Math 137 Problem Set 7

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April 25, 2022

I collaborated with AJ LaMotta for this problem set.

Problem 1. Which of the following morphisms are finite? (Say for $K = \mathbb{C}$.)

(a) The morphism $\varphi : K^2 \to K$ sending (x, y) to $x^3y + xy^3 + 3x + 1$.

(b) The morphism $\varphi : K \to K^2$ sending x to (x^2, x^3).

(a) Notice that $x^3y + xy^3 + 3x + 1 = 1$ has infinitely many solutions; an infinite family can be given by $(0, y)$ for any $y \in \mathbb{R}$. Thus φ cannot be finite since a point in K has infinitely many preimages.

(b) Note that $\Gamma(K) = K[t]$, and $\varphi^*(\Gamma(K^2)) = \varphi^*(K[x, y]) = K[t^2, t^3] = K[t^2]$. Then $K[t]$ has a natural $K[t^2]$-module structure, and is generated by $\{1, t\}$. Since $\Gamma(K)$ is a finitely generated $\varphi^*(\Gamma(K^2))$-module, the morphism φ is finite.

Problem 2.

(a) Let $\varphi : V \to W$ be a morphism. Show that if V is the union of algebraic subsets $V_1, \ldots, V_n$ and each restriction $\varphi : V_i \to W$ is a finite morphism, then φ is a finite morphism.

(b) Let $V \subseteq K^n$ be a finite set and let W be any algebraic set. Show that every map $\varphi : V \to W$ is a finite morphism.

(a) By induction, we can assume without loss of generality that $n = 2$, so let $V = A \cup B$. We'll show that $\Gamma(A \cup B)$ is integral over $\varphi^*(\Gamma(W))$. Let $f \in \Gamma(A \cup B)$ be some function. Then $f\big|_A$ and $f\big|_B$ are integral over $\varphi^*(\Gamma(W))$. We thus have monic polynomials $\alpha, \beta \in \varphi^*(\Gamma(W))[x]$ with $\alpha(f) = 0$ in $\Gamma(A)$ and $\beta(f) = 0$ in $\Gamma(B)$. Then $(\alpha \cdot \beta)(f) = 0$ on $\Gamma(A \cup B) = \Gamma(V)$ so f is integral, which then implies that φ is finite.

(b) Again by induction and the argument in (a), we can assume that V is a single point. Then $\varphi^* : \Gamma(W) \to K$ is surjective and so φ is finite.

Problem 3. Let $\varphi : V \to W$ be a dominant morphism between irreducible algebraic sets. Assume $\Gamma(V)$ is generated by n elements as a $\varphi^*(\Gamma(W))$-module. Show that the preimage of any point $Q \in W$ has size at most n.

Since $\Gamma(V)$ is a finite $\varphi^*(\Gamma(W))$ extension with a generating set of size n, we can choose some generating set $f_1, \ldots, f_n \in \Gamma(V)$ such that any element $f \in \Gamma(V)$ can be written as

$$f = (g_1 \circ \varphi) \cdot f_1 + \cdots + (g_n \circ \varphi) \cdot f_n$$

for some $g_i \in \Gamma(W)$. Note that the restriction map $\Gamma(V) \to \Gamma(\varphi^{-1}(Q))$ gives a spanning set for $\Gamma(\varphi^{-1}(Q)$ of elements of the form $f_i\big|_{\varphi^{-1}(Q)}$ and we can choose arbitrary K-coefficients since $g_i(Q)$ can take on any value. Since $\Gamma(\varphi^{-1}(Q)) \cong K^{|\varphi^{-1}(Q)|}$, we thus have that $|\varphi^{-1}(Q)| \leq n$ by basic linear algebra.

Problem 4. Let L be a finitely generated field extension of K with $n = \operatorname{trdeg}(L/K)$ and let $R \subseteq L$ be a finitely generated ring extension of K whose field of fractions is L. Show that there are elements $a_1, \ldots, a_n$ of R such that R is an integral ring extension of $K[a_1, \ldots, a_n]$.

Recall the Noether normalization lemma:

Lemma (Noether normalization)**.** If V is irreducible algebraic, there is a finite morphism $V \to K^{\dim V}$.

We will show that this lemma implies our result. Note that since R is a finitely generated ring extension of K, it must be a finitely generated K-algebra so it is isomorphic to some $K[x_1, \ldots, x_m]/I$ for an ideal I. Since R must be an integral domain, (otherwise we wouldn't be able to take the field of fractions) it follows that I must be a prime ideal. However by lecture we can always find an irreducible algebraic set $V \subset K^m$ with $\mathcal{I}(V) = I$. Then since $\operatorname{trdeg}(\operatorname{Frac}(R)/K) = n$, V is n dimensional so by Noether's normalization lemma there is some finite morphism $\varphi : V \to K^n$. Equivalently, $R = \Gamma(V)$ is integral over $\varphi^*(K[t_1, \ldots, t_n])$. This second K-algebra can be identified with $K[a_1, \ldots, a_n]$ where $a_i = \varphi_i \in \Gamma(V)$ is the i-th coordinate function. This is what we were looking for.

Problem 5. Say $K = \mathbb{C}$. Construct a surjective but nonfinite morphism $\varphi : V \to W$ between irreducible algebraic sets such that every $P \in W$ has only finitely many preimages.

Let $V = \mathcal{V}(x^2y + z - x) \subset \mathbb{C}^3$ and let $W = \mathbb{C}^2$. Construct the morphism $\varphi : V \to W$ as taking a point (x, y, z) and sending it to (y, z). First, let's show that every point in W has a finite, nonempty preimage, this will show that φ is surjective. Notice that when $y = 0$ we have $\varphi(-z, 0, z) = (0, z)$, so we have a single preimage. When $y \neq 0$, we have

$$\varphi\left(\frac{1 \pm \sqrt{1 - 4yz}}{2y}, y, z\right) = (z, y)$$

so we have two preimages in this case. Now recall from lecture that finite morphisms are closed maps in the Zariski topology, so suppose for the sake of contradiction that φ is finite. Then consider the closed set $\mathcal{V}(z) \cap \mathcal{V}(x^2y + z - x)$. This is the algebraic set $\mathcal{V}(x(xy - 1))$. Note that the image of this set under φ is $(\mathbb{C} \setminus 0) \times \mathbb{C}$, which isn't closed (if it was, $\mathbb{C}^2$ would be reducible), a contradiction to the closedness of φ.