

# Duality Theory

## Connecting Logic, Algebra, and Topology

### A Reader

Draft from December 5, 2023.

Please don't cite or distribute without permission.

Comments welcome!

Levin Hornischer

`Levin.Hornischer@lmu.de`

# Contents

<b>Preface</b>	<b>1</b>
<b>1 Introduction: the key idea of duality</b>	<b>4</b>
1.1 Intuitive examples of duality . . . . .	4
1.1.1 Metaphysics: Properties vs objects . . . . .	5
1.1.2 Semantics: Propositions vs possible worlds . . . . .	7
1.1.3 Logic: models vs formulas . . . . .	10
1.1.4 Further examples in physics and computer science	12
1.2 Towards characterizing duality . . . . .	13
1.3 Exercises . . . . .	14
<b>2 The algebraic side: distributive lattices</b>	<b>16</b>
2.1 Order theory . . . . .	16
2.1.1 Objects: Partial orders . . . . .	16
2.1.2 Morphisms: Monotone maps . . . . .	20
2.2 Lattices . . . . .	22
2.2.1 Objects: lattices . . . . .	22
2.2.2 Morphisms: lattice homomorphisms . . . . .	24
2.2.3 Constructions: products, sublattices, homomorphic images, congruences . . . . .	25
2.3 Distributive lattices and Boolean algebras . . . . .	26
2.3.1 Distributive lattices . . . . .	26
2.3.2 Boolean algebras . . . . .	27
2.4 Duality for finite distributive lattices and finite Boolean algebras . . . . .	28
2.4.1 From properties/lattice to recovered objects/spaces and back . . . . .	29
2.4.2 The recovered lattices and spaces are isomorphic to the original ones . . . . .	30
2.4.3 Also including morphisms . . . . .	32
2.4.4 The duality and the special case of Boolean algebras	34
2.5 Exercises . . . . .	35
<b>3 The spatial side: topological spaces</b>	<b>39</b>
3.1 Introduction to topological spaces . . . . .	39
3.1.1 Objects: topological spaces . . . . .	40
3.1.2 Morphisms: continuous functions . . . . .	43
3.2 Orders on topological spaces . . . . .	45
3.3 Compact ordered spaces and stably compact spaces . . . .	45
3.4 Exercises . . . . .	45
<b>4 Two sides of the same coin: Priestley and Stone duality</b>	<b>47</b>

<b>Bibliography</b>	<b>47</b>
<b>Index</b>	<b>49</b>

## Preface

This is the reader for the course “Duality Theory: Connecting Logic, Algebra, and Topology” given during the winter semester 2023/24 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. The reader is written as the course progresses. A website (or rather git repository) with all the course material is found at

<https://github.com/LevinHornischer/DualityTheory>.

**Comments** I’m happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just send an informal email to [Levin.Hornischer@lmu.de](mailto:Levin.Hornischer@lmu.de).

**Course description and objectives** This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone’s duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (totally disconnected compact Hausdorff spaces). This has been extended in many ways. The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,
- open sets of a space vs. points of the space,
- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications that make use of dualities by combining the often opposing advantages of the two perspectives.

**Prerequisites** An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts

of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course “Philosophical Logic”, which I taught in the summer semester 2023. In that course, I stressed two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. We’ve seen that these semantic approaches are often equivalent, and this is a special case of the more general phenomenon of duality.

**Contents** We start with an informal chapter describing the key idea of duality. The rest of the course is about developing this key idea precisely. For this, we follow the recent textbook Gehrke and van Gool 2023. We first precisely define the algebraic structure (lattices) and then topological structures (topological spaces), and we finally prove the duality result. The remainder of the course is about deepening this result and applying it in logic and computer science.

**Layout** These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you’d rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end.

*This is a margin note.*

This is a todo note

**Study material** The main textbook that we use is by Gehrke and van Gool (2023). And informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press
- B. A. Davey and H. A. Priestley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press
- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press
- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag
- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer

- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser
- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory are

- P. T. Johnstone (1982). *Stone Spaces*. Cambridge studies in advanced mathematics 3. Cambridge: Cambridge University Press
- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press
- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: [10.1017/9781316543870](https://doi.org/10.1017/9781316543870)
- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press
- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser
- S. Abramsky and A. Jung (1994). "Domain Theory." In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at <http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf> (last checked 24 January 2018). Oxford: Oxford University Press

**Notation** Throughout, 'iff' abbreviates 'if and only if'.

# 1 Introduction: the key idea of duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

*Advice on how to read this chapter.* Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, going in different directions, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the ‘rhyme and reason’ to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

Duality theory can be quite abstract. The advantage of this is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. So before attempting any general definition of duality, let us consider several examples (section 1.1). From those we can generalize an informal characterization of duality (section 1.2). This then hints at how duality theory is formalized mathematically and how it can be applied. Finally, in section 1.3 we list some exercises.

## 1.1 Intuitive examples of duality

We present several examples of duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

*For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).*

*To use the words of Abramsky (2023).*

*I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.*

### 1.1.1 Metaphysics: Properties vs objects

When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon. Properties are, for example, being red, being higher than 300m, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Let us write  $\mathcal{O}$  for the set of all objects and  $\mathcal{P}$  for the set of all properties. Crucially, observe that there is a certain dependency between  $\mathcal{O}$  and  $\mathcal{P}$ :

$(\mathcal{O} \rightarrow \overline{\mathcal{P}})$  Each object  $x \in \mathcal{O}$  determines a set of properties  $F_x \subseteq \mathcal{P}$  consisting of precisely those properties that  $x$  has.

(The bar in ' $\overline{\mathcal{P}}$ ' indicates that we assign to each  $x$  a *set* of elements in  $\mathcal{P}$  rather than a single element of  $\mathcal{P}$ .) So we might wonder whether we can also go in the opposite direction ( $\overline{\mathcal{P}} \rightarrow \mathcal{O}$ )? Does a subset  $F$  of properties also determine an object, i.e., the unique object that has exactly the properties in  $F$ ? Actually, no: some sets of properties might not be satisfied by any object (e.g.,  $F = \{\text{being exactly 300m high, being exactly 200m high}\}$ ) or by more than one (e.g.,  $F = \{\text{being exactly 300m high}\}$ ).

But let us not give up too early. After all, the set  $F_x$  is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if  $F$  is a set of properties with these nice features, that then it determines a unique object.)

1. Assume  $a, b \in \mathcal{P}$  are two properties such that having  $a$  implies having  $b$ ; we abbreviate this as  $a \leq b$ . For example,

$$a = \text{being higher than 300m} \leq \text{being higher than 200m} = b.$$

So if our object  $x$  has property  $a$ , then it also has property  $b$ , i.e., if  $a \in F_x$ , then  $b \in F_x$ . We may express this as:  $F_x$  is closed under implication.

2. Assume  $a, b \in \mathcal{P}$  are two properties. Note that then there is another property: namely, the property of having both property  $a$  and property  $b$ . We denote this property  $a \wedge b$ . So  $a \wedge b$  is again in  $\mathcal{P}$  and we have  $a \wedge b \leq a$  and  $a \wedge b \leq b$ . Moreover, if our object  $x$  has property  $a$  and it has property  $b$ , then it has property  $a \wedge b$ , i.e., if  $a, b \in F_x$ , then  $a \wedge b \in F_x$ . We may express this as:  $F_x$  is closed under conjunction.

*Philosophers know phrases of the form 'The F' (referring to the unique object satisfying F) as definite description. For their important role in philosophy, see e.g. Ludlow (2022).*

*Later we will say  $F_x$  is an upset. This sounds funny now, but by the end of the course, you will have said this so often that you won't even notice.*



3. Similarly, if  $a, b \in \mathcal{P}$  are two properties, there also is the property of having either property  $a$  or property  $b$  (or both). We denote this property  $a \vee b$ . So  $a \vee b$  is again in  $\mathcal{P}$  and we have  $a \leq a \vee b$  and  $b \leq a \vee b$ . Moreover, if our object  $x$  has property  $a \vee b$ , then either it has property  $a$  or it has property  $b$ , i.e., if  $a \vee b \in F_x$ , then either  $a \in F_x$  or  $b \in F_x$ . Later, we express this as  $F_x$  being prime.
4. Note that  $\mathcal{P}$  also contains the trivial property like being identical to oneself. We denote this property  $\top$ . In particular, our object  $x$  has it, i.e.,  $\top \in F_x$ .
5. Similarly, note that  $\mathcal{P}$  also contains the inconsistent property like not being identical to oneself. We denote this property  $\perp$ . In particular, our object  $x$  does not have it, i.e.,  $\perp \notin F_x$ .

Now, we can ask our question again: If  $F$  is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in  $F$ ? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

$(\overline{\mathcal{P}} \rightarrow \mathcal{O})$  Each set of properties  $F \subseteq \mathcal{P}$  satisfying (1)–(5) determines an object  $x \in \mathcal{O}$ , namely, the unique object having exactly the properties in  $F$ .

The uniqueness part is close to Leibniz's principle about the **identity of indiscernibles**: if two objects  $x$  and  $x'$  have exactly the properties in  $F$ , they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description  $F$  of an object, there in fact is a (possible) object that has these properties. For this, we should consider  $\mathcal{O}$  to contain not only the objects in our world, but all possible objects. After all, the actual world need not be ontologically complete:  $F$  might consistently describe a unicorn, even if this does not exist in the actual world.

We will see that this bidirectional determination ( $\mathcal{O} \rightarrow \overline{\mathcal{P}}$ ) and  $(\overline{\mathcal{P}} \rightarrow \mathcal{O})$  is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensional-ity principle: two properties  $a$  and  $b$  are identical if they apply to exactly

*I will always read 'either A or B' as inclusive-or (either only A is the case, or only B is the case, or both A and B are the case) Cf. a number  $p > 1$  is prime iff (that is **Euclid's lemma**), for all numbers  $a$  and  $b$ , if  $a \times b$  is divided by  $p$ , then either  $a$  is divided by  $p$  or  $b$  is divided by  $p$ ).*

*Or is the list (1)–(5) not complete because we should also add a principle concerning negation: you can think about this in exercise 1.b.*

*Actually, I don't know if a principle like this is considered in metaphysics: if you do, please let me know :-). Also see exercise 1.c asking for a comparison to formal concept analysis.*

*Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements.*

the same possible objects (i.e., for all  $x \in \mathcal{O}$ ,  $x$  has  $a$  iff  $x$  has  $b$ ). Each property  $a$  determines a set of objects: namely, the set of those objects that have property  $a$ . This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

*Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.*

### 1.1.2 Semantics: Propositions vs possible worlds

The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is 400m high. So the proposition  $a$  expressed by the sentence ‘The Eiffel Tower is 330m high’ contains the actual world  $x_0$  (i.e.,  $x_0 \in a$ ) but not the just described possible world  $x_1$  (i.e.,  $x_1 \notin a$ ). Some common notation for the phrase ‘world  $x$  makes true proposition  $a$ ’ is  $x \models a$ ; so possible world semantics analyses  $\models$  as elementhood  $\in$ .

There is much debate in philosophy what the set  $\mathcal{W}$  of possible worlds is (Menzel 2021) and what the set  $\mathcal{P}$  of propositions is (McGrath and Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds (‘worlds first, propositions later’)? Or is it rather that worlds are maximally consistent sets of propositions (‘propositions first, worlds later’)? The latter goes by the name ‘ersatzism’ since full-blown possible worlds are substituted by something constructed out of linguistic entities—and ‘Ersatz’ is German for substitute.

We won’t enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

*World individuation* Possible worlds are individuated by the propositions they make true: if two possible worlds  $x$  and  $y$  make true exactly the same propositions (i.e., for every proposition  $a$ , we have  $x \models a$  iff  $y \models a$ ), then  $x = y$ .

*Cf. Leibniz's above principle about the identity of indiscernibles.*

*Proposition individuation* Propositions are individuated by the possible worlds at which they are true: if two propositions  $a$  and  $b$  are true at exactly the same possible worlds (i.e., for every possible world  $x$ , we have  $x \models a$  iff  $x \models b$ ), then  $a = b$ .

*A hyperintensional account of propositions would contest this; see Berto and Nolan (2021).*

And there is more. Just like properties, also the set of propositions has logical structure: If  $a$  and  $b$  are propositions, there also are the propositions  $a \wedge b$  (conjunction),  $a \vee b$  (disjunction),  $\neg a$  (negation),  $\top$  (logical truth), and  $\perp$  (logical falsity). With this we can also express implications between propositions: proposition  $a$  implies proposition  $b$ , written  $a \leq b$ , precisely if  $a \wedge b = a$ . The proposition expressed by 'I am in Munich' implies the proposition expressed by 'I am in Germany' because the sentence 'I am in Munich and I am in Germany' is equivalent to the sentence 'I am in Munich', i.e., they express identical propositions.

Thus, given a possible world  $x \in \mathcal{W}$ , we can again consider the set of propositions  $F_x \subseteq \mathcal{P}$  that are true in  $x$  (i.e.,  $F_x = \{a \in \mathcal{P} : x \models a\}$ ). And  $F_x$  again satisfies the features (1)–(5) above: If  $a \in F_x$ , i.e.,  $x \models a$ , and  $a$  implies  $b$ , i.e.,  $a \leq b$ , then  $x \models b$ , i.e.,  $b \in F_x$ . If  $a, b \in F_x$ , then  $x$  makes true both  $a$  and  $b$ , so  $a \wedge b \in F_x$ . As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

*Metaphysical completeness* Each set of propositions  $F \subseteq \mathcal{P}$  satisfying (1)–(5) determines a possible world  $x \in \mathcal{W}$ , namely, the unique possible world making true exactly the propositions in  $F$ .

