Mathematical formulae

Algebra

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, a, b, c are real numbers.

$y = ax^2 + bx + c$ represents a parabola.

- If a > then the parbola is concave upwards.
- If a < 0 then the parabola is concave downwards.

Roots of the quadratic equation are given by $x = (-b \pm \sqrt{(b^2 - 4ac)})/2a$.

$D = b^2 - 4ac$ is called **discriminant.**

The roots are (i) real and distinct if D > 0 (ii) real and equal if D = 0 and (iii) imaginary if D < 0.

Logarithm:

Let a and x be real numbers such that $a \neq 1$, $a > 0$.

Let $y = a^x$. Then x is called the logarithm of y to the base a and we write $x = \log_a y$.

$\therefore x = \log_a y \Leftrightarrow a^x = y$

Since $a^x > 0$, for every real x, logarithm of negative real numbers and logarithm of zero are not defined.

Arithmetic Progression with first term a and common difference d is a sequence of the form

a, a + d, a + 2d, … , a + (n-)d, …

- n^{th} term, $t_n = a + (n - 1)\, d$.
- Sum of the first n terms, $S_n = \frac{n}{2}\left[2a + (n - 1)\, d\right] = \frac{n}{2}\left[a + \ell\right]$ where *l* is the last term.

Geometric Progression with first term a & constant ratio r is a sequence of the form

$a, ar, ar^2, \ldots, ar^n, \ldots$

- n^{th} term = $t_n = a\, r^{n-1}$
- The sum of the first n terms, $S_n = \frac{a\,(1 - r^n)}{1 - r}$, $r < 1$ or $\frac{a\,(r^n - 1)}{r - 1}$, $r > 1$ or na if r = 1
- If $|\, r \,| < 1$, then the sum to infinity of the G.P. = $\frac{a}{1 - r}$

Harmonic Progression is a sequence in which the reciprocals of the terms are in arithmetic progression.

Given n numbers $a_1, a_2, a_3, \ldots, a_n$, we define

- Arithmetic Mean: AM = $\frac{a_1+a_2+\cdots+a_n}{n}$
- Geometric Meam: GM= $(a_1 a_2 \dots a_n)^{1/n}$
- Harmonic Mean: HM = $\frac{(a_1^{-1}+a_2^{-1}+\cdots+a_n^{-1})^{-1}}{n}$

Matrix: A system of mn numbers arranged in a rectangular array of m rows & n columns is called a matrix of order m x n. If a_{ij} denotes the element in the i^{th} row and j^{th} column, then the matrix is denoted by A = (a_{ij}).

The matrix is said to be a

- ✓ **row matrix**, if number of rows is 1.
- ✓ **column matrix**, if number of columns is 1.
- ✓ **square matrix**, if number of rows = number of columns.
- ✓ **diagonal matrix,** if it is square and all its elements are zero except the principal diagonal elements.
- ✓ **scalar matrix,** if it is diagonal and the diagonal entries are equal.
- ✓ **unit matrix,** if it is diagonal and each diagonal entry is unity.
- ✓ **null matrix**, if all the elements are zero.
- ✓ **upper triangular**, if it is square and $a_{ij} = 0$, $i > j$.
- ✓ **lower triangular,** if it is square and $a_{ij} = 0$, $i < j$.

Transpose: Let A be an m × n matrix. The matrix of order n × m obtained by interchanging the rows and columns of A is called the **transpose of A** and is denoted by A′.

A matrix A is said to be (i) **symmetric** if A′ = A (ii) **skew symmetric** if A′ = - A.

- A + A′ and AA′ are symmetric (ii) A – A′ is skew symmetric.

Every square matrix can be uniquely expressed as sum of two square matrices of which one is symmetric and the other is skew symmetric. i.e. A = (A + A′)/2 + (A-A′)/2

A matrix A is

(i) orthogonal if AA′= I i.e., $A^{-1} = A'$, (ii) **an involutory** matrix if $A^2 = I$ i.e. $A^{-1} = A$, (iii) **an idempotent** matrix if $A^2 = A$, (iv) **a nilpotent** if there is a positive integer m such that $A^m = 0$.

Determinant:

Let $A=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, then the determinant of A id defined as $|A| = a_1b_2 - b_1 a_2$.

If $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, then $|A| = a_1(b_2 c_3 - c_2 b_3) - a_2(b_1 c_3 - c_1 b_3) + a_3(b_1 c_2 - c_1 b_2)$.

A matrix is said to be **singular** if |A| = 0. It is **non-singular** if |A| ≠ 0.

A matrix A is said to be invertible if there exists a matrix B such that AB = BA = I.

B is denoted by A^{-1}. Inverse of a matrix A exists iff A is non-singular.

