Math 231a Problem Set 7

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Problem 1. Let X be a finite CW complex. Show that for any field F,

$$\chi(X) = \sum_k (-1)^k \dim_F H_k(X; F).$$

Recall cellular homology, given as the homology of the chain complex $C_k(X) = \mathbb{Z}I_k$ with boundary maps given by the degree formula. Then we showed that $H_n(C_*(X)) \cong H_n(X)$ and

$$\chi(X) = \sum_k (-1)^k |I_k|.$$

This construction can be generalized to an arbitrary field F by setting $C_k(X; F) = FI_k$. Again, the same proof will hold, since whenever we have an exact sequence

$$0 \to U \to V \to W \to 0$$

of vector spaces over a field, we have $\dim_F V = \dim_F U + \dim_F W$. Thus we get $H_n(C_*(X; F)) \cong H_n(X; F)$ and similarly we'll get

$$\sum_k (-1)^k \dim_F H_k(X; F) = \sum_k (-1)^k \mathrm{rank}_F H_k(X; F) = \sum_k (-1)^k |I_k| = \chi(X).$$

Problem 2. Let p be a prime number. Give an example of two maps $f, g : X \to Y$ inducing the same map on integral homology but not homology with coefficients in $\mathbb{F}_p$ (and that are therefore not homotopic).

Let $L(p)$ be the quotient of D^2 by the map which identifies the boundary by a degree p map. Using cellular homology, we quickly see that the cellular chain complex $C_*(L(p); R)$ for $R = \mathbb{Z}$ or $\mathbb{Z}_p$ is:

$$0 \longleftarrow R \xleftarrow{0} R \xleftarrow{p} R \longleftarrow 0$$

Thus the homology groups are:

$$H_k(L(p); \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/p & k = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad H_k(L(p); \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & 0 \le k \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the constant map $f : L(p) \to S^2$ which sends everything to some point $x \in S^2$. This induces an isomorphism on $H_0(-; \mathbb{Z})$, and the zero map everywhere else. The same thing happens in $H_*(-; \mathbb{Z}/p)$. Next we have the map $g : L(p) \to S^2$ which collapses the boundary of D^2 to a single point. As before, this induces an isomorphism on $H_0(-; \mathbb{Z})$ and $H_0(-; \mathbb{Z}/p)$. We get a zero map on $H_1(-; \mathbb{Z})$ and $H_1(-; \mathbb{Z}/p)$, yet a nonzero map on $H_2(-; \mathbb{Z}/p)$. So the two maps are not homotopic, but they induce the same maps of integral homology.

Problem 3. Let m, n be positive integers and consider the cyclic groups $\mathbb{Z}/m$ and $\mathbb{Z}/n$. Compute the tensor product $\mathbb{Z}/m \otimes \mathbb{Z}/n$.

We claim that $\mathbb{Z}/m \otimes \mathbb{Z}/n \cong \mathbb{Z}/\gcd(n, m)$. Firstly, we have a bilinear map $\mathbb{Z}/n \times \mathbb{Z}/m \to \mathbb{Z}/\gcd(n, m)$ which sends (a, b) to $ab \mod \gcd(n, m)$. Since $\gcd(n, m) \mid n$ and $\gcd(n, m) \mid m$, this map is well defined. It's easy to check that it is bilinear.

Now suppose there is some other ring P with a bilinear map $f : \mathbb{Z}/m \times \mathbb{Z}/n \to P$. This means that $(1, 1)$ must get sent to some $f(1, 1) \in P$ such that $n \cdot f(1, 1) = f(n, 1) = 0$ and $m \cdot f(1, 1) = f(1, m) = 0$, so $\gcd(n, m) \cdot f(1, 1) = 0$. Clearly there is a unique linear map $\widetilde{f} : \mathbb{Z}/\gcd(n, m) \to P$ which sends 1 to $f(1, 1)$, and so we are done.