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions  $F$  satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to  $F$  without making it inconsistent (i.e., making it contain  $\perp$ ). Ersatzism not only claims that then there is a world  $x$  which makes true exactly the propositions in  $F$ , it even identifies this world  $x$  with  $F$ . The metaphysic completeness claim only follows along with the existence claim, and the uniqueness of  $x$  follows from the world individuation principle above.

*This is assuming that the set of propositions forms what is known as a Boolean algebra.*

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective

correspondence between the set  $\mathcal{W}$  of possible worlds and the set  $\overline{\mathcal{P}}$  of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets  $F \in \overline{\mathcal{P}}$  will be called *prime filters* and  $\overline{\mathcal{P}}$  will be called the *spectrum* of the algebra of propositions.)

$$\mathcal{W} \xrightarrow{\sim} \overline{\mathcal{P}}$$

$$x \mapsto F_x = \{a \in \mathcal{P} : x \models a\}$$

the  $x$  making true exactly the  $a \in F \leftrightarrow F$

Let us verify that this really is a bijection: We have already checked that the function  $f : \mathcal{W} \rightarrow \overline{\mathcal{P}}$  mapping  $x$  to  $F_x$  is well-defined. It is injective by the world individuation principle: if  $x \neq y$ , then there is a proposition  $a$  with  $x \models a$  and  $y \not\models a$  (or vice versa), so  $a \in F_x$  and  $a \notin F_y$  (or vice versa), so  $F_x \neq F_y$ . It is surjective by metaphysical completeness: Given  $F \in \overline{\mathcal{P}}$ , let  $x$  be the unique world in  $\mathcal{W}$  making true exactly the propositions in  $F$ . Then  $F = F_x$  because:  $a \in F$  iff  $x \models a$  iff  $a \in F_x$ .

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of  $\mathcal{W}$ ) and their ersatz constructions as sets of propositions (the elements of  $\overline{\mathcal{P}}$ ). But what about the other side: How do full-blown propositions (the elements of  $\mathcal{P}$ ) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition  $a \in \mathcal{P}$  determines the set of worlds  $\llbracket a \rrbracket := \{x \in \mathcal{W} : x \models a\}$  where  $a$  is true. This is also known as the *truthset* of  $a$ . And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write  $\overline{\mathcal{W}}$  for the sets of worlds that determine propositions and  $2^{\mathcal{W}}$  for the set of all sets of worlds. So our assumption for now is that  $\overline{\mathcal{W}} = 2^{\mathcal{W}}$ . Analogous to the previous case, we want to know if the function

$$\llbracket \cdot \rrbracket : \mathcal{P} \rightarrow 2^{\mathcal{W}}$$

$$a \mapsto \llbracket a \rrbracket = \{x \in \mathcal{W} : x \models a\}$$

is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if  $a \neq b$ , there is a world  $x$  with  $x \models a$  and  $x \not\models b$  (or vice versa), so  $\llbracket a \rrbracket \neq \llbracket b \rrbracket$ . In fact, it also preserves the logical structure:  $\llbracket a \wedge b \rrbracket = \llbracket a \rrbracket \cap \llbracket b \rrbracket$ ,  $\llbracket \perp \rrbracket = \emptyset$ , etc. (Later we formalize this as  $\llbracket \cdot \rrbracket$  being a Boolean algebra homomorphism.) However, the issue is

*A function  $f : X \rightarrow Y$  is injective if  $x \neq y$  implies  $f(x) \neq f(y)$ , it is surjective if for every  $y \in Y$  there is  $x \in X$  with  $f(x) = y$ , and it is bijective if it is both injective and surjective.*

*If  $X$  is a set, the powerset of  $X$  is the set of all subsets of  $X$  and it is denoted  $2^X$  or  $\mathcal{P}(X)$ .*

surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why  $\llbracket \cdot \rrbracket$  is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a ‘finitistic’ grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset  $2^{\mathcal{W}}$  of  $\mathcal{W}$  is uncountable. So  $\mathcal{P}$  and  $2^{\mathcal{W}}$  have different cardinalities, which means there cannot be a bijection between, hence the already injective function  $\llbracket \cdot \rrbracket$  cannot be surjective.

*That is Cantor’s diagonal argument.*

So actually not any set of worlds determines a proposition, i.e.,  $\overline{\mathcal{W}}$  is a proper subset of  $2^{\mathcal{W}}$ . The ingenious insight of Stone, who discovered the Stone duality, was to realize how to precisely describe this special subset  $\overline{\mathcal{W}}$  of  $2^{\mathcal{W}}$ . The key idea is to realize that there is some additional structure on the set of worlds  $\mathcal{W}$  that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in the formal chapters.

*Also see exercise 1.d.*

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

### 1.1.3 Logic: models vs formulas

Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches, which are very different in spirit, actually are equivalent. This also is a form of duality. Let’s explore this concretely.

Consider the language of classical propositional logic: sentences are formed from atomic sentences  $p_0, p_1, \dots$  using the connectives  $\wedge, \vee, \neg$  and the constants  $\perp$  and  $\top$ . And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence calculus for classical logic. It consists of various axioms and rules to define the relation  $\Gamma \vdash \varphi$ , i.e., when the sentence  $\varphi$  is derivable in the proof-system  $S$  using as axioms the sentences in the set  $\Gamma$ . This is the syntactic description of the logic.

The model-theoretic description of the logic defines the relation  $\Gamma \models \varphi$ ,

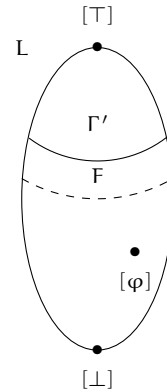
i.e., that the sentence  $\varphi$  is a logical consequence of the sentences in  $\Gamma$ . This is done as follows. A valuation is a function  $v : \{p_0, p_1, \dots\} \rightarrow \{0, 1\}$  that assigns each atomic sentences a truth-value, i.e., true (1) or false (0). This can be extended to all sentences:  $v(\varphi \wedge \psi) = 1$  iff  $v(\varphi) = 1$  and  $v(\psi) = 1$ ;  $v(\neg\varphi) = 1$  iff  $v(\varphi) = 0$ ;  $v(\perp) = 0$ ; etc. Then  $\Gamma \models \varphi$  is defined as: for all valuations  $v$ , if  $v(\psi) = 1$  for all  $\psi \in \Gamma$ , then  $v(\varphi) = 1$ . Thus, logical consequence is truth-preservation.

Now, the completeness theorem for classical propositional logic states that:  $\Gamma \vdash \varphi$  iff  $\Gamma \models \varphi$ . To be more precise, one often only calls the right-to-left implication ‘completeness’, and the left-to-right implication ‘soundness’. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserves logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences  $\varphi$  and  $\psi$  equivalent, written  $\varphi \equiv \psi$ , iff both  $\varphi \vdash \psi$  and  $\psi \vdash \varphi$ . An equivalence class of a sentence  $\varphi$  is the set of sentences that are equivalent to it:  $[\varphi] := \{\psi : \varphi \equiv \psi\}$ . Write  $L$  for the set of all equivalence classes. It also has logical structure:  $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$ ;  $\neg[\varphi] = [\neg\varphi]$ , etc.  $L$  is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each a valuation  $v$  determines a subset  $F_v \subseteq L$ : namely, those equivalence classes  $[\varphi]$  with  $v(\varphi) = 1$ . Note again that  $F_v$  has features (1)–(5): If  $[\varphi] \in F_v$  and  $[\varphi] \leq [\psi]$  (i.e.,  $[\varphi \wedge \psi] = [\varphi]$ ), then  $\varphi \vdash \psi$ , so, by soundness,  $\varphi \models \psi$ , so, since  $v(\varphi) = 1$ , also  $v(\psi) = 1$ , so  $[\psi] \in F_v$ . If  $[\varphi], [\psi] \in F_v$ , then  $v(\varphi) = 1$  and  $v(\psi) = 1$ , so  $v(\varphi \wedge \psi) = 1$ , so  $[\varphi \wedge \psi] \in F_v$ . Etc. Conversely, if  $F \subseteq L$  satisfies (1)–(5), then  $v_F$  is a valuation mapping  $\varphi$  to 1 iff  $[\varphi] \in F$ . So, again, the set  $X$  of valuations is in bijective correspondence with the set  $\bar{L}$  of subsets of  $L$  satisfying (1)–(5).

But how does completeness follow? For this, first note that subsets of  $L$  are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if  $\Gamma \not\vdash \varphi$ , consider the deductive closure  $\Gamma'$  of  $\Gamma$ , i.e., the set of all sentences that can be derived from  $\Gamma$ , so also  $\Gamma' \not\vdash \varphi$ . When we regard  $\Gamma'$  as a subset of  $L$ , this is, in formal terminology, a filter of  $L$  that does not intersect the ideal of all equivalence classes that imply  $[\varphi]$ . Now one only needs one formal result, namely Stone’s Prime Filter Theorem (which we prove later on in the course), which says that we can extend this filter to a prime filter



$F$  which still does not intersect this ideal. Then  $v_F$  is a valuation that makes true all the premises in  $\Gamma$  but not the conclusion  $\varphi$ , hence  $\Gamma \not\models \varphi$ , as desired.

#### 1.1.4 Further examples in physics and computer science

We sketch two further examples, one in physics and one in computer science.

*Physics: states vs observations.* Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). A physical system comes both with a *state space*  $X$  and an algebra  $A$  of *observations* and these two again are dual in the sense that

- the states are determined by the observations that they give rise to,
- the observations are determined by the states that give rise to them.

The observations have logical structure: in a classical (as opposed to quantum) system, observing  $A \wedge B$  means observing  $A$  and observing  $B$ , observing  $A \vee B$  means observing  $A$  or observing  $B$ , etc. Each state  $x$  of the system determines a set of observations: namely, those that can be made if the system is in that state. Conversely, we can also start with the algebra of observations (they are empirically more accessible anyway) and postulate the states of the system as theoretical entities corresponding to certain subsets of observations.

*Computation: denotations of programs vs observable properties.* Computer programs are written in a programming language, and so, much like for sentences written in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation*. For example, the denotation of a program could be the (partial) function that it computes. Domain theory is the mathematical theory to systematically describe these meanings. There again also is a side that is dual to the side of meanings, and this was a crucial discovery in the development of domain theory by Abramsky (1991, p. 16). This is the side of *observable properties* of the computer programs. For example, it could be the property that, on input  $x = 3$ , the program halts and outputs  $f(x) = 5$ . Again, we would hope for a bidirectional determination in the sense that the meaning of a program is completely determined by its observable properties, and that these observable properties are determined by the denotations that have them.

## 1.2 Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will work toward doing this formally for a good part of the course. But let's already give it an informal try here.

We had the following components in the examples:

- On the one side, we have a set  $X$ , e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.
- On the other side, we have a set  $A$ , e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction ( $\wedge$ ), disjunction ( $\vee$ ), logical falsity ( $\perp$ ), logical truth ( $\top$ ), and possibly negation ( $\neg$ ).
- And we have a way to go from the spatial side to the algebraic side, and we also have a way to go in the other direction. In particular, we have:
  - A canonical way to determine from subsets of  $A$  with certain nice features an element from  $X$ , i.e., a function  $\epsilon : \overline{A} \rightarrow X$ .
  - A canonical way to assign to each element from  $A$  a subset of  $X$ , i.e., a function  $\eta : A \rightarrow \overline{X}$ .

Finally, we want the translation manual to be *formulaic* in  $X$  and  $A$ : i.e., it should not depend on the idiosyncrasies of the specific  $X$  and  $A$ ; rather, it should work for all  $X$ 's and  $A$ 's of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we do not want the above data for specific  $X$  and  $A$ . Rather, we want it to hold for any  $X$  that is a candidate set for the spatial side, and for any  $A$  that is a candidate for the algebraic side. And hence we also want the ways of going back and forth between the two sides to respect the relations between these candidates for the spatial side and the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates  $X$ , which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category



consists of the algebraic candidates  $A$  and their relations. Then we will see that all the above components of the duality is succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

### 1.3 Exercises

**Exercise 1.a.** Complete the left-out details in the main text. For example, why, for a possible world  $x$  the set of propositions  $F_x$  really satisfied properties (1)–(5). Similarly for valuations  $v$ .

**Exercise 1.b.** Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If  $a \in \mathcal{P}$  is a property, then there also is the property  $\neg a$  of not having property  $a$ . It seems plausible to require that either a given object  $x \in \mathcal{O}$  has a property or it does not. In other words, either  $a \in F_x$  or  $\neg a \in F_x$ . Do you think this is plausible to require? What about vague properties? (Later we see that this if if we have a negation operator on our set of properties obeying the Boolean laws, than  $F$  being prime is equivalent to having the just mentioned negation property.)

**Exercise 1.c** (More of a research project than an exercise). Consider to what extend the first example (objects vs properties) can be developed along the lines of **formal concept analysis**.

**Exercise 1.d.** Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

**Exercise 1.e.** For a logico-philosophical discussion of the principle of indiscernibly, see Ladyman et al. (2012). How does this inform the above philosophical discussion (section 1.1.1)? This paper is in the context of model theory, what does the above duality-theoretic perspective add?

**Exercise 1.f.** Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states

(computable theory of mind vs **connectionism**). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)? Or are these better seen as relations of supervenience rather than duality? What about the infamous Cartesian duality between the physical and the mental world?

**Exercise 1.g.** Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

## 2 The algebraic side: distributive lattices

This chapter introduces formally the algebraic side of duality, which, for us, will be distributive lattices. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. And we end with section 2.4, where we already establish a duality between finite sets (resp. finite partial orders) on the one hand and finite Boolean algebras (resp. finite distributive lattices) on the other hand. This will provide a good idea of the more general case of Stone (resp. Priestley) duality. The main missing ingredient for the general case is topology, which will be the topic of the next chapter.