Eigen Values: Let A be a square matrix and I be the identity matrix of the sane order then,

1) A - λI is called the characteristic matrix of A.
2) |A - λI| is called the characteristic polynomial of A.
3) |A - λI| = 0 is called the characteristic equation of A.
4) The roots of |A -λI| = 0, are called the characteristic roots or eigen values or latent roots of A. If A is of order n, then it has n eigen values.

Properties of eigen values::

Let A be an $n \times n$ matrix. Assume that A has n distinct eigen values say, $\lambda_1, \lambda_2, ..., \lambda_n$ then

- the eigen values of A^T are $\lambda_1, \lambda_2, ..., \lambda_n$
- the eigen values of A^{-1} (if it exists) are $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, ..., \dfrac{1}{\lambda_n}$
- the eigen values of the matrix $A - \alpha I$ are $\lambda_1 - \alpha, \lambda_2 - \alpha, ..., \lambda_n - \alpha$
- for any non negative integer k, the eigen values of A^k are $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$

Property 2.2.

- *For any square matrix A, the sum of the eigen values of A is equal to the sum of the diagonal elements of A. The sum of the eigen values of A is called the **trace** of A, denoted by trace(A).*
- *For any square matrix A, the product of the eigen values of A is equal to the **determinant** A, denoted by* $\det(A)$.
- *If X is an eigen vector of a matrix A corresponding to an eigen value λ then kX is also an eigen vector of A for λ where k is any non-zero number.*

- *If X_1 & X_2 are non zero-eigen vectors of a matrix A corresponding to an eigen value λ then $k_1X_1 + k_2X_2$ is also an eigen vector of A for λ where k_1, k_2 are non-zero numbers.*
- *The eigen vectors corresponding to distinct eigen values of a matrix A are linearly independent.*

Eigen values of some special matrices

- **Diagonal matrix:** The eigen values of a diagonal matrix A are the diagonal entries.

- **Triangular matrix (upper or lower):** The eigen values of a triangular matrix A are the diagonal entries.

- **Symmetric matrix:** The eigen values of a symmetric matrix A are real numbers.
- **Non-symmetric matrix:** The eigen values of a non-symmetric matrix A are either real number or complex conjugate pairs.

Definition 3.1.

- **Orthogonal matrix:** A matrix A is said to be an orthogonal if $AA^T = A^T A = I$.
- **Orthogonal vectors:** The set of n-component vectors $\{v_1, v_2, ..., v_n\}$ is said to be orthogonal if the dot product $v_i \cdot v_j = 0$ if $i \neq j$..
- **Orthonormal vectors:** The set of n-component vectors $\{v_1, v_2, ..., v_n\}$ is said to be orthonormal if $v_i \cdot v_j = 0$ if $i \neq j$. and $v_i.v_i = 1$.

Theorem 4.1. ***The Diagonalization Theorem:*** *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen vectors.*

Theorem 4.2. *An $n \times n$ matrix with n distinct eigen values are diagonalizable.*

Linear Transformation: Let V and W be two vector spaces over a field F. A mapping $T: V \to W$ is said to be a Linear Transformation (L.T.) if T preserves addition and scalar multiplication.

That is T satisfies the following conditions: For every $u, v \in V$ and $\alpha \in F$

1. $T(u + v) = T(u) + T(v)$
2. $T(\alpha u) = \alpha T(u)$.

Remark 1.1. The above definition is equivalent to the following:

T is a linear transformation from V to W if and only if $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for every $u, v \in V$ and $\alpha, \beta \in F$.

Linear Operator: Let V be a vector space over F. Then a linear transformation from V to V is called a **linear operator** on V.

Remark 1.2. Linear transformations are also known as **vector space homomorphisms**.

Linear transformation associated with a matrix: Let A be an $n \times n$ matrix. The linear transformation associated with A is the function $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(v) = Av$ for all $v \in \mathbb{R}^n$.

1. **Range space of a linear transformation**: The **range** of T is a subspace of W, defined as

$$\mathcal{R}_T = \{\, w \in W \mid T(v) = w \text{ for some } v \in V \,\}.$$

2. **Null space of a linear transformation**: The **null space** of T is defined as the set of all vectors in V such that $T(v) = 0$ where 0 is the zero vector in W. That is,

$$\mathcal{N}_T = \{\, v \in V \mid T(v) = 0 \,\}.$$

Remark 3.1.

1. The range space $\mathcal{R}_T$ and the null space $\mathcal{N}_T$ are subspaces of the vector spaces W and V respectively.
2. The null space of T is also known as the **kernel** of T.
3. If $A_{n\times n}$ is the matrix corresponding to a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ then
   (a) the null space of A is $\mathcal{N}_A = \{x \in \mathbb{R}^n \mid Ax = 0\}$.
   (b) the range space of A is $\mathcal{R}_A = \{b \in \mathbb{R}^n \mid Ax = b \text{ for some } x \in \mathbb{R}^n\}$.

