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AMATH 573, Homework 1, Due 10/4/2024

Problem 1.8.2

The KdV equation $U_t = UU_x + U_{xxx}$ is often written with different coefficients. By using a scaling transform, show that the choice of coefficients is arbitrary.

Can you say the same for mKdV $U_t = U^2U_x + U_{xxx}$?

Solution

First, we show that the coefficients in KdV are arbitrary. Consider the scalings

$U = a\omega$, $x = by$, and $t = c\tau$. This gives $\partial_y = b\partial_x$ and $\partial_t = c\partial_\tau$. After this transformation, the KdV equation becomes

$$ac\omega_t = a^2 b\,\omega\omega_y + (ab)^3\omega_{yyy}. \quad (1)$$

Now, consider the KdV equation $A\omega_\tau = B\omega\omega_y + C\omega_{yyy}$ with $A, B, C \in \mathbb{R}\setminus\{0\}$. Choosing

$$\boxed{a = BC^{-1/3},\ b = C^{2/3}B^{-1},\ c = AC^{1/3}B^{-1},}$$ eq. (1) becomes

$$\cancel{B}\cancel{C^{-1/3}}A\cancel{C^{1/3}}\cancel{B^{-1}}\,w_\tau = B^{2}\cancel{C^{-2/3}}\cancel{C^{2/3}}B^{-1}\,ww_y + (\cancel{B}\cancel{C^{-1/3}}C^{\cancel{2/3}}\cancel{B^{-1}})^3 w_{yyy}$$

$\Rightarrow$ $Aw_\tau = Bww_y + Cw_{yyy}$, as needed. Thus, the scalings can be chosen to match arbitrary, nonzero coefficients. If $BC = 0$ but $|B| + |C| \neq 0$, no choice of scaling would work since $BC = 0 \Rightarrow ab = 0 \Rightarrow B, C = 0$.

This is fine, because zero coefficients would yield a non-KdV equation anyway!

SEE NEXT

Problem 1.8.2 cont.

Now, we show that the same is untrue for mKdV. Again using the scalings: $U = aw$, $x = by$, and $t = c\tau$, we obtain

$$acw_\tau = a^3 b w^2 w_y + (ab)^3 w_{yyy}. \quad (2)$$

Noting that $a, b, c \in \mathbb{R} \setminus \{0\}$, we have

$\dfrac{(ab)^3}{a^3 b} = b^2 > 0$. Thus, the coefficients of the $w w_y$ and w_{yyy} terms must be the same.

$\Rightarrow$ No scaling can transform mKdV to

$$w_\tau = w^2 w_y - w_{yyy}.$$

$\Rightarrow$ $\boxed{\text{The coefficients of mKdV matter!}}$

However, if we wish to transform mKdV to $Aw_\tau = Bw^2 w_y + Cw_{yyy}$ where $BC > 0$, we may choose

$$\boxed{a = |B|^{-1/2} |C|^{-1/6}, \quad b = |C|^{1/2} |B|^{-1/2}, \quad c = A|B|^{-1/2} |C|^{1/6} \operatorname{sgn}(C).}$$

LCB

Problem 1.8.5

Show that every nonzero solution of the heat equation $\theta_t = \nu \theta_{xx}$ gives rise to a solution of the dissipative Burgers' equation $U_t + UU_x = \nu U_{xx}$ through the mapping $U = -2\nu\theta_x/\theta$

Solution

We simply verify that the transform works. We compute

$$U_x = -2\nu \frac{\partial}{\partial x}\frac{\theta_x}{\theta} = -2\nu\,\theta_{xx}\theta^{-1} + 2\nu\,\theta_x^2\theta^{-2}$$

$$= -2\theta_t\theta^{-1} + \frac{1}{2\nu}\left(-2\nu\frac{\theta_x}{\theta}\right)^2 = -2\theta_t\theta^{-1} + \frac{1}{2\nu}U^2$$

$$U_{xx} = -2\theta_{xt} + 2\theta_t\theta_x\theta^{-2} + \frac{1}{\nu}UU_x$$

$$U_t = -2\nu\frac{\partial}{\partial t}\frac{\theta_x}{\theta} = -2\nu\,\theta_{xt}\theta^{-1} + 2\nu\,\theta_x\theta_t\theta^{-2}$$

Finally, we have

$$U_t + UU_x = -2\nu\,\theta_{xt}\theta^{-1} + 2\nu\,\theta_x\theta_t\theta^{-2} + UU_x$$

$$= \nu\left(-2\theta_{xt}\theta^{-1} + 2\theta_x\theta_t\theta^{-2} + \frac{1}{\nu}UU_x\right)$$

$= \nu U_{xx}$, as needed.