### 2.1 Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The ‘structure-preserving’ maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

We follow one of the keys lessons of category theory: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism). We will introduce basic notions from category theory later when we need them. For now we only foreshadow it with the ‘objects’ and ‘morphisms’ distinction.

Probably we add an extra chapter on it. If so, link to it here.

#### 2.1.1 Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than.

But the things could also be the dishes offered at your go-to lunch place with the sense of ‘better’ given by your preferences. The formal definition goes as follows.

**Definition 2.1.** A *partial order* (or *partially ordered set*, or *poset*) is a pair  $(P, \leq)$  where  $P$  is a (possibly empty) set and  $\leq$  is a binary relation on  $P$  such that

1. *Reflexive*: For all  $a \in P$ , we have  $a \leq a$ .
2. *Transitive*: For all  $a, b, c \in P$ , if  $a \leq b$  and  $b \leq c$ , then  $a \leq c$ .
3. *Anti-symmetric*: For all  $a, b \in P$ , if  $a \leq b$  and  $b \leq a$ , then  $a = b$ .

A binary relation  $R$  on a set  $P$  is simply a subset of  $P \times P = \{(a, b) : a, b \in P\}$ . For  $a, b \in P$ , one writes  $aRb$  for  $(a, b) \in R$ .

If we do not require axiom 3, we speak of a *preorder*. We say  $\leq$  is a (partial or pre-) order on  $P$ . If the order  $\leq$  is clear from context, we often simply speak of the (partial or pre-) order  $P$ .

The name ‘partial’ is to indicate that not all elements need to be comparable: Formally, for  $a, b \in P$ , we say that  $a$  and  $b$  are *comparable*, if either  $a \leq b$  or  $b \leq a$ ; otherwise they are *incomparable*. If all elements are comparable, we say  $(P, \leq)$  is a *linear* (or *total*).

Formally, the example of the numbers is  $(\mathbb{N}, \leq)$  where  $\mathbb{N}$  is the set  $\{0, 1, 2, \dots\}$  and, for  $n, m \in \mathbb{N}$ , the relation  $n \leq m$  is defined as:  $n$  is smaller or equal to  $m$  (equivalently, there is  $k \in \mathbb{N}$  such that  $n + k = m$ ). Hence this a linear order. In the example of your lunch place, if you have two dishes  $a$  and  $b$  that you find equally tasty—or, more precisely, none tastier than the other, i.e.,  $a$  and  $b$  are incomparable—, then your preference order is only partial and not linear.

Check that this satisfies the axioms.

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder  $(P, \leq)$  into a partial order  $(\bar{P}, \bar{\leq})$  as follows. For  $a, b \in P$ , define  $a \equiv b$  as  $a \leq b$  and  $b \leq a$ . This is an equivalence relation (reflexive, symmetric, and transitive). Equivalence classes are the sets  $[a] := \{b \in P : a \equiv b\}$  for  $a \in P$ . The quotient of  $P$  under  $\equiv$  is  $\bar{P} := P / \equiv := \{[a] : a \in P\}$ . Define  $[a] \bar{\leq} [b]$  by  $a \leq b$  (note that this is independent of the representatives  $a$  and  $b$ ). This renders  $(\bar{P}, \bar{\leq})$  a partial order. It is also called the *poset reflection* of  $P$ . Exercise 2.a makes formally precise in what sense it is the canonical or best possible poset approximating the preorder  $P$ .

There is a nice visualization of partial orders. They are known as Hasse diagrams. An example is in figure 2.1. It depicts the partial order  $(P, \leq)$

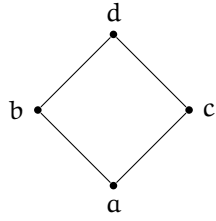


Figure 2.1: The ‘diamond’ as an example of a partial order.

with  $P = \{a, b, c, d\}$  and

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of  $P$  and the edges are the minimal information to recover the order:

- if there is an edge between  $x$  and  $y$  and  $x$  is lower (on the page) than  $y$ , then  $x \leq y$ .
- we do not need to draw an edge from one node to itself because for all nodes  $x$  we have  $x \leq x$ .
- we do not need to draw edges that result from composing existing edges: for example, we have an edge from  $a$  to  $b$  and an edge from  $b$  to  $d$ , so we already know that  $a \leq d$ , hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order  $(P, \leq)$  is as follows. For  $a, b \in P$ , we say that  $b$  *covers*  $a$  (short  $a < b$ ) if  $a \leq b$  and for all  $c \in P$ , if  $a \leq c \leq b$ , then  $c = a$  or  $c = b$ . The elements of  $P$  are the nodes of the Hasse diagram, and an edge is drawn from node  $a$  to node  $b$  whenever  $b$  covers  $a$ . The direction of the edge is indicated by drawing  $b$  higher up in the diagram than  $a$ . So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

**Definition 2.2.** Let  $(P, \leq)$  be a partial order and  $A \subseteq P$ .

- An element  $b \in P$  is a *lower bound* of  $A$  if, for all  $a \in A$ , we have  $b \leq a$ .

*They can be confusing at first, but they really are worth learning. Make sure to draw little Hasse diagrams to illustrate the concepts and how they differ from each other.*

- An element  $b \in P$  is an *upper bound* of  $A$  if, for all  $a \in A$ , we have  $a \leq b$ .
- An element  $c \in P$  is an *infimum* or *greatest lower bound* of  $A$  if (1)  $c$  is a lower bound of  $A$ , and (2), for all lower bounds  $b$  of  $A$ , we have  $b \leq c$ .
- An element  $c \in P$  is a *supremum* or *least upper bound* of  $A$  if (1)  $c$  is an upper bound of  $A$ , and (2), for all upper bounds  $b$  of  $A$ , we have  $c \leq b$ .
- An element  $b \in P$  is a *least* or *bottom* or *minimum* element of  $P$ , if, for all  $a \in P$ , we have  $b \leq a$  (i.e.,  $b$  is the supremum of  $A = \emptyset$ ).
- An element  $b \in P$  is a *greatest* or *top* or *maximum* element of  $P$ , if, for all  $a \in P$ , we have  $a \leq b$  (i.e.,  $b$  is the infimum of  $A = \emptyset$ ).
- An element  $b \in P$  is *minimal* if, for all  $a \in P$ , if  $a \leq b$ , then  $a = b$ .
- An element  $b \in P$  is *maximal* if, for all  $a \in P$ , if  $b \leq a$ , then  $b = a$ .
- An element  $b \in P$  is *minimal in  $A$*  if (1)  $b \in A$  and (2) for all  $a \in A$ , if  $a \leq b$ , then  $a = b$ .
- An element  $b \in P$  is *maximal in  $A$*  if (1)  $b \in A$  and (2) for all  $a \in A$ , if  $b \leq a$ , then  $b = a$ .
- $A$  is an *upset* if for all  $a, b \in P$ , if  $a \in A$  and  $a \leq b$ , then  $b \in A$ .
- $A$  is a *downset* if for all  $a, b \in P$ , if  $b \in A$  and  $a \leq b$ , then  $a \in A$ .
- $A$  is *directed* (aka up-directed) if it is nonempty and for any  $a, b \in A$ , there is  $c \in A$  with  $a \leq c$  and  $b \leq c$ . (Equivalently, all finite subsets of  $A$  have an upper bound in  $A$ .)
- $A$  is *filtered* (aka filtering or down-directed) if it is nonempty and for any  $a, b \in A$ , there is  $c \in A$  with  $c \leq a$  and  $c \leq b$ . (Equivalently, all finite subsets of  $A$  have a lower bound in  $A$ .)

(These notions also make sense in a preorder  $(P, \leq)$ , but if  $P$  is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted  $\bigwedge A$ , called the *meet* of  $A$ ; and the supremum is denoted  $\bigvee A$ , called the *join* of  $A$ . If  $A = \{a_1, \dots, a_n\}$  is finite and nonempty, we write  $\bigwedge A = a_1 \wedge \dots \wedge a_n$  and  $\bigvee A = a_1 \vee \dots \vee a_n$ . In particular,  $\bigwedge \{a, b\} = a \wedge b$  and  $\bigvee \{a, b\} = a \vee b$ . The bottom element, if it exists, is denoted  $\perp$  or  $0$ ; and the top element by  $\top$  or  $1$ . We write  $\min(A)$  (resp.  $\max(A)$ ) for

*It is a good exercise to prove this.*

the elements that are minimal (resp. maximal) in  $A$ . A *directed join* is the supremum of a directed set.

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcp*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can 'turn them upside down' and get another preorder. Formally, if  $(P, \leq)$  is a preorder, define the preorder  $\leq'$  on  $P$  by  $a \leq' b$  iff  $b \leq a$ . We write  $P^{\text{op}}$  for this preorder.

*Verify that this again is a preorder (resp. partial order), and draw some Hasse diagram example to see that this really turns things upside down.*

### 2.1.2 Morphisms: Monotone maps

What maps between partial orders should be considered to be 'structure preserving'? Surely they should preserve the order structure. This yields the concept of a monotone map, and is the standard choice. But there also are other ones, which we mention as well.

*We consider the words 'map' and 'function' as synonymous.*

**Definition 2.3.** Let  $(P, \leq_P)$  and  $(Q, \leq_Q)$  be two preorders and  $f : P \rightarrow Q$  a function. We say  $f$  is

- *monotone* or *order preserving* if, for all  $a, b \in P$ , if  $a \leq_P b$ , then  $f(a) \leq_Q f(b)$ .
- *order reflecting* if, for all  $a, b \in P$ , if  $f(a) \leq_Q f(b)$ , then  $a \leq_P b$ .
- an *order-embedding* if  $f$  is both order preserving and order reflecting.
- an *order-isomorphism* if  $f$  is monotone with a monotone inverse (further comments below).

*Order-embeddings between posets are injective, but the converse fails (i.e., there are injective order preserving maps between posets which are not order-embeddings).*

If  $P$  and  $Q$  are posets, an equivalent condition for  $f$  being an order-isomorphism is that  $f$  is a surjective order-embedding. (It's a good exercise to verify this.) In practice, this is often easier to check, although the definition via a monotone inverse better captures the (category-theoretic) concept of an isomorphism. In full, the latter says: A monotone function  $f : P \rightarrow Q$  between two preorders is an order-isomorphism if there is a monotone function  $g : Q \rightarrow P$  such that

- for all  $a \in P$ , we have  $a = g(f(a))$ , i.e.,  $a$  is the  $g$ -inverse of  $f(a)$  (in short,  $\text{id}_P = g \circ f$ ), and

*Here  $\text{id}_X$  denotes the identity function on set  $X$ . And if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are functions,  $g \circ f$  ( $g$  after  $f$ ) denotes their composition, which maps  $x \in X$  to  $g(f(x)) \in Z$ .*

- for all  $b \in Q$ , we have  $f(g(b)) = b$ , i.e., mapping the  $g$ -inverse of  $b$  along  $f$  yields  $b$  (in short,  $f \circ g = \text{id}_Q$ ).

If two preorders are isomorphic (i.e., there is an order isomorphism between them), we can consider them to be essentially identical. This is difficult to achieve, so it makes sense to look for a *generalization* of the concept of an isomorphism. The key idea is to still require a monotone function  $g : Q \rightarrow P$  in the other direction, but it need not be the *true* inverse but only the best possible *approximation* to an inverse:

- for all  $a \in P$ , we have  $a \leq_P g(f(a))$ , i.e., the  $g$ -inverse of  $f(a)$  is at least as good as  $a$ , and
- for all  $b \in Q$ , we have  $f(g(b)) \leq_Q b$ , i.e., mapping the  $g$ -inverse of  $b$  along  $f$  approximates  $b$ .

Exercise 2.b shows why this approximation then really is best possible; and it also provides the following equivalent definition.

**Definition 2.4.** Let  $(P, \leq_P)$  and  $(Q, \leq_Q)$  be preorders, and let  $f : P \rightarrow Q$  and  $g : Q \rightarrow P$  be monotone functions. The pair  $(f, g)$  is called an *adjunction*, with  $f$  the *left* or *lower adjoint* and  $g$  the *right* or *upper adjoint*, if, for all  $a \in P$  and  $b \in Q$ ,

$$f(a) \leq_Q b \text{ iff } a \leq_P g(b).$$

We also write this as  $\mathbf{l} : P \rightleftharpoons Q : \mathbf{u}$ . An adjunction between  $P^{\text{op}}$  and  $Q$  is called a *Galois connection* or *contravariant adjunction*.

It is best to illustrate this abstract concept with examples. An important template of how Galois connections arise is the following (which includes the instance coining them).

**Lemma 2.5.** Let  $R \subseteq X \times Y$  be a relation between two sets. For any  $a \subseteq X$  and  $b \subseteq Y$ , define

$$\begin{aligned} \mathbf{u}(a) &:= \{y \in Y : \forall x \in a. xRy\} \subseteq Y \\ \mathbf{l}(b) &:= \{x \in X : \forall y \in b. xRy\} \subseteq X \end{aligned}$$

Then  $\mathbf{l} : \mathcal{P}(Y) \rightleftharpoons \mathcal{P}(X) : \mathbf{u}$  forms a Galois connection between the posets  $(\mathcal{P}(X), \subseteq)$  and  $(\mathcal{P}(Y), \subseteq)$ , i.e., for any  $b \subseteq Y$  and  $a \subseteq X$ , we have  $a \subseteq \mathbf{l}(b)$  (i.e.,  $\mathbf{l}(b) \subseteq^{\text{op}} a$ ) iff  $b \subseteq \mathbf{u}(a)$ .