Matrix representation of a linear transformation: Let V and W be two vector spaces over the field F and $T\colon V \to W$ be a linear transformation. Let $B = \{e_1, e_2, \ldots, e_n\}$ and $B' = \{f_1, f_2, \ldots, f_m\}$ be the standard bases of V and W respectively.

Suppose that

$$\begin{aligned}
T(e_1) &= \lambda_{11} f_1 + \lambda_{12} f_2 + \cdots + \lambda_{1m} f_m \\
T(e_2) &= \lambda_{21} f_1 + \lambda_{22} f_2 + \cdots + \lambda_{2m} f_m \\
T(e_3) &= \lambda_{31} f_1 + \lambda_{32} f_2 + \cdots + \lambda_{3m} f_m \\
&\vdots \\
T(e_n) &= \lambda_{n1} f_1 + \lambda_{n2} f_2 + \cdots + \lambda_{nm} f_m.
\end{aligned}$$

Then the matrix representation of T w.r.t. the bases B and B' is $[T] = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nm} \end{pmatrix}^T$.

Calculus

Graph of a function y = f(x) is the set given by g = {(x, y)| y = f(x)}.

A function f is said to be differentiable at a point a, if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists. The value of the limit is then called derivative of f at a and is denoted by f ′(a).

Geometrically f ′(a) represents the slope of the tangent to the curve y = f(x) at x = a.

- Linearity Property: $\frac{d(af(x)+bg(x))}{dx} = a\frac{d(f(x))}{dx} + b\frac{d(g(x))}{dx}$, a and b are constants.
- **Product Rule:** $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$. **Quotient Rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- **Chain Rule for Composite Functions:** If y = f (z) and z = g (x), then $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

Derivatives of some standard functions

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	nx^{n-1}	sin x	cos x	$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$, |x |< 1
√x	$\frac{1}{2\sqrt{x}}$	cos x	- sin x	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$, |x |< 1
a^x	$a^x \log a$ $(a > 0, a \neq 1)$	tan x	$\sec^2 x$	$\tan^{-1} x$	$\frac{1}{1+x^2}$
e^x	e^x	sec x	sec x tan x	$\cot^{-1} x$	$\frac{-1}{1+x^2}$

logx	1/x	cosec x	- cosec x cot x	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert \sqrt{x^2-1}}$, $\lvert x\rvert > 1$
		cot x	$-\operatorname{cosec}^2 x$	$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert \sqrt{x^2-1}}$, $\lvert x\rvert > 1$

Mean Value Theorems :

Rolle's Theorem: If f(x) is continuous in closed interval [a, b], differential in open interval (a, b) and if f(a)=f(b) then, there exists atleast one value c of x in (a, b) such that $f'(c) = 0$.

Lagrange Mean value Theorem: If f(x) is continuous in a closed interval [a, b] and differentiable in the open interval (a, b), then there exists atleast one value c of x in (a, b) such that $f'(x) = \frac{f(b)-f(a)}{b-a}$

Cauchy's Mean Value Theorem : Suppose that two functions f(x) and g(x) are such that

1. f(x) and g(x) are continuous in a closed interval [a, b]
2. f(x) and g(x) are differential in the open interval (a, b)
3. $g'(x) \neq 0$, for all x $\in$(a, b).

Then there exists at least one value c $\in$ (a, b) such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Indeterminate forms :

Suppose that $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$. Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ is said to be an indeterminate form of the type 0/0.

L` Hospital's Rule :

Suppose that $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$. In this case,

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = l \quad \Rightarrow \quad \lim_{x \to a} \frac{f(x)}{g(x)} = l.$$

If $\lim_{x \to a} f'(x) = 0$, $\lim_{x \to a} g'(x) = 0$, then $\lim_{x \to a} \frac{f''(x)}{g''(x)} = l \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = l$ and so on.

The other indeterminate forms are $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞. Each such form can be reduced to 0/0 form.

Taylor's series for a function of one variable: If $f(x)$ has derivatives of all orders in an interval containing a, then f(x) can be expressed as series in powers of (x- a) as

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Maclaurin's series is a particular case of Taylor's series with a = 0 and is given by .

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Some standard series

i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

iii) $\tan x = x + \frac{x^3}{3!} + \frac{2}{15}x^5 + \cdots$

iv) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Maxima and Minima of a function of single variable:

A necessary condition for $f(c)$ to be an extreme value of $f(x)$ is that $f'(c) = 0$.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a minimum value of $f(x)$.

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a maximum value of $f(x)$.

Taylor's theorem for function of two variables: $f(a+h, b+k) = f(a,b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots$

Maclaurin's series for a function of two variables:

$$f(x,y) = f(0,0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

Maxima and Minima of a function of two variables:

Let $f(x,y)$ be a function of two variables x, y such that it is continuous and finite for all values of x and y in the neighbourhood of a point (a, b). Then the value of $f(a,b)$ is called maximum or minimum value of $f(x,y)$ according as $f(a+h, b+k), < or >$ $f(a,b)$ for all finite and sufficiently small values of h and k.

