Math 137 Problem Set 4

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I collaborated with AJ LaMotta for this problem set.

Throughout, K is assumed to be an algebraically closed field.

Problem 1. Let $K = \mathbb{C}$ and for any integers $a, b \geq 1$, consider the algebraic subset $V_{a,b} = \mathcal{V}(X^b - Y^a)$ of $\mathbb{C}^2$ and the morphism $\varphi_{a,b} : \mathbb{C} \to V_{a,b}$ sending t to (t^a, t^b).

(a) For which pairs (a, b) is $\varphi_{a,b}$ injective?

(b) For which pairs (a, b) is $\varphi_{a,b}$ surjective?

(c) For which pairs (a, b) is $\varphi_{a,b}$ an isomorphism?

(d) (bonus) For which pairs (a, b) is $V_{a,b}$ isomorphic to K?

(a) Suppose $\varphi_{a,b}$ is a morphism. Then $\varphi_{a,b}$ is injective if and only if whenever $(t_1^a, t_1^b) = (t_2^a, t_2^b)$ we have $t_1 = t_2$. So letting $z = t_1/t_2$, this is equivalent to saying $z^a = 1$ and $z^b = 1$ implies $z = 1$. Finding a $z \neq 1$ with $z^a = 1$ and $z^b = 1$ can only happen if a and b have a common divisor d, then z is a d-th root of unity. So $\varphi_{a,b}$ is injective if and only if a and b are coprime.

(b) This is only true if a and b are relatively prime. Note that if a and b are not relatively prime, say $d|a, b$, then $(1, e^{2\pi i/a})$ has no preimage. This is because for some $t^a = 1$ and $t^b = e^{2\pi i/a}$, so t is an a-th and b-th root of unity. This can't happen since $d|a, b$.

(c) $\varphi_{a,b}$ is an isomorphism if and only if the pullback map $\widetilde{\varphi_{a,b}} : \Gamma(V_{a,b}) \to \Gamma(\mathbb{C})$ is an isomorphism. Note that $\widetilde{\varphi_{a,b}} : \mathbb{C}[x, y]/(x^b - y^a) \to \mathbb{C}[t]$ maps $f(x, y) \mapsto f(t^a, t^b)$. So the image of $\widetilde{\varphi_{a,b}}$ is $\mathbb{C}[t^a, t^b]$. Thus for $\varphi_{a,b}$ to be an isomorphism, either a or b must be 1. Conversely, if $a = 1$ (without loss of generality), we have an inverse morphism $\varphi_{a,b}^{-1} : (t, t^b) \mapsto t$. So $\varphi_{a,b}$ is an isomorphism if and only if $a = 1$ or $b = 1$.

Problem 2.

(a) Consider the algebraic set

$$V = \{(x, y, z) \in K^3 \mid x^2 + y^2 = z^2\}.$$

Find a non-constant morphism $\varphi : K \to V$.

(b) Consider the algebraic set

$$W = \{(x, y) \in K^2 \mid x^2 + y^2 = 1\}.$$

Assuming that the field K has characteristic zero, show that there is no nonconstant morphism $\psi : K \to W$.

(a) Note that for any $m, n \in K$, we have the relation

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2.$$

Then for any $\alpha \in K$, we have the morphism $\varphi_\alpha : K \to V$ which takes x to $(x^2 - \alpha^2, 2x\alpha, x^2 + \alpha^2)$.

(b) We'll prove that any morphism $\psi : K \to W$ must be constant, so let ψ be some morphism. Now consider the pullback map $\widetilde{\psi} : \Gamma(W) \to \Gamma(K)$. However $\Gamma(W) = K[x, y]/(x^2 + y^2 - 1)$ and $\Gamma(K) = K[t]$. Let i be a root of $x^2 - 1$ in K (guaranteed because K is algebraically closed field of characteristic zero), and consider the map $f : K[z, z^{-1}] \to K[x, y]/(x^2 + y^2 - 1)$ where z maps to $y - ix$ and z^{-1} maps to $y + ix$.

This is clearly an isomorphism because it has inverse given by x mapping to $(z - z^{-1})/2i$ and y mapping to $(z + z^{-1})/2$. So $\widetilde{\psi} : \Gamma(W) \to \Gamma(K)$ can be thought of as a k-algebra map $K[z, z^{-1}] \to K[t]$. Such a map is determined by where z goes. However z is a unit in $K[z, z^{-1}]$, so it must map to $K \subset K[t]$. Thus the pullback map is constant, and so every morphism $\psi : K \to W$ must also be constant.

Problem 3.

(a) Find algebraic subsets V_1, V_2 of $\mathbb{C}^2$ and functions $f_1 \in \Gamma(V_1)$ and $f_2 \in \Gamma(V_2)$ such that $f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2}$ but there is no function $f \in \Gamma(V_1 \cup V_2)$ with $f|_{V_1} = f_1$ and $f|_{V_2} = f_2$.

