Duality Theory
Connecting Algebra and Topology via Logic

Lecture Notes

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Comments welcome!

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Contents

Preface

This is the reader for the course "Duality Theory: Connecting Logic, Algebra, and Topology" given during the winter semester 2024/5 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. These lecture notes are updated as the course progresses. A website with all the course material is found at

> `https://levinhornischer.github.io/DualityTheory/`.

Comments   I'm happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just send an informal email to `Levin.Hornischer@lmu.de`.

Course description and objectives   This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone's duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (totally disconnected compact Hausdorff spaces). The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,

- open sets of a space vs. points of the space,

- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory, after a philosophical motivation. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications—especially to philosophical phenomena—that make use of dualities by combining the often opposing advantages of the two perspectives.

Prerequisites   An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course Philosophical Logic. In that course, I stress two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. These approaches are often equivalent, which is a special case of duality.

Contents   We start with an informal chapter describing the key idea of duality. The rest of the course is about developing this key idea precisely. We first precisely define the algebraic structure (Boolean algebras) and the topological structures (topological spaces), and then prove the duality result. The remainder of the course is about applying this result to modal logic (and sketching applications in computer science) and generalizing the result to Priestley duality.

Layout   These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you'd rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end. Exercises are at the end of each chapter.

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Study material   The main textbook that we use is by Gehrke and van Gool (2023). And informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press

- B. A. Davey and H. A. Pristley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press

- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press

- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag

- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer

- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser

- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory are

- P. T. Johnstone (1982). *Stone Spaces*. Cambrdige studies in advanced mathematics 3. Cambridge: Cambridge University Press

- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press

- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: 10.1017/9781316543870

- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press

- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser

- S. Abramsky and A. Jung (1994). "Domain Theory." In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf (last checked 24 January 2018). Oxford: Oxford University Press

- E. Orłowska et al. (2015). *Dualities for Structures of Applied Logic*. Studies in Logic 56. College Publications

Notation Throughout, 'iff' abbreviates 'if and only if'.

1. Introduction: the key idea of duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

Advice on how to read this chapter. Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the 'rhyme and reason' to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).

To use the words of Abramsky (2023).

Duality theory can be quite abstract. The advantage of this is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. So before attempting any general definition of duality, let us consider several examples (section 1.1). From those we can generalize an informal characterization of duality (section 1.2). This then hints at how duality theory is formalized mathematically and how it can be applied. Finally, in section 1.3 we list some exercises.

1.1. Intuitive examples of duality

We present several examples of duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.

1.1.1. Metaphysics: Properties vs objects

When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon, and we will here also include merely possible objects like unicorns. Properties are, for example, being red, being higher than 300m, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Philosophically, it is difficult to make this talk of objects and properties precise (e.g., if we are too permissive about what counts as a property, Russell's paradox creeps in). For now, let us just rely on our everyday intuitions about these concepts. Once we see where this will lead, the exercises at the end of this chapter will ask you to come back and scrutinize the concept of object and property at play (see exercise 1.c).

If 'being a property' is a property, consider the property p of 'not being a property'. Then p has p iff p does not have p, contradiction. But, arguably, 'being a property' is not an 'everyday' property.

Let us write $\mathcal{O}$ for the set of all possible objects and $\mathcal{P}$ for the set of all properties. Crucially, observe that there is a certain dependency between $\mathcal{O}$ and $\mathcal{P}$:

($\mathcal{O} \to \overline{\mathcal{P}}$) Each object $x \in \mathcal{O}$ determines a set of properties $F_x \subseteq \mathcal{P}$ consisting of precisely those properties that x has.

Philosophers also call F_x the role *of the individual x (McMichael 1983, p. 57).*

(The bar in '$\overline{\mathcal{P}}$' indicates that we assign to each x a *set* of elements in $\mathcal{P}$ rather than a single element of $\mathcal{P}$.) So we might wonder whether we can also go in the opposite direction ($\overline{\mathcal{P}} \to \mathcal{O}$)? Does a subset F of properties also determine an object, i.e., the unique object that has exactly the properties in F? Actually, no: some sets of properties might not be satisfied by any object (e.g., $F = \{being\ exactly\ 300m\ high, being\ exactly\ 200m\ high\}$) or by more than one (e.g., $F = \{being\ exactly\ 300m\ high\}$).

Philosophers know phrases of the form 'The F' (referring to the unique object satisfying F) as definite description. *For their important role in philosophy, see e.g. Ludlow (2022).*

But let us not give up too early. After all, the set F_x is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if F is a set of properties with these nice features, that then it determines a unique object.)

1. Assume $a, b \in \mathcal{P}$ are two properties such that having a implies having b; we abbreviate this as $a \leqslant b$. For example,

$$a = being\ higher\ than\ 300m \leqslant being\ higher\ than\ 200m = b.$$

So if our object x has property a, then it also has property b, i.e., if $a \in F_x$, then $b \in F_x$. We may express this as: F_x is closed under implication.

Later we will say F_x is an upset. *This sounds funny now, but by the end of the course, you will have said this so often that you won't even notice.*

2. Assume $a, b \in \mathcal{P}$ are two properties. Note that then there is another property: namely, the property of having both property a and property b. We denote this property $a \wedge b$. So $a \wedge b$ is again in $\mathcal{P}$ and we have $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$. Moreover, if our object x has property a and it has property b, then it has property $a \wedge b$, i.e., if $a, b \in F_x$, then $a \wedge b \in F_x$. We may express this as: F_x is closed under conjunction.

3. Similarly, if $a, b \in \mathcal{P}$ are two properties, there also is the property of having either property a or property b (or both). We denote this property $a \vee b$. So $a \vee b$ is again in $\mathcal{P}$ and we have $a \leqslant a \vee b$ and $b \leqslant a \vee b$. Moreover, if our object x has property $a \vee b$, then either it has property a or it has property b, i.e., if $a \vee b \in F_x$, then either $a \in F_x$ or $b \in F_x$. Later, we express this as F_x being prime.

4. Note that $\mathcal{P}$ also contains the trivial property like being identical to oneself. We denote this property $\top$. In particular, our object x has it, i.e., $\top \in F_x$.

5. Similarly, note that $\mathcal{P}$ also contains the inconsistent property like not being identical to oneself. We denote this property $\bot$. In particular, our object x does not have it, i.e., $\bot \notin F_x$.

Now, we can ask our question again: If F is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in F? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

($\overline{\mathcal{P}} \to \mathcal{O}$) Each set of properties $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines an object $x \in \mathcal{O}$, namely, the unique object having exactly the properties in F.