Necessary and sufficient conditions for the existence of Maximum and Minimum value of a function f(x,y) :

The necessary condition for an extremum is $\frac{\partial f(a,b)}{\partial x} = 0, \quad \frac{\partial f(a,b)}{\partial y} = 0$

Sufficient Conditions: Let A = f_{xx} (a, b), B = f_{xy} (a, b), C = f_{yy} (a, b).

(i) If AC – B^2 > 0 and A > 0, then f (a, b) is minimum

(ii) If AC – B^2 > 0 and A < 0, then f (a, b) is maximum
(iii) If AC – B^2 < 0, there is neither maximum nor minimum at (a, b)
(iv) If AC – B^2 = 0, then the analysis depends on third degree terms.

1. **Homogeneous function**: $x^n\varphi\left(\frac{y}{x}\right)$ or $y^n\varphi\left(\frac{x}{y}\right)$
2. **Euler theorem for a homogeneous function with degree *n***:
   $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$
3. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$

Errors and Approximations

Let $f(x,y)$ be a continuous function of x and y. Then $\delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y.$

If f is a function of several variables $x,\ y,\ z,\ t$ ..., then $\delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial z}\,\delta z + \frac{\partial f}{\partial t}\,\delta t +$...

Basic Integrals

∫a dx	ax + C	
Variable	∫x dx	$x^2/2$ + C
Square	∫x^2 dx	$x^3/3$ + C
Reciprocal	∫(1/x) dx	ln|x| + C
Exponential	∫e^x dx	e^x + C
	∫a^x dx	a^x/ln(a) + C
	∫ln(x) dx	x ln(x) − x + C
Trigonometry (x in radians)	∫cos(x) dx	sin(x) + C
	∫sin(x) dx	-cos(x) + C
	∫$\sec^2$(x) dx	tan(x) + C

Rules	Function	Integral
Multiplication by constant	∫cf(x) dx	c∫f(x) dx
Power Rule (n≠-1)	∫x^n dx	x^{n+1}**n+1** + C
Sum Rule	∫(f + g) dx	∫f dx + ∫g dx
Difference Rule	∫(f - g) dx	∫f dx - ∫g dx
Integration by Parts	∫u v dx = u∫v dx −∫u' (∫v dx) dx	

Multiple Integrals:

1. Area of the region in the Cartesian form $= \iint_R dx\ dy$
2. Area of the region in the Polar form $= \iint_R r\,dr\,d\theta$
3. Volume $= \iiint_V dx\ dy\,dz = \iint_R z\,dx\,dy.$

4. Volume of a solid obtained by the revolution of a curve enclosing on area A about the intial line is given by the formula $V = \iint_A 2\pi r^2 \, sin\theta \, drd\theta$
5. The change of Cartesian coordinate to cylindrical polar coordinates

$$\iiint_{R_{xyz}} f(x,y,z)dx\,dy\,dz = \iiint_{RI_{r\emptyset z}} f(rcos\emptyset, rsin\emptyset, z)r\,drd\emptyset dz$$

6. The change of Cartesian coordinate to Spherical polar coordinates

$$\iiint_{R_{xyz}} f(x,y,z)dx\,dy\,dz = \iiint_{RI_{r\theta\emptyset}} f(r\,sin\theta cos\emptyset\,, rsin\theta sin\emptyset, rcos\theta)r^2 sin\theta\,drd\theta d\emptyset$$

Beta , gamma functions and Jacobians:

1) $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$
2) $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$
3) $\Gamma(n) = (n-1)\Gamma(n-1)$
4) $\Gamma(n) = (n-1)!$, if n is a positive integer
5) Legendre's duplication formula $\Gamma(n)\Gamma(n+1/2) = \dfrac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$
6) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
7) $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
8) Jacobian J = $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$
9) If $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ $\quad and \quad$ $J^1 = \dfrac{\partial(x,y)}{\partial(u,v)}$ then $JJ^1 = 1$
10) If $J_1 = \dfrac{\partial(u,v)}{\partial(x,y)}$ $\quad and \quad$ $J_2 = \dfrac{\partial(x,y)}{\partial(z,w)}$ then $J_1J_2 = \dfrac{\partial(u,v)}{\partial(z,w)}$

Interpolation

Interpolation formulae:

Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ be a set of tabulated points satisfying y = f(x), where explicit nature of *y* is not known and values of *x* are equally spaced,

Newton-Gregory Forward Difference Interpolation Formula

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n y_0$$

where $x = x_0 + ph$.

