Math 114 Problem Set 8

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Problem 1. Let $\mathcal{H}$ denote a Hilbert space, and $\mathcal{L}(\mathcal{H})$ the vector space of all bounded linear operators on $\mathcal{H}$.

Given $T \in \mathcal{L}(\mathcal{H})$, we define the operator norm

$$\|T\| = \inf\{B : |Tv| \leq B|v|, \quad \text{for all } v \in \mathcal{H}\}.$$

a. Show that $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$ whenever $T_1, T_2 \in \mathcal{L}(\mathcal{H})$.

Recall that we have

$$\|T\| = \sup_{|v|=1} |Tv| = \sup_{v \neq 0} |Tv|/|v|.$$

Thus for any $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and some vector $v \in \mathcal{H}$ we have

$$|T_1 v + T_2 v| \leq |T_1 v| + |T_2 v| \leq (\|T_1\| + \|T_2\|)|v| \implies \|T_1 + T_2\| \leq \|T_1\| + \|T_2\|.$$

b. Prove that

$$d(T_1, T_2) = \|T_1 - T_2\|$$

defines a metric on $\mathcal{L}(\mathcal{H})$.

Nonnegativity is obvious, and we proved the triangle inequality in the previous problem. For definiteness, suppose $\|T\| = 0$. For all $v \in \mathcal{H}$, this implies that $|Tv| \leq 0$ so $T = 0$. Laslty, for any $c \in k$ we have $\|cT\| = |c|\|T\|$.

c. Show that $\mathcal{L}(\mathcal{H})$ is complete in the metric d.

Suppose we have a Cauchy sequence $\{T_i\}_{i=1} \subset \mathcal{L}(\mathcal{H})$. For all $v \in \mathcal{H}$ we have a Cauchy sequence $\{T_i v\}_{i=1}$ since $|T_n v - T_m v| \leq \|T_n - T_m\||v|$. Since $\mathcal{H}$ is a complete space by definition, the sequence $\{T_i v\}_{i=1}$ converges. Thus let's define $Tv = \lim T_i v$.

Since addition and scalar multiplication are continuous, we obviously have linearity of T. All we need to show is that $T \in \mathcal{L}(\mathcal{H})$ and $\lim T_i = T$. First of all, since T_i is a Cauchy sequence, there is some M with $\|T_n\| \leq M$ so for all $v \in \mathcal{H}$ we have $|T_i v| \leq M|v|$. Thus we have $|Tv| \leq |Tv - T_i v| + M|v|$. As $i \to \infty$, we get $|Tv| \leq M|v|$ so $\|T\| \leq M$ and thus $T \in \mathcal{L}(\mathcal{H})$.

Lastly, let $\epsilon > 0$. Since T_i is a Cauchy sequence, there is some N with $n, m \geq N \implies \|T_m - T_n\| < \epsilon/2$. For any unit vector v we have $m \geq N$ with $|Tv - T_m v| < \epsilon/2$. Thus

$$|(T - T_n)v| = |Tv - T_m v + T_m v - T_n v| \leq |Tv - T_m v| + |(T_m - T_n)v| < \epsilon \quad \forall n \geq N.$$

Thus $\|T - T_n\| \leq \epsilon$ for all $n \geq N$ so $\lim T_n = T$.

Problem 2. Prove that the operator
$$Tf(x) = \frac{1}{\pi}\int_0^\infty \frac{f(y)}{x+y}dy$$
is bounded on $L^2(0,\infty)$ with norm $\|T\| \le 1$.

The result of Homework 7 Problem 4 would hold for $(0,\infty)$, so we only need to find some (measurable) function $0 < w(x) < \infty$ on $(0,\infty)$ with
$$\frac{1}{\pi}\int_0^\infty \frac{w(y)}{x+y}\,dy \le w(x) \quad \text{a.e.} \quad x > 0.$$
To apply Problem 4, set $K(x,y) = 1/\pi(x+y)$ and $a = 1$. Letting $w(y) = y^{-1/2}$, we get
$$\frac{1}{\pi}\int_0^\infty \frac{y^{-1/2}}{x+y}\,dy = x^{-1/2} \implies \|T\| \le 1.$$

Problem 3. Let $\mathcal{H}$ be a Hilbert space.

