Vector and Matrix Algebra

Definition. A **scalar** is a quantity with size, often called magnitude, but no direction.

For example, the set of real numbers $\mathbb{R}$ are scalars!

Definition. A **vector** is a quantity with size and direction.

(1) Draw examples of vectors in the Cartesian plane $\mathbb{R}^2$ and 3-dimensional $\mathbb{R}^3$.

The point that a vector starts at is often called the "tail" of the vector and where it ends is referred to as the "head".

Definition. Let n be a positive integer. The set of points $(x_1, x_2, \ldots, x_n)$ where each $x_1, x_2, \ldots, x_n$ are real numbers is an n-tuple of real numbers. We denote the set of all n-tuples of real numbers by $\mathbb{R}^n$.

For example, $\mathbb{R}$ is the set of all 1-tuples and $\mathbb{R}^2$ is the set of all 2-tuples or more familiarly, xy-pairs.

(2) Notice that vectors are independent of position because they only depend on size and direction. What would be a good way to distinguish between vectors? What do you notice if you were to put every vector's tail at the origin–the point $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$?

Note: We often use a general subscript n, as in $\mathbb{R}^n$, when discussing a fact or property that holds for any real space $\mathbb{R}^n$.

Definition. A vector is in **standard position** if its tail is on the origin.

Definition. Consider a vector $\mathbf{v}$ in standard position and suppose the head of the vector is at the point $(v_1, v_2, \ldots, v_n)$. The **component form** of $\mathbf{v}$ is

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Note. We are using "column vector" notation, but we can also write vectors as "row vectors" like $[v_1, v_2, \ldots, v_n]$.

"Head−Tail" Formula in $\mathbb{R}^n$. Suppose a vector's tail is at a point $A = (a_1, a_2, \ldots, a_n)$ and head is at a point $B = (b_1, b_2, \ldots, b_n)$, then the coponent form of this vector $\vec{AB}$ is

$$\vec{AB} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{bmatrix}$$

(3) Let $A = (0, -1, 3, 7)$ and $B = (2, 1, -4, 1)$. Compute $\vec{AB}$

Notation. We usually write vectors in bold like $\mathbf{v}$ or with an arrow over a letter like $\vec{v}$. We can do lots of things with vectors! In particular, we can scale and add them:

Definition. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$, and let k be a scalar, then

$$k\mathbf{u} = k \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}$$

Definition. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. We define $\mathbf{u} + \mathbf{v}$ by

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

(4) Let $\mathbf{u} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$. Compute the following:
- (a) $2\mathbf{u}$
- (b) $\mathbf{u} + \mathbf{v}$
- (c) $3\mathbf{u} - \frac{1}{2}\mathbf{v}$

Theorem 1. The following properties hold for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars k and p in $\mathbb{R}$.

(a) Commutative Property of Addition

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

(b) Associative Property of Addition

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

(c) Existence of Additive Identity: There exists a vector $\mathbf{0}$ such that

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

(d) Existence of Additive Inverse: For every vector $\mathbf{u}$, there exists a vector $-\mathbf{u}$ such that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

(e) Distributive Property over Vector Addition

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$$

(f) Distributive Property over Scalar Addition

$$(k + p)\mathbf{u} = k\mathbf{u} + p\mathbf{u}$$

(g) Associative Property for Scalar Multiplication

$$k(p\mathbf{u}) = (kp)\mathbf{u}$$

(h) Multiplication by 1

$$1\mathbf{u} = \mathbf{u}$$

(5) Let $\mathbf{a} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute the following:

(a) $5(\mathbf{a} + 2\mathbf{b} - \mathbf{c})$

(b) $3\mathbf{a} - 7\mathbf{a}$

Matrices

Definition. A **matrix** is a rectangular array of numbers. The plural form of matrix is **matrices**. Here is some vocabulary that will help us talk about matrices:

- The dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns.
- A **column vector** in $\mathbb{R}^m$ is an $m \times 1$ matrix.
- A **row vector** in $\mathbb{R}^n$ is an $1 \times n$ matrix.
- We refer to the individual entries of the matrix by their position. The (i, j)**-entry** of a matrix is the entry in the i^{th} row and j^{th} column.

(6) Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix}$$

(a) What are the dimensions of M?

(b) What is the second row of M?

(c) The entry 8 is in which position?

(d) What value is in the $(3, 2)$ position?

Notation. We denote the entry in the i^{th} row and the j^{th} column of an arbitrary $m \times n$ matrix A by a_{ij}, and write A in terms of its entries as

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

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Occasionally it will be convenient to talk about columns and rows of a matrix A as vectors. We will use the following notation:

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \\ | & | & & | \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{bmatrix}$$

and

$$A = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_m & - \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$$

Definition (The Zero Matrix)**.** The $m \times n$ **zero matrix** is the $m \times n$ matrix having every entry equal to zero. The zero matrix is denoted by O.

