1.10.1] To show $\rho = \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix} = I - 2uu^T$

Given $= I - 2uu^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\begin{bmatrix} d_1 & d_2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix}$

$= \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & -2d_2^2 \end{bmatrix}$ LHS = RHS

Hence proven

1.10.2] (a) $u = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$\rho = I - 2uu^T$

$\rho = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}\begin{bmatrix} 3/5 & 4/5 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix} = \begin{bmatrix} 1-18/25 & -24/25 \\ -24/25 & 1-32/25 \end{bmatrix}$

$= \begin{bmatrix} 7/25 & -24/25 \\ -24/25 & -7/25 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 7 & -24 \\ -24 & -7 \end{bmatrix}$

(b) $x = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ $u = \frac{x-y}{\|x-y\|}$

$\therefore u = \frac{1}{\left\|\begin{bmatrix} 4 \\ -2 \end{bmatrix}\right\|}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{20}}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \frac{1}{2\sqrt{5}}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$

$$Q = I - 2uu^T$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8/5 & -4/5 \\ -4/5 & 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

1.10.3 (a) $Q = I - 2uu^T$ If symmetric $Q^T = Q$

$$Q^T = (I - 2uu^T)^T = I^T - (2uu^T)^T$$
$$= I - 2(u^T)^T u^T$$
$$= I - 2uu^T = Q$$

$\therefore$ Household matrix $Q$ is symmetric

(b) $Q$ is orthogonal if $Q^T Q = I$

$$\therefore Q^T Q = (I - 2uu^T)^T (I - 2uu^T)$$
$$= (I - 2uu^T)(I - 2uu^T)$$
$$= (I - 2uu^T)^2 = I^2 - 4uu^T + (2uu^T)^2$$
$$= I - 4uu^T + (2uu^T)(2uu^T)$$
$$= I - 4uu^T + 4(u)(u^T u)(u^T)$$
$$= I - 4uu^T + 4\|u\|^2 uu^T \qquad [\|u\|^2 = 1]$$
$$= I - \cancel{4uu^T} + \cancel{4uu^T}$$
$$= I \qquad \therefore Q \text{ is orthogonal}$$

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c) $Q^2 = I$

$Q^2 = (I - 2uu^T)^2$

$(I - 2uu^T)(I - 2uu^T)$

$= I^2 - 4uu^T + 4(uu^T)^2$

$= I^2 - 4uu^T + 4(uu^T)(uu^T)$

$= I^2 - 4uu^T + 4\|u\|^2(uu^T)$

$= I - 4uu^T + 4(1)(uu^T)$

$= I - 4uu^T + 4uu^T$

$= I$ Hence proven

1.10.4] To prove $Qv = \begin{cases} -v & \text{if } v \text{ is in span}\{u\} \\ v & \text{if } v \cdot u = 0 \end{cases}$

Given $Q = I - 2uu^T$

•) $Qv = (I - 2uu^T)v$

$= v - 2u(u^Tv)$

if $v \in \text{span}\{u\}$ $v = \alpha u$

$Qv = \alpha u - 2\alpha u^T u\, u$

$= \alpha u - 2\alpha\|u\|^2 u$ $[u^Tu = \|u\|^2 = 1]$

$= \alpha u - 2\alpha u$

$= -\alpha u = -v$ $\therefore Qv = -v$ if $v \in \text{span}\{u\}$

•) if $v \cdot u = 0$ $\therefore u^Tv = 0$

$Qv = v - 2u(u^Tv)$

$= v - 2u(0)$

$= v$ $\therefore Qv = v$ if $v \cdot u = 0$

1.10.5) $x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ , $y = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

To prove $Qx = y$ with $\|x\| = \|y\|$

$Q = I - 2uu^T$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2uu^T$ where $u = \dfrac{x-y}{\|x-y\|}$

$\|x-y\| = \left\| \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} \right\| = \sqrt{4+4+4} = \sqrt{12}$

$x - y = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$ $\therefore u = \dfrac{1}{\sqrt{12}} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$

$\therefore Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} -2/\sqrt{12} \\ 2/\sqrt{12} \\ 2/\sqrt{12} \end{bmatrix} \begin{bmatrix} -2/\sqrt{12} & 2/\sqrt{12} & 2/\sqrt{12} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 4/12 & -4/12 & -4/12 \\ -4/12 & 4/12 & 4/12 \\ -4/12 & 4/12 & 4/12 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/3 & -2/3 & -2/3 \\ -2/3 & 2/3 & 2/3 \\ -2/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

$$\varphi x = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 + 4/3 + 4/3 \\ 2/3 + 2/3 - 4/3 \\ 2/3 - 4/3 + 2/3 \end{bmatrix} = \begin{bmatrix} 9/3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = y$$

