Frequency Analysis
Lecture -3
Spring 2019,
Prof. Megherbi

ECE 5810/ECE 4840
Digital Image Processing and Computer Vision
Spring 2019
Lecture-3Topics:

1. The Fourier Transform
   - 1-D continuous FT, 1-D inverse FT
   - 2-D continuous forward and inverse FT
   - 1- and 2-D discrete FT (DFT) and its inverse
   - 2. Properties of the FT
     - Separability
     - translation
     - periodicity
     - conjugate symmetry
     - distributivity
     - rotation
     - scaling
     - average value
     - convolution and correlation
     - Sampling
     - The Fast Fourier Transform (FFT)
     - Frequency Domain Processing
     - the ideal low-pass and high-pass filters
     - Butterworth filters

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Fourier Transform, What does it mean? What and how is it used in signal and mage processing?

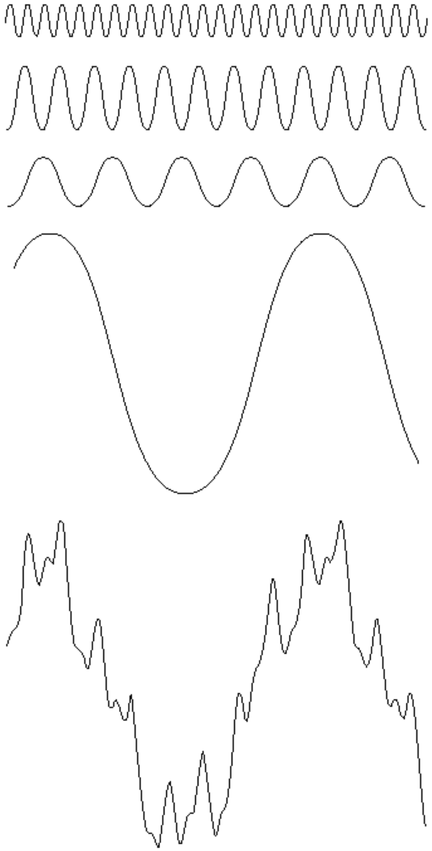


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Fourier Transform. (FT)

1- 1-D continuous FT, 1-D inverse FT

$$\begin{cases} F(u) = \int_{-\infty}^{+\infty} f(x)\, e^{-j 2\pi u x}\, dx. \\ f(x) = \int_{-\infty}^{\infty} F(u)\, e^{j 2\pi u x}\, du. \end{cases}$$

Notation: ✓ $F(u) = R(u) + j\, I(u)$

$$= \sqrt{R^2(u) + I^2(u)}\; e^{j\phi(u)}$$

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

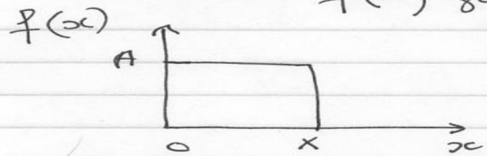
✓ $|F(u)|$ is the Fourier Spectrum of $f(x)$

✓ $\phi(u)$ is the phase angle (Fourier)

✓ $P(u) = |F(u)|^2$ the square of the spectrum ≡ power spectrum ≡ density.

✓ u is the frequency variable.

Example: The FFT of the function $f(x)$ given by



$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi u x}\, dx.$$

$$= A \int_{0}^{X} e^{-j2\pi u x}\, dx$$

$$= \frac{-A}{j2\pi u}\left[e^{-j2\pi u X} - 1\right]$$

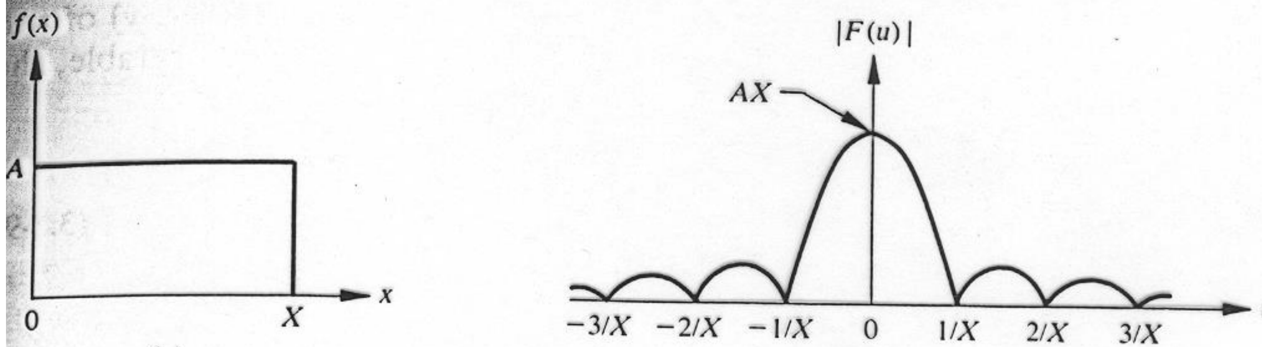
$$= \frac{-A}{j2\pi u} e^{-j\pi u X}\left[e^{-j\pi u X} - e^{j\pi u X}\right]$$

since $\sin(y) = \dfrac{e^{jy} + e^{-jy}}{2j} \Rightarrow$

$$F(u) = \frac{A}{j2\pi u} e^{-j\pi u X}(2j)\left[\frac{e^{j\pi u X} - e^{-j\pi u X}}{2j}\right]$$

$$= \frac{A}{\pi u} e^{-j\pi u X} \sin(\pi u X)$$

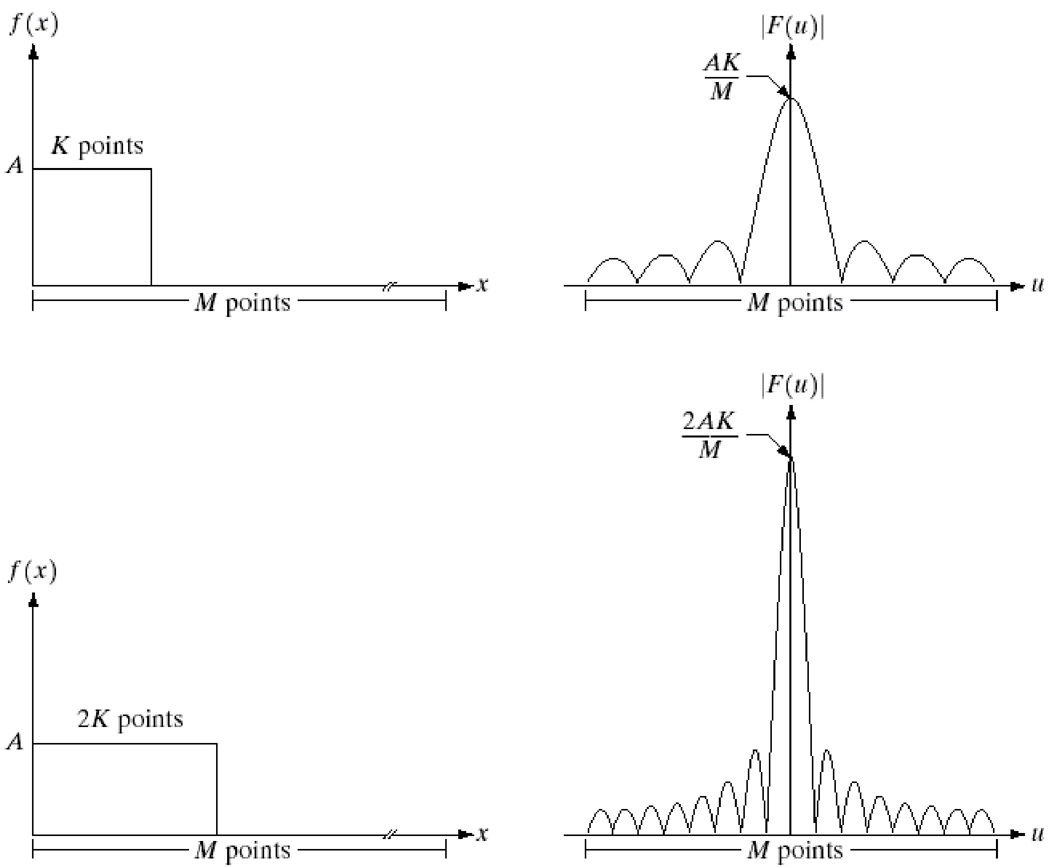
$$|F(u)| = \frac{A \sin(\pi u X)}{\pi u}.$$



(a)

(b)

Fourier Transform



a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Why?

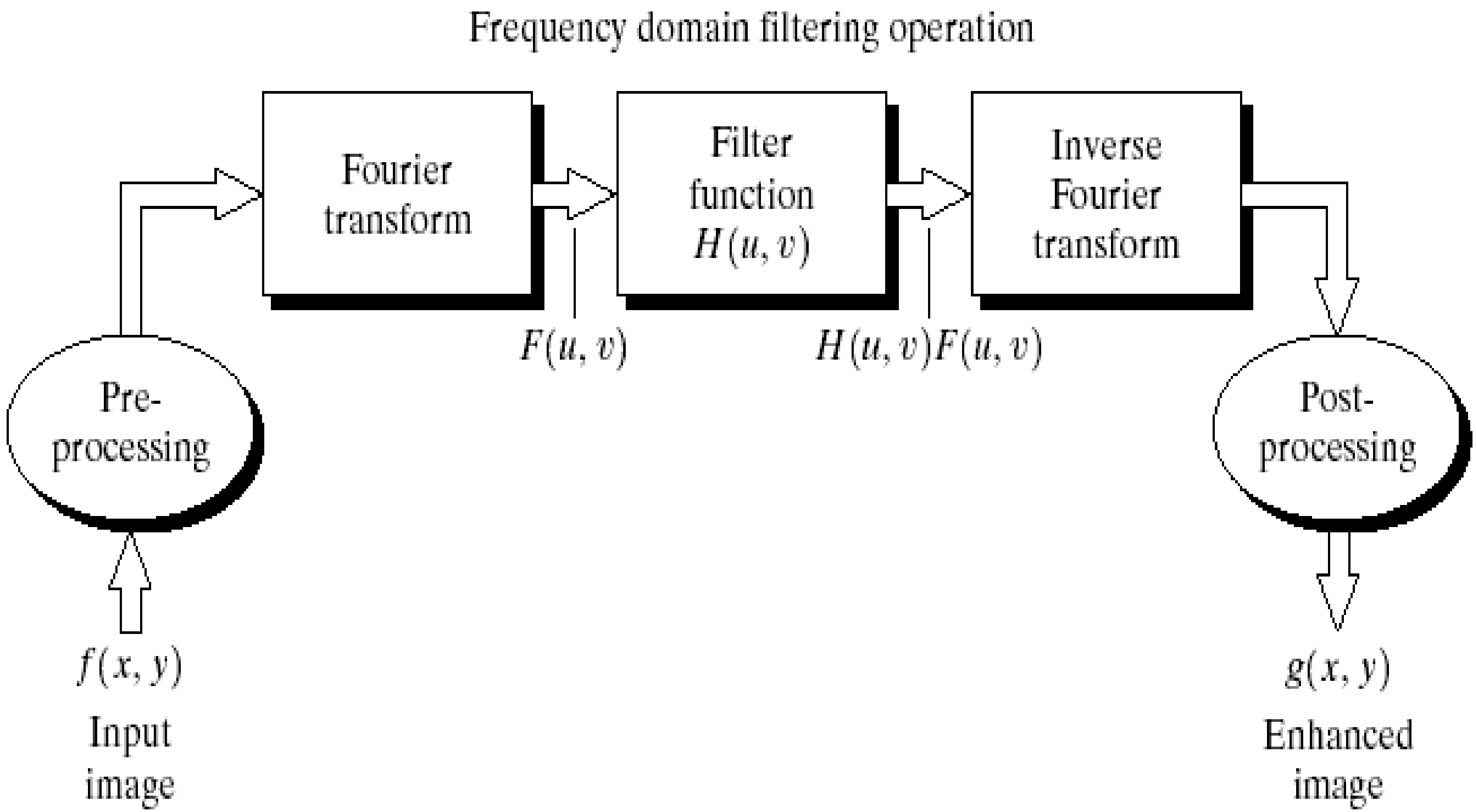


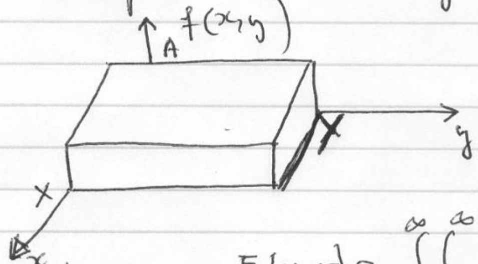
FIGURE 4.5 Basic steps for filtering in the frequency domain.

(2) 2-D Continuous Forward and Inverse

$$\begin{cases} F(u,v) = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{j2\pi(ux+vy)}\, dx\, dy. \\ f(x,y) = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{-j2\pi(ux+vy)}\, du\, dv. \end{cases}$$

- u, v frequency variables
- $|F(u,v)| = R^2(u,v) + I^2(u,v)$
- $\phi(u,v) = \tan^{-1}\left(\frac{I(u,v)}{R(u,v)}\right)$

<u>Example:</u> The FT of the function shown.