(b) Corollary 6.2 from class can fail when K is not algebraically closed: Find disjoint algebraic subsets V_1, V_2 of $\mathbb{R}^2$ and functions $f_1 \in \Gamma(V_1)$ and $f_2 \in \Gamma(V_2)$ such that there is no function $f \in \Gamma(V_1 \cup V_2)$ such that $f|_{V_1} = f_1$ and $f|_{V_2} = f_2$.

(c) Show that Corollary 6.3 from class still holds when K is not algebraically closed: If $V \subseteq K^n$ is a finite set and $f : V \to K$ any function, there is a polynomial $g \in K[X_1, \ldots, X_n]$ such that $f(P) = g(P)$ for all $P \in V$.

(a) Let $V_1 = \mathcal{V}(y)$ and $V_2 = \mathcal{V}(y - x^2)$. Define functions $f_1(x, y) = y \in \Gamma(V_1)$ and $f_2(x, y) = x \in \Gamma(V_1)$. The only place where these algebraic sets intersect is $(0, 0)$, and $f_1(0, 0) = f_2(0, 0) = 0$ so these satisfy the conditions of the problem. Now suppose for the sake of contradiction that $f \in \Gamma(V_1 \cup V_2)$ with $f\big|_{V_1} = f_1$ and $f\big|_{V_2} = f_2$. This means that $f(x, 0) = 0$ and $f(x, x^2) = x$. Since $f(x, 0) = 0$, it follows that $f(x, y) = yg(x, y)$ for some polynomial $g(x, y)$. Yet $f(x, x^2) = x^2 g(x, y) = x$. This is clearly impossible, just by looking at the degree of x in both sides. So no such function exists.

(b) Let $V_1 = \mathcal{V}(x^2 - y + 1)$, $V_2 = \mathcal{V}(y)$, with $f_1 = x^2 + x \in \Gamma(V_1)$ and $f_2 = x \in \Gamma(V_2)$. Suppose f agrees with f_1, f_2 on V_1, V_2 respectively. Then $f(x, 0) = x$ so $f(x, y) = x + yg(x, y)$. Thus $f(x, x^2 + 1) = x + (x^2 + 1)g(x, x^2 + 1) = x^2 + x$, however this implies that $x^2 | x^2 + 1$, a contradiction.

(c) Let $\{P_1\}, \{P_2\}, \ldots, \{P_n\}$ be the set of points in V. Note that they are all algebraic sets. Let $I_i = \mathcal{I}(\{P_i\})$. Note that $I_i + I_j = K[X_1, \ldots, X_n]$, and since all of I_i are radical ideals, the Chinese remainder theorem implies that $\Gamma(V) \cong \Gamma(\{P_1\}) \times \cdots \times \Gamma(\{P_n\})$ by the map sending f to $(f(P_1), \ldots, f(P_n))$. This completes the proof, since we can pick a $f_i \in \Gamma(\{P_i\})$ taking on any value at P_i and lift this to a polynomial $f \in \Gamma(V)$.

Problem 4. Identify the space $M_n(K)$ of $n \times n$-matrices with entries in K with the vector space K^{n^2} (by sending a matrix A to a vector consisting of its entries). For any $r \leq n$, consider the subset $V_r \subseteq M_n(K) = K^{n^2}$ of matrices of rank at most r.

(a) Show that V_r is an algebraic subset of K^{n^2}.

(b) Show that V_r is an irreducible subset of K^{n^2}.

(a) Note that $V_r = \{A \in M_n(K) \mid \det(U) = 0 \text{ where } U \text{ is a } (r+1) \times (r+1) \text{ submatrix of } A\}$. Since $\det(U)$ is a polynomial in the terms of A for each $(r+1)$-submatrix U. Thus V_r is the intersection of finitely many algebraic sets and so is an algebraic subset of K^{n^2}.

(b) Let $W_r \subset M_n(K)$ be the algebraic set of matrices of rank exactly r. Consider the irreducible algebraic subset $\mathrm{GL}_n(K) \times \mathrm{GL}_n(K)$ of K^{2n^2}. There is a Zariski continuous map $f : \mathrm{GL}_n(K) \times \mathrm{GL}_n(K) \to W_r$ given by

$f(g,h) \mapsto gAh^{-1}$ for any rank r matrix $A \in W_r$. Note that this map is surjective. Then by Problem 7 on Set 3, $\overline{f(\mathrm{GL}_n(K) \times \mathrm{GL}_n(K))}$ is irreducible. But $\overline{f(\mathrm{GL}_n(K) \times \mathrm{GL}_n(K))} = \overline{W_r} = W_r \sqcup W_{r-1} \sqcup \cdots \sqcup W_0 = V_r$. So V_r is irreducible.