*This is an advanced concept. Give yourself the time to let it sink in by coming back to it over and over again.*

*Note that  $f$  occurs on the left of ' $\leq$ ' and  $g$  on the right.*

*Here  $\mathcal{P}(X)$  is the set of all subsets of the set  $X$ .*



*Proof.* ( $\Rightarrow$ ). Assume  $a \subseteq l(b)$ . To show  $b \subseteq u(a)$ , let  $y \in b$  and show  $y \in u(a)$ . So let  $x \in a$  and show  $xRy$ . By the assumption,  $x \in l(b)$ , so for our  $y \in b$  we have  $xRy$ .

( $\Leftarrow$ ). Assume  $b \subseteq u(a)$ . To show  $a \subseteq l(b)$ , let  $x \in a$  and show  $x \in l(b)$ . So let  $y \in b$  and show  $xRy$ . By the assumption,  $y \in u(a)$ , so for our  $x \in a$  we have  $xRy$ .  $\square$

Here are three instances of this lemma.

1. Maybe you know the name ‘Galois’ from the theory of fields in algebra. Then you know **Galois theory** as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation  $R$  between the set  $X$  of subfields of a given field and the set  $Y$  of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.
2. If  $X$  is a set and  $R \subseteq X \times X$  is a preorder, then  $u(a)$  is the set of upper bounds of  $a \subseteq X$ , and  $l(b)$  is the set of lower bounds of  $b \subseteq X$ .
3. Consider a class of structures  $\mathcal{C}$  (in, say, a first-order signature) and a class  $\mathcal{F}$  of formulas (of this signature). Let  $\models$  be the *interpretation* relation: For  $M \in \mathcal{C}$  and  $\varphi \in \mathcal{F}$  means that structure  $M$  makes true formula  $\varphi$ . Then for a set of models  $a$ ,  $u(a)$  is the theory of  $a$ , i.e., the set of formulas that are true in all those models. And for a theory  $b \subseteq \mathcal{F}$ ,  $l(b)$  is the class of models of  $b$ , i.e., the set of models which make true all the sentences in  $b$ .

*But you don't need to know this for the course. If you'd like an accessible introduction, have a look, e.g., at [this](#) or [this](#) video, or at [these](#) great lecture notes by Tom Leinster.*

*Also recall the examples from section 1.1.*

## 2.2 Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

### 2.2.1 Objects: lattices

The order-theoretic definition of a lattice goes as follows.

**Definition 2.6** (Lattice, order-theoretic). A (*bounded*) *lattice* is a partial order  $L$  in which every finite subset has a supremum and an infimum.

For example, the diamond of figure 2.1 is a lattice. Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.
2. Often a lattice is defined as a partial order in which all binary suprema and infima exist (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exists (i.e., which have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word ‘bounded’ (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.
3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

*As an exercise, prove this.*

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

**Definition 2.7** (Lattice, algebraic). A lattice is a tuple  $(L, \vee, \wedge, \perp, \top)$  where  $\vee$  (pronounced *join*) and  $\wedge$  (pronounced *meet*) are binary operations on  $L$  (i.e., functions  $L \times L \rightarrow L$ ), and  $\perp$  (pronounced *bottom*) and  $\top$  (pronounced *top*) are elements of  $L$ , such that the following axioms holds:

1. *commutative*: for all  $a, b \in L$ , we have  $a \vee b = b \vee a$  and  $a \wedge b = b \wedge a$ .
2. *associative*: for all  $a, b, c \in L$ , we have  $(a \vee b) \vee c = a \vee (b \vee c)$  and  $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ .
3. *idempotent*: for all  $a \in L$ , we have  $a \vee a = a$  and  $a \wedge a = a$ .
4. *absorption*: for all  $a, b \in L$ , we have  $a \wedge (a \vee b) = a$  and  $a \vee (a \wedge b) = a$ .
5. *neutrality*: for all  $a \in L$ , we have  $\perp \vee a = a$  and  $\top \wedge a = a$ .

For example, if  $X$  is a set, then the powerset  $2^X$  forms a lattice in this algebraic sense with union  $\cup$  as join, intersection  $\cap$  as meet,  $\emptyset$  as bottom, and  $X$  as top. This also provides my mnemonic for remembering what ‘join’ and what ‘meet’ is. Think of  $X$  as a set of propositions, and let  $a \in 2^X$  be the beliefs (opinions, values, etc.) that Alice holds, and let  $b \in 2^X$  be the beliefs that Bob holds. Then the meet of  $a$  and  $b$ —i.e.,  $a \wedge b = a \cap b$ —is where Alice and Bob can meet: the common (meeting) ground, the set of

*Prove this. (Hint: think about the supremum of all lower bounds.)*

*Though I’m happy to learn about a better one :-)*

beliefs they agree on. And the join of  $a$  and  $b$ —i.e.,  $a \vee b = a \cup b$ —is the result of joining Alice and Bob together: their joint beliefs, taking together all of their beliefs even if incoherent.

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.c asks you to prove it: that is a bit tedious, but quite instructive.

- Theorem 2.8.** 1. *Given a lattice  $(L, \vee, \wedge, \perp, \top)$  according to the algebraic definition, define  $a \leq_L b$  as  $a \wedge b = a$ . Then  $(L, \leq_L)$  is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by  $\vee$  and  $\wedge$ .*
2. *Given a lattice  $(L, \leq)$  according to the order-theoretic definition, define the binary operations  $\vee$  and  $\wedge$  as binary supremum and infimum, and take  $\perp$  and  $\top$  to be the least and greatest element of  $L$ . Then  $(L, \vee, \wedge, \perp, \top)$  is a lattice according to the algebraic definition, with  $a \wedge b = a$  iff  $a \leq b$  iff  $a \vee b = b$ .*

From now on, we will often just speak of a lattice  $L$  and both use its order-theoretic definition (taking  $\leq$  to be implicitly given) and its algebraic definitions (taking  $\vee, \wedge, \perp, \top$  to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure  $(L, \cdot, 1)$ , where  $\cdot$  is a commutative, associative, and idempotent binary operation on  $L$ , and  $1$  is a neutral element for the operation. The operation  $\cdot$  can then either be seen as the binary infimum for the partial order defined by  $a \leq b$  iff  $a \cdot b = a$  (the join semilattice), or as the binary supremum for the opposite partial order defined by  $a \leq b$  iff  $a \cdot b = b$  (the meet semilattice).

## 2.2.2 Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

**Definition 2.9.** A function  $f : L \rightarrow M$  between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all  $a, b \in L$ , we have  $f(a \vee_L b) = f(a) \vee_M f(b)$
2. for all  $a, b \in L$ , we have  $f(a \wedge_L b) = f(a) \wedge_M f(b)$
3.  $f(\perp_L) = \perp_M$
4.  $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*.

*Prove this.*

If a function  $f : L \rightarrow M$  between lattices preserves  $\perp$  and  $\vee$ , then it preserves all finite joins. This does, in general, *not* imply any preservation of arbitrary existing joins or preservation of infima. The analog statement is true for  $\top$  and  $\wedge$  and preservation of all finite meets.

*Prove this.*

### 2.2.3 Constructions: products, sublattices, homomorphic images, congruences

We introduce several common constructions on lattices. They are common algebraic operations that you might have seen already in other contexts (e.g., group theory); and, in any case, they are worth knowing as they come up quite often.

*Products.* Given a family  $(L_i)_{i \in I}$  of lattices, we can define a lattice  $L = \prod_{i \in I} L_i$  on the Cartesian product where the operations are defined component-wise: e.g., for  $a = (a_i)_i$  and  $b = (b_i)_i$  in  $L$ , we define  $a \leq_L b$  as  $\forall i \in I : a_i \leq_{L_i} b_i$ , and  $(a \wedge b)_i = a_i \wedge b_i$  (similarly for  $\vee$ ), and  $(\perp_L)_i = \perp_{L_i}$  (similarly for  $\top$ ). The projection maps  $\pi_i : L \rightarrow L_i$ , which map  $a = (a_i)_i$  to  $a_i$ , is a surjective lattice homomorphism.

*Recall that the Cartesian product of a family of sets is the set of functions  $a$  that map each  $i \in I$  to an element  $f(i) \in L_i$ . We often write such a function as  $a = (a_i)_{i \in I}$ .*

*Sublattices.* A sublattice of a lattice  $L$  is a subset  $L'$  of  $L$  that contains  $\perp$  and  $\top$  and that is closed under  $\wedge$  and  $\vee$  (i.e., if  $a, b \in L'$ , then  $a \wedge b, a \vee b \in L'$ ). Then  $L'$  is a bounded lattice in its own right and the inclusion map  $\iota : L' \rightarrow L$ , which maps  $a \in L'$  to  $a \in L$ , is a lattice embedding. If we do not require  $\perp$  and  $\top$  to be in  $L'$ , we speak of an *unbounded sublattice*. And if we require  $L'$  to be closed under all suprema and infima, we call it a *complete sublattice*. If  $f : L \rightarrow M$  is a lattice homomorphism, then the direct image  $L' := f[L] = \{f(a) : a \in L\}$  is a sublattice of the lattice  $M$ .

*Homomorphic images.* A lattice  $L'$  is a *homomorphic image* of a lattice  $L$  if there is a surjective lattice homomorphism  $f : L \rightarrow L'$ .

*Congruences.* A congruence on a lattice  $L$  is an equivalence relation  $\vartheta$  on  $L$  that respects the lattice operations, i.e., for all  $a, a', b, b' \in L$ , if  $a \vartheta a'$  and  $b \vartheta b'$ , then also  $a \vee b \vartheta a' \vee b'$  and  $a \wedge b \vartheta a' \wedge b'$ . For an intuitive example, think of the elements of  $L$  as propositions and of  $\vartheta$  as having the same subject matter. The quotient  $L/\vartheta$  carries a unique lattice structure that turns the quotient map  $p : L \rightarrow L/\vartheta$ , which maps  $a \in L$  to its equivalence class  $[a]_\vartheta$  under  $\vartheta$ , into a lattice homomorphism; concretely, this lattice structure

*Birkhoff's famous theorem in universal algebra says that a class of algebraic structures (like lattices) is closed under Homomorphic images, Subalgebras, and Products iff it is definable by equations (hence aka 'HSP theorem').*

is given by  $[a]_{\vartheta} \vee [b]_{\vartheta} := [a \vee b]_{\vartheta}$  (similarly for  $\wedge$ ) with bottom element  $[\perp]_{\vartheta}$  (similarly for  $\top$ ). Note how this is reminiscent of the Lindenbaum–Tarski algebra from the introduction (section 1.1.3).

*The first isomorphism theorem for lattices.* This says that any lattice homomorphism  $f : L \rightarrow M$  can be factored as a surjective lattice homomorphism  $p$  followed by a lattice embedding  $e$  (i.e.,  $f = e \circ p$ ). These are given as follows. The *kernel* of  $f$  is the congruence relation

$$\ker f := \{(a, a') \in L \times L : f(a) = f(a')\}.$$

Choose  $p : L \rightarrow L/\ker f$  (mapping  $a$  to  $[a]$ ) and  $e : L/\ker f \rightarrow M$  (mapping  $[a]$  to  $f(a)$ ). In particular,  $L/\ker f$  is isomorphic to  $f[L]$  (take  $M := f[L]$ , so  $e$  also is surjective); hence the homomorphic images of  $L$  are, up to isomorphism, the quotients of  $L$ .

## 2.3 Distributive lattices and Boolean algebras

We get further subclasses of lattices by requiring that  $\vee$  and  $\wedge$  interact nicely, which is made precise as distributive lattices (section 2.3.1), and by additionally requiring that there is a sense of negation, which is made precise as Boolean algebras (section 2.3.2).

### 2.3.1 Distributive lattices

The idea  $\vee$  and  $\wedge$  interact nicely is made precise as follows.

**Definition 2.10.** A lattice  $L$  is distributive if,

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad (2.1)$$

or, equivalently,

$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \quad (2.2)$$

For example, the four diamond from figure 2.1 is distributive (check why), as is any powerset  $2^X$ .

The equivalence of 2.1 and 2.2 implies that  $L$  is distributive iff  $L^{\text{op}}$  is distributive. So distributivity is a so-called self-dual property. Moreover, homomorphic images, sublattices, and products of distributive lattices are again distributive. (This follows from the ‘HSP theorem’ and the fact that distributive lattices are defined equationally.)

*To exciting thing about this is that lattice homomorphism can be very complicated, but this tells us that they can be broken down into two much simpler things: surjective lattice homomorphisms and injective lattice homomorphisms!*

*Again, the quotients of  $L$  intuitively are much simpler: to determine them, we only have to look at  $L$ , while for homomorphic images we also need to consider other lattices  $M$ .*

*Cf. distributivity from high school:  
 $x \times (y + z) = (x \times y) + (x \times z)$*

*Proving the equivalence of 2.1 and 2.2 is a good exercise!*

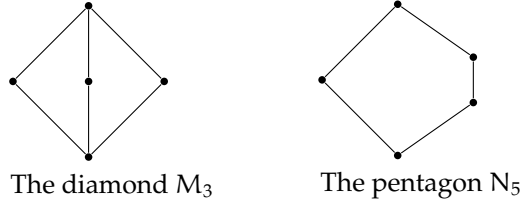


Figure 2.2: The forbidden substructures for distributivity.

Again important special cases are as follows: A *frame* is a complete lattice  $L$  satisfying the join infinite distributive law (JID)

$$\text{for any } a \in L \text{ and } B \subseteq L, a \wedge \bigvee_{b \in B} b = \bigvee_{b \in B} (a \wedge b). \quad (2.3)$$

*In case you have heard of this: A frame is the same thing as a complete Heyting algebra, but their respective choice of morphisms differ.*

In a distributive lattice this, in general, only holds for all *finite*  $B \subseteq L$ .

