Math 129 Problem Set 9

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Problem 6.1. Fill in the proof of Theorem 39:

(a) Prove Lemma 1.

(b) Verify that D is homogenous when $\upsilon = (1, \dots, 2)$.

(c) Be sure you believe that f is Lipschitz.

(d) Verify that f is the composition of the f_i and that the f_i preserve open sets.

(a) The statement of Lemma 1 is:

Lemma. Let $f : G \to G'$ be a homomorphism is abelian groups and let S be a subgroup of G which is carried isomorphically onto a subgroup $S' \subset G'$. Suppose D' is a set of coset representatives for S' in G'. Then its total inverse image $D = f^{-1}(D')$ is a set of coset representatives for S in G.

Proof. We'll prove this in two parts, first showing that no two elements in D represent the same coset for S in G, then showing that every coset for S in G is represented by an element of D. Let $d_1, d_2 \in D$ be distinct, and suppose for the sake of contradiction that $d_1 + S = d_2 + S$, or equivalently $d_2 - d_1 \in S$. Then $f(d_2 - d_1) = f(d_2) - f(d_1) \in S'$, so $f(d_2) + S' = f(d_1) + S'$. Since f is an isomorphism when restricted to S, these are distinct elements of D', contradicting the fact that D' are coset representatives for S' in G'. Next, to prove that every coset for S in G is represented by an element of D, let $g + S$ be any coset. Let d' be a representative of the coset $f(g) + S'$, then clearly $f^{-1}(d') + S = g + S$ and $f^{-1}(d') \in D$ so we are done. ☐

(b) Recall that the logarithm function is a function $\log : (\mathbb{R}^*)^r \times (\mathbb{C}^*)^s \to \mathbb{R}^{r+s}$, given by

$$\log(x_1, \dots, x_r, z_1, \dots, z_s) = (\log|x_1|, \dots, \log|x_r|, 2\log|z_1|, \dots, 2\log|z_s|).$$

Thus for any vector $x \in (\mathbb{R}^*)^r \times (\mathbb{C}^*)^s$ and $a \in \mathbb{R}^*$, we have $\log(ax) = a\upsilon + \log(x)$, where $\upsilon = (1, \dots, 2)$. Recall now that D is the set

$$D = \{x \in (\mathbb{R}^*)^r \times (\mathbb{C}^*)^s \mid \log x \in F \oplus \mathbb{R}\upsilon\}$$

where F is a fundamental parallelotype for Λ_U. Thus for any $a \in \mathbb{R}^*$ and $x \in D$, we have $\log(ax) = a\upsilon + \log(x) = \upsilon(a + t) + f$ for some $t \in \mathbb{R}$ and $f \in F$, where $\log(x) = f + \upsilon t$. So $ax \in D$ and so D is homogenous.

(c) As the proof notes, $f : [0,1]^n \to \mathbb{R}^r \times \mathbb{C}^s$ has all of its partial derivatives, and since $[0,1]^n$ is a compact space, the partial derivatives must be bounded. We just have to show that this implies that f is a Lipschitz function, which means that there exists some bound $B > 0$ such that for any $x, y \in [0,1]^n$ we have

$$\frac{|f(x) - f(y)|}{|x - y|} \le B.$$

By the triangle inequality it suffices to show that f is Lipschitz (with independent Lipschitz constant) on every line parallel to a coordinate axis. However this is clear to see by the one dimensional mean value theorem that this is the case.

(d) The fact that f is the composition of all of the f_i is obvious, and follows straight from the definitions of f_i as well as the parametrization. f_1 is an open map because it is the identity everywhere except for the $(r+s)$th coordinate, and log is an open map on $(0,1)$. f_2 is an open map because it is a linear transformation of rank n, so it is invertible. f_3 is again an open map because it applies open maps to each of the coordinates, same for f_4.

Problem 6.2. Fill in the proof of Theorem 40:

(a) Verify that the $\frac{\partial w_j}{\partial t_k}$ are as claimed.

(b) Verify that $J(t_1, \ldots, t_n)$ is as claimed.

(c) Verify that $x_1 \cdots x_r \rho_1^2 \cdots \rho_s^2 = t_{r+s}^n$

(a) Recall that the w_j are defined as

$$w_j = \begin{cases} t_{r+s} \exp\left(\sum_{k=1}^{r+s-1} t_k \upsilon_k^{(j)}\right) & j \leq r \\ t_{r+s} \exp\left(\frac{1}{2} \sum_{k=1}^{r+s-1} t_k \upsilon_k^{(j)}\right) & r < j \leq r+s \\ 2\pi t_j & j > r+s \end{cases}$$

Thus is it quite easy to see that for $k < r+s$ we have

$$\frac{\partial w_j}{\partial t_k} = \begin{cases} \upsilon_k^{(j)} t_{r+s} \exp\left(\sum_{i=1}^{r+s-1} t_i \upsilon_i^{(j)}\right) = \upsilon_k^{(j)} w_j & j \leq r \\ \frac{1}{2} \upsilon_k^{(j)} t_{r+s} \exp\left(\sum_{i=1}^{r+s-1} t_i \upsilon_i^{(j)}\right) = \frac{1}{2} \upsilon_k^{(j)} w_j & r < j \leq r+s \\ 0 & j > r+s \end{cases}$$

