Homework 1

Problem 1. Decide whether the following is a λ term (or an abbreviation of a λ term). If it is not, explain the reason.

(a) $\lambda x.xxx$
(b) $\lambda\lambda x.x$
(c) $\lambda y.(\lambda x.x)$
(d) $\lambda uv.((xxy)xy)y)$

Solution. (a) Yes
(b) No, the first λ is not matched.
(c) Yes.
(d) No, the number of parentheses are not matched.

Problem 2. Compute the terms represented by the following substitutions:

(a) $(xyz)[y/z]$.
(b) $(\lambda x.x)[y/z]$.
(c) $(\lambda y.xy)[yy/x]$.

Solution. (a) xyy
(b) $\lambda x.x$
(c) $\lambda z.yyz$

Problem 3. Prove the following equalities in the theory of $\lambda\beta$. You need to draw the "proof tree" using the rules we defined in the lecture.

(a) $\lambda uv.v = \lambda u.(\lambda x.x)(\lambda u.u)$.
(b) $\lambda uv.(\lambda xy.y)uv = \lambda ab.b$.

Solution. (a)

$$\cfrac{\cfrac{\cfrac{}{(\lambda x.x)(\lambda v.v) = \lambda v.v}\,(\beta)}{\lambda u.(\lambda x.x)(\lambda v.v) = \lambda u.(\lambda v.v)}\,(\text{abs})}{\lambda u.(\lambda v.v) = \lambda u.(\lambda x.x)(\lambda v.v)}\,(\text{sym})$$

Use α conversion to obtain the final result.

(b)

$$
\dfrac{\dfrac{\dfrac{}{(\lambda xy.y)u = \lambda y.y}\,(\beta) \qquad \dfrac{}{v = v}\,(\text{refl})}{(\lambda xy.y)uv = (\lambda y.y)v}\,(\text{app}) \qquad \dfrac{}{(\lambda y.y)v = v}\,(\beta)}{\dfrac{\dfrac{(\lambda xy.y)uv = v}{\lambda v.(\lambda xy.y)uv = \lambda v.v}\,(\text{abs})}{\lambda uv.(\lambda xy.y)uv = \lambda uv.v}\,(\text{abs})}\,(\text{tran})
$$

Use α conversion to obtain the final result.

Please note that when writing proof trees, each reduction should be of the *exactly* same structure as the five rules. Operations like merging multiple abs together/ only applying β rule to the first term is not allowed.

Problem 4.

(a) Find a λ term s such that the equality $stu = ut$ holds in $\lambda\beta$ for all terms t and u.

(b) Show that there is a λ term s such that for all term t, $\lambda\beta \vdash st = ss$.

Solution. (a) $s = \lambda yx.xy$

(b) two possible solutions:

1. $\lambda x.s$, $x \notin \mathrm{FV}(s)$.

2. Note that we are trying to find s that satisfies $s = \lambda y.ss$, or equivalently, s is the fixed point for $\lambda xy.xx$. Take any fixed point combinator $\mathbf{\Theta}$, and set $s = \mathbf{\Theta}(\lambda xy.xx)$.

Problem 5. Show that there is a term G such that all fixed-point combinators can be *characterized* as the fixed points of G. That is, s is a fixed-point combinator if and only if $\lambda\beta \vdash Gs = s$.

Solution. $G = \lambda yx.x(yx)$.

If: $Gs = s$ implies $s = \lambda x.x(sx)$. Thus for any f, $sf = f(sf)$.

Only if: First we assume $s = \lambda x.t$.

$$Gs = \lambda x.x(sx) = \lambda x.sx = \lambda x.((\lambda y.t[y/x])x) = \lambda x.(t[x/x]) = s$$

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First equality by def of G and β conversion, second by that s is a fixed point. third by α conversion, fourth by β conversion.

To provide a full proof for the only if direction, we need to use the Church-Rosser theorem from Lecture 2. Here we give some hints, and left the full proof as an exercise.

Note that we want to prove $s = \lambda x.t$ without using the η rule. Since β reduction satisfies the Church-Rosser property, we have that $u = v$ iff there exists some w such that $u \twoheadrightarrow w$ and $v \twoheadrightarrow w$. Now we set $u = f(sf)$, $v = sf$, and discuss the structure of w. Hint: Consider u first. Since f is an arbitrary term, checking through all rules for $\twoheadrightarrow_\beta$, what should w looks like?