A seemingly magic characterization of distributive lattices is the following.

**Theorem 2.11** (The  $M_3$ – $N_5$  theorem). *Let  $L$  be a lattice. Then  $L$  is distributive iff  $L$  does not contain an unbounded sublattice which is isomorphic to  $M_3$  or  $N_5$ , depicted in figure 2.3.1.*

*For a proof, see, e.g., Davey and Priestley (2002, 89 ff.).*

### 2.3.2 Boolean algebras

So far, we have seen the order  $\leq$  and the operations  $\vee$  and  $\wedge$  in a lattice, which act like implication, disjunction, and conjunction, respectively. So you might have wondered: what about negation? Especially since this also played a role in our motivating introduction (chapter 1). The (or, more precisely, a) idea of negation is made precise as follows.

**Definition 2.12.** Let  $L$  be a lattice and  $a$  an element of  $L$ . A *complement* of  $a$  is an element  $b$  of  $L$  such that  $a \wedge b = \perp$  and  $a \vee b = \top$ . A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element  $a$  in a distributive lattice is unique, if it exists, and denoted  $\neg a$ .

For example, again the four diamond from figure 2.1 is a Boolean algebra (check why), as is any powerset  $2^X$ . Some further comments:

1. Usually, the negation is then taken into the signature: so a Boolean algebra is a tuple  $(B, \wedge, \vee, \perp, \top, \neg)$  such that  $(B, \wedge, \vee, \perp, \top)$  is a distributive lattice and  $\neg : B \rightarrow B$  a unary function such that, for all  $a \in B$ , we have  $a \wedge \neg a = \perp$  and  $a \vee \neg a = \top$ .

2. But if we have an additional operation around, shouldn't we require the morphisms to preserve it? Fortunately, they already do: If  $f : B \rightarrow A$  is a lattice homomorphism between Boolean algebras, then, for all  $a \in B$ , we have  $f(\neg a) = \neg f(a)$ . We often still refer to them as *Boolean algebra homomorphisms* just to emphasize that we are dealing with Boolean algebras.
3. However, with the notion of a sublattice we need to be more careful: A Boolean algebra may have many sublattices that themselves are not Boolean algebras; so by a (*Boolean*) *subalgebra* of a Boolean algebra  $B$  we mean a sublattice which is also closed under  $\neg$ .
4. If you like ring theory, a Boolean algebra can equivalently be defined as a commutative ring with unit in which all elements are idempotent, see exercise 2.d.
5. There is a best way to turn a distributive lattice  $L$  into a Boolean algebra  $B$ . This  $B$  is called the *Boolean envelope* or *free Boolean extension* of  $L$ . More precisely, this means that for every distributive lattice  $L$  there is a Boolean algebra  $B$  and an injective homomorphism  $e : L \rightarrow B$  such that for any other lattice homomorphism  $h : L \rightarrow A$  into a Boolean algebra  $A$ , there is a unique Boolean algebra homomorphism  $\bar{h} : B \rightarrow A$  such that  $\bar{h} \circ e = h$ . As a diagram:

$$\begin{array}{ccc} L & \xrightarrow{e} & B \\ & \searrow h & \downarrow \bar{h} \\ & & A \end{array}$$

We later will prove this theorem using duality theory.

*The fact that we can use the same morphisms is expressed in categorical terms as the category of Boolean algebras and Boolean algebra homomorphisms being a full (as opposed to any) subcategory of the category of distributive lattices and lattice homomorphisms.*

*In categorical terms this means the category of Boolean algebras is a full reflective subcategory of the category of distributive lattices.*

## 2.4 Duality for finite distributive lattices and finite Boolean algebras

In this section, we prove a 'baby version' of the duality result that we are working toward. Of course, the baby version will follow from the full version, but we prove it now already mostly for pedagogical reasons (1) to already reap some benefits of the build-up of theory so far and (2) to already get used to how a duality theorem looks like.

So let's see to what extent we can formalize the intuitions from the introduction (chapter 1). For concreteness, let's work with the first example

of ‘properties vs objects’ (section 1.1.1), but all we will say will also work for the other examples; so feel free to swap things to your favorite example.

#### 2.4.1 From properties/lattice to recovered objects/spaces and back

So let  $L$  be a *finite* set of properties with the logical operations  $\vee$  (disjunction of properties),  $\wedge$  (conjunction of properties),  $\perp$  (the inconsistent property),  $\top$  (the trivial property). So, plausibly,  $L$  is a finite distributive lattice. Recall how we recovered what the set  $X$  of possible objects must be: They are subsets  $F$  of  $L$  with the following properties, for all  $a, b \in L$ :

1. If  $a \in F$  and  $a \leq b$ , then  $b \in F$ .
2. If  $a, b \in F$ , then  $a \wedge b \in F$ .
3. If  $a \vee b \in F$ , then either  $a \in F$  or  $b \in F$ .
4.  $\top \in F$
5.  $\perp \notin F$ .

Call a subset  $F \subseteq L$  with properties 1, 2, and 4 a *filter*. A filter is *proper* if it also has property 5. And it is *prime* if it additionally has property 3.

Since  $L$  is finite, filters can be described more concretely: they all must have a least element  $c \in F$ , so they are of the form  $\uparrow c = \{a \in L : c \leq a\}$ . Here is why: Since  $F$  is nonempty, let  $c_0 \in F$ . If  $c_0$  is not the least element of  $F$ , there must be  $c'_1 \in F$  with  $c_0 \not\leq c'_1$ . So  $c_1 := c_0 \wedge c'_1 \in F$  which is strictly below  $c_0$  (as infimum it is below, and if we had  $c_0 = c_1 = c_0 \wedge c'_1$ , then  $c_0 \leq c'_1$ ). If  $c_1$  is not the least element of  $F$ , there again must be  $c'_2 \in F$  with  $c_1 \not\leq c'_2$ , so  $c_1 > c_2 := c_1 \wedge c'_2 \in F$ . And so on. However, since  $F \subseteq L$  is finite, this process cannot go on forever, and whenever it stops, we have found the least element of  $F$ .

So a filter  $F$  can more concisely be described by its least element  $c$ . What, then, are our desired prime filters? To say that  $F$  is proper is to say that  $c \neq \perp$ , and to say that  $F$  is prime is to say that also the least element  $c$  has the following property:

$$\forall a, b \in L : c \leq a \vee b \Rightarrow c \leq a \text{ or } c \leq b.$$

An element  $c$  of  $L$  that is not  $\perp$  and that has this property is called *join-prime*.

So, our recovered version  $\mathcal{J}(L)$  of the set  $X$  of possible objects is the set of join-prime elements of  $L$ . So our recovered objects  $\mathcal{J}(L) \subseteq L$  are ordered by

*NB: The book defines  $\mathcal{J}(L)$  to be the join-irreducible elements of  $L$ . For general lattices, these differ from join-prime elements; but for distributive lattices, they coincide (and this is equivalent to being distributive). See exercise 2.f.*



the order inherited from  $L$ . And this makes sense: also our real objects  $X$  are ordered by *generalization*: For  $x, y \in X$ , say  $y$  is more or equally general compared to  $x$  (written  $y \succeq x$ ), if for all properties  $a \in L$ , if  $y$  has  $a$ , then also  $x$  has  $a$  (but  $x$  might have more properties, and thus be more special). The properties of a recovered object  $c \in \mathcal{J}(L)$  are precisely the elements of the filter it represents, i.e., the  $a \in \uparrow c$ . So for two recovered objects  $c, d \in \mathcal{J}(L)$ , we have that  $d$  is more general than  $c$  (i.e.,  $d \succeq c$ ) precisely if for all  $a \in L$ , if  $a \geq d$ , then  $a \geq c$ . But this is equivalent to  $d \geq c$ . So the generalization order  $\succeq$  of  $\mathcal{J}(L)$  really is the order  $\geq$  inherited from  $L$ .

Okay, so far we started with a finite distributive lattice  $L$  of properties and recovered the finite set of possible objects as  $\mathcal{J}(L)$ . But what if we go in the other direction and start with a finite set of possible objects  $X$ , which—as we just saw—comes with a generalization order  $\preceq$ , and we want to recover the properties  $L$ ?

So let  $X$  be a finite set of possible objects with generalization order  $\preceq$ . What are the properties of  $X$ ? They are described by their extension, i.e., the subset  $a$  of  $X$  consisting of the objects in  $X$  which have this property. These properties will respect the generalization order: For  $x, y \in X$ , if  $y$  is more general than  $x$ , i.e.,  $x \preceq y$ , then if  $y$  has property  $a$  (i.e.,  $y \in a$ ), then also  $x$  has property  $a$  (i.e.,  $x \in a$ ). So  $a$  is a downset of the generalization order! So we identify the recovered properties of  $X$  with the downsets of  $(X, \preceq)$ . We write  $\mathcal{D}(X)$  for the set of downsets of  $X$ . It's not hard to check that  $\mathcal{D}(X)$  is a (even complete) sublattice of the powerset lattice  $2^X$ , and hence also distributive.

*The opposite of the generalization order is the specialization order (which might be better known and which we see in chapter 3):  $y$  is more special than  $x$  iff  $x$  is more general than  $y$  iff any property of  $x$  is also a property of  $y$ .*

*See exercise 2.g.*

## 2.4.2 The recovered lattices and spaces are isomorphic to the original ones

Next we said that we would expect (1) that the set of objects with a generalization order is isomorphic to the objects recovered from the recovered properties (the 'double dual' properties), and that (2) the set of properties is isomorphic to the properties recovered from the recovered objects (the 'double dual' objects). Let's verify these in turn.

Concerning the isomorphism between double dual and original objects, recall from section 1.1.1, that our guiding principle was this: Each set of properties  $F \subseteq L$  that is a prime filter determines an object  $x \in X$ , namely, the unique object having exactly the properties in  $F$ . Translated into our formal setting, the recovered objects  $A$  (corresponding to  $F$ ) are join-prime elements of the recovered properties, which in turn are downsets of  $X$  (the set of downsets corresponds to  $L$ ). The set of properties encoded by  $A$  is

the set of all downsets  $D$  that contain  $A$ . So an original object  $x \in X$  has all those properties if it is in all  $D$ , i.e., it is in  $A$ . To have a canonical choice, we would want to pick the most general object of  $A$ , i.e., we hope that  $A$  has a greatest element—fortunately this is the case.

**Proposition 2.13.** *Let  $(X, \preceq)$  be a finite partial order. Then the following is well-defined and an order-isomorphism.*

$$\begin{aligned} \hat{\cdot} : \mathcal{J}(\mathcal{D}(X)) &\rightarrow X \\ A &\mapsto \text{the greatest element of } A \end{aligned}$$

*Proof.* We first show that  $A$  indeed has a greatest element. Since  $A \subseteq X$  is a downset, we have (writing  $\downarrow a := \{x \in X : x \preceq a\}$ )

$$A = \bigcup_{a \in A} \downarrow a.$$

Since  $X$  is finite, this is a finite join in the lattice  $\mathcal{D}(X)$  that is (greater-or) equal to  $A$ . Since  $A$  is join-prime, there hence is  $a \in A$  such that  $\downarrow a \supseteq A$ , so, since we have the other inclusion since  $a \in A$ , we have  $\downarrow a = A$ , hence  $a$  is the greatest element of  $A$ .

So  $\hat{\cdot}$  is a well-defined function. It remains to show that it is an order-isomorphism.

**Order-preservation and -reflection:** Let  $A, B \in \mathcal{J}(\mathcal{D}(X))$ . If  $A \subseteq B$ , then the greatest element  $a$  of  $A$  is in  $B$ , and hence it is below the greatest element  $b$  of  $B$ . Conversely, if  $a \leq b$ , then  $A = \downarrow a \subseteq \downarrow b = B$ .

**Surjectivity:** If  $x \in X$ , consider  $A := \downarrow x$ . This downset is join-prime: If  $A_1$  and  $A_2$  are two downsets of  $X$  with  $A_1 \cup A_2 \supseteq A$ , then  $x$  is in  $A_i$  for  $i$  either 1 or 2, and since  $A_i$  is a downset,  $A = \downarrow x \subseteq A_i$ . Finally, we clearly have  $\hat{A} = x$ , as needed.  $\square$

Concerning the isomorphism between double dual and original properties, the guiding principle was the following. (We did not discuss it in section 1.1.1, but as the truthset function in section 1.1.2.) Each property  $a$  is mapped to its *extension*, i.e., the set of objects that have this property. Translated into our formal setting, the recovered objects are prime filters, represented as join-prime elements  $j$ , and them having property  $a$  means that  $a$  is in the filter, i.e.,  $j \leq a$ .

**Proposition 2.14.** *Let  $L$  be a finite distributive lattice. Then the following defines*

*This is known as Birkhoff's representation theorem (from 1937).*

a lattice isomorphism.

$$\begin{aligned}\widehat{\cdot} : L &\rightarrow \mathcal{D}(\mathcal{J}(L)) \\ a &\mapsto \widehat{a} := \{j \in \mathcal{J}(L) : j \leq_L a\}\end{aligned}$$

*Proof.* Note that, by construction,  $\widehat{a}$  is a downset of  $\mathcal{J}(L)$ , so this is well-defined. We show that  $\widehat{\cdot}$  is a bijective lattice homomorphism.

Concerning meet, we have to show  $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$ . Indeed, for  $j \in \mathcal{J}(L)$ , we have  $j \leq a \wedge b$  iff  $j \leq a$  and  $j \leq b$ .

