La convolución de dos funciones $f, g : \mathbb{R} \to \mathbb{R}$ continuas es la función $f * g : \mathbb{R} \to \mathbb{R}$ mediante

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

En nuestro problema tenemos:

$$f(x) = \exp\left[-\pi \frac{x^2}{w_a^2}\right] ; \quad g(x) = \exp\left[-\pi \frac{x^2}{w_b^2}\right]$$

Hallamos $f*g$

$$(f*g)(x) = \int_{-\infty}^{\infty} \exp\left[-\frac{\pi t^2}{w_a^2}\right] \exp\left[-\pi \frac{(x-t)^2}{w_b^2}\right] dt.$$

Operamos primero

$$\exp\left[-\pi \frac{(x-t)^2}{w_b^2}\right] = \exp\left[\frac{-\pi}{w_b^2}\left(x^2 - 2xt + t^2\right)\right]$$

Ya que la integral es sobre la variable t, lo que depende solamente de x lo podemos quitar.

$$(f*g)(x) = \exp\left[-\frac{\pi}{w_b^2} x^2\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\pi t^2}{w_a^2}\right] \exp\left[-\frac{\pi}{w_b^2}\left(-2xt + t^2\right)\right] dt$$

$$= \exp\left[-\frac{\pi}{w_b^2} x^2\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\pi}{w_a^2} t^2 + \frac{2\pi}{w_b^2} xt - \frac{\pi}{w_b^2} t^2\right] dt.$$

$$= \exp\left[-\frac{\pi}{w_b^2} x^2\right] \int_{-\infty}^{\infty} \exp\left[-\pi t^2\left(\frac{1}{w_a^2} + \frac{1}{w_b^2}\right) + \frac{2\pi}{w_b^2} xt\right] dt.$$

$$= \exp\left[-\frac{\pi}{w_b^2} x^2\right] \int_{-\infty}^{\infty} \exp\left[-\pi t^2\left(\frac{w_a^2 + w_b^2}{w_a^2 w_b^2}\right) + \frac{2\pi}{w_b^2} xt\right] dt.$$

Realicamos el cambio de variable:

$$\mu = \frac{\sqrt{\pi} \cdot \sqrt{w_a^2 + w_b^2}}{w_a w_b} t \quad \rightarrow d\mu = \frac{\sqrt{\pi} \cdot \sqrt{w_a^2 + w_b^2}}{w_a w_b} dt.$$

$$= \exp\left[-\frac{\pi}{w_b^2} x^2\right] \int_{-\infty}^{\infty} \exp\left[-\mu^2 + \frac{2\pi}{w_b^2}\left(\frac{w_a w_b}{\sqrt{\pi} \cdot \sqrt{w_a^2 + w_b^2}}\right) x\right] \frac{w_a w_b}{\sqrt{\pi}\sqrt{w_a^2 + w_b^2}} d\mu.$$

$$= \frac{w_a w_b}{\sqrt{\pi}\sqrt{w_a^2 + w_b^2}} \exp\left[-\frac{\pi}{w_b^2} x^2\right] \int_{-\infty}^{\infty} \exp\left[-\mu^2 + p\mu\right] d\mu$$

$$\text{donde } p = 2\sqrt{\pi} \cdot \frac{w_a}{w_b} \cdot \frac{1}{\sqrt{w_a^2 + w_b^2}} x$$

Tenemos que

$$\int_{-\infty}^{\infty} \exp\left\{-\mu^2 + p\mu\right\} d\mu = \exp\left\{\frac{p^2}{4}\right\}\sqrt{\pi} = \sqrt{\pi} \exp\left\{\pi \frac{w_a^2}{w_b^2} \cdot \frac{1}{(w_a^2 + w_b^2)} x^2\right\}$$

Nos queda

$$= \frac{w_a w_b}{\sqrt{w_a^2 + w_b^2}} \exp\left[-\frac{\pi}{w_b^2} x^2\right] \exp\left[\pi \frac{w_a^2}{w_b^2} \cdot \frac{1}{(w_a^2 + w_b^2)} x^2\right]$$

$$= \frac{w_a w_b}{\sqrt{w_a^2 + w_b^2}} \exp\left[\underbrace{-\frac{\pi}{w_b^2} x^2 + \pi \frac{w_a^2}{w_b^2} \cdot \frac{1}{w_a^2 + w_b^2} x^2}\right]$$

$$\frac{\pi}{w_b^2}\left(-1 + w_a^2 \cdot \frac{1}{w_a^2 + w_b^2}\right) x^2$$

$$\frac{\pi}{w_b^2}\left(\frac{-w_a^2 - w_b^2 + w_a^2}{w_a^2 + w_b^2}\right) x^2$$

$$-\pi\left(\frac{1}{w_a^2 + w_b^2}\right) x^2$$

Finalmente encontramos

$$\boxed{(f*g)(x) = \frac{w_a w_b}{\sqrt{w_a^2+w_b^2}} \exp\left[-\pi\left(\frac{1}{w_a^2+w_b^2}\right)x^2\right]}$$

Cálculo de la analítico de la convolución entre dos funciones gaussianas.