Math 55b Problem Set 3

Lev Kruglyak

February 16, 2022

Problem 1. Let X be a metric space d; let $A \subset X$ be nonempty.

(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define an ϵ-neighborhood of A in X to be the set

$$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.

(e) Show the result in (d) need not hold if A is closed but not compact.

(a) Let $B = \{x \in X \mid d(x, A) = 0\}$. Clearly, $A \subset B$, since $d(a, A) = 0$ for all $a \in A$. Note that $B = d(\cdot, A)^{-1}(\{0\})$ is the inverse image of a closed set under a continuous function, B is closed. Since B is a closed set containing A, by definition of closure it follows that $\overline{A} \subset B$. To prove the other direction, suppose for the sake of contradiction that $x \in B$ yet $x \notin \overline{A}$. Then there exists some $B_\epsilon(x)$ with $B_\epsilon(x) \cap A = \varnothing$, so $d(x, a) > \epsilon$ for all $a \in A$. This is a contradiction because this implies that $d(x, A) \geq \epsilon$, yet we assumed that $d(x, A) = 0$. Thus $B \subset \overline{A}$ so $\overline{A} = B$.

(b) Fix some $x \in X$. Since A is compact, the function $d(x, \cdot) : A \to \mathbb{R}_{\geq 0}$ must achieve its minimum, say $d(x, a)$. Then clearly $d(x, A) = \inf_{a \in A} d(x, a) = d(x, a)$.

(c) First, we'll show that $U(A, \epsilon) \subset \bigcup_{a \in A} B_\epsilon(a)$. Let $x \in U(A, \epsilon)$. Then $d(x, A) < \epsilon$ so there must be some $a \in A$ such that $d(x, a) < \epsilon$. Then $x \in B_\epsilon(a) \subset \bigcup_{a \in A} B_\epsilon(a)$. Conversely, suppose $x \in B_\epsilon(a)$ means that $d(x, a) < \epsilon$ so $d(x, A) \leq d(x, a) < \epsilon$ so $x \in U(A, \epsilon)$. So $U(A, \epsilon) = \bigcup_{a \in A} B_\epsilon(a)$.

(d) Consider the continuous function $d(\cdot, X - U) : X \to \mathbb{R}_{\geq 0}$. Since $A \subset U$ and $X - U$ is a closed set, it follows that $d(x, X - U) > 0$ for all $x \in A$. Since A is compact, $d(\cdot, X - U)$ must achieve its minimum on A, say $a \in A$ with $d(a, X - U) = \epsilon$ minimal. Then clearly an ϵ-neighborhood of A is contained in U.

(e) Consider the closed but not compact set $A = \bigcup_{n=0}^{\infty} \left[n, n + \frac{1}{2}\right] \subset \mathbb{R}$. For any $\alpha < \frac{1}{2}$ the open set $U = \bigcup_{n=0}^{\infty} \left(n - \alpha^n, n + \frac{1}{2} + \alpha^n\right)$ clearly contains A, yet no ϵ-neighborhood of A is contained in U because for any $\epsilon > 0$, there will always be an n such that $\left(n - \alpha^n, n + \frac{1}{2} + \alpha^n\right) \subset U(A, \epsilon)$.

Problem 2. Let (X, d) be a metric space. A map $f : X \to X$ is a *shrinking map* if, $\forall x, y \in X$, $x \neq y \Rightarrow d(f(x), f(y)) < d(x, y)$. There is also a slightly stronger notion: f is a *contraction* if there exists a real number $\alpha < 1$ such that, for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha\, d(x, y)$. Finally, we say a point $p \in X$ is a *fixed point* of f if $f(p) = p$.

(a) Show that shrinking maps are continuous.

(b) Show that, if (X, d) is complete, then every contraction has a unique fixed point. Show that the result is in general false for shrinking maps.

(c) Show that, if (X, d) is compact, then every shrinking map has a unique fixed point.

(a) Let $p \in X$ be some arbitrary point, and $\epsilon > 0$. Then setting $\delta = \epsilon$, it follows that for any $x \in X$, whenever $d(p, x) < \epsilon$ we have $d(f(p), f(x)) < d(p, x) < \epsilon$, so the function is continuous at p. Since p was arbitrary, it is a continuous function. This implies that every contraction is also continuous because every contraction is a shrinking map.

