SIG / Maven Spring Week 2025

Probability & Trading Interview Prep

Contents

1 Mental Math & Quick Conversions

Essential quick calculations

- **Probability to Odds:** If $P(A) = p$, then odds FOR are $p : (1-p)$ or $\frac{p}{1-p}$ to 1
- **Odds to Probability:** If odds are $a : b$, then $P(A) = \frac{a}{a+b}$
- **Common fractions:** $1/6 \approx 0.167$, $1/7 \approx 0.143$, $1/8 = 0.125$, $1/9 \approx 0.111$
- **Powers of 2:** $2^{10} = 1024$, $2^{20} \approx 10^6$
- **Factorials:** $5! = 120$, $6! = 720$, $7! = 5040$, $10! = 3{,}628{,}800$
- **Common binomial coefficients:** $\binom{52}{5} = 2{,}598{,}960$, $\binom{10}{3} = 120$, $\binom{n}{2} = \frac{n(n-1)}{2}$

Trading-specific conversions

- **Fair price from probability:** If $P(\text{win}) = p$ and payoff is \$1, fair price $= p \times 1$
- **Expected profit:** $E[\text{profit}] = P(\text{win}) \times \text{payoff} - \text{cost}$
- **Breakeven probability:** To break even on a \$1 bet paying \$X: need $P(\text{win}) \geq \frac{1}{X}$

2 Counting & Combinatorics

Core formulas

- **Permutations** (order matters): $P(n,k) = \frac{n!}{(n-k)!}$
- **Combinations** (order doesn't): $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Stars and bars** (distribute n identical items into k bins): $\binom{n+k-1}{k-1}$
- **Inclusion-exclusion (2 sets):** $|A \cup B| = |A| + |B| - |A \cap B|$
- **Inclusion-exclusion (3 sets):** $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- **Derangements** (permutations with no fixed points): $D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \approx \frac{n!}{e}$

Practice problems

1. **5-digit codes with no repeated digits:** $P(10,5) = \frac{10!}{5!} = 30{,}240$
2. **Committee of 3 from 12 people:** $\binom{12}{3} = 220$
3. **Poker hands:** Number of different 5-card hands: $\binom{52}{5} = 2{,}598{,}960$
4. **Anagrams of MISSISSIPPI:** $\frac{11!}{4!4!2!} = 34{,}650$ (divide by repeated letters)
5. **Distributing 10 identical candies to 4 children:** $\binom{10+4-1}{4-1} = \binom{13}{3} = 286$

3 Probability Fundamentals

Core formulas

- **Conditional probability:** $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- **Multiplication rule:** $P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$

- **Independence:** A, B independent $\Leftrightarrow P(A \cap B) = P(A)P(B) \Leftrightarrow P(A \mid B) = P(A)$
- **Law of total probability:** $P(A) = \sum_i P(A \mid B_i)P(B_i)$ where $\{B_i\}$ partitions the sample space
- **Bayes' theorem:** $P(B_j \mid A) = \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}$
- **Complement rule:** $P(A^c) = 1 - P(A)$
- **At least one:** $P(\text{at least one}) = 1 - P(\text{none})$

Practice problems with solutions

1. **Two cards without replacement, both Aces:**
   $P(\text{1st Ace}) = \frac{4}{52}$, $P(\text{2nd Ace} \mid \text{1st Ace}) = \frac{3}{51}$
   Answer: $\frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$
2. **Monty Hall problem:** 3 doors (1 car, 2 goats). You pick door 1. Host opens door 3 (goat). Should you switch?
   Solution: Always switch! $P(\text{win if switch}) = \frac{2}{3}$
   Reasoning: Initially $P(\text{car behind 1}) = \frac{1}{3}$, $P(\text{car behind 2 or 3}) = \frac{2}{3}$. After host reveals goat at door 3, all $\frac{2}{3}$ probability concentrates on door 2.
3. **Two coins (fair and biased $P(H) = 0.8$):** Pick randomly, flip once, get Heads. Probability it's biased?
   Solution using Bayes:
   $P(B) = 0.5$, $P(F) = 0.5$, $P(H \mid B) = 0.8$, $P(H \mid F) = 0.5$
   $P(H) = 0.8(0.5) + 0.5(0.5) = 0.65$
   $P(B \mid H) = \frac{0.8 \times 0.5}{0.65} = \frac{0.4}{0.65} = \frac{8}{13} \approx 0.615$
4. **Disease testing:** Disease affects 1% of population. Test is 99% accurate (both sensitivity and specificity). You test positive. What's $P(\text{disease} \mid \text{positive})$?
   Solution:
   $P(D) = 0.01$, $P(+ \mid D) = 0.99$, $P(+ \mid D^c) = 0.01$
   $P(D \mid +) = \frac{0.99 \times 0.01}{0.99 \times 0.01 + 0.01 \times 0.99} = \frac{0.0099}{0.0099 + 0.0099} = 0.5$
   Only 50%! (Common interview question)

