Math 55b Problem Set 8

Lev Kruglyak

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I did not collaborate with anyone on this problem set.

Problem 1. Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\int_0^1 x^n\, f(x)\, dx = 0$ for all $n \in \mathbb{N}$, then $f(x) = 0$ on $[0, 1]$.

For clarity, we'll rephrase this problem. Consider the bilinear form defined on $C^0([0, 1])$ by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx.$$

This clearly satisfies the bilinearity condition by elementary properties of the Riemann integral. The condition that $\int_0^1 x^n f(x)\, dx = 0$ then becomes that $\langle p, f\rangle = 0$ for any $p \in \mathbb{R}[x]$. We claim that $\langle f, f\rangle = 0$. Let $\epsilon > 0$. Then by the Weierstrass theorem, there is some polynomial $p \in \mathbb{R}[x]$ with $|f - p| < \epsilon$ on $[0, 1]$. This means that $|\langle f, f\rangle| = |\langle f - p, f\rangle| \leq \langle |f - p|, |f|\rangle < \langle \epsilon, |f|\rangle = \epsilon\|f\|_1$. So $\langle f, f\rangle = 0$ since it can be bound arbitrarily small.

We claim that this implies that $f = 0$. Consider the function

$$F(x) = \int_0^x |f(x)|^2\, dx.$$

Since $|f(x)|^2 \geq 0$, $F(x)$ is an increasing function on $[0, 1]$. Since $F(0) = F(1) = 0$, $F(x)$ vanishes on $[0, 1]$. This means that $F'(x) = 0$ on $[0, 1]$. However by the fundamental theorem of calculus, $F'(x) = |f(x)^2|$, so $f(x) = 0$ on $[0, 1]$.

Problem 2. Let c_n be the Fourier coefficients of a continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, and $k \in \mathbb{N}$ an integer.

(a) Show that if $\sum |c_n|$ is convergent then the Fourier series $\sum c_n e^{inx}$ converges uniformly to f.

(b) Show that if $\sum |n|^k |c_n|$ is convergent then $f \in C^k$.

(c) Conversely, show that if $f \in C^k$ then $\sum n^{2k}|c_n|^2$ converges, and in particular $n^k|c_n| \to 0$.

(d) Deduce Dirichlet's theorem: if $f \in C^1$ then $\sum c_n e^{inx}$ converges uniformly to f.

(a) Suppose $\sum_{k\in\mathbb{Z}} |c_k|$ converges, and let $f_n(x) = \sum_{|k|\leq n} c_k e^{ikx}$ be the partial sum. Since $|c_k e^{ikx}| \leq |c_k|$, the sum $f_\infty(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$ is defined everywhere. By the triangle inequality we have

$$|f_\infty(x) - f_n(x)| = \left|\sum_{|k|>n} c_k e^{ikx}\right| \leq \left|\sum_{|k|>n} c_k\right| \leq \sum_{|k|>n} |c_k|.$$

This right sum can be made arbitrarily small since $\sum_{k\in\mathbb{Z}} |c_k|$ converges, so for every $\epsilon > 0$, there is some N such that whenever $n \geq N$ we have $|f_\infty(x) - f_n(x)| < \epsilon$ for all $x \in \mathbb{R}$. So $\sum c_n e^{inx}$ converge uniformly to f_∞.

Next we claim that $f_\infty(x) = f(x)$ for all $x \in \mathbb{R}$. Since f_∞ is a uniform limit of continuous functions, it must be continuous. (and similarly 2π-periodic). Then notice that

$$\langle f_\infty, e_n \rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k\in\mathbb{Z}} c_k e^{ikx}\right) e^{-inx}\, dx = \sum_0^{2\pi} c_k \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}e^{-inx}\, dx = c_n.$$

So f_∞ and f have the same Fourier coefficients. We claim that $f_\infty - f = 0$. Let b_n be the Fourier coefficients of $f_\infty - f$. Then by Parseval's theorem,

$$0 = \sum_{k\in\mathbb{Z}} |b_k| = \|f_\infty - f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f_\infty(x) - f(x)|^2\, dx.$$

This shows that $f_\infty = f$.

