Problem Set 6

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1. Suppose we have the sets $A = \{1, 2, 3\}$ and $B = \{\pi, 7\}$. How many possible relations are there on A? How many possible functions are there from A to B?

A relation is defined as the subset of the Cartesian product of two sets, which in this case are the same set: $A \times A$. Furthermore, all the possible relations would thus be contained of in the power set of $A \times A$. So, the total number of possible relations in A would be the cardinality of that power set:

$$|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{|A| \times |A|} = 2^9 = 512$$

As for the second part of the question, we know that because we are looking for functions, each preimage (in set A) needs to correspond to exactly one image (in set B). So each of the 3 elements in A could correspond to either of the 2 elements in B, so there are $2^3 = 8$ possible functions.

2. Can you invent a strict partial order on the complex numbers?

Let us define the relation $\vdash$ on the set of all complex numbers C, where $\vdash$ orders the complex numbers by the their real component, in ascending order. For example, take $z_1 = a + bi$ and $z_2 = c + di$, for $z_1, z_2 \in C$. When $a < c$, $z_1 \vdash z_2$ and when $c < a$, $z_2 \vdash z_1$. We will now prove that $\vdash$ is a strict partial order.

Proof. In order to prove that $\vdash$ is a strict partial order, we will show that it satisfies the three necessary conditions:

1. Nonreflexivity

Let $z = a + bi$, for $z \in C$. Clearly, $z \nvdash z$, as $a \not< a$. Thus, $\vdash$ is nonreflexive.

2. Antisymmetry

Let $z_1 = a + bi$ and $z_2 = c + di$ for $z_1, z_2 \in C$, where $a < c$. Therefore, $z_1 \vdash z_2$. But clearly, $z_2 \nvdash z_1$, because $c \not< a$. So, $\vdash$ is antisymmetric.

3. Transitivity

Let $z_1 = a + bi$, $z_2 = c + di$, and $z_3 = e + fi$ for $z_1, z_2, z_3 \in C$. If $z_1 \vdash z_2$ and $z_2 \vdash z_3$, then by the definition of the relation $a < c$ and $c < e$. So then $a < e$ and $z_1 \vdash z_3$. Thus, $\vdash$ is transitive.

Since we have proved that $\vdash$ satisfies all three conditions, it has been shown that it is a strict partial order on the complex numbers. □

3. Find the Image (i.e., the set of all "images" or range) of the function $f : \mathbf{R} \to \mathbf{R}$ given by the polynomial $x^4 + x^3 - x^2 - 1$. Also find the preimage(s) of 1.

The image of the function can simply be derived by finding the range of the polynomial.
First we check the endpoints, finding that clearly both go to $+\infty$:

$$\lim_{x \to \infty} x^4 + x^3 - x^2 - 1 = \infty$$

$$\lim_{x \to -\infty} x^4 + x^3 - x^2 - 1 = \infty$$

Next, we find the critical and inflection points, and then locate the absolute minimum:

$$\frac{d}{dx}(x^4 + x^3 - x^2 - 1) = 4x^3 + 3x^2 - 2x = 0$$

$$x \approx -1.175, 0, 0.425$$
$$\frac{d^2}{dx^2}(4x^3 + 3x^2 - 2x) = 12x^2 + 6x - 2$$
$$f''(-1.175) = 7.5175, f''(0) = -2, f''(0.425) = 2.175$$
$$f(-1.175) = -2.097, f(0.425) = -1.071$$
$$min \approx -2.097$$

So, the image is approximately $[-2.1, \infty)$

We can find the preimage(s) of 1 by setting:

$$1 = x^4 + x^3 - x^2 - 1$$

Using a calculator, we get the preimages of approximately 1.157 and −1.854.

4. Consider the set E^* of all points in the Euclidean plane except for the origin. Define a relation on E^* by declaring that $P_1 \sim P_2$ whenever $x_1y_2 = x_2y_1$. Prove that $\sim$ is an equivalence relation, and describe its equivalence classes.

Proof. In order to prove that $\sim$ is an equivalence relation, we must prove that it satisfies all three properties of an equivalence relation, as follows:

1. Reflexivity

Take the points $P_1 = (x, y)$ and $P_2 = (x, y)$ for $P_1, P_2 \in E^*$. Clearly, $xy = xy$ ($P_1 \sim P_2$), so $\sim$ is reflexive.

2. Symmetry

Take the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ for $P_1, P_2 \in E^*$. If $P_1 \sim P_2$, then we have that $x_1y_2 = x_2y_1$. This is clearly means that similarly, $x_2y_1 = x_1y_2$, thus establishing that $P_2 \sim P_1$. Therefore, $\sim$ is symmetric.

3. Transitivity

Take the points, $P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)$ for $P_1, P_2, P_3 \in E^*$. Suppose that $P_1 \sim P_2$ and $P_2 \sim P_3$, so $x_1y_2 = x_2y_1$ and $x_2y_3 = x_3y_2$. If we multiply the two equations, we get:

$$x_1y_2x_2y_3 = x_2y_1x_3y_2$$
$$x_1y_3 = x_3y_1$$

Which shows that $P_1 \sim P_3$, so $\sim$ is transitive.

Thus, we have proved that $\sim$ has satisfied all three properties of an equivalence relation. □

Now for the equivalence classes. For the class $[(x_2, y_2)]$, where $(x_2, y_2) \in E^*$, we a member of that class, $(x_1, y_1) \in E^*$, so that $x_1y_2 = x_2y_1$ as defined. Rewriting this equation to isolate the constants, we have: $\frac{x_1}{y_1} = \frac{x_2}{y_2}$, or $\frac{x_1}{y_1} = z$ for $z \in \mathcal{Z}$. This equation clearly falls into the form of a line through the origin (but not including the origin because the E^* excludes the origin). Thus describes the equivalence classes of $\sim$.

5. Suppose we define R^{-1} to be the inverse of the relation R on a set A. For example, if $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 3)\}$ then $R^{-1} = \{(1, 1), (3, 1)\}$. If a relation R is transitive, is R^{-1} necessarily transitive? What would need to be true about R and its inverse to guarantee that they are both symmetric?

Take the elements x_1 and x_2, where $x_1 = x_2$. As relation R is defined as transitive, x_1 relates to x_2 under R. Given R^{-1} is the inverse of R, then x_2 relates to x_1, so R^{-1} is also transitive.

Moving on, take two elements, x and y. If we say that under R, x relates to y, y would relate to x under R^{-1}. For R to be symmetric, then y would also need to relate to x under it. And for R^{-1} to be symmetric, x would need to relate y under it. So, would have that under R, x relates to y and y relates to x, and under R^{-1}, y relates to x and x relates to y. So clearly, for R and R^{-1} to both by symmetric, they would have to be equivalent to eachother.