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Homework 1

Problem 1. Decide whether the following is a λ term (or an abbreviation of a λ term). If it is not, explain the reason.

(a) $\lambda x.xxx$

(b) $\lambda\lambda x.x$

(c) $\lambda y.(\lambda x.x)$

(d) $\lambda uv.((xxy)xy)y)$

Solution. (a) Yes.

(b) No. The reason is that there are 2 consecutive λ s, but there is only 1 dot(.).

(c) Yes.

(d) No. The reason is that the rightmost parenthesis does not match.

Problem 2. Compute the terms represented by the following substitutions:

(a) $(xyz)[y/z]$.

(b) $(\lambda x.x)[y/z]$.

(c) $(\lambda y.xy)[yy/x]$.

Solution. (a) $(xyz).[y/z] = xyy$.

(b) $(\lambda x.x)[y/z] = \lambda x.x$.

(c) $(\lambda y.xy)[yy/x] = (\lambda u.xu)[yy/x] = \lambda u.(xu[yy/x]) = \lambda u.yyu$.

Problem 3. Prove the following equalities in the theory of $\lambda\beta$. You need to draw the "proof tree" using the rules we defined in the lecture.

(a) $\lambda uv.v = \lambda u.(\lambda x.x)(\lambda u.u)$.

(b) $\lambda uv.(\lambda xy.y)uv = \lambda ab.b$.

Solution. (a)

$$\text{(abst)}\ \dfrac{\text{(trans)}\ \dfrac{(\alpha)\ \dfrac{}{\lambda v.v = \lambda u.u} \qquad (\beta)\ \dfrac{}{(\lambda x.x)(\lambda u.u) = \lambda u.u}}{\lambda v.v = (\lambda x.x)(\lambda u.u)}}{\lambda uv.v = \lambda u.(\lambda x.x)(\lambda u.u)}$$

Therefore $\lambda\beta \vdash \lambda uv.v = \lambda u.(\lambda x.x)(\lambda u.u)$.

(b)

$$\dfrac{\text{(abst)}\,\dfrac{(\beta)\,\dfrac{}{(\lambda xy.y)uv = v}}{\lambda uv(\lambda xy.y)uv = \lambda uv.v} \qquad (\alpha)\,\dfrac{}{\lambda uv.v = \lambda ab.b}}{\lambda uv.(\lambda xy.y)uv = \lambda ab.b}\ \text{(trans)}$$

Therefore $\lambda\beta \vdash \lambda uv.(\lambda xy.y)uv = \lambda ab.b$.

Problem 4.

(a) Find a λ term s such that the equality $stu = ut$ holds in $\lambda\beta$ for all terms t and u.

(b) Show that there is a λ term s such that for all term t, $\lambda\beta \vdash st = ss$.

Solution. (a) Let $s = \lambda xy.yx$, then

$$\begin{aligned} stu &= (\lambda xy.yx)tu \\ &= (\lambda x.(\lambda y.yx))tu \\ &= (\lambda y.yx)[t/x]u \\ &= (\lambda y.yt)u \\ &= yt[u/y] \\ &= ut. \end{aligned}$$

Therefore $stu = ut$ holds in $\lambda\beta$ for all terms t and u, and thus $s = \lambda xy.yx$ is what we want.

(b) If $s = \lambda x.ss$, then for all term t,

$$\lambda\beta \vdash st = (\lambda x.ss)t = ss[t/x] = ss.$$

So we need to solve the equation $s = \lambda x.ss$, and it is a fixed point of the function

$$f = \lambda yx.yy,$$

because $fs = (\lambda yx.yy)s = (\lambda x.yy)[s/y] = \lambda x.ss = s$. Using the Y combinator, we can construct one possible answer:

$$
\begin{aligned}
s &= \mathbf{y}f \\
&= (\lambda u.f(uu))(\lambda u.f(uu)) \\
&= (\lambda u.(\lambda yx.yy)(uu))(\lambda u.(\lambda yx.yy)(uu)).
\end{aligned}
$$

Problem 5. Show that there is a term G such that all fixed-point combinators can be *characterized* as the fixed points of G. That is, s is a fixed-point combinator if and only if $\lambda\beta \vdash Gs = s$.

Solution. Let $G = \lambda yx.x(yx)$.

Suppose s is a fixed-point combinator, then the equation $sf = f(sf)$ holds for all terms f. If we bind f as λx, s satisfies the equation $s = \lambda x.x(sx)$. And we can verify that

$$
\begin{aligned}
Gs &= (\lambda yx.x(yx))s \\
&= (\lambda x.x(yx))[s/y] \\
&= \lambda x.x(sx) \\
&= s.
\end{aligned}
$$

On the other hand, suppose $Gs = s$, then

$$
\begin{aligned}
s &= Gs \\
&= (\lambda yx.x(yx))s \\
&= (\lambda x.x(yx))[s/y] \\
&= \lambda x.x(sx).
\end{aligned}
$$

Hence for all terms f, $sf = (\lambda x.x(sx))f = x(sx)[f/x] = f(sf)$. That is, s is a fixed-point combinator.

QED.