Задание 3

$$\left\{\begin{aligned} u_{xx} &= f(x), \qquad x \in (0,1) \\ u|_{t=0} &= 2, \\ u|_{t=0} &= 0, \\ u_x|_{x=0} &= t, \\ u_x|_{x=l} = -1 \end{aligned}\right\}$$

$$u = v + \omega$$

$$\omega = \frac{-t-1}{l}x^2 + tx$$

$$\begin{cases} v_{tt} - v_{xx} = t^2 - \frac{t+1}{l} - x \\ v|_{t=0} = 2 + \frac{x^2}{l}, \\ v_t|_{t=0} = 0, \\ v_x|_{x=0} = 0, \\ v_x|_{x=l} = 0 \end{cases}$$

Будем искать решение в виде: $v = TX$

$$T''X = a^2TX''$$

$$\frac{T''}{a^2T} = \frac{X''}{X} = -\lambda$$

$$X_k = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

$$X'_k|_{x=0} = 0 \Rightarrow C_2 = 0$$

$$X'_k|_{x=\ell} = C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) = 0 \Rightarrow \sin(\sqrt{\lambda}\ell) = 0 \Rightarrow \sqrt{\lambda_k} = \frac{\pi k}{\ell}$$

$$X_k = \cos\left(\frac{\pi k}{\ell}x\right)$$

$$P = \sum_{k=1}^{\infty} T_k \cos\left(\frac{\pi k}{\ell}x\right)$$

Подставим:

$$\sum_{k=1}^{\infty}\left(T''_k + \left(\frac{\pi k}{\ell}\right)^2 T_k\right)\cos\left(\frac{\pi k}{\ell}x\right) = t^2 - \frac{t+1}{l} - x$$

Разложим в ряд Фурье:

$$t^2 - \frac{t+1}{l} - x = \sum_{k=1}^{\infty} T_k X_k = \sum_{k=1}^{\infty} \frac{2(1-\cos k\ell)}{\pi k}\cos\left(\frac{\pi k}{\ell}x\right)$$

Имеем:

$$T''_k + \left(\frac{\pi k}{\ell}\right)^2 T_k = \frac{2(1-\cos(k\ell))}{\pi k}$$

Разложим $\phi = 2 + \dfrac{x^2}{l}$ в ряд Фурье

$$\phi = \sum_{k=1}^{\infty} \frac{2l\cos(\pi k)}{(\pi k)^2} \cos\left(\frac{\pi k}{l} x\right)$$

Подставим в начальные условия:

$$\sum_{k=1}^{\infty} T_k cos(\frac{\pi k}{l} x)|_{t=0} = \sum_{k=1}^{\infty} \frac{2lcos(\pi k)}{(\pi k)^2} cos(\frac{\pi k}{l} x)$$

Отсюда имеем:

$$T_k|_{t=0} = \frac{2lcos(\pi k)}{(\pi k)^2}$$

$$\begin{cases} T_k'' + \frac{\pi k}{l} T_k = \frac{2(1-\cos(k\ell))}{\pi k} \\ T_k|_{t=0} = \frac{2lcos(\pi k)}{(\pi k)^2} \\ T_k'|_{t=0} = 0 \end{cases}$$

$$T_k = A_k cos(\frac{\pi k t}{l}) + B_k sin(\frac{\pi k t}{l})$$

$$T_k|_{t=0} = \frac{2lcos(\pi k)}{(\pi k)^2} \Rightarrow A_k = \frac{2lcos(\pi k)}{(\pi k)^2}$$

$$T_k'|_{t=0} = \frac{\pi k}{l} B_k = 0 \Rightarrow B_k = 0$$

Таким образом:

$$T_k = \frac{2lcos(\pi k)}{(\pi k)^2} cos(\frac{\pi k t}{l})$$

$$v = \sum_{k=1}^{\infty} X_k T_k = \sum_{k=1}^{\infty} \frac{2lcos(\pi k)}{(\pi k)^2} cos(\frac{\pi k t}{l}) cos(\frac{\pi k x}{l})$$

$$u = \sum_{k=1}^{\infty} X_k T_k = \sum_{k=1}^{\infty} \frac{2lcos(\pi k)}{(\pi k)^2} cos(\frac{\pi k t}{l}) cos(\frac{\pi k x}{l}) - \frac{t+1}{l} x^2 + tx$$

Визуализация 1

$$u(x,t) = \sum_{k=1}^{\infty} \left(\frac{16gl^2}{a^2(\pi + 2xk)^3} \sin \frac{a\sqrt{(\pi + 2xk)^2}}{2l} t \right) \sin \frac{\pi - 2k}{2l} x$$