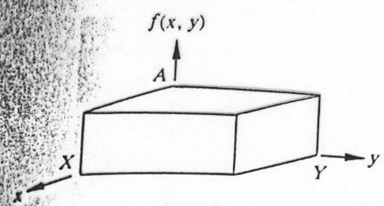


$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, d$$

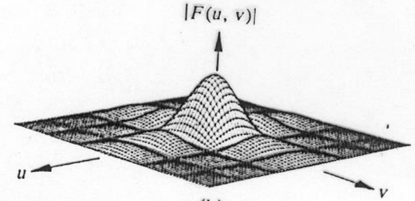
$$= \int_0^X \int_0^Y A\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$= AXY \left[\frac{\sin(\pi u X)\, e^{-j\pi u X}}{\pi u X}\right]\left[\frac{\sin(\pi v Y)\, e^{-j\pi v Y}}{\pi v Y}\right]$$

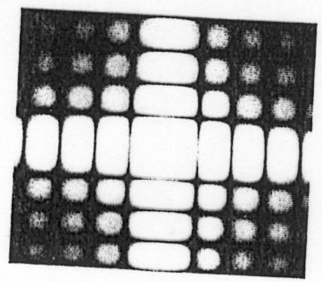
$$|F(u,v)| = AXY \left|\frac{\sin(\pi u X)}{\pi u X}\right| \left|\frac{\sin(\pi v Y)}{\pi v Y}\right|$$



(a)



(b)



(c)

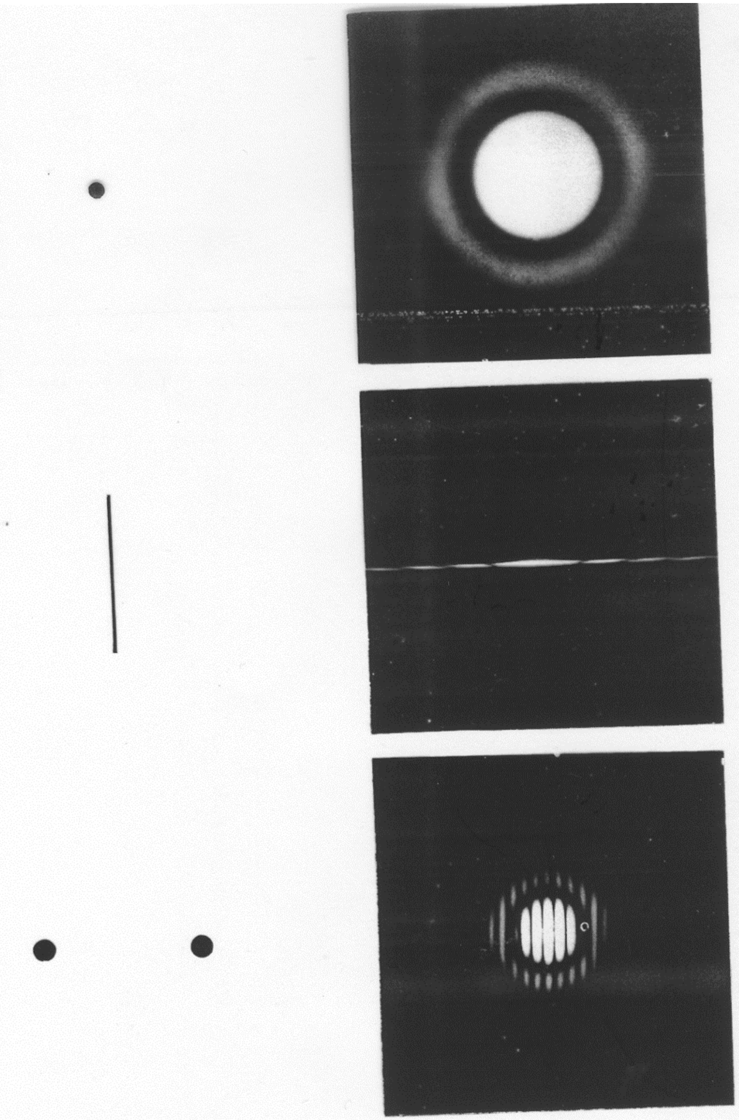
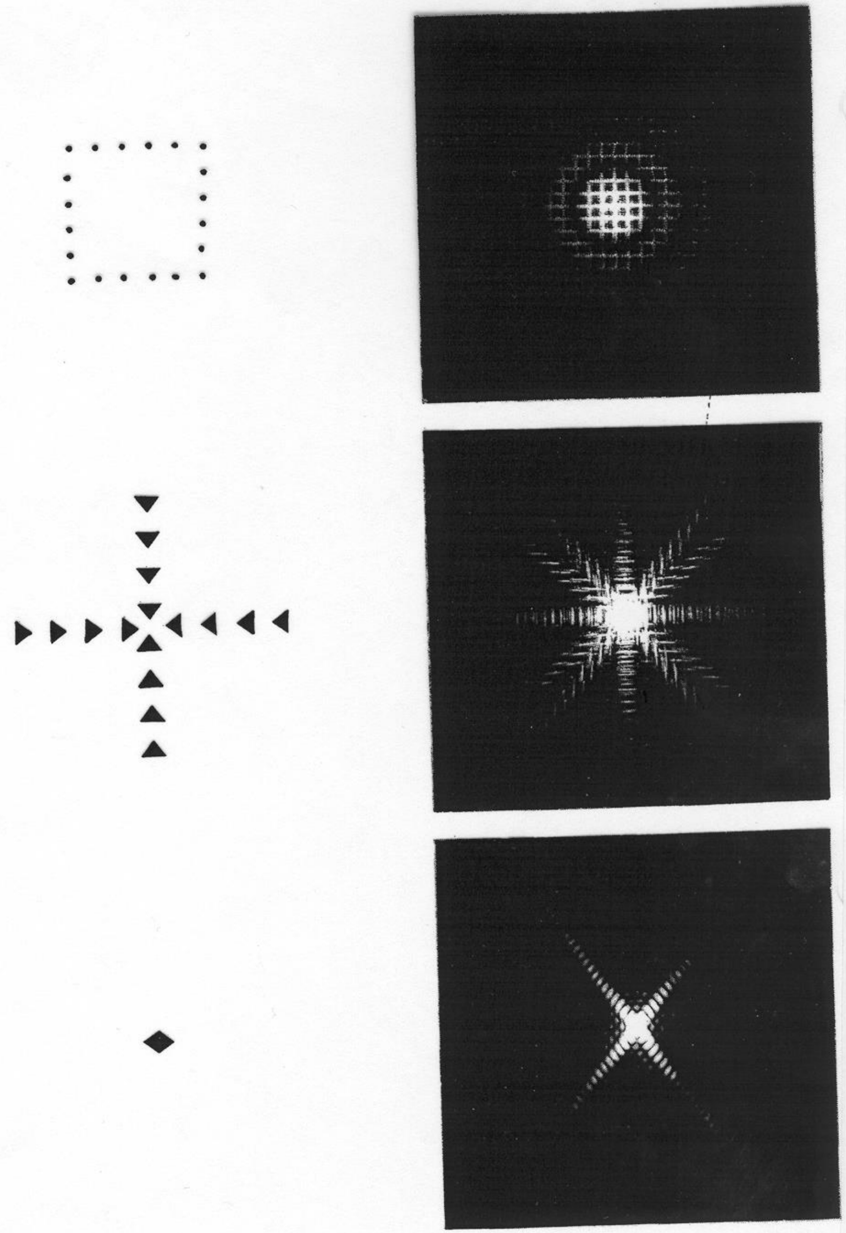
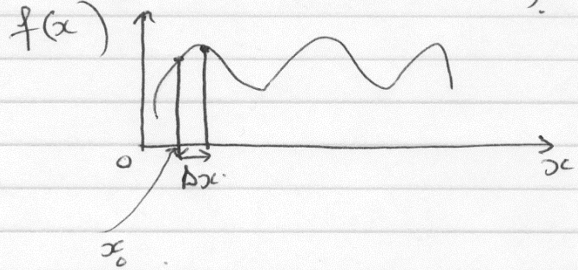


Figure 3.3 *Some 2-D functions and their Fourier spectra.*



③ the 1-D discrete Fourier Transform: DFT

$$f(x) = f(x_0 + x\,\Delta x)$$



$$f(x) = f(x_0 + x\,\Delta x) \qquad x = 0, \dots, N-1$$

$$\{ f(x) : f(0), f(1), \dots f(n), \dots f(N-1) \}$$

Definition: the DFT is

$$\begin{cases} F(u) = \dfrac{1}{N} \displaystyle\sum_{x=0}^{N-1} f(x)\, e^{-j 2\pi u x} \\ u = 0, 1, \dots, N-1 \\ x = 0, 1, \dots, N-1 \end{cases}$$

$$\boxed{\Delta u = \frac{1}{N \Delta x}}$$

$$\begin{cases} f(x) = \displaystyle\sum_{u=0}^{N-1} F(u)\, e^{j \frac{2\pi}{N} x u} \\ x = 0, \dots, N-1 \\ u = 0, \dots, N-1 \end{cases}$$

(4) 2-D Discrete Fourier Transform: DFT

$$
\begin{cases}
F(u,v) = \dfrac{1}{MN} \displaystyle\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(k,\ell)\, e^{-j\frac{2\pi}{1}\left[\frac{uk}{M} + \frac{v}{N}\right.} \\[2ex]
f(k,\ell) = \displaystyle\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{+j2\pi\left[\frac{uk}{M} + \frac{v\ell}{N}\right.}
\end{cases}
$$

$u = 0, \ldots, M-1$
$v = 0, \ldots, N-1$
$k = 0, \ldots, M-1$
$\ell = 0, \ldots, N-1$

$$
\boxed{\begin{aligned} \Delta u &= \frac{1}{M\Delta x} \\ \Delta v &= \frac{1}{N\Delta y}. \end{aligned}}
$$

if $N = M$ we can rewrite as:

$$
\begin{cases}
F(u,v) = \dfrac{1}{N} \displaystyle\sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy} \\[2ex]
f(x,y) = \dfrac{1}{N} \displaystyle\sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v)\, e^{j\frac{2\pi}{N}(ux+vy}
\end{cases}
$$

(5) Properties of 2-D DFT

5_1 Separability

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy)}$$

$u, v = 0, 1, \ldots N-1$

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} e^{-j\frac{2\pi}{N}ux} \sum_{y=0}^{N-1} f(x,y)\, e^{-j\frac{2\pi}{N}vy}$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} e^{-j\frac{2\pi}{N}ux} \cdot N \cdot \underbrace{\frac{1}{N} \sum_{y=0}^{N-1} f(x,y)\, e^{-j\frac{2\pi}{N}vy}}_{F(x,v)}$$

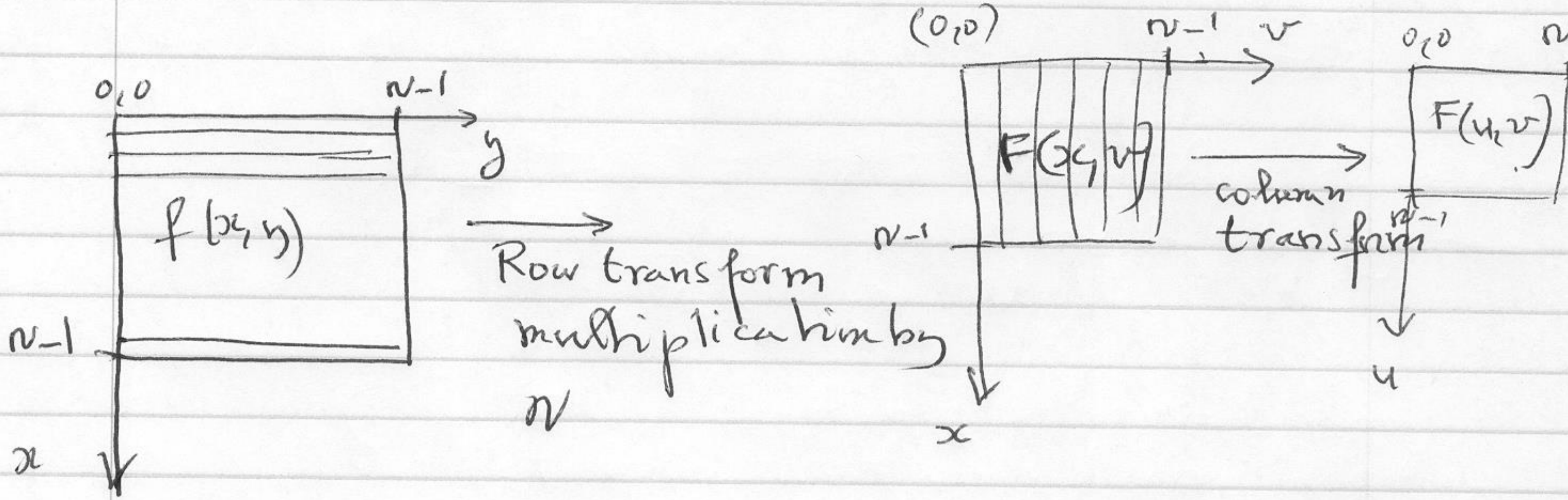
$F(x,v) \leftarrow$ 1D DFT

$$F(u,v) = \underbrace{\frac{1}{N} \sum_{x=0}^{N-1} F(x,v)\, e^{-j\frac{2\pi}{N}ux}}_{\text{1-D DFT}} \cdot N$$

* $F(u,v)$ is obtained by taking the following steps:

① – take a transform along each row of $f(x,y)$ and multiplying the result by N

② – $F(u,v)$ is then obtained by taking a transform along each column of $F(x,v)$



* Computing 2D_DFT $\Longrightarrow$ series of 1_D DFT

5-2 Translation

① $$f(x,y)\, e^{j\frac{2\pi}{N}(x u_0 + v_0 y)} \Longleftrightarrow F(u-u_0, v-v_0)$$

② $$f(x-x_0, y-y_0) \Longleftrightarrow F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0 + v y_0)}.$$

Ex: to moves the origin of the spectrum to the ~~origin~~ center of the image.

$$f(x,y)\, e^{j\frac{2\pi}{N}\left(\frac{u_0 x}{N} + \frac{v_0 x}{N}\right)}$$ with

$$u_0 = v_0 = \frac{N}{2}$$

$$\Rightarrow \quad f(x,y)\, e^{j 2\pi\left(\frac{1}{2}(x+y)\right)} = f(x,y)(-1)^{x+y} \quad \checkmark$$

$$f(x,y)(-1)^{x+y} \longrightarrow F\left(u-\frac{N}{2}, v-\frac{N}{2}\right)$$

origine of spectrum @ $\left(\frac{N}{2}, \frac{N}{2}\right) \longleftarrow$ center of imag

Remark: from ② shift in $f(x,y)$ does not affect the spectrum.

