Homework 10

Problem 1.

(a) Let X be a random variable taking values in $[0,1]$. Prove that if $\mathbb{E}(X)=\varepsilon$, then

$$\Pr\Big(X \geq \frac{\varepsilon}{2}\Big) \geq \frac{\varepsilon}{2}.$$

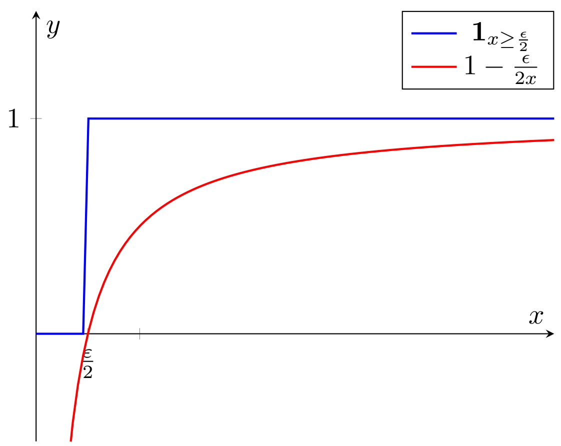
(b) Let $X \geq 0$ be a random variable. Prove that

$$\Pr(X=0) \leq \frac{\mathrm{Var}(X)}{\big(\mathbb{E}(X)\big)^2}.$$

Solution. (a)

Let $\mathbf{1}_{x\geq\frac{\varepsilon}{2}}$ be the indicator function of the event $x \geq \frac{\varepsilon}{2}$. Then from the figure of the indicator function, we have

$$\mathbf{1}_{x\geq\frac{\varepsilon}{2}} \geq 1-\frac{\varepsilon}{2x}.$$



Therefore, we have

$$\Pr\Big(X \geq \frac{\varepsilon}{2}\Big) = \mathbb{E}\big(\mathbf{1}_{x\geq\frac{\varepsilon}{2}}\big) \geq \mathbb{E}\big(1-\frac{\varepsilon}{2x}\big) = 1-\frac{\varepsilon}{2\,\mathbb{E}(x)} = \frac{1}{2} \geq \frac{\varepsilon}{2}.$$

(b)

From the Chebyshev's inequality, for any $t>0$, we have

$$\Pr(|X-\mathbb{E}(X)| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

Since $X \in [0,1]$, then $\mathbb{E}(X) \geq 0$. Let $t = \mathbb{E}(X)$, then

$$\begin{aligned}
\Pr(X = 0) &= \Pr(X \leq 0) \\
&\leq \Pr(X \leq 0 \text{ or } X \geq 2\,\mathbb{E}(X)) \\
&= \Pr(|X - \mathbb{E}(X)| \geq \mathbb{E}(X)) \\
&\leq \frac{\mathrm{Var}(X)}{\big(\mathbb{E}(X)\big)^2}.
\end{aligned}$$

Problem 2. Let $\mathrm{RandomSign}(n)$ be the distribution of vectors of n entries where each entry is independently chosen to be ± 1 with probability $\frac{1}{2}$. Sample m vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)} \sim \mathrm{RandomSign}(n)$. Define the normalized vectors $\mathbf{w}^{(i)} = \mathbf{v}^{(i)}/\sqrt{n}$ so that $\|\mathbf{w}^{(i)}\| = 1$ for all $i = 1, 2, \ldots, m$. Prove the following claims:

(a) For all $i \neq j$, the inner product $\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \rangle = \sum_k \mathbf{w}_k^i \mathbf{w}_k^j$ is small with high probability. That is,

$$\Pr\Big(\Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| \geq \delta\Big) \leq \exp\Big(-\Omega\big(\delta^2 n\big)\Big).$$

(b) There exists some $m = \exp(\Omega(\delta^2 n))$ such that the m vectors are pairwise almost-orthogonal with high probability. More precisely,

$$\Pr\Big(\Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| \leq \delta \text{ for all pairs } i \neq j\Big) \geq 0.99.$$

(Note: By probabilistic method, this proves that there are exponentially many almost-orthogonal unit vectors in $\mathbb{R}^n$ even though there are at most n exactly orthogonal vectors.)