Ниже приведен код на языке Python, который изображает график данной функции

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

g = 9.8
l = 1
a = 1

x = np.linspace(0, 2 * l, 200)
```

```
t = np.linspace(0, 5, 200)
X, T = np.meshgrid(x, t)

U = np.zeros_like(X)
for k in range(1, 101):
    coef = (16 * g * l**2) / (a**2 * (np.pi + 2*k)**3)
    omega_k = a * (np.pi + 2*k) / (2 * l)
    U += coef * np.sin(omega_k * T) * np.sin((np.pi - 2*k) * X / (2 * l))

fig = plt.figure(figsize=(12, 6))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(X, T, U, cmap='viridis')

ax.set_title('u(x, t)')
ax.set_xlabel('x')
ax.set_ylabel('t')
ax.set_zlabel('u(x, t)')

plt.savefig("graph1.png", dpi=300)
plt.show()
```

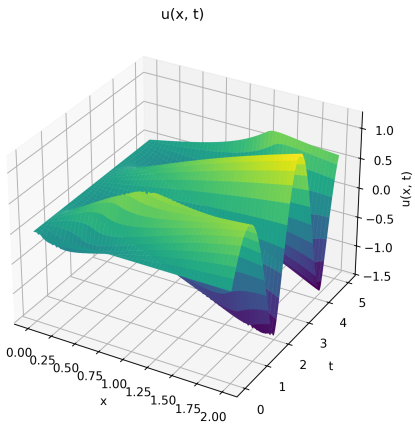

Рис. 1: График функции $u(x, t)$, построенный по приближённой сумме ряда

Код проверки

```
a = 1;
l = 1;
g = 9.8;
v = 1;
solution = DSolve[{
    D[u[x, t], {t, 2}] == a^2*D[u[x, t], {x, 2}] + g,
    u[0, t] == 0,
    u[l, t] == 0,
    u[x, 0] == 0,
    Derivative[0, 1][u][x, 0] == v
```

```
}, u[x, t], {x, t}]

simplifiedSolution = FullSimplify[u[x, t] /. solution[[1]]]
```

Результат проверки

```
    u[x_, t_] := Sum[
    (16 g l^2)/(a^2 (\pi + 2 x k)^3) *
    Sin[(a Sqrt[(\pi + 2 x k)^2]/(2 l) t] *
    Sin[(\pi - 2 k)/(2 l) x],
    {k, 1, \infty}
];
```

Визуализация 2

$$u(x,t) = \left(1 - \frac{lt}{2at} + \left(\frac{l}{2at}\right)^2 \sin\frac{2a\pi t}{e}\cos\frac{a\pi t}{e} - \left(\frac{l}{at}\right)^2 \cdot \frac{1}{2}\sin^3\frac{a\pi t}{e}\right) \cdot \sin\frac{\pi x}{e}$$
$$+ \sum_{k=1}^{\infty}\left(-2 \cdot \frac{\cos kl}{k\cos kl}\cos\frac{ak\pi t}{e} + \sin\frac{kl}{e}x\right)$$

Ниже приведен код на языке Python, который изображает график данной функции

```
  import numpy as np
import matplotlib.pyplot as plt

from mpl_toolkits.mplot3d import Axes3D

a = 1
l = 1
e = 1
N = 100
x = np.linspace(0, e, 200)
t = np.linspace(0.1, 5, 200)
X, T = np.meshgrid(x, t)

term1 = (1 - l / (2 * a) + ((l / (2 * a * T))**2) * np.sin(2 * a * np.pi * T
         0.5 * (l / (a * T))**2 * np.sin(a * np.pi * T / e)**3) * np.sin(np.p

term2 = np.zeros_like(X)

for k in range(1, N + 1):
    denom = k * np.cos(k * l)
    if np.any(np.isclose(denom, 0)):
        continue
    term2 += (-2 * np.cos(k * l) / denom) * np.cos(a * k * np.pi * T / e) + n

U = term1 + term2

fig = plt.figure(figsize=(12, 6))
```

```
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(X, T, U, cmap='plasma')

ax.set_title("u(x, t)")
ax.set_xlabel("x")
ax.set_ylabel("t")
ax.set_zlabel("u(x, t)")

plt.tight_layout()
plt.savefig("graph2.png", dpi=300)
plt.show()
```

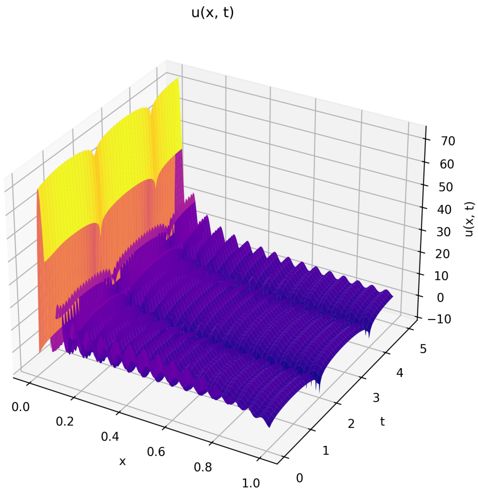

Рис. 2: График функции $u(x, t)$, построенный по приближённой сумме ряда

Код проверки:

```
a = 1;
l = Pi;

solution = DSolve[{
    D[u[x, t], {t, 2}] == a^2*D[u[x, t], {x, 2}] + Sin[Pi*x/l]*Sin[Pi*t/l],
    u[0, t] == 0,
    u[l, t] == 0,
    u[x, 0] == 2*x,
    Derivative[0, 1][u][x, 0] == 0
}, u[x, t], {x, t}]

simplifiedSolution = FullSimplify[u[x, t] /. solution[[1]]]
```

Результат проверки:

```
    u[x_, t_] := (1 − (l*t)/(2*a*t) + (l/(2*a*t))^2*Sin[(2*a*Pi*t)/E]*Cos[(a*
               (l/(a*t))^2*(1/2)*Sin[(a*Pi*t)/E]^3)*Sin[(Pi*x)/E] +
               Sum[(−2*(Cos[k*l]/(k*Cos[k*l]))*Cos[(a*k*Pi*t)/E] + Sin[(k*l*x)/

TraditionalForm[u[x, t]]
```

Визуализация 3

$$u = \sum_{k=1}^{\infty} X_k T_k = \sum_{k=1}^{\infty} \frac{2lcos(\pi k)}{(\pi k)^2} cos(\frac{\pi k t}{l}) cos(\frac{\pi k x}{l}) - \frac{t+1}{l} x^2 + tx$$

Ниже приведен код на языке Python, который изображает график данной функции

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

l = 1
N = 100
x = np.linspace(0, l, 200)
t = np.linspace(0, 5, 200)
X, T = np.meshgrid(x, t)

U = np.zeros_like(X)

for k in range(1, N + 1):
    coef = (2 * l * np.cos(np.pi * k)) / ((np.pi * k)**2)
    U += coef * np.cos(np.pi * k * T / l) * np.cos(np.pi * k * X / l)

U += -((T + 1) / l) * X**2 + T * X

fig = plt.figure(figsize=(12, 6))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(X, T, U, cmap='inferno')

ax.set_title("u(x, t)")
ax.set_xlabel("x")
ax.set_ylabel("t")
ax.set_zlabel("u(x, t)")

plt.tight_layout()
plt.savefig("graph3.png", dpi=300)
plt.show()
```

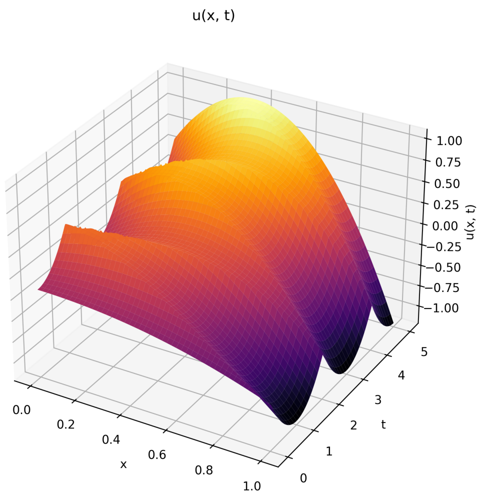

Рис. 3: График функции $u(x,t)$, построенный по приближённой сумме ряда

Код проверки

```
eqn = D[u[x, t], {x, 2}] == f[x];
bc1 = u[0, t] == 2;
bc2 = Derivative[1, 0][u][0, t] == t;
bc3 = Derivative[1, 0][u][l, t] == -1;
ic1 = u[x, 0] == 0;
ic2 = Derivative[0, 1][u][x, 0] == 0;

solution = DSolve[{eqn, bc1, bc2, bc3, ic1, ic2}, u[x, t], {x, t}]

simplifiedSolution = Simplify[u[x, t] /. solution[[1]]]
```

Результат проверки

```
    u[x_, t_, l_] :=
  Sum[(2 l Cos[Pi k])/((Pi k)^2) * Cos[(Pi k t)/l] * Cos[(Pi k x)/l],
      {k, 1, Infinity}] - ((t + 1)/l) x^2 + t x
```