Manuscript Number:	SG-D-21-00130
Article Type:	Original article
Keywords:	Fractals; Generalised Extreme Value (GEV) distributions; Fréchet-, Weibull-, Gumbel-distributions; box counting; nearest neighbour-distributions; mineralisation growth models.
Corresponding Author:	Alison Ord The University of Western Australia Crawley, Western Australia AUSTRALIA
First Author:	Bruce Edward Hobbs, PhD
Order of Authors:	Bruce Edward Hobbs, PhD
	Alison Ord
	Thomas Blenkinsop, PhD
Abstract:	The concept of fractal spatial distributions of mineralisation has been widely proposed since Mandelbrot (1965) who emphasised the stable Pareto-Lévy distribution as the relevant distribution. The fractal concept is used as a basis for estimating endowment and for erecting exploration models based on self-organised criticality. This paper explores the proposition that the growth kinetics for a mineralising system are reflected in the probability distributions that describe the spatial patterns of mineralisation. We revisit the data sets and ask the question: What are the best fit probability distributions for the spatial distribution of mineralisation ? The answer is: members of the Extreme Value Distribution family (Gumbel-, Fréchet- and Weibull- distributions) and not the Pareto distribution. Thus, the spatial distribution of mineralisation is not a fractal although the tails of the distributions can be or resemble power-laws. The standard box counting procedure for a spatial point distribution establishes a nearest neighbour distribution and hence, by definition, the resulting distribution is Weibull and not Pareto. The mass distributions are Fréchet and not Pareto. The extreme end members are Gumbel. We discuss the implications of these distributions for models that generate mineralisation sites within a system and for the underlying thermodynamics.
Suggested Reviewers:	Julian Vearncombe, PhD Director, SJS Resource Management julian@vearncombe.com Julian is an expert in the application of effective structural geology to mineral exploration and mining.
	Jorn Kruhl, PhD Professor Emeritus, Technische Universität München: Technische Universitat Munchen kruhl@tum.de Jorn is an expert structural geologist with special interests in nonlinear systems.
	Peter Sorjonen-Ward, PhD mineral systems researcher, Geological Survey of Finland: Geologian tutkimuskeskus peter.sorjonen-ward@gsf.fi Peter is a superb geologist, specialising in structure and mineralising systems.
	Steve Henley, PhD President, International Raw Materials Observatory steve@vmine.net Steve is expert in the development of alternatives to geostatistics for mineral deposit modelling and exploration.
	Cees Passchier Professor Emeritus, Johannes Gutenberg University: Johannes Gutenberg Universitat Mainz cpasschi@uni-mainz.de

	Cees is an expert geologist with special interests in the application of nonlinear theories to deforming rocks.

26 May 2021

Dear Editors,

Journal of Structural Geology

I would like to submit our manuscript entitled *The Spatial Distributions of Mineralisation* by Bruce E. Hobbs, Alison Ord and Thomas Blenkinsop, to the Journal of Structural Geology.

This paper determines members of the Extreme Value Distribution family to best fit the spatial distribution of mineralisation, and discusses the implications of these distributions for models that generate mineralisation sites within a system and for the underlying thermodynamics.

Neither the manuscript itself nor any of the content are submitted elsewhere. We have no conflicts of interest.

BEH conceived the study and drafted the manuscript. AO developed the study through Wolfram Language probability distribution analyses, and critically revised the manuscript. TB provided field data and critically revised the manuscript. All authors gave final approval for publication.

Sincerely yours,

Alison Ord

The University of Western Australia

+61 41 993 1276

alison.ord@uwa.edu.au

1   **The Spatial Distributions of Mineralisation.**

2   **Bruce E. Hobbs, Alison Ord and Thomas Blenkinsop.**

3

4   **Abstract.**

5   The concept of fractal spatial distributions of mineralisation has been widely proposed since

6   Mandelbrot (1965) who emphasised the stable Pareto-Lévy distribution as the relevant

7   distribution. The fractal concept is used as a basis for estimating endowment and for erecting

8   exploration models based on self-organised criticality. This paper explores the proposition

9   that the growth kinetics for a mineralising system are reflected in the probability distributions

10   that describe the spatial patterns of mineralisation. We revisit the data sets and ask the

11   question: *What are the best fit probability distributions for the spatial distribution of*

12   *mineralisation*? The answer is: *members of the Extreme Value Distribution family* (*Gumbel-,*

13   *Fréchet- and Weibull- distributions*) *and not the Pareto distribution.* Thus, the spatial

14   distribution of mineralisation is not a fractal although the tails of the distributions can be or

15   resemble power-laws. The standard box counting procedure for a spatial point distribution

16   establishes a nearest neighbour distribution and hence, by definition, the resulting distribution

17   is Weibull and not Pareto. The mass distributions are Fréchet and not Pareto. The extreme

18   end members are Gumbel. We discuss the implications of these distributions for models that

19   generate mineralisation sites within a system and for the underlying thermodynamics.

20   **Keywords:** *Fractals, Generalised Extreme Value (GEV) distributions, Fréchet-, Weibull-,*

21   *Gumbel-distributions, box counting, nearest neighbour-distributions, mineralisation growth*

22   *models.*

23   **1.   Introduction.**

24   This paper concerns process based probability distributions that characterise the

25   spatial distribution of mineralisation at the regional scale. The emphasis is on orogenic gold

26   deposits. The spatial distribution at all length scales is intrinsically heterogeneous and there

27   are a number of ways of characterising such heterogeneity. One way is to use correlation

28   functions of various kinds (Kroner, 1971; Torquato, 2002; Kalidindi, 2015); this commonly

29   employs Fourier transforms to establish n-point correlation functions (Kalidindi, 2015). A

30   second way is to use wavelet transforms (Arneodo et al., 1995) in association with Hurst

31   exponents (Ord et al., 2016; Munro et al., 2018; Doutre et al., 2015; Doutre, 2018) to

32   establish spatial patterns, correlations and multi-fractal spectra. A third way is to establish the

33   recurrence patterns and associated quantitative measures for the system (Ord et al., 2018;

34   Hobbs and Ord., 2021). Another is to establish the probability functions that characterise the

35   heterogeneity. This paper concentrates on the latter. Although all four methods have a

36   thermodynamic underpinning (Beck and Schlögl, 1995; Arneodo et al., 1995), the statistical

37   functions are directly related to the thermodynamics of the system (Lavenda, 1995) and hence

38   place important constraints on models for the formation of the mineralisation. In fact all four

39   approaches are related through the attractor for the system. The attractor is the N-dimensional

40   topological surface (which can be fractal) that describes the geometry of physical and

41   chemical states that the system can occupy (Sprott, 2003). N is the number of independent

42   degrees of freedom for the system. Nonlinear systems involving similar physical and

43 chemical processes evolve with time towards similar attractors for a large range of initial
44 conditions. The attractor is characterised by many quantitative measures: The large number
45 of (in principle, indefinite) states on the attractor of a nonlinear system gives rise to the
46 multifractal nature of the system (Beck and Schlögl, 1995) together with the observed
47 probability distributions (Lucarini et al., 2016). Recurrence arises from the system repeatedly
48 visiting neighbouring states on the attractor (Marwan et al., 2007b). The correlations measure
49 how the states on the attractor for the system are inter-related (Sethna, 2006, Chapter 10). The
50 probability distributions are measures of the geometry of the attractor (Lucarini et al., 2016;
51 Bodai, 2016).

52 A quote from Savageau (1979) illustrates the concept explored in this paper: *Any*
53 *system that grows into a stable mature form has a growth curve that is a legitimate*
54 *cumulative probability distribution*. Hence the observed probability distributions not only
55 contain information on the geometry of the attractor resulting from the nonlinear dynamics of
56 a mineralising system but have the potential to constrain the growth kinetics of the
57 mineralising system.

58 Traditionally, two empirical probability distributions have been used to characterise
59 mineralisation patterns. These are the log-normal (Singer, 2013; Singer and Menzie, 2010)
60 and the power-law (Mandelbrot, 1965) distributions although only the latter are commonly
61 used for spatial patterns of mineralisation. The power-law distribution is written:

62 $$N(r) = Ar^{-D} \qquad (1)$$

63 where $N(r)$ is the number of boxes or spheres with dimension, r, occupied by mineralisation
64 sites in a region of interest, A is a constant and D is a characteristic *power law exponent*,
65 commonly referred to as the *fractal dimension*. Although fractal behaviour is implied
66 explicitly by many authors, power law behaviour can commonly be demonstrated over only a
67 limited range in r and commonly is an approximate fit to the data (Corral and Gonzalez,
68 2019). (1) is known as a *power-law distribution* (physics), a *Pareto distribution* (finance), a
69 *hyperbolic distribution* (social sciences) and as a stable[1] *Pareto-Lévy distribution*
70 (Mandelbrot, 1960, 1961, 1963). These distributions have been popular because they have
71 simple mathematical expressions and have heavy right hand tails so that considerable
72 departures from the sample mean can be accommodated. In particular the Pareto-Lévy
73 distribution was favoured by Mandelbrot because he showed it was able to characterise
74 considerable volatility (departures from the mean) in commodity and financial markets. Both
75 he and Fama (1963, 1965) identified the Pareto distribution as a best fit for the data, capable
76 of accommodating skewness and the slowly decaying tails of the empirical distributions. It is
77 clear though that Fama (1965) was aware of the Generalised Extreme Value (GEV)
78 distributions being studied by Tippet (1925), Gnedenko and Kolmogorov (1968) and Gumbel
79 (1954). Mandelbrot (1956) refers to the work of Gnedenko and Kolmogorov (1968) but does
80 not follow up on the extreme distributions. The GEV distributions are the topic of this paper,

81 We note that although both the log-normal and power law distributions have heavy
82 right handed tails only the log-normal distribution has a well-defined mean and variance so

---

[1] A probability distribution is said to be *stable* if linear combinations of that distribution add together to produce the same kind of distribution. The stable distributions are the Gaussian, Cauchy, Lévy, Gumbel, Weibull and Fréchet distributions (Nair et al., 2021).

83   that the mean of the (logarithm of) observations tends to the mean of the distribution with

84   increasing numbers of observations as defined by the Law of Large Numbers (Dekking,

85   2005, p181 - 190). In contrast, the power-law distribution lacks a mean or variance and the

86   sample arithmetic mean diverges as the number of observations increases.

87   Mandelbrot (1997) credits Fréchet (1941) with the first observation that the spatial

88   distribution of some natural features follows the hyperbolic relation, (1). Mandelbrot realised

89   that many natural phenomena are characterised by statistical distributions where the variance

90   is infinite. To represent such phenomena he emphasised the power-law or Pareto distribution

91   which became identified with the concept of fractals; the Pareto distribution has infinite mean

92   and variance and has a fat tail so that it can represent extreme events. However he was aware

93   of other fat tailed distributions such as those arising from Lévy stochastic processes

94   (Mandelbrot, 1963). Even though the stable extreme distributions had been identified many

95   years earlier (Fisher and Tippett, 1928; Gnedenko, 1943; Gnedenko and Kolmogorov, 1949

96   1968; 1949 in Russian), Mandelbrot did not use these distributions and his excursions into the

97   statistics of extremes ended with some Lévy distributions capable of accommodating

98   skewness and the sub-exponential decaying tails of observed distributions, in contrast to the

99   thin and rapidly decaying tails of the normal distribution.

100   Authors in the geological literature became infatuated with the concept of fractals,

101   encouraged by the book by Mandelbrot (1982) and influential works by Mandelbrot (1965),

102   Barton and Scholtz (1965), Pointe (1965) and Turcotte (1986, 1989). The result is that the

103   Pareto distribution has become the heavy tailed distribution of choice in the geological

104   literature especially if the distribution is power-law like in appearance. Moreover, the

105   concepts of nonlinear behaviour, critical behaviour and fractal geometry have become widely

106   inter-linked with the implication that any nonlinear behaviour automatically leads to power-

107   law probability distributions for variants produced by such behaviour. This is especially

108   emphasised in the notion that fractal geometries and power-law distributions arise from self-

109   organised criticality (Hronsky and Groves, 2008; McCuaig and Hronsky, 2014). This

110   association of Pareto distributions with nonlinear systems has not proved correct and the

111   literature on GEV distributions arising from nonlinear dynamical systems is now immense

112   (see Lucarini et al., 2016 for examples). It is the Generalised Extreme Value family of

113   probability functions that characterise nonlinear systems rather than Pareto distributions

114   (Nicolis et al., 2006; Lucarini et al., 2012, 2014, 2016). In saying this one should also

115   appreciate that the extremes of the GEV distributions are commonly (see Lucarini et al., 2016

116   as an example) represented by Generalised Pareto Distributions (GPD) which include the

117   exponential distribution (as the tail of a Gumble distribution) and the Pareto distribution (as

118   the tail of a Fréchet distribution). A system at criticality (that is, a system that has evolved to

119   be scale free) is by definition characterised by a Pareto distribution.