Problem 4. The goal of this problem is to prove the Borsuk-Ulam theorem, which states that for every map

$$g : S^n \to \mathbb{R}^n,$$

there is a point $x \in S^n$ with $g(x) = g(-x)$. Along the way we will prove that any *odd* map $f : S^n \to S^n$ has odd degree.

Let $p : \widetilde{X} \to X$ be a two sheeted covering.

a. Prove that singular n-simplex $\sigma : \Delta^n \to X$ admits exactly two lifts $\sigma_1, \sigma_2 : \Delta^n \to \widetilde{X}$.

This follows because Δ^n is a simply connected and locally path connected space.

b. Prove that there is a short exact sequence of chain complexes

$$0 \longrightarrow S_*(X; \mathbb{F}_2) \xrightarrow{\tau} S_*(\widetilde{X}; \mathbb{F}_2) \xrightarrow{p_*} S_*(X; \mathbb{F}_2) \longrightarrow 0$$

where the *transfer map* τ is defined on n-simplices σ by $\tau(\sigma) = \sigma_1 + \sigma_2$. This gives rise to the long exact *transfer sequence*:

$$\cdots \longrightarrow H_n(X; \mathbb{F}_2) \xrightarrow{\tau_*} H_n(\widetilde{X}; \mathbb{F}_2) \xrightarrow{p_*} H_n(X; \mathbb{F}_2) \longrightarrow H_{n-1}(X; \mathbb{F}_2) \longrightarrow \cdots$$

First suppose $\sigma \in S_*(X; \mathbb{F}_2)$ with $\tau(\sigma) = 0$. This means that $\tau(\sigma) = 2\omega$, so $\sigma_1 + \sigma_2 = 2\omega$. Since σ_1, σ_2 are disjoint chains, we must have $\omega = \omega' + \omega''$ such that $\sigma_1 = 2\omega' = 0$ and $\sigma_2 = 2\omega'' = 0$. Since both lifts are zero, the original chain must be zero. So the sequence is exact at $S_*(X; \mathbb{F}_2)$.

Next, let $\sigma \in S_*(X; \mathbb{F}_2)$. Then $p_*(\tau(\sigma)) = p \circ \sigma_1 + p \circ \sigma_2 = 2\sigma = 0$, so $\mathrm{Im}(\tau) \subset \mathrm{Ker}(p_*)$. To prove the converse, suppose $p_*(\omega) = 0$ for some $\omega \in S_*(\widetilde{X}; \mathbb{F}_2)$. Since we're working in the chain complex, it follows that $\omega = \sigma_1 + \sigma_2$ for some chain σ in X.

Lastly, p_* is surjective because for any chain $\sigma \in S_*(X; \mathbb{F}_2)$ we have $\sigma = p \circ \sigma_1$. So the short sequence of chain complexes is exact, and so we have a long exact sequence of homology groups.

c. Given an odd map $f : S^n \to S^n$, there is an induced map $\overline{f} : \mathbb{RP}^n \to \mathbb{RP}^n$. Prove that there is a commutative diagram of transfer sequences of the form:

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r, "\tau_*"] \arrow[d, "\overline{f}_*"] & H_k(S^n; \mathbb{F}_2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] \arrow[d, "\overline{f}_*"] & H_{k-1}(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] \arrow[d, "\overline{f}_*"] & \cdots \\
\cdots \arrow[r] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r, "\tau_*"] & H_k(S^n; \mathbb{F}_2) \arrow[r, "p_*"] & H_k(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] & H_{k-1}(\mathbb{RP}^n; \mathbb{F}_2) \arrow[r] & \cdots
\end{tikzcd}$$