(b) Pick some arbitrary $x \in X$, and consider the sequence $\{f^n(x)\}$. Note that by a simple repeated application of the contraction rule, we get for any $n \geq 1$

$$d(f^{n+1}(x), f^n(x)) \leq \alpha^n d(f(x), x).$$

To find a general bound for $d(f^m(x), f^n(x))$ for $m > n$, observe that by the triangle inequality

$$\begin{aligned} d(f^m(x), f^n(x)) &\leq (\alpha^m + \alpha^{m-1} + \cdots + \alpha^n) d(f(x), x) \\ &\leq \left(\frac{\alpha^n}{1-\alpha}\right) d(f(x), x). \end{aligned}$$

So for any $N > 0$, it follows that

$$d(f^m(x), f^n(x)) \leq \left(\frac{\alpha^n}{1-\alpha}\right) d(f(x), x) \leq \left(\frac{\alpha^N}{1-\alpha}\right) d(f(x), x).$$

Since $d(f(x), x)$ is constant and $\frac{\alpha^N}{1-\alpha}$ gets arbitrarily small, it follows that $\{f^n(x)\}$ is a Cauchy sequence. Since X is a complete space, it must converge to some $L \in X$. Then $f(L) = \lim_n f^{n+1}(x) = L$ so L is a fixed point of f.

Next we'll show uniqueness. Suppose L_1 and L_2 are fixed points of f, so $f(L_1) = L_1$ and $f(L_2) = L_2$. Then $d(f(L_1), f(L_2)) \leq \alpha d(L_1, L_2)$, so $d(L_1, L_2) \leq \alpha d(L_1, L_2)$. This is impossible unless $d(L_1, L_2) = 0$ and $L_1 = L_2$. So fixed points are unique.

A counterexample to show that this is false for general shrinking maps: Consider the complete metric space $[0, \infty)$ with the Euclidean metric and the map $f(x) = x + e^{-x}$. Then if $x, y \in [0, \infty)$ with $y > x$ we have $y - x > y + e^{-y} - x - e^{-x} = (y - x) + (e^{-y} - e^{-x})$ since $e^{-y} - e^{-x} < 0$. So this is a shrinking map, yet it has no fixed point because if $x = x + e^{-x}$, then $e^{-x} = 0$ which is impossible.

(c) Let $A = \bigcap_{n>0} f^n(X)$. A is clearly nonempty and compact because it is an intersection of an infinite chain of descending compact sets. (Note that $f^n(X) \subset f^{n-1}(X)$)

Next we claim that $A = f(A)$. In one direction $A \subset f(A)$ because $f(A) = \bigcap_{n>0} f^{n+1}(X)$ which contains $A = \bigcap_{n>0} f^n(X)$. Conversely, suppose $x \in A$. Then by definition of A, there exists x_n such that $f^{n+1}(x_n) = x$. Now consider the limit $y \in A$ of a convergent subsequence of $f^n(x_n)$. Then $f(y) = x$, so $x \in f(A)$.

Now since A is nonempty, assume for the sake of contradiction that A contains at least two elements. Since A is compact it must have a diameter which is achieved by two points, say $a, b \in A$ with $\operatorname{diam}(A) = d(a, b)$. However since $f(A) = A$ we can find $f^{-1}(a), f^{-1}(b) \in A$ with $d(f^{-1}(a), f^{-1}(b)) > d(a, b)$, a contradiction to the maximality of the diameter. So A must consist of a single point $a \in A$ with $f(a) = a$.

To prove uniqueness, suppose $a, b \in X$ are two fixed points $f(a) = a, f(b) = b$. Then $d(a, b) > d(f(a), f(b)) = d(a, b)$, a contradiction. So $a = b$, and we have a unique fixed point.

Problem 3.

(a) Let $\mathcal{B}$ be the space of bounded continuous functions from $\mathbb{R}$ to itself, equipped with the uniform metric $d(f_1, f_2) = \sup_{x \in \mathbb{R}} |f_1(x) - f_2(x)|$. Show that the composition map,

$$\begin{aligned} \mathcal{B} \times \mathcal{B} &\to \mathcal{B} \\ (f, g) &\mapsto f \circ g, \end{aligned}$$

is continuous.

(b) Does the result remain true if do not restrict ourselves to bounded functions? Namely: denoting by $\mathcal{C}$ the space of all continuous functions from $\mathbb{R}$ to itself, with the uniform topology, does $(f, g) \mapsto f \circ g$ define a continuous map from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$?

(a) Fix some $(f, g) \in \mathcal{B} \times \mathcal{B}$ and let $\epsilon > 0$. g is bounded, so its range fits into some compact interval $C = [-L, L] \subset \mathbb{R}$. By the uniform convergence theorem, f must be uniformly continuous on C, so there exists a $\delta > 0$ such that for all $x, y \in C$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon/2$. Consider the open neighborhood $U = B_{\epsilon/2}(f) \times B_\delta(g)$ of (f, g) in $\mathcal{B} \times \mathcal{B}$. For any $(\alpha, \beta) \in U$ and $x \in \mathbb{R}$ we have $|f(x) - \alpha(x)| \leq d(f, \alpha) < \epsilon/2$ and $|g(x) - \beta(x)| \leq d(g, \beta) < \delta$. By uniform continuity, this implies that $|f(g(x)) - f(\beta(x))| < \epsilon/2$. So $d(f \circ g, \alpha \circ \beta)$ is the supremum of

$$|f(g(x)) - f(\beta(x)) + f(\beta(x)) - \alpha(\beta(x))| \leq |f(g(x)) - f(\beta(x))| + |f(\beta(x)) - \alpha(\beta(x))|.$$

However $|f(g(x)) - f(\beta(x)) + f(\beta(x)) - \alpha(\beta(x))| \leq \epsilon/2$ and $|f(\beta(x)) - \alpha(\beta(x))| \leq \epsilon/2$ so $d(f \circ g, \alpha \circ \beta) < \epsilon$, and so the composition map is continuous.