The transform is defined so long as the solution θ is nonzero, and it corresponds to a solution of Burgers' equation. ∎

AMATH 573 - Homework 1

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Due 10/1/2024

Abstract

Many non-linear partial differential equations tend to evolve nice initial conditions into gobbledygook nonsense. Yet, the physical world is full of beauty and *coherent structures*. How can this be? Certain nonlinear PDE's have a propensity for creating and maintaining structure. These equations and their solutions are physically relevant and mathematically rich. There is so very much to learn from numerical experimentation, though not all phenomena can easily be made rigorous. In this project, we explore the features and dynamics of solutions to Burgers' Equation and the KdV Equation.

1 Introduction

We set out to explore soliton solutions to two equations of physical importance, namely the Korteweg-de Vries (KdV) and Burgers' equations. The name 'soliton' was selected in connection to physics (electron, positron, neutron, etc.), and it persists to this day. To some extent, the naming is apt - solitons are akin to particles that do their own thing and occasionally collide with other solitons. Pertinently, these interactions don't destroy the structure!

Manyof the symbolic calculations were handled by Mathematica, and all of that code can be found publicly in on my Github at https://github.com/Levent-Batakci/AMATH-573---HW-1.

2 One and Two soliton solutions to KdV (1.8.4)

The Korteweg-de Vries (KdV) equation is typically used to model small-amplitude, long wavelength water waves in shallow water. It is given by

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

We begin by establishing a one-soliton solution. Consider $u(x,t) = 12\partial_x^2 \ln(1 + e^{k_1 x - k_1^3 t + \alpha})$. Using Mathematica (notebook is attached to submission), we find that

$$u(x,t) = 12k_1^2 \frac{e^{k_1 x - k_1^3 t + \alpha}}{e^{k_1^3 t} + e^{k_1 x + \alpha}} = 3k_1^2 \Big(\frac{2}{e^{\frac{1}{2}(k_1 x - k_1^3 t + \alpha)} + e^{-\frac{1}{2}(k_1 x - k_1^3 t + \alpha)}} \Big)^2 = 3k_1^2 \text{sech}(\frac{1}{2}(k_1 x - k_1^3 t + \alpha)).$$

Equation (1.4) from the notes provides the familiar form of 1 soliton solutions:

$$u(x,t) = 12\kappa^2 \delta^2 \text{sech}^2(\kappa(x - 4\kappa^2\delta^2 t + \phi)). \tag{2}$$

In our case, $\delta^2 = 1$ is given as the coefficient of the $u_x xx$ term in the KdV equation. Furthermore, we rearrange our expression for $u(x,t)$ to find that

$$u(x,t) = 12 \cdot (\frac{1}{2}k_1)^2 \text{sech}(\frac{1}{2}k_1(x - 4(\frac{1}{2}k_1)^2 t + \frac{\alpha}{k_1})). \tag{3}$$

That is, our solution matches 2 with $\delta = 1, \kappa = \frac{1}{2}k_1$, and $\phi = \frac{\alpha}{k_1}$. For ease of reference, we show this solution in figure 1.

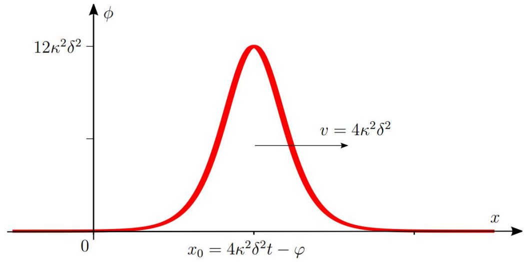

Figure 1: One-soliton solution to $u_t + uu_x + \delta^2 u_{xxx} = 0$. *This graphic is taken directly from page 13 of Bernard Deconinck's notes on nonlinear waves.*

