Math 55b Problem Set 11

Lev Kruglyak

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I collaborated with AJ LaMotta and Marvin Li for this problem set.

Problem 1. Find Laurent series expressions for the function $f(z) = \dfrac{1}{z(z-1)(z-2)}$:

(i) in the region $\{1 < |z| < 2\}$,

(ii) in the region $\{|z| > 2\}$.

We begin by using partial fractions to decompose this function into a sum of simple fractions, i.e.

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2}\left(\frac{1}{z}\right) - \frac{1}{z-1} + \frac{1}{2}\left(\frac{1}{z-2}\right).$$

(i) Since $1 < |z| < 2$, we have $|1/z| < 1$ and $|z/2| < 1$ so by geometric series we have

$$f(z) = \frac{1}{2}\left(\frac{1}{z}\right) - \frac{1}{z(1-1/z)} - \frac{1}{4}\left(\frac{1}{1-z/2}\right) = \frac{1}{2z} - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = \sum_{n=-\infty}^{\infty} a_n z^n$$

where the coefficients a_n of the Laurent series expansion are

$$a_n = \begin{cases} 1/2^{n+2} & n \geq 0 \\ -1/2 & n = -1 \\ -1 & n \leq -2 \end{cases}.$$

(ii) Here $|z| > 2$ so $|1/z|, |2/z| < 1$. Thus by geometric series we can similarly expand

$$f(z) = \frac{1}{2}\left(\frac{1}{z}\right) - \frac{1}{z(1-1/z)} + \frac{1}{2z}\left(\frac{1}{1-2/z}\right) = \frac{1}{2z} - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} + \frac{1}{2z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = \sum_{n=-\infty}^{\infty} a_n z^n$$

where the Laurent coefficients are given by

$$a_n = \begin{cases} 0 & n \geq -1 \\ 1/2^{n+2} - 1 & n \leq -2 \end{cases}.$$

Problem 2. Let f be an analytic function over a domain U which contains the closed disc $\{z \in \mathbb{C},\ |z| \leq 3\}$, and suppose that $f(1) = f(i) = f(-1) = f(-i) = 0$. Show that $|f(0)| \leq \dfrac{1}{80}\max_{|z|=3}|f(z)|$.

Let's consider the magically chosen function $g(z) = f(z)/(1-z^4)$. Notice that it is analytic over $U \setminus \{\pm 1, \pm i\}$, and undefined at those points. However $f(z)$ vanishes on $\{\pm 1, \pm i\}$, so we can extend $g(z)$ to an analytic function on all of U. Now apply the maximum-modulus principle to $g(z)$ on the disk $B_3(0)$ of radius 3 centered at the origin. We get that

$$|f(0)| = |g(0)| \leq \max_{z \in \partial B_3(0)} |g(z)| = \max_{z \in \partial B_3(0)} \frac{|f(z)|}{|1-z^4|}.$$

However on the radius 3 circle $\partial B_3(0)$, we have $|z^4 - 1| \geq |z^4| - 1 = 80$, so we get the desired claim that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|.$$

Problem 3. Let $f(z)$ be an analytic function over the annulus $R_1 < |z| < R_2$, and let $M(r) = \sup_{z \in S^1(r)} |f(z)|$. Prove that $\log M(e^s)$ is a convex function of $s \in (\log R_1, \log R_2)$.

Suppose for the sake of contradiction that $\log M(e^s)$ is not a convex function of $s \in (\log R_1, \log R_2)$. This means that there exists a $0 < t < 1$ and disctinct $s_1, s_2 \in (\log R_1, \log R_2)$ such that if $u = ts_1 + (1-t)s_2$ we have

$$\log(M(e^u)) > t \log(M(e^{s_1})) + (1-t) \log(M(e^{s_2})),$$

and raising e to the power of both sides we get

$$M(e^u) > M(e^{s_1})^t M(e^{s_2})^{1-t}.$$

We can assume without loss of generality that $M(e^s) > 0$ for all $s \in (\log R_1, \log R_2)$, (otherwise f would be identically zero) so we can consider the quantity

$$\Delta = \frac{M(e^u)}{M(e^{s_1})^t M(e^{s_2})^{1-t}} - 1 > 0.$$

Next, for any integers m, n with $n > 0$ and $r \in (R_1, R_2)$, we can define the notation