$\therefore \varphi x = y$

1.10.6

(a) $H_1 = I - \frac{2uu^T}{u^Tu}$ where $u = \frac{x-y}{\|x-y\|}$

$H_1$ is orthogonal if $H_1^T H_1 = I$

$$H_1^T H_1 = \left(I - \frac{2uu^T}{u^Tu}\right)^T \left(I - \frac{2uu^T}{u^Tu}\right)$$

$$= \left(I - \frac{2uu^T}{u^Tu}\right)\left(I - \frac{2uu^T}{u^Tu}\right)$$

$$= \left(I - \frac{2uu^T}{u^Tu}\right)^2 = I - \frac{4uu^T}{u^Tu} + 4\left(\frac{uu^T}{u^Tu}\right)^2$$

$$= I - \frac{4uu^T}{u^Tu} + 4\frac{(uu^T)^2}{(u^Tu)^2} = I - \frac{4uu^T}{u^Tu} + 4\frac{(uu^T)(uu^T)}{(u^Tu)(u^Tu)}$$

$$= I - \frac{4uu^T}{u^Tu} + \frac{4u\cancel{(u^Tu)}u^T}{\cancel{(u^Tu)}(u^Tu)} = I - \cancel{\frac{4uu^T}{u^Tu}} + \cancel{\frac{4uu^T}{u^Tu}}$$

$$= I$$

$\therefore$ $H_1$ is orthogonal

$H_1$ is symmetric as $H_1^T = H_1$

$$\left(I - \frac{2uu^T}{u^Tu}\right)^T = I^T - \frac{2(uu^T)^T}{(u^Tu)^T}$$

$$= I^T - \frac{2uu^T}{u^Tu} = I - \frac{2uu^T}{u^Tu}$$

$\therefore$ $H_1$ is orthogonal and symmetric

(b) To show that $A_1 = H_1 A H_1$ has the same eigenvalues as $A$

Since $H_1$ is an orthogonal matrix: $H_1^T = H_1^{-1}$, $H_1 H_1^T = I$

we know $Av = \lambda v$ ($\lambda$ is eigenvalue of $A$) ($v$ is eigenvector)

$$\therefore H_1 A v = H_1 \lambda v$$

$$H_1 A v = \lambda H_1 v$$

Take $w = H_1 v$ [change of basis where $v \neq 0$, $w \neq 0$]

$$A_1 = H_1 A H_1$$

$A_1 w = H_1 A H_1 w$ [multiply both sides by $w$]

$A_1 (H_1 v) = H_1 A H_1 (H_1 v) \Rightarrow A_1 (H_1 v) = H_1 A H_1 w$

where $H_1 w = H_1 (H_1 v) = v$ as $v$ is aligned with $H_1$ so applying $H_1$ again will reflect it back to its original position

$$\therefore A_1 (H_1 v) = H_1 A v \Rightarrow A_1 (H_1 v) = H_1 (\lambda v)$$

$$A_1 (H_1 v) = \lambda (H_1 v)$$

$\therefore \lambda$ is an eigenvalue of both $A$ and $A_1$

(c) To show $A_1 = H_1 A H_1$

$$\begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q_{11} & q_{12} & q_{13} \\ 0 & q_{21} & q_{22} & q_{23} \\ 0 & q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q_{11} & q_{12} & q_{13} \\ 0 & q_{21} & q_{22} & q_{23} \\ 0 & q_{31} & q_{32} & q_{33} \end{bmatrix}$$

$$RHS = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}$$

$A_{12}$, $A_{21}$, $A_{22}$

$\Rightarrow$ Block Multiplication for $H_1 A$:

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \left(Q_1 \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}\right) & \left(Q_1 \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}\right) \end{bmatrix}$$

$\Rightarrow$ Block Multiplication for $(H_1 A) \cdot H_1$

$A_{12}$

$$= \begin{bmatrix} a_{11} & a_{12} \;\; a_{13} \;\; a_{14} \\ Q_1 \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} & Q_1 \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}$$