The uniqueness part is close to Leibniz's principle about the identity of indiscernibles: if two objects x and x' have exactly the properties in F, they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description F of an object, there in fact is a (possible) object that has these properties. This is why we consider the set $\mathcal{O}$ of all possible objects. The actual world need not be ontologically complete: F might consistently describe a unicorn, even if this does not exist in the actual world.

We will see that this bidirectional determination $(\mathcal{O} \to \overline{\mathcal{P}})$ and $(\overline{\mathcal{P}} \to \mathcal{O})$ is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensionality principle: two properties a and b are identical if they apply to exactly the same possible objects (i.e., for all $x \in \mathcal{O}$, x has a iff x has b). Each property a determines a set of objects: namely, the set of those objects that have property a. This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements. Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.

1.1.2. Semantics: Propositions vs possible worlds

The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is 400m high. So the proposition a expressed by the sentence 'The Eiffel Tower is 330m high' contains the actual world x_0 (i.e., $x_0 \in a$) but not the just described possible world x_1 (i.e., $x_1 \notin a$). Some common notation for the phrase 'world x makes true proposition a' is $x \vDash a$; so possible world semantics analyses $\vDash$ as elementhood $\in$.

There is much debate in philosophy what the set $\mathcal{W}$ of possible worlds is (Menzel 2021) and what the set $\mathcal{P}$ of propositions is (McGrath and Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds ('worlds first, propositions later')? Or is it rather that worlds are maximally consistent sets of propositions ('propositions first, worlds later')? The latter goes by the name 'ersatzism' since full-blown possible worlds are substituted by something constructed out of linguistic entities—

and 'Ersatz' is German for substitute.

We won't enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

World individuation Possible worlds are individuated by the propositions they make true: if two possible worlds x and y make true exactly the same propositions (i.e., for every proposition a, we have $x \vDash a$ iff $y \vDash a$), then $x = y$.

Proposition individuation Propositions are individuated by the possible worlds at which they are true: if two propositions a and b are true at exactly the same possible worlds (i.e., for every possible world x, we have $x \vDash a$ iff $x \vDash b$), then $a = b$.

And there is more. Just like properties, also the set of propositions has logical structure: If a and b are propositions, there also are the propositions $a \wedge b$ (conjunction), $a \vee b$ (disjunction), $\neg a$ (negation), $\top$ (logical truth), and $\bot$ (logical falsity). With this we can also express implications between propositions: proposition a implies proposition b, written $a \leqslant b$, precisely if $a \wedge b = a$. The proposition expressed by 'I am in Munich' implies the proposition expressed by 'I am in Germany' because the sentence 'I am in Munich and I am in Germany' is equivalent to the sentence 'I am in Munich', i.e., they express identical propositions.

Thus, given a possible world $x \in \mathcal{W}$, we can again consider the set of propositions $F_x \subseteq \mathcal{P}$ that are true in x (i.e., $F_x = \{a \in \mathcal{P} : x \vDash a\}$). And F_x again satisfies the features (1)–(5) above: If $a \in F_x$, i.e., $x \vDash a$, and a implies b, i.e., $a \leqslant b$, then $x \vDash b$, i.e., $b \in F_x$. If $a, b \in F_x$, then x makes true both a and b, so $a \wedge b \in F_x$. As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

Metaphysical completeness Each set of propositions $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines a possible world $x \in \mathcal{W}$, namely, the unique possible world making true exactly the propositions in F.

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions F satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to F without

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making it inconsistent (i.e., making it contain $\bot$). Ersatzism not only claims that then there is a world x which makes true exactly the propositions in F, it even identifies this world x with F. Hence both the existence and the uniqueness of x follows.

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective correspondence between the set $\mathcal{W}$ of possible worlds and the set $\overline{\mathcal{P}}$ of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets $F \in \overline{\mathcal{P}}$ will be called *prime filters* and $\overline{\mathcal{P}}$ will be called the *spectrum* of the algebra of propositions.)

$$\mathcal{W} \leftrightarrows \overline{\mathcal{P}}$$

$$x \mapsto F_x = \{a \in \mathcal{P} : x \vDash a\}$$

the x making true exactly the $a \in F \leftarrowtail F$

Let us verify that this really is a bijection: We have already checked that the function $f : \mathcal{W} \to \overline{\mathcal{P}}$ mapping x to F_x is well-defined. It is injective by the world individuation principle: if $x \neq y$, then there is a proposition a with $x \vDash a$ and $y \nvDash a$ (or vice versa), so $a \in F_x$ and $a \notin F_y$ (or vice versa), so $F_x \neq F_y$. It is surjective by metaphysical completeness: Given $F \in \overline{\mathcal{P}}$, let x be the unique world in $\mathcal{W}$ making true exactly the propositions in F. Then $F = F_x$ because: $a \in F$ iff $x \vDash a$ iff $a \in F_x$.

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of $\mathcal{W}$) and their ersatz constructions as sets of propositions (the elements of $\overline{\mathcal{P}}$). But what about the other side: How do full-blown propositions (the elements of $\mathcal{P}$) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition $a \in \mathcal{P}$ determines the set of worlds $[\![a]\!] := \{x \in \mathcal{W} : x \vDash a\}$ where a is true. This is also known as the *truthset* of a. And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write $\overline{\mathcal{W}}$ for the sets of worlds that determine propositions and $2^{\mathcal{W}}$ for the set of all sets of worlds. So our assumption for now is that $\overline{\mathcal{W}} = 2^{\mathcal{W}}$. Analogous to the previous case, we want to know if the function

$$[\![\cdot]\!] : \mathcal{P} \to 2^{\mathcal{W}}$$

$$a \mapsto [\![a]\!] = \{x \in \mathcal{W} : x \vDash a\}$$

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is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if $a \neq b$, there is a world x with $x \vDash a$ and $x \nvDash b$ (or vice versa), so $[\![a]\!] \neq [\![b]\!]$. In fact, it also preserves the logical structure: $[\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$, $[\![\bot]\!] = \emptyset$, etc. (Later we formalize this as $[\![\cdot]\!]$ being a Boolean algebra homomorphism.) However, the issue is surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why $[\![\cdot]\!]$ is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a 'finitistic' grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset 2^W of W is uncountable. So $\mathcal{P}$ and 2^W have different cardinalities, which means there cannot be a bijection between them, hence the already injective function $[\![\cdot]\!]$ cannot be surjective.

