MATH 231BR: ADVANCED ALGEBRAIC TOPOLOGY
PROJECT 2: HOMOTOPY GROUPS OF SPHERES

In this project, you will compute the first eight 2-primary stable homotopy groups of spheres using Serre's method. To do this, you will inductively build up the mod $\mathcal{C}_2$ Postnikov tower of S^n in a range. Throughout this project, assume that $n \gg 0$, so that you are in the stable range.

Here's the idea: start with $S^n \to K(\mathbb{Z}, n)$ as your approximation. You will inductively kill the 2-local homology groups of this approximation to obtain a tower whose 2-local homology groups are isomorphic to those of S^n in an increasing range. For example, the first few 2-local homology groups of $K(\mathbb{Z}, n)$ are $H^n \cong \mathbb{Z}_{(2)}\{\iota_n\}, H^{n+1} \cong 0, H_{n+2} \cong \mathbb{Z}/2$. The generator of H_{n+2} is detected by $\mathrm{Sq}^2\,\iota_n$ in the mod 2 cohomology of $K(\mathbb{Z}, n)$, so we begin by taking the homotopy fiber X_1 of the map $K(\mathbb{Z}, n) \xrightarrow{\mathrm{Sq}^2} K(\mathbb{Z}/2, n+2)$. Then $S^n \to K(\mathbb{Z}, 2)$ factors as $S^n \to X_1 \to K(\mathbb{Z}, n)$, since $H^{n+2}(S^n; \mathbb{Z}/2) \cong 0$.

The map $S^n \to X_1$ is then a mod $\mathcal{C}_2$ $(n+2)$-equivalence, so that $\pi_{n+1}(S^n) \cong \pi_{n+1}(X_1) \cong \mathbb{Z}/2$ mod $\mathcal{C}_2$. To continue in this way, you will need to make constant use of the action of the Steenrod operation on mod 2 cohomology. On the other hand, you will need to know the 2-local homology to know what to kill. For this reason, it will be convenient to use the *Bockstein spectral sequence* to relate the mod 2 and 2-local cohomology.

(1) Given a complex of free $\mathbb{Z}_{(2)}$-modules C_*, construct the Bockstein spectral sequence

$$E_1^{s,t} = H_t(C_* \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}/2) \Rightarrow H_t(C_*).$$

Identify the first differential with the Bockstein operation. Supposing that each $H_k(C_*)$ is finitely generated, explain the sense in which the Bockstein sequence converges.

(2) Prove the Bockstein lemma, whose statement follows: Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration and $u \in H^n(F; \mathbb{Z}/2)$ be a transgressive class. Suppose that there exists $v \in H^n(B; \mathbb{Z}/2)$ and $i \geq 1$ such that $d_i v = \tau(u)$, where d_i is the ith differential in the Bockstein spectral sequence. Then $p^* v$ survives to the E_{i+1}-page of the Bockstein spectral sequence, and

$$j^* d_{i+1} p^*(v) = d_1(u).$$

(Hint: reduce to the universal case of $K(\mathbb{Z}/2, n) \to K(\mathbb{Z}/2^{i+1}, n) \to K(\mathbb{Z}/2^i, n)$.)

(3) Compute the mod 2 cohomology of $X_1 = \mathrm{fib}(K(\mathbb{Z}, n) \xrightarrow{\mathrm{Sq}^2} K(\mathbb{Z}/2, n+2))$ through degree 11. Keep track of the action of the Steenrod algebra and the Bockstein spectral sequence as well as you can. You will need to use the Bockstein lemma to compute some Bockstein differentials.

(4) Inductively define X_2, X_3, X_4, X_5 by killing the first 2-local homology group that differs from $H^*(S^n)$. You will have inductively compute the mod 2 cohomology of each X_i in a range and determine enough of the Bockstein spectral sequence to obtain the desired 2-local homology group.

You will need to use the fact that the transgression in the Serre spectral sequence commutes with Bockstein differentials when computing the mod 2 cohomology.

(5) Conclude that the first eight stable mod $\mathcal{C}_2$ homotopy groups of spheres are given by[1]

$$\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/8, 0, 0, \mathbb{Z}/2, \mathbb{Z}/16.$$

In conclusion, you should have constructed a tower of fiber sequences:

[1] It is much easier to compute the odd-primary contributions using analogous methods.

$$\begin{tikzcd}
K(\mathbb{Z}/16, n+7) \arrow[r] & X_5 \arrow[d] & \\
K(\mathbb{Z}/2, n+6) \arrow[r] & X_4 \arrow[r] \arrow[d] & K(\mathbb{Z}/16, n+8) \\
K(\mathbb{Z}/8, n+3) \arrow[r] & X_3 \arrow[r] \arrow[d] & K(\mathbb{Z}/2, n+7) \\
K(\mathbb{Z}/2, n+2) \arrow[r] & X_2 \arrow[r] \arrow[d] & K(\mathbb{Z}/8, n+4) \\
K(\mathbb{Z}/2, n+1) \arrow[r] & X_1 \arrow[r] \arrow[d] & K(\mathbb{Z}/2, n+3) \\
 & K(\mathbb{Z}, n) \arrow[r] & K(\mathbb{Z}/2, n+2),
\end{tikzcd}$$

and the first nine 2-local homology groups of X_5 (starting at H_n) should be $\mathbb{Z}_{(2)}, 0, 0, 0, 0, 0, 0, 0, 0$.

It is very easy to make mistakes when doing this kind of computation! For this reason, I have included the dimensions of the first several mod 2 cohomology groups of the X_i, so that you may check your work:

X_1 : 1 0 0 1 2 1 2 3 2 2 4 5
X_2 : 1 0 0 0 1 1 1 2 2 1 1
X_3 : 1 0 0 0 0 0 0 1 2 2
X_4 : 1 0 0 0 0 0 0 0 1
X_5 : 1 0 0 0 0 0 0 0 0