120   In summary, Mandelbrot (1963) and Fama (1965) proposed the Lévy stable

121   distribution to represent skewed heavy tailed distributions as an alternative to the Gaussian

122   distribution to characterise commodity and financial systems. Although there are many

123   heavy-tailed (see Figure 1) alternatives to the Gaussian distribution (see Nair et al., 2021), the

124   fundamental reason for selecting Lévy stable distributions is that they arise from the

125   Generalized Central Limit Theorem, which states that stable laws are the only possible limit

126   distributions for properly normalized and centred sums of independent, identically distributed

random variables. An important further outcome of this theorem extended to distributions that lack a mean or variance is that there are just three extreme stable distributions that are members of the *Generalised Extreme Value* (GEV) distribution family. These are the Gumbel, Weibull and Fréchet distributions. In this paper we show that these distributions characterise the spatial distribution of mineralisation far better than the Lévy-Pareto distribution. Moreover, not only do the GEV distributions arise directly from the Generalised Central Limit Theorem, they have a strong foundation in thermodynamics and arise directly both from realistic growth laws for mineralising systems and from the nonlinear dynamics of these systems.

The structure of this paper is as follows. In Section 2, we examine the statistical basis for and the overall characteristics of the GEV distributions. In Section 3 we give examples of spatial distributions for mineralisation that have formerly been interpreted as fractal (or bi-fractal) distributions and show that they are better represented as GEV distributions rather than the traditional Pareto distributions. In Section 4 we consider the thermodynamic basis for GEV distributions and indicate the significance for mineralising systems. Basically the generation of GEV distributions for a mineralising system is an expression of the partitioning of energy that has been input to the open system. We also show that realistic growth models for mineralising systems lead to GEV distributions for the spatial arrangement of mineralising sites within a system. We also indicate some directions for future work and point to other distributions that may characterise mineralised systems other than orogenic gold deposits. Finally we draw some conclusions in Section 5.
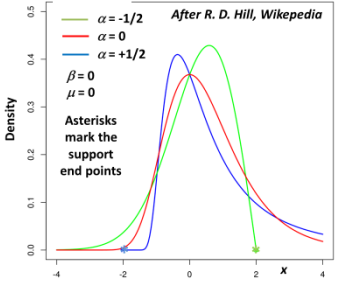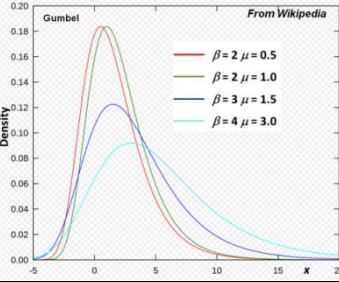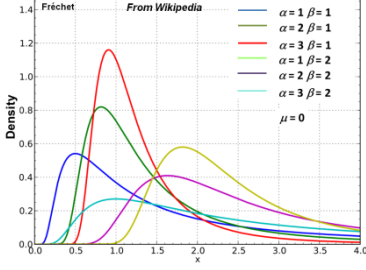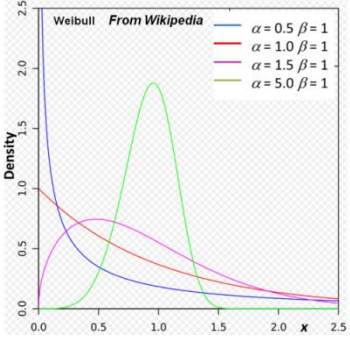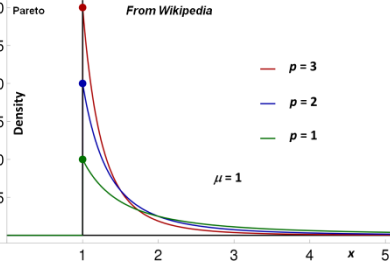
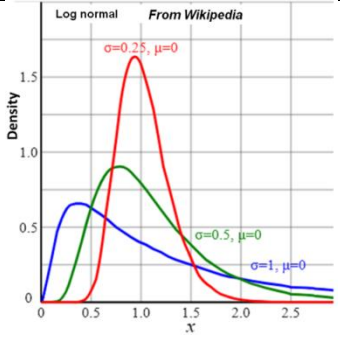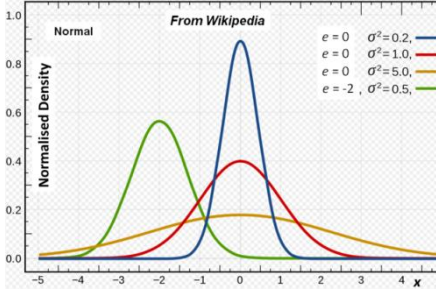## 2. Probability distributions for mineralising systems.

We would like an understanding of the statistics that result from the various processes that operate in mineralising systems with the perhaps optimistic view that the measured statistics of such systems may improve predictability, discovery and extractability, and provide constraints on the processes that operated to form these systems. To this end we seek an understanding of the observed probability distributions in terms of physical and chemical processes and if possible of thermodynamics rather than pragmatic statements regarding the class of probability distribution displayed, such as log-normal or Pareto, with no associated physical, chemical or geologic insight. Interestingly, we find that log-normal and fractal (Pareto) distributions play only subsidiary roles in this framework despite the wide emphasis on such distributions in the literature. In order to progress and understand any relationship to the processes involved in mineralisation we need to look at the physical basis for many of the distributions used in the literature.

In advance, we present in Table 1 some of the probability distributions of interest with respect to spatial distributions along with their associated physical processes and constraints that result in the distribution maximising entropy. For details see Frank (2009, 2019).

Table 1. Some probability distributions pertinent to spatial distributions. In this table, α is the shape factor, β is the scale factor, μ is the position factor, m is the mean and σ^2 is the variance. $<a>$ denotes the average of the quantity, a.

Distribution	Probability Distribution Function, $y(x)$.	Physical significance	Maximum entropy constraints	Graphical density distributions

4

Generalised Extreme Value	$y = f(x;\alpha,\beta,\mu) = \dfrac{1}{\beta} t(x)^{\alpha+1} \exp(-t(x))$ $\text{where } t(x) = \left(1+\alpha\left(\dfrac{x-\mu}{\beta}\right)\right)^{-\frac{1}{\alpha}} \text{ if } \alpha \neq 0$ $= \exp\left(-\dfrac{x-\mu}{\beta}\right) \text{ if } \alpha = 0$ **If $\alpha = 0$ this is a Gumbel distribution. If $\alpha > 0$ this is a Fréchet distribution. If $\alpha < 0$ this is a Weibull distribution.**	**A family of extreme value stable distributions. The Gumbel (Type I), Fréchet (Type II) and Weibull (Type III) as specific examples.**	**Types I, II and III arise from two constraints. One constraint sets the average location; the other is a measure of the average tail weighting.**	
Gumbel	$y = f(x;\beta,\mu) = \dfrac{1}{\beta}\exp(-(z+\exp(-z)))$ $\text{where } z = \dfrac{x-\mu}{\beta}$ **Tail of distribution is an exponential distribution**	**Corresponds to systems that nucleate, grow competitively and have moderate extinction times due to competition between processes.**	**To maximise entropy average location parameter is $\langle x \rangle$. Average tail weighting is $\langle \exp(-x) \rangle$**	
Fréchet	$y = f(x;\alpha,\beta,\mu) = \dfrac{\alpha}{\beta}\left(\dfrac{x-\mu}{\beta}\right)^{-1-\alpha} \exp\left(-\left(\dfrac{x-\mu}{\beta}\right)\right)$ **Tail of distribution is a power law distribution**	**Corresponds to systems that nucleate quickly, grow competitively and have long extinction times due to relatively weak competition.**	**To maximise entropy average location parameter is $\langle \log x \rangle$. Average tail weighting is $\langle x^{-\delta} \rangle$ where δ is the power law exponent for the tail**	
Weibull	$y = f(x;\alpha,\beta) = \dfrac{\alpha}{\beta}\left(\dfrac{x}{\alpha}\right)^{\beta-1}\exp\left(-\left(\dfrac{x}{\alpha}\right)^{\beta}\right); x \geq 0$ $= 0; x < 0$ **If X is Weibull then $\sqrt{X}$ is Exponential$\left(\dfrac{1}{\sqrt{X}}\right)$** **If X is Weibull then $\dfrac{1}{X^m}$ is Fréchet**	**Corresponds to systems that nucleate quickly, grow competitively and have short extinction times due to relatively strong competition.**	**To maximise entropy average location parameter is $\langle \log x \rangle$. Average tail weighting is $\langle (M-x)^{\delta} \rangle$ where M is the maximum value of x and δ is the power law exponent for the truncated tail**	
Pareto or power law	$y = f(x;\alpha,\mu) = \dfrac{\alpha\mu^p}{x^{p+1}}$	**A Pareto distribution often arises for the last few percentiles of a log normal distribution.**	**A special case of the log normal distribution for systems where the growth in the value of a variable in successive time intervals is proportional to the value of the variable in the previous**	

			time interval
Log-normal	$y = f\left(x;e,\sigma\right) = \dfrac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\left(\dfrac{\left(\ln x - e\right)^2}{2\sigma^2}\right)\right)$	**Closely related to GEV-distributions but can differ in detail. Those distributions arise from growth-competitive (birth-death) processes. Also can be coupled to diffusion processes (Crescenzo and Paraggio, 2019).**	**Maximises entropy for fixed geometric mean. If a system grows such that the growth in successive time intervals is a proportion of that in previous time interval the probability distribution is likely to be log normal.**
Normal	$y = f\left(x;e,\sigma^2\right) = \dfrac{1}{\sigma\sqrt{2\pi}}\exp\left(-\dfrac{1}{2}\left(\dfrac{x-e}{\sigma}\right)^2\right)$	**Typical of systems at equilibrium. If a system is controlled by processes that result in small variations from the mean then the probability distribution is likely to be normal.**	**Maximises entropy for fixed variance. Many distributions closely resemble normal distributions for some range of parameters as shown in above diagrams**

168

2.1. Closed systems at equilibrium.

In closed systems at equilibrium the entropy is maximised (Gibbs, 1875-1878; Callen, 1960). Thus the governing principle for such systems is that the processes that lead to equilibrium produce probability distributions that maximise the entropy. Other constraints also maximise the entropy for other distributions. There is a zoo of statistical functions (see: https://en.wikipedia.org/wiki/Relationships_among_probability_distributions and https://www.itl.nist.gov/div898/handbook/eda/section3/eda366.htm). The classical approach has been to select the simplest of these, namely the Gaussian distribution, the log-normal distribution or the power-law and to regard any other kind of distribution as something pathological.

One of the simplest statistical distributions after the uniform distribution is the Gaussian distribution which results from stochastic processes operating with no correlation between events. The underlying thermodynamic constraint for this distribution is that differential entropy is maximized for a given variance. A Gaussian random variable has the largest entropy amongst all random variables of equal variance, or, alternatively, the maximum entropy distribution under constraints of mean and variance is the Gaussian. This distribution follows directly from the *Law of Large Numbers* (which states that for data with a well-defined mean and variance the mean of the distribution will approach that of the sample as the sample size increases) and the Central Limit Theorem which states that independent and identically distributed (i.i.d.) random variables with finite non-zero

variances will tend to a Gaussian distribution as the number of variables grows. Thus the Gaussian distribution is precisely defined by the Central Limit Theorem with a mean specified by the Law of Large Numbers and a variance that maximises the differential entropy. It is a thin tailed distribution. The Gaussian distribution results from processes where there are strong physical, chemical or genetic controls that limit the variation around a mean value such as in the probability distribution for the heights of adult humans.

The log-normal distribution is defined in a similar manner to the Gaussian distribution; the Central Limit Theorem now applies to the logarithms of the variates rather than to the variates themselves. The log-normal distribution is the maximum entropy probability distribution for a random variate, X, for which both the mean and variance of $\ln(X)$ are specified. The log-normal distribution is fat tailed and so is used to model some fat tailed distributions. As indicated, since both the Gaussian and log-normal distributions maximise the entropy they are relevant to closed systems at equilibrium.

Other distributions follow from different constraints (Frank, 2009, 2019). Fixing the *geometric mean* attracts a distribution towards a *power law* distribution as the population increases whilst fixing the *arithmetic mean* attracts a distribution towards an *exponential* distribution. Frank (2009) speculates that the geometric mean is representative of most of the information in many systems and hence power laws or distributions close to power laws are common in Nature.