So actually not any set of worlds determines a proposition, i.e., $\overline{W}$ is a proper subset of 2^W. The ingenious insight of Stone, who discovered Stone duality, was to realize how to precisely describe this special subset $\overline{W}$ of 2^W. The key idea is to realize that there is some additional structure on the set of worlds W that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in chapter 3.

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

1.1.3. Logic: formulas/syntax vs models/semantics

Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches—that are very different in spirit—actually are equivalent. This also is a form of duality. Let's explore this concretely.

Consider the language of classical propositional logic: sentences are formed from atomic sentences $p_0, p_1, \ldots$ using the connectives $\wedge, \vee, \neg$ and the constants $\bot$ and $\top$. And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence

calculus for classical logic—whichever you prefer. It consists of various axioms and rules to define the relation $\Gamma \vdash \varphi$, i.e., when the sentence φ is derivable in the proof-system S using as axioms the sentences in the set Γ. This is the syntactic description of the logic.

The model-theoretic description of the logic defines the relation $\Gamma \vDash \varphi$, i.e., that the sentence φ is a logical consequence of the sentences in Γ. This is done as follows. A valuation is a function $v : \{p_0, p_1, \ldots\} \to \{0, 1\}$ that assigns each atomic sentences a truth-value, i.e., true (1) or false (0). This can be extended to all sentences: $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$; $v(\neg\varphi) = 1$ iff $v(\varphi) = 0$; $v(\bot) = 0$; etc. Then $\Gamma \vDash \varphi$ is defined as: for all valuations v, if $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\varphi) = 1$. Thus, logical consequence is truth-preservation.

Now, the completeness theorem for classical propositional logic states that: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$. To be more precise, one often only calls the right-to-left implication 'completeness', and the left-to-right implication 'soundness'. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserves logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences φ and ψ equivalent, written $\varphi \equiv \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. An equivalence class of a sentence φ is the set of sentences that are equivalent to it: $[\varphi] := \{\psi : \varphi \equiv \psi\}$. Write L for the set of all equivalence classes. It also has logical structure: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$; $\neg[\varphi] = [\neg\varphi]$, etc. L is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each valuation v determines a subset $F_v \subseteq L$: namely, those equivalence classes $[\varphi]$ with $v(\varphi) = 1$. Note again that F_v has features (1)–(5): If $[\varphi] \in F_v$ and $[\varphi] \leqslant [\psi]$ (i.e., $[\varphi \wedge \psi] = [\varphi]$), then $\varphi \vdash \psi$, so, by soundness, $\varphi \vDash \psi$, so, since $v(\varphi) = 1$, also $v(\psi) = 1$, so $[\psi] \in F_v$. If $[\varphi], [\psi] \in F_v$, then $v(\varphi) = 1$ and $v(\psi) = 1$, so $v(\varphi \wedge \psi) = 1$, so $[\varphi \wedge \psi] \in F_v$. Etc. Conversely, if $F \subseteq L$ satisfies (1)–(5), then v_F is a valuation mapping φ to 1 iff $[\varphi] \in F$. So, again, the set X of valuations is in bijective correspondence with the set $\overline{L}$ of subsets of L satisfying (1)–(5).

But how does completeness follow? For this, first note that subsets of L are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if $\Gamma \nvdash \varphi$, consider the deductive closure Γ' of Γ, i.e., the set of all sentences

that can be derived from Γ, so also Γ′ ⊬ φ. When we regard Γ′ as a subset of L, this is, in formal terminology, a filter of L that does not intersect the ideal of all equivalence classes that imply [φ]. Now we use Stone's Prime Filter Theorem, which is at the heart of Stone duality and which we prove later on in the course. It says that we can extend this filter to a prime filter F which still does not intersect that ideal. Then v_F is a valuation that makes true all the premises in Γ but not the conclusion φ, hence Γ ⊬ φ, as desired.

1.1.4. Further examples

Physics: observations vs states. Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). A physical system comes both with a *state space* X and an algebra $\mathcal{A}$ of *observations* and these two again are dual in the sense that

- the states are determined by the observations that they give rise to,

- the observations are determined by the states that give rise to them.

The observations have logical structure: in a classical (as opposed to quantum) system, observing $A \wedge B$ means observing A and observing B, observing $A \vee B$ means observing A or observing B, etc. Each state x of the system determines a set of observations: namely, those that can be made if the system is in that state. Conversely, we can also start with the algebra of observations (they are empirically more accessible anyway) and postulate the states of the system as theoretical entities corresponding to certain subsets of observations.

Computation: observable properties vs denotations of programs. Computer programs are written in a programming language, and so, much like for sentences written in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation*. For example, the denotation of a program could be the (partial) function that it computes. Domain theory is the mathematical theory to systematically describe these meanings. There again also is a side that is dual to the side of meanings, and this was a crucial discovery in the development of domain theory by Abramsky (1991, p. 16). This is the side of *observable properties* of the computer programs. For example, it could be the property that, on input x = 3, the program halts and outputs f(x) = 5. Again, we would hope for a bidirectional determination in the sense that the meaning of a program is complete determined by its observable properties, and that these observable properties are determined by the denotations that have them.
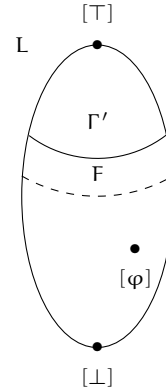
Philosophy of science: observations vs theories. _____ to be added

Time: duration vs points. _____ to be added

1.2. Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will do this formally in the next chapters, but let's already give it an informal try here.

We had the following components in the examples:

- On the 'spatial' side, we have a set X, e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.

- On the 'algebraic' side, we have a set A, e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction ($\wedge$), disjunction ($\vee$), logical falsity ($\bot$), logical truth ($\top$), and possibly negation ($\neg$).

- We have a way to go from the spatial side to the algebraic side: each element of X is determined by a subsets of A with certain nice features, i.e., we have a bijective function $\epsilon : \overline{A} \to X$.

- We have a way to go from the algebraic side to the spatial side: each element from A can be assigned to a subset of X, so we have a bijective function $\eta : A \to \overline{X}$.