Concerning join, we have to show  $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$ . Indeed, for  $j \in \mathcal{J}(L)$ , we have that if  $j \leq a$  or  $j \leq b$ , also  $j \leq a \vee b$ . And if  $j \leq a$  and  $j \leq b$ , then, since  $j$  is join-prime, either  $j \leq a$  or  $j \leq b$ .

Concerning bottom, we have to show  $\widehat{\perp} = \emptyset$ . Indeed, otherwise there is  $j \in \mathcal{J}(L)$  with  $j \leq \perp$ , so  $j = \perp$ , but the bottom element never can be join-prime.

Concerning top, we have to show  $\widehat{\top} = \mathcal{J}(L)$ . Indeed, for any  $j \in \mathcal{J}(L)$ , we have  $j \leq \top$ .

Concerning injectivity, if  $a \neq b$ , then either  $a \not\leq b$  or  $b \not\leq a$ . Without loss of generality, assume  $a \not\leq b$ . We find  $j \in \mathcal{J}(L)$  such that  $j \leq a$  but  $j \not\leq b$ , hence  $\widehat{a} \neq \widehat{b}$ . Indeed, consider the set  $A := \{c \in L : c \leq a \text{ and } c \not\leq b\}$ . It is nonempty, since  $a \in A$ . Since  $L$  is finite, it has a minimal element  $j$ . So it remains to show that  $j$  is join-prime. Now,  $j$  cannot be  $\perp$ , since  $\perp \leq b$ . So assume  $j \leq c \vee c'$ , i.e.,  $j = j \wedge (c \vee c') = (j \wedge c) \vee (j \wedge c')$ . We have that either  $j \wedge c \not\leq b$  or  $j \wedge c' \not\leq b$ , because otherwise  $b$  is an upper bound of both, so  $j = (j \wedge c) \vee (j \wedge c') \leq b$ . Without loss of generality, say  $j \wedge c \not\leq b$ . Since also  $j \wedge c \leq j \leq a$ , we have  $j \wedge c \in A$ . Since  $j \wedge c \leq j$  and  $j$  is minimal in  $A$ , we have  $j \wedge c = j$ , so  $j \leq c$ , as needed.

Concerning surjectivity, let  $A \in \mathcal{D}(\mathcal{J}(L))$  and find  $a \in L$  such that  $\widehat{a} = A$ . Since  $A$  is finite,  $a := \bigvee A \in L$  exists. To show  $\widehat{a} = A$ , let  $j \in \mathcal{J}(L)$ . Indeed, we have:  $j \in \widehat{a}$  iff  $j \leq \bigvee A$  iff (since  $j$  is join-prime)  $j \leq b$  for some  $b \in A$  iff (since  $A$  is a downset)  $j \in A$ .  $\square$

### 2.4.3 Also including morphisms

Finally, to complete the duality, we also want a correspondence between the morphisms of the algebraic and the spatial side. After all, we should not only be able to relate objects, but also their interconnections!

More concretely, we now know that there is an exact correspondence between finite partial orders  $X$  (the spatial side) and finite distributive lattices  $L$  (the algebraic side): we can relate  $X$  to the lattice  $\mathcal{D}(X)$ , and this

is ‘bijective’ correspondence because every lattice  $L$  is of this form (since  $L \cong \mathcal{D}(\mathcal{J}(L))$ ) and distinct posets/spaces are related to distinct lattices (if  $\mathcal{D}(X) \cong \mathcal{D}(Y)$ , then  $X \cong \mathcal{J}(\mathcal{D}(X)) \cong \mathcal{J}(\mathcal{D}(Y)) \cong Y$ ).

So, do we have a similar bijective correspondence for the morphisms? Do order-preserving maps between posets correspond bijectively to lattice homomorphisms between their dual lattices? Yes, as the following shows.

**Proposition 2.15.** *Let  $(X, \preceq)$  and  $(Y, \preceq)$  be finite partial orders. Let  $f : X \rightarrow Y$  be order-preserving. Then*

$$\begin{aligned} \mathcal{D}(f) : \mathcal{D}(Y) &\rightarrow \mathcal{D}(X) \\ D &\mapsto f^{-1}(D) \end{aligned}$$

*is a lattice homomorphism. And if  $h : \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$  is a lattice homomorphism, there is a unique order-preserving function  $f : X \rightarrow Y$  such that  $\mathcal{D}(f) = h$ .*

Before we consider the proof, some comments. Note that  $f$  and  $\mathcal{D}(f)$  go in opposite directions:  $f$  goes from  $X$  to  $Y$ , but  $\mathcal{D}(f)$  goes from  $\mathcal{D}(Y)$  to  $\mathcal{D}(X)$ . This is a characteristic feature of dualities: it is precisely the reason why, in the category-theoretical terminology, we say that we have a *dual* equivalence and not just an equivalence between the spatial and the algebraic side. One reason is that the inverse image map  $f^{-1}(\cdot)$  has much better preservation properties than the direct image map  $f[\cdot]$ , which would go in the same direction: For example, the inverse of an intersection is the intersection of the inverses, but the image of an intersection need not be the intersection of the images.

We won’t consider the proof in full detail, but we stress the use of an adjunction in it.

*Proof sketch.* Proving that  $\mathcal{D}(f)$  is a lattice homomorphism is a very good exercise it precisely brings about the strong preservation properties of the inverse image map. For the second claim, one convenient fact about homomorphisms  $h$  between finite lattices is this: Since they preserve all finite meets, and since there are only finitely many meets, they in fact preserve all meets. We saw in exercise 2.b that this is characteristic of upper adjoints, and one can show that  $h$  indeed must have a lower adjoint  $g$ . Since  $h$  also preserves joins, it is not difficult to show—using the defining feature of an adjunction—that then  $g$  maps join-prime elements to join-prime elements.

With this, we can define  $f : X \rightarrow Y$  as follows. Given  $x \in X$ , we find  $y = f(x)$  as follows: The downset  $\downarrow x \in \mathcal{D}(X)$  is join-prime, so  $g(\downarrow x)$  is

*Technically, this relates class-many structures to class-many structures, so we cannot speak of a bijective functions, and it is only up to isomorphism—hence the quotation marks.*

*See exercise 2.h.*

join-prime in  $\mathcal{D}(Y)$ , so it is of the form  $\downarrow y$ , with  $y \in Y$ , and we choose this  $y$  as the value of  $x$  under  $f$ .

That  $f$  is order preserving can be seen from  $g$  being order preserving. Finally, for  $E \in \mathcal{D}(Y)$ , we need to show  $h(E) = f^{-1}(E)$ . Indeed, for  $x \in X$ , we have, by the adjunction, that

$$\begin{aligned} x \in h(E) &\xLeftrightarrow{\text{downset}} \downarrow x \subseteq h(E) \xLeftrightarrow{\text{adjunction}} g(\downarrow x) \subseteq E \\ &\xLeftrightarrow{\text{def. of } f} \downarrow f(x) \subseteq E \xLeftrightarrow{\text{downset}} f(x) \subseteq E. \end{aligned}$$

For uniqueness, if  $f, f' : X \rightarrow Y$  are such that  $\mathcal{D}(f) = \mathcal{D}(f')$ , we leave it as an exercise to show that  $f = f'$ .  $\square$

See exercise 2.i.

#### 2.4.4 The duality and the special case of Boolean algebras

If we take everything together, we get our desired duality. With  $\mathcal{D}$ , we have a way to go from the spatial side to the algebraic side: map a poset  $X$  to the lattice  $\mathcal{D}(X)$ , and a order preserving map  $f : X \rightarrow Y$  to lattice homomorphism  $\mathcal{D}(f) : \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$ . With  $\mathcal{J}$ , we have a way to go from the algebraic side to the spatial side: map a lattice  $L$  to the poset  $\mathcal{J}(L)$ , and a lattice homomorphism  $h : L \rightarrow M$  to the unique order preserving map  $f : \mathcal{J}(L) \rightarrow \mathcal{J}(M)$  with  $\mathcal{D}(f) = h$  (up to isomorphism of the domain and codomain). And going back and forth in this way cancels out:  $\mathcal{D}(\mathcal{J}(L)) \cong L$  and  $\mathcal{J}(\mathcal{D}(X)) \cong X$ . To summarize:

**Theorem 2.16.** *The facts expressed by propositions 2.13–2.15 is expressed category-theoretically as: The functors  $\mathcal{D}$  and  $\mathcal{J}$  constitute a duality between the category  $\mathbf{DL}_f$  of finite distributive lattices with lattice homomorphisms and the category  $\mathbf{Pos}_f$  of finite posets with order-preserving functions.*

It remains to see what happens if we restrict us from distributive lattices to Boolean algebras: i.e., distributive lattice where we also have a notion of complement. What are the corresponding spaces?

Let's see: If  $L$  is a finite Boolean algebra, what can we say about the dual space  $(X, \preceq) := \mathcal{J}(L)$ ? Specifically, how does the partial order look like? Surprisingly, it must be the identity relation: If  $a$  and  $b$  are join-prime elements of the Boolean algebra  $A$  with  $a \leq b$ , then  $a = b$ . Proof: We have  $b \leq \top = a \vee \neg a$ , so either  $b \leq a$  or  $b \leq \neg a$ . The latter cannot be: Otherwise  $a \leq b \leq \neg a$ , so  $a = a \wedge \neg a = \perp$ , but  $a \neq \perp$  qua join-prime element. Hence  $b \leq a$ . Since also  $a \leq b$ , the claim  $a = b$  follows.

*In fact, finite Boolean algebras are characterized by this, see Gehrke and van Gool 2023, prop. 1.26.*

In short: Once the properties of our space that we consider have complements (i.e., can be negated), the generalization order on the points of

the space becomes trivial.

Conversely, if  $(X, \preceq)$  is a finite partial order where  $\preceq$  is the identity relation, then any subset is a downset, so  $\mathcal{D}(X) = 2^X$  is a Boolean algebra! So the dual spaces of finite Boolean algebras are precisely the posets where the order is the identity relation. But then we might as well just forget the order, as it adds no information. Hence our duality cuts down to a duality between finite Boolean algebras and finite sets:

**Theorem 2.17.** *The functors  $\mathcal{D}$  and  $\mathcal{J}$  cut down to a duality between the category  $\mathbf{BA}_f$  of finite Boolean algebras with homomorphisms and the category  $\mathbf{Set}_f$  of finite sets with functions.*

At the end of section 1.1.2, we already hinted at the fact that, to generalize this result beyond the finite, we to add some further structure on the spaces: namely, a topology. This is what we will do in the next chapter.

## 2.5 Exercises

**Exercise 2.a.** Recall that for a preorder  $(P, \leq)$ , we have defined the poset reflection  $(\bar{P}, \bar{\leq})$ . This exercise makes precise in which sense this is the best possible poset approximating the preorder  $(P, \leq)$ .

*Exercise 1.1.5 in Gehrke and van Gool (2023), with small changes.*

1. Prove that  $\equiv$  is an equivalence relation.
2. Prove that the definition of  $\bar{\leq}$  is independent of the representatives: If  $a' \in [a]$  and  $b' \in [b]$ , then  $a \leq b$  iff  $a' \leq b'$ .
3. Prove that  $(\bar{P}, \bar{\leq})$  is indeed a partial order.
4. Prove that  $\bar{\leq}$  is the smallest partial order on  $\bar{P} = P / \equiv$  such that the quotient map  $f : P \rightarrow P / \equiv$ , which maps  $a$  to  $[a]$ , is order preserving: That is, if  $\leq'$  is another such partial order on  $P / \equiv$ , then  $\bar{\leq} \subseteq \leq'$ .
5. Prove that, for any order preserving  $g : P \rightarrow Q$  into a poset  $Q$ , there exists a unique order preserving  $\bar{g} : P / \equiv \rightarrow Q$  such that  $\bar{g} \circ f = g$ . As a diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & P / \equiv \\ & \searrow g & \downarrow \bar{g} \\ & & Q \end{array}$$

*The category-theoretic formulation of this fact is: the inclusion of the category of partial orders and monotone maps in the category of preorders and monotone maps has a left adjoint. Adjoint functors can be interpreted as formalizing the idea of finding a best possible approximation.*

Think about how the last item formalizes the idea that  $(\bar{P}, \bar{\leq})$  is the best possible poset approximating the preorder  $(P, \leq)$ .

**Exercise 2.b.** Let  $(P, \leq_P)$  and  $(Q, \leq_Q)$  be two preorders, and let  $f : P \rightarrow Q$  and  $g : Q \rightarrow P$  be monotone maps.

*Exercise 1.1.8 in Gehrke and van Gool (2023).*

1. Prove that  $(f, g)$  is an adjunction iff for all  $a \in P$  we have  $a \leq_P g(f(a))$  and for all  $b \in Q$  we have  $f(g(b)) \leq_Q b$ .

For the rest of this exercise, assume that  $(f, g)$  is an adjunction.

2. Prove that  $f \circ g \circ f(a) \equiv f(a)$  and  $g \circ f \circ g(b) \equiv g(b)$  for every  $a \in P$  and  $b \in Q$  (and  $a \equiv b$  iff  $a \leq b$  and  $b \leq a$ ).
3. Conclude that, in particular, if  $P$  and  $Q$  are posets, then  $fgf = f$  and  $gfg = g$ .
4. Prove that, if  $P$  is a poset, then for any  $a \in P$ ,  $gf(a)$  is the least element above  $a$  that lies in the image of  $g$ .
5. Formulate and prove a similar statement to the previous item about  $fg(b)$ , for  $b \in Q$ .
6. Prove that, for any subset  $A \subseteq P$ , if the supremum of  $A$  exists, then  $f(\bigvee A) = \bigvee f(A)$  (where  $f(A) = \{f(a) : a \in A\}$  is the image of  $A$  under  $f$ ).
7. Prove that, for any subset  $B \subseteq Q$ , if the infimum of  $B$  exists, then  $g(\bigwedge B) = \bigwedge g(B)$ .