It suffices to show that we have a commutative diagram:

$$\begin{tikzcd}
S_*(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau"] \arrow[d, "\overline{f}_*"] & S_*(S^n;\mathbb{F}_2) \arrow[r, "p_*"] \arrow[d, "f_*"] & S_*(\mathbb{RP}^n;\mathbb{F}_2) \arrow[d, "\overline{f}_*"] \\
S_*(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau"] & S_*(S^n;\mathbb{F}_2) \arrow[r, "p_*"] & S_*(\mathbb{RP}^n;\mathbb{F}_2)
\end{tikzcd}$$

To prove commutativity of the first square, first note that for any $\sigma \in S_*(\mathbb{RP}^n;\mathbb{F}_2)$ we have $\overline{f}_*(\tau(\sigma)) = \overline{f}_*(\sigma_1+\sigma_2) = \overline{f}_*(\sigma_1) + \overline{f}_*(\sigma_2)$. On the other side, we have $\tau(\overline{f}_*(\sigma))$. However the lifts of $\overline{f}_*(\sigma)$ are exactly $\overline{f}_*(\sigma_1)$ and $\overline{f}_*(\sigma_2)$ since $\overline{f} \circ p = p \circ f$.

Commutativity of the second square follows simply because $S_*(-;\mathbb{F}_2)$ is a functor and since $\overline{f} \circ p = p \circ f$.

d. Using (c), prove that any odd map $f : S^n \to S^n$ has odd degree.

Let $(\overline{f}_*)_k : H_k(\mathbb{RP}^n;\mathbb{F}_2) \to H_k(\mathbb{RP}^n;\mathbb{F}_2)$ be the natural induced map. Recall that $H_k(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2$ for all $k \le n$. We first claim that $(\overline{f}_*)_k$ is the identity isomorphism between $\mathbb{F}_2$ for all $k \le n$. Let's proceed by induction. When $k = 0$, this is clearly the case, since $\mathbb{RP}^n$ is path connected. Now suppose $(\overline{f}_*)_{k-1}$ is the identity isomorphism for some $0 \le k-1 < n-1$. Look at a slice of the diagram from (c):

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "p_*"] \arrow[d, "(\overline{f}_*)_k"] & H_{k-1}(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau_*"] \arrow[d, "(\overline{f}_*)_{k-1}"] & H_{k-1}(S^n;\mathbb{F}_2) \arrow[r] \arrow[d, "(f_*)_{k-1}"] & \cdots \\
\cdots \arrow[r] & H_k(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "p_*"] & H_{k-1}(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau_*"] & H_{k-1}(S^n;\mathbb{F}_2) \arrow[r] & \cdots
\end{tikzcd}$$

Since $k \neq 0, n$, we have $H_{k-1}(S^n;\mathbb{F}_2) = 0$ so p_* is surjective and hence an isomorphism. Since $p_* \circ (\overline{f}_*)_k = (\overline{f}_*)_{k-1} \circ p_*$, it follows that $(\overline{f}_*)_k$ is an isomorphism as well.

Now lets look at the "end" of the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau_*"] \arrow[d, "(\overline{f}_*)_n"] & H_n(S^n;\mathbb{F}_2) \arrow[r, "p_*"] \arrow[d, "(f_*)_n"] & H_n(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\partial"] \arrow[d, "(\overline{f}_*)_n"] & H_{n-1}(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau_*"] \arrow[d, "(\overline{f}_*)_{n-1}"] & H_{n-1}(S^n;\mathbb{F}_2) \arrow[d, "(f_*)_{n-1}"] \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{F}_2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\partial"] & H_{n-1}(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\tau_*"] & H_{n-1}(S^n;\mathbb{F}_2)
\end{tikzcd}$$

Note that $H_{n-1}(S^n;\mathbb{F}_2) = 0$ so ∂ is an isomorphism, which makes p_* the zero map, and so τ_* is an isomorphism. By commutativity, $\tau_* \circ (\overline{f}_*)_n = (f_*)_n \circ \tau_*$ and since $(\overline{f}_*)_n$ is an isomorphism, it follows that $(f_*)_n$ is as well.