Definition (Equality of Matrices)**.** Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be two $m \times n$ matrices. Then $A = B$ means that $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition (Addition of Matrices)**.** Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be two $m \times n$ matrices. Then the **sum of matrices** A and B, denoted by $A + B$, is an $m \times n$ matrix given by

$$A + B = \left[a_{ij} + b_{ij}\right]$$

(7) Find the sum of A and B, if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Definition (Scalar Multiplication)**.** If $A = \left[a_{ij}\right]$ and k is a scalar, then $kA = \left[ka_{ij}\right]$.

(8) Find $7A$ if

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Theorem 2 (Properties of Matrix Addition)**.** Let A, B and C be matrices. Then, the following properties hold.

(a) Commutative Law of Addition

$$A + B = B + A$$

(b) Associative Law of Addition

$$(A + B) + C = A + (B + C)$$

(c) Additive Identity

There exists a zero matrix such that
$$A + O = A$$

(d) Additive Inverse

There exists a matrix, $-A$, such that
$$A + (-A) = O$$

Theorem 3 (Properties of Scalar Multiplication)**.** Let A, B be matrices, and k, p be scalars. Then, the following properties properties of scalar multiplication hold.

(a) Distributive Law over Matrix Addition

$$k\,(A + B) = kA + kB$$

(b) Distributive Law over Scalar Addition

$$(k+p)\,A = kA + pA$$

(c) Associative Law for Scalar Multiplication

$$k\,(pA) = (kp)\,A$$

(d) Multiplication by 1

$$1A = A$$

(9) If

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$$

then compute
(a) 2(A+B)
(b) 3A-2B

Definition. Let A be an $m \times n$ matrix with columns $\mathbf{a_1}, \cdots, \mathbf{a_n}$ and let $\mathbf{x}$ be a column vector in $\mathbb{R}^n$. We define the **product** of A and $\mathbf{x}$ as follows:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n}$$

(10) Determine which of the following products are defined. If the product is defined, compute it. Otherwise, explain why the product is undefined.

$$\begin{bmatrix} 6 & 5 \\ 2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad\qquad \begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$$

(11) Compute $A\mathbf{x}$ if

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -2 & 1 \\ -2 & 4 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

Definition. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \cdot \vec{v}$, is given by

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \ldots + u_nv_n.$$

(12) Compute the following dot products:

(a) $\begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 4 \\ 7 \\ 13 \end{bmatrix} =$

(b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} =$

(c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} =$

(d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

(13) Sketch the vectors in parts 12b and 12c. Make a conjecture about when the dot product of two vectors is 0.

Definition. Let A be an $m \times n$ matrix whose rows are vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$. Let B be an $n \times p$ matrix with columns $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p$. Then the entries of the matrix product AB are given by the dot products

$$AB = \begin{bmatrix} - & \vec{r}_1 & - \\ - & \vec{r}_2 & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_m & - \end{bmatrix} \begin{bmatrix} | & | & & | & & | \\ \vec{b}_1 & \vec{b}_2 & \ldots & \vec{b}_j & \ldots & \vec{b}_p \\ | & | & & | & & | \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{b}_1 & \vec{r}_1 \cdot \vec{b}_2 & \ldots & \vec{r}_1 \cdot \vec{b}_j & \ldots & \vec{r}_1 \cdot \vec{b}_p \\ \vec{r}_2 \cdot \vec{b}_1 & \vec{r}_2 \cdot \vec{b}_2 & \ldots & \vec{r}_2 \cdot \vec{b}_j & \ldots & \vec{r}_2 \cdot \vec{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \vec{r}_i \cdot \vec{b}_1 & \vec{r}_i \cdot \vec{b}_2 & \ldots & \vec{r}_i \cdot \vec{b}_j & \ldots & \vec{r}_i \cdot \vec{b}_p \\ \vdots & \vdots & & \vdots & & \vdots \\ \vec{r}_m \cdot \vec{b}_1 & \vec{r}_m \cdot \vec{b}_2 & \ldots & \vec{r}_m \cdot \vec{b}_j & \ldots & \vec{r}_m \cdot \vec{b}_p \end{bmatrix}$$

(14) Suppose A is any 3×2-matrix, B is a 2×2-matrix and C is a 2×3-matrix. Which of the products AB, BC and CA are defined? What is the general rule for the product of two matrices to be defined?

(15) Compute the matrix product $AB = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

(16) If a and b are real numbers and we know $ab = 0$, then either $a = 0$ or $b = 0$ (or both). This is not true for matrices. Give an example of two 2×2-matrices A and B such that $AB = 0$.

(17) Provide an example of two 3×3 matrices A and B such that AB does not equal BA.