4 Discrete Distributions

Binomial: $X \sim \textbf{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad E[X] = np, \quad \text{Var}(X) = np(1-p)$$

Use when: n independent trials, each with success probability p

Geometric: $X \sim \textbf{Geom}(p)$

$$P(X = k) = (1-p)^{k-1} p, \quad E[X] = \frac{1}{p}, \quad \text{Var}(X) = \frac{1-p}{p^2}$$

Use when: Counting trials until first success
Memoryless property: $P(X > n + m \mid X > n) = P(X > m)$

Negative Binomial: $X \sim \textbf{NB}(r, p)$

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad E[X] = \frac{r}{p}$$

Use when: Trials until r-th success

Hypergeometric: $X \sim \textbf{Hyp}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad E[X] = n\frac{K}{N}$$

Use when: Sampling without replacement from finite population

Poisson: $X \sim \textbf{Pois}(\lambda)$

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad E[X] = \mathrm{Var}(X) = \lambda$$

Use when: Rare events, arrival processes, or $n \to \infty$, $p \to 0$ with $np = \lambda$ fixed

Practice problems

1. **Fair coin 10 times, exactly 6 heads:** $\binom{10}{6}(0.5)^{10} = \frac{210}{1024} \approx 0.205$
2. **Roll die until first 6:** $E[\text{rolls}] = \frac{1}{1/6} = 6$
3. **Draw 5 cards, probability of exactly 2 hearts:**
   $\frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} = \frac{78 \times 9139}{2{,}598{,}960} \approx 0.274$
4. **Trading scenario:** Stock moves up with $p = 0.55$ each day. Over 100 days, expected up days? $E[X] = 100 \times 0.55 = 55$

5 Expectation & Variance

Core formulas

- **Linearity of expectation:** $E\left[\sum_i a_i X_i\right] = \sum_i a_i E[X_i]$ (NO independence needed!)
- **Indicator trick:** If $I = \mathbf{1}\{\text{event}\}$, then $E[I] = P(\text{event})$
- **Variance:** $\mathrm{Var}(X) = E[X^2] - E[X]^2$
- **Variance of sum (independent):** $\mathrm{Var}\left(\sum_i X_i\right) = \sum_i \mathrm{Var}(X_i)$
- **Variance of linear transform:** $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
- **Law of total expectation:** $E[X] = E[E[X \mid Y]]$
- **Covariance:** $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$
- **Variance formula:** $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$

Classic problems

1. **Expected number of 6's in 30 rolls:**
   Let $I_i = \mathbf{1}\{\text{roll } i \text{ is } 6\}$, then $E[\sum I_i] = \sum E[I_i] = 30 \times \frac{1}{6} = 5$
2. **Coupon collector:** n coupon types, how many purchases until you have all?
   $E[\text{coupons}] = n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = nH_n \approx n \ln n$
   For $n = 10$: $E \approx 29.3$ purchases
3. **Matching problem (derangement):** n people randomly pick from n hats. Expected number with correct hat?
   Let $I_i = \mathbf{1}\{\text{person } i \text{ gets own hat}\}$, then $E[\sum I_i] = n \times \frac{1}{n} = 1$ (always 1!)

4. **Balls and bins:** Throw m balls into n bins uniformly. Expected empty bins?
   $E[\text{empty}] = n\left(1 - \frac{1}{n}\right)^m \approx ne^{-m/n}$ for large n
5. **Trading P&L:** You buy stock at \$100. It goes up \$1 with prob 0.6, down \$1 with prob 0.4.
   $E[\text{change}] = 0.6(1) + 0.4(-1) = 0.2$, so expected price after 1 period: \$100.20

6 Conditional Expectation & Trading Applications

First-step analysis

Set up equation by conditioning on first outcome:

$$E[\text{cost to reach goal}] = 1 + \sum_i p_i E[\text{cost from state } i]$$

Classic examples

1. **Expected rolls to see first 6:**
   Let E = expected rolls. After first roll: either get 6 (prob $\frac{1}{6}$) or back to start (prob $\frac{5}{6}$)
   $E = 1 + \frac{5}{6}E \Rightarrow E = 6$
2. **Gambler's ruin:** Start with \$5, bet \$1 each round, win with prob $p = 0.4$. Expected rounds until broke or reach \$10?
   Let E_i = expected rounds from \$i. Then $E_i = 1 + pE_{i+1} + (1-p)E_{i-1}$ with $E_0 = E_{10} = 0$
   (Solve system of equations)
3. **Two dice until sum 7:** Expected rolls?
   $P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}$, so $E = \frac{1}{1/6} = 6$
4. **Unfair coin until 2 heads in a row:** $p = 0.6$ for heads. Expected flips?
   Let E_0 = start, E_1 = after 1 head. Then:
   $E_0 = 1 + 0.6E_1 + 0.4E_0$
   $E_1 = 1 + 0.6 \times 0 + 0.4E_0$
   Solving: $E_1 = 1 + 0.4E_0$ and $0.4E_0 = 1 + 0.6E_1$
   $0.4E_0 = 1 + 0.6(1 + 0.4E_0) = 1.6 + 0.24E_0 \Rightarrow E_0 = 10$