$$M_{m,n}(r) = \sup_{z \in S^1(r)} |z^m f(z)^n| = r^m M(r)^n.$$

We can similarly define $\Delta_{m,n}$ in the obvious way. We can check that $\Delta_{m,n} = (\Delta + 1)^n - 1 \geq \Delta$. Now apply the maximum-modulus principle to the function $z^m f(z)^n$ over the annulus $e^{s_1} < |z| < e^{s_2}$, which yields

$$e^{um} M(e^u)^n \leq \max(e^{s_1 m} M(e^{s_1})^n, e^{s_2 m} M(e^{s_2})^n).$$

This means that

$$\frac{M_{m,n}(e^u)}{\max(M_{m,n}(e^{s_1}), M_{m,n}(e^{s_2}))} \leq 1.$$

We will obtain a contradiction by showing that this ratio is in fact bigger than 1. We'll do this by showing that given any $\delta > 0$, we can find a tuple (m, n) such that $|M_{m,n}(e^{s_1}) - M_{m,n}(e^{s_2})| < \delta$, this would mean that

$$\left| \frac{M_{m,n}(e^u)}{M_{m,n}(e^{s_1})^t M_{m,n}(e^{s_2})^{1-t}} - \frac{M_{m,n}(e^u)}{\max(M_{m,n}(e^{s_1}), M_{m,n}(e^{s_2}))} \right| = \left| \Delta_{m,n} - \frac{M_{m,n}(e^u)}{\max(M_{m,n}(e^{s_1}), M_{m,n}(e^{s_2}))} \right|$$

becomes arbitrarily small, in fact lest that $(1+\Delta)^n - 1$, which implies that

$$\frac{M_{m,n}(e^u)}{\max(M_{m,n}(e^{s_1}), M_{m,n}(e^{s_2}))} > 1.$$

To find such a tuple, observe that

$$\frac{M_{m,n}(e^{s_1})}{M_{m,n}(e^{s_2})} = e^{m(s_1-s_2)}\left(\frac{M(e^{s_1})}{M(e^{s_2})}\right)^n.$$

Since we're trying to get the left side to be as close as possible to 1, set it equal to $1+\gamma$. Then

$$\log(1+\gamma) = m(s_1-s_2) + n\log\left(\frac{M(e^{s_1})}{M(e^{s_2})}\right) \implies \frac{m}{n} = \frac{n\log\left(\frac{M(e^{s_1})}{M(e^{s_2})}\right) - \log(1+\gamma)}{s_2-s_1}.$$

However for any $\gamma \approx$, we can always find an arbitrarily good rational approximation of $\frac{m}{n}$, so we can make the ratio approach 1, completing the contradiction.

Problem 4.

(a) We consider the following three domains in $\mathbb{C}$: $D = \{|z| < 1\}$, $H = \{\Re(z) > 0\}$, and $S = \{0 < \Im(z) < 1\}$ (the unit disc, the right half-plane, and an infinite horizontal strip), and their closures in $\mathbb{C}$. Find explicit homeomorphisms $\overline{D} - \{\pm 1\} \simeq \overline{H} - \{0\} \simeq \overline{S}$ whose restrictions to the interior are biholomorphisms (i.e. analytic maps with analytic inverses) $D \simeq H \simeq S$.

(b) Use this to find a continuous function $u : \overline{D} - \{\pm 1\} \to \mathbb{R}$ such that u is harmonic in D, $u(z) = 1$ on the upper half of the unit circle ($|z| = 1$ and $\Im(z) > 0$), and $u(z) = 0$ on the lower half of the unit circle ($|z| = 1$ and $\Im(z) < 0$).

(a) We'll first consider the Möbius transformation $f : \overline{D} \setminus \{\pm 1\} \to \overline{H} \setminus \{0\}$ given by $z \mapsto (z+1)/(1-z)$. Clearly,

$$\Re(f(z)) = \frac{1-|z|^2}{|1-z|^2},$$

so $f(z)$ is indeed a map from $\overline{D}\setminus\{\pm 1\} \to \overline{H}\setminus\{0\}$. Notice that f has an inverse given by $f^{-1}(z) = (z-1)/(z+1)$. The inverse clearly maps $\overline{H} \setminus \{0\}$ into $\overline{D} \setminus \{\pm 1\}$ since

$$|f^{-1}(z)| = \left|\frac{z-1}{z+1}\right| = \frac{|z|^2 + 1 - 2\Re(z)}{|z|^2 + 1 + 2\Re(z)}.$$

Since both f and f^{-1} are continuous, the map is a homeomorphisms. Similarly, $f\big|_D$ and its inverse are given by rational analytic functions so $f\big|_D$ is a biholomorphism.