(b) No. Let $f = e^x$ and $g = x$. Suppose for the sake of contradiction that composition is continuous at (e^x, x). Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d((f, g), (\alpha, \beta)) < \delta$ implies that $d(f \circ g, \alpha \circ \beta) < \epsilon$. Let $\alpha = e^x$ and $\beta = x + \frac{\delta}{2}$. Clearly $d((f, g), (\alpha, \beta)) = \frac{\delta}{2} < \delta$. Then $d(f \circ g, \alpha \circ \beta) = d(e^x, e^{x + \delta/2}) = d(e^x, e^{\delta/2} e^x) = 1 > \epsilon$. This is a contradiction so composition is not continuous at (e^x, x).

Problem 4. Show that the rationals $\mathbb{Q}$ are not locally compact.

Suppose for the sake of contradiction that $\mathbb{Q}$ is locally compact. Then there is a compact set C containing $U = (a, b) \cap \mathbb{Q}$. Let $r \in (a, b) \cap (\mathbb{R} - \mathbb{Q})$ be irrational, and suppose r_i is a sequence in U converging to r. This clearly has no convergent subsequence in $\mathbb{Q}$, which violates the compactness of C. So $\mathbb{Q}$ cannot be locally compact.

Problem 5. Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with subspace $\{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

Consider the map $f : \widehat{\mathbb{Z}}_+ \to \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ given by $f(t) = \frac{1}{t}$ if $t \neq \infty$ and $f(\infty) = 0$. To prove it's continuous suppose $U \subset \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ is an open interval (intersected in the subspace topology). There are two cases. If $0 \in U$, then $U = \{0\} \cup \{\frac{1}{n}, \frac{1}{n+1}, \ldots\}$ for some $n > 0$. Then $f^{-1}(U) = \{\infty\} \cup (\mathbb{Z}_+ - [0, n-1])$, which is open in $\widehat{\mathbb{Z}}_+$. Otherwise if $0 \notin U$ and U is nonempty, then $U = \{\frac{1}{a}, \ldots, \frac{1}{b}\}$. Then $f^{-1}(U) = [b, a]$ which is open in $\mathbb{Z}_+$ and so in $\widehat{\mathbb{Z}}_+$. Proving that the inverse is continuous follows in basically the same exact way, so the map is a homemorphism.

Problem 6. Show that every compact metrizable space X has a countable basis.

For each $n \in \mathbb{Z}_+$, consider the open cover of X given by $\bigcup_{x \in X} B_{1/n}(x)$. Since X is compact this has a finite subcover, let's call it $\mathcal{B}_n$. We claim that $\mathcal{B} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{B}_n$ is a countable basis for X. There are two axioms we must check to ensure that this is a basis. Clearly $\mathcal{B}$ is an open cover for X, since every $\mathcal{B}_n$ is an open cover for X. The second axiom is harder to check. Suppose we had two balls $B_{1/n}(x)$ and $B_{1/m}(y)$ which intersect nontrivially, so there exists some point $z \in B_{1/n}(x) \cap B_{1/m}(y)$. Let $\epsilon > 0$ be some ϵ such that $B_\epsilon(z) \subset B_{1/n}(x) \cap B_{1/m}(y)$. Pick some $k \in \mathbb{Z}_+$ such that $1/k < \epsilon/2$ and let $B_{1/k}(z_0) \in \mathcal{B}$ be a basis element containing z. Then by basic geometry, $B_{1/k}(z_0) \subset B_\epsilon(z)$ so $B_{1/k}(z_0) \subset B_{1/n}(x) \cap B_{1/m}(y)$. This completes the proof.

Problem 7. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Recall from Munkres that X is normal if and only if for every closed set $A \subset X$ and open set $U \supset A$, there exists an open set $V \supset A$ such that $\overline{V} \subset U$. So given two disjoint closed sets $A, B \subset X$, we can first use the normal property to find open $U_1 \supset A, U_2 \supset B$ with $U_1 \cap U_2 = \varnothing$ and then applying the equivalent normal property we can find $V_1 \supset A$ and $V_2 \supset B$ with $V_1 \cap V_2 = \varnothing$ and $\overline{V_1} \subset U_1, \overline{V_2} \subset U_2$. Then clearly $\overline{V_1} \cap \overline{V_2} = \varnothing$, so we are done.