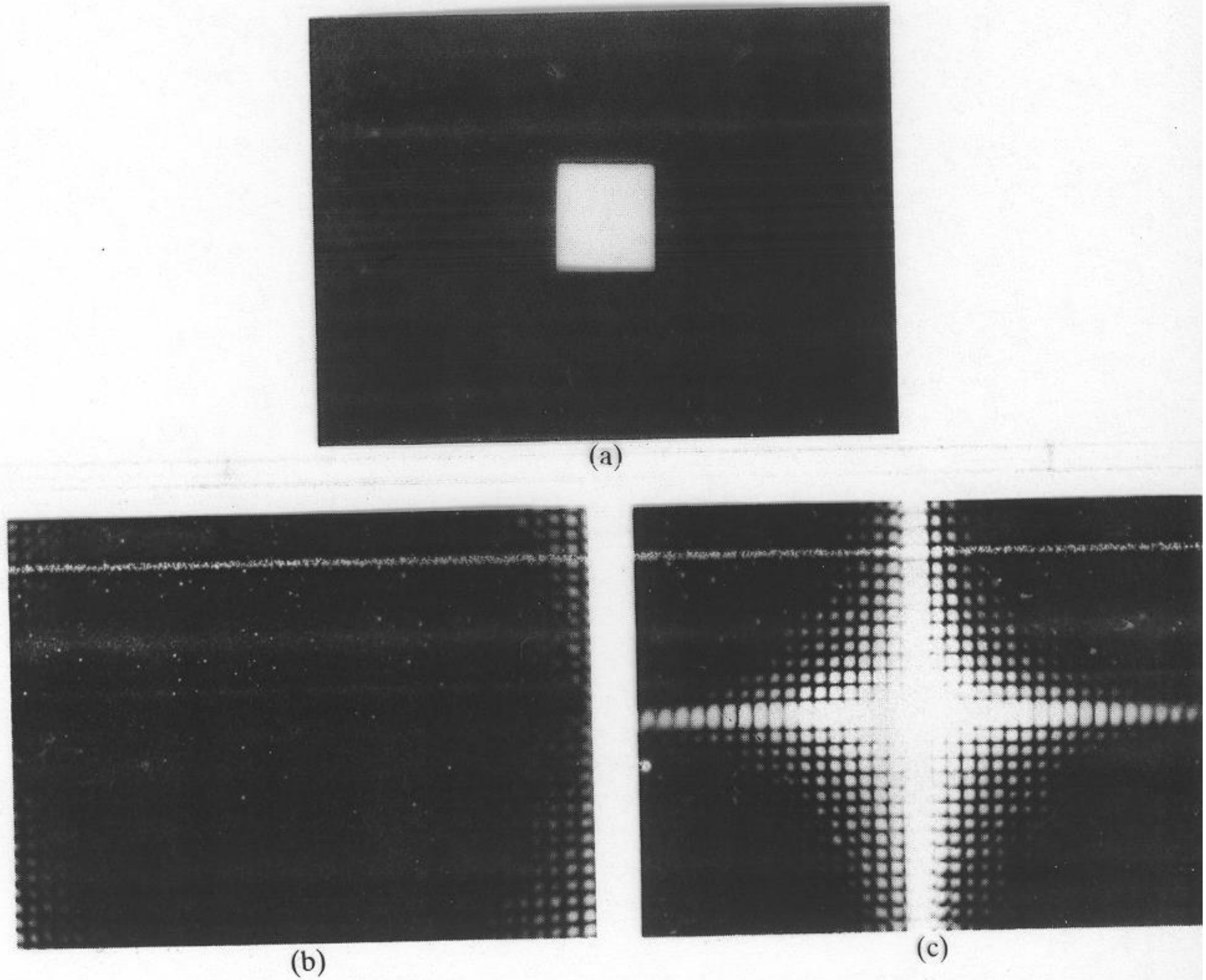


Figure 3.9 *(a) A simple image; (b) Fourier spectrum without shifting; (c) Fourier spectru*
shifted to the center of the frequency square.

5-3 Periodicity and Conjugate Symmetry

- $$F(u,v) = F(u+N,v) = F(u,v+N)$$
$$= F(u+N,\ v+N)$$

- if $f(x,y)$ real $\Rightarrow \begin{cases} F(u,v) = F^*(-u,-v) \\ |F(u,v)| = |F(-u,-v) \end{cases}$

F^*: complex conjugate o

5-4 Rotation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{aligned} u &= w\cos\phi \\ v &= w\sin\phi \end{aligned}$$

$$f(r, \theta+\theta_0) \Longleftrightarrow F(w, \phi+\theta_0)$$

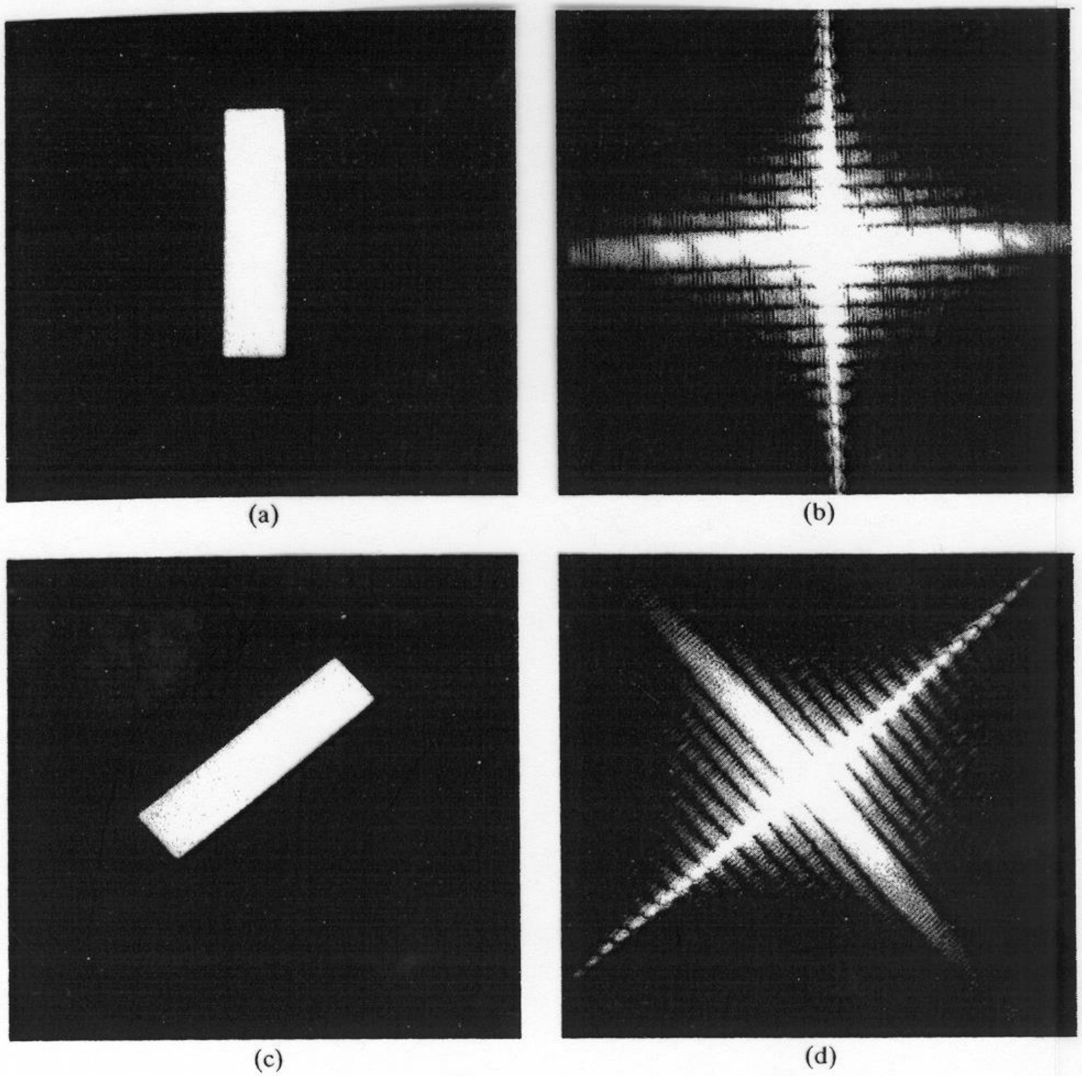


Figure 3.10 — Rotational properties of the Fourier transform: (a) a simple image; (b) spectrum; (c) rotated image; (d) resulting spectrum.

5.4 Distributivity & scaling

$* \; f(x,y) + g(x,y) \Longleftrightarrow F(u,v) + G(u,v)$

$* \; f(x,y) \cdot g(x,y) \;\not\Longleftrightarrow\; F(u,v) \cdot G(u,v)$

$* \; a\, f(x,y) \Longleftrightarrow a\, F(u,v)$

$* \; f(ax, by) \Longleftrightarrow \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right).$

5.5 Average value

Average value of 2-D function $\bar{f}(x,y)$

$$\bar{f}(x,y) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y).$$

$$F(0,0) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)$$

$$\boxed{\bar{f}(x,y) = \frac{1}{N} F(0,0)}$$

5-6 Convolution theorem

$$f(x,y) \circledast g(x,y) \Longleftrightarrow F(u,v) \cdot G(u,v)$$

convolution

$$f(x,y) \cdot g(x,y) \Longleftrightarrow F(u,v) \circledast G(u,v).$$

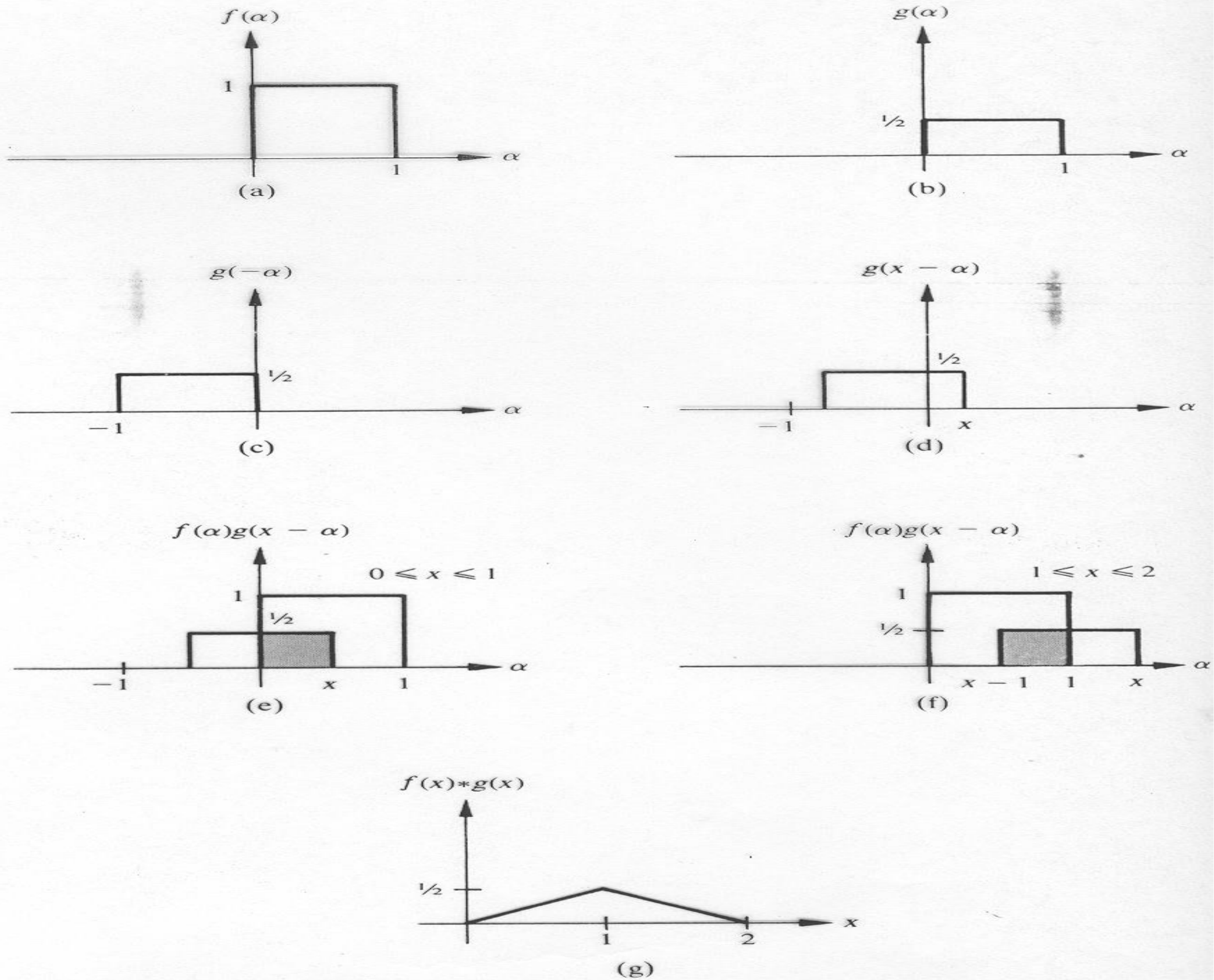


Figure 3.11 *Graphic illustration of convolution. The shaded areas indicate regions the product is not zero.*