A distribution that defines the distribution or partitioning of energy in an isolated system is the classical Boltzmann distribution which proposes that the probability, p_i, that a given state, i, should have an energy, E_i, is given in terms of the fraction of states, $\frac{N_i}{N}$, that have an energy, E_i, as

$$p_i = \frac{N_i}{N} = \frac{\exp\left(-E_i / kT\right)}{Z} \qquad (2)$$

where N_i is he number of states that have energy, E_i, N is the total number of states, k is Boltzmann's constant and T is the absolute temperature. Z (*the canonical partition function*) is present to ensure all accessible states add up to 1 and is given by

$$Z = \sum_{j-1}^{M} \exp\left(-E_j / kT\right) \qquad (3)$$

The Boltzmann distribution, (2), maximises the entropy of the system, $S = -\sum_{i=1}^{M} p_i \log p_i$, subject to the constraint that the mean energy of the system is $\sum p_i E_i$. The Boltzmann distribution is the basis of classical statistical thermodynamics and hence has particular relevance to the discussion of equilibrium states in isolated thermodynamic systems.

Of course there are many other distributions that maximise entropy for various constraints (Dawson and Wragg, 1973; Frank, 2009) but many are relevant to systems at equilibrium. The relevant distribution for a system at equilibrium depends on the scale of the distribution, the number of ways in which the events can occur, and the moments of the distribution (Lienhard and Myer, 1967).

226   For spatial systems at equilibrium, similar statements hold. For instance (McFadden,
227 1965) the point distribution that corresponds to maximum entropy for a given density is the
228 Poisson distribution. As we have indicated, natural examples are commonly proposed to be
229 the fractal random fields represented by a fractal dimension (Carlson, 1991; Blenkinsop,
230 1994, 1995, 2014; Hodkiewicz et al., 2005). Methods (other than box counting) of
231 establishing the fractal dimension of random spatial fields with scale invariance are given by
232 Biermé (2017).

2.2. Open systems far from equilibrium.

234   There is of course an enormous range of distributions other than those considered
235 above but few have any thermodynamic basis or seem capable of reflecting the processes that
236 operate during the formation of the systems involved. It is important to note that the
237 Gaussian, log-normal and Boltzmann distributions are relevant to isolated systems or closed
238 systems in contact with a heat bath; all such systems are at equilibrium. However,
239 mineralising systems, while they operate, are far from equilibrium and are open systems (Ord
240 et al., 2012). It seems logical that the distributions in Section 2.1 may not be relevant to open
241 systems far from equilibrium and we ask: *what statistical distributions might be relevant to*
242 *far from equilibrium systems where variants are not independent of each other and both long*
243 *and short range interactions and correlations exist*? Such systems are commonly
244 characterised by long, fat tails where the mean of the distribution diverges as the sample size
245 increases and the variance may be infinite. Some of these distributions are classified in Figure
246 1 which distinguishes distributions according to the nature of the tails of the distributions. If
247 the tails approach a power law for large populations they are said to be *regularly varying*.
248 There are many distributions in this category other than the Pareto distribution which is
249 distinguished by being also scale invariant.

250
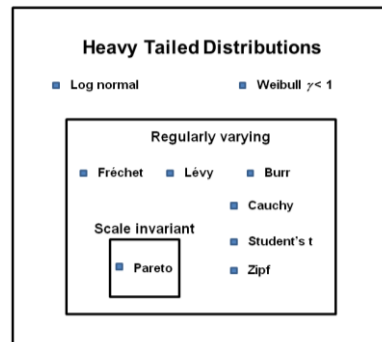
251 Figure 1. Classification of the heavy tailed distributions of interest. The *regularly varying*
252 distributions are those that behave close to power laws as the population size increases. For
253 details of this diagram and of the distributions within it see Nair et al. (2021). The γ quoted
254 for the Weibull distribution is the extreme value index of (4) and Figure 2.

255   A generalisation of the Central Limit Theorem by Gnedenko and Kolmogorov (1968)
256 states that the sum of a number of random variables with symmetric distributions having
257 infinite variance, will tend to a *stable* distribution as the population increases. The family of
258 distributions corresponding to this limit are the *Generalised Extreme Value* (GEV)
259 distributions. The important concept here is that the extreme values of an initial distribution
260 (namely, those in the tail of the initial distribution, Figure 3a) have their own probability

261 distributions known as extreme value distributions. The initial distributions for relatively
262 small populations are asymptotic to the extreme distributions as the population increases.
263 These asymptotic distributions are said to be *attractors* for the initial distribution. The only
264 stable distributions that correspond to the tails of other distributions (those that belong to the
265 family of generalised extreme value (GEV) distributions) and that have a formal
266 mathematical expression are the *Weibull*, *Gumbel* and *Fréchet* distributions (Figure 2).
267   The generalised extreme value distributions are given by

$$H(x) = \exp\left[-\left(1 + \gamma\left(\frac{x-\mu}{\varphi}\right)\right)_+^{-\frac{1}{\gamma}} \right] \qquad (4)$$

269   The family of extreme value distributions (EVD) is given by

$$H_\gamma(x) = \exp\left[-(1+\gamma x)_+^{-\frac{1}{\gamma}} \right] \text{ if } \gamma \neq 0^2 \qquad (5)$$

$$H_\gamma(x) = \exp\left[-\exp^{-x} \right] \text{ if } \gamma = 0 \qquad (6)$$

272 where $H_\gamma(x)$ is the cumulative distribution function of the Extreme Value Distribution of the

273 random variable, *x*, with $\gamma = \frac{1}{\alpha}$ the *extreme-value index*. For a detailed discussion of these

274 distributions see Nair et al. (2021).
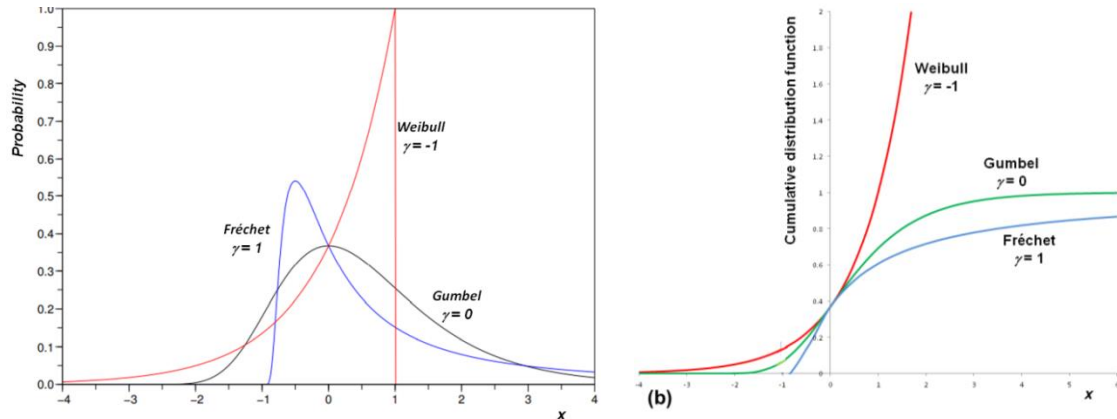
275
276 Figure 2. Probability distributions for members of the extreme value distributions with values
277 of the extreme value index, γ, indicated. (a) Density distribution functions. (b) Cumulative
278 distribution functions.
279   Depending on the sign of γ, three maximum domains of attraction are defined (Figure
280 2):
281   If $\gamma > 0$, *H*(*x*) belongs to the Fréchet maximum domain of attraction. This domain of
282 attraction includes distributions with *heavy* tails, i.e. their initial distribution functions
283 decrease as a power function or more slowly.
284   If $\gamma = 0$, *H*(*x*) belongs to the Gumbel maximum domain of attraction. This domain of
285 attraction includes distributions with *light* tails, i.e. their initial distribution functions decrease
286 as an exponential function.

---

² The symbol $(y)_+$ means that the quantity inside the brackets takes the maximum value between 0 and *y*.

287   If $\gamma < 0$, $H(x)$ belongs to the Weibull maximum domain of attraction. This domain of

288   attraction includes distributions with *short* tails, i.e. they have a finite endpoint. Although the

289   Weibull distribution is short tailed it can be either heavy-tailed or light-tailed.

290   **(a)**   **Fracture length**   **(b)**

291   Figure 3. Asymptotic distributions. (a) An initial distribution shown as the full line. This

292   could for instance represent the probability distribution of all fractures in an area. The

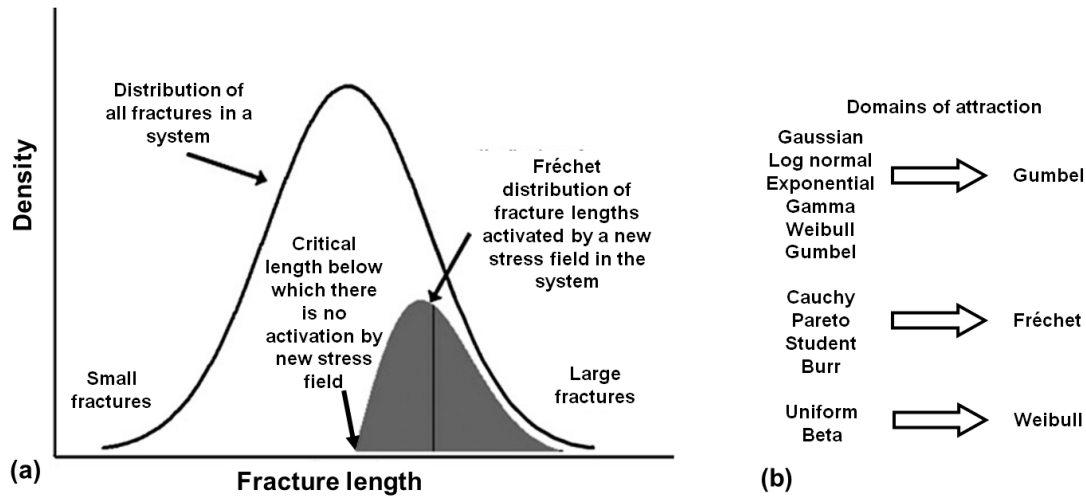293   distribution of fractures activated by a new stress field is shown in gray. Griffith theory

294   predicts there is a critical length below which reactivation does not occur. The probability

295   distribution of activated fracture is Fréchet (although the distribution of stress sites is

296   Weibull; Lavenda, 1995, p180). In this case the Fréchet distribution is the domain of

297   attraction of the initial distribution. (b) Common initial distributions and their domains of

298   attraction. After Embrechts et al.(1997, Chapter 3) and Quinn and Quinn (2010).

299   The following relations exist between the three stable extreme distributions:

300   If X has a Fréchet distribution (commonly written Φ_α), then $-1/X$ has the Weibull

301   distribution (commonly written Ψ_α) and $\log(X^\alpha)$ has a Gumbel distribution (commonly

302   written, Λ_α).

303   Lavenda (1995, pp 6, 78 - 79) presents a thermodynamic explanation for the stable

304   distributions. For a closed system the entropy must always increase as the system evolves to

305   maximum entropy at equilibrium. However an open system can exchange entropy with its

306   environment. So that although the total entropy, S, of the system plus its environment must

307   always increase towards equilibrium to satisfy the second law of thermodynamics, the

308   internal entropy can decrease as long as the decrease in internal entropy, $\Delta S_{internal}$, is

309   compensated by an increase in entropy of the environment, $\Delta S_{external}$. Thus the generation of

310   ordered structures and patterns in systems far from equilibrium is associated with an internal

311   entropy reduction, $\Delta S_{internal}$ (Kondepudi and Prigogine, 1998).

312   Lavenda shows that

$$\Delta S_{internal}(x) = -(1-\alpha)\left(\frac{\beta}{x}\right)^{\frac{\alpha}{1-\alpha}}$$

314   where α is a constant $0 < \alpha < 1$. This has the form of a Fréchet distribution:

$$\Delta S_{internal}(x) = \ln G(x) = -cx^{-D}$$

The Fréchet distribution can be considered as the generalisation for extreme value distributions of the Boltzmann principle (Lavenda , 1965, pp 6, 49-51, 149; Lavenda and Florio, 1992). Hence in an open system far from equilibrium we expect the energy to be partitioned so as to follow a Fréchet distribution just as in a closed system at equilibrium the equipartition of energy follows a Boltzmann distribution as discussed in Section 2.1.