(b) First of all, since $\sum_{n\in\mathbb{Z}} |n|^k|c_n|$ converges, $\sum_{n\in\mathbb{Z}} |c_n|$ must converge by the comparison test, so by (a) we can write $f(x) = \sum_{n\in\mathbb{Z}} c_n e_{inx}$. Now for any $\ell \le k$, consider the function $D_\ell f(x) = \sum_{n\in\mathbb{Z}} (in)^\ell c_n e^{inx}$. This is a continuous function because $\sum_{n\in\mathbb{Z}} |in^\ell c_n| = \sum_{n\in\mathbb{Z}} |n^\ell||c_n| \le \sum_{n\in\mathbb{Z}} |n^k||c_n|$, so by (a), $D_\ell f(x)$ is a uniform limit of continuous functions and hence continuous.

We claim that $D_\ell(x)$ is an ℓ-th derivative of f. Note that as long as $\ell + 1 \le k$, $D_1 D_\ell f = D_{\ell+1} f$ so we can proceed by induction, i.e. it suffices to prove that $D_1 f$ is the first derivative of f. Let $f_N(x) = \sum_{|n|\le N} c_n e^{inx}$ be the partial sum. Note that $f_N'(x) = \sum_{|n|\le N} (in) c_n e^{inx}$. This is exactly a partial sum of $D_1 f$, and by (a) f_N' converge uniformly to $D_1 f$. This meaans that $D_1 f = f'$ since uniform limits preserve differentiation. This completes the proof.

(c) First, suppose $f \in C^1$. Then using integration by parts we have

$$\langle f', e_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-inx}\, dx = \frac{1}{2\pi}\left(f'(x)e^{-inx}\right)\Big|_0^{2\pi} + in\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}\, dx = in\langle f, e_n\rangle.$$

Then by Parseval's theorem, the sum $\sum_{n\in\mathbb{Z}} |inc_n|^2 = \sum_{n\in\mathbb{Z}} n^2|c_n|^2$ converges. By induction, and using the above identity for the coefficients of a derivative, we get that $\sum_{n\in\mathbb{Z}} n^{2k}|c_n|^{2k}$ converges. In particular, $n^{2k}|c_n|^{2k} \to 0$ and so $n^k|c_n|^k \to 0$ as well.

(d) First, we note the inequality $2x \le 1/n^2 + n^2x^2$, which is implied by $(nx - 1/n)^2 = n^2x^2 - 2x + 1/n^2 \ge 0$. Now if $f \in C^1$ then by (c), $\sum_{n\in\mathbb{Z}} n^2|c_n|^2$ converges. Yet $n^2|c_n|^2 \ge 2|c_n| - 1/n^2$ so by the comparison test, $2\sum_{n\in\mathbb{Z}} |c_n| - \sum_{n\in\mathbb{Z}} \frac{1}{n^2}$ converges. This of course means that $\sum_{n\in\mathbb{Z}} |c_n|$ converges, which means that f is a uniform limit of its Fourier series.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be the unique 2π-periodic function such that $f(x) = (x - \pi)^2$ for $x \in [0, 2\pi]$.

(a) Show that the Fourier coefficients of f are $c_n = \dfrac{2}{n^2}$ for $n \ne 0$, and $c_0 = \dfrac{\pi^2}{3}$.

(b) Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

(a) We'll calculate the Fourier coefficients using the projection formula

$$c_n = \langle f, e_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx.$$

For the first coefficient c_0, we calculate

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} (x-\pi)^2\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) = \frac{\pi^2}{3}.$$

For the other coefficients c_n for $|n| \geq 1$, we use integration by parts:

$$\begin{aligned} c_n = \frac{1}{2\pi}\int_0^{2\pi} (x-\pi)^2 e^{-inx}\,dx &= \frac{1}{2\pi}\left(\frac{(x-\pi)^2}{-in} e^{-inx}\Big|_0^{2\pi} + \frac{2}{-in}\int_0^{2\pi} (x-\pi)e^{-inx}\,dx \right) \\ &= \frac{2}{in}\left(\frac{x-\pi}{-in} e^{-inx}\Big|_0^{2\pi} - \frac{1}{in\pi}\int_0^{2\pi} e^{-inx}\,dx \right) \\ &= \frac{1}{in\pi}\left(\frac{2\pi}{-in}\right) = \frac{2}{n^2}. \end{aligned}$$

Note that $\sum_{n\in\mathbb{Z}} |c_n| = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$ which is finite by a simple p-series test. Then by Problem 2a, the Fourier series $\sum c_n e^{inx}$ converges uniformly to f.