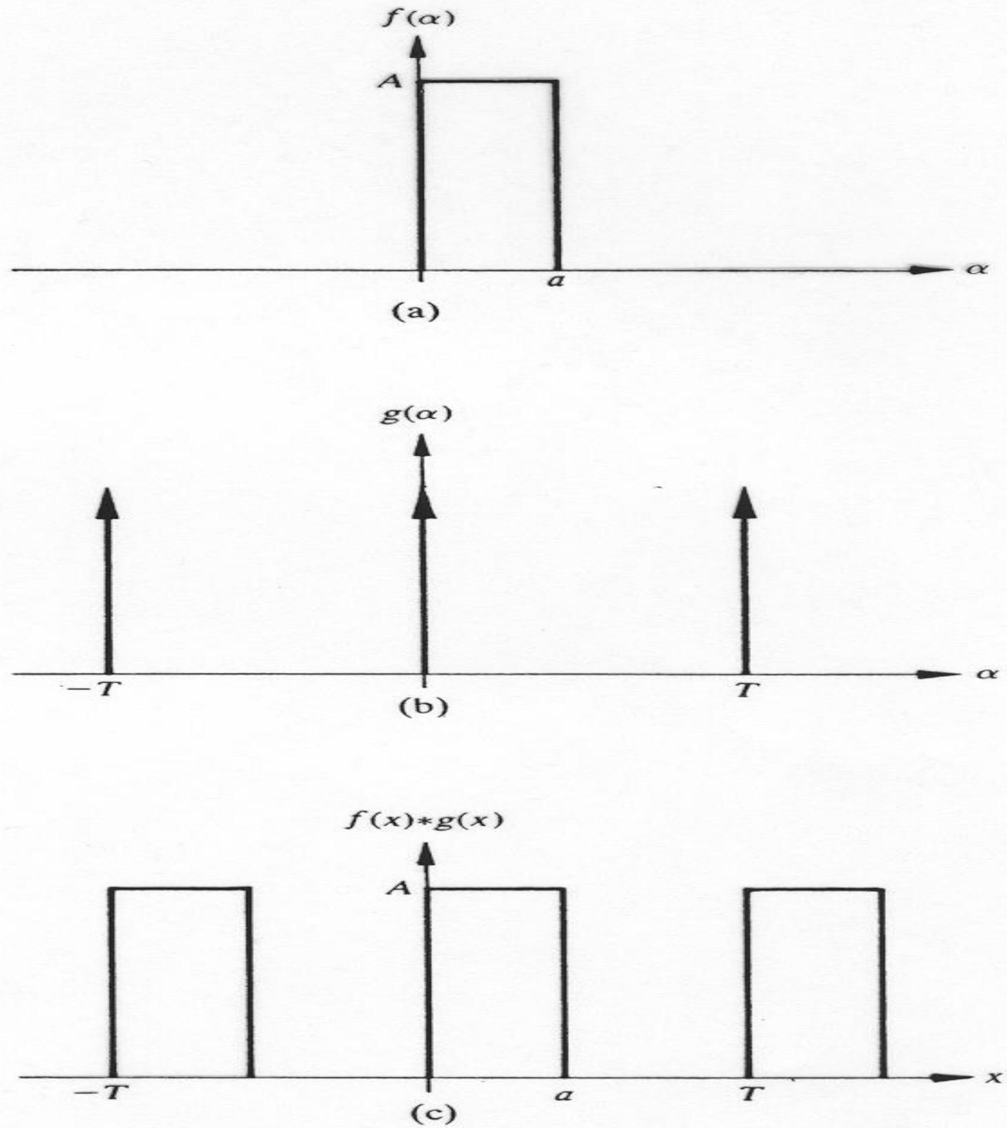


Figure 3.13 *Convolution involving impulse functions.*

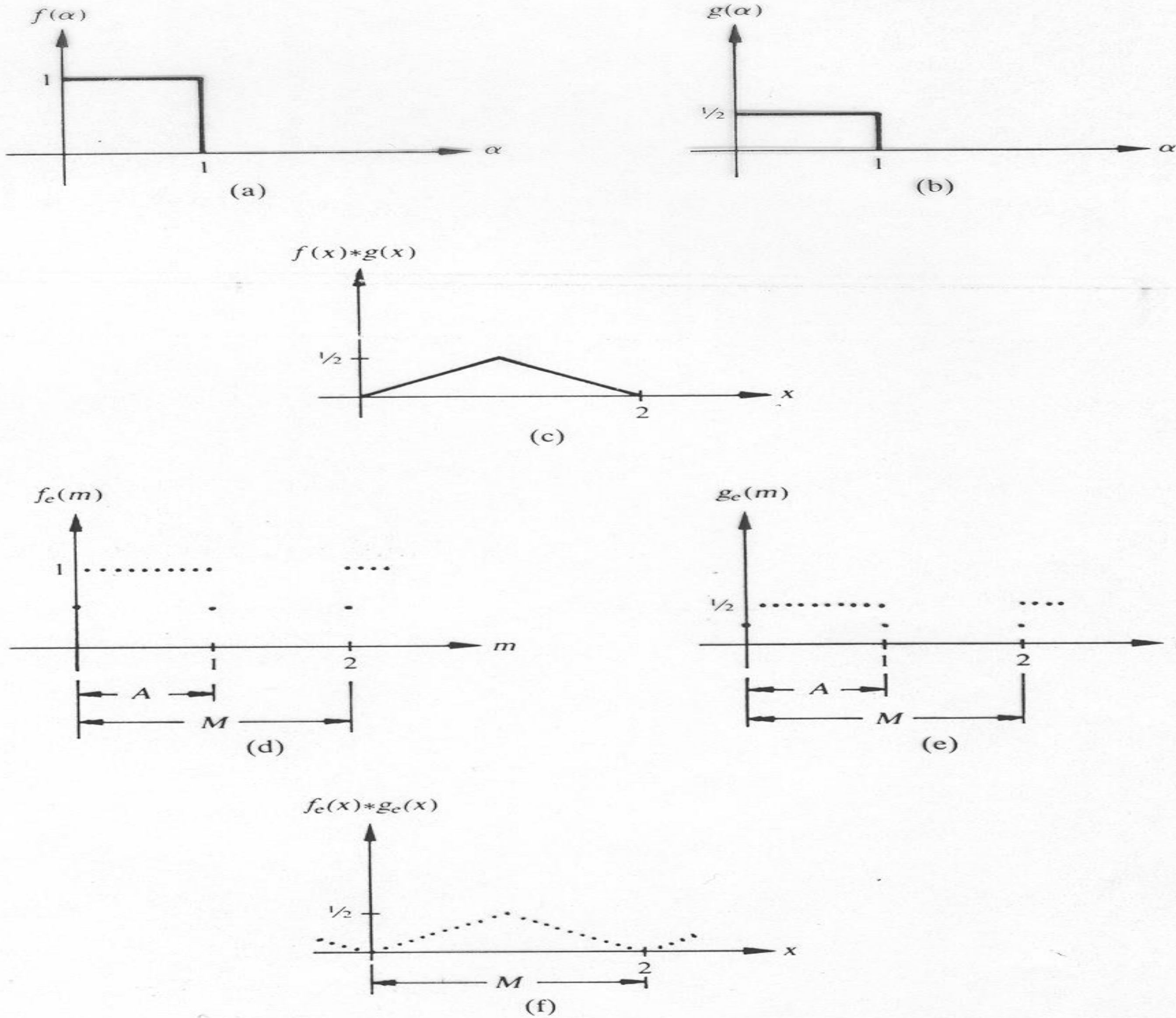


Figure 3.14 *Comparison between continuous and discrete convolution.*

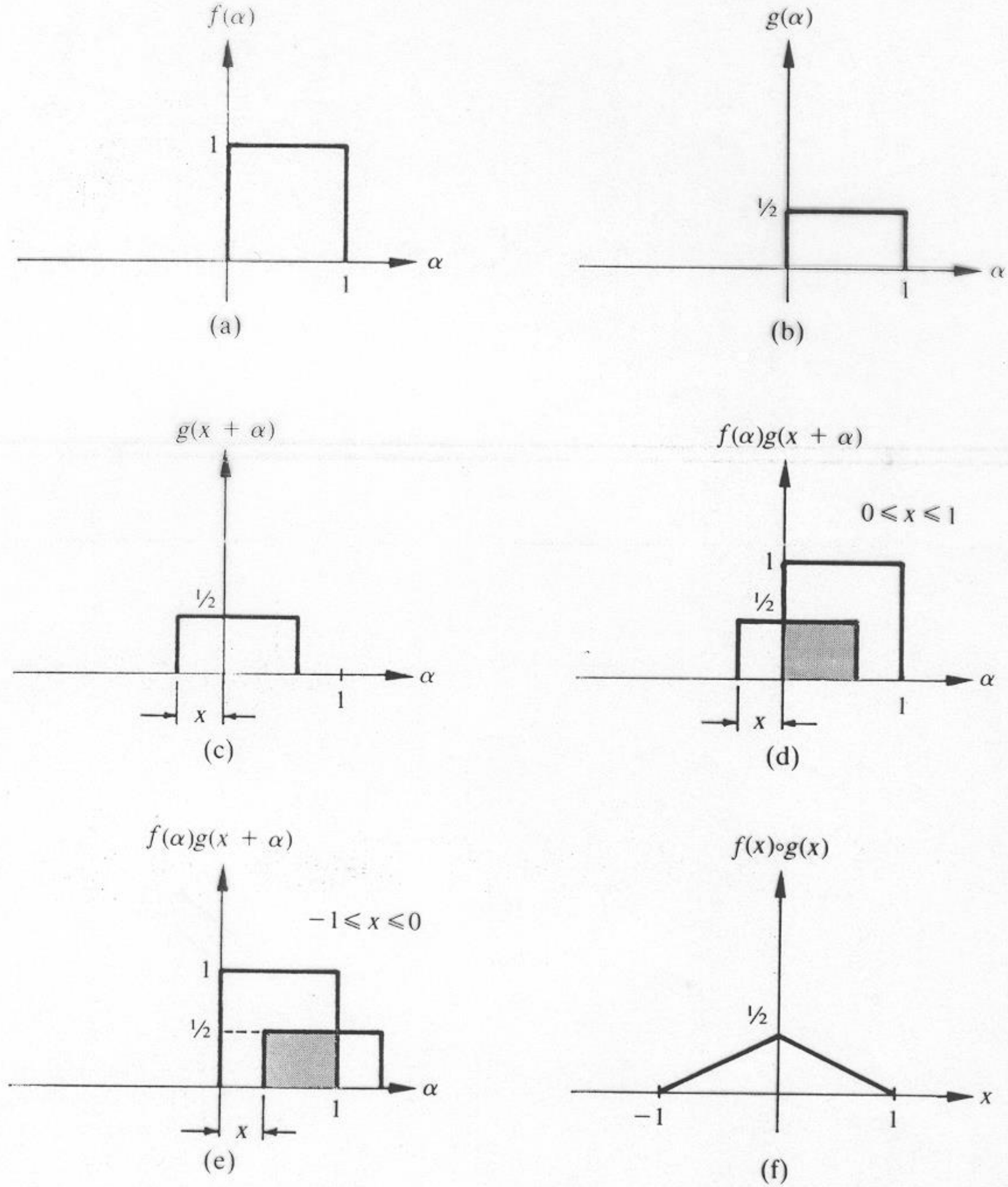


Figure 3.16 *Graphic illustration of correlation. The shaded areas indicate regions where t product is not zero.*

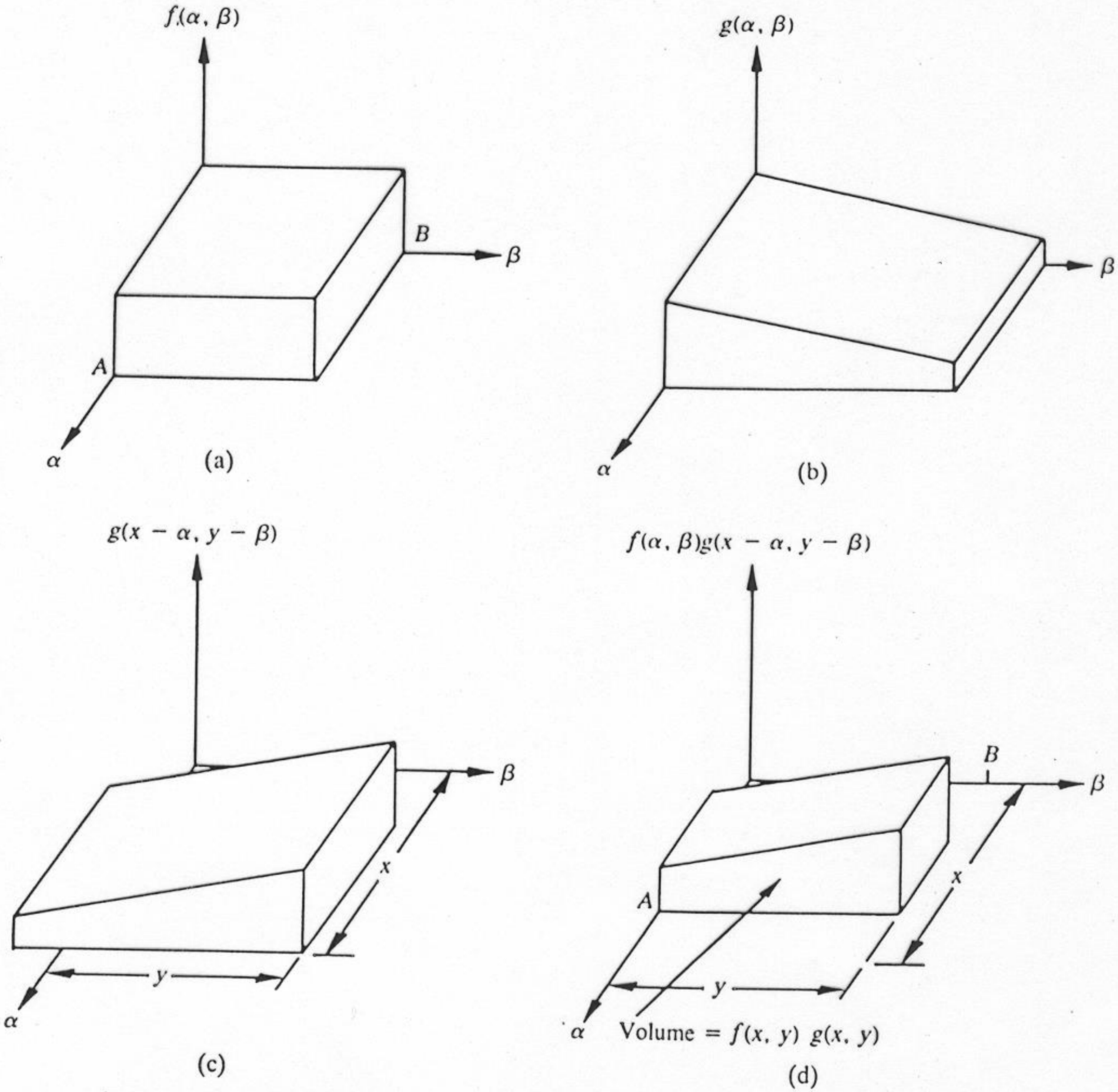


Figure 3.15 *Illustration of the folding, displacement, and multiplication steps needed to perform two-dimensional convolution.*