Δ is the forward difference operator and the r^{th} order forward differences are obtained using the relation $\Delta^r y_k = \Delta^{r-1} y_{k+1} - \Delta^{r-1} y_k$, r = 1, 2,…

Newton-Gregory Backward Difference Interpolation Formula

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^n y_n$$

where $x = x_n + ph$

The operator ∇ is called the backward difference operator and the r^{th} order backward differences are obtained using the relation

$\nabla^r y_k = \nabla^{r-1} y_k - \nabla^{r-1} y_{k-1}$, r = 1, 2, …

Interpolation for unevenly spaced values of x:

Lagrange's formula

$$y_n(x) = \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots$$
$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n.$$

Newton's divided difference formula

$$y = y_0 + (x-x_0)[x_0,\ x_1] + (x-x_0)(x-x_1)[x_0,\ x_1,\ x_2]$$
$$+(x-x_0)(x-x_1)(x-x_2)\,[x, x_0, x_1,\ x_2] + \ldots$$
$$+(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})\,[x_0, x_1, \ldots, x_n]$$
$$+(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)\,[x, x_0, x_1, \ldots, x_n]$$

where $[x_0,\ x_1] = \dfrac{y_1-y_0}{x_1-x_0}$, $[x_1,\ x_2] = \dfrac{y_2-y_1}{x_2-x_1}$, etc., are the first order divided differences, $[x_0, x_1, x_2] = \dfrac{[x_1,\ x_2]-[x_0,\ x_1]}{x_2-x_0}$, $[x_1, x_2, x_3] = \dfrac{[x_2,\ x_3]-[x_1,\ x_2]}{x_3-x_1}$, … are the second order divided differences and so on.

Numerical Differentiation :

Derivatives using Newton's forward interpolation formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots\right]$$
$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \cdots\right]$$

where $p = \frac{x-x_0}{h}$.

At $x = x_0, p = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \ldots\right]$$

Derivatives using Newton's backward interpolation formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{4!}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{12p^2+36p+22}{4!}\nabla^4 y_n + \cdots\right]$$

where $p = \frac{x-x_n}{h}$.

At $x = x_0, p = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n \ldots\right]$$

Numerical Integration :

Newton-Cotes quadrature formula

$$\int_{x_0}^{x_0+nh} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \ldots\right]$$

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots \ldots + y_{n-1})]$$

Simson's one-third rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots \ldots + y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots + y_{n-2})]$$

Simson's three-eigth rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[\begin{array}{c}(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \ldots + y_{n-1}) \\ +2(y_3 + y_6 + \cdots + y_{n-3})\end{array}\right]$$

Techniques for finding roots of Algebraic and Transcendental Equations

Bisection Method: Let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a and b and its approximate value be given by $x_0 = \frac{a+b}{2}$.

Regula Falsi method / Method of false position: Let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a and b and its approximate value be given by $x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

Newton- Raphson Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0,1,2,3.....$

Non-linear simultaneous equations by Newton-Raphson (or Newton's) method:

$f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = 0, \quad g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0$

Taylor Series Method: $y = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + ...$

Euler's Method: $y_{n+1} = y_n + hf(x_n, y_n)$

Modified Euler's method: $\boldsymbol{y_1^{(0)}} = y_0 + hf(x_0, y_0)$

$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$, $n = 0,1,2,...$

Runge-Kutta Method of Second Order: $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$,

where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k_1)$.

Runge-Kutta Method of Fourth Order:

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, where $k_1 = hf(x_0, y_0)$,

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$, $k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$, $k_4 = hf(x_0 + h, y_0 + k_3)$.

Analytical Geometry

For two dimension :

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ then:

1. The distance formula: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. The midpoint formula: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
3. The point *R(x, y)* dividing *PQ* in the ratio $\frac{k1}{k2}$ is: $x = \frac{k_1x_2+k_2x_1}{k_1+k_2}, y = \frac{k_1y_2+k_2y_1}{k_1+k_2}$

4. The Slope of PQ is: $m = \frac{y_2 - y_1}{x_2 - x_1}$
5. The Slope of the X-axis and a line parallel to X-axis = 0
6. The Slope of the Y-axis and a line parallel to Y-axis is not defined, i.e ∞.
   The equation of the X-axis is ***y* = 0** and the equation of Y-axis is $x = 0$.
7. The equation of the line parallel to the Y-axis and is at a distance 'a' is ***x* = *a*.**
8. The equation of the line parallel to the X-axis and is at a distance '**a**' is ***y* = *a*.**
9. The equation of the line with slope **m** and Y-intercept **c** is **y = mx + c**, which is called the slope intercept form.
10. The equation of the line passing through (x_1, y_1) and having slope m is $\mathbf{y - y_1 = m(x - x_1)}$, which is called the slope-point form.
11. The equation of the line passing through two points (x_1, y_1) and (x_2, y_2) is $\frac{y - y1}{y2 - y1} = \frac{x - x1}{x2 - x1}$
12. The equation of the line having a and b as the x-intercept and y-intercept is $\frac{x}{a} + \frac{y}{b} = 1$ and is called the equation of the line intercept form.
13. The normal form of the straight line is $x \cos \alpha + y \sin \alpha = p$, where p is the length of the perpendicular from O (0,0) to the line, α is the inclination of the perpendicular.
14. The general form of the equation of a straight line is ax + by + c = 0.
15. The distance of the point (x_1, y_1) from the line ax + by + c = 0 is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$
16. If ax + by + c = 0 with b > 0, is the equation of the line l, then P (x_1, y_1) lies:
    i. Above the line l if $ax_1 + by_1 + c > 0$
    ii. Below the line l if $ax_1 + by_1 + c < 0$
    iii. On the line l if $ax_1 + by_1 + c = 0$
17. Consider two lines l_1 and l_2 having the slopes m_1 and m_2 respectively.
    If two lines l_1 and l_2 are parallel, then $m_1 = m_2$.
    If two lines l_1 and l_2 are perpendicular, then $m_1 \times m_2 = -1$.
    The angle θ from l_1 to l_2 is $\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