Finally, suppose that f had even degree. Since $H_*(S^n;-)$ is a functor as well, we'd have a commutative square:

$$\begin{tikzcd}
H_n(S^n;\mathbb{Z}) \arrow[r, "f'_*"] \arrow[d] & H_n(S^n;\mathbb{Z}) \arrow[d] \\
H_n(S^n;\mathbb{F}_2) \arrow[r, "f_*"] & H_n(S^n;\mathbb{F}_2)
\end{tikzcd}$$

Here the vertical projections are modulo two, so f would induce the zero map on $H_n(S^n;\mathbb{F}_2)$, which contradicts the fact that we showed it was an isomorphism. So the (integral) degree of f is odd.

e. Given a map $g : S^n \to \mathbb{R}^n$, prove that the odd map $f : S^n \to \mathbb{R}^n$ given by $f(x) = g(x) - g(-x)$ must have a zero. Deduce the Borsuk-Ulam theorem.

Suppose for the sake of contradiction that the odd map f does not have a zero. This gives us an continuous map $\widehat{f} : S^n \to S^{n-1}$ by composing with the normalization map. In particular, this map is still odd. If we further compose with the equatorial inclusion $S^{n-1} \subset S^n$, we get an odd map $\widehat{f} : S^n \to S^n$. By part (d), it follows that $\widehat{f}$ must have odd degree. But $\widehat{f}$ isn't surjective, so it must have degree zero. This is a contradiction, so f must have a zero and so there exists a x such that $g(x) = g(-x)$.

Problem 5. Computation of the homology of $\mathbb{RP}^n$ using transfer sequences.

The computation of the homology of $\mathbb{RP}^n$ via cellular homology presented in class depended on a careful analysis of orientations and signs. In this problem, you will use the *transfer sequence* introduced in the previous problem to recompute the homology of $\mathbb{RP}^n$ in a way that is less vulnerable to sign errors.

a. Given the fact that $\mathbb{RP}^n$ is an n-dimensional CW complex, use the transfer sequence associated to the cover $p : S^n \to \mathbb{RP}^n$ to compute $H_*(\mathbb{RP}^n; \mathbb{F}_2)$.

Since $\mathbb{RP}^n$ is an n-dimensional CW complex, we know that $H_k(\mathbb{RP}^n; \mathbb{F}_2) = 0$ for all $k > n$. Also by the results of the previous sections, we've shown that there is an isomorphism $H_k(\mathbb{RP}^n; \mathbb{F}_2) \cong H_{k-1}(\mathbb{RP}^n; \mathbb{F}_2)$ for all $0 < k \leq n$. Since $\mathbb{RP}^n$ is path connected, we get the following homology:

$$\boxed{H_k(\mathbb{RP}^n; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}}$$

b. The transfer map τ may be defined at the level of integral chains by the same formula as in Problem 4b. Verify that the induced composite

$$H_n(X; \mathbb{Z}) \xrightarrow{\tau_*} H_n(\widetilde{X}; \mathbb{Z}) \xrightarrow{p_*} H_n(X; \mathbb{Z})$$

is multiplication by 2.

For any $\sigma \in H_n(X; \mathbb{Z})$ we have $p_*(\tau_*(\sigma)) = p_*(\sigma_1) + p_*(\sigma_2) = 2\sigma$, so this is the multiplication by 2 map.

c. Using the pushout squares

$$\begin{tikzcd} S^{n-1} \arrow[r, "p"] \arrow[d] & \mathbb{RP}^{n-1} \arrow[d] \\ D^n \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

and induction on n, reduce the computation of $H_*(\mathbb{RP}^n; \mathbb{Z})$ to the statement that, when n is odd, $p_* : H_n(S^n; \mathbb{Z}) \to H_n(\mathbb{RP}^n; \mathbb{Z})$ sends a generator to ± 2 times a generator.

