MATH 231A: ALGEBRAIC TOPOLOGY
HOMEWORK 9
DUE: WEDNESDAY, NOVEMBER 9 AT 10:00PM ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Size of mod p homology vs integral homology (5 points)

Do Exercise 24.3 of LAT. Note that "Betti numbers" and "torsion coefficients" are defined on the bottom half of page 54 of LAT.

2. Problem 2: Chain homotopy (10 points)

Let I_* denote the following chain complex of abelian groups:

$$\cdots \to 0 \to \mathbb{Z}\{f\} \xrightarrow{f \mapsto e_1 - e_0} \mathbb{Z}\{e_0, e_1\} \to 0 \to \dots,$$

where f lies in degree 1 and e_0, e_1 lie in degree 0. Regarding $\mathbb{Z}$ as a chain complex concentrated in degree 0, there are chain maps $e_0, e_1 : \mathbb{Z} \to I_*$ given by sending 1 to e_0, e_1. Prove that there is a natural bijection between chain homotopies $f_0 \simeq f_1 : C_* \to D_*$ and chain maps

$$I_* \otimes_{\mathbb{Z}} C_* \to D_*$$

for which the compositions

$$C_* \cong \mathbb{Z} \otimes_{\mathbb{Z}} C_* \xrightarrow{e_i \otimes \mathrm{id}_{C_*}} I_* \otimes_{\mathbb{Z}} C_* \to D_*$$

are equal to f_i.

Remark: This defintion is very similar to the definition of homotopy of continuous maps! It also gives rise to a rather straightforward way to phrase the proof of homotopy invariance of homology: I_* may be viewed as the semisimplicial chains on the usual semisimplicial structure on I. This gives rise to a chain inclusion $I_* \hookrightarrow S_*(I)$ given explicitly by sending e_i to c_i^0 and f to a (correctly oriented) identification $\Delta^1 \xrightarrow{\cong} I$. Then, given a homotopy $I \times X \to Y$, the composition

$$I_* \otimes_{\mathbb{Z}} S_*(X) \hookrightarrow S_*(I) \otimes_{\mathbb{Z}} S_*(X) \xrightarrow{\times} S_*(I \times X) \to S_*(Y)$$

is the corresponding chain homotopy.

3. Problem 3: Relative Künneth formula (20 points)

The goal of this problem is to prove the relative Künneth formula, whose statement follows. Let X, Y denote spaces and suppose that $A \subset X$ and $B \subset Y$ are open subspaces.[1] Moreover, let R denote a PID. Then there is a natural short exact sequence

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} H_p(X, A; R) \otimes_R H_q(Y, B; R) &\to H_n(X \times Y, A \times Y \cup X \times B; R) \\ \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X, A; R), H_q(Y, B, R)) &\to 0 \end{aligned}$$

which is non-naturally split.

[1] The theorem also holds in more general circumstances, for example when A and B are subcomplexes of cell structures on CW complexes X and Y. The precise condition we need is that homology should satisfy locality in the sense of lecture 11 of LAT with respect to the cover $\{A \times Y, X \times B\}$ of $A \times Y \cup X \times B$.

To prove this, it suffices to prove the relative Eilenberg–Zilber theorem: there is a natural chain map
$$S_*(X \times Y, A \times Y \cup X \times B) \to S_*(X, A) \otimes_{\mathbb{Z}} S_*(Y, B)$$
which induces an isomorphism on homology.

(1) Consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & S_*(A) \otimes_{\mathbb{Z}} S_*(Y) + S_*(X) \otimes_{\mathbb{Z}} S_*(B) \arrow[r] & S_*(X) \otimes_{\mathbb{Z}} S_*(Y) \arrow[r] \arrow[d, shift right] & S_*(X, A) \otimes_{\mathbb{Z}} S_*(Y, B) \arrow[r] & 0 \\ 0 \arrow[r] & S_*^{\mathcal{A}}(A \times Y \cup X \times B) \arrow[r] & S_*(X \times Y) \arrow[r] \arrow[u, shift right] & S_*(X \times Y)/S_*^{\mathcal{A}}(A \times Y \cup X \times B) \arrow[r] & 0, \end{tikzcd}$$

where $\mathcal{A} = \{A \times Y, X \times B\}$ and the vertical maps are inverse chain homotopy equivalences coming from the Eilenberg–Zilber theorem. Note that $S_*^{\mathcal{A}}(A \times Y \cup X \times B)$ is defined as in Lecture 11 of LAT, specifically the middle of page 32. Prove that the top row is exact. (The bottom row is exact by defintion.) (Hint: you will need to use the fact that $S_*(X, A)$ and $S_*(Y, B)$ are chain complexes of flat $\mathbb{Z}$-modules.)

(2) Using the naturality of the chain maps in the Eilenberg–Zilber theorem, prove that this extends to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S_*(A) \otimes_{\mathbb{Z}} S_*(Y) + S_*(X) \otimes_{\mathbb{Z}} S_*(B) \arrow[r] \arrow[d, shift right] & S_*(X) \otimes_{\mathbb{Z}} S_*(Y) \arrow[r] \arrow[d, shift right] & S_*(X, A) \otimes_{\mathbb{Z}} S_*(Y, B) \arrow[r] \arrow[d, shift right] & 0 \\ 0 \arrow[r] & S_*^{\mathcal{A}}(A \times Y \cup X \times B) \arrow[r] \arrow[u, shift right] & S_*(X \times Y) \arrow[r] \arrow[u, shift right] & S_*(X \times Y)/S_*^{\mathcal{A}}(A \times Y \cup X \times B) \arrow[r] \arrow[u, shift right] & 0. \end{tikzcd}$$

(Hint: you just need to check that the middle vertical maps preserve the subcomplexes $S_*(A) \otimes_{\mathbb{Z}} S_*(Y) + S_*(X) \otimes_{\mathbb{Z}} S_*(B)$ and $S_*^{\mathcal{A}}(A \times Y \cup X \times B)$.)

(3) Using the naturality of the chain homotopies in the Eilenberg–Zilber theorem, prove that the first and third pairs of vertical maps are also inverse chain homotopy equivalences.

(4) Using locality (cf. Lecture 11 of LAT), prove that the natural map
$$S_*(X \times Y)/S_*^{\mathcal{A}}(A \times Y \cup X \times B) \to S_*(X \times Y, A \times Y \cup X \times B)$$
induces an isomorphism on homology. Putting this together with (3), deduce the relative Eilenberg–Zilber theorem.

4. Problem 4: $\mathbb{R}^{2n+1}$ does not admit a square root (15 points)

The goal of this problem is to prove that there is no space X such that $X \times X$ is homeomorphic to $\mathbb{R}^{2n+1}$. Assume that we have a space X satisfying $X \times X \cong \mathbb{R}^{2n+1}$.

(1) Prove that any such space X must be T_1: in other words, for any $x \in X$, the singleton subset $\{x\} \subseteq X$ is closed.

(2) Pick $x \in X$. Using the relative Künneth formula, prove that $H_*(X \times X, X \times X - \{(x, x)\}; \mathbb{Q})$ must be nonzero in some even degree.

(3) Using (2), conclude that there is a contradiction: $X \times X$ cannot be homeomorphic to $\mathbb{R}^{2n+1}$.