For a Weibull distribution Lavenda (1995, pp79) shows that

$$\Delta S_{internal}(x) = -(\alpha - 1)\left(\frac{x}{\beta}\right)^{\frac{\alpha}{\alpha-1}}$$

2.3. Spatial point distributions.

The traditional box counting procedure in two dimensions for an array of points consists of scanning the array with a series of "boxes" (circles or squares) with varying dimension, r, and counting the number, $N(r)$, of boxes that contain at least one point for each value of r. If $N(r)$ is related to r by (1) then $N(r)$ follows a power law and fractal geometry is implied with a fractal dimension, D. The normal procedure is to plot log $N(r)$ against log r and if a straight line ensues the geometry of the point array is implied to be fractal. The following argument follows Blenkinsop and Sanderson (1999). If the sampling of points is a Poisson process so that n boxes are distributed randomly over the space then the probability of a box containing x points is

$$P(x) = \frac{\lambda^x}{x!}\exp(-\lambda)$$

where λ is the mean number of points per box.

The probability of a box containing at least one point is (remembering that $0! = 1$)

$$P(x \geq 1) = 1 - P(0) = 1 - \exp(-\lambda)$$

If we take $\lambda = nr^2$ (Blenkinsop and Sanderson, 1999) then

$$P(x) = 1 - \exp(-nr^2) \qquad (7)$$

which is the equation for a cumulative Weibull distribution with $\gamma = 0.5$. The corresponding probability density distribution is

$$G(x) = 2nx\exp(-nx^2) \qquad (8)$$

This distribution is illustrated in Figure 4 for the number of mineralised sites, $n = 1000$ and a square sample area 1000 km across. The result is a curve that could be fitted by two linear regions representing two fractal dimensions, one at spatial scales below 100km and the other at larger spatial scales. This is the kind of interpretation made by Raines (2008).
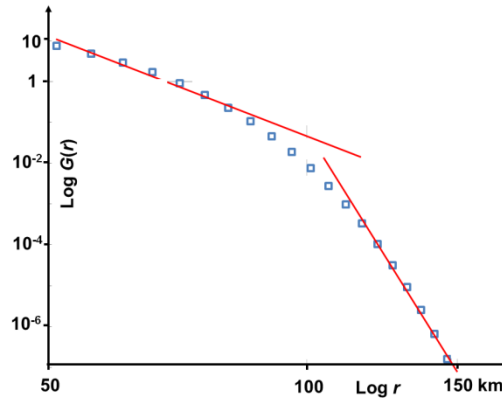
Figure 4. The nearest neighbour density distribution, (8), as a log-log plot with $n = 1000$ and a square sample area 1000 km across. This is a Weibull density distribution with γ in (5) equal to 0.5. The two red lines are attempts to define two linear regions with different "fractal dimensions".

The traditional box counting procedure for a two dimensional array of points is in fact a means of calculating a nearest neighbour distribution (Cressie, 1993, his Figure 8.9) which is known to be a Weibull distribution (Chandrasekhar, 1943; Lavenda, 1995, pp 164, 182; Carpena and Coronado, 2019). This distribution closely resembles a Pareto distribution for small values of α but for large values of α might be interpreted as a *bifractal* distribution (Figure 4) as does Blenkinsop (1994), Carlson (1991) and many others for their data sets. Hodkiewicz et al. (2005) document fractal distributions of damage in the Yilgarn of WA. Fractal dimensions vary between 1.5 and 1.9. It would be interesting to test if this is an expression of a Weibull nearest neighbour distribution as predicted by theory.

Carpena and Coronado (2019) show that if the probability distribution of spacings, s, between consecutive members of a point set is $P(s)$ [this is the nearest neighbour distribution] then the box counting dimension, D_{box}, is related to $P(s)$ by

$$P(s) \sim s^{-(1+D_{box})} \qquad (8)$$

So that

$$D_{box} = -\left(1 + \frac{\log P(s)}{\log s}\right) \qquad (9)$$

Thus D_{box} is a constant only if $\log P(s)$ is a linear function of $\log s$.

This means that if the nearest neighbour distribution is a power law (albeit perhaps only over a limited range) then the data are fractal (over that range) and D_{box} can be defined. Otherwise the data are not fractal. It is possible then to define a fractal dimension at each point in the distribution; this is called a *local fractal dimension*. Further discussion in some detail is in Carpena and Coronado (2019, p 02205-3).

3. Examples of mineralised systems.

In this section we take the data from three well documented examples (Agterberg, 2013; Blenkinsop and Sanderson, 1999 and Blenkinsop, 2014) that have been discussed in terms of fractal (or power-law) distributions and reconsider them in terms of best fit probability distributions: in all cases these turn out to be extreme value distributions.

 3.1. Example from Kirkland Lake, Canada.

The first example is from Agterberg (2013, his Figure 4) and is a reprocessed version of data given in Raines (2008) where the interpretation was that the data are bi-fractal. Agterberg fits a line with slope 1.528 to these data in a log-log plot (Figure 5a). These same data on a linear-linear plot are shown in Figure 5(b) with a power-law fit shown. One sees that the entire data set can be represented with a power-law exponent of 1.206 whereas Agterberg (2014) fits eight data points to a power law with exponent 1.528. As indicated, Raines (2008) proposed a bi-fractal fit to these data. An attempt to fit two Pareto distributions to the data of Figure 5(a) using Mathematica failed. Results for a single Pareto fit are moderately good as suggested by Figure 5(b) and are given in Figure 6.

These same data are shown in Figure 5(c) as a log-log plot where a good parabolic fit is shown. Figure 5(d) shows the local box-counting dimension calculated from (9). One sees that values close to that proposed by Agterberg for D_{box} are obtained only for the last two data points. Most values are greater than two and hence are physically unrealistic. The conclusion is that for this example, the nearest neighbour distribution, $P(s)$, is not fitted well by a power law (although Figure 5b looks good to the eye) and the data set is not a fractal. In fact the data set is well fitted by a Weibull distribution (Figures 5 e, f) as is expected of a nearest neighbour distribution.
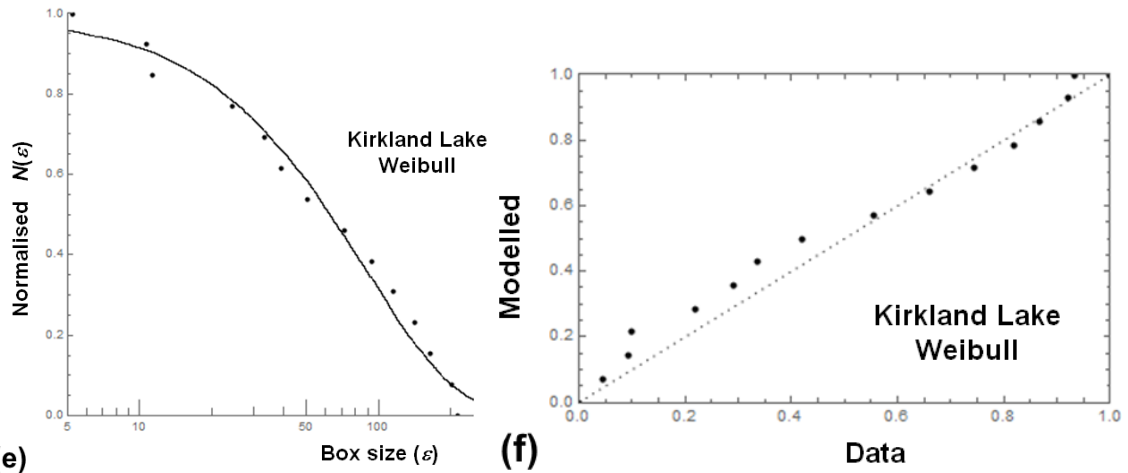

13

(e) **(f)**
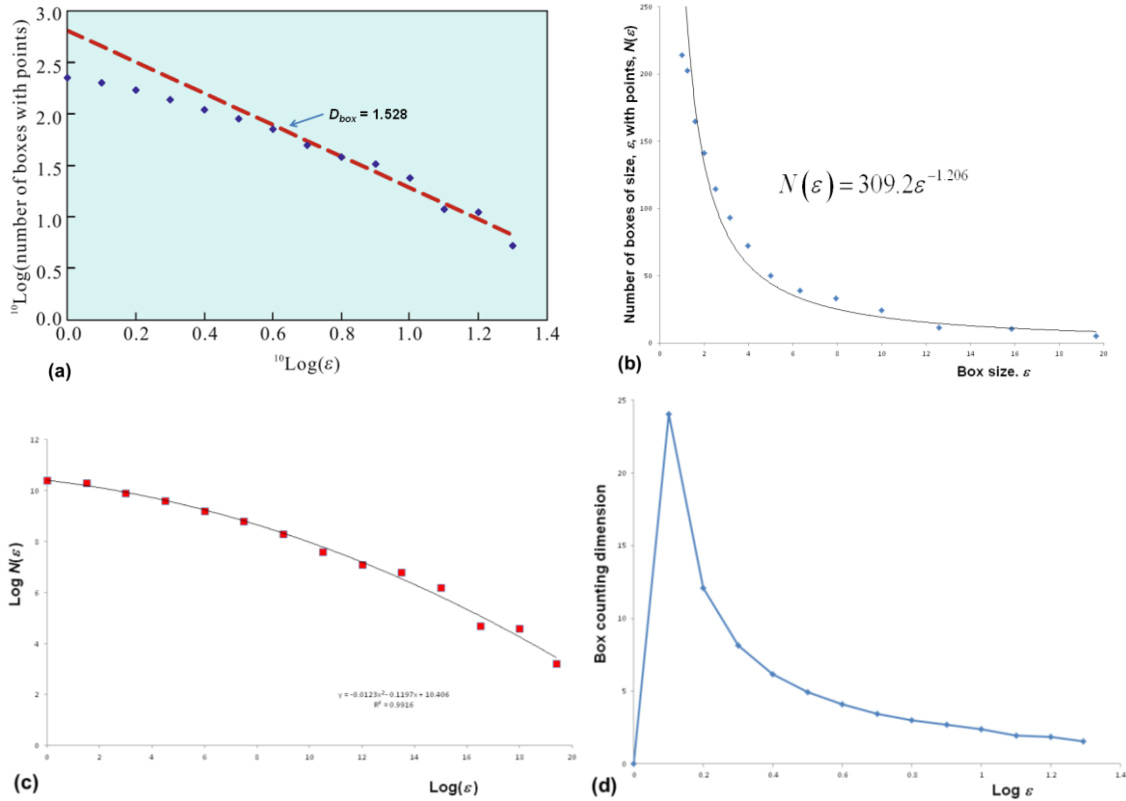
Figure 5. Analysis of data from Figure 4 of Agterberg (2013). (a) Reproduction of Figure 4 from Agterberg (2013) with his box count fit of $D_{box} = 1.528$ shown. (b) The data of Figure 4 shown on a linear-linear plot with a power law fit, $N(\varepsilon) = 309.2\ \varepsilon^{-1.206}$ shown. (c) Figure 4 of Agterberg (2013) reproduced showing that the data define a continuous parabolic curve. (d) D_{box} calculated from (c) using (2). (e) Weibull2 fit to data of Figure 4 using Mathematica. (f) Probability plot showing departures of data of Agterberg's Figure 4 from the Weibull2 fit.

(a) **(b)**
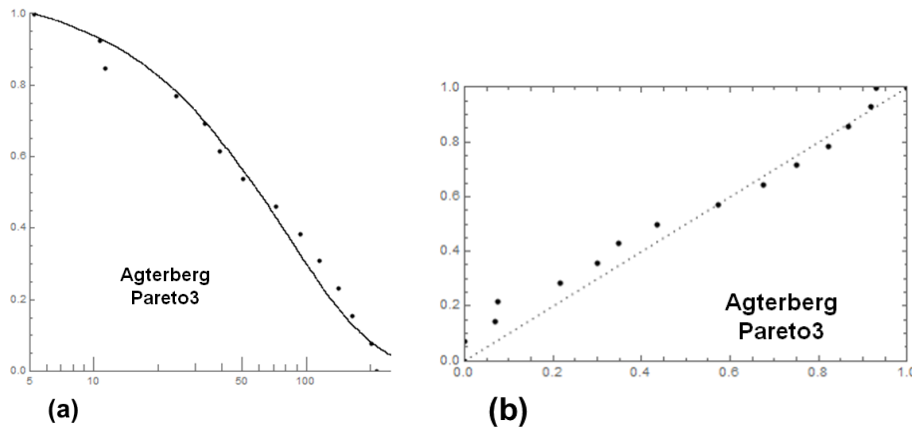
Figure 6. Mathematica fits to Agterberg data. (a) Single Pereto3 fit. (b) Probability plot for single Pareto3 fit.

From a purely pragmatic view of the Agterberg data set, another distribution (the gamma distribution) is just as likely as the Weibull distribution. In order of fit, Mathematica identifies and ranks the distributions in Table 2 where a value of 1.0 is a perfect fit and zero means Mathematica cannot find a fit. The fit for a gamma distribution is shown in Figure 7 for comparison with Figures 5 (e and f). The fit for a single Pareto distribution (Figure 6) is moderate.