Recall that we have a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & S_*(S^{n-1}) \arrow[r] \arrow[d, "p_*"] & S_*(D^n) \arrow[r] \arrow[d] & S_*(D^n/S^{n-1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & S_*(\mathbb{RP}^{n-1}) \arrow[r] & S_*(\mathbb{RP}^n) \arrow[r] & S_*(\mathbb{RP}^n/\mathbb{RP}^{n-1}) \arrow[r] & 0 \end{tikzcd}$$

This gives us long exact sequences of (reduced) homology groups:

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H_k}(S^{n-1}) \arrow[r] \arrow[d, "p_*"] & \widetilde{H_k}(D^n) \arrow[r] \arrow[d] & \widetilde{H_k}(D^n/S^{n-1}) \arrow[r, "\partial_k"] \arrow[d, "\alpha_k"] & \widetilde{H_{k-1}}(S^{n-1}) \arrow[r] \arrow[d, "p_*"] & \cdots \\ \cdots \arrow[r] & \widetilde{H_k}(\mathbb{RP}^{n-1}) \arrow[r, "\omega_k"] & \widetilde{H_k}(\mathbb{RP}^n) \arrow[r, "\gamma_k"] & \widetilde{H_k}(\mathbb{RP}^n/\mathbb{RP}^{n-1}) \arrow[r, "\beta_k"] & \widetilde{H_{k-1}}(\mathbb{RP}^{n-1}) \arrow[r] & \cdots \end{tikzcd}$$

In particular, note that since $\widetilde{H_k}(D^n) = 0$ for all n, k, it follows that ∂_k is an isomorphism for all k. We've already shown that α_k is an isomorphism for all k in our proof of cellular homology.

We'll first prove that:

$$\widetilde{H}_n(\mathbb{RP}^n) \cong \begin{cases} 0 & n \text{ even}, \\ \mathbb{Z} & n \text{ odd}. \end{cases}$$

Assume the statement that $p_* : H_n(S^n; \mathbb{Z}) \to H_n(\mathbb{RP}^n; \mathbb{Z})$ is the multiplication by ± 2 map. When $n = 0$, we have the reduced homology $\widetilde{H}_k(\mathbb{RP}^0) = 0$ for all k, since $\mathbb{RP}^0$ is a path connected CW complex of dimension zero, so this is clearly true.

Now suppose the claim is true for $n-1$. Looking at the bottom row of the commutative diagram, we have the exact sequence:

$$\widetilde{H}_n(\mathbb{RP}^{n-1}) \longrightarrow \widetilde{H}_n(\mathbb{RP}^n) \xrightarrow{\gamma_n} \widetilde{H}_n(\mathbb{RP}^n/\mathbb{RP}^{n-1}) \xrightarrow{\beta_n} \widetilde{H_{n-1}}(\mathbb{RP}^{n-1})$$

Recall that $\mathbb{RP}^n/\mathbb{RP}^{n-1}$ is homeomorphic to S^n by the pushout square, so $\widetilde{H}_n(\mathbb{RP}^n/\mathbb{RP}^{n-1}) \cong \mathbb{Z}$. Additionally $\widetilde{H}_n(\mathbb{RP}^{n-1}) = 0$ since $\mathbb{RP}^{n-1}$ is n-dimensional. So γ_n is injective. We now have two cases. If n is odd, then $n-1$ is even so $\widetilde{H_{n-1}}(\mathbb{RP}^{n-1}) = 0$ and γ_n becomes an isomorphism. Thus $\widetilde{H}_n(\mathbb{RP}^n) \cong \mathbb{Z}$ as desired.

In the case when n is even, it's a little bit more tricky because then $n-1$ is odd so $\widetilde{H_{n-1}}(\mathbb{RP}^{n-1}) \cong \mathbb{Z}$. Since the sequence is exact, we have $\widetilde{H}_n(\mathbb{RP}^n) \cong \mathrm{Im}(\gamma_n) = \mathrm{Ker}(\beta_n)$. But $\beta_n \circ \alpha_k = p_* \circ \partial_k$. Since all the maps involved are injective, so is β_n and so $\widetilde{H}_n(\mathbb{RP}^n) \cong 0$ as desired.

