Math 132
Problem Set 8
Spring, 2023

This problem set is due on Friday, April 7th Please make your answers as complete and clear as possible. You are allowed to discuss these problems with others in the class, but your writing should be your own.

You are only required to turn in four of these problems.
(though it would be a good idea to understand how to work all of them)

I've been lecturing from Chapters VII and VIII of Part II the lecture notes. I've also been skipping a lot of proofs. You can find the definitions I've been using in Sections 2 and 3 of Chapter VII.
The following are exercises 3.1-3.3 in the lecture notes.

1. Suppose that $f_0 : M \to X$ and $f_1 : M \to X$ are homotopic maps. Show that f_0 and f_1 are cobordant when regarded as manifolds over X.

2. Suppose that $g : X \to Y$ is a continuous map of topological spaces. Show that the map sending $f : M \to X$ to $g \circ f : M \to Y$ induces a linear transformation $f_* : MO_n(X) \to MO_n(Y)$. Show that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are two composable maps then $(g \circ f)_* = g_* \circ f_*$, or in other words that the diagram

$$\begin{tikzcd} MO_d(X) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & MO_dY \arrow[d, "g_*"] \\ & MO_dZ \end{tikzcd}$$

commutes.

3. Suppose that M is a manifold. Show that every continuous function $f : \partial M \to \mathbb{R}$ extends to a continuous function $g : M \to \mathbb{R}$. Using this for every k, the map

$$MO_k(\mathbb{R}^n) \to MO_k(\text{pt}) = MO_k$$

is an isomorphism. (Hint: Use a collar neighborhood).

For the following problems you will need the Moving Lemma VII.2.1

Theorem (Moving Lemma)**.** *Any 1-manifold $f : M \to X$ over X is cobordant one which is transverse to Z and has the property that*

$$\#\{f^{-1}(Z)\} < 2.$$

4. Work Exercise IX.1.2 of Part II of the lecture notes.

5. Work Exercise IX.1.3 of Part II of the lecture notes.

6. Compute $MO_1\mathbf{RP}^2$.

7. Compute MO_1K, where K is the Klein bottle, defined as the quotient of $\mathbb{R}^2$ by the equivalence relation

$$(x, y) = (x + 1, y)$$
$$(x, y) = -(x, y + 1).$$

Hint: The set of points with $y = 0$ is a circle. Every point equivalent to one with $y \neq 0$ is equivalent to a unique (x, y) with $0 < y < 1$ and this set is diffeomorphic to with

$$S^1 \times (0, 1)$$

by the map

$$(x, y) \mapsto (e^{2\pi i x}, y).$$

8. Suppose that Y is a smooth, compact, closed manifold of dimension n and $i : Z \to Y$ is the inclusion of a closed submanifold of dimension k. Recall that if X is a smooth compact closed manifold of dimension m, and $f : X \to Y$ is a smooth map which is transverse to Z, then $f^{-1}Z$ is a submanifold of X of dimension $m + k - n$. Show that this operation defines a linear map

$$MO_n(Y) \to MO_{m+k-n}(X).$$

For this problem you will need Proposition 3.9 of Part II, Chapter VII of the lecture notes.