Table 2. Ranking of probability distribution fits for Agterberg data.

Gamma	0.92
Weibull2[3]	0.89
Log-normal	0.76

---

[3] Weibull2, Weibull3 refer to two and three parameter distributions respectively. A 2-parameter representation involves only the scale and shape factors in Table 1; the 3-parameter representation involves the position factor in addition. The same notation applies to the other distributions.

416	Weibull 3	0.49
417	Pareto 3	0.49
418	Fréchet 3	0.49
419	Fréchet 2	0.43
420	Inverse Gaussian	0.40
421	Gumbel	0.20

(a)   (b)
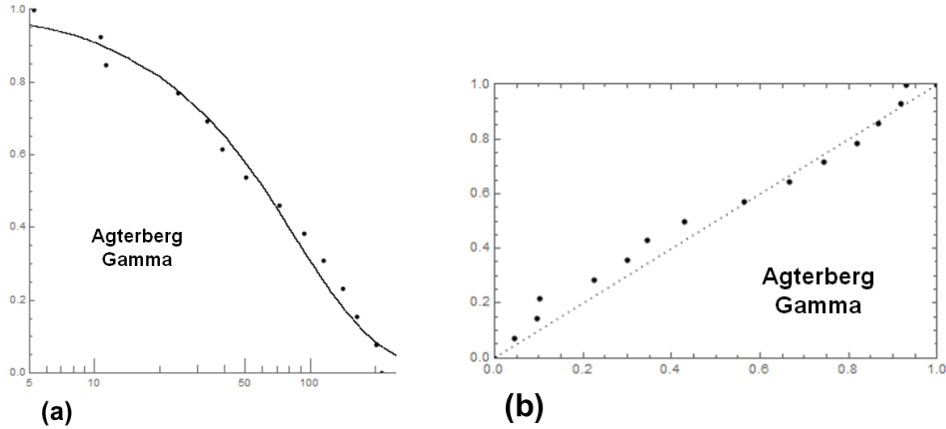
423 Figure 7. Gamma distribution fit to Agterberg (2013, his Figure 4) data. Mathematica
424 identifies this as a slightly better fit to the data than Weibull2.

3.2. Spatial data from Zimbabwe.

426   The second example is from Blenkinsop and Sanderson (1999) and consists of spatial
427 distribution data from the Zimbabwe craton. Log-log plots of $N(r)$ against r are nonlinear
428 (Figure 8) similar to the Agterberg data in Figure 5 (a, c). This nonlinearity reduces the
429 estimated fractal dimension and is interpreted by Blenkinsop and Sanderson (1999) as a roll-
430 off effect.

431   In Figure 9 we show best fit distributions to the Blenkinsop and Sanderson spatial
432 data. Mathematica choses a Weibull distribution as best fit for most situations. The exception
433 is the Shamva data set where the best fit is a Gumbel distribution.
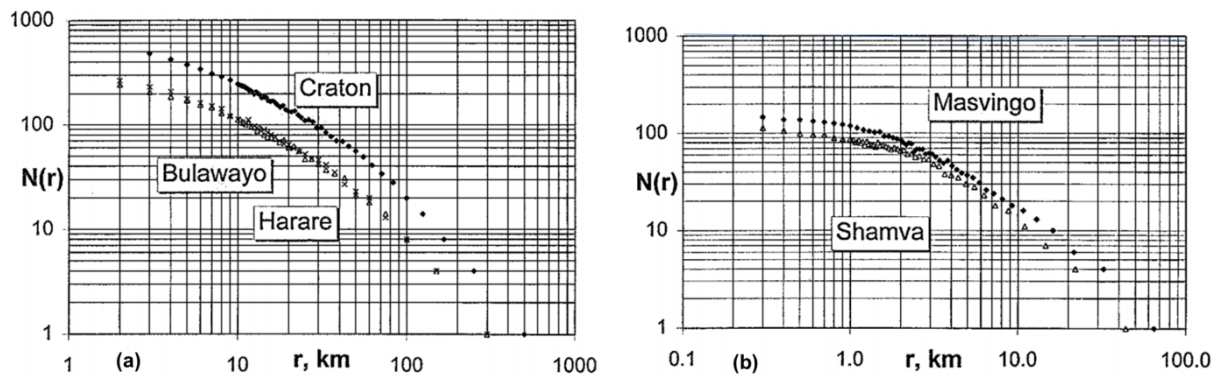
435 Figure 8. Reproduction of Figure 7 from Blenkinsop and Sanderson (1999).

436 (a) (b) (c) (d)

437 (e) (f) (g) (h)

438 (i) (j) (k) (l)
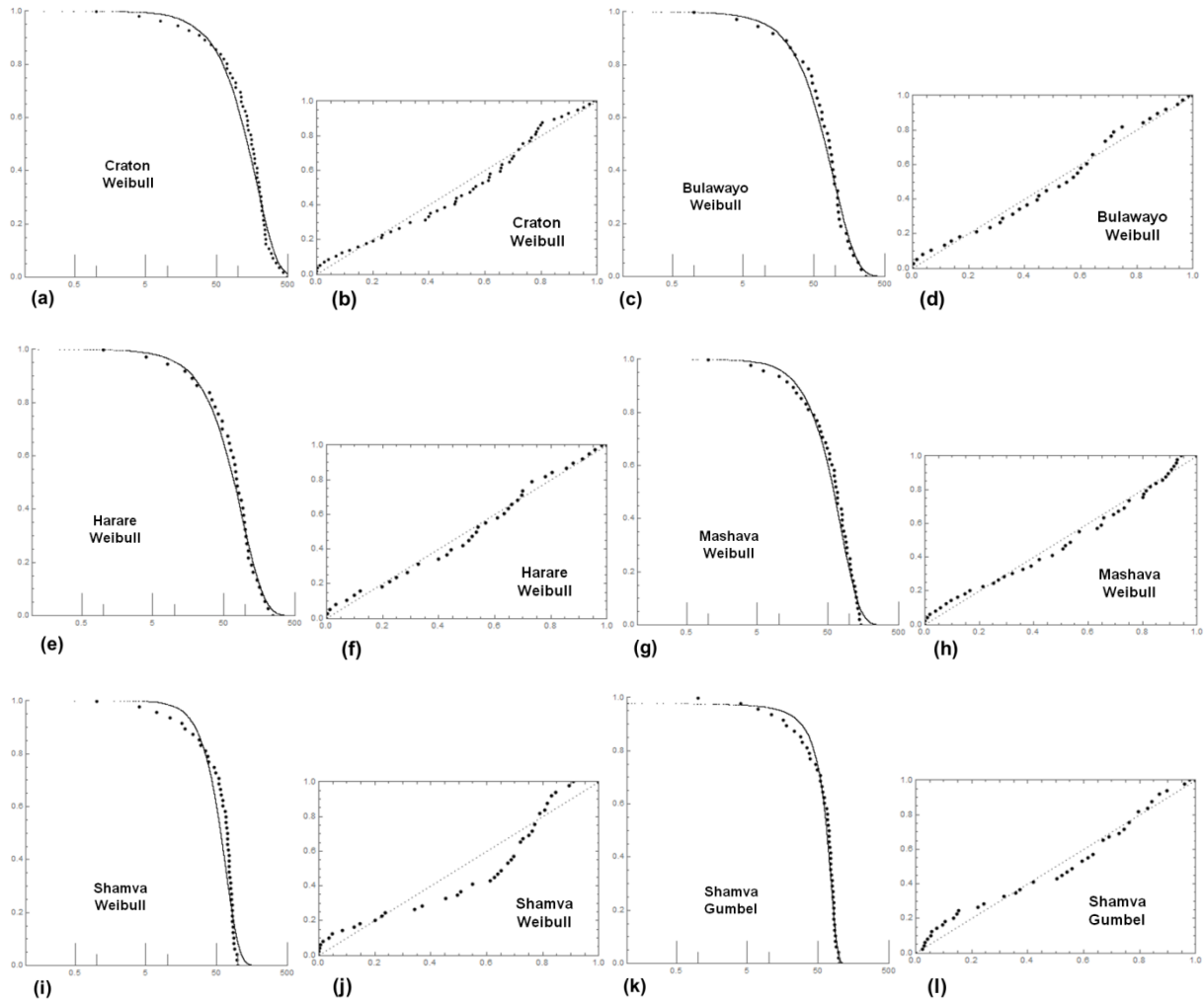
439 Figure 9. Best fit distributions for the data sets shown in Figure 8. The location and relevant
440 distribution is shown on each figure. For each location the probability distribution is shown
441 followed by the probability plot which shows how good the fit is. The plots for Shamva data (i,
442 j, k, l) show that a Gumbel distribution is a better fit than a Weibull distribution.

3.3. Mass data from Zimbabwe.

444   The third example (from Blenkinsop, 2014, his Figure 7) consists of mass
445 distributions (Figure 10) for some of the Zimbabwe examples given above in example 2.

446
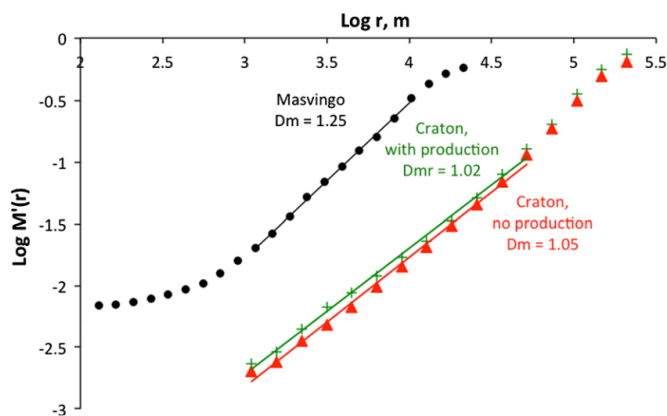
447 Figure 10. Reproduction of Figure 7 from Blenkinsop (2014).

(a)
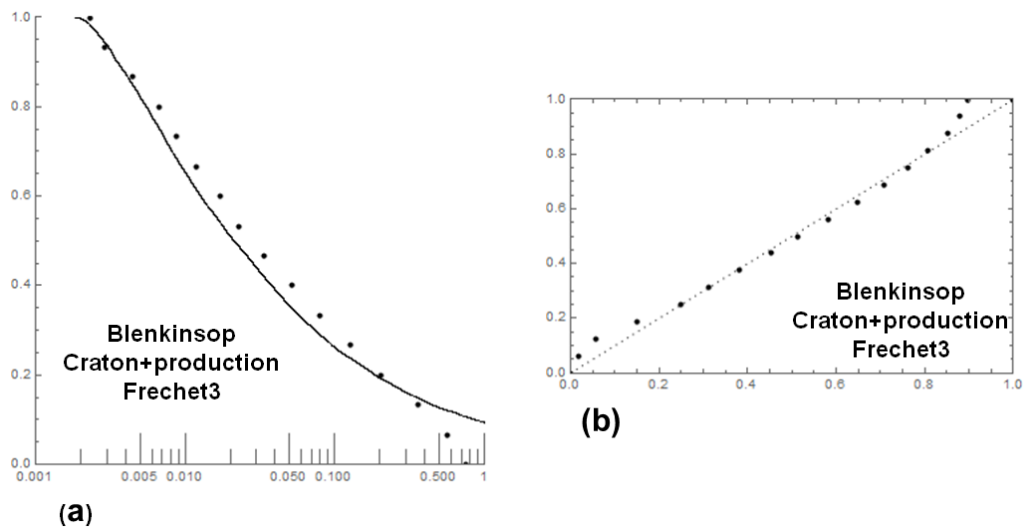
Figure 11. Best fit probability distribution for green data in Figure 10. This is a Fréchet3 distribution although there is departure from Fréchet3 at high mass values, The probability distribution for the red data in Figure 10 is almost identical to the distribution shown here for the green data.

Figure 12. Best fit probability distribution for data from the Masvingo area (the black, straight line, data in Figure 10). This is a Weibull2 distribution with some departure at low mass levels.