(b) We know that $f(0) = \pi^2$, so applying the Fourier series, we get

$$f(0) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{n^2} = \pi^2 \implies \sum_{k=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

For the other identity, we'll use Parseval's theorem to get

$$\begin{aligned} \frac{\pi^4}{9} + \sum_{n=1}^{\infty} \frac{8}{n^4} = \sum_{n\in\mathbb{Z}} |c_n|^2 = \frac{\|f\|^2}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} (x-\pi)^4\,dx = \frac{\pi^4}{5} \\ \implies \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}. \end{aligned}$$

This completes the proof.

Problem 4. Recall that the Dirichlet kernel is $D_n(x) = \sum_{|k|\leq n} e^{ikx} = \dfrac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}}$. The Féjer kernel is defined to be $K_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x)$.

(a) Show that $K_N(x) = \dfrac{(e^{iNx/2} - e^{-iNx/2})^2}{N(e^{ix/2} - e^{-ix/2})^2} = \dfrac{\sin^2(Nx/2)}{N\sin^2(x/2)}$.

(b) Show that K_N approximates identity, in the sense that: (i) $K_N \geq 0$, (ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N\,dx = 1$, and (iii) $\forall \delta > 0$, $\int_{\delta\leq|x|\leq\pi} K_N\,dx \to 0$ (in fact, K_N converges uniformly to 0 on $[-\pi,-\delta]\cup[\delta,\pi]$).

(c) Let f be a continuous 2π-periodic function, and denote by $s_n = \sum_{k=-n}^{n} c_k e^{ikx}$ the partial sums of the Fourier series of f. We consider the arithmetic mean $\sigma_N = \frac{1}{N}(s_0 + \cdots + s_{N-1})$. Show that σ_N is the convolution of f with K_N, in the sense that

$$\sigma_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,K_N(t)\,dt.$$

(d) Deduce Féjer's theorem: for any continuous 2π-periodic function f, the sequence σ_N converges uniformly to f.

(a) By rearranging terms, we have the expression

$$D_n(x) = \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} \cdot \frac{e^{-ix/2}}{e^{-ix/2}} = \frac{e^{-inx} - e^{-i(n+1)x}}{1 - e^{-ix}}.$$

Then we have the relations

$$\sum_{n=0}^{N-1} e^{-inx} = 1 + \frac{1}{e^{inx}} + \cdots + \frac{1}{e^{i(N-1)x}} = \frac{1 - e^{-iNx}}{1 - e^{-ix}} = e^{ix}\left(\frac{e^{-iNx} - 1}{1 - e^{ix}}\right)$$

and

$$\sum_{n=0}^{N-1} e^{i(n+1)x} = e^{ix}(1 + e^{inx} + \cdots + e^{i(N-1)x}) = \frac{1 - e^{-iNx}}{1 - e^{-ix}} = e^{ix}\left(\frac{1 - e^{iNx}}{1 - e^{ix}}\right).$$

Putting this together with the definition of a Féjer kernel, we get

$$K_N(x) = \frac{1}{N}\left(\frac{\sum_{n=0}^{N-1} e^{-inx} + \sum_{n=0}^{N-1} e^{i(n+1)x}}{1 - e^{ix}}\right) = \frac{1}{N}\left(\frac{e^{ix}(e^{iNx} - 2 + e^{-iNx})}{(1 - e^{ix})^2}\right)$$

$$= \frac{1}{N}\left(\frac{e^{iNx/2} - e^{iNx/2}}{e^{ix/2} - e^{-ix/2}}\right)^2 = \frac{\sin^2(Nx/2)}{N\sin^2(x/2)}.$$