For three dimension :

Direction Cosines of a line:

1. If α, β, γ be the angles which a given line makes with the positive directions of co-ordinate axes, then $l = \cos \alpha, \mathrm{m} = \cos \beta, \mathrm{n} = \cos \gamma$ are called the direction cosines (d.c.'s) of the given line.
2. If l, m, n are the direction cosines of a line PQ then the d.c's of QP are $-l, -m, -n$.
3. If l, m, n be the d.c.'s of a line OP and OP=r then the co-ordinates of P are (lr, mr, nr).

4. If l, m, n be the d.c.'s of a line, then $l^2 + m^2 + n^2 = 1$.

Direction Ratios of a line:

1. Three numbers a, b, c which are propotional to the directional cosines l, m, n respectively of a line are called direction ratios (d.r's)of a line.
2. If a, b, c are the d.r.'s of a line then the direction cosines are
$$l = \pm\frac{a}{\sqrt{a^2+b^2=c^2}},\ m = \pm\frac{b}{\sqrt{a^2+b^2=c^2}}\ ,\ n = \pm\frac{c}{\sqrt{a^2+b^2=c^2}}$$
3. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$be two given points then the direction ratios of PQ are, $x_2 - x_1, y_2 - y_1, z_2 - z_1$.
4. The angle between the lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2 is
$$\theta = \cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2)$$
5. If the d.c.'s of a line are l_1, m_1, n_1 and l_2, m_2, n_2then
6. Condition for perpendicularity; $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$
7. Condition for parallelism
8. The angle between the lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2 is $\cos\theta = \frac{(a_1a_2+b_1b_2+c_1c_2)}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$.
9. If the d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2 then the lines are perpendicular if $(a_1a_2 + b_1b_2 + c_1c_2) = 0$. and parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
10. The projection of the join of two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ on a line whose d.c.'s are l, m, n is $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.
11. **General Form**: The two linear equations in x, y and z together represent a straight line $ax + by + cz + d = 0\ \ and\ a'x + b'y + c'z + d' = 0$.
12. **Symmetrical Form**: Equations of the line through the point $A(x_1, y_1, z_1)$ and having direction cosines l, m, n are $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.
13. Any point on the symmetrical form of a line is $(x_1 + kl, y_1 + km, z_1 + kn)$.
14. The equations of the line joining the points $(x_1, y_1, z_1) and\ (x_2, y_2, z_2)$ are $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ for the direction ratios of the line joining the given points are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.
15. The condition that the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ may lie in the plane $ax + by + cz + d = 0$ are (i) the line should be parallel to the plane and (ii) a point of line should lie in the plane.
16. The equation of any plane through the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where $al + bm + cn = 0$.
17. The equation of any plane through the line of intersection of the planes $ax + by + cz + d = 0\ \ and\ a'x + b'y + c'z + d' = 0$ is $ax + by + cz + d + k(a'x + b'y + c'z + d') = 0$.
18. Conditions for the two lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2}$ to intersect is $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$=0.

19. The equation of the plane containing the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2}=\frac{y-y_1}{m_2}=\frac{z-z_1}{n_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$=0.

20. The length of the shortest distance between two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2}=\frac{y-y_1}{m_2}=\frac{z-z_1}{n_2}$ is $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$.

21. The equation of the line of shortest distance between two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2}=\frac{y-y_1}{m_2}=\frac{z-z_1}{n_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix}$=0 and $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$=0.

22. The length of the perpendicular from (x_1, y_1) to the line $ax+by+c=0$ is $\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$.

23. The conditions that the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ may lie in the plane $ax+by+cz+d=0$ are $al+bm+cn=0$ and $ax_1+by_1+cz_1+d=0$.

24. The condition that the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is perpendicular to the plane $ax+by+cz+d=0$ is $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$.