Now, we examine a two-soliton solution given by

$$u(x,t) = 12\partial_x^2 \ln\Big(1 + e^{k_1 x - k_1^3 t + \alpha} + e^{k_2 x - k_2^3 t + \beta} + (\frac{k_1 - k_2}{k_1 + k_2})^2 e^{k_1 x - k_1^3 t + \alpha + k_2 x - k_2^3 t + \beta}\Big). \tag{4}$$

This expression, while beautiful in its own sense, doesn't make clear any properties of the solution. To better understand what's going on, we compute and plot the function at several times. We examine the simple case of $\alpha = \beta = 0$. In fact, these parameters only the affect horizontal displacement of the solitons. Thus, we may safely disregard other choices of α, β in our analysis of the solution dynamics.

We found that the relative speeds and amplitudes of the solitons are dictated by the ratio $\frac{k_1}{k_2}$, we fix $k_2 = 1$ for simplicity and experiment over several regions for that ratio. **(a)** The first region we check is $\frac{k_1}{k_2} > \sqrt{3} \approx 1.732$. We produced all of the plots by using Desmos, and the graph is publicly available at https://www.desmos.com/calculator/gkscef4x6k. Our results are seen below:

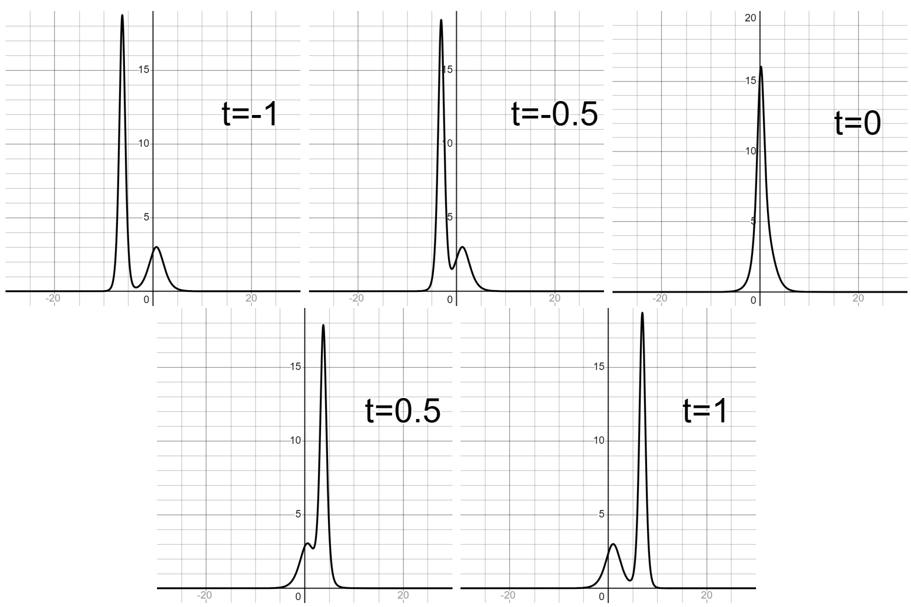

Figure 2: Two-soliton collision with $k_2 = 1$, $\frac{k_1}{k_2} = 2.5$, and $\alpha = \beta = 0$.

Immediately, we notice that the two solitons are clearly visible at large (negative or positive) times. Despite the collision, the solitons re-form and retain their original amplitudes and shape. This is a nontrivial outcome since the interaction is non-linear (it is not just a superposition of two solitons). The nonlinearity is indeed noticeable - the smaller soliton is 'pulled' back and the larger soliton is 'pushed' forward during the interaction. This can be seen clearly when the plots for $t = \pm 0.5$ are compared in figure 2.

Furthermore, the push-pull seems to be the *only* lasting effect of the collision. After separating, the solitons resume travel at their original speeds. The collision gives the impression that the larger soliton is passing over the smaller soliton. We perceive the interaction in this way since the peak of the tall soliton dips but is never overtaken. We will see that not all two-soliton collisions are of this nature!

(b) The next parameter region we check is $\sqrt{3} > \frac{k_1}{k_2} > \sqrt{(3+\sqrt{5})/2} \approx 1.618$. The plots are shown in figure 3.