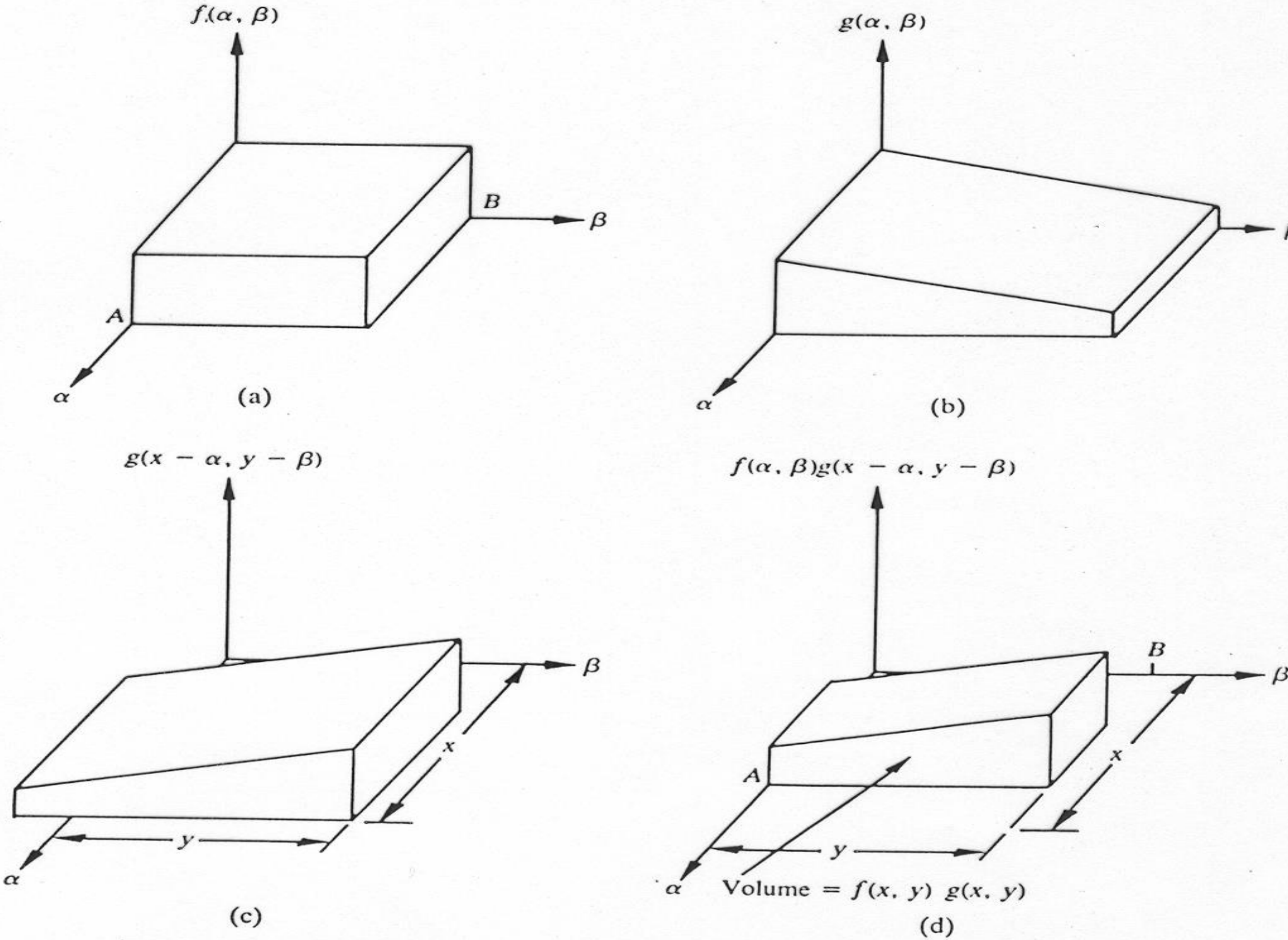
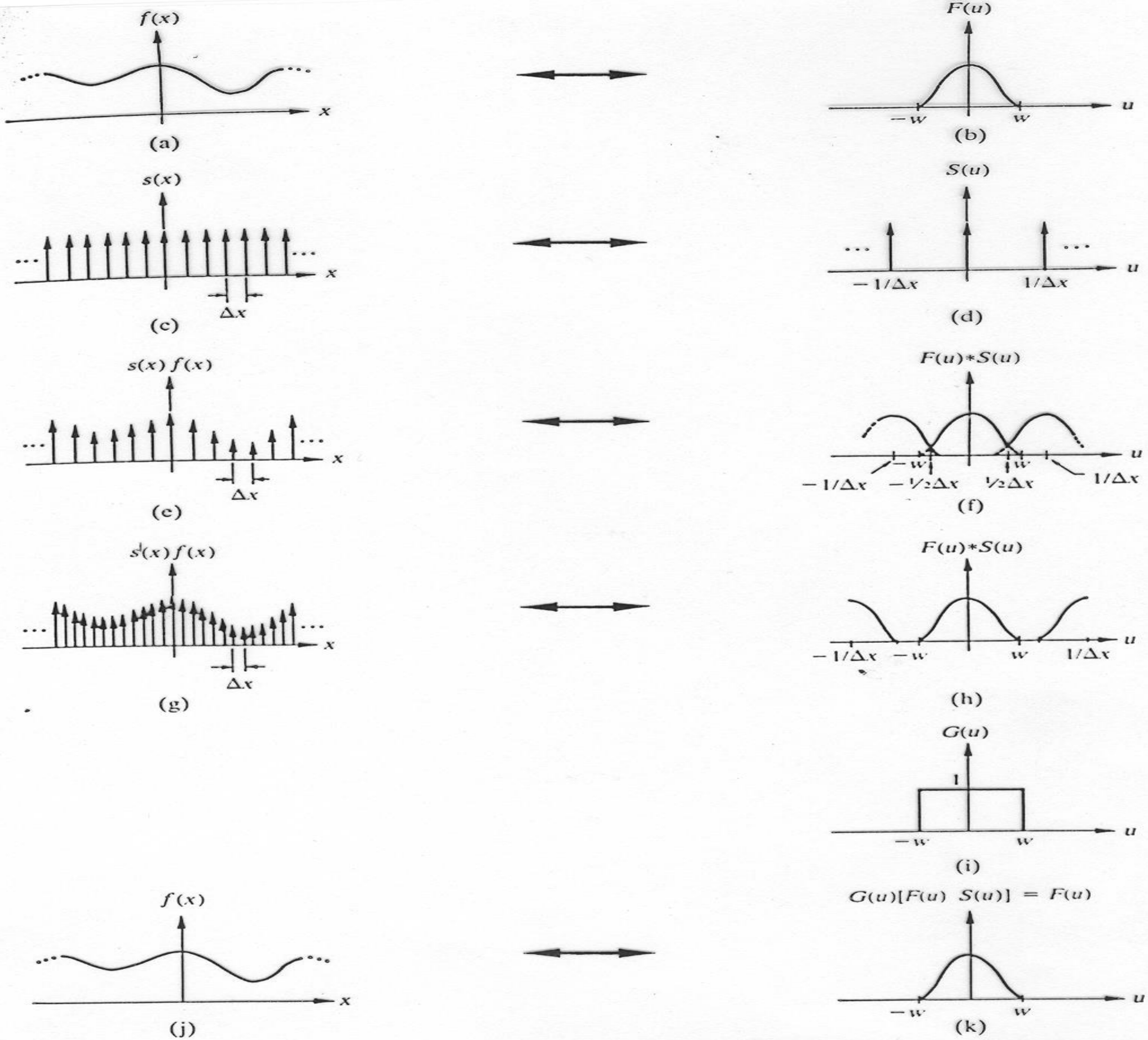


Figure 3.15 *Illustration of the folding, displacement, and multiplication steps needed to perform two-dimensional convolution.*



f(x)
x
(a)
F(u)
u
−w
w
(b)
s(x)
x
Δx
(c)
S(u)
u
−1/Δx
1/Δx
(d)
s(x) f(x)
x
Δx
(e)
F(u)*S(u)
u
−w
w
−1/Δx
−½Δx
½Δx
1/Δx
(f)
s(x) f(x)
x
Δx
(g)
F(u)*S(u)
u
−1/Δx
−w
w
1/Δx
(h)
G(u)
1
u
−w
w
(i)
f(x)
x
(j)
G(u)[F(u) S(u)] = F(u)
u
−w
w
(k)

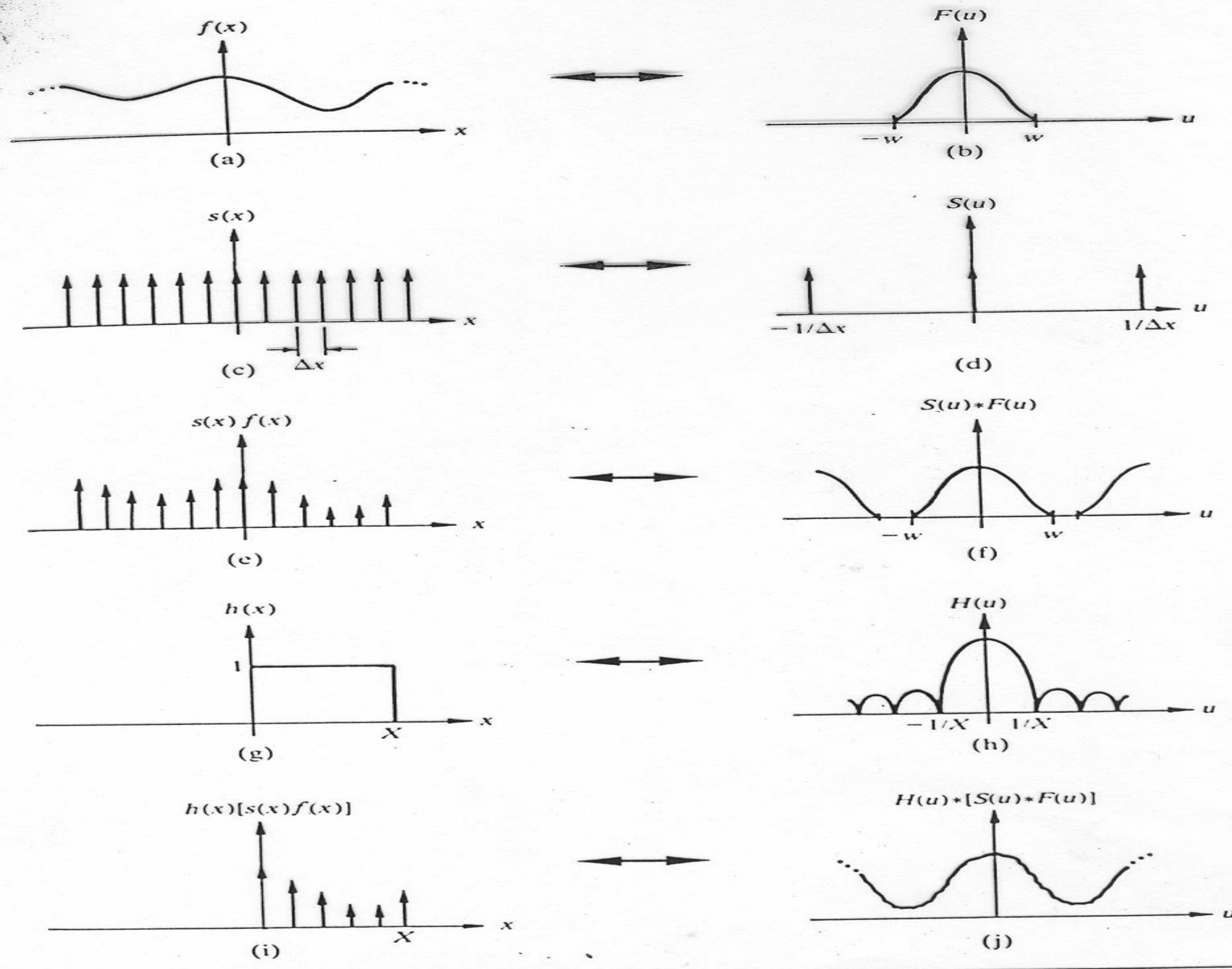


Figure 3.18 *Graphic illustration of finite-sampling concepts.*

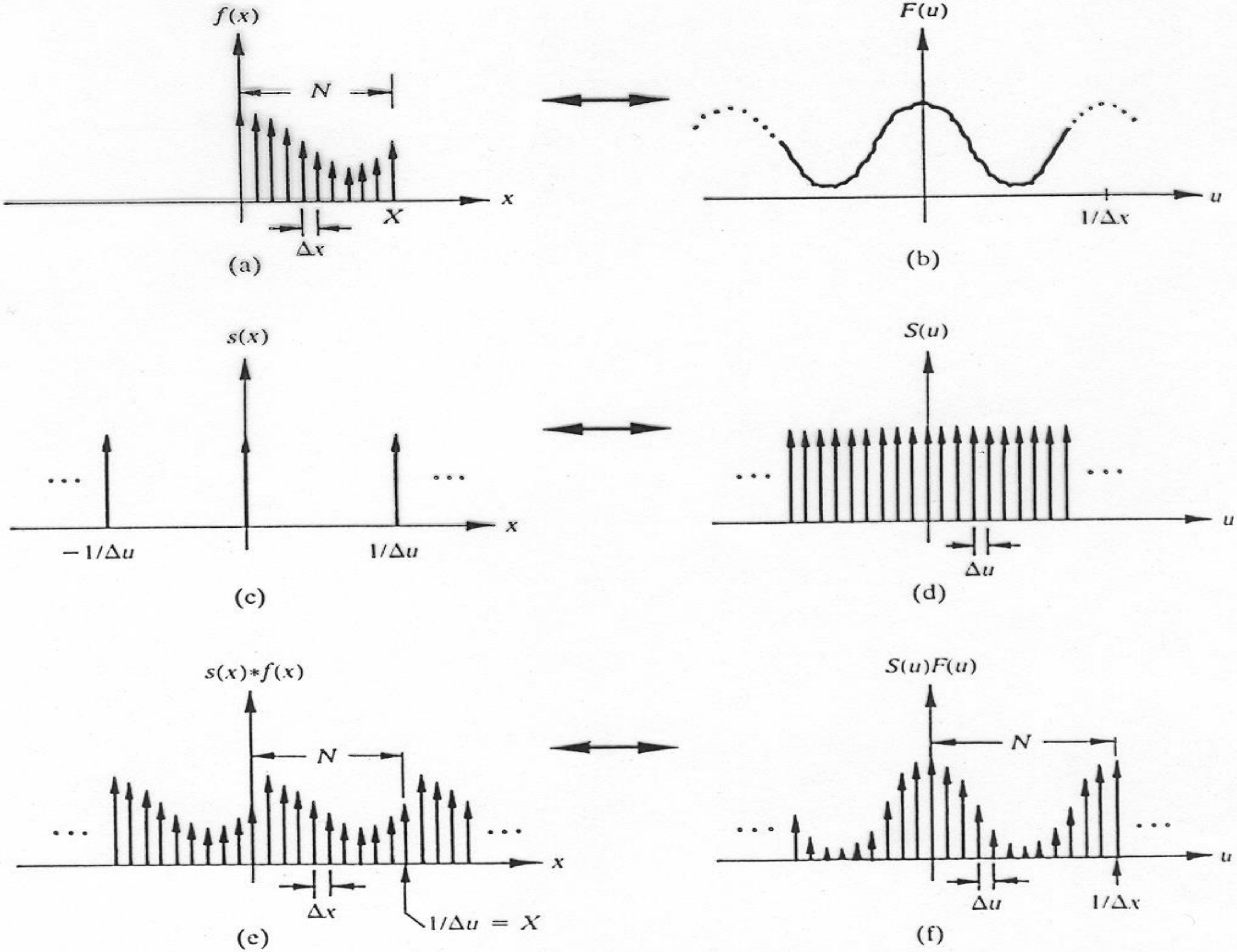


Figure 3.19 *Graphic illustration of the discrete Fourier transform.*

Fast Fourier Transform FFT:

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, e^{-j\frac{2\pi}{N}ux} \qquad N = 2^n.$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, W_N^{ux}$$

$$W_N = e^{-j\frac{2\pi}{N}}.$$

$$F(u) = \frac{1}{2M} \sum_{x=0}^{2M-1} f(x)\, W_{2M}^{ux} \qquad M = \frac{N}{2}$$

$$= \frac{1}{2}\left[\frac{1}{M} \sum_{x=0}^{M-1} f(2x)\, W_{2M}^{u2x} + \frac{1}{M} \sum_{x=0}^{M-1} f(2x+1)\, W_{2M}^{u(2x+}\right.$$

$$W_{2M}^{2xu} = W_M^{ux} \qquad \text{(Fourier periodic).}$$

$$\Rightarrow \quad F(u) = \frac{1}{2}\left[\underbrace{\frac{1}{M} \sum_{x=0}^{M-1} f(2x)\, W_M^{ux}}_{\text{even}} + \underbrace{\frac{1}{M} \sum_{x=0}^{M-1} f(2x+1)\, W_{2M}^{u(2x}}_{\text{odd.}}\right.$$

$\Rightarrow$

$$F(u) = \frac{1}{2}\left[\frac{1}{M} \sum_{x=0}^{M-1} f(2x)\, W_M^{ux} + \frac{1}{M} \sum_{x=0}^{M-1} f(2x+1)\, W_M^{ux}\, W_{2M}^{u}\right.$$

$$F_{even} = \frac{1}{M} \sum_{x=0}^{M-1} f(2x)\, W_M^{ux}$$

$$F_{odd} = \frac{1}{M} \sum_{x=0}^{M-1} f(2x+1)\, W_M^{ux}.$$

$$W_M^{u+M} = W_M^{u} \quad ; \quad W_{2M}^{u+M} = -W_{2M}^{u}.$$

$$\begin{cases} F(u+M) = \frac{1}{2}\left[F_{even} - F_{odd}\, W_{2M}^{u}\right] \\ F(u) = \frac{1}{2}\left[F_{even} + F_{odd}\, W_{2M}^{u}\right. \end{cases}$$

$$\begin{cases} F_{even} = \frac{1}{M} \sum_{x=0}^{M-1} f(2x)\, W_M^{ux} \\ F_{odd} = \frac{1}{M} \sum_{x=0}^{M-1} f(2x+1)\, W_N^{ux} \end{cases} \qquad u = 0, 1, \ldots, M-1.$$

N points FFT $\longrightarrow$ $2 \times \frac{N}{2}$ points FFT.

$$2 \times M$$

$$M = 2 \times P \qquad \underbrace{2 \times 2 \times \frac{M}{2} \times \cdots\cdots}_{M \times N.}$$

$$N \log N$$

Table 3.1 A Comparison of N^2 versus $N \log_2 N$ for Various Values of N

N	N^2 (*Direct* FT)	$N \log_2 N$ (FFT)	*Computational Advantage* ($N/\log_2 N$)
2	4	2	2.00
4	16	8	2.00
8	64	24	2.67
16	256	64	4.00
32	1,024	160	6.40
64	4,096	384	10.67
128	16,384	896	18.29
256	65,536	2,048	32.00
512	262,144	4,608	56.89
1024	1,048,576	10,240	102.40
2048	4,194,304	22,528	186.18
4096	16,777,216	49,152	341.33
8192	67,108,864	106,496	630.15

Frequency Domain Processing

Foundation: the convolution theorem

$$g(x,y) = h(x,y) * f(x,y) \Longleftrightarrow G(u,v) = H \cdot F$$

$$H(u,v) = \text{transfer function.}$$

$$\underset{\text{input}}{f(x,y)} \longrightarrow \boxed{h(x,y)} \longrightarrow \underset{g(x,y)}{\text{output}} \Longleftrightarrow$$

$$g = h * f(x,y)$$

$$\equiv \quad F(u,v) \longrightarrow \boxed{H(u,v)} \longrightarrow G(u,v).$$

$$G = H F$$

$$\frac{\text{Input}}{\text{Output}} \quad \text{or} \quad \frac{\text{output}}{\text{input}} = H$$

* Goal of Image enhancement is to select $h(x,y)$ or $H(u,v)$ so that the output $g(x,y)$ exhibits some features of $f(x,y)$.

* <u>Example:</u> Edges of $f(x,y)$ can be accentuated by using a function $H(u,v)$ that emphasizes the high frequency components of $F(u,v)$

② The ideal low pass and high pass filter

— Low pass filter.

- edges and sharp transitions (e.g. noise) in the gray ~~area~~ levels $\Longrightarrow$ contribute to the high frequency content of its Fourier transfr.

- blurring (smoothing) in the frequency domain by alternating high frequency components of the image spectrum.

$$G(u,v) = H(u,v) \cdot F(u,v)$$

$G(u,v)$ → filtered image

$H(u,v)$ → desired filter to be designed

$F(u,v)$ → given spectrum of an image.

Problem: select (design) $H(u,v)$ that yields the desired output $G(u,v)$ by alternating high-frequency components of $F(u,v)$

Why?

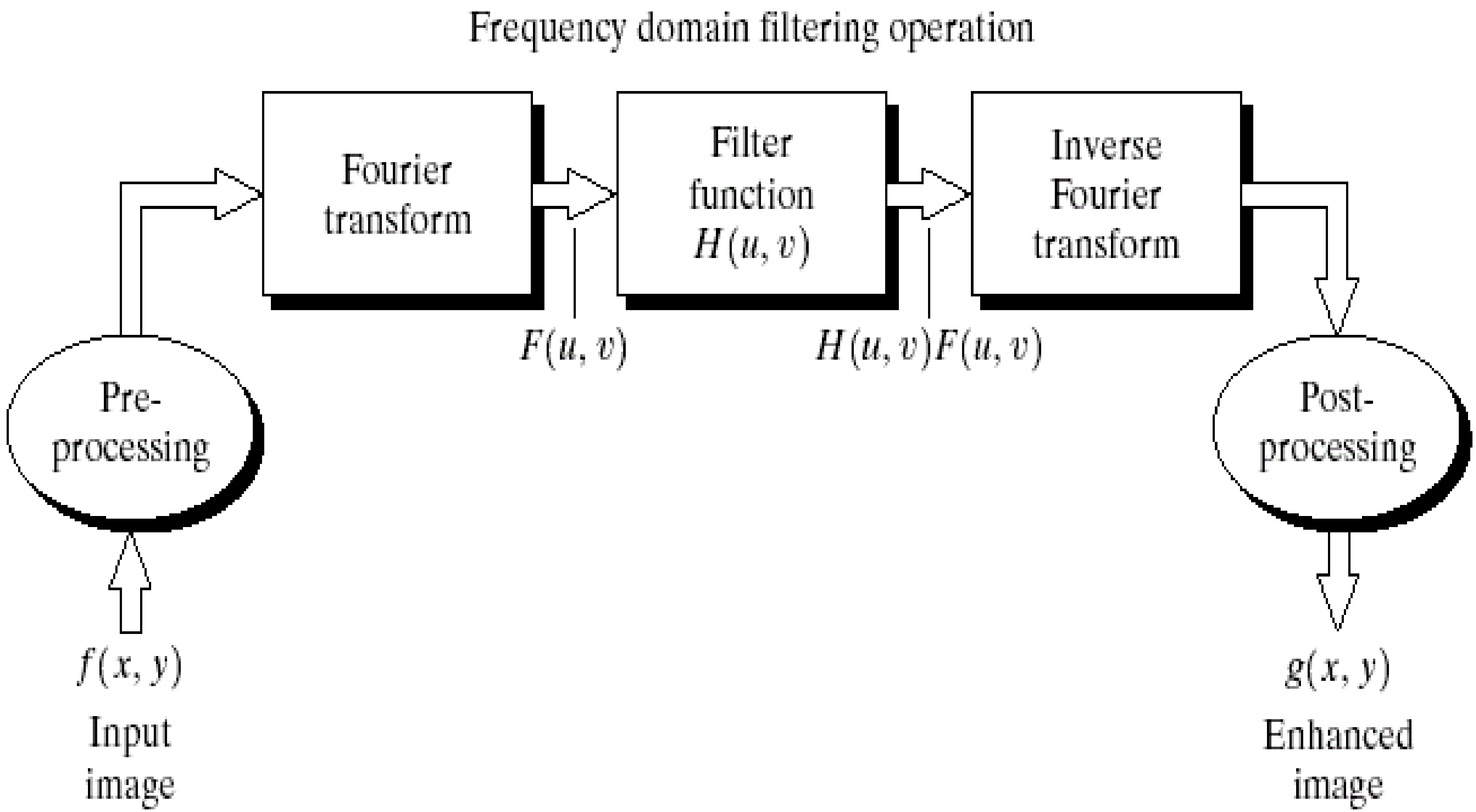


FIGURE 4.5 Basic steps for filtering in the frequency domain.

<u>Definition:</u> zero-shift filters: filter transfer function that affects the real and imaginary parts of $F(u,v)$ in the same manner.

<u>2-D Ideal low pass filter: ILPF</u>

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \sqrt{u^2+v^2}$$

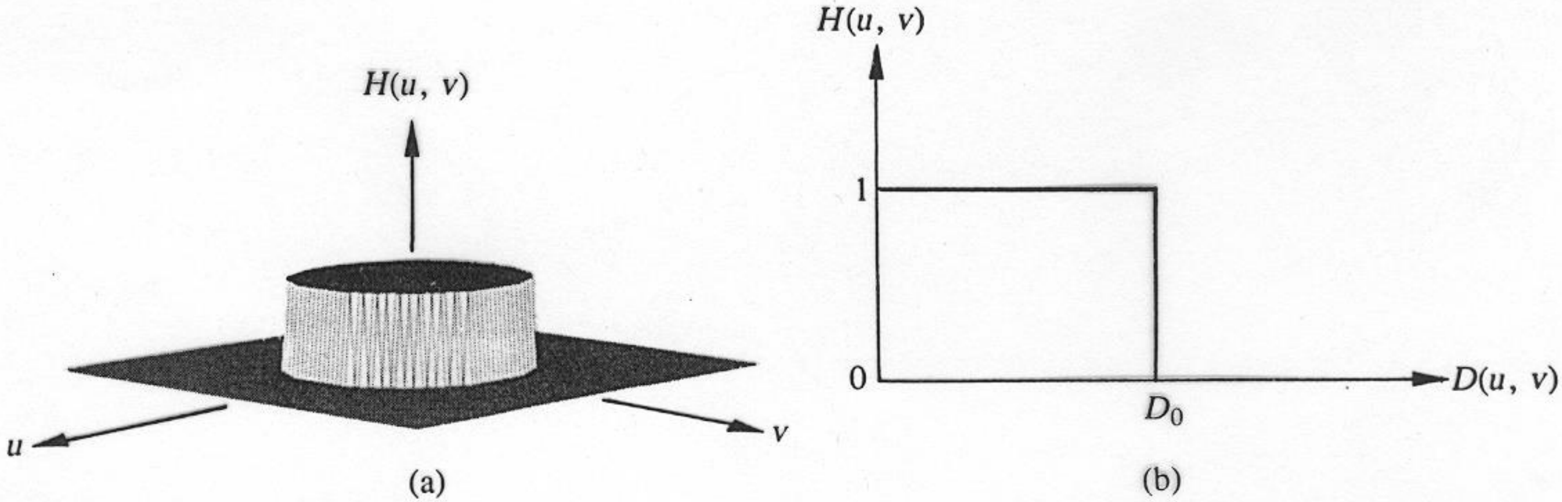


Figure 4.30 *(a) Perspective plot of an ideal lowpass filter transfer function; (b) filter cross section.*

* All frequencies inside a circle in (u,v) plane centred @ 0 and centered with radius D_0 are passed without attenuation whereas all frequencies outside this circle are completely attenuated.

* D_0: cut-off frequency.

<u>Remark</u>: The sharp cut-off frequencies on an ideal low pass filter cannot be realized with electronic components although they can be simulated on a computer (transistor: rising time is not zero)

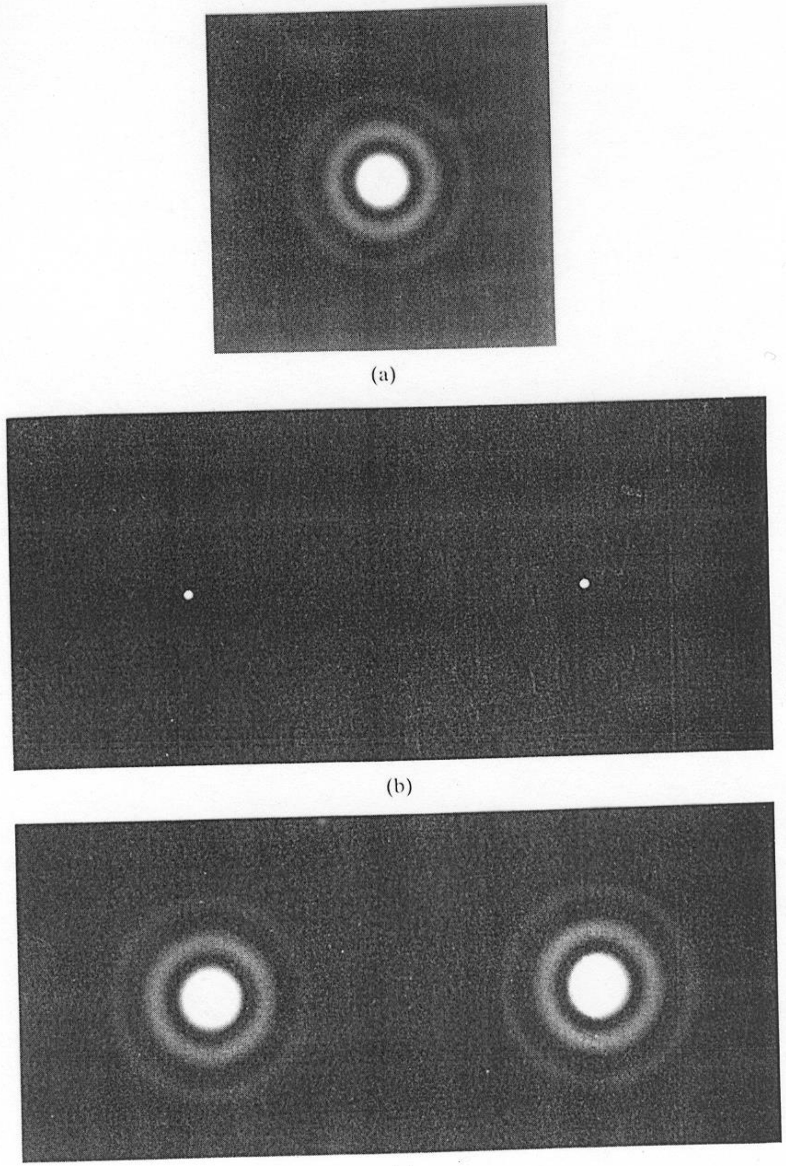


Figure 4.33 Illustration of the blurring process in the spatial domain: (a) blurring function $h(x, y)$ for an ideal lowpass filter; (b) a simple image composed of two bright dots; (c) convolution of $h(x, y)$ and $f(x, y)$.