25. The angle between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$ is $\theta=\sin^{-1}\left(\frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}\sqrt{a^2+b^2+c^2}}\right)$.

26. The equation of any plane through the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is $A(x_2-x_1)+B(y_2-y_1)+C(z_2-z_1)=0$ where $Al+Bm+Cn=0$.

Figure	Area	Perimeter
Equilateral triangle with side a	$\sqrt{3}\,a^2/4$	3a
Triangle with sides a, b, c	(base ×height)/2	a + b + c
Rectangle with sides l and b	l × b	2(l + b)
Parelleogram	base × height	2(l + b)
Trapezoid	(sum of parallel sides)×h/2	
Circle with radius r	πr^2	$2\pi r$

	Volume	Surface Area
sphere with radius r	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Cube with side a	a^3	$6a^2$
Right circular cylinder	$\pi r^2 h$	$2\pi r(r+h)$

Cuboid	l ×b×h	2(l×b + b×h + h×l)
Right circular cone	$\frac{1}{3} * \pi r^2 h$	S = π r *l* +π r^2

1. For a spherical shell, if R and r are the outer and inner radii respectively, then the volume of the shell is $= \frac{4}{3}\pi(R^3 - r^3)$
2. The equation of a sphere of radius R centred at the point (x_0, y_0, z_0) is given by: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$

Measurements of Cone

A cone is a solid figure generated by a line, one end of which is fixed, and the other end describes a closed curve in a plane.

A right circular cone is a cone whose base is a circle and whose axis is perpendicular to the base. Such a cone can also be described as a solid formed by a right triangle rotated about one of its sides as an axis.

1. A right cone of height *h* and base radius *r* oriented along the z – axis, with vertex pointing up, and with the base located at z = 0 can be described with parametric equations: $x = \frac{h-u}{h} r\cos\theta$, $y = \frac{h-u}{h} r\sin\theta$, z = u, for u ϵ [0, h] and θ ϵ [0, 2π)
2. The opening angle of a right cone is the vertex angle made by a cross section through the apex and centre of the base. For a cone of height *h* and radius *r*, it is given by $\vartheta = 2\tan^{-1}\frac{r}{h}$

Measurements of Cylinders

A cylinder is a solid figure generated by a straight line moving to its original position, while its end describes a closed figure in a plane.

1. Volume of a Cylinder $V = \pi r^2 h$ (r being the radius) or $V = \frac{\pi}{4} d^2 h$ (d being the diameter)
2. For a Hollow Cylinder, if R is the outer radius and r is the inner radius, then $V = \pi R^2 h - \pi r^2 h = \pi(R^2 - r^2)h$
3. Curved surface area $S = 2\pi r h$
4. The Total surface area $= 2\pi r h + 2\pi r^2 = 2\pi r(r + h)$
5. The equation of a cylinder centred at (a, b) having radius *r* and its axis parallel to z – axis is given by: $(x - a)^2 + (y - b)^2 = r^2$

6. The lateral surface of a cylinder of height h and radius r can be described parametrically by: $x = r\cos\theta$, y$= r\sin\theta$, $z = z$

Infinite Series

1. **Geometric series test:** The series $\sum_{r=0}^{\infty} r^n$
   (i) Converges if $|r| < 1$
   (ii) Diverges if $r \geq 1$
   (iii) Oscillates finitely, if $r = -1$ and oscillates infinitely if $r < -1$.
2. **Comparison Test:**
   Let $\sum u_n$ and $\sum v_n$ be two positive term series
   (i) If $\sum v_n$ is convergent and $u_n \leq v_n$, for all n, then $\sum u_n$ is also convergent.
   (ii) If $\sum v_n$ is divergent and $u_n \geq v_n$, for all n, then $\sum u_n$ is also divergent.
3. **Quotient test:**
   Suppose $\sum u_n$ and $\sum v_n$ be two positive term series. If $\lim_{n\to\infty} \frac{u_n}{v_n} = k(\neq 0$ a constant), then $\sum u_n$ and $\sum v_n$ converge or diverge together.
4. **Integral Test:**
   A positive term series $f(1) + \cdots + f(n)+$.. where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^{\infty} f(x)dx$ is finite or infinite.
5. **p-series (harmonic series) test:**

A positive term series $\sum u_n = \sum \frac{1}{n^p}$ is (i) convergent if $p > 1$ (ii) divergent if $p \leq 1$.

6. **D'Alemberts;s Ratio test:** A positive term series $\sum u_n$

Converges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, Diverges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$, Test fails if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

7. **Raabe's Test:** A positive term series $\sum u_n$

Converges if $\lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) > 1$, Diverges if $\lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) < 1$, fails if the limit = 1

8. **CAUCHY'S ROOT TEST:** In a positive series $\sum u_n$, if $\lim_{n\to\infty} (u_n)^{1/n}$ =λ, the series

converges for $\lambda < 1$, diverges for $\lambda > 1$, test fails for $\lambda = 1$.