Prove the following variants of the spectral theorem.

a. If T_1 and T_2 are two linear symmetric and compact operators on $\mathcal{H}$ that commute (that is, $T_1T_2 = T_2T_1$), show that they can be diagonalized simultaneously. In other words, there exists an orthonormal basis for $\mathcal{H}$ which consists of eigenvectors for both T_1 and T_2.

First, let's show that T_1 and T_2 contain a common eigenvector. Since T_1 is a compact symmetric operator, it has an eigenvalue λ with an eigenvector, so the eigenspace $V_\lambda^{T_1}$ is a nontrivial subspace of $\mathcal{H}$. For some $v \in V_\lambda^{T_1}$, we have $T_1T_2v = T_2T_1v = T_2(\lambda v) = \lambda T_2 v$. Thus $T_2(V_\lambda^{T_1}) \subset V_\lambda^{T_1}$. So T_2 is a compact symmetric linear operator on $V_\lambda^{T_1}$ so it has a nonzero eigenvector in $v \in V_\lambda^{T_1} \cap V_\lambda^{T_2}$.

Let $\mathcal{S}$ be the closure of $V_\lambda^{T_1} \cap V_\lambda^{T_2}$. This is nontrivial, so we have $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. Note that T_1 and T_2 are invariant on the space $\mathcal{S}^\perp$, and since they're compact symmetric linear operators, they have a common eigenvector $v \in \mathcal{S}^\perp$. Then $v \in \mathcal{S} \cap \mathcal{S}^\perp$ which is a contradiction.

b. A linear operator on $\mathcal{H}$ is *normal* if $TT^* = T^*T$. Prove that if T is normal and compact, then T can be diagonalized.

Consider the compact operators
$$T_1 = \frac{T+T^*}{2}, \quad T_2 = \frac{T-T^*}{2i} \quad \text{so that} \quad T = T_1 + iT_2.$$
These operators are also symmetric since
$$T_1^* = \left(\frac{T+T^*}{2}\right)^* = \frac{T^*+T^{**}}{2} = \frac{T^*+T}{2} = T_1.$$
The same follows for T_2. Lastly, we have
$$T_1T_2 = \frac{(T+T^*)(T-T^*)}{4i} = \frac{T^2-(T^*)^2}{4i} = \frac{(T-T^*)(T+T^*)}{4i} = T_2T_1$$
so we can apply (a) to simultaneously diagonalize T_1 and T_2. Say v_i is an orthonormal basis for $\mathcal{H}$ with eigenvalues λ_i, ζ_i for T_1 and T_2 respectively. Then $(\lambda_i + i\zeta_i)v_i$ diagonalizes T.

Problem 4. Suppose ν, ν_1, and ν_2 are signed measures on $(X, \mathcal{M})$ and μ is a (positive) measure on $\mathcal{M}$.

Using the symbols $\perp$ and $\ll$ defined in Section 4.2, prove:

a. If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.

Suppose $A_1, A_2, B \in \mathcal{M}$ are such that A_1 and A_2 are both disjoint from B, and ν_i is supported on A_1 and A_2. Let μ be supported on B. Now let Let $A = A_1 \cup A_2 \in \mathcal{M}$. Then $A \cap B = (A_1 \cap B) \cup (A_2 \cap B) = \varnothing$. Then for any $E \in \mathcal{M}$, we have

$$\begin{aligned}(\nu_1 + \nu_2)(E \cap A) &= \nu_1(E \cap A \cap A_1) + \nu_2(E \cap A \cap A_2) \\ &= \nu_1(E \cap A_1) + \nu_2(E \cap A_2) = (\nu_1 + \nu_2)(E).\end{aligned}$$

So $\nu_1 + \nu_2$ is supported on A, and thus $\nu_1 + \nu_2 \perp \mu$.

b. If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.

