Physics 212 Problem Set 1

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Problem 1. Show explicitly that the 00 and ij component of the Einstein equations are given by

$$G^0_0 = -\frac{3}{a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + K\right] \quad \text{and} \quad G^i_j = -\frac{1}{a^2}\left[2\left(\frac{\ddot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2 + K\right]\delta^i_j$$

The dots here correspond to derivatives with respect to conformal time.

The Friedman-Roberston-Walker metric can be written as

$$g_{\mu\nu} = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega\right]$$

where $d\Omega = d\theta^2 + \sin^2(\theta)d\phi^2$ is the metric on a 2-dimensional sphere. The dual metric is then given by

$$g^{\mu\nu} = -dt^2 + \frac{1-Kr^2}{a^2(t)}dr^2 + \frac{1}{r^2a^2(t)}d\theta^2 + \frac{1}{r^2a^2(t)\sin^2(\theta)}d\phi^2.$$

We'll denote our coordinate system by $\{x^0, x^1, x^2, x^3\} = \{t, r, \theta, \phi\}$ to make some formulas simpler, but we will use these symbols interchangeable. Let's now calculate the Christoffel symbols. Let's adopt the convention that a' denotes differentiation with respect to coordinate time t and $\dot{a}$ denotes differentiation with respect to conformal time η.

Firstly, let's expand the metric derivatives $\partial g_{\mu\nu}/\partial x^\kappa$. It suffices to consider $\mu = \nu$ since the metric is diagonal in our chosen coordinate system.

Note that, $\partial g_{00}/\partial x^\nu = 0$ since g_{00} is constant. The other derivatives are given in the table:

$$\begin{aligned}
&\frac{\partial g_{11}}{\partial x^0} = a'(t)\cdot 2a(t)\cdot\frac{1}{1-Kr^2}, && \frac{\partial g_{11}}{\partial x^1} = a^2(t)\cdot -2Kr\cdot\frac{-1}{(1-Kr)^2}, && \frac{\partial g_{11}}{\partial x^2} = 0,\\
&\frac{\partial g_{22}}{\partial x^0} = a'(t)\cdot 2a(t)\cdot r^2, && \frac{\partial g_{22}}{\partial x^1} = a^2(t)\cdot 2r, && \frac{\partial g_{22}}{\partial x^2} = 0,\\
&\frac{\partial g_{33}}{\partial x^0} = a'(t)\cdot 2a(t)\cdot r^2\sin^2(\theta), && \frac{\partial g_{33}}{\partial x^1} = a^2(t)\cdot 2r\cdot\sin^2(\theta), && \frac{\partial g_{33}}{\partial x^2} = \cos(\theta)\cdot 2\sin(\theta)\cdot a^2(t)r^2.
\end{aligned} \tag{1}$$

Next, recall that the Christoffel symbols are defined to be

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right].$$

Since the metric is diagonal, we can rewrite this formula as

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\mu}}{2}\left[\frac{\partial g_{\alpha\mu}}{\partial x^\beta} + \frac{\partial g_{\beta\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right]. \tag{2}$$

By the torsion-free property of the Levi-Civita connection, we have the symmetry $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$. At this point, we'll drop the function dependence of $a(t)$, $\sin(\theta)$, and $\cos(\theta)$ for brevity since we aren't taking derivatives any longer and since the variables of these functions are unambiguous. We can now plug the expressions in (1) into (2). To speed things up, we note that whenever α or β are zero, $\Gamma^\mu_{\alpha\beta} = 0$ since the derivatives of g_{00} are all zero. Also, the indices μ, α, β must have at least one

When $\mu = 0$, the only nonzero Christoffel symbols are

$$\Gamma^0_{\alpha\alpha} = -\frac{1}{2}\left[-\frac{\partial g_{\alpha\alpha}}{\partial x^0}\right] \implies \Gamma^0_{11} = \frac{a'a}{1-Kr^2}, \quad \Gamma^0_{22} = a'ar^2, \quad \Gamma^0_{33} = a'ar^2\sin^2. \tag{3}$$