> *What's 'algebraic' about this? Algebra is the study of rules for combining objects or symbols. In school, this means combining symbols like $ax^2 + bx + c$ and studying when they equal another, like 0. Here it is about combining elements of A with the operations $\wedge, \vee$, etc.*

Finally, we want this translation manual to be *formulaic* in X and A: i.e., it should not depend on the idiosyncrasies of the specific X and A; rather, it should work for all X's and A's of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we want the above data for any X that is a candidate for the spatial side and for any A that is a candidate for the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates X, which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category consists of the algebraic candidates A and their relations. Then we will see

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that all the above components of the duality is succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

1.3. Exercises

Exercise 1.a. Complete the left-out details in the main text. For example, why, for a possible world x the set of propositions F_x really satisfied properties (1)–(5). Similarly for valuations v.

Exercise 1.b. Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If $a \in \mathcal{P}$ is a property, then there also is the property $\neg a$ of not having property a. It seems plausible to require that either a given object $x \in \mathcal{O}$ has a property or it does not. In other words, either $a \in F_x$ or $\neg a \in F_x$. Do you think this is plausible to require? What about vague properties? (Later we see that this if if we have a negation operator on our set of properties obeying the Boolean laws, than F being prime is equivalent to having the just mentioned negation property.)

Exercise 1.c. As promised in the first example (section 1.1.1), this exercise asks you to scrutinize the concepts of objects and properties. Here are two questions and what duality theory might respond. Philosophically evaluate these answers.

1. *Russell's paradox*: We mentioned the worry that if we are too permissive in our conception of properties, Russell's paradox might creep in.

   Response: For this paradox to occur, we would need the property a of being a property. However, such higher-order properties (i.e., properties of properties) are not considered in Stone duality, for the following reasons. First, the set $\mathcal{O}$ of objects is considered to be disjoint from the set $\mathcal{P}$ of properties. After all, the properties are 'isomorphic' to the (clopen) sets of objects, but sets of objects are not objects; similarly, objects are 'isomorphic' to sets of properties (that are prime filters), but sets of properties are not properties. Second, if we had the higher-order property a of being a property, we would have $\mathcal{P} \subseteq \mathcal{O}$. This is because the property a determines its extension

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$\llbracket a \rrbracket \subseteq \mathcal{O}$, though, by definition of a, $\llbracket a \rrbracket = \mathcal{P}$. But $\mathcal{P} \subseteq \mathcal{O}$ contradicts disjointness (since $\mathcal{P}$ is nonempty).

2. *Leibniz indiscernibility*: A trivial way to satisfy the principle of identity of indiscernibles is by acknowledging, for every object, the property of being x. This property is also known as the haecceity of the object x (see, e.g., Ladyman et al. 2012 for more on this). But, according to duality theory, are all haecceities really properties?

   Response: When we introduce topology, we will see that (1) the space of objects is compact and (2) the extensions of properties are clopen sets. The extension of the haecceity of object x is the singleton $\{x\}$. So, if the haecceity of every object is a property, then, by (2), all singletons are clopen, which, by (1), can only happen if there only are finitely many objects to start with. In other words, according to duality theory, if there are infinitely many objects, not all haecceities can be properties. (Also see exercise 1.f.)

Exercise 1.d (More of a research project than an exercise)**.** Consider to what extend the first example (objects vs properties) can be developed along the lines of formal concept analysis.

Exercise 1.e. Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

Exercise 1.f. For a logico-philosophical discussion of the principle of indiscernibility, see Ladyman et al. (2012). How does this inform the above philosophical discussion (section 1.1.1)? This paper is in the context of model theory, what does the above duality-theoretic perspective add? For a start, see exercise 1.c (2).

Exercise 1.g. Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states (computable theory of mind vs connectionism). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)? Or are these better seen as relations of supervenience rather than duality? What about the infamous Cartesian duality between the physical and the mental world?

Exercise 1.h. Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

2. The algebraic side: Boolean algebras

This chapter introduces formally the algebraic side of duality, which, for us, will be Boolean algebras. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. The next chapter will deal with the other, spatial side of the duality. If you would like to refresh the standard set-theoretic terminology (which we use through-out the course), see appendix A.1.

2.1. Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The 'structure-preserving' maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

Even if we do not need category theory, we follow one of its keys lessons: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism).

For those interested in further reading on category theory, see, e.g., Leinster 2014, ch. 1. Eventually, I might add this into appendix A.2.

2.1.1. Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than. But the things could also be the dishes offered at your go-to lunch place with the sense of 'better' given by your preferences. The formal definition goes as follows.

Definition 2.1. A *partial order* (or *partially ordered set*, or *poset*) is a pair $(P, \leqslant)$ where P is a (possibly empty) set and $\leqslant$ is a binary relation on P

such that

1. *Reflexive*: For all $a \in P$, we have $a \leqslant a$.

2. *Transitive*: For all $a, b, c \in P$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

3. *Anti-symmetric*: For all $a, b \in P$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

A binary relation R on a set P is simply a subset of $P \times P = \{(a, b) : a, b \in P\}$. For $a, b \in P$, one writes $a \, R \, b$ for $(a, b) \in R$.

If we do not require axiom 3, we speak of a *preorder*. We say $\leqslant$ is a (partial or pre-) order on P. If the order $\leqslant$ is clear from context, we often simply speak of the (partial or pre-) order P. We write $a < b$ if $a \leqslant b$ and $a \neq b$.

The name 'partial' is to indicate that not all elements need to be comparable: Formally, for $a, b \in P$, we say that a and b are *comparable*, if either $a \leqslant b$ or $b \leqslant a$; otherwise they are incomparable. If all elements are comparable, we say $(P, \leqslant)$ is *linear* (or *total*).

Formally, the example of the numbers is $(\mathbb{N}, \leqslant)$ where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ and, for $n, m \in \mathbb{N}$, the relation $n \leqslant m$ is defined as: n is smaller or equal to m (equivalently, there is $k \in \mathbb{N}$ such that $n + k = m$). Hence this a linear order. In the example of your lunch place, if you have two dishes a and b that you find equally tasty—or, more precisely, none tastier than the other, i.e., a and b are incomparable—, then your preference order is only partial and not linear.

Check that this satisfies the axioms.

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder $(P, \leqslant)$ into a partial order $(\overline{P}, \overline{\leqslant})$ as follows. For $a, b \in P$, define $a \equiv b$ as $a \leqslant b$ and $b \leqslant a$. This is an equivalence relation. Equivalence classes are the sets $[a] := \{b \in P : a \equiv b\}$ for $a \in P$. The quotient of P under $\equiv$ is $\overline{P} := P/{\equiv} := \{[a] : a \in P\}$. Define $[a] \overline{\leqslant} [b]$ by $a \leqslant b$ (note that this is independent of the representatives a and b). This renders $(\overline{P}, \overline{\leqslant})$ a partial order. It is also called the *poset reflection* of P. Exercise 2.c makes formally precise in what sense it is the canonical or best possible poset approximating the preorder P.