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

*We will see that the converse holds for complete lattices. This is a special case of the Adjoint Functor Theorem.*

**Exercise 2.c.** Prove theorem 2.8.

*From Gehrke and van Gool 2023, ex. 1.2.13.*

**Exercise 2.d.** This exercise shows that Boolean algebras and Boolean rings are equivalent.

1. Let  $(B, +, \cdot, 0, 1)$  be a Boolean ring, i.e., a commutative ring with unit in which  $a \cdot a = a$  for all  $a \in B$ . Define  $a \leq b$  if  $a \cdot b = a$ . (We often write  $ab$  for  $a \cdot b$ .) Prove that  $\leq$  is a distributive lattice order on  $B$  where
  - 1 is the greatest element and 0 is the least element,
  - meet is given by  $ab$  and join is given by  $a + b + ab$ , and
  - every element  $a$  of has the complement  $1 + a$  with respect to  $\leq$ .

*Hint:* First show that  $a + a = 0$  for all  $a \in B$ .

2. Conversely, let  $(B, \wedge, \vee, \perp, \top, \neg)$  be a Boolean algebra. Define, for any  $a, b \in B$ ,

$$a + b := (a \wedge \neg b) \vee (\neg a \wedge b)$$

$$0 := \perp$$

$$a \cdot b := a \wedge b$$

$$1 := \top.$$

*The operation  $+$  is known as symmetric difference.*

Prove that  $(B, +, \cdot, 0, 1)$  is a Boolean ring.

3. Finally, show that the composition of these two assignments in either order yields the identity.

**Exercise 2.e.** To provide examples and non-examples about the notion of complement:

1. Show that the  $n$ -chain  $\mathbf{n} = \{0, 1, \dots, n-1\}$  is a distributive lattice.
2. Show that, for  $n = 2$ , the two-element lattice  $\mathbf{2} = \{0, 1\}$  is a Boolean algebra.
3. But for  $n \geq 3$ ,  $\mathbf{n}$  is not a Boolean algebra. (Complements don't always exist.)
4. Show that the distributive lattices  $M_3$  and  $N_5$  from figure 2.3.1 are not Boolean algebras. (Complements exists but are not unique.)
5. Show that the complement of an element of a distributive lattice is unique if it exists.

**Exercise 2.f.** This exercise explores the difference and similarities of join-prime and join-irreducible elements. Let  $L$  be a lattice. Recall that an element  $a \in L$  is

- join-prime if  $a \neq \perp$  and for all  $b, c \in L$ , if  $a \leq b \vee c$ , then either  $a \leq b$  or  $a \leq c$ .
- join-irreducible if  $a \neq \perp$  and for all  $b, c \in L$ , if  $a = b \vee c$ , then either  $a = b$  or  $a = c$ .

Now, establish the following relationships between these concepts.

1. Show that  $a$  is join-prime iff for all finite  $S \subseteq L$ , if  $a \leq \bigvee S$ , then there is  $s \in S$  such that  $a \leq s$ .
2. Convince yourself that you can analogously show that  $a$  is join-irreducible iff for all finite  $S \subseteq L$ , if  $a = \bigvee S$ , then there is  $s \in S$  such that  $a = s$ .



3. Show that if  $a$  is join-prime, then  $a$  is join-irreducible.
4. Show that the converse holds if  $L$  is distributive.
5. Provide an example of a lattice  $L$  which has a join-irreducible element  $a$  which is not join-prime.

**Exercise 2.g.** Let  $X$  be a poset and  $\mathcal{D}(X)$  the set of its downsets. Show that  $\mathcal{D}(X)$  is a sublattice of  $2^X$ . In particular, intersections and unions of downsets are again downsets.

**Exercise 2.h.** Let  $f : X \rightarrow Y$  be a function between two sets  $X$  and  $Y$ . Show that the function from  $2^Y$  to  $2^X$  defined by

$$B \mapsto f^{-1}(B)$$

is a Boolean algebra homomorphism. Show that if  $f$  additionally is a monotone function between two posets  $X$  and  $Y$ , then this mapping restricts to a lattice homomorphism  $\mathcal{D}(Y) \rightarrow \mathcal{D}(X)$ .

**Exercise 2.i.** In the context of proposition 2.15, show that if  $f, f' : X \rightarrow Y$  are such that  $\mathcal{D}(f) = \mathcal{D}(f')$ , then  $f = f'$ .

## 3 The spatial side: topological spaces

This chapter introduces formally the spatial side of duality, which, for us, will be certain ordered topological spaces, i.e., topological spaces that also have a partial order on their points. This naturally structures this chapter: In section 3.1, we provide a general introduction to topological spaces. In section 3.2, we discuss how partial orders can be added to topological spaces (like the generalization order that we have already seen). And in section 3.3, we define the particular ordered topological spaces that we will consider: the compact ordered spaces. And we show that they are equivalently described without the order as stably compact spaces. Then we have both the algebraic and the spatial side together, so we can prove the duality result in the next chapter.

### 3.1 Introduction to topological spaces

When we hear of ‘space’, we naturally think of the three-dimensional space we live in. And this indeed is an example of a topological space. It is the three-dimensional Euclidean space  $\mathbb{R}^3$  whose points  $x = (x_1, x_2, x_3)$  are described by the values on the  $x$ -axis, the  $y$ -axis, and the  $z$ -axis. From high-school we also know what lines and planes are in this space, and what their geometry is.

But there also are other spaces. For example, the surface of a sphere. Its points are those  $(x_1, x_2, x_3)$  with  $x_1^2 + x_2^2 + x_3^2 = 1$ . But its geometry is different: for instance, the angles of a triangle add up to more than 180 degrees. Yet another space is the spacetime that we live in according to general relativity. Its points  $x = (x_1, x_2, x_3, x_4)$  are four-dimensional—with three spatial and one temporal component—and its geometry is given by a metric tensor. And there are even wilder spaces, where it might not even make sense to speak of a ‘geometry’ (e.g., angles between lines), but only of ‘spatial’ properties (e.g., continuous paths from one point to another).

After much research, mathematicians—most notably Felix Hausdorff in 1914—came up with a general definition of a topological space that includes all these examples. When one first reads this rather abstract definition, one wonders how it possibly can cover all the relevant spatial concepts of the specific examples. But we see how, just from this parsimo-

*Here we only refer to an intuitive difference between ‘geometric’ vs ‘spatial’ (or topological) properties: the latter are invariant under stretching and squishing the space, but the former are not. This is why topology is colloquially also described as rubber sheet geometry.*

nious definition of a topological space, we can define many of the common spatial concepts. Again, we split this discussion into objects (topological spaces) and morphisms (continuous functions between spaces).

### 3.1.1 Objects: topological spaces

Without further ado, here is the abstract definition of a topological space.

**Definition 3.1.** A *topological space* is a pair  $(X, \tau)$  where  $X$  is a nonempty set and  $\tau$  is a collection of subsets of  $X$  such that

1.  $\emptyset$  and  $X$  are in  $\tau$
2. If  $U, V \in \tau$ , then  $U \cap V \in \tau$
3. If  $U_i \in \tau$  is a collection of sets indexed by a set  $I$ , then  $\bigcup_{i \in I} U_i \in \tau$ .

We also call  $\tau$  a topology on  $X$ . And we call the elements of  $X$  *points*. The elements of  $\tau$  are called *open sets* (or *opens*). Their complements, i.e., sets of the form  $C = X \setminus U$  for  $U \in \tau$ , are called *closed sets*. A subset  $K \subseteq X$  that is both open and closed (i.e.,  $K \in \tau$  and  $K^c \in \tau$ ) is called *clopen*. We just speak of the topological space  $X$  if  $\tau$  is clear from context. Then we write  $\Omega(X)$  for the opens of  $X$ . The collection of closed (resp. clopen) subsets of  $X$  is denoted  $\mathcal{C}(X)$  (resp.  $\text{Clp}(X)$ ).

Here are some examples:

1. Two trivial examples: For any nonempty set  $X$ , the set  $\tau := 2^X$  is a topology on  $X$ . It is called the *discrete* topology. Also  $\tau := \{\emptyset, X\}$  is a topology on  $X$ . It is called the *indiscrete* topology.
2. Our three-dimensional space as a topological space: the underlying set is  $X := \mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$  and the opens are those subsets  $U \subseteq \mathbb{R}^3$  that allow some ‘wiggle-room’, which is made precise as follows. Recall that the usual distance between two points  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3)$  is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

So a subset  $U \subseteq \mathbb{R}^3$  is defined to be open precisely if:

for all  $x \in \mathbb{R}^3$ , if  $x \in U$ , then there is  $\epsilon > 0$  such that for all  $x' \in \mathbb{R}^3$  with  $d(x, x') < \epsilon$ , we have  $x' \in U$ .

This is called the Euclidean topology on  $\mathbb{R}^3$ .

*Mathematics provides many formal notions of space (e.g., Euclidean space, vector space, Hilbert space, probability space, Banach space, etc.). But topological spaces are a very general such notion.*

*Some also allow the empty topological space.*

*Equivalently:  $\tau$  is closed under finite intersection and arbitrary union (which includes the empty intersection  $X$  and the empty union  $\emptyset$ ). In particular,  $\tau$  is a sublattice of  $2^X$ .*

Next, we define some central topological concepts for a topological space  $X$ .

1. *Interior and closure.* If  $S \subseteq X$  is a subset, there is a largest open set contained in  $S$ , which is called the *interior* of  $S$ :

$$\text{Int}(S) := \bigcup \{U \in \tau : U \subseteq S\}.$$

There also is a smallest set closed set containing  $S$ , which is called the *closure* of  $S$ :

$$\text{Cl}(S) := \bigcap \{C \in \mathcal{C}(X) : S \subseteq C\}.$$

2. *Neighborhood.* A subset  $S \subseteq X$  is a *neighborhood* of a point  $x \in X$  if  $x \in \text{Int}(S)$ . Accordingly, an *open neighborhood* of a point is an open set containing this point. (If it's clear we're talking about an open neighborhood, we might drop the adjective 'open'.)

3. *Dense.* A subset  $S \subseteq X$  is *dense* (in  $X$ ) if for all points  $x \in X$  and open neighborhoods  $U$  of  $x$ , there is a point  $s \in S$  with  $s \in U$ . So the points of  $X$  can be approximated arbitrarily closely by points in  $S$ . An equivalent formulation is:  $\text{Cl}(S) = X$ .

4. *Convergence.* A sequence  $(x_n)_{n \in \mathbb{N}}$  of points in  $X$  *converges* to a point  $x \in X$  if for all open neighborhoods  $U$  of  $x$ , there is  $N \geq 0$  such that, for all  $n \geq N$ , we have  $x_n \in U$ .

5. *(Sub)base.* Given a nonempty set  $X$ , any collection  $\mathcal{S}$  of subsets of  $X$  *generates* a topology  $\langle \mathcal{S} \rangle$ : namely, the smallest topology that contains  $\mathcal{S}$ . This exists because an arbitrary intersection of topologies on  $X$  is again a topology on  $X$ . Concretely,  $\langle \mathcal{S} \rangle$  is the set of arbitrary unions of finite intersections of elements of  $\mathcal{S}$ .

If  $\tau$  is a topology on  $X$ , a collection  $\mathcal{S}$  of subsets of  $X$  is called a *subbase* of  $\tau$  if  $\tau = \langle \mathcal{S} \rangle$ . So the opens of  $\tau$  are arbitrary unions of finite intersection of subbasic elements.

Finally, a *base* for the topology  $\tau$  is a subbase  $\mathcal{S}$  such that for every open neighborhood  $U$  of any point  $x \in X$ , there is  $V \in \mathcal{S}$  such that  $x \in V \subseteq U$ . Hence the opens of  $\tau$  are arbitrary unions of basic elements.

Coming to constructions with topological spaces, we have the following.

*Convince yourself that (a) this is an open set, (b) it is contained in  $S$ , and (c) it is the largest such set.*

*Convince yourself that closed sets are closed under arbitrary intersection, so this is indeed a closed set.*

*For example, the (countable) set  $S$  of all points in  $\mathbb{R}^3$  with rational coordinates is dense in  $\mathbb{R}^3$ .*

*The point (no pun intended) of (sub)bases is to have a more succinct description of the topology. For example, a base for the Euclidean topology on  $\mathbb{R}^3$  is given by the open balls  $B_\epsilon(x) := \{y \in \mathbb{R}^3 : d(x, y) < \epsilon\}$  for  $x \in \mathbb{R}^3$  and  $\epsilon > 0$ .*

*Cf. the trinity of sublattice, product, and homomorphic images/quotient for lattices.*

1. Subspace. Given a topological space  $(X, \tau)$ , any nonempty subset  $Y \subseteq X$  can be naturally made into a topological space by equipping it with the *subspace topology*

$$\tau \upharpoonright Y := \{U \cap Y : U \in \tau\}.$$

2. Product topology. If  $(X_i)_{i \in I}$  is a collection of topological spaces indexed by a set  $I$ , the product space  $\prod_{i \in I} X_i$  has as underlying set the Cartesian product of the sets  $X_i$  and its topology is generated by the subbase of sets of the form  $\{x \in \prod_{i \in I} X_i : x_j \in V\}$  for  $j \in I$  and  $V \subseteq X_j$  open.
3. Quotient space. If  $X$  is a topological space and  $\equiv$  an equivalence relation on  $X$ , the quotient space has as underlying set  $X/\equiv$  and the opens are those sets  $U \subseteq X/\equiv$  such that  $\{x \in X : [x]_{\equiv} \in U\}$  is open.