(b) Clearly $K_N(x) \geq 0$ since it a ratio of two square functions. To see that the integral is one, we'll compute directly:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x) = \frac{1}{2\pi N}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi} D_n(x)\,dx = \frac{1}{2\pi N}\sum_{n=0}^{N-1}\left(2\pi + \sum_{1\leq|k|\leq N}\int_{-\pi}^{\pi} e^{ikx}\,dx\right)$$

$$= \frac{1}{2\pi N}\sum_{n=0}^{N-1} 2\pi = 1$$

Lastly, let $0 < \delta \leq \pi$. Notice that for any $\delta \leq |x| \leq \pi$ we have $\frac{1}{\sin^2 x} \leq \frac{1}{\sin^2(\delta/2)}$. Thus $0 \leq F_n(x) \leq \frac{1}{n}\left(\frac{1}{\sin^2(\delta/2)}\right)$ since $\sin^2(nx/2) \leq 1$. Then

$$\lim_{n\to\infty}\int_{\delta\leq|x|\leq\pi} K_n(x)\,dx \leq \lim_{n\to\infty}\left(\frac{2(\pi - \delta)}{n\sin^2(\delta/2)}\right) = 0.$$

(c) Recall that we have the following convolution expression for the partial sums s_n:

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x - t)D_n(t)\,dt.$$

Plugging this into the expression for the $\sigma_n(x)$, we get

$$\sigma_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} s_n(x) = \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x - t)D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x - t)\frac{1}{N}\sum_{n=0}^{N-1} D_n(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x - t)K_N(t)\,dt.$$

(d) Using the triangle inequality, (b), and (c), we get

$$|\sigma_n(x) - f| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x - t) - f(x)|K_n(t)\,dt.$$

Since $[-\pi, \pi]$ is compact, continuous functions on it are uniformly continuous, so let $\epsilon > 0$ and $\delta > 0$ be the corresponding value such that $|x - y| \leq \delta$ implies $|f(x) - f(y)| \leq \epsilon$. Then for any $\delta > 0$,

$$\frac{1}{2\pi}\int_{|t|\leq\delta} |f(x - t) - f(x)|K_n(t)\,dt \leq \frac{\epsilon}{2\pi}\int_{|t|\leq\delta} K_n(t)\,dt \leq \frac{\epsilon}{2\pi}.$$

Similarly,

$$\frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi}|f(x-t)-f(x)|K_n(t)\,dt \leq \frac{\sup_{t\in[-\pi,\pi]}|f(t)|}{\pi}\int_{\delta\leq|t|\leq\pi}K_n(t)\,dt.$$

Putting everything together, it follows that

$$|\sigma_n(x)-f| \leq \frac{\epsilon}{2\pi}+\frac{\sup_{t\in[-\pi,\pi]}|f(t)|}{\pi}\int_{\delta\leq|t|\leq\pi}K_n(t)\,dt.$$

Since the right hand side of this inequality is not a function of x and can be made arbitrarily small as $n\to\infty$ and $\epsilon\to 0$ by (iii), we have uniform continuity.

Problem 5. Let f be a real-valued function on $\mathbb{R}^2$, and suppose the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist for every (x,y). Prove or disprove each of the following assertions:

(a) f is continuous.

(b) if the partial derivatives of f are bounded (i.e. $\exists M>0$ such that $|\partial f/\partial x|\leq M$ and $|\partial f/\partial y|\leq M$ everywhere), then f is continuous.

(c) if the partial derivatives of f are bounded, then f is differentiable.

(a) This is false. Consider the function

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

Clearly this function is not continuous at 0, since $f(x,x)=\frac{1}{2}$ everywhere except $x=0$, where it is equal to zero. Yet for every $(x,y)\neq(0,0)$, we can easily calculate the partial derivatives using standard rules of differentiation, they are

$$\frac{\partial f}{\partial x}=\frac{y^3-x^2y}{(x^2+y^2)^2},\quad\text{and}\quad\frac{\partial f}{\partial y}=\frac{x^3-xy^2}{(x^2+y^2)^2}.$$

At the point $(0,0)$, note that the f is differentiable because

$$\lim_{\|(x,y)\|\to 0}\frac{f(x,y)-L(x,y)}{\|(x,y)\|}=\lim_{\|(x,y)\|\to 0}\frac{\frac{xy}{\|(x,y)\|^2}-L(x,y)}{\|(x,y\|}=\lim_{\|(x,y)\|\to 0}\frac{xy-\|(x,y)\|L(x,y)}{\|(x,y)\|^3}$$

should be zero for some linear map L. However taking $L(x,y)=0$ for all x,y, we get

$$\lim_{\|(x,y)\|\to 0}\frac{xy}{\|(x,y)\|^3}=0.$$

This shows that the derivative is defined everywhere, yet the function isn't continuous.