See appendix A.1 for terminology around equivalence classes.

There is a nice visualization of partial orders. They are known as *Hasse diagrams*. An example is in figure 2.1. It depicts the partial order $(P, \leqslant)$ with $P = \{a, b, c, d\}$ and

$$\leqslant := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of P and the edges are the minimal information to recover the order:

- if there is an edge between x and y and x is lower (on the page) than y, then $x \leqslant y$.
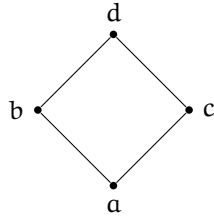
Figure 2.1.: The 'diamond' as an example of a partial order.

- we do not need to draw an edge from one node to itself because for all nodes x we have $x \leqslant x$.

- we do not need to draw edges that result from composing existing edges: for example, we have an edge from a to b and an edge from b to d, so we already know that $a \leqslant d$, hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order $(P, \leqslant)$ is as follows. For $a, b \in P$, we say that b *covers* a (short $a \lessdot b$) if $a < b$ and for all $c \in P$, if $a \leqslant c \leqslant b$, then $c = a$ or $c = b$. The elements of P are the nodes of the Hasse diagram, and an edge is drawn from node a to node b whenever b covers a. The direction of the edge is indicated by drawing b higher up in the diagram than a. So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

Definition 2.2. Let $(P, \leqslant)$ be a partial order and $A \subseteq P$.

- An element $b \in P$ is a *lower bound* of A if, for all $a \in A$, we have $b \leqslant a$.

- An element $b \in P$ is an *upper bound* of A if, for all $a \in A$, we have $a \leqslant b$.

- An element $c \in P$ is an *infimum* or *greatest lower bound* of A if (1) c is a lower bound of A, and (2), for all lower bounds b of A, we have $b \leqslant c$.

- An element $c \in P$ is a *supremum* or *least upper bound* of A if (1) c is an upper bound of A, and (2), for all upper bounds b of A, we have $c \leqslant b$.

- An element $b \in P$ is a *least* or *bottom* or *minimum* element of P, if, for all $a \in P$, we have $b \leqslant a$ (i.e., b is the supremum of $A = \emptyset$).

- An element $b \in P$ is a *greatest* or *top* or *maximum* element of P, if, for all $a \in P$, we have $a \leqslant b$ (i.e., b is the infimum of $A = \emptyset$).

- An element $b \in P$ is *minimal* if, for all $a \in P$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal* if, for all $a \in P$, if $b \leqslant a$, then $b = a$.

- An element $b \in P$ is *minimal in A* if (1) $b \in A$ and (2) for all $a \in A$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal in A* if (1) $b \in A$ and (2) for all $a \in A$, if $b \leqslant a$, then $b = a$.

- A is an *upset* if for all $a, b \in P$, if $a \in A$ and $a \leqslant b$, then $b \in A$.

- A is a *downset* if for all $a, b \in P$, if $b \in A$ and $a \leqslant b$, then $a \in A$.

- A is *directed* (aka up-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. (Equivalently, all finite subsets of A have an upper bound in A.)

- A is *filtered* (aka filtering or down-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $c \leqslant a$ and $c \leqslant b$. (Equivalently, all finite subsets of A have a lower bound in A.)

(These notions also make sense in a preorder $(P, \leqslant)$, but if P is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted $\bigwedge A$, called the *meet* of A; and the supremum is denoted $\bigvee A$, called the *join* of A. If $A = \{a_1, \ldots, a_n\}$ is finite and nonempty, we write $\bigwedge A = a_1 \wedge \ldots \wedge a_n$ and $\bigvee A = a_1 \vee \ldots \vee a_n$. In particular, $\bigwedge\{a, b\} = a \wedge b$ and $\bigvee\{a, b\} = a \vee b$. The bottom element, if it exists, is denoted $\perp$ or 0; and the top element by $\top$ or 1. We write $\min(A)$ (resp. $\max(A)$) for the elements that are minimal (resp. maximal) in A. A *directed join* is the supremum of a directed set.

It is a good exercise to prove this.

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcop*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can 'turn them upside down' and get another preorder. Formally, if $(P, \leqslant)$ is a preorder, define the preorder $\leqslant'$ on P by $a \leqslant' b$ iff $b \leqslant a$. We write P^{op} for this preorder.

2.1.2. Morphisms: Monotone maps

What maps between partial orders should be considered to be 'structure preserving'? They should preserve the order structure, which yields the concept of a monotone map.

Definition 2.3. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders and $f : P \to Q$ a function. We say f is

- *monotone* or *order preserving* if, for all $a, b \in P$, if $a \leqslant_P b$, then $f(a) \leqslant_Q f(b)$.

The converse notion is:

- *order reflecting* if, for all $a, b \in P$, if $f(a) \leqslant_Q f(b)$, then $a \leqslant_P b$.

We call f an *order-embedding* if it is both order preserving and order reflecting. Finally, f is an *order-isomorphism* if it is monotone and it has a *monotone inverse*, i.e., there is a monotone function $g : Q \to P$ such that

- for all $a \in P$, we have $a = g(f(a))$, i.e., a is the g-inverse of $f(a)$ (in short, $\mathrm{id}_P = g \circ f$), and

- for all $b \in Q$, we have $f(g(b)) = b$, i.e., mapping the g-inverse of b along f yields b (in short, $f \circ g = \mathrm{id}_Q$).

We say two preorders are *isomorphic* if there is an order isomorphism between them.

We can consider two isomorphic preorders to be essentially identical (because any order-theoretic property that one has, the other has, too). For partial orders, the notion of isomorphism can be simplified. The above definition captures the general (category-theoretic) concept of an isomorphism, but in practice the following is often easier to check.

Proposition 2.4. *Let $f : P \to Q$ be a monotone function between posets. Then f is an isomorphism iff f is a surjective order-embedding.*

Proof. Exercise 2.d. □

In exercise 2.e, we show how the notion of an isomorphism can be generalized to that of an adjunction. This provides yet another notion of morphism between posets. We do not need it here, but since it is a very useful (but also abstract) concept in the vicinity of the presented concepts, we include it as an exercise. Exercise 2.f shows how such adjunction naturally occur once one has a relation between two sets.

2.2. Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

2.2.1. Objects: lattices

The order-theoretic definition of a lattice goes as follows.