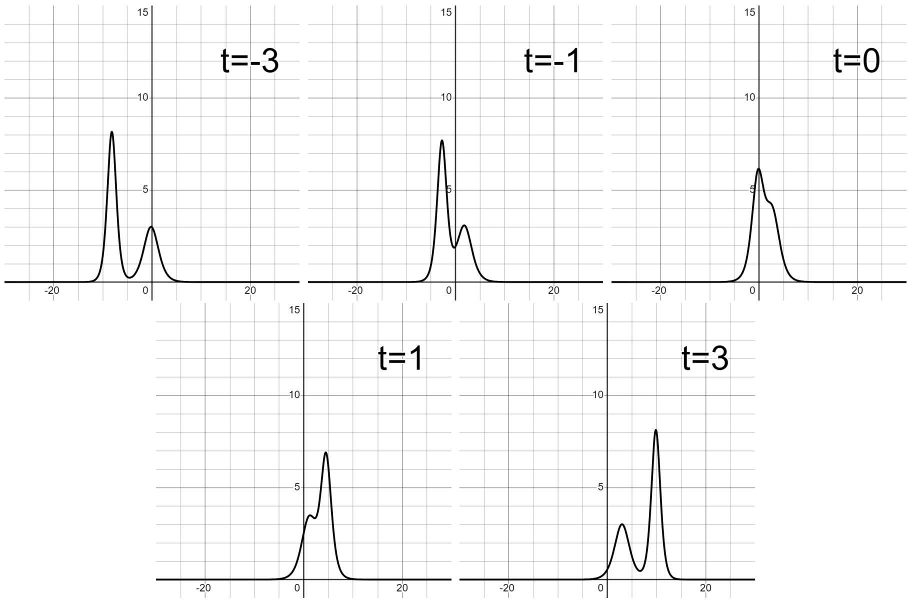

Figure 3: Two-soliton collision with $k_2 = 1$, $\frac{k_1}{k_2} = 1.65$, and $\alpha = \beta = 0$.

The first key observation is that one thing remained the same: the solitons regenerate with their original amplitudes and shapes. Moreover, watching an animation of the collision (as can be done using the Desmos link) makes it evident that the speeds are not diminished.

Examining the times around the collision, we see that pull-back we witnessed before is much less visible. There does seem to be a delay, but the slow wave isn't behind where it was at the start of the collision. More interestingly, the peak of the tall soliton at $t = 3$ doesn't come from tracing the peak of the original tall soliton. While there is 'mixing' during the interaction, the notion of the larger soliton 'passing over' is no longer reasonable. In some sense, there is some kind of transference as the solitons swap properties.

It may be worth taking a step back and remembering that these are just equations. There is no water/mass/particles involved. However, it would be asinine to fully disregard phsyical intuition when trying to understand solutions to models developed to explain the physical world... For this reason, we continue to imagine particles, water, and other nonsense for the duration of our analysis.

(c) Next, we test in the parameter region $\sqrt{(3+\sqrt{5})/2} > \frac{k_1}{k_2}$. Our results are seen in figure 4.

As before, the interaction has no lasting effect on the shapes, speeds, or heights of the solitons except for moving the faster soliton to the front. Unlike in the previous cases, however, the interaction isn't *really* a collision. The (originally) faster soliton never catches up to the slower one; there is persistently a valley between the two peaks.

Since the larger soliton does move faster, this valley begins to close up. As the valley narrowers, the tall peak falls and the short peak rises. In tandem, the fast soliton slows and the slow soliton