4. Discussion.

4.1. Growth and probability distributions.

Section 3 has considered spatial-point and spatial-mass data sets from two different locations. In each case the spatial data sets conform to a Weibull distribution rather than a Pareto. This is to be expected since the standard box counting procedure for spatial data points is a way of determining a nearest neighbour distribution which is by definition a Weibull distribution. The spatial-mass distributions are Fréchet or Weibull distributions.
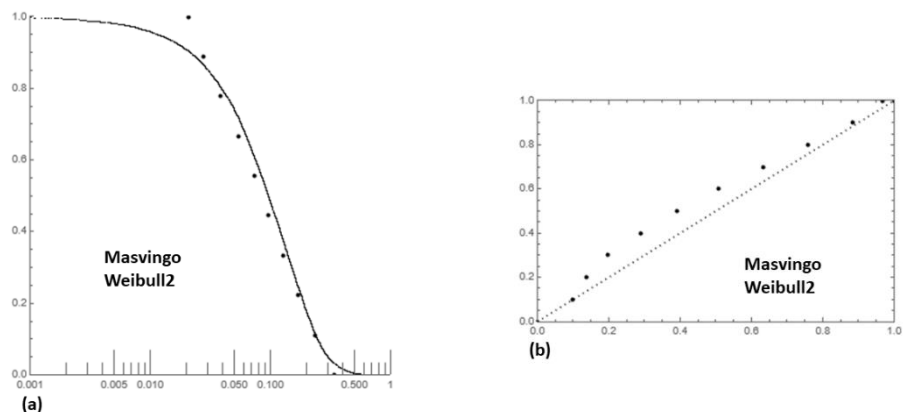
Analysis, using Carpena and Coronado (2019), shows that the underlying spatial distributions are not fractal-like. In this section we attempt a rationalisation of these observations in terms of the probability distributions to be expected from the growth of a mineralising system.

17

467   Consider a deforming segment of the Earth's lithosphere that is subjected to energy
468   and mass input in the form of hot reactive fluids bearing dissolved metals (Figure 13). We
469   suppose the deformation takes place by brittle processes that generate increased permeability.
470   At these sites energy is further dissipated by the initiation of mineral reactions that result in
471   alteration and mineralisation. The question is: *What controls the spatial distribution of*
472   *mineralising sites which in this context we identify with sites where the energy input to the*
473   *system is dissipated*?
474   In classical statistical thermodynamics, Boltzmann statistics describes the energy
475   partitioning in an isolated system and for a system in contact with a heat bath. This arises
476   because the partition of an isolated thermodynamic system into a number of components at
477   equilibrium leads to sharing of the system's energy in such a way that the most probable
478   value of the energy of any component is the mean value of the energy. The result is a
479   Boltzmann distribution in three dimensions or a Rayleigh distribution in two dimensions.
480   However for an open system far from equilibrium the internal entropy decrease associated
481   with the ordering represented by the spatial patterning of energy leads to stable extreme
482   distributions where there is an overwhelming probability for one component to take a much
483   larger share of the energy than all others (Mandelbrot, 1956; Lavenda and Florio, 1992;
484   Lavenda, 1995, pp 11 - 15, 80 - 83). The distribution of energy now follows an arcsine
485   distribution (Mandelbrot, 1956; Lavenda and Florio, 1992; Lavenda, 1995, pp 11 -15, 80 -
486   83) and can also be described by one of the Extreme Value probability distributions. The
487   distribution represented in any particular situation is a function of the growth mechanisms
488   associated with energy dissipation. For instance, if there is some kind of cut off with respect
489   to the stress required for fracture propagation (as there is in the classical Griffith theory of
490   fracture; Lavenda, 1965, p 180)) a Weibull distribution is expected for stress sites and a
491   Fréchet distribution for fracture length (Lavenda, 1965, p 180). We explore such concepts
492   below in terms of the growth of a mineralising system from the initial input of energy to the
493   dissipation of this energy by the formation of alteration and mineralising sites to the final
494   extinction of the system due to a lack of supply of energy (heat or fluid) to the system or
495   depletion in reactive components.
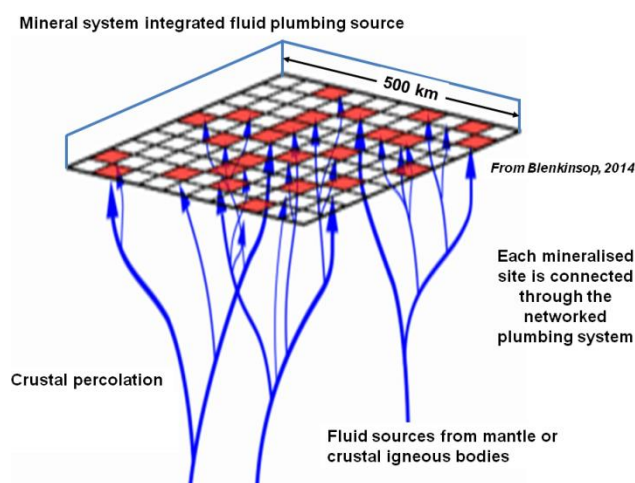
496
497   Figure 13. A crustal scale mineral system where energy and mass are added from the lower
498   crust or mantle. Some sites are places of energy dissipation and others are not. We identify

499 the dissipation sites with mineralised sites. We want to know the spatial distribution of these
500 energy dissipation sites. After Blenkinsop (2014).

501 The following discussion is motivated by the observations of Savageau (1979, 1980),
502 Frank (2009, 2019) and Rocha and Aleixo (2013) that the cumulative probability distribution
503 for a quantity, X, reflects aspects of the curve that characterises the growth kinetics of X. In
504 addition, Nair et al. (2021, Chapter 5) show that non-equilibrium conditions ("external
505 influences") are necessary to generate members of the generalised extreme value family.
506 Savageau (1979, 1980) show that for interacting nonlinear systems, a general equation can be
507 derived that describes the generation of a quantity of interest, X, and competition with other
508 processes to produce a generalised growth law for X. This equation includes many of the
509 common growth laws (logarithmic, power law, Weibull, stochastic, Compertz and Lotka-
510 Volterra) as special cases. His analysis emphasises that although a large number of processes
511 may operate to produce the growth of a system, an overall simple pattern of growth may
512 result.

513 A process oriented discussion of the significance of the three stable extreme
514 distributions is given by Rocha and Aleixo (2013). That paper (although directed at the
515 growth of tumours) provides a model for the growth of a mineralising system or any other
516 system growth that comprises nucleation, growth and extinction phases. The model is
517 relevant at all scales so we expect it to be applicable to the regional scale for the distribution
518 of metal endowment and at smaller scales where one is concerned with the distribution of
519 alteration assemblages or of ore grade in a single deposit or for the spatial distribution of
520 mineralisation.

521 Rocha and Aleixo (2013) explore a generalised growth model that describes the
522 progressive evolution of a system where growth nucleates, and subsequent growth follows a
523 symmetrical or asymmetrical sigmoidal curve to ultimate extinction. This is the Gompertz
524 law:

525
$$f_{r,q,p}\left(x\right) = rx^{p-1}(1-x)^{q-1}$$

526 which is a generalisation of the simple logistic equation, widely used in population dynamics,
527 for which $q = p = 2$. The Gompertz law describes the competition between an accelerating
528 growing process and processes that tend to inhibit growth; it is attractive from a process point
529 of view since it is used in various forms in material science (in the form of Kolmogorov–
530 Avrami kinetics for recrystallisation; Martyushev and Axelrod, 2003) and as a form of
531 kinetics for non-equilibrium chemical systems with coupling to both heat and fluid supply
532 (Ord et al., 2012; Hobbs and Ord, 2018). It is also one member of the more general growth
533 laws discussed by Savageau (1979, 1980)

534 With the approximation, $\ln(x) \approx -(1-x)$ for $0 \le x \le 2$, and t a normalised time,
535 Rocha and Aleixo show that the Gompertz law can be expressed as

536
$$\frac{df_N\left(t\right)}{dt} = bf_N\left(t\right)^{p-1}\left(-\ln f_N\left(t\right)\right)^{q-1}$$

537 With $p = 2$ and $1 < q < 2$, this equation leads to Weibull distributions. If $p = 2$ and $q > 2$ then
538 Freshet distributions result. For $b = 1$, the Weibull distributions are of the form

539
$$f_N(t) = \exp\left(-(-t)^\alpha\right) \text{ with } 0 < \frac{1}{\alpha} < 1$$

540 The Fréchet distributions are of the form

541
$$f_N(t) = \exp\left((-t)^{-\alpha}\right) \text{ with } \frac{1}{\alpha} > 0$$

542 Plots of these distributions are shown in Figure 14.
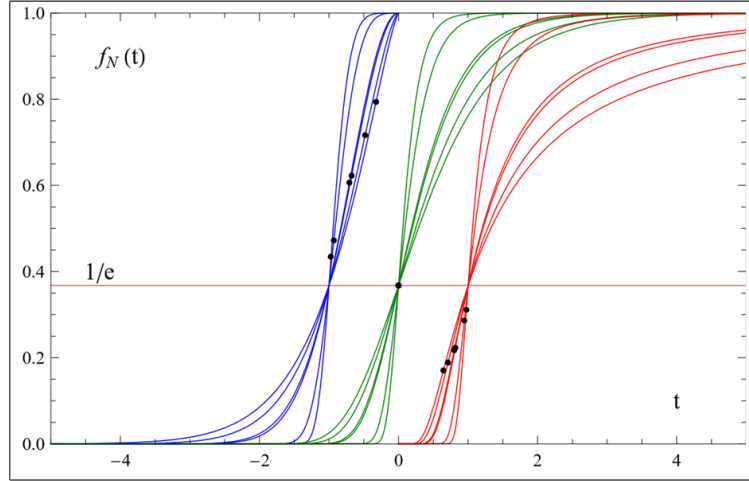
543

544

545 Figure 14. Normalised sigmoidal curves Weibull (blue), Gompertz (green) and Fréchet (red)

546 with $\alpha = b = 1.3, 1.5, 1.9, 2, 4, 6$. The normalised Weibull curves are for $f_N(t) = \exp(-(-t)^\alpha)$

547 and have inflection points (black dots) for $f_N(t) > \dfrac{1}{e}$. The inflection point measures the

548 number of mineralising sites developed at the peak of growth rate. The normalised Gompertz

549 curves are for $f_N(t) = \exp(-\exp(-bt))$ and there is just one inflection point for the family at

550 $f_N(t) = \dfrac{1}{e}$. The normalised Fréchet curves are for $f_N(t) = \exp(-t^{-\alpha})$ and have inflection

551 points for $f_N(t) < \dfrac{1}{e}$. From Rocha and Aleixo (2013).

552 • The Weibull type models describe mineralising site growths (at all scales) in which the
553 initial growth phase is long. After the initial growth phase, the period of time of the
554 mineralising site development until death is very short. During this short growth time the
555 maximum number of growth sites appears. Martyushev and Axelrod (2003) propose that
556 these kinetics are associated with the shortest time for system development and maximum
557 entropy production.
558 • The Fréchet type models represent mineralising site growths (at all scales) in which the
559 initial growth phase is short. During this short growth time the maximum number of
560 growth sites appears. The period of the mineralising site development to the death is
561 highly variable, i.e., the mineralising growth can stabilize near reaching the maximum
562 capacity or can take a long time to reach this value.
563 The growth rates, $g_{\lambda,q}(x)$, for various conditions and distributions are shown in Figure 15.
564 Here the growth rate is given by

$$g_{\lambda,q}(x) = \lambda x \left(-\ln x\right)^{q-1}$$

where $x = f_N(t)$ and $\lambda > 0$ is an intrinsic growth rate factor for mineralising sites that expresses the mutual inhibition of sites and the competition for energy and/or nutrients.
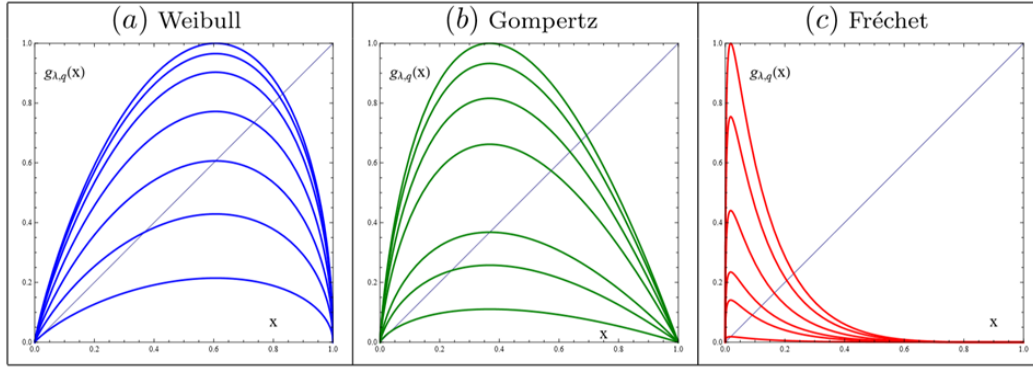

Figure 15. Growth rates of mineralising sites or of mineralising site sizes defined by $g_{\lambda,q}(x)$. For Weibull, $q = 1.5$ and $\lambda = 0.5, 1, 1.4, 1.8, 2.1, 2.25, 2.33$. For Gompertz, $q = 2$ and $\lambda = 0.3, 0.7, 1.0, 1.8, 2.22, 2.54, 2.72$. For Fréchet, $q = 5$ and $\lambda = 0.004, 0.03, 0.05, 0.094, 0.161, 0.213$. These values of λ are selected to illustrate various dynamic behaviour discussed by Rocha and Aleixo (see their Figure 4 and discussion). From Rocha and Aleixo (2013).