7 Game Theory & Strategic Thinking

Key concepts

- **Nash equilibrium:** No player can improve by unilaterally changing strategy
- **Dominant strategy:** Best response regardless of opponent's action
- **Mixed strategy:** Randomize between pure strategies
- **Expected value matching:** In equilibrium, make opponent indifferent

Examples

1. **Penalty kick:** Kicker chooses left/right, goalie chooses left/right. If goalie guesses correctly, saves 80% of time.
   Equilibrium: Both randomize 50-50 (make opponent indifferent)
2. **Rock-Paper-Scissors:** Nash equilibrium is $\frac{1}{3}$ each option

3. **Matching pennies:** Two players simultaneously show heads or tails. Player 1 wins if match, Player 2 wins if different.
   Equilibrium: Both play 50-50 mixed strategy

8 Advanced Probability Topics

Markov chains

Definition: $P(X_{n+1} \mid X_n, X_{n-1}, \ldots, X_0) = P(X_{n+1} \mid X_n)$ (memoryless)

Stationary distribution π: Satisfies $\pi = \pi P$ where P is transition matrix

Example: Weather model: Rainy $\to$ Rainy (0.7), Rainy $\to$ Sunny (0.3), Sunny $\to$ Rainy (0.4), Sunny $\to$ Sunny (0.6)

Random walks

Simple random walk: $S_n = \sum_{i=1}^{n} X_i$ where $X_i \in \{-1, +1\}$

Properties:

- $E[S_n] = (2p-1)n$ where $p = P(X_i = 1)$
- $\mathrm{Var}(S_n) = 4np(1-p)$
- If $p = 0.5$: symmetric, will return to 0 with probability 1 (recurrent)
- If $p \neq 0.5$: will drift to $\pm\infty$ (transient)

Kelly criterion (bet sizing)

Optimal fraction to bet: $f^* = \frac{p(b+1)-1}{b}$ where p = win probability, b = odds ratio

Example: Bet pays 2:1 (win \$2 for \$1 bet), you have edge with $p = 0.6$
$f^* = \frac{0.6(3)-1}{2} = \frac{0.8}{2} = 0.4 \Rightarrow$ bet 40% of bankroll

Half-Kelly: Conservative traders use $\frac{f^*}{2}$ to reduce volatility

Birthday problem variants

1. **Classic:** 23 people, $P(\text{collision}) \approx 0.507$
2. **General:** n items, k samples: $P(\text{all distinct}) = \prod_{i=0}^{k-1} \frac{n-i}{n}$
3. **Approximation:** $P(\text{collision}) \approx 1 - e^{-k^2/(2n)}$

Memoryless property

Geometric: $P(X > n + m \mid X > n) = P(X > m)$

Example: Already rolled die 10 times without seeing 6. Expected additional rolls until 6? Still 6!

9 Classic Trading Interview Puzzles

Card and dice problems

1. **Draw cards until Ace:** Expected draws from 52-card deck?
   Solution: By symmetry, 5 groups of cards (before 1st Ace, between Aces, after last Ace). Average position of first Ace is $\frac{53}{5} \approx 10.6$
2. **Higher card:** You draw a card. Opponent draws. You win if higher (Aces high). What's your probability?
   Solution: By symmetry, $P(\text{win}) = P(\text{lose})$ and $P(\text{tie}) = \frac{1}{13}$ (same rank)
   $P(\text{win}) = \frac{1-1/13}{2} = \frac{6}{13} \approx 0.462$
3. **Roll until decrease:** Roll die repeatedly. Stop when roll is less than previous. Expected rolls?
   Solution: Let E_k = expected additional rolls when last roll was k
   $E_k = 1 + \frac{6-k}{6} \times$ average of $E_{k+1}, \ldots, E_6$
   $E_6 = 1$ (always stop next), work backwards: $E_1 \approx 2.45$
4. **Three of a kind:** Roll 5 dice. Probability of exactly three dice showing same number?
   Solution: Choose which number appears 3 times (6 ways), choose 3 dice for it $\binom{5}{3}$, other 2 dice different from it and each other
   $\frac{6\times\binom{5}{3}\times5\times4}{6^5} = \frac{6\times10\times20}{7776} = \frac{1200}{7776} \approx 0.154$