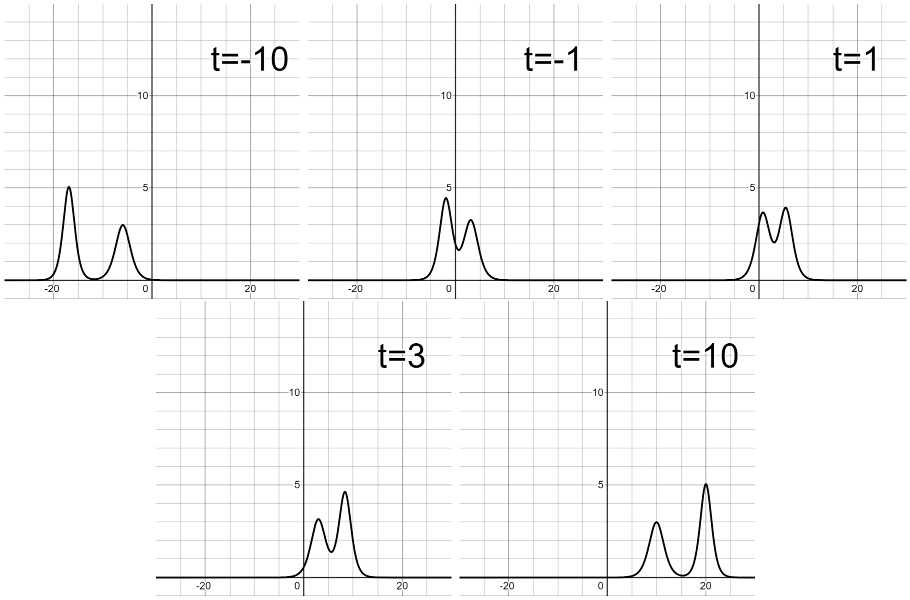

Figure 4: Two-soliton collision with $k_2 = 1$, $\frac{k_1}{k_2} = 1.3$, and $\alpha = \beta = 0$.

speeds up. All of this continues until the solitons have completely swapped their original properties!

If we are to appeal to the physical world, it almost seems as if the faster moving crest approaches the small crest, but not fast enough to overtake it. Instead, it transfers some of its energy to the crest in front of it - growing it and speeding it up. The marvel in all of this is that the so called 'transference' perfectly swaps the properties of the solitons. This can be gleaned from figure 4, but it is more amusing to watch the animation on our Desmos graph!

In conclusion, two soliton solutions to the KdV equation feature a variety of interactions and incredible coherence. All of the cases we examined support that the solitons regenerate and persist after some kind of interaction. Moreover, the faster soliton always moves to the front, but does so in several different fashions. We expect a lot of these ideas and findings to carry over to the many-soliton case.

3 Solutions to Burgers' equation (1.8.6)

Burgers' equation, used as a simplification of the Navier-Stokes equation, is given by

$$u_t + uu_x = \nu u_{xx}. \tag{5}$$

We will generate solutions to this equation by using the Cole-Hopf transform, $u = -2\nu \frac{\theta_x}{\theta}$, on solutions to the heat equation $\theta_t = \nu\theta_{xx}$.

The graphs in this section can be recreated at https://www.desmos.com/calculator/rlvw4dkjht.

(a) The first solution that is given to us is $\theta = 1 + \alpha e^{-kx+\nu k^2 t}$. It is easy to verify that this is indeed a solution to the heat equation, and the verification is present in the Mathematica notebook submitted alongside this project.

To see which solution of Burgers' equation this corresponds to, we compute the Cole-Hopf transform:

$$u(x,t) = -2\nu\frac{\theta_x}{\theta} = \frac{2\nu k}{1 + \alpha^{-1}e^{kx-\nu k^2 t}}, \tag{6}$$

given that $\alpha \neq 0$. If $\alpha = 0$, then $u(x,t) = 0$ and we will not discuss this case (as there is nothing to discuss).

We first restrict our attention to the case $\alpha > 0$, as this corresponds to bounded solutions. Indeed, in this case we have

$$u(x,t) = f_1(x - \nu k t), \text{ where } f_1(y) = \frac{2\nu k}{1 + e^{ky - \ln\alpha}}. \tag{7}$$

We recognize f_1 as a logistic function. **Hence, the solution is a traveling logistic with speed νk, amplitude $2\nu k$, steepness k, and sigmoid point $x = \nu k t + \ln a$.** The shape of the solution is shown in figre 5.

