Math 132 Problem Set 8

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Problem 1. Suppose that $f_0: M \to X$ and $f_1: M \to X$ are homotopic maps. Show that f_0 and f_1 are cobordant when regarded as manifolds over X.

A homotopy between f_0 and f_1 is some map $H: M \times I \to X$ with $H(-,0) = f_0$ and $H(-,1) = f_1$. Notice that $M \times I$ is an (n+1)-manifold over X, with boundary $\partial M \times I = M \sqcup M$. Clearly $H|_{\partial (M \times I)}$ is a cobordism between f_0 and f_1 so we are done.

Problem 2. Suppose that $g: X \to Y$ is a continuous map of topological spaces. Show that the map sending $f: M \to X$ to $g \circ f: M \to Y$ induces a linear transformation $f_*: MO_n(X) \to MO_n(Y)$. Show that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are two composable maps then $(g \circ f)_* = g_* \circ f_*$, or in other words that the diagram

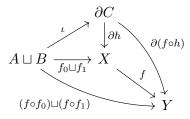
$$MO_d(X) \xrightarrow{f_*} MO_d(Y)$$

$$\downarrow^{g_*}$$

$$MO_d(Z)$$

commutes.

This problem essentially amounts to proving that the pushforward $MO_d(-): \mathbf{Top}^{co} \to \mathbf{Vect}_{\mathbb{F}_2}$ is a well-defined functor, where \mathbf{Top}^{co} is the category of cobordism classes of topological manifolds. First, let's show that for any $f: X \to Y$ $MO_d(f)$ is well-defined on cobordism classes. Suppose $f_0: A \to X$ and $f_1: B \to X$ are cobordant d-manifolds over X, so there is some map $h: C \to X$ where C is a (d+1)-manifold, and there is a diffeomorphism $\iota/X: f_0 \sqcup f_1 \to \partial h$. Note that $\partial(f \circ h) = f \circ \partial h$ and $f \circ (f_0 \sqcup f_1) = (f \circ f_0) \sqcup (f \circ f_1)$, so we have a commutative diagram:



This proves that $(f \circ f_0)$ is cobordant to $(f \circ f_1)$ and so f_* is well defined. The above diagram also proves that $MO_d(f)$ is a linear map, since $MO_d(f)(f_0) + MO_d(f)(f_1) = MO_d(f)(f_0 \sqcup f_1) = MO_d(f)(f_0 + f_1)$, and $MO_d(-)$ is an \mathbb{F}_2 -vector space. The preservation of identities is obvious, and composition follows since $MO_d(f)$ is a pushforward.

Problem 3. Suppose that M is a manifold. Show that every continuous function $f: \partial M \to \mathbb{R}$ extends to a continuous function $g: M \to \mathbb{R}$. Using this for every k, the map

$$MO_k(\mathbb{R}^n) \to MO_k(\mathrm{pt}) = MO_k$$

is an isomorphism. (Hint: Use a collar neighborhood.)

Suppose we are given $f: \partial M \to \mathbb{R}$. Let U be some collar neighborhood of ∂M in M, this comes with a diffeomorphism $\psi: U \to \partial M \times [0,1)$, with $\psi \circ (1_{\partial M} \times 0)$ a the identity map. Consider the projections $\psi_{\partial M} = \pi_{\partial M} \circ \psi$ and $\psi_{[0,1)} = \pi_{[0,1)} \circ \psi$. Consider now the continuous $\alpha: U \to \mathbb{R}$ given by

$$g(x) = \max(0, 1 - 2\psi_{[0,1)}(x))(f \circ \psi_{\partial M})(x).$$

Let V be the complement of $\psi^{-1}(\partial M \times [0,1/2])$ in M. Then let $\beta: V \to \mathbb{R}$ to be the zero function on V. Since $\alpha|_{\partial M} = f$, $U \cup V = M$, and $\alpha|_{U \cap V} = \beta|_{U \cap V} = 0$, it follows that we can extend f to some $g: M \to \mathbb{R}$. This also works for extending $f: \partial M \to \mathbb{R}^n$ to maps $f: M \to \mathbb{R}^n$ by the universal property of a product.

Let's now prove that $MO_k(\mathbb{R}^n) \to MO_k$ is an isomorphism. Recall that MO_k is a functor. We're interested in the induced map $\mathbb{R}^n \to \operatorname{pt}$. First of all, note that it has a left inverse given by the zero map $\operatorname{pt} \to \mathbb{R}^n$, and this gives a left inverse to $MO_k(\mathbb{R}^n) \to MO_k$. The right inverse follows because there is a cobordism $f: M \to \mathbb{R}^n$ with $0: M \to \mathbb{R}^n$. Indeed, consider $f \sqcup 0: \partial(M \times I) = M \sqcup M \to \mathbb{R}^n$. By the first part of the problem, let's extend this to some $g: M \times I \to \mathbb{R}^n$, which is a cobordism between f and 0. Thus $MO_k(\mathbb{R}^n) \to MO_k$.

Problem 4. Suppose that V_1 , V_2 , and V_3 are vector spaces over a field k. A sequence of linear transformations

$$V_1 \stackrel{p}{\longrightarrow} V_2 \stackrel{q}{\longrightarrow} V_3$$

is exact if the image of p is equal to the kernel of q. Note that this implies that the composition $q \circ p$ is zero. Show that in this situation one has dim $V_2 \leq \dim V_1 + \dim V_3$.

By the rank-nullity theorem, we have rank $p \leq \dim V_1$ and rank $q \leq \dim V_3$, By exactness, we get Im $p = \ker q$ so dim $\ker q = \operatorname{rank} p$ so again by the rank-nullity theorem we get dim $V_2 = \operatorname{rank} p + \operatorname{rank} q \leq \dim V_1 + \dim V_2$.