When $\mu \in \{1, 2, 3\}$ we have a slightly more complicated pattern. Here, the nonzero Christoffel symbols are

$$\begin{aligned}
\Gamma^1_{01} &= \frac{1-Kr^2}{2a^2} \cdot \frac{2a'a}{1-Kr^2} = \frac{a'}{a}, & \Gamma^1_{11} &= \frac{1-Kr^2}{2a^2} \cdot \frac{2Kra^2}{(1-Kr)^2} = \frac{Kr}{1-Kr^2}, \\
\Gamma^1_{22} &= \frac{1-Kr^2}{2a^2} \cdot (-2ra^2) = -r(1-Kr^2), & \Gamma^1_{33} &= \frac{1-Kr^2}{2a^2} \cdot 2ra^2\sin^2 = \sin^2 r(1-Kr^2), \\
\Gamma^2_{02} &= \frac{1}{2r^2a^2} \cdot 2a'r^2 = \frac{a'}{a}, & \Gamma^2_{12} &= \frac{1}{2r^2a^2} \cdot 2ra^2 = \frac{1}{r}, \\
\Gamma^2_{33} &= \frac{1}{2r^2a^2} \cdot 2a^2r^2\cos\sin = \cos\sin, & \Gamma^3_{03} &= \frac{1}{2r^2a^2\sin^2} \cdot 2a'ar^2\sin^2 = \frac{a'}{a} \\
\Gamma^3_{13} &= \frac{1}{2r^2a^2\sin^2} \cdot 2r\sin^2 a^2 = \frac{1}{r}, & \Gamma^3_{23} &= \frac{1}{2r^2a^2\sin^2} \cdot 2\sin\cos a^2r^2 = \cot.
\end{aligned} \tag{4}$$

Now that we've calculated all of the Christoffel symbols, let's calculate the Ricci curvature tensor. In terms of the Christoffel symbols, the Ricci tensor can be written as

$$R_{\mu\nu} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}. \tag{5}$$

At this point, the computations get a bit tedious, so we omit the details and only state the results. If we plug the equations (3) and (4) into (5), we get the following nonzero components for the Ricci tensor

$$R_{00} = -\frac{3a''}{a}, \quad R_{ii} = g_{ii}\left(\frac{2K + 2(a')^2 + aa''}{a^2}\right). \tag{6}$$

Here, we've simplified the expression R_{ii} by factoring out terms which also appear in the metric tensor. Finally, let's compute the Ricci scalar.

$$R = g^{00}R_{00} + g^{ii}R_{ii} = 3\left(\frac{a''}{a}\right) + 3\left(\frac{2K + 2(a')^2 + aa''}{a^2}\right) = \frac{6(K + (a')^2 + aa'')}{a^2}.$$

Since we have both the Ricci tensor and scalar, we can now compute the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \implies G_{00} = -\frac{3a''}{a} - \frac{3(K + (a')^2 + aa''^2)}{a^2} = \frac{3(K + (a')^2)}{a^2},$$

Contracting to get G^0_0, we get

$$G^0_0 = g^{0\alpha}G_{0\alpha} - G_{00} = \frac{-3(K + (a')^2)}{a^2}$$

The remaining terms of the Einstein tensor are

$$\begin{aligned}
G_{ii} = g_{ii}\left[\frac{2K + 2(a')^2 + aa''}{a^2}\right] - \frac{1}{2}g_{ii}\left[\frac{6(K + (a')^2 + aa'')}{a^2}\right] &= -g_{ii}\left(\frac{K + (a')^2 + 2aa''}{a^2}\right) \\
\implies \quad G^i_i &= -\frac{1}{a^2}(K + (a')^2 + 2aa'').
\end{aligned}$$

Putting these results together, we get the expressions

$$G_0^0 = \frac{-3K}{a^2} - 3\left(\frac{a'}{a}\right)^2 \quad \text{and} \quad G_i^i = -\frac{K}{a^2} - \left(\frac{a'}{a}\right)^2 - 2\left(\frac{a''}{a}\right).$$

Finally, we transform derivatives to with respect to conformal time by the transformation law

$$\frac{\partial a}{\partial t} = \frac{1}{a}\frac{\partial a}{\partial \eta} \quad \text{and} \quad \frac{\partial^2 a}{\partial t^2} = \frac{\partial^2 a}{\partial \eta^2}\frac{1}{a^2} - \left(\frac{1}{a}\frac{\partial a}{\partial \eta}\right)^2.$$

In abbreviated notation, this means $a' = \dot{a}a$ and $a'' = \ddot{a}/a^2 - (\dot{a}/a)^2$. This results in the desired formulas

$$\boxed{G_0^0 = -\frac{3}{a^2}\left[K + \left(\frac{\dot{a}}{a}\right)^2\right]}, \quad \text{and} \quad \boxed{G_i^j = -\frac{1}{a^2}\left[K + 2\left(\frac{\ddot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2\right]\delta_i^j}.$$