Solution. (a) Suppose $\mathbf{w}^{(i)} = (w_1^{(i)}, w_2^{(i)}, \ldots, w_n^{(i)})$ and $\mathbf{w}^{(j)} = (w_1^{(j)}, w_2^{(j)}, \ldots, w_n^{(j)})$. For a fixed $\mathbf{w}^{(j)}$, each entry $w_k^{(j)}$ is chosen to be $\frac{1}{\sqrt{n}}$ or $-\frac{1}{\sqrt{n}}$ with equal probability $\frac{1}{2}$. Therefore, for any $k \in \{1, 2, \cdots, n\}$, we have $w_k^{(i)} \cdot w_k^{(j)} = \pm\frac{1}{n}$ with equal probability $\frac{1}{2}$.

Let $X_k = w_k^{(i)} \cdot w_k^{(j)}$ for $k \in \{1, 2, \cdots, n\}$. Then X_k is a random variable taking values in $\{\frac{1}{n}, -\frac{1}{n}\}$. It is easy to see that $\mathbb{E}(X_k) = 0$ and $\mathrm{Var}(X_k) = \frac{1}{2} \cdot (1 - \frac{1}{2}) = \frac{1}{4}$.

Let $X = \sum_k X_k = \sum_k w_k^{(i)} \cdot w_k^{(j)} = |\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \rangle|$. Then $\mathbb{E}(X) = \sum_k \mathbb{E}(X_k) = 0$.

From the Hoeffding's inequality, we have for any $\delta > 0$, if $X_k \in [a_k, b_k]$ for all k, then

$$\Pr\big(|X - \mathbb{E}(X)| \geq \delta\big) \leq 2\exp\Big(-\frac{2\delta^2}{\sum_k (b_k - a_k)^2}\Big).$$

Since $\mathbb{E}(X) = 0$, and $X_k \in \{\frac{1}{n}, -\frac{1}{n}\}$, then

$$\Pr\big(|X| \geq \delta\big) \leq 2\exp\Big(-\frac{2\delta^2}{\sum_k (\frac{1}{n} - (-\frac{1}{n}))^2}\Big).$$

That is to say,

$$\Pr\left(\left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| \geq \delta\right) \leq 2\exp\left(-\frac{n\delta^2}{2}\right) = \exp\left(-\Omega\left(\delta^2 n\right)\right).$$

(b) From (a), we have,

$$\Pr\left(\left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| \geq \delta\right) \leq 2\exp\left(-\frac{n\delta^2}{2}\right), \text{ for all } i \neq j.$$

Using the union bound, we have

$$\begin{aligned}
&\Pr\left(\left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| \leq \delta \text{ for all pairs } i \neq j\right) \\
&\geq 1 - \binom{m}{2} \cdot 2\exp\left(-\frac{n\delta^2}{2}\right) \\
&> 1 - m^2 \exp\left(-\frac{n\delta^2}{2}\right)
\end{aligned}$$

So when $m = \frac{1}{100}\exp\left(\frac{n\delta^2}{4}\right) = \exp\left(\Omega(\delta^2 n)\right)$, we have

$$\Pr\left(\left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| \leq \delta \text{ for all pairs } i \neq j\right) > 1 - m^2 \exp\left(-\frac{n\delta^2}{2}\right) = 1 - 10^{-4} > 0.99.$$

Problem 3. Let RandomGraph(n, p) be the distribution of random graphs of n vertices where, for each pair of vertices u, v, $\{u, v\}$ is chosen as an edge of the graph independently with probability p. Prove the following for such a random graph $G \sim \text{RandomGraph}(n, p)$.

(a) If $p = o(n^{-2/3})$,

$$\lim_{n \to \infty} \Pr(G \text{ contains a 4-clique}) = 0.$$

(b) If $p = \omega(n^{-2/3})$,

$$\lim_{n \to \infty} \Pr(G \text{ does not contain a 4-clique}) = 0.$$

(Hint: Use Part (b) of Problem 1 and you need to carefully calculate the probability of 4-cliques occurring simultaneously on vertex sets A and B when $|A \cap B| \geq 2$.)