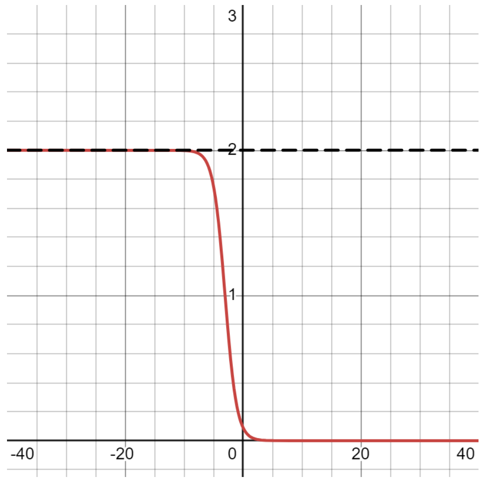

Figure 5: Traveling logistic solution to Burgers' equation with parameters $\nu = k = \alpha = 1$. The solution is plotted at time $t = -3$.

We now consider $\alpha < 0$. In this case, there is a singularity for $x - \nu k t = \frac{1}{k}\ln(-\frac{1}{\alpha})$. The end behavior of the function is given by

$$\lim_{x \to \operatorname{sgn}(k)\infty} = 0, \quad \text{and} \quad \lim_{x \to -\operatorname{sgn}(k)\infty} = 2\nu k. \tag{8}$$

Again, this is a traveling solution with speed νk. This solution is shown in figure 6

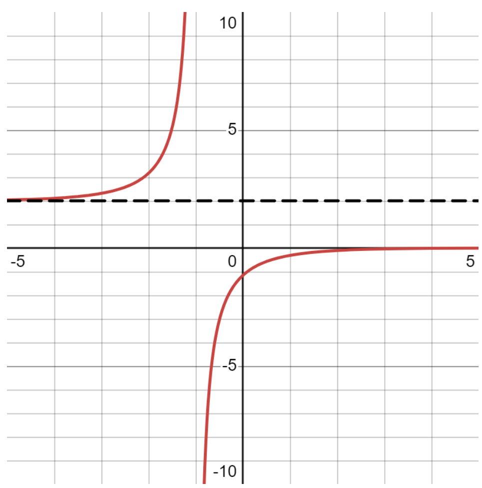

Figure 6: Singular traveling solution with parameters $\nu = k = -\alpha = 1$. The solution is plotted at time $t = -1$.

(b) We now use the solution to the heat equation $\theta = 1 + \alpha e^{-k_1 x + \nu k_1^2 t} + \beta e^{-k_2 x + \nu k_2^2 t}$. This indeed solves $\theta_t = \nu \theta_{xx}$, as we've verified with Mathematica. We compute the Cole-Hopf transform of this function to find

$$u(x,t) = 2\nu \frac{\alpha k_1 e^{-k_1 x + \nu k_1^2 t} + \beta k_2 e^{-k_2 x + \nu k_2^2 t}}{1 + \alpha e^{-k_1 x + \nu k_1^2 t} + \beta e^{-k_2 x + \nu k_2^2 t}}. \tag{9}$$

First, we will examine the case $\alpha, \beta > 0$, as this corresponds to a bounded solution. The evolution of the solution is shown in figure 7.

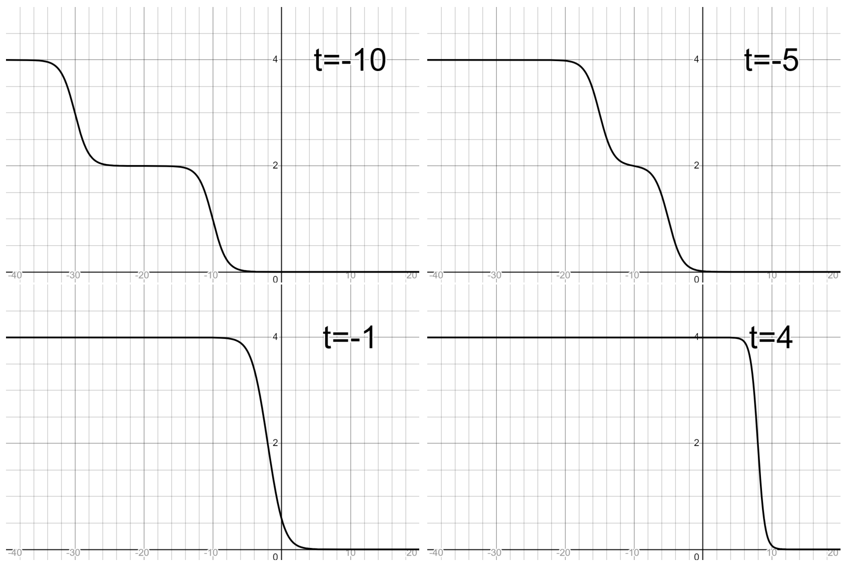

Figure 7: Two wave solution to Burgers' equation with parameters $\alpha = \beta = \nu = k_1 = \frac{1}{2}k_2 = 1$.