(b) This is true. Let (x,y) be some point in $\mathbb{R}^2$. Then for any $\epsilon>0$ let $B_\epsilon(x,y)$ be some open ball around (x,y). Let

$$S=\sup_{(x',y')\in\mathbb{R}^2}\frac{\partial f(x',y')}{\partial x}+\sup_{(x',y')\in\mathbb{R}^2}\frac{\partial f(x',y')}{\partial y},$$

which is finite since the partial derivatives exist everywhere and are bounded. Then for any $(x',y')\in B_\epsilon$, we have

$$\begin{aligned}|f(x,y)-f(x',y')| &\leq |f(x,y)-f(x,y')|+|f(x,y')-f(x',y')|\\ &\leq S|x-x'|+S|y-y'|\leq 4S\epsilon\end{aligned}.$$

So $\|(x,y)-(x',y')\| \le \epsilon$ implies that $|f(x,y)-f(x',y')| \le 2S\epsilon$. This proves that f is continuous.

(c) This is false. Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Notice that this function is continuous. Then the partial derivatives at any $(x,y) \neq (0,0)$ are given by

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \begin{cases} \left(\dfrac{2xy^3}{(x^2+y^2)^2}, \dfrac{x^2(x^2-y^2)}{(x^2+y^2)^2}\right) & \text{if } (x,y) \neq (0,0) \\ (0,0) & \text{if } (x,y) = (0,0) \end{cases}.$$

Note that by some elementary algebra these functions are bounded. We claim that the function isn't differentiable at $(x,y) = (0,0)$. At $(0,0)$ we have partial derivatives $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (0,0)$. If f were differentiable at $(0,0)$, this would imply that all partial derivatives at $(0,0)$ would be zero. Yet $f(x,x) = \frac{x}{2}$ which has derivative $\frac{1}{2}$ everywhere, so the partial derivative in the $(1,1)$ direction is nonzero. This is a contradiction so f isn't differentiable at $(0,0)$.

Problem 6.

(a) Give an example showing that a differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\det D_f(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ does not need to be injective.

(b) Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable and $\sup_{(x,y)\in\mathbb{R}^2} |D_f(x,y) - I| \le \alpha$ for some constant $\alpha < 1$, then f is a bijection.

(c) Consider the statement: if $U \subset \mathbb{R}^2$ is a connected open subset, and $f : U \to \mathbb{R}^2$ is a differentiable map which satisfies $\sup_{(x,y)\in U} |D_f(x,y) - I| \le \alpha < 1$, then f is injective. Show that this statement is true if U is an open ball, but false for some other connected open subsets of $\mathbb{R}^2$ (give an example).

(a) Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (e^x \cos(y), e^x \sin(y))$. Then the function is differentiable with total derivative:

$$D_f(x,y) = \begin{bmatrix} e^x \cos(y) & e^x \sin(y) \\ -e^x \sin(y) & e^x \cos(y) \end{bmatrix}.$$

The determinant of this total derivative is $\det D_f(x,y) = e^{2x}$, which is always nonzero. Yet $f(0,0) = f(0,2\pi) = (1,0)$ which means that f isn't injective.

(b) The proof is very similar to the proof of the inverse function theorem. Given some differentiable $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $\sup_{(x,y)\in\mathbb{R}^2} |D_f(x,y) - I| \le \alpha$ for $\alpha < 1$, consider for any $(x_0,y_0) \in \mathbb{R}^2$ the function $f' : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f'(x,y) = (x,y) + (x_0,y_0) - f(x,y)$. We claim that f' has a unique fixed point over $\mathbb{R}^2$, which would imply that every $(x_0,y_0) \in \mathbb{R}^2$ has a unique $(x,y) \in \mathbb{R}^2$ such that $(x_0,y_0) = f(x,y)$. Notice that $D_{f'} = I - D_f$ so by assumption $|D_f| \le \alpha < 1$. Then for points $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$, we can use the mean value inequality to get $|f'(x_1,y_1) - f'(x_2,y_2)| \le \alpha|(x_1,y_1) - (x_2,y_2)|$ so f' is a contraction mapping, and so it has a unique fixed point by a previous problem set.

(c) When U is an open ball, it is convex so the argument from (a) applies. Not sure about the rest.