When $k = r+s$, w_j is linear in t_k so we have

$$\frac{\partial w_j}{\partial t_k} = \begin{cases} \exp\left(\sum_{i=1}^{r+s-1} t_i \upsilon_i^{(j)}\right) = \frac{w_j}{t_{r+s}} & j \leq r+s \\ 0 & j > r+s \end{cases}$$

Lastly, for $k > r+s$ we also have the simple linear relation

$$\frac{\partial w_j}{\partial t_k} = \begin{cases} 2\pi & j = k \\ 0 & \text{otherwise} \end{cases}$$

This is exactly the form given in the proof so we are done.

(b) Recall that the Jacobian has the form

$$J(t_1, \ldots, t_n) = \begin{vmatrix} \frac{\partial w_1}{\partial t_1} & \cdots & \frac{\partial w_n}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_1}{\partial t_n} & \cdots & \frac{\partial w_n}{\partial t_n} \end{vmatrix} = \begin{vmatrix} A & B & 0 \\ & C & \\ 0 & D & 0 \end{vmatrix}.$$

Here A is the Jacobian matrix $\left|\frac{\partial x_j}{t_k}\right|$, $B = \left|\frac{\partial \rho_j}{\partial t_k}\right|$, C is the single row consisting of w_j / t_{r+s}, and D is a single vertical column consisting of 2π. Factoring out constant terms and through a sequence of row operations, we get

$$J(t_1, \ldots, t_n) = \frac{\pi^s x_1 \cdots x_r \rho_1 \cdots \rho_s}{t_{r+s}} |M|$$

where M has equal determinant to the M matrix from earlier in the proof.

(c) Notice that

$$x_1 \cdots x_r \rho_1^2 \cdots \rho_s^2 = t_{r+s}^{r+2s} \exp\left(\sum_{j=1}^{r+s} \sum_{k=1}^{r+s-1} t_k v_k^{(j)}\right) = t_{r+s}^n$$

since the sum of the coordinates of v_k must be zero.

Problem 5.48. For $m \geq 3$, set $\omega = e^{2\pi i/m}$ and $\alpha = e^{\pi i/m}$.

(a) Show that

$$1 - \omega^k = -2i\alpha^k \sin(k\pi/m)$$

for all $k \in \mathbb{Z}$; conclude that

$$\frac{1-\omega^k}{1-\omega} = \alpha^{k-1} \frac{\sin(k\pi/m)}{\sin(\pi/m)}.$$

(b) Show that if k and m are not both even, then $\alpha^{k-1} = \pm\omega^h$ for some $h \in \mathbb{Z}$.

(c) Show that if k and m are relatively prime, then

$$u_k = \frac{\sin(k\pi/m)}{\sin(\pi/m)}$$

is a unit in $\mathbb{Z}[\omega]$.

(a) Recall that for real θ, we have

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Then $1-\omega^k = \alpha^k\alpha^{-k} - \alpha^{2k} = -\alpha^k(\alpha^k - \alpha^{-k})$. Using the definition of sin, this in turn is equal to $-2i\alpha^k(e^{k\pi i/m} - e^{-k\pi i/m}) = -2i\alpha^k \sin(k\pi/m)$ which naturally implies

$$\frac{1-\omega^k}{1-\omega} = \alpha^{k-1} \frac{\sin(k\pi/m)}{\sin(\pi/m)}.$$

(b) Let h be some solution to the modular equation $k - 1 \equiv 2h \mod m$. Notice that this equation has a solution since k and m are not both even. Then

$$\alpha^{k-1} = \alpha^{2h+mt} = \alpha^{mt}\omega^h = e^{t\pi i}\omega^h = \pm\omega^h$$

for some $t \in \mathbb{Z}$.

(c) Notice that by (b), α^{k-1} is a unit, so it suffices to show that $(1-\omega^k)/(1-\omega)$ is a unit. To do this, we'll prove that its norm is 1. Recalling that $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = (\mathbb{Z}/m\mathbb{Z})^*$, we have

$$\mathrm{N}_{\mathbb{Q}}^{\mathbb{Q}(\omega)}\left(\frac{1-\omega^k}{1-\omega}\right) = \prod_{a=1}^{m-1}\left(\frac{1-\omega^{ak}}{1-\omega^a}\right) = \prod_{z \in \mu_m^*}\left(\frac{1-z^k}{1-z}\right) = \prod_{z \in \mu_m^*}\sum_{a=0}^{k-1} z^a.$$

However since k is relatively prime to m, the terms cancel by the property $1 + \omega + \omega^2 + \cdots + \omega^{m-1} = 1$, and we are left with 1. So the u_k must be a unit.