At large negative time, we see that there are two fronts. The respective amplitudes of the fronts correspond precisely to the logistic solutions from part (a) of the same k, ν parameters. It appears that the larger wave dominates, not changing its speed or height as it overtakes the smaller front. Eventually, the wave profile matches (up to negligible error) that of the logistic solution corresponding to the larger front (in this case, $k_2 = 2$).

We also check an interesting case where $k_1 k_2 < 0$. The results are shown in figure 8

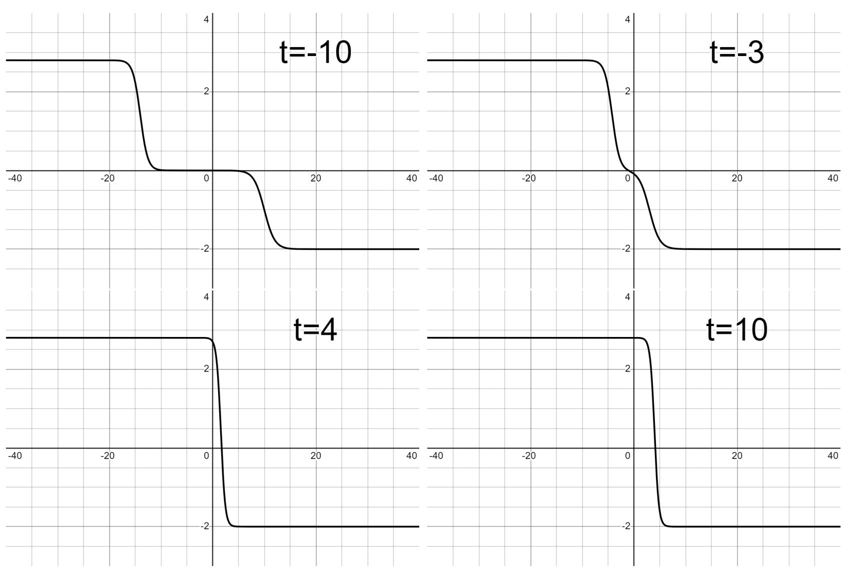

Figure 8: Two wave solution to Burgers' equation with parameters $\alpha = \beta = \nu = 1$, $k_1 = -1$, and $k_2 = 1.4$.

This outcome of this case is slightly different. Ultimately the amplitude of the wave grows, and as is evident from the $t = 4, 10$ graphs, the wave gets much slower. This is due to the collision of two waves that are going in opposite directions of one another. We do not include it here, but in a similar case with equal-amplitude waves, the solution (essentially) comes to a halt in finite time.

Now, we consider some silly, physically infeasible cases. The conquering of infinity is shown in figure 9

