MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY
HOMEWORK 2
DUE: TUESDAY, FEBRUARY 14 AT 12:00AM (MIDNIGHT) ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: `https://math.mit.edu/~hrm/papers/lectures-905-906.pdf`.

1. Problem 1: Homotopy smash product (10 points)

Do Exercise 41.6 of LAT.

2. Problem 2: Homotopy properties of the cofiber (10 points)

Do Exercise 45.4 of LAT.

3. Problem 3: Mod p homotopy groups (10 points)

Let p denote a prime number and $n \geq 2$. Let $M(\mathbb{Z}/p\mathbb{Z}, n) = S^{n-1} \cup_p D^n$ denote the n-dimensional mod p Moore space, and define the *mod p homotopy groups* of a pointed space X to be $\pi_n(X; \mathbb{Z}/p\mathbb{Z}) = [M(\mathbb{Z}/p\mathbb{Z}, n), X]_*$.

Since $M(\mathbb{Z}/p\mathbb{Z}, n) \simeq \Sigma^{n-2} M(\mathbb{Z}/p\mathbb{Z}, 2)$, this is a group for $n \geq 3$ and is an abelian group for $n \geq 4$.

When $n \geq 3$, prove that there is a short exact sequence

$$0 \to \pi_n(X)/p \to \pi_n(X; \mathbb{Z}/p\mathbb{Z}) \to \mathrm{tor}_p \pi_{n-1}(X) \to 0.$$

This is the analogue of the universal coefficients theorem for homotopy groups.

Remark: Unlike in homology, this sequence need not split when $p = 2$! In fact, it is not necessarily the case that 2 acts by zero on the mod 2 homotopy groups. For example, $\pi_{n+1}(S^n) \cong \pi_{n+2}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$, and the sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_{n+2}(S^n; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/\mathbb{Z} \to 0$$

does not split: $\pi_{n+2}(S^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ for $n \gg 0$.

4. Problem 4: Homotopy cofiber of chain complexes (20 points)

Let R denote a commutative ring. Given a chain complex of R-modules C and integer $i \in \mathbb{Z}$, let $C[i]$ denote the chain complex with $C[i]_n = C_{n-i}$ and boundary maps $d_n^{C[1]} = (-1)^i d_{n-i}^C$.

Given a map $f : C \to D$ of chain complexes of R-modules, define the homotopy cofiber $i(f) : D \to C(f)$ and construct a map $\pi(f) : C(f) \to C[1]$ by analogy with the case of spaces.

Prove that applying H_0 to the bi-infinite sequence

$$\dots \xrightarrow{f[-1]} D[-1] \xrightarrow{i(f)[-1]} C(f)[-1] \xrightarrow{\pi(f)[-1]} C \xrightarrow{f} D \xrightarrow{i(f)} C(f) \xrightarrow{\pi(f)} C[1] \xrightarrow{f[1]} D[1] \xrightarrow{i(f)[1]} \dots$$

gives rise to a long exact sequence

$$\dots \to H_{-1}D \to H_{-1}C(f) \to H_0 C \to H_0 D \to H_0 C(f) \to H_1 C \to H_1 D \to \dots.$$

(Hint: the analogue of the interval I in the category of chain complexes is the complex

$$\dots \to 0 \to R\{f\} \xrightarrow{f \mapsto e_0 - e_1} R\{e_1, e_2\} \to 0 \to \dots,$$

where e_i lie in degree 0 and f lies in degree 1.)