Transformations

Note: Remember that a point in $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ corresponds to the vector in $\mathbb{R}^n$: $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

From now on we will refer to points and vectors in $\mathbb{R}^n$ interchangeably.

Definition. A **transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$** is a function that maps vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$.

Notation: We denote a "transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$" by $T : \mathbb{R}^n \to \mathbb{R}^m$. Note that the *domain* of T is $\mathbb{R}^n$ and the *codomain* of T is $\mathbb{R}^m$.

(1) Consider the transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

For this transformation to be defined, what must the domain and codomain of T be?

(2) In general, if A is a $r \times c$ matrix, what is the domain and codomain of the transformation $T(\mathbf{x}) = A\mathbf{x}$?

Definition. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation and suppose for some $\mathbf{x}$ in $\mathbb{R}^n$ and $\mathbf{y}$ in $\mathbb{R}^m$ that $T(\mathbf{x}) = \mathbf{y}$. Then we call $\mathbf{y}$ the **image** of $\mathbf{x}$ under T.

(3) Consider the square in $\mathbb{R}^2$ with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$. For each of the transformations $\mathbb{R}^2 \to \mathbb{R}^2$ below, plot the square and the image of the square under the transformation.

(a) $R(\mathbf{x}) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \mathbf{x}$

(b) $S(\mathbf{x}) = \mathbf{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(4) Geometrically, what are each of the transformations in the previous problem?

Definition. The standard unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}_n$ are the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \ldots, \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(5) Write the arbitrary vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a linear combination of standard unit vectors in $\mathbb{R}^3$.

(6) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(\mathbf{x}) = B\mathbf{x}$ where $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Evaluate T at each of the three standard unit vectors in $\mathbb{R}^3$ and compare the results to $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$.

(7) Determine transformations $T_x : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_y : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_x(\mathbf{x})$ is the reflection of $\mathbf{x}$ across the x-axis and $T_y(\mathbf{x})$ is the reflection of $\mathbf{x}$ across the y-axis for any point $\mathbf{x}$ in $\mathbb{R}^2$.

(8) Determine a transformation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $R_\theta(\mathbf{x})$ is a rotation of the point $\mathbf{x}$ about the origin $(0, 0)$ by the angle θ for any point $\mathbf{x}$ in $\mathbb{R}^2$.