Next, consider the map $g : \overline{H} \setminus \{0\} \to \overline{S}$ given by $z \mapsto \frac{1}{\pi}\log z + \frac{i}{2}$, where we take the branch of log with $\Im(\log z) \in (-\pi, \pi)$. Notice that it maps into $\overline{S}$ because $\arg(z) \in [-\pi/2, \pi/2]$ for all $z \in \overline{H} \setminus \{0\}$. (This can be easily seen geometrically.) More so, g has continuous inverse $g^{-1}(z) = \exp(\pi(z - \frac{1}{2}))$ so g is a homeomorphism. g and its inverse are also clearly analytic when restricted to the interior, so g is a biholomorphism.

(b) If we compose the functions from (a), we get a continuous map $g \circ f : \overline{D} \setminus \{\pm 1\}$ which becomes analytic when we restrict to D. Because it is analytic, the real-valued functions $\Re(g \circ f)$ and $\Im(g \circ f)$ are harmonic in D. The imaginary part of this function $\Im(g \circ f)$ then clearly satisfies the conditions of the problem.

Problem 5. How many roots of the equation $z^7 + 7z^4 - 3z^2 + 2 = 0$ satisfy $1 < |z| < 2$?

Let $f(z) = z^7 + 7z^4 - 3z^2 + 2$. We'll first count the number of roots with $|z| \leq 1$. On the boundary of this region, when $|z| = 1$, we have

$$|z^7 - 3z^2 + 2| \leq 1 + 3 + 2 = 6 < 7 = |7z^4|$$

so by Rouche's theorem, there are exactly 4 roots of $f(z)$ with $|z| \leq 1$. If we instead consider the region given by $|z| \leq 2$, on the boundary $|z| = 2$ we have

$$|7z^4 - 3z^2 + 2| \leq 7 \cdot 16 + 3 \cdot 4 + 2 \leq 126 < 128 = |z^7|$$

so by Rouche's theorem, there are exactly 7 roots in the region. Thus, there are $7 - 4 = 3$ roots in the region $1 < |z| < 2$.

Problem 6. Prove or disprove: if $f(z)$ is analytic on the unit disc D and has n zeros in D, then $f'(z)$ has at least $n-1$ zeros in D. What happens if "at least" is replaced by "at most"?

Recall that $\cos z$ has no zeroes in the unit disk, yet its derivative $\sin z$ has a zero in D at the origin. So $f'(z)$ does not necessarily have at most $n-1$ zeroes in D. Conversely, consider the function $f(z) = e^{8\pi i z^2} - 1$. $f(z)$ has at least 3 zeroes in the unit disk, namely $0, \pm 1/2$, yet its derivative $f'(z) = 16\pi i z(f(z)+1)$ has only one zero in the unit disk at the origin. This contradcits the claim that $f'(z)$ should have at least 2 zeroes in the unit disk.

Problem 7. Find all the singularities of $f(z) = z/(e^{z^2} - 1)$, and determine the residues at each of its poles.

We first must find the singularities of $f(z)$. Solving $e^{z^2} - 1 = 0$, we get that $z^2 \equiv 0 \mod 2\pi i$ so $z = \pm\sqrt{2\pi i k}$ for all $k \in \mathbb{Z}$. The simple pole at zero has residue one because

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z^2}{e^{z^2} - 1} = \lim_{c \to 0} \frac{c}{e^c - 1} = \left(\lim_{c \to 0} \frac{e^c - 1}{c} \right)^{-1} = 1.$$

For other singularities, of the form $z_0 = \pm\sqrt{2\pi i k}$, notice that the coefficient of $z - z_0$ in the Taylor expansion of $e^{z^2} - 1$ is $2z_0$, so we have $e^{z^2} - 1 = 2z_0(z - z_0) + (z - z_0)^2 g(z)$ for some analytic $g(z)$. Since this pole is simple, the residue is thus

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{z}{2z_0 + (z - z_0)g(z)} = \frac{1}{2}.$$