Let $E \in \mathcal{M}$ be a μ-measure zero set. Then we have $\mu_1(E) = \mu_2(E) = 0$ and so $(\mu_1 + \mu_2)(E) = \mu_1(E) + \mu_2(E) = 0$. This implies $\mu_1 + \mu_2 \ll \mu$.

c. $\nu_1 \perp \nu_2$ implies $|\nu_1| \perp |\nu_2|$.

Let $A_1, A_2 \in \mathcal{M}$ be disjoint sets with ν_1 supported on A_1 and ν_2 supported of A_2. Let $E \in \mathcal{M}$. For any partition $\{E_j\}$ of E, we get a partition $\{E_j \cap A_i\}$ of $E \cap A$ and so

$$\sum_{j=1}^{\infty} |\mu_i(E_j)| = \sum_{j=1}^{\infty} |\mu_i(E_j \cap A)| \leq |v|(E \cap A_i).$$

Thus $|\mu_i|(E) \leq |\mu_i|(E \cap A_i)$. The other direction follows trivially, so $|\mu_i|(E) = |\mu_i|(E \cap A_i)$. As a result, we have $|\mu_1| \perp |\mu_2|$.

d. $\nu \ll |\nu|$.

Let $E \in \mathcal{M}$ be a set with $|\mu|$-measure zero. Then $|\nu(E)| \leq |\nu|(E) = 0$ by taking the partition $(E, \varnothing, \varnothing, \ldots)$. Thus $\nu(E) = 0$ and so $\nu \ll |\nu|$.

e. If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

Let $A, B \in \mathcal{M}$ be disjoing and ν and μ supported on A and B respectively. Suppose $E \in \mathcal{M}$. Then $\mu(E \cap A) = \mu(E \cap A \cap B) = \mu(\varnothing) = 0$, so $\nu \ll \mu$ implies $\nu(E) = \nu(E \cap A) = 0$. Thus $\nu = 0$.

Problem 5. Examples of compactly supported functions in $\mathcal{S}(\mathbb{R})$ are very handy in many applications in analysis.

Some examples are:

a. Suppose $a < b$, and f is the function such that $f(x) = 0$ if $x \le a$ or $x \ge b$ and

$$f(x) = e^{-1/(x-a)} e^{-1/(b-x)} \quad \text{if } a < x < b.$$

Show that f is infinitely differentiable on $\mathbb{R}$.

Consider the function $g(x)$ given by

$$g(x) = \begin{cases} e^{-1/x} & x > 0, \\ 0 & x \le 0. \end{cases}$$

We'll begin by showing that g is C^∞. This function is obviously C^∞ when $x \neq 0$. Thus we only need to show that $g^{(n)}(x) = 0$ as $x \to 0^+$. Note that repeated differentiation gives us $g^{(n)}(x) = p(1/x)e^{-1/x}$ for some polynomial $p(x) \in \mathbb{R}[x]$. Since $e^{-1/x}/x^n \to 0$ as $x \mapsto 0^+$, we are done, since $f(x) = g(x-a)g(b-x)$.

b. Prove that there exists an infinitely differentiable function F on $\mathbb{R}$ such that $F(x) = 0$ if $x \le a$, $F(x) = 1$ if $x \ge b$, and F is strictly increasing on $[a, b]$.

Let $c = \int_{-\infty}^{\infty} f(t)\ dt$ and let $F(x) = \frac{1}{c} \int_{-\infty}^{x} f(t)\ dt$. Note that F is C^∞ and strictly increasing on $[a, b]$. By the fundamental theorem of calculus, $F'(x) = f(x)$. The rest follows obviously from the fact that f is supported on $[a, b]$ and from our choice of c.

c. Let $\delta > 0$ be so small that $a + \delta < b - \delta$. Show that there exists an infinitely differentiable function g such that g is 0 if $x \le a$ or $x \ge b$, g is 1 on $[a + \delta, b - \delta]$, and g is strictly monotonic on $[a, a + \delta]$ and $[b - \delta, b]$.

Construct functions F and G as tin the previous part on the intervals $[a, a+\delta]$ and $[-b, \delta-b]$ respectively, and let $g(x) = F(x)G(-x)$. This clearly C^∞. The required facts follow immediately from the previous problem.