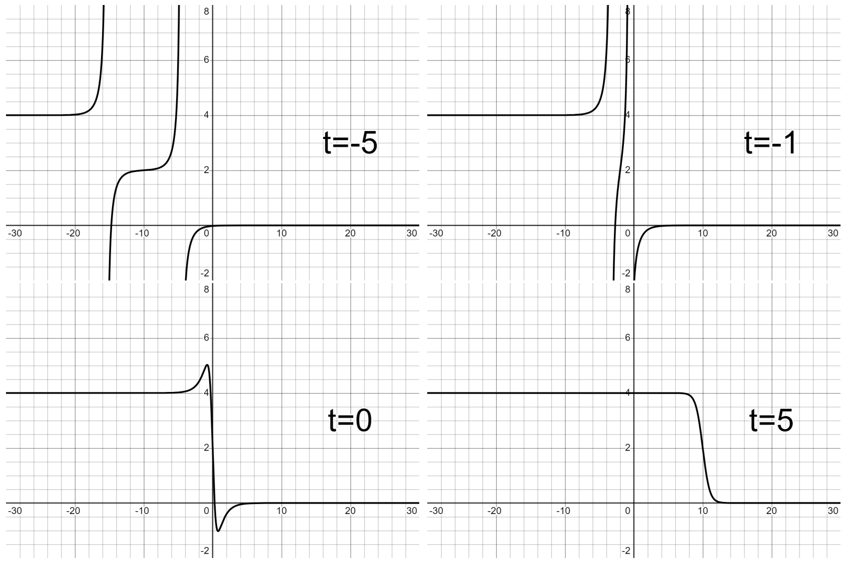

Figure 9: Silly solution to Burgers' equation with parameters $-\alpha = 1.5$, $\beta = \nu = k_1 = \frac{1}{2}k_2 = 1$.

This solution is infeasible due to the singularity. However, it is funny to see a 'larger' wave eat it up. The man born at time $t = 5$ is afforded the luxury of oblivion. Other nonsensical choices of parameters lead to similarly wacky outcomes. These are not physically feasible, but they are indeed solutions.

In conclusion, these solutions to Burgers' equation all turn into traveling solutions. The common theme is that even if we start with multiple modes, the dominant one (*i.e.* the one with greater amplitude and speed) will overtake the smaller one. Unlike in the case of KdV, the smaller front gets totally swallowed up and lost.

HW1.nb - Wolfram Mathematica

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Levent Batakci - 9/25/2024

AMATH 573 - Homework 1

Problem 1.8.4

In[•]:=
```
f1[x_, t_] = Log[1 + Exp[k1 * x - k1^3 * t + a]];
u1[x_, t_] = 12 * D[f1[x, t], {x, 2}];
u1[x, t]
Simplify[u1[x, t]]
```

Out[•]= $12\left(-\frac{e^{2a-2k1^3 t+2k1x}k1^2}{\left(1+e^{a-k1^3 t+k1x}\right)^2}+\frac{e^{a-k1^3 t+k1x}k1^2}{1+e^{a-k1^3 t+k1x}}\right)$

Out[•]= $\frac{12\,e^{a+k1^3 t+k1x}k1^2}{\left(e^{k1^3 t}+e^{a+k1x}\right)^2}$

In[•]:=
```
f2[x_, t_] = Log[1 + Exp[k1 * x - k1^3 * t + a] + Exp[k2 * x - k2^3 * t + b] + ((k1 - k2) / (k1 + k2))^2 * Exp[k1 * x - k1^3 * t + a + k2 * x - k2^3 * t + b]];
u2[x_, t_] = 12 * D[f2[x, t], {x, 2}];
u2[x, t];
Simplify[D[u2[x, t], t] + u2[x, t] * D[u2[x, t], x] + D[u2[x, t], {x, 3}]]
```

Out[•]= 0

Problem 1.8 .6

In[46]:=
```
theta1[x_, t_] = 1 + a * Exp[-k1 * x + n * k1^2 * t];
Simplify[D[theta1[x, t], t] - n * D[theta1[x, t], {x, 2}]]

u1[x_, t_] = -2 * n * D[theta1[x, t], x] / theta1[x, t];
u1[x, t]
Simplify[D[u1[x, t], t] + u1[x, t] * D[u1[x, t], x] - n * D[u1[x, t], {x, 2}]]

theta2[x_, t_] = 1 + a * Exp[-k1 * x + n * k1^2 * t] + b * Exp[-k2 * x + n * k2^2 * t];
Simplify[D[theta2[x, t], t] - n * D[theta2[x, t], {x, 2}]]

u2[x_, t_] = -2 * n * D[theta2[x, t], x] / theta2[x, t]
Simplify[D[u2[x, t], t] + u2[x, t] * D[u2[x, t], x] - n * D[u2[x, t], {x, 2}]]
```

Out[47]= 0

Out[49]= $\frac{2\,a\,e^{k1^2 n t-k1x}k1\,n}{1+a\,e^{k1^2 n t-k1x}}$

Out[50]= 0

Out[52]= 0

Out[53]= $-\frac{2\left(-a\,e^{k1^2 n t-k1x}k1-b\,e^{k2^2 n t-k2x}k2\right)n}{1+a\,e^{k1^2 n t-k1x}+b\,e^{k2^2 n t-k2x}}$

Out[54]= 0