Definition 2.5 (Lattice, order-theoretic). A *(bounded) lattice* is a partial order L in which every finite subset has a supremum and an infimum.

For example, the diamond of figure 2.1 is a lattice. Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.

   As an exercise, prove this.

2. Often a lattice is defined as a partial order in which all binary suprema and infima exist (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exits (i.e., which a have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word 'bounded' (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.

3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

   Prove this. (Hint: think about the supremum of all lower bounds.)

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

Definition 2.6 (Lattice, algebraic)**.** A lattice is a tuple $(L, \vee, \wedge, \bot, \top)$ where $\vee$ (pronounced *join*) and $\wedge$ (pronounced *meet*) are binary operations on L (i.e., functions $L \times L \rightarrow L$), and $\bot$ (pronounced *bottom*) and $\top$ (pronounced *top*) are elements of L, such that the following axioms holds:

1. *commutative*: for all $a, b \in L$, we have $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

2. *associative*: for all $a, b, c \in L$, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

3. *idempotent*: for all $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$.

4. *absorption*: for all $a, b \in L$, we have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

5. *neutrality*: for all $a \in L$, we have $\bot \vee a = a$ and $\top \wedge a = a$.

For example, if X is a set, then the powerset 2^X forms a lattice in this algebraic sense with union $\cup$ as join, intersection $\cap$ as meet, $\emptyset$ as bottom, and X as top. This also provides my mnemonic for remembering what 'join' and what 'meet' is. Think of X as a set of propositions, and let $a \in 2^X$ be the beliefs (opinions, values, etc.) that *A*lice holds, and let $b \in 2^X$ be the beliefs that *B*ob holds. Then the meet of a and b—i.e., $a \wedge b = a \cap b$—is where Alice and Bob can meet: the common (meeting) ground, the set of beliefs they agree on. And the join of a and b—i.e., $a \vee b = a \cup b$—is the result of joining Alice and Bob together: their joint beliefs, taking together all of their beliefs even if incoherent.

Though I'm happy to learn about a better one :-)

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.g asks you to prove it: that is a bit tedious, but quite instructive.

Theorem 2.7.   *1. Given a lattice $(L, \vee, \wedge, \bot, \top)$ according to the algebraic definition, define $a \leqslant_L b$ as $a \wedge b = a$. Then $(L, \leqslant_L)$ is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by $\vee$ and $\wedge$.*

2. *Given a lattice $(L, \leqslant)$ according to the order-theoretic definition, define the binary operations $\vee$ and $\wedge$ as binary supremum and infimum, and take $\bot$ and $\top$ to be the least and greatest element of L. Then $(L, \vee, \wedge, \bot, \top)$ is a lattice according to the algebraic definition, with $a \wedge b = a$ iff $a \leqslant b$ iff $a \vee b = b$.*

From now on, we will often just speak of a lattice L and both use its order-theoretic definition (taking $\leqslant$ to be implicitly given) and its algebraic definitions (taking $\vee, \wedge, \bot, \top$ to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure $(L, \cdot, 1)$, where $\cdot$ is a commutative, associative, and idempotent binary operation on L, and 1 is a neutral element for the operation. The operation $\cdot$ can then either be seen as the binary infimum for the partial order defined by $a \leqslant b$ iff $a \cdot b = a$ (the join semilattice), or as the binary supremum for the opposite partial order defined by $a \leqslant b$ iff $a \cdot b = b$ (the meet semilattice).

2.2.2. Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

Definition 2.8. A function $f : L \to M$ between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all $a, b \in L$, we have $f(a \vee_L b) = f(a) \vee_M f(b)$

2. for all $a, b \in L$, we have $f(a \wedge_L b) = f(a) \wedge_M f(b)$

3. $f(\bot_L) = \bot_M$

4. $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*.

Prove this.

If a function $f : L \to M$ between lattices preserves $\bot$ and $\vee$, then it preserves all finite joins. This does, in general, *not* imply any preservation of arbitrary existing joins or preservation of infima. The analog statement is true for $\top$ and $\wedge$ and preservation of all finite meets.

Prove this.

At this point, one would usually also introduce common constructions on lattices, which also are common for other algebraic structures (e.g., you might have seen them for groups). We do not need them here, so we only briefly mention them for completeness. These are:

- products: if we have a family of lattices $(L_i)_{i \in I}$, we can canonically define lattice operations on the Cartesian product $\prod_{i \in I} L_i$.

- sublattices: a subset of a lattice that contains the top and bottom element and is closed under the lattice operations.

- homomorphic images: the image of a lattice homomorphism.

- congruences: an equivalence relation ϑ on a lattice L that respects the lattice operations, i.e., for all $a, a', b, b' \in L$, if $a\vartheta a'$ and $b\vartheta b'$, then also $a \vee b\vartheta a' \vee b'$ and $a \wedge b\vartheta a' \wedge b'$.

2.3. Distributive lattices and Boolean algebras

We get further subclasses of lattices by requiring that $\vee$ and $\wedge$ interact nicely, which is made precise as distributive lattices (section 2.3.1), and by additionally requiring that there is a sense of negation, which is made precise as Boolean algebras (section 2.3.2).

2.3.1. Distributive lattices

The idea $\vee$ and $\wedge$ interact nicely is made precise as follows.

Definition 2.9. A lattice L is distributive if,

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{2.1}$$

or, equivalently,

$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{2.2}$$

Cf. distributivity from high school:
$$x \times (y + z) = (x \times y) + (x \times z)$$

Proving the equivalence of 2.1 and 2.2 is exercise 2.h.

For example, the four diamond from figure 2.1 is distributive, as is any powerset 2^X, and also any chain $n = \{0, 1, \ldots, n-1\}$ with the usual ordering (see exercise 2.i). The equivalence of 2.1 and 2.2 implies that L is distributive iff L^{op} is distributive. So distributivity is a so-called self-dual property.

There also are strengthenings of the distributivity law. We mentioned one example here for context, but do not need it later. A *frame* is defined as a complete lattice L satisfying the join infinite distributive law (JID)

$$\text{for any } a \in L \text{ and } B \subseteq L, \ a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b). \tag{2.3}$$

In case you have heard of this: A frame is the same thing as a complete Heyting algebra, but their respective choice of morphisms differ.

In a distributive lattice this, in general, only holds for all *finite* $B \subseteq L$.

A seemingly magic characterization of distributive lattices is the following.