It is important to note that the Gompertz growth law undergoes a bifurcation (Rocha and Aleixo (2013) well known in the chemical reactor industry (Rawlings and Ekerdt, 2013, Chapter 6)) which can lead to unstable growth and is the basis for understanding the formation of giant mineral deposits. Such an analysis is the subject of a future paper.

As we have indicated, for spatial distributions of mineralisation the Weibull fit is a direct arithmetic outcome of the fact that when one does a classical box count for point distributions then one is automatically carrying out a nearest neighbour analysis (as is proved implicitly in Blenkinsop and Sanderson, 1999). A nearest neighbour distribution is a special case of a Weibull distribution so it is inescapable that a Weibull distribution will result from a box-counting exercise; it does not necessarily have any physical significance. However some Weibull distributions (those with small γ) are close to a power-law as (it turns out) is the situation for the Agterberg data set.

What is of fundamental importance is the answer to the question: *What is the underlying statistical distribution that describes the spatial distribution of mineralised sites*? This is not obvious from the nearest neighbour analysis although such relations are considered by Sakhr and Nieminen (2018) and Carpena and Coronado (2019).

In order to understand what the spatial probability distribution might be one needs a relation between fracture or shear zone spacing and stress or some other parameter (assuming fractures and/or shear zones are the sites for mineralisation). Veveakis and Regenauer-Lieb (2015) and Alevizos et al. (2016) derive an expression for the spacing, h, of fractures and shear zones in a deforming fluid saturated material with a capped yield surface. The fractures result from stress singularities whose spacing is controlled by the fluid diffusivity; this means the fracture spacing is controlled by heterogeneities in the permeability in many instances. The argument is very general and includes opening/shear and compaction/shear bands as well

599 as opening mode fractures both parallel and normal to (compressive) σ_1, together with pure

600 compaction bands normal to σ_1. The overall behaviour of the system is governed by a

601 dimensionless parameter, λ, defined as

602
$$\lambda = \frac{mechanical\ diffusivity}{fluid\ diffusivity} = \frac{\dot{\varepsilon}\mu_{fluid}}{kp'}L^2 \qquad (10)$$

603 where μ_{fluid} is the fluid viscosity, $\dot{\varepsilon}$ is the strain rate, p' is the mean stress and k is the

604 permeability. L is the length of the system parallel to σ_1.

605   An informative view is to express the fracture spacing, h, in terms of λ (Alevizos et

606 al. 2016, Figure 13):

607
$$h = \left(\frac{H}{0.26\sqrt{\lambda}}\right) = \sqrt{\frac{kp'}{\dot{\varepsilon}\mu_{fluid}}}$$

608   Localisation of fractures does not occur until λ reaches a value of 13 (see Figure 16).

609 This means that there is a lower cut off for λ, below which fractures do not occur. This is the

610 type of behaviour for λ to be expected (Lavenda, 1965) of Weibull statistics and hence h

611 (which is inversely proportional to $\lambda^{-1/2}$) is expected to have Fréchet statistics, This is

612 identical to the classical distribution of fracture *lengths* derived from Weibull theory where

613 the distribution of *stress* is Weibull but the distribution of fracture *lengths* is Fréchet (Figure

614 3a and Lavenda, 1995, p180). The relation for Griffith cracks is $\sigma_y = \dfrac{C}{Y\sqrt{l}}$ where C is the

615 fracture toughness, Y is a constant that depends on the crack shape and the stress/strain field

616 and l is the fracture length. σ_y is the stress required to initiate fracture. There is a minimum

617 stress below which fractures do not initiate. Thus the relation between σ_y and l for Griffith

618 crack length is the same form as the relation between fracture spacing, h, and λ for

619 generalised fracture development and the prediction is Fréchet distributions for crack length

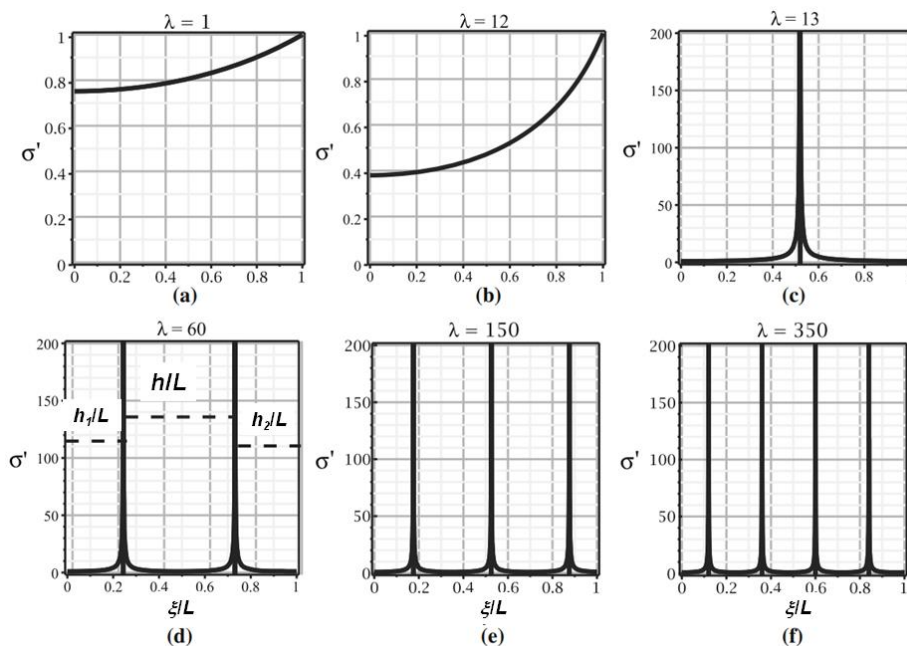620 and crack spacing respectively.

621

622

Figure 16. Plots of localised effective stress against normalised distance, ξ/L, for various values of λ. Below $\lambda = 13$ no localisation occurs so there is a lower limit for λ where fractures do not form suggesting Weibull statistics for l. For $\lambda \geq 13$, the spacing follows the relation $h = L/\left(0.26\sqrt{\lambda}\right)$. Weibull statistics for λ results in Fréchet statistics for the spacing.

We conclude that we expect the underlying statistical distribution for mineralised sites to be a Generalised Extreme Value distribution and its precise form depends on whether the distribution of λ is a Fréchet or a Weibull distribution. From (10) the distribution of mineralised sites is Fréchet if the distribution of strain rate is Weibull. The distribution of mineralised sites is Weibull if the distribution of permeability is Weibull. This conclusion is based on some physical arguments and is independent of the fact that the box counting algorithm always gives a nearest neighbour or Weibull distribution by definition. This is supported by the observed Fréchet and Weibull mass distributions from Zimbabwe.

4.2. Future work.

In order to establish the true spatial probability distributions for mineralised sites one approach is to employ *n*-point correlation functions (Kroner, 1971; Torquato, 2002; Kalidindi, 2015). Although such methods are widespread in the astronomy, materials and microstructure literature (and especially in the petroleum industry) we know of no applications in the characterisation of mineralisation. Such methods enable fractal or departures from fractal geometry to be quantified (Peebles, 1989, 1980; Jones et al., 2004) and should be pursued with vigour.

This paper has examined only data from orogenic gold deposits which (at least for vein-hosted deposits) we envisage as undergoing a nucleation-growth-death evolutionary history. For deposits where replacement processes or reaction-diffusion reactions dominate as in Pb-Zn deposits, or in gold deposits where gold/iron oxide replacement reactions or growth involving incorporation of gold in arseno-pyrite predominates, probability distributions other than GEV distributions are likely to be involved. These include Gamma and InverseGaussian distributions. Work in this regard is in progress.

5. Conclusions.

The spatial distributions of mineralisation (at least for the examples examined in this paper) are not fractal but belong to members of the Generalised Extreme Value (GEV) family of distributions. These members include the Weibull, Gumbel and Fréchet distributions. For some ranges of parameters some of these distributions can approximate scale invariant power-laws. The classical box counting method for spatial distributions of points is a method for calculating the nearest neighbour distribution which is a Weibull distribution. Hence all box counting methods produce a Weibull distribution which may sometimes appear to be a bifractal but such an appearance does not have any physical significance. The box counting procedure only gives a fractal dimension if the underlying distribution is a pure fractal. Otherwise the procedure gives a local fractal dimension which generally is spurious. Most examples examined here are best represented by Weibull distributions with a single Gumbel distribution. The underlying probability distribution is not a pure fractal. The best way of characterising the underlying probability distribution is to establish the mass distribution. For the examples examined this can be a Fréchet or Weibull distribution. Both these distributions

are to be expected from current theories of fracture spacing. The cumulative distributions to be expected for spatial distributions are direct reflections of the growth kinetics of the mineralising system; nonlinear growth kinetics for nucleation-growth-extinction systems frequently lead to GEV probability distributions.

References.

Agterberg, F, P. 2013. Fractals and spatial statistics of point patterns. Journal of Earth Science. 24, 001–011.

Agterberg, F. 2014. Geomathematics: Theoretical Foundations, Applications and Future Developments. Springer.

Alevizos, S., Poulet, T., Sari, M., Lesueur, M., Regenauer-Lieb. K., Veveakis. M. 2016. A framework for fracture network formation in overpressurised impermeable shale: deformability versus diagenesis. Rock Mech. Rock Eng. DOI 10.1007/s00603-016-0996-y.

Arneodo A, Bacry E, Muzy JF. 1995 The thermodynamics of fractals revisited with wavelets. Physica A 213, 232–275. (doi:10.1016/0378-4371(94)00163-N)

Barton. C. C., and Scholtz, C. H. 1965. The fractal size and spatial distribution of hydrocarbon accumulations. Chapter 2 In: Fractals in Petroleum and Earth Processes. Eds: CC. Barton, PR Pointe. Pleneum Press. 13 – 34.

Beck C, Schlögl F. 1995 Thermodynamics of chaotic systems. New York, NY: Cambridge Universit Press.

Biermé, H. 2017. Introduction to random fields and scale invariance. Hal-01493834v2. https://hal.archives-ouvertes.fr/hal-01493834v2.

Blenkinsop, T. 2014, Scaling laws for the distribution of gold, geothermal, and gas resources. Pure Appl. Geophys. DOI 10.1007/s00024-014-0909-5

Blenkinsop, T. G., 1994. The fractal distribution of gold deposit: Two examples from the Zimbabwe Archean craton. In: Fractals and Dynamic Systems in Geoscience. Ed: J. H. Krühl. Springer. 247 – 258.

Blenkinsop, T. G., 1995. Fractal Measures for Size and Spatial Distributions of Gold Mines: Economic Implications. In: Blenkinsop, T. G., Tromp, P. L., eds., Sub-Saharan Economic Geology. Special Publication Geological Society of Zimbabwe, 3: 177–186.

Blenkinsop, T. G., and Sanderson, D. J. 1999. Are gold deposits in the crust fractals? A study of gold mines in the Zimbabwean craton. In." McCaffrey, K. J. W., Lonergan, L. & Wilkinson, J. J. (eds) Fractures, Fluid Flow and Mineralization. Geological Society, London, Special Publications, 155, 141-151.

Bódai, T. 2016. Extreme value analysis in dynamical systems: Two case studies. Nonlinear and Stochastic Climate Dynamics, Editors: Christian Franzke, Terence O'Kane. Cambridge University Press.

Callen, H.B., 1960. Thermodynamics: An Introduction to the Physical Theories of

Equilibrium Thermostatics and Irreversible Thermodynamics. John Wiley and Sons, New York.

Carlson, C. A. 1991. Spatial distribution of ore deposits. Geology. 19, 111 -114.

Carpena, P., Coronado, A. V. 2019. Connection of the nearest-neighbour spacing and the local box-counting dimension for discrete sets. Phys. Rev. E. 100, 022205.

Chandrasekar, S. 1943. Stochastic problems in physics and astronomy. Reviews of Modern Physics 15, 89 pp.

Corral, A., González, A. 2019. Power law size distributions in geoscience revisited. Earth and Space Science. 6, 673–697

Crescenzo, A. Di, Paraggio, P. 2019. Logistic growth described by birth-death and diffusion processes. Mathematics 7, 489. doi:10.3390/math7060489.

Cressie, N. A. C. 1993. Statistics for Spatial Analysis. John Wiley.

Dawson, D. C., Wragg, A. 1973. Maximum-entropy distributions having prescribed first and second moments. IEEE Trans. Info. Theory, 19, 689 − 693.