Next, we claim that

$$\widetilde{H_{n-1}}(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2 & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

We don't need induction for this. For any $n \geq 1$, we take a slice of the diagram to get:

$$\begin{tikzcd}
\widetilde{H}_n(D^n/S^{n-1}) \arrow[r, "\partial_n"] \arrow[d, "\alpha_n"] & \widetilde{H}_n(S^{n-1}) \arrow[d, "p_*"] & & \\
\widetilde{H}_n(\mathbb{RP}^n/\mathbb{RP}^{n-1}) \arrow[r, "\beta_n"] & \widetilde{H_{n-1}}(\mathbb{RP}^{n-1}) \arrow[r, "\omega_{n-1}"] & \widetilde{H_{n-1}}(\mathbb{RP}^n) \arrow[r] & 0
\end{tikzcd}$$

This last zero in the bottom row occurs because $\widetilde{H_{n-1}}(\mathbb{RP}^n/\mathbb{RP}^{n-1}) = 0$. This makes ω_{n-1} surjective. Now when n is even, $\widetilde{H_{n-1}}(\mathbb{RP}^{n-1}) \cong \mathbb{Z}$. Since α_n and ∂_n are isomorphisms and p_* is the multiplication by ± 2 map, it follows that β_n must be the multiplication by ± 2 map as well. So $\mathrm{Im}(\beta_n) = 2\mathbb{Z}$ and so $\widetilde{H_{n-1}}(\mathbb{RP}^n) \cong \mathbb{Z}/2$ as desired. If instead n were odd, then $\widetilde{H_{n-1}}(\mathbb{RP}^{n-1}) \cong 0$ so ω_{n-1} would be the zero map, and so $\widetilde{H_{n-1}}(\mathbb{RP}^n)$.

Finally, since $\mathbb{RP}^0 \subset \mathbb{RP}^1 \subset \cdots \subset \mathbb{RP}^n$ is a CW decomposition of $\mathbb{RP}^n$, we get the full homology of $\mathbb{RP}^n$:

$$\boxed{\widetilde{H}_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2 & k < n, k \text{ odd}, \\ \mathbb{Z} & k = n, k \text{ odd}, \\ 0 & \text{otherwise}, \end{cases}} \quad \text{or} \quad \boxed{H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2 & k < n, k \text{ odd}, \\ \mathbb{Z} & k = n, k \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}}$$

d. Using (a) and (b), prove this statement.

We have the commutative diagram:

$$\begin{tikzcd}
H_n(\mathbb{RP}^n;\mathbb{Z}) \arrow[r, "\tau_*"] \arrow[d] & H_n(S^n;\mathbb{Z}) \arrow[r, "p_*"] \arrow[d] & H_n(\mathbb{RP}^n;\mathbb{Z}) \arrow[d] \\
H_n(\mathbb{RP}^n;\mathbb{F}_2) \arrow[r, "\overline{\tau_*}"] & H_n(S^n;\mathbb{F}_2) \arrow[r, "\overline{p_*}"] & H_n(\mathbb{RP}^n;\mathbb{F}_2)
\end{tikzcd}$$

First, we recall from Problem 4 that $\overline{\tau_*}$ is injective, and hence an isomorphism because $H_n(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2$ and $H_n(S^n;\mathbb{F}_2) \cong \mathbb{F}_2$. Since $\mathbb{RP}^n$ has a cell structure with one cell in each dimension, $H_n(\mathbb{RP}^n;\mathbb{Z})$ has a single generator, so by commutativity of the diagram, τ_* must send it to an odd element of $H_n(S^n;\mathbb{Z})$. However $p_* \circ \tau_*$ is the multiplication by ± 2 map, so p_* must be a multiplication by ± 2 map since τ_* has odd image.