Theorem 2.10 (The M_3–N_5 theorem). *Let L be a lattice. Then L is distributive iff L does not contain an unbounded sublattice which is isomorphic to M_3 or N_5, depicted in figure 2.2.*

For a proof, see, e.g., Davey and Pristley (2002, 89 ff.).

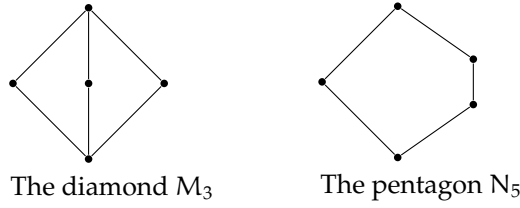The diamond M_3 The pentagon N_5

Figure 2.2.: The forbidden substructures for distributivity.

2.3.2. Boolean algebras

So far, we have seen the order $\leqslant$ and the operations $\vee$ and $\wedge$ in a lattice, which act like implication, disjunction, and conjunction, respectively. So you might have wondered: what about negation? Especially since this also played a role in our motivating introduction (chapter 1). The—or, more precisely, a—idea of negation is made precise as follows.

Definition 2.11. Let L be a lattice and a an element of L. A *complement* of a is an element b of L such that $a \wedge b = \bot$ and $a \vee b = \top$. A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element a in a distributive lattice is unique, if it is exist, an denoted $\neg a$.

Prove this! Note that in non-distributive lattices, like M_3 and N_5 from figure 2.2, elements can have multiple complements.

For example, again the four diamond from figure 2.1 is a Boolean algebra, as is any powerset 2^X; but, for $n > 2$, the chain n is not a Boolean algebra (see exercise 2.i). Some further comments:

1. Usually, the negation is then taken into the signature: so a Boolean algebra is a tuple $(B, \wedge, \vee, \bot, \top, \neg)$ such that $(B, \wedge, \vee, \bot, \top)$ is a distributive lattice and $\neg : B \to B$ a unary function such that, for all $a \in B$, we have $a \wedge \neg a = \bot$ and $a \vee \neg a = \top$.

2. But if we have an additional operation around, shouldn't we require the morphisms to preserve it? Fortunately, they already do: If $f : B \to A$ is a lattice homomorphism between Boolean algebras, then, for all $a \in B$, we have $f(\neg a) = \neg f(a)$. We often still refer to them as *Boolean algebra homomorphisms* just to emphasize that we are dealing with Boolean algebras.

The fact that we can use the same morphisms is expressed in categorical terms as the category of Boolean algebras and Boolean algebra homomorphisms being a full (as opposed to any) subcategory of the category of distributive lattices and lattice homomorphisms.

3. However, with the notion of a sublattice we need to be more careful: A Boolean algebra may have many sublattices that themselves are not Boolean algebras; so by a (*Boolean*) *subalgebra* of a Boolean algebra B we mean a sublattice which is also closed under $\neg$.

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4. If you like ring theory, a Boolean algebra can equivalently be defined as a commutative ring with unit in which all elements are idempotent, see exercise 2.j.

5. There is a best way to turn a distributive lattice L into a Boolean algebra B. This B is called the *Boolean envelope* or *free Boolean extension* of L. More precisely, this means that for every distributive lattice L there is a Boolean algebra B and an injective homomorphism $e : L \to B$ such that for any other lattice homomorphism $h : L \to A$ into a Boolean algebra A, there is a unique Boolean algebra homomorphism $\overline{h} : B \to A$ such that $\overline{h} \circ e = h$. As a diagram:

In categorical terms this means the category of Boolean algebras is a full reflective subcategory of the category of distributive lattices.

$$L \xrightarrow{\ e\ } B$$
$$h \searrow \quad \downarrow \overline{h}$$
$$A$$

This will be a corollary from Stone duality theorem.

2.4. Exercises

Exercise 2.a. Show that the following are partial orders and draw their Hasse diagrams:

- The chain $n = \{0, \ldots, n-1\}$ with the usual order. Draw it for, say, $n = 10$.

- The set $N = \{1, \ldots, n\}$ with the order defined by $n \leqslant m$ iff n divides m (why is 0 excluded from the set?). Draw it for, say, $n = 10$.

- The powerset 2^X for some set X ordered by the subset relation, i.e., $A \leqslant B$ iff $A \subseteq B$. Draw it for, say, $X = \{4, 7\}$.

Exercise 2.b. Go through the partial order concepts defined in definition 2.2 and pick a few of them and draw (minimal) Hasse diagrams to show how they differ. For example, a maximal element that is not a greatest element; an upper bound that is not a greatest upper bound; or an upset that is not directed.

Exercise 2.c. Recall that for a preorder $(P, \leqslant)$, we have defined the poset reflection $(\overline{P}, \overline{\leqslant})$. This exercise makes precise in which sense this is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 1.1.5 in Gehrke and van Gool (2023), with small changes.

1. Prove that $\equiv$ is an equivalence relation.

2. Prove that the definition of $\overline{\leqslant}$ is independent of the representatives: If $a' \in [a]$ and $b' \in [b]$, then $a \leqslant b$ iff $a' \leqslant b'$.

3. Prove that $(\overline{P}, \overline{\leqslant})$ is indeed a partial order.

4. Prove that $\overline{\leqslant}$ is the smallest partial order on $\overline{P} = P/\equiv$ such that the quotient map $f : P \to P/\equiv$, which maps a to $[a]$, is order preserving: That is, if $\leqslant'$ is another such partial order on $P/\equiv$, then $\overline{\leqslant} \subseteq \leqslant'$.

5. Prove that, for any order preserving $g : P \to Q$ into a poset Q, there exists a unique order preserving $\overline{g} : P/\equiv \to Q$ such that $\overline{g} \circ f = g$. As a diagram:

$$P \xrightarrow{\ f\ } P/\equiv$$
$$\quad \searrow_{g} \quad \downarrow_{\overline{g}}$$
$$\qquad Q$$

Think about how the last item formalizes the idea that $(\overline{P}, \overline{\leqslant})$ is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 2.d. Prove proposition 2.4.

The next two exercises introduce the notion of an order adjunction (this is a special case of the notion of an adjoint functor). The first states the general definition and the second a common situation how they occur.

Exercise 2.e (Order adjunction). Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone maps. The pair (f, g) is called an *adjunction*, with f the *left* or *lower adjoint* and g the *right* or *upper adjoint*, if, for all $a \in P$ and $b \in Q$,

$$f(a) \leqslant_Q b \text{ iff } a \leqslant_P g(b).$$

We also write this as $l : P \leftrightarrows Q : u$. An adjunction between P^{op} and Q is called a *Galois connection* or *contravariant adjunction*.