Non-ideal Low Pass filter: Butterworth filter.

$$H(u,v) = \frac{1}{1+\left(\frac{D(u,v)}{D_0}\right)^{2n}}$$

Low P. F Butterworth filter of order n.

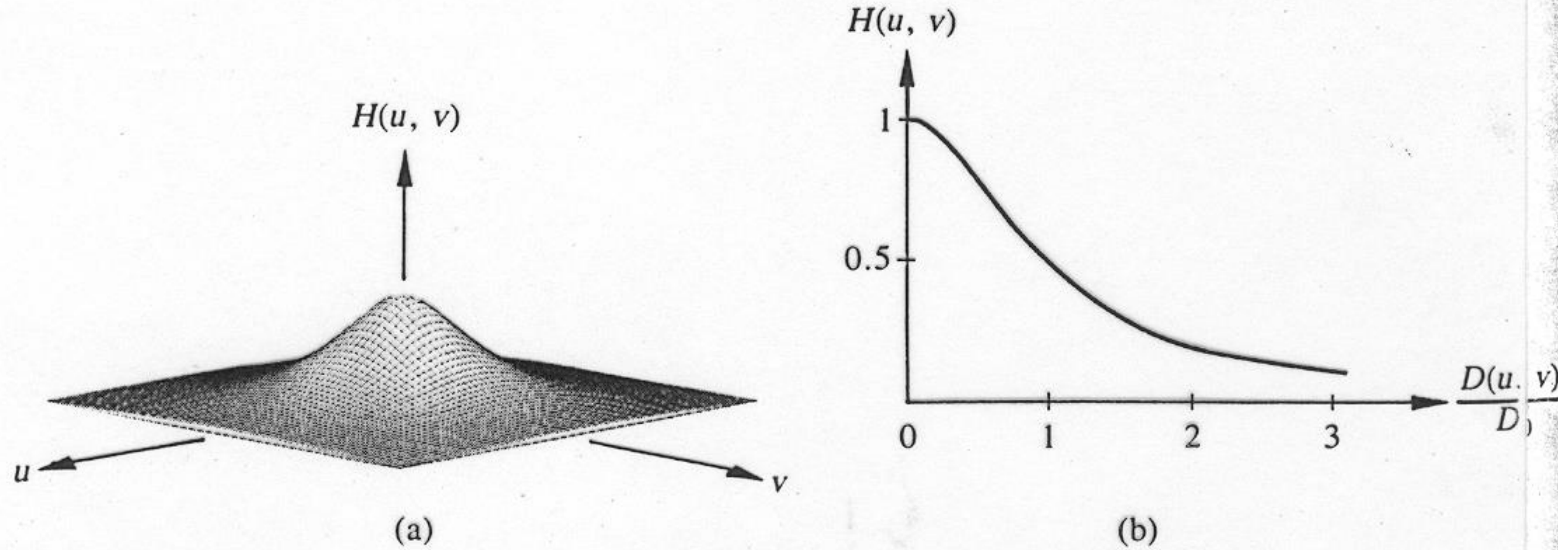
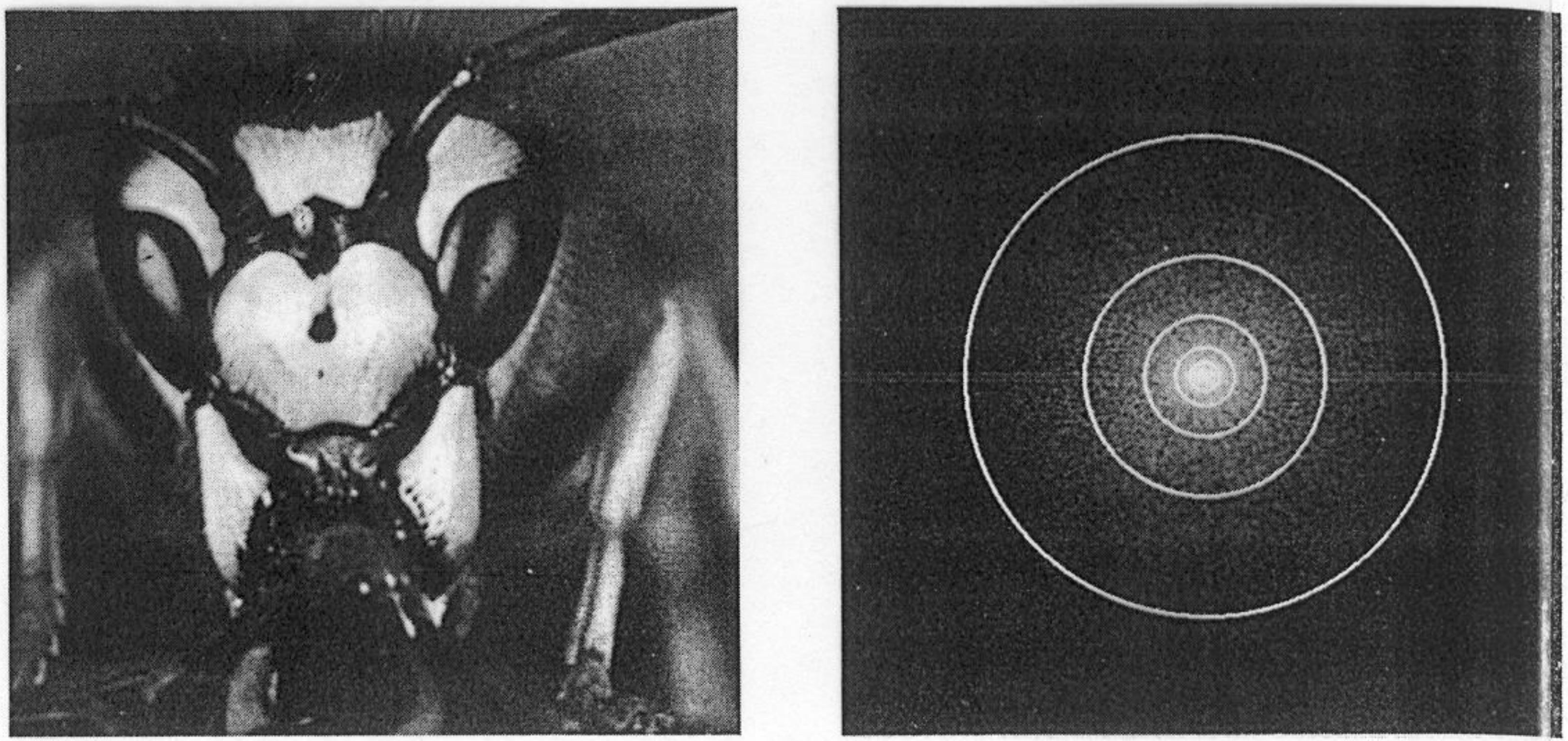


Figure 4.34 *(a) A Butterworth lowpass filter; (b) radial cross section for n = 1.*



(a) (b)

Figure 4.31 *(a) 512 × 512 image and (b) its Fourier spectrum. The superimposed circles, which have radii equal to 8, 18, 43, 78, and 152, enclose 90, 93, 95, 99, and 99.5 percent of the image power, respectively.*

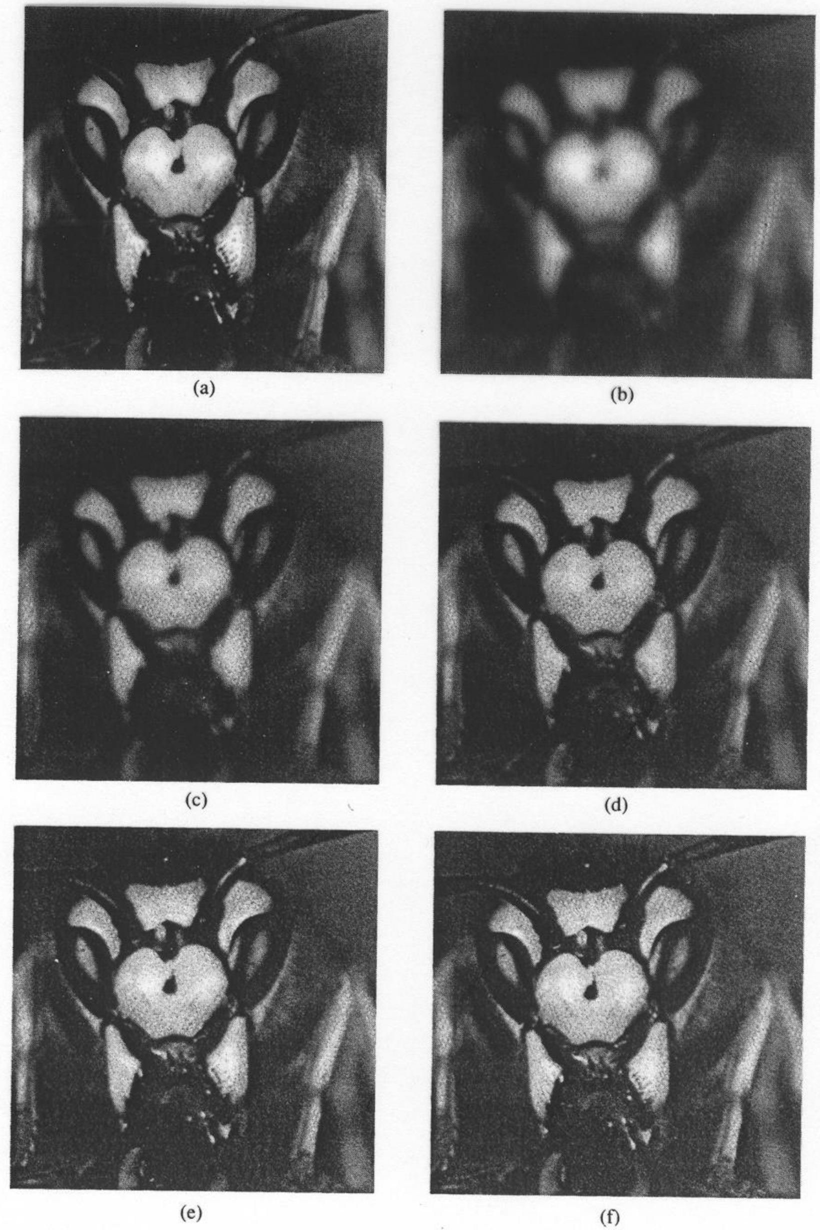


Figure 4.35 *(a) Original image; (b)–(f) results of Butterworth lowpass filtering with the cutoff point set at the radii shown in Fig. 4.31(b).*

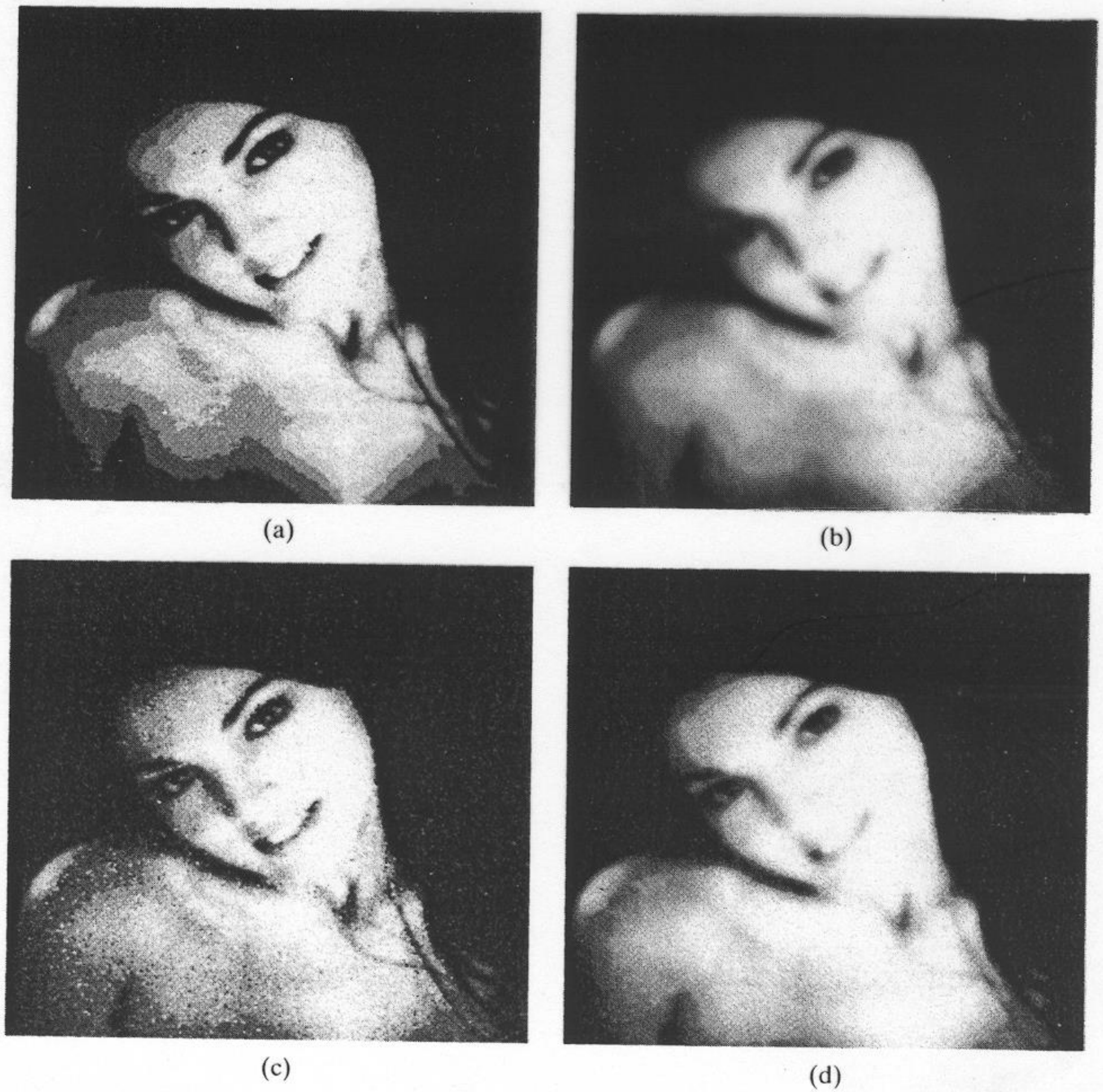


Figure 4.36 Two examples of image smoothing by lowpass filtering (see text).