$A_{21}$, $A_{22}$

$$= \begin{bmatrix} a_{11} & [a_{12} \;\; a_{13} \;\; a_{14}] Q_1^T \\ Q_1 \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} & Q_1 \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} Q_1^T \end{bmatrix} = \begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

where $\|x\| = \sqrt{a_{21}^2 + a_{31}^2 + a_{41}^2}$

as $Q_1$ zeroes out subdiagonal entries and $Q_1 A_{22} Q_1^T = B_{22}$

To find upper Hessenberg form   __/__/__

(d) $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ -2 & -1 & -5 & -1 \\ 4 & -3 & 0 & 2 \\ 4 & 2 & 3 & 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}$   $\vec{w} : \begin{bmatrix} \pm\|x\| \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$   $\vec{v} = \vec{w} - \vec{x} = \begin{bmatrix} 8 \\ -4 \\ -4 \end{bmatrix}$

$$P = \frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} = \frac{1}{96}\begin{bmatrix} 64 & -32 & -32 \\ -32 & 16 & 16 \\ -32 & 16 & 16 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 1/6 & 1/6 \\ -1/3 & 1/6 & 1/6 \end{bmatrix}$$

$$\hat{H}_1 = I - 2P = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3 & 2/3 & 2/3 \\ 0 & 2/3 & 2/3 & -1/3 \\ 0 & 2/3 & -1/3 & 2/3 \end{bmatrix}$$

$$H_1AH_1 = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 6 & -1/3 & 11/3 & 7/3 \\ 0 & -10/3 & -13/3 & 1/3 \\ 0 & 5/3 & -4/3 & -2/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3 & 2/3 & 2/3 \\ 0 & 2/3 & 2/3 & -1/3 \\ 0 & 2/3 & -1/3 & 2/3 \end{bmatrix}$$

$$H_1 A H_1 = \begin{bmatrix} 1 & 5/3 & -4/3 & 11/3 \\ 6 & 37/9 & 13/9 & 1/9 \\ 0 & -14/9 & -47/9 & -5/9 \\ 0 & -17/9 & 4/9 & 10/9 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -14/9 \\ -17/9 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} \pm\|x\| \\ 0 \end{bmatrix}$$

$$\|x\| = \sqrt{\left(\frac{-14}{9}\right)^2 + \left(\frac{-17}{9}\right)^2} = \frac{\sqrt{485}}{9} = 2.446$$

$$\vec{w} = \begin{bmatrix} 2.446 \\ 0 \end{bmatrix} \qquad \vec{v} = \vec{w} - \vec{x} = \begin{bmatrix} 4 \\ 1.888 \end{bmatrix}$$

$$P = \frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} = \begin{bmatrix} 16 & 7.552 \\ 7.552 & 3.564 \end{bmatrix} \cdot \frac{1}{19.564} = \begin{bmatrix} 0.817 & 0.386 \\ 0.386 & 0.182 \end{bmatrix}$$

$$\hat{H}_2 = I - 2P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.634 & 0.772 \\ 0.772 & 0.364 \end{bmatrix} = \begin{bmatrix} -0.634 & -0.772 \\ -0.772 & 0.636 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.634 & -0.772 \\ 0 & 0 & -0.772 & 0.636 \end{bmatrix}$$

$$H_2 (H_1 A H_1) H_2 = \begin{bmatrix} 1 & 1.\overline{66} & -1.9853 & 3.361 \\ 6 & 4.\overline{11} & -1.001\overline{5} & -1.0\overline{4} \\ 0 & 2.\overline{4} & -1.4912 & -2.612 \\ 0 & 0 & -3.6118 & -2.608 \end{bmatrix}$$

(e) To explain why any upper Hessenberg form of a symmetric matrix is tridiagonal

If we have symmetric matrix $A \Rightarrow A = A^T$ must hold at every step of the transformation

we know $H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & h_{33} & \cdots & h_{3n} \\ \vdots & 0 & h_{43} & \ddots & h_{4n} \\ 0 & 0 & 0 & \cdots & h_{nn} \end{bmatrix}$ $h_{ij} = 0 \ \forall \ i > j+1$

Since A is symmetric $a_{ij} = a_{ji}$

$a_{ij} = 0$ for $i > j+1$ (below first subdiagonal)

$\Rightarrow a_{ji} = 0$ for $j > i+1$ [since symmetric]

(above first superdiagonal)

- Upper Hessenberg condition zeroes all entries below first subdiagonal

$a_{ij} = 0 \ \forall \ i > j+1$

- Symmetry forces all corresponding entries above the first superdiagonal to also be zero

$a_{ji} = 0 \ \forall \ j > i+1$

- Potentially nonzero entries are:

• diagonal entries $a_{ii}$ for $i = 1, 2, \ldots n$

• first subdiagonal entries $a_{i+1,i}$ for $i = 1, 2, \ldots n-1$

• first superdiagonal entries $a_{i,i+1}$ for $i = 1, 2, \ldots n-1$

$\therefore A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ 0 & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & & \ddots & a_{n-1,n} \\ 0 & 0 & 0 & a_{n,n-1} & a_{nn} \end{bmatrix}$ $\therefore$ Resulting matrix has to be tridiagonal