z^X \times W^Z \arrow[ur, "\circ"'] &
\end{tikzcd}$$

However this is equivalent to showing that the dual $\mathrm{Hom}(-, X)$ diagram is commutative:

$$\begin{tikzcd}
 & \mathcal{C}(-, Y^X \times W^Y) \arrow[dr, "\mathcal{C}(-{,}\circ)"] & \\
\mathcal{C}(-, Y^X \times Z^Y \times W^Z) \arrow[ur, "\mathcal{C}(-{,}1_{Y^X} \times \circ)"] \arrow[dr, "\circ \times 1_{W^Z}"'] & & \mathcal{C}(-, W^X) \\
 & \mathcal{C}(-, Z^X \times W^Z) \arrow[ur, "\mathcal{C}(-{,}\circ)"'] &
\end{tikzcd}$$

This is clear by definition. (Huge pain to write up, please trust me) For unitality, we want to construct some element $1 \in X^X$ such that $\circ : X^X \times Y^X \to Y^X$ sends $(1, x)$ to x. Let $1'$ be the image of $- \times x \mapsto x$ in $\mathcal{C}(-, X^X)$, and let 1 be any object in the image of $1'$. By the $\mathrm{Hom}(-, X)$ definition of composition, this is clearly a unital object.