1. Prove that (f, g) is an adjunction iff

- for all $a \in P$, we have $a \leqslant_P g(f(a))$, i.e., the g-inverse of $f(a)$ is at least as good as a, and

- for all $b \in Q$, we have $f(g(b)) \leqslant_Q b$, i.e., mapping the g-inverse of b along f approximates b.

For the rest of this exercise, assume that (f, g) is an adjunction.

2. Prove that $f \circ g \circ f(a) \equiv f(a)$ and $g \circ f \circ g(b) \equiv g(b)$ for every $a \in P$ and $b \in Q$ (and $a \equiv b$ iff $a \leqslant b$ and $b \leqslant a$).

3. Conclude that, in particular, if P and Q are posets, then $fgf = f$ and $gfg = g$.

4. Prove that, if P is a poset, then for any $a \in P$, $gf(a)$ is the least element above a that lies in the image of g.

5. Formulate and prove a similar statement to the previous item about $fg(b)$, for $b \in Q$.

6. Prove that, for any subset $A \subseteq P$, if the supremum of A exists, then $f(\bigvee A) = \bigvee f(A)$ (where $f(A) = \{f(a) : a \in A\}$ is the image of A under f).

7. Prove that, for any subset $B \subseteq Q$, if the infimum of B exists, then $g(\bigwedge B) = \bigwedge g(B)$.

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

Exercise 2.f (Galois connection from a relation). Let $R \subseteq X \times Y$ be a relation between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define

$$u(a) := \{y \in Y : \forall x \in a.xRy\} \subseteq Y$$
$$l(b) := \{x \in X : \forall y \in b.xRy\} \subseteq X$$

Show that $l : \mathcal{P}(Y) \leftrightarrows \mathcal{P}(X) : u$ forms a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, i.e., for any $b \subseteq Y$ and $a \subseteq X$, we have $a \subseteq l(b)$ (i.e., $l(b) \subseteq^{op} a$) iff $b \subseteq u(a)$.

Here $\mathcal{P}(X)$ is the set of all subsets of the set X.

For those interested in further reading, here are three instances of this.

1. Maybe you know the name 'Galois' from the theory of fields in algebra. Then you know Galois theory as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation R between the set X of subfields of a given field and the set Y of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.

For an accessible introduction, take a look, e.g., at this or this video, or at these great lecture notes by Tom Leinster.

2. If X is a set and $R \subseteq X \times X$ is a preorder, then $u(a)$ is the set of upper bounds of $a \subseteq X$, and $l(b)$ is the set of lower bounds of $b \subseteq X$.

3. Consider a class of structures $\mathcal{C}$ (in, say, a first-order signature) and a class $\mathcal{F}$ of formulas (of this signature). Let $\vDash$ be the *interpretation* relation: For $M \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ means that structure M makes true formula φ. Then for a set of models a, $u(a)$ is the theory of a, i.e., the set of formulas that are true in all those models. And for a a theory $b \subseteq \mathcal{F}$, $l(b)$ is the class of models of b, i.e., the set of models which make true all the sentences in b.

Exercise 2.g. Prove theorem 2.7.

Exercise 2.h. Prove the equivalence of the two ways of defining distributivity: 2.1 and 2.2.

Exercise 2.i. Show that the following are distributive lattices:

- The four diamond from figure 2.1.

- The powerset 2^X, for any set X.

- The chain $n = \{0, \dots, n-1\}$ with the usual ordering, for any n.

Show that the first two also are Boolean algebras. Show that the last one is a Boolean algebra if $1 \leqslant n \leqslant 2$, and not if $n > 2$.

You might have had the suspicion that the join $\vee$ acts quite like addition $+$ and the meet $\wedge$ quite like the multiplication $\cdot$. If so, you might like the next exercise, which makes this precise.

Exercise 2.j. This exercise shows that Boolean algebras and Boolean rings are equivalent.

1. Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, i.e., a commutative ring with unit in which $a \cdot a = a$ for all $a \in B$. Define $a \leqslant b$ if $a \cdot b = a$. (We often write ab for $a \cdot b$.) Prove that $\leqslant$ is a distributive lattice order on B where

   - 1 is the greatest element and 0 is the least element,

   - meet is given by ab and join is given by $a + b + ab$, and

   - every element a of has the complement $1 + a$ with respect to $\leqslant$.

   Hint: First show that $a + a = 0$ for all $a \in B$.

2. Conversely, let $(B, \wedge, \vee, \bot, \top, \neg)$ be a Boolean algebra. Define, for any $a, b \in B$,

$$a + b := (a \wedge \neg b) \vee (\neg a \wedge b) \qquad a \cdot b := a \wedge b$$

$$0 := \bot \qquad\qquad\qquad 1 := \top.$$

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Prove that $(B, +, \cdot, 0, 1)$ is a Boolean ring.

3. Finally, show that the composition of these two assignments in either order yields the identity.

The following is a fact that we will later use a lot.

Exercise 2.k. Let $f : X \to Y$ be a function between two sets X and Y. Show that the function from 2^Y to 2^X defined by

$$B \mapsto f^{-1}(B)$$

is a Boolean algebra homomorphism.

3. The spatial side: topological spaces

4. Two sides of the same coin: Stone duality

5. Application: (Re)discovering semantics for modal logic

6. Generalization: Priestley duality

A. Appendix

A.1. Set-theoretic terminology

We use standard set-theoretic terminology as it is common in mathematics. A set is a collection of objects. We write $a \in A$ to say that object a is in (or is an element of, or is a member of) the set A. If $a_1, \ldots, a_n$ are objects, we write $\{a_1, \ldots, a_n\}$ for the set of these objects. Sets do not count 'order' and 'multiplicities', so $\{1, 0, 2, 2\} = \{0, 1, 2\}$. A set with just one element is called a singleton, and if a is an object, $\{a\}$ is the singleton of a (note $\{a\} \neq a$). The set without any elements is called the empty set and is denoted $\emptyset$.

If A and B are sets, we say A is a subset of B (written $A \subseteq B$) if every element of A is an element of B. So the empty set trivially is a subset of any set. And two sets A and B are identical iff $A \subseteq B$ and $B \subseteq A$. If A and B are sets, then the union of A and B (written $A \cup B$) is the set containing exactly those objects that either are in A or in B (or both). The intersection of A and B (written $A \cap B$) is the set containing exactly those objects that are both in A and in B. The complement of a set A relative to a set B (written $B \setminus A$) is