Dekking, M. 2005. A Modern Introduction to Probability and Statistics. Springer.

Doutre, R. C. 2018. Multi-scale organisation and geological controls on large orogenic gold deposits. PhD thesis, University of Western Australia.

Doutre, R., Micklethwaite, S., Kovesi, P., McCuaig. C. T., Ford, A., Hayward, N. 2015. Multi-scale spacing and endowment of orogenic gold deposits. Mineral Resources in a Sustainable world. 13th SGA Biennial Meeting Proceedings. 1, 81 − 84.

Embrechts, P., Klüppelberg, C., Mikosch, T. 1997. Modelling Extreme Events for Insurance and Finance. Springer.

Fama, E, F. 1963. Mandelbrot and the stable Paretian hypothesis. The Journal of Business. 36, 420 − 429.

Fama, E, F, 1965. The behavior of stock-market prices. The Journal of Business. 38, 34-105.

Fisher, R. A., Tippett, L. H. C. 1928. Limiting forms of the frequency distribution of the largest and smallest member of a sample. Proceedings of the Cambridge Philosophical Society, 24, 180–190.

Frank, S, A. 2009. The common patterns of Nature. J Evol. Biol. 22, 1563 − 1585.

Frank, S. A. 2019. The common patterns of abundance: the log series and Zipf's law. F1000Research, 8, 334.

Fréchet, M. 1941. Sur la loi de répartition de certaines grandeurs géographiques. J. Soc. Statistique de Paris. 82, 114 − 122.

Freudenthal, A. M. 1968. Statistical approach to brittle fracture. In Fracture, H. Liebowitz, Ed., vol. 2, 591–619

Gibbs, J.W., 1875–1878. On the equilibrium of heterogeneous substances. Trans. Conn. Acad. Arts Sci. 3, 108–248 ((1875–6); 343–524 (7–8)).

Gnedenko, B. V. 1943. Sur la distribution limite du terme maximum d'une serie aleatoire.

Annals of Mathematics, 44:423–453.

Gnedenko, B.V., Kolmogorov, A.N. 1968. Limit distributions for sums of independent random variables (Translated from the Russian, annotated, and revised by K.L. Chung with appendices by J.L. Doob and P.L. Hsu). Addison-Wesley, Reading.

Gumbel, E. J. 1954. Statistical theory of extreme values and some practical applications. Applied Mathematics Series, 33, Washington, D.C.: National Bureau of Standards, February 12, 1954.

Hobbs, B. E., Ord, A. 2021. Orogenic gold systems as nonlinear systems: Nonlinear analysis of data. In preparation.

Hobbs, B.E. and Ord, A. 2018. Episodic modes of operation in hydrothermal gold systems: Part II. A model for gold deposition. Geological Society, London, Special Publications, 453, https://doi.ord/10.1144.SP453.15

Hodkiewicz, P. F., Weinberg, R. F., Gardoll, S. J., Groves, D. I. 2005. Complexity gradients in the Yilgarn Craton: fundamental controls on crustal-scale fluid flow and the formation of world-class orogenic-gold deposit. Australian Journal of Earth Sciences. 52, 831–841.

Hronsky, J. N. A., Groves, D. I. 2008. Science of targeting: definition, strategies, targeting and performance measurement. Australian Journal of Earth Sciences, 55, 3-12.

Kalidindi, S. R. 2015. Hierarchical Materials Informatics. Elsevier.

Jones, B, J. T., Martínez, V. J., Saar, E., Tribble, V. 2004. Scaling laws in the distribution of galaxies. Rev. Modern Physics. 76, 1211 – 1266.

Kondepudi,D., Prigogine, I. 1998. Modern Thermodynamics, Wiley.

Kroner, E. 1971. Statistical Continuum Mechanics. Courses and Lectures, 92, International Centre for Mechanical Sciences, Udine. Springer.

La Pointe, P. R. 1965. Estimation of undiscovered hydrocarbon potential through fractal geometry. Chapter 3 In: Fractals in Petroleum and Earth Processes. Eds: CC. Barton, PR Pointe. Pleneum Press. 35 – 57.

Lavenda, B. H. 1995. Thermodynamics of Extremes. Albion Publishing.

Lavenda, B. H., Florio, A. 1992. Thermodynamics of rare events. International Journal of Theoretical Physics. 31, 1445 – 1475.

Lienhard, J. H., Myer, P. L. 1967. A physical basis for the generalised gamma distribution. Quart. Appl. Math.25, 330 – 334.

Lucarini, V., Faranda, D., de Freitas, A. C. G. M. M., de Freitas, J. M. M., Holland, M., Kuna, T., Nicol, M., Todd, M., Vaienti, S. 2016. Extremes and Recurrence in Dynamical Systems. Wiley.

Lucarini, V., Faranda, D., Wouters, J., Kuna, T. 2014. Towards a general theory of extremes for observables of chaotic dynamical systems. Journal of Statistical Physics, 154, 723–750.

780 Lucarini, V., Faranda, D., Turchetti, G., Vaienti, S. 2012. Extreme value theory for singular
781   measures. Chaos, 22(2).

782 Mandelbrot, B. B. 1997. Fractals and Scaling in Finance. Springer.

783 Mandelbrot, B. B. 1982. The fractal geometry of nature. San Francisco, Freeman.

784 Mandelbrot, B. 1965. The statistics of natural resources and the law of Pareto. Chapter 1 In:
785   Fractals in Petroleum and Earth Processes. Eds: CC. Barton, PR Pointe. Pleneum
786   Press.

787 Mandelbrot, B. 1963. New methods in statistical economics. The Journal of Political
788 Economy. 71, 421– 440.

789 Mandelbrot, B. 1960. The Pareto–Lévy Law and the distribution of income, International
790   Economic Review. https://www.jstor.org/stable/2525289.

791 Mandelbrot, B. 1961. Stable Paretian random functions and the multiplicative variation of
792   income. Econometrica. https://www.jstor.org/stable/pdfplus/1911802.pdf.

793 Mandelbrot, B, 1956. On the language of taxonomy: An outline of a "thermostatistical"
794   theory of systems of categories with Willis (natural) structure. In; Information Theory:
795   The third London symposium, Ed. C. Cherry, 135 – 145, Academic Press.

796 Martyushev, L.M., Axelrod, E.G. 2003. From dendrites and S-shaped growth curves to the
797   maximum entropy production principle. JETP Letters. 78, 476-479.

798 Marwan N, Roman MC, Thiel M, Kurths J. 2007b. Recurrence plots for the analysis of
799   complex systems. Phys. Rep. 438, 237–329. (doi:10.1016/j.physrep.2006.11.001)

800 McCuaig. T.C., Hronsky. J. M, A. 2014. The mineral system concept: the key to exploration
801   targeting. Soc Econ Geol. Spec. Pub. 18, 153–175.

802 McFadden, J.A. 1965. The mixed Poisson process. Sankhya: The Indian Journal of Statistics
803   A, 27, 83-92.

804 Munro MA, Ord A, Hobbs BE. 2018 Spatial organization of gold and alteration mineralogy
805   in hydrothermal systems: wavelet analysis of drillcore from Sunrise Dam Gold Mine,
806   Western Australia. In Characterization of ore-forming systems from geological,
807   geochemical and geophysical studies. Geological Society, London, Special
808   Publications, vol. 453 (eds K Gessner, TG Blenkinsop, P Sorjonen-Ward). London,
809   UK: Geological Society. (doi:10.1144/ SP453.10).

810 Nair, J., Wierman, A., Zwart, B. 2021. The Fundamentals of Heavy Tails: Properties,
811   Emergence, and Estimation. Unpublished but available at:
812   https://adamwierman.com/book/. To be published by Cambridge University Press.

813 Nicolis, C., Balakrishnan, V., and Nicolis, G. (2006). Extreme events in deterministic
814   dynamical systems. Physical Review Letters, 97:210602.

815 Ord. A., Hobbs, B., Dering, G., Gessner, K. 2018. Nonlinear analysis of natural folds using
816   wavelet transforms and recurrence plots. Phil. Trans. R. Soc. A 376: 20170257.

817 Ord A, Munro M, Hobbs BE. 2016 Hydrothermal mineralising systems as chemical reactors:
818   wavelet analysis, multifractals and correlations. Ore Geol. Rev. 79, 155–179.

819   (doi:10.1016/j.oregeorev.2016.03.026)

820 Ord, A., Hobbs, B. E., Lester, D, R, 2012. The mechanics of hydrothermal systems: I. Ore
821   systems as chemical reactors. Ore Geology Reviews. 49, 1 - 44.

822 Peebles, P, J, E. 1989. The fractal galaxy distribution. Physica D. 38, 273-278.

823 Peebles, P. J. E. 1980. The Large Scale Structure of the Universe. Princeton University Press.

824 Quinn, J. B., Quinn, G. D. 2010. A practical and systematic review of Weibull statistics for
825   reporting strengths of dental materials. Dental Materials. 26, 135–147.

826 Raines, G. L., 2008. Are fractal dimensions of the spatial distribution of mineral deposits
827   meaningful ? Natural Resources Research, 17(2): 87–97.

828 Rawlings, J.B., Ekerdt, J.G. 2013. Chemical Reactor Analysis and Design Fundamentals. 2nd
829   edition. Madison, Wisconsin: Nob Hill Publishing. 633 pp.

830 Rocha, J. L., Aleixo, S. M. 2013. An extension of Gompertzian growth dynamics; Weibull
831   and Fréchet models. Mathematical Biosciences and Engineering. 10, 379 – 398.

832 Sakhr, J., Nieminen, J. M. 2018. Local box-counting dimensions of discrete quantum
833   eigenvalue spectra: Analytical connection to quantum spectral statistics. Phys Rev
834   E.97, 030202.

835 Savageau, M. A. 1979. Growth of complex systems can be related to the properties of their
836   underlying determinants. Proc. Nat. Acad. Sci. USA. 76, 5413-5417.

837 Savageau, M. A. 1980. Growth equations: A general equation and a survey of special cases.
838   Mathematical Biosciences. 48, 267 -278.

839 Sethna, J. P. 2006. Statistical Mechanics: Entropy, Order Parameters, and Complexity.
840   Oxford University Press.

841 Singer, D. A. 2013. The lognormal distribution of metal resources in mineral deposits. Ore
842   Geology Reviews, 55, 80 - 86.

843 Singer, D. A., Menzie, W. D. 2010. Quantitative Mineral Resource Assessments. An
844   Integrated Approach. Oxford University Press.

845 Sprott, J. C. 2003. Chaos and Time Series Analysis. Oxford.

846 Tippet, L. H. C. 1925. On the extreme individuals and the range of samples taken from a
847   normal population. Biometrika, XVII, 364-87.

848 Torquato, S. 2002. Random Heterogeneous Materials. Springer.

849 Turcotte, D. L. 1986. A fractal approach to the relationship between ore grade and tonnage.
850   Econ. Geol. 81, 1528 – 1532.

851 Turcotte, D. L. 1989. Fractals in geology and geophysics. PAGEOPH. 131, 171 – 196.

852 Veveakis, E., Regenauer-Lieb, K. 2015. Cnoidal waves in solids. Journal of the Mechanics
853   and Physics of Solids.78, 231 -248.

854 Wolfram Research, Inc., Mathematica, Version 12.2, Champaign, IL (2020).

855   http://reference.wolfram.com/language/guide/ParametricStatisticalDistributions.html

Highlights

- The spatial distributions of mineralisation are not fractal but belong to members of the Generalised Extreme Value (GEV) family (Gumbel-, Fréchet- and Weibull- distributions).

- The cumulative distributions to be expected for spatial distributions are direct reflections of the growth kinetics of the mineralising system.

- Nonlinear growth kinetics for nucleation-growth-extinction systems frequently lead to GEV probability distributions.

26 May 2021

Dear Editors,

Journal of Structural Geology

The Spatial Distributions of Mineralisation by Bruce E. Hobbs, Alison Ord and Thomas Blenkinsop

Neither the manuscript itself nor any of the content are submitted elsewhere. We have no conflicts of interest.

Sincerely yours,

Alison Ord

The University of Western Australia

+61 41 993 1276

alison.ord@uwa.edu.au

26 May 2021

Dear Editors,

Journal of Structural Geology

The Spatial Distributions of Mineralisation by Bruce E. Hobbs, Alison Ord and Thomas Blenkinsop

BEH conceived the study and drafted the manuscript. AO developed the study through Wolfram Language probability distribution analyses, and critically revised the manuscript. TB provided field data and critically revised the manuscript. All authors gave final approval for publication.

Sincerely yours,

Alison Ord

The University of Western Australia

+61 41 993 1276

alison.ord@uwa.edu.au