2-D ideal High-Pass Filter: IHPF

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{elsewhere.} \end{cases}$$

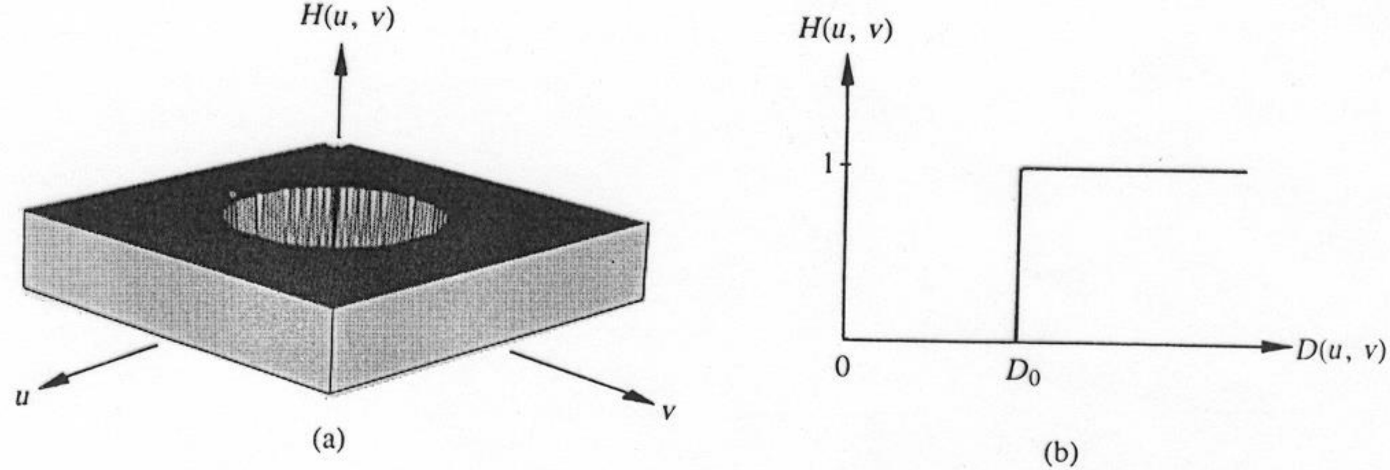


Figure 4.37 *Perspective plot and radial cross section of ideal highpass filter.*

Non-ideal 2D High Pass Butterworth Filter

$$H(u,v) = \begin{cases} \dfrac{1}{1+\left(\dfrac{D_0}{D(u,v)}\right)^{2n}} & \text{for } D \geq D_0 \\ 0 & \text{elsewhere.} \end{cases}$$

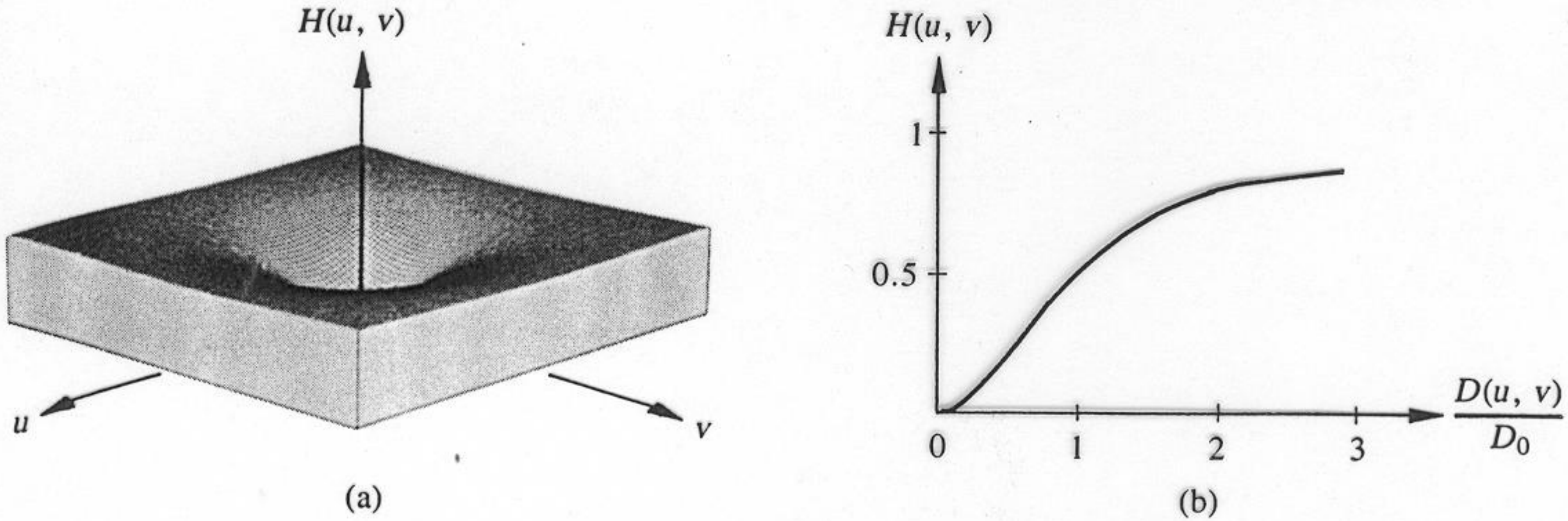


Figure 4.38 *Perspective plot and radial cross section of Butterworth highpass filter for n = 1.*

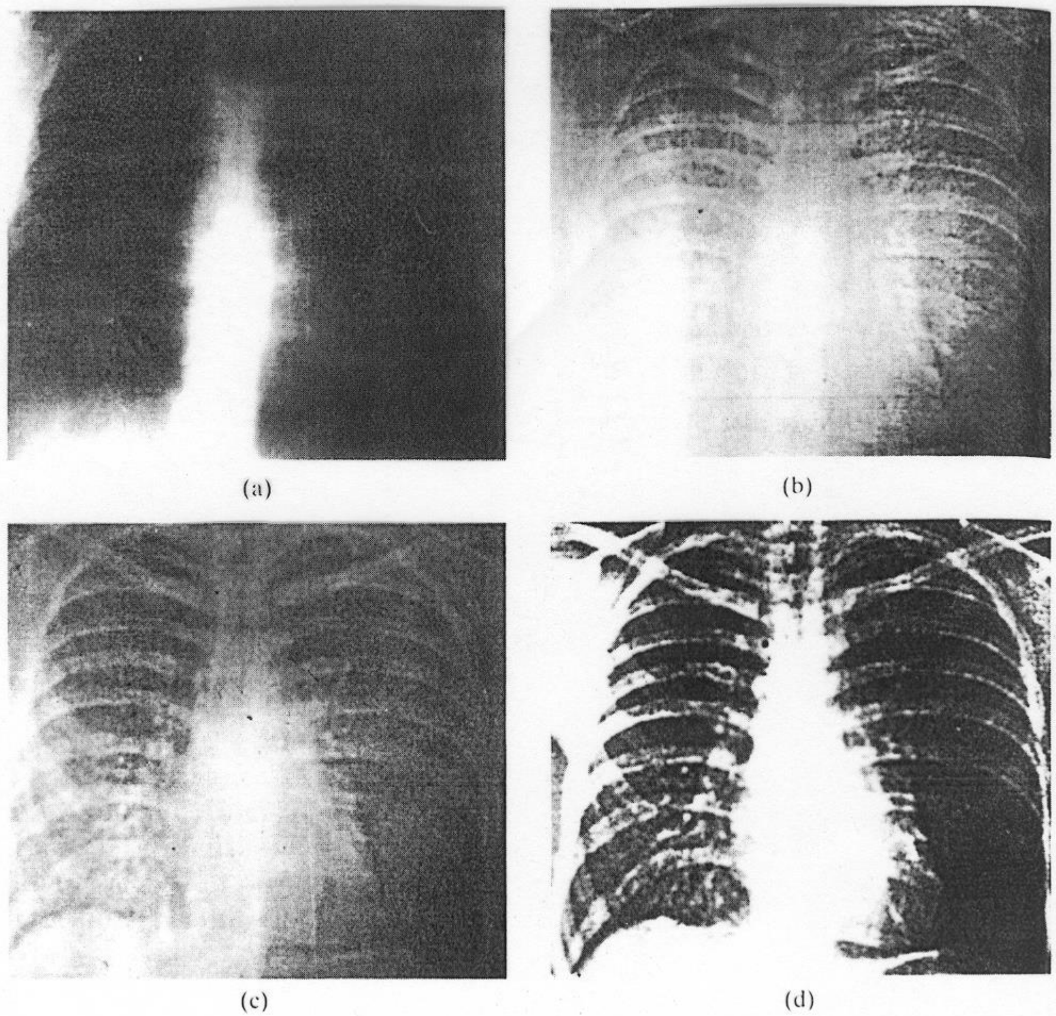
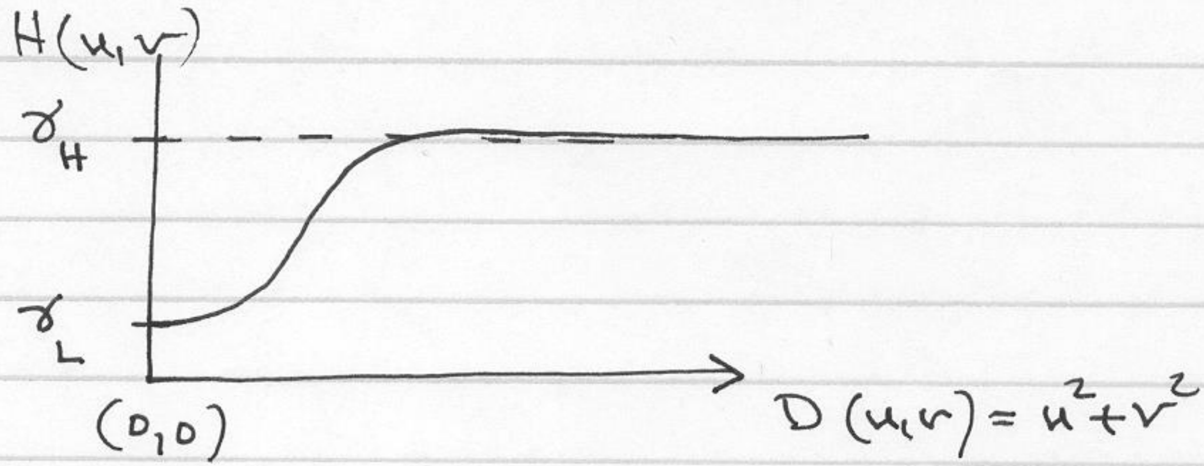
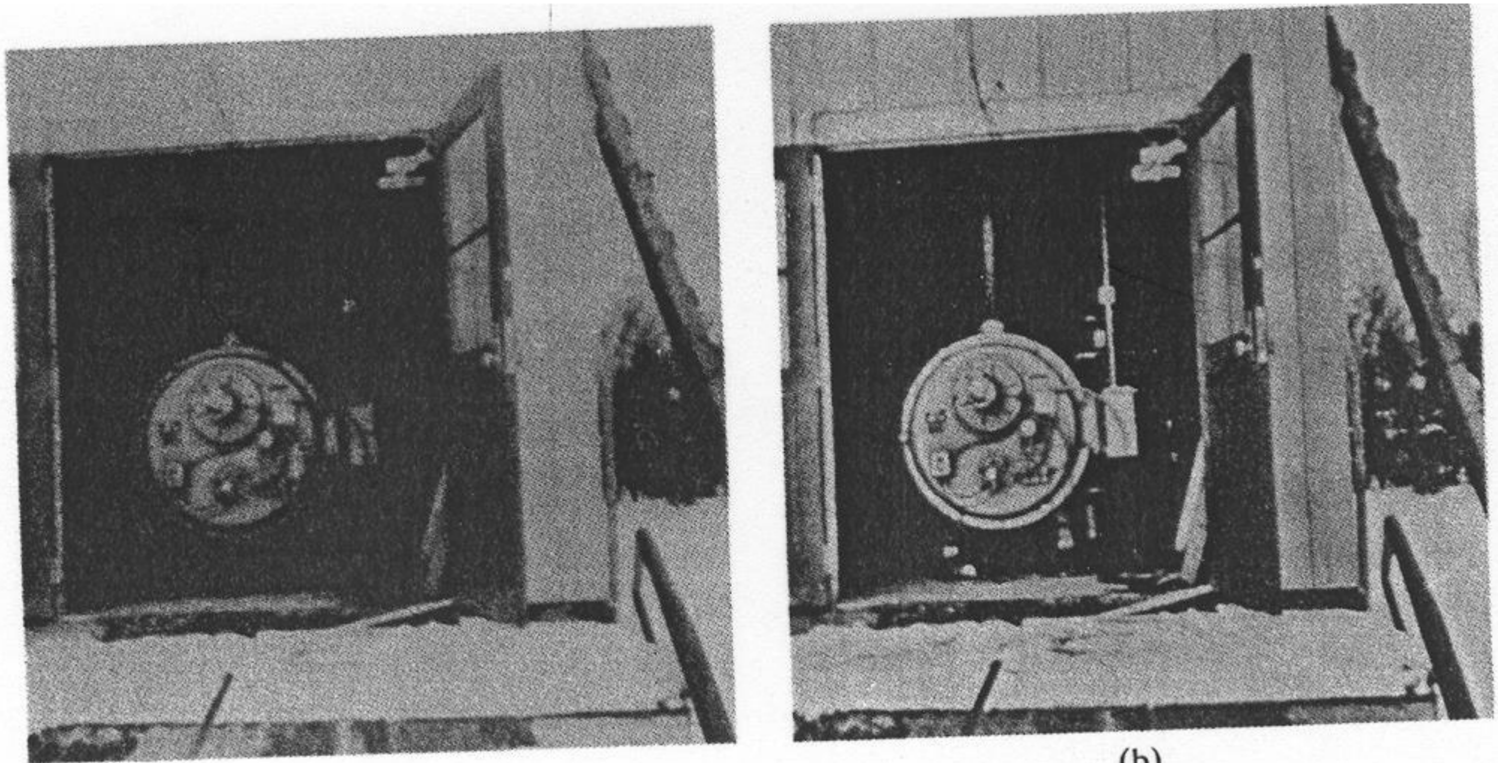


Figure 4.39 *Example of highpass filtering: (a) original image; (b) image processed with a highpass Butterworth filter; (c) result of high-frequency emphasis; (d) high-frequency emphasis and histogram equalization. (From Hall et al. [1971].)*

<u>Example</u>: <u>another filter</u>: homomorphic





(a) (b)

Figure 4.42 *(a) Original image; (b) image processed by homomorphic filtering to achieve simultaneous dynamic range compression and contrast enhancement. (From Stockham [1972].)*