Solution. Let X be the number of 4-cliques in the graph. Let X_S be the indicator random variable of the event that a vertex set S of size 4 is a 4-clique. Then $X = \sum_S X_S$. Since $X_S \in \{0, 1\}$, then $\mathbb{E}(X_S) = p^6$. So we have

$$\mathbb{E}(X) = \sum_S \mathbb{E}(X_S) = \binom{n}{4} \cdot (p^6 - p^{12}).$$

(a) The probability that there exists a 4-clique in the graph is

$$\Pr(G \text{ contains a 4-clique}) = \Pr(X > 0) = \Pr(X \geq 1).$$

By Markov's inequality, we have

$$\Pr(X \geq 1) \leq \mathbb{E}(X) = \binom{n}{4} \cdot (p^6 - p^{12}) = \Theta(n^4)(p^6 - p^{12}).$$

Since $p = o(n^{-2/3})$, then $\lim_{n\to\infty} \mathbb{E}(X) = 0$. Therefore,

$$\lim_{n\to\infty} \Pr(G \text{ contains a 4-clique}) = 0.$$

(b) From Part (b) of Problem 1, we have

$$\Pr(X = 0) \leq \frac{\operatorname{Var}(X)}{\left(\mathbb{E}(X)\right)^2}.$$

Now we need to calculate $\operatorname{Var}(X)$. Rewrite $\operatorname{Var}(X)$ as follows:

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \sum_S \mathbb{E}(X_S^2) + \sum_{S\neq T} \mathbb{E}(X_S X_T) - \sum_S \mathbb{E}(X_S)^2 - \sum_{S\neq T} \mathbb{E}(X_S)\,\mathbb{E}(X_T) \\
&= \sum_S \operatorname{Var}(X_S) + \sum_{S\neq T} \operatorname{Cov}(X_S, X_T).
\end{aligned}$$

Since $X_S \in \{0,1\}$, then $\operatorname{Var}(X_S) = \mathbb{E}(X_S^2) - \mathbb{E}(X_S)^2 = p^6 - p^{12}$. So

$$\sum_S \operatorname{Var}(X_S) = \binom{n}{4} \cdot (p^6 - p^{12}).$$

Now we need to calculate $\operatorname{Cov}(X_S, X_T)$ for $S \neq T$. There are 4 cases for S, T:

1. S, T have 0 common vertex: It easy to see that $\operatorname{Cov}(X_S, X_T) = 0$.
2. S, T have 1 common vertex: Still, $\operatorname{Cov}(X_S, X_T) = 0$. This is because there are no common edges between S and T.
3. S, T have 2 common vertices: In this case, $\mathbb{E}(X_S X_T) = p^{11}$. So

   $$\operatorname{Cov}(X_S, X_T) = \mathbb{E}(X_S X_T) - \mathbb{E}(X_S)\,\mathbb{E}(X_T) = p^{11} - p^{12}.$$

   The number of ways to choose S, T for this case is

   $$\binom{n}{6} \cdot \binom{6}{2} \cdot \binom{4}{2} = \Theta(n^6).$$

4. S, T have 3 common vertices: In this case, $\mathbb{E}(X_S X_T) = p^9$. So

$$\mathrm{Cov}(X_S, X_T) = \mathbb{E}(X_S X_T) - \mathbb{E}(X_S)\,\mathbb{E}(X_T) = p^9 - p^{12}.$$

The number of ways to choose S, T for this case is

$$\binom{n}{5} \cdot \binom{5}{3} \cdot \binom{2}{1} = \Theta(n^5).$$

By summing up all four cases together, we have

$$\begin{aligned}\mathrm{Cov}(X_S, X_T) &= \Theta(n^6)(p^{11} - p^{12}) + \Theta(n^5)(p^9 - p^{12}) \\ &\le \Theta(n^6)p^{11} + \Theta(n^5)p^9\end{aligned}$$

So the variance of X is

$$\begin{aligned}\mathrm{Var}(X) &= \sum_S \mathrm{Var}(X_S) + \sum_{S \neq T} \mathrm{Cov}(X_S, X_T) \\ &\le O(n^4 p^6) + \Theta(n^6)p^{11} + \Theta(n^5)p^9\end{aligned}$$

Therefore, when $p = \omega(n^{-2/3})$,

$$\begin{aligned}&\lim_{n\to\infty} \frac{\mathrm{Var}(X)}{\left(\mathbb{E}(X)\right)^2} \\ \le\ &\lim_{n\to\infty} \frac{O(n^4p^6) + \Theta(n^6)p^{11} + \Theta(n^5)p^9}{\Theta(n^4)(p^6 - p^{12})} \\ =\ &\lim_{n\to\infty} \frac{O(1) + \Theta(n^2)p^5 + \Theta(n)p^3}{1 - p^6} \\ =\ &0.\end{aligned}$$

Since $\frac{\mathrm{Var}(X)}{\left(\mathbb{E}(X)\right)^2} \ge 0$, hence, we have

$$\lim_{n\to\infty} \Pr(G \text{ does not contain a 4-clique}) = 0.$$