Probability-based games

1. **Dice betting game:** Roll die. Pay \$1 to play. Win amount shown on die (in dollars). Fair game?
   Solution: $E[\text{winnings}] = \frac{1+2+3+4+5+6}{6} = 3.5$. Yes, fair if cost is \$3.50.
2. **Red or black:** Bet on red or black in roulette (18 red, 18 black, 2 green). Pays 1:1. Expected return on \$1 bet?
   Solution: $E = \frac{18}{38}(2) + \frac{20}{38}(0) = \frac{36}{38} \approx 0.947$. House edge $\approx 5.3\%$
3. **Double or nothing:** Start with \$1. Each round: double with prob 0.5, lose all with prob 0.5. Expected value after n rounds?
   Solution: $E_n = 2^n \times (0.5)^n = 1$ always! But median $\to 0$ (bad for Kelly)
4. **Offer game:** Receive offers of \$1, \$2, ..., \$n uniformly random order. Accept one (no going back). Maximize expected value?
   Solution: Secretary problem variant. Reject first $\frac{n}{e}$, then accept next one better than all previous. $E[\text{payoff}] \approx 0.58n$

Sampling and estimation

1. **Pick random point in unit square. Expected distance from origin?**
   Solution: $E[D] = \iint \sqrt{x^2+y^2}\,dx\,dy \approx 0.382$ (requires integration)
2. **Breaking stick:** Break stick randomly in 2 places. Probability the 3 pieces form triangle?
   Solution: Need all pieces $< \frac{1}{2}$. By geometric probability, answer is $\frac{1}{4}$
3. **Meeting probability:** Two people arrive uniformly random in [0,1] hour. Each waits 15 min. Probability they meet?
   Solution: Need $|X - Y| < 0.25$. Geometric area: $1 - 2 \times \frac{1}{2}(0.75)^2 = 0.4375$

10 Common Paradoxes & Pitfalls

Simpson's paradox

Trend appears in groups but reverses when combined.

Example: Drug A has better rate than Drug B in both men and women, but Drug B has better overall rate.

False positive paradox

Rare disease + accurate test $\Rightarrow$ most positives are false positives (see Section 3 example)

Berkson's paradox

Selection bias creates spurious negative correlation.

Example: Among hospitalized patients, diseases appear negatively correlated (hospitalization is conditioning)

St. Petersburg paradox

Fair game with infinite expected value, but no one would pay much to play.

Setup: Flip coin until tails. Win $\$2^n$ where n = number of flips. $E[\text{winnings}] = \infty$ but bounded utility explains low willingness-to-pay.

11 Mental Math Practice

Quick calculations to practice

1. $\binom{7}{3} = 35$
2. $\binom{9}{4} = 126$
3. $\binom{12}{2} = 66$
4. $(0.5)^{10} = \frac{1}{1024} \approx 0.001$
5. $(0.9)^{10} \approx 0.349$
6. $(0.95)^{20} \approx 0.358$
7. $e^{-2} \approx 0.135$
8. $\ln(2) \approx 0.693$
9. $\sqrt{2} \approx 1.414$
10. $\frac{1}{52} \times \frac{1}{51} = \frac{1}{2652} \approx 0.000377$

12 Spring Week Strategy

During probability interviews

- **Talk through your reasoning** — don't solve silently
- **Identify the distribution** — binomial? geometric? hypergeometric?

- **Check edge cases** — what if $n = 1$? $p = 0$ or $p = 1$?
- **Use complement** — often easier to find $P(\text{none})$ than $P(\geq 1)$
- **Draw pictures** — Venn diagrams, probability trees, sample spaces
- **Verify reasonableness** — does answer make intuitive sense?
- **Know when to approximate** — large n small $p \Rightarrow$ Poisson

Trading game tips

- **Calculate expected value quickly** — is this +EV or -EV?
- **Update beliefs** — use Bayes' rule as information arrives
- **Consider risk** — Kelly criterion for sizing
- **Don't chase losses** — each bet is independent
- **Track your P&L mentally** — quick addition/subtraction
- **Think adversarially** — what are others likely to do?

Final checklist

1. Can you compute $\binom{n}{k}$ quickly for small n, k?
2. Do you know all distributions and when to use each?
3. Can you set up Bayes' theorem correctly?
4. Can you solve recurrences with first-step conditioning?
5. Have you practiced converting between odds and probabilities?
6. Can you calculate expected values with linearity?
7. Do you know the indicator trick cold?
8. Have you reviewed classic puzzles (Monty Hall, birthday, etc.)?

Good luck with your spring week interviews!
Practice problems daily, think probabilistically, and trust your preparation.