MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY HOMEWORK 3

DUE: TUESDAY, FEBRUARY 21 AT 12:00AM (MIDNIGHT) ON CANVAS

In the below, I use LAT to refer to Miller's *Lectures on Algebraic Topology*, available at: https://math.mit.edu/~hrm/papers/lectures-905-906.pdf.

1. Problem 1: Closure properties of Cofibrations (10 points)

Prove that cofibrations are closed under the following operations:

- (1) Products with a space Y: if $A \to X$ is a cofibration, then the induced map $A \times Y \to X \times Y$ is as well.
- (2) Compositions: if $A \to B$ and $B \to X$ are cofibrations, then so is $A \to B \to X$.
- (3) Countably transfinite compositions: if $X_i \to X_{i+1}$ are cofibrations for integers $i \ge 0$, then $X_0 \to \varinjlim_n X_n$ is a cofibration.
 - 2. Problem 2: Homotopy pushouts (15 points)

Given a diagram of spaces of the form

$$\begin{array}{c} A \longrightarrow X \\ \downarrow \\ B, \end{array}$$

construct the homotopy pushout, which is universal for diagrams of the form

$$\begin{array}{ccc}
A & \longrightarrow X \\
\downarrow & & \downarrow \\
B & \longrightarrow Y.
\end{array}$$

Furthermore:

- Explain how to modify your construction to apply to the setting of pointed spaces and express the homotopy cofiber as a special case of a homotopy pushout of pointed spaces.
- Give an example where the homotopy pushout (of unpointed spaces) is *not* a pushout in the homotopy category. (Hint: consider the case where X = B = *.)
 - 3. Problem 3: Space of Little cubes (15 points)

Let $\square^n = (0,1)^n$ denote the open unit cube of dimension n. A map $f: \square^n \hookrightarrow \square^n$ is said to be a rectilinear embedding if it is of the form

$$f(x_1,\ldots,x_n) = (a_1x_1 + b_1,\ldots,a_nx_n + b_n)$$

for real numbers $a_i > 0$ and b_i . A map $\coprod_{i=1}^k \Box^n \to \Box^n$ is said to be a rectilinear embedding if it is an open embedding and its restriction to each \Box^n in the domain is a rectilinear embedding in the above sense.

We let $C_k(n)$ denote the space of rectilinear embeddings of k disjoint n-cubes $\coprod_{i=1}^k \square^n \to \square^n$, topologized as an open subspace of $(\mathbb{R}^{2n})^k$. As we discussed in class for k=2, there is a continous map $C_k(n) \times (\Omega^n X)^k \to \Omega^n X$, which allows us to view $C_k(n)$ as a space parametrizing k-ary products on n-fold loop spaces.

¹This is true for general transfinite compositions as well, with the same proof.

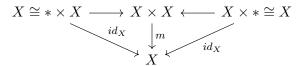
On the other hand, let $\operatorname{Conf}_k(\mathbb{R}^n)$ denote the space of k disjoint ordered points in \mathbb{R}^n , viewed as an open subspace of $(\mathbb{R}^n)^k$.

- (1) Prove that there are homotopy equivalences $C_k(n) \simeq \operatorname{Conf}_k(\mathbb{R}^n)$.
- (2) Prove that there are homotopy equivalences $\operatorname{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$. (Hint: one direction is given by the map $S^{n-1} \simeq \mathbb{R}^n \{0\} \to \operatorname{Conf}_2(\mathbb{R}^n)$ which sends $x \in \mathbb{R}^n \{0\}$ to the ordered pair of points (0, x).)

Remark: The maps $C_k(n) \times (\Omega^n X)^k \to \Omega^n X$ fit together into what is called an *action of the* \mathbb{E}_{n} - or *little n-cubes operad* on an *n*-fold loop space $\Omega^n X$. In *The Geometry of Iterated Loop Spaces*, May proved that this action completely charactertizes connected *n*-fold loop spaces: any connected space Y with an action of the \mathbb{E}_n -operad is homotopy equivalent to $\Omega^n X$ for a space X called the n-fold delooping of Y.

4. Problem 4: Eckmann-Hilton argument (10 points)

An *H*-space is pointed space (X,*) equipped with a multiplication map $m: X \times X \to X$ which is homotopy unital in the sense that the diagram



commutes up to homotopy. (We do *not* include the homotopies as part of the data of an H-space.) Prove that if X is an H-space, then $\pi_1(X,*)$ is an abelian group. Moreover, prove that the map $\pi_1(X,*) \times \pi_1(X,*) \to \pi_1(X,*)$ induced by m is equal to the group operation. (Hint: make an argument in pure algebra using the fact that m induces a unital group homomorphism $\pi_1(X,*) \times \pi_1(X,*) \to \pi_1(X,*)$.)

Remark: Taking $X = \Omega^{n-1}Y$, this gives a different proof that $\pi_n(Y, *)$ is abelian for $n \geq 2$, as well as the fact that the group operation doesn't depend on which coordinate we use. It also proves that for an H-space X (for example, a topological group), the group operation in $\pi_n(X, *)$ is equal to the operation given by pointwise multiplication of two representatives $S^n \to X$ using the H-space product.