*Equivalently, this is the smallest topology making continuous all the projections  $\pi_i : \prod_{i \in I} X_i \rightarrow X_i$  mapping  $x$  to its  $i$ -th component  $x_i$ .*

There is an important classification of topological spaces according to which increasingly stronger so-called *separation axioms* they satisfy. There are many such axioms and they all are of the form that two distinct points can be—in various senses—separated by the topology. The five main ones are the following for a topological space  $X$ .

1.  $X$  is  $T_0$  (aka *Kolmogorov*) if, for all  $x \neq y$  in  $X$ , there is an open  $U \subseteq X$  such that  $U$  contains exactly one of  $x$  and  $y$ .
2.  $X$  is  $T_1$  (aka *Fréchet*) if, for all  $x \neq y$  in  $X$ , there is an opens  $U \subseteq X$  such that  $x \in U$  and  $y \notin U$ .
3.  $X$  is  $T_2$  (aka *Hausdorff*) if, for all  $x \neq y$  in  $X$ , there are disjoint opens  $U, V \subseteq X$  such that  $x \in U$  and  $y \in V$ .
4.  $X$  is  $T_3$  (aka *regular*) if  $X$  is  $T_1$  and, for all  $x \in X$  and closed  $C \subseteq X$  with  $x \notin C$ , there are disjoint opens  $U, V \subseteq X$  with  $x \in U$  and  $C \subseteq V$ .
5.  $X$  is  $T_4$  (aka *normal*) if  $X$  is  $T_1$  and, for all disjoint closed  $C, D \subseteq X$ , there are disjoint opens  $U, V \subseteq X$  with  $C \subseteq U$  and  $D \subseteq V$ .

Finally, we define the concept of compactness. It formalizes the intuition that the space does not extend infinitely but has finite bounds.

**Definition 3.2** (Compactness). Let  $X$  be a topological space. If  $S \subseteq X$  is a subset, an *open cover*  $\mathcal{U}$  of  $S$  is a collection of open sets such that  $S \subseteq \bigcup_{U \in \mathcal{U}} U$ . A subset  $S \subseteq X$  is *compact* if every open cover  $\mathcal{U}$  of  $S$  contains a finite subcover, i.e., there is a finite subset  $\mathcal{U}_0 \subseteq \mathcal{U}$  such that  $\mathcal{U}_0$  is an open cover of  $S$ . The space  $X$  is called *compact* if  $S := X$  is compact.

Two examples:

1. Closed boxes in Euclidean space like this one

$$[0, 1] \times [0, 1] \times [0, 1] = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_1, x_2, x_3 \leq 1\}$$

are compact.

2. Equip  $2 := \{0, 1\}$  with the discrete topology. Define  $X := 2^{\mathbb{N}}$  to be the product space  $\prod_{\mathbb{N}} 2$  of  $\mathbb{N}$ -many copies of 2. This is known as the Cantor space. It is compact.

A local version of compactness is this: A topological space  $X$  is *locally compact* if, for any open neighborhood  $U$  of any point  $x \in X$ , there is an open  $V \subset X$  and compact  $K \subseteq X$  such that  $x \in V \subseteq K \subseteq U$ . If  $X$  is Hausdorff, then compactness implies local compactness, but this is not true in general.

Three useful results regarding compactness are the following.

1. Finite intersection property characterization. Let  $X$  be a topological space and  $S \subseteq X$  a subset. A collection  $\mathcal{A}$  of closed sets has the *finite intersection property* with respect to  $S$  if for every finite subcollection  $\mathcal{A}_0$ , there is  $x \in S$  such that  $x \in \bigcap \mathcal{A}_0$ . Then  $S$  is compact iff, for every collection  $\mathcal{A}$  of closed sets with the finite intersection property with respect to  $S$ , there is  $x \in S$  with  $x \in \bigcap \mathcal{A}$ .
2. Alexander Subbase Theorem. Let  $X$  be a topological space and  $S$  a subbase. If every cover  $\mathcal{U} \subseteq S$  of  $X$  has a finite subcover, then  $X$  is compact.
3. Tychonoff's Theorem. The product of compact spaces is again compact.

If  $S = X$ , we omit the 'with respect to  $S$ '.

The proof of this requires a non-constructive principle, i.e., a version of the axiom of choice. As this is an axiom of standard set theory, we assume this throughout in this course.

### 3.1.2 Morphisms: continuous functions

Now that we know what topological spaces are, what are the structure-preserving mappings between them? Again, the abstract definition is this.

**Definition 3.3.** Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  a function. We say  $f$  is continuous if, for all open subsets  $V$  of  $Y$ , the preimage  $f^{-1}(V) = \{x \in X : f(x) \in V\}$  is an open subset of  $X$ .

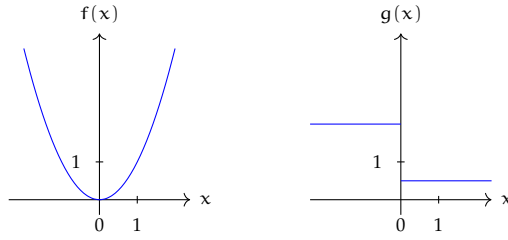


Figure 3.1: A continuous function  $f$  (left) and a non-continuous function  $g$  (right).

**Example 3.4.** As an example to illustrate this definition, consider the usual, so-called epsilon–delta definition of a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . This definition says that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous if

1. For every  $x \in \mathbb{R}$  and every  $\epsilon > 0$ , there is  $\delta > 0$  such that, for all  $y \in \mathbb{R}$  with  $|x - y| < \delta$ , we have  $|f(x) - f(y)| < \epsilon$ .

This captures the idea that, to draw the graph of the function, you do not have to lift your pen: If you want to continue drawing the graph a bit to the left or right of an argument  $x$ , the value outputted by the function will not ‘jump away’ but be close to the value at point  $x$ . To illustrate, consider the following two functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) := x^2 \qquad f(x) := \begin{cases} 2 & \text{if } x < 0 \\ 0.5 & \text{if } x \geq 0. \end{cases}$$

When drawing their graphs, as in figure 3.1, we can do this for  $f$  without lifting the pen, while for  $g$  we have to lift it at  $x = 0$ . And indeed, for  $\epsilon := \frac{1}{4} > 0$ , we cannot find the required  $\delta > 0$ .

*Verify this for yourself.*

It is a good exercise (see exercise 3.b) to show that this ‘hands-on’ definition of continuity is equivalent to—and hence generalized by—the abstract topological definition. For this, we have to define the standard topology on the real line  $\mathbb{R}$ . This is done just like in the three-dimensional case, except that the distance function now simplifies: Here, since  $\mathbb{R}$  has just one dimension,  $d(x, y) = \sqrt{(x - y)^2} = |x - y|$ . So the opens of the real line are those subsets  $U \subseteq \mathbb{R}$  such that, for all  $x \in \mathbb{R}$ , if  $x \in U$ , then there is  $\epsilon > 0$  such that, for all  $x' \in \mathbb{R}$  with  $d(x, x') < \epsilon$ , we have  $x' \in U$ .  $\perp$

*Okay, this time the pun is intended*

Some further useful terminology around continuous functions is the following. A continuous function  $f : X \rightarrow Y$  between topological spaces is

- *open* if, for all open  $U \subseteq X$ , the image  $f[U] = \{f(x) : x \in U\}$  is an open subset of  $Y$ .
- *closed* if, for all closed  $C \subseteq X$ , the image  $f[C] = \{f(x) : x \in C\}$  is a closed subset of  $Y$ .
- an *embedding*, if  $f^{-1} : f[X] \rightarrow X$ , which maps  $y \in f[X]$  to the unique  $x \in X$  with  $f(x) = y$ , is continuous when equipping  $f[X] \subseteq Y$  with the subspace topology.
- a *homeomorphism*, if  $f$  has a continuous inverse, i.e.,  $f$  is a bijection and both  $f$  and  $f^{-1}$  are continuous. (Equivalently, as exercise 3.c shows,  $f$  is a continuous and closed bijection.)

*Note the additional 'e': it is not 'homomorphism' as with lattices.*

Homeomorphisms are the isomorphisms of spaces: If there is a homeomorphism between spaces they are classed homeomorphic and hence are topologically the same. The standard example is that a donut and a coffee mug are homeomorphic: you can obtain one from the other by squishing and squeezing, but—importantly—without breaking and tearing.

*Hence the common joke that topologists cannot tell them apart.*

## 3.2 Orders on topological spaces

## 3.3 Compact ordered spaces and stably compact spaces

## 3.4 Exercises

**Exercise 3.a.** Let  $X$  be a topological space. Prove that the interior map  $\text{Int} : 2^X \rightarrow \Omega(X)$  is upper adjoint to the inclusion map  $\iota : \Omega(X) \rightarrow 2^X$ , and that the closure map  $\text{Cl} : 2^X \rightarrow \mathcal{C}(X)$  is lower adjoint to the inclusion  $\iota' : \mathcal{C}(X) \rightarrow 2^X$ .

*Exercise 2.1.3 in Gehrke and van Gool (2023)*

**Exercise 3.b.** Show that for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  the topological sense of continuity from definition 3.3 is equivalent to the epsilon–delta definition of continuity from example 3.4.

*Exercise 2.1.1 (c) in Gehrke and van Gool (2023)*

**Exercise 3.c.** Let  $f : X \rightarrow Y$  be a continuous bijection. Show that the following are equivalent.

*Exercise 2.1.4 (c) in Gehrke and van Gool (2023)*

1.  $f$  is a homeomorphism (i.e., its inverse is continuous)
2.  $f$  is open (i.e., maps open sets to open sets)
3.  $f$  is closed (i.e., maps closed sets to closed sets).



**Exercise 3.d.** This exercise asks you to prove the following useful facts about compact spaces.

1. A closed subset of a compact space is compact.
2. A compact subset of a Hausdorff space is closed. This need not be true without the Hausdorffness assumption.
3. The image of a compact subset under a continuous function is compact.
4. Conclude that a continuous function from a compact space to a Hausdorff space is closed.
5. Conclude with exercise 3.c that a continuous bijection between compact Hausdorff spaces is a homeomorphism.

#### **4 Two sides of the same coin: Priestley and Stone duality**

## Bibliography

- Abramsky, S. (1991). “Domain theory in logical form.” In: *Annals of pure and applied logic* 51.1-2, pp. 1–77 (cit. on pp. 4, 12).
- (2023). “Logical Journeys: A Scientific Autobiography.” In: *Samson Abramsky on Logic and Structure in Computer Science and Beyond*. Springer, pp. 1–38 (cit. on p. 4).
- Abramsky, S. and A. Jung (1994). “Domain Theory.” In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum. Corrected and expanded version available at <http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf> (last checked 24 January 2018). Oxford: Oxford University Press (cit. on p. 3).
- Balbes, R. and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press (cit. on p. 2).
- Berto, F. and D. Nolan (2021). “Hyperintensionality.” In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. Summer 2021. Metaphysics Research Lab, Stanford University (cit. on p. 8).
- Davey, B. A. and H. A. Priestley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press (cit. on pp. 2, 27).
- Dickmann, M., N. Schwartz, and M. Tressl (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: [10.1017/9781316543870](https://doi.org/10.1017/9781316543870) (cit. on p. 3).
- Gehrke, M. (2009). *Duality*. Inaugurale Rede. URL: <http://hdl.handle.net/2066/83300> (cit. on pp. 2, 4).
- Gehrke, M. and S. van Gool (2023). *Topological Duality for Distributive Lattices: Theory and Applications*. arXiv: 2203.03286 [math.LO] (cit. on pp. 2, 34 sqq., 45).
- Gierz, G. et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press (cit. on p. 3).

- Givant, S. (2014). Ed. by D. theories for Boolean algebras with operators. Springer (cit. on p. 2).
- Givant, S. and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag (cit. on p. 2).
- Goubault-Larrecq, J. (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press (cit. on p. 3).
- Grätzer, G. (2003). *General Lattice Theory*. 2nd ed. Birkhäuser (cit. on p. 3).
- (2011). *Lattice Theory: Foundation*. Birkhäuser (cit. on p. 3).
- Johnstone, P. T. (1982). *Stone Spaces*. Cambridge studies in advanced mathematics 3. Cambridge: Cambridge University Press (cit. on p. 3).
- Ladyman, J., Ø. Linnebo, and R. Pettigrew (2012). “Identity and Discernibility in Philosophy and Logic.” In: *The Review of Symbolic Logic* 5.1, pp. 162–186. DOI: [10.1017/S1755020311000281](https://doi.org/10.1017/S1755020311000281) (cit. on p. 14).
- Ludlow, P. (2022). “Descriptions.” In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta and U. Nodelman. Winter 2022. Metaphysics Research Lab, Stanford University (cit. on p. 5).
- McGrath, M. and D. Frank (2023). “Propositions.” In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta and U. Nodelman. Winter 2023. Metaphysics Research Lab, Stanford University (cit. on p. 7).
- Menzel, C. (2021). “Possible Worlds.” In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. Fall 2021. Metaphysics Research Lab, Stanford University (cit. on p. 7).
- Picado, J. and A. Pultr (2012). *Frames and Locales*. Birkhäuser (cit. on p. 3).
- Strocchi, F. (2008). *An introduction to the mathematical structure of quantum mechanics: a short course for mathematicians*. 2nd ed. Singapore: World Scientific (cit. on p. 12).
- Vickers, S. (1989). *Topology via Logic*. Cambridge: Cambridge University Press (cit. on pp. 2, 4).