Leibnitz's rule: An alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$
converges if

(i) each term is numerically less than its preceding term, and

(ii) $\lim_{n\to\infty} u_n$=0.

If $\lim_{n\to\infty} u_n \neq 0$, the given series is oscillatory.

Absolutely convergent

If the series of arbitrary terms $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ be such that the series $|u_1|$+$|u_2|$+$|u_3|$+…..+$|u_n|$+….. is convergent, then the series $\sum u_n$ is said to be absolutely convergent.

Conditionally convergent

If $\sum|u_n|$ is divergent but $\sum u_n$ is convergent, then $\sum u_n$ is said to be conditionally convergent.

Note: An absolutely convergent series is necessarily convergent but not conversely.

Laplace Transform

$$\mathcal{L}\{f(t)\} \equiv f(s) = \int_0^\infty f(t)e^{-st}dt$$

Properties of Laplace transform

1. $L\{af(t)+bg(t)-ch(t)\}=aL\{f(t)\}+bL\{g(t)\}-cL\{h(t)\}$ **(Linearity Property)**
2. If L{f (t)} = F(s) then $L\{e^{at}f(t)\} = F(s-a)$. **(First Shifting property)**
3. If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}F(s)$
4. If $L\{f(t)\}=F(s)$ then $L\{t^n f(t)\}=(-1)^n \frac{d^n}{ds^n}F(s),\quad n=1,2,3,...$
5. If $L\{f(t)\} = F(s)$ then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty F(s)ds$
6. If $F(t)$ has a Laplace Transform and if

$$F(t+\omega)=F(t), L\{F(t)\} = \frac{\int_0^\omega e^{-s\beta}F(\beta)d\beta}{1-e^{-sw}}.$$

Laplace transforms of standard function

f(t)	L (f(t))
I	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
sin k t	$\frac{k}{s^2+k^2}$
cos k t	$\frac{s}{s^2+k^2}$
sinh k t	$\frac{k}{s^2-k^2}$

cosh kt	$\frac{s}{s^2 - k^2}$
e^{at} sin kt	$\frac{k}{(s-a)^2 + k^2}$
e^{at} cos kt	$\frac{s-a}{(s-a)^2 + k^2}$
e^{at} sinh kt	$\frac{k}{(s-a)^2 - k^2}$
e^{at} cosh kt	$\frac{s-a}{(s-a)^2 - k^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, n a positive integer
H (t-a)=u_a(t)	$\frac{e^{-as}}{s}, s > 0$
δ (t)	1
δ (t-t_o)	e^{-st_0}
e^{at} f(t)	F(s-a)
f(t-a) H(t-a)	e^{-as}F(s)
t^nf(t)	$(-1)^n \frac{d^n}{ds^n} F(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) \ldots\ldots - f^{(n-1)}(0)$
$\int_0^t f(u)\, g(t-u) du$	F(s) G(s)

Inverse Laplace transforms:

Let L{f(t)} = F(s). Then f(t) is defined as the inverse Laplace transform of F(s) and is denoted by L^{-1} {F(s)}. **Thus** L^{-1} F(s) = f(t)

Linearity Property

Let L^{-1} {F(s)} = f(t) and L^{-1} {G(s)} = g(t) and a and b be any two constants.

Then L^{-1} [a F(s) ± b G(s)] = a L^{-1} {F(s)} ± b L^{-1} { G(s)}

Shifting Property

If L^{-1} {F(s)}=f(t) then L^{-1} [F(s-a)]= e^{at} L^{-1}{F(s)}

Inverse transform of derivative

$L^{-1}\{F^{n}(s)\} = (-1)^n t^n L^{-1}\{F(s)\}$

$L^{-1}[e^{-as} F(s)]$ = f(t-a) H(t-a)

$$L^{-1}[F(s)G(s)] = \int_0^t f(t-u)g(u)du = f(t) * g(t)$$

Inverse Laplace Transforms of some standard functions

F(s)	$f(t) = L^{-1}F(s)$
$\frac{1}{s}, s > 0$	1
$\frac{1}{s-a}, s > a$	e^{at}
$\frac{s}{s^2+a^2}, s > 0$	Cos at
$\frac{1}{s^2+a^2}, s > 0$	$\frac{\text{Sin at}}{\text{a}}$
$\frac{1}{s^2-a^2}, s > \|a\|$	$\frac{\text{Sin h at}}{\text{a}}$
$\frac{s}{s^2-a^2}, s > \|a\|$	Cos h at
$\frac{1}{s^{n+1}}, s > 0$ *n = 0, 1, 2, 3, . . .*	$\frac{t^n}{n!}$
$\frac{1}{s^{n+1}}, s > 0$ *n > -1*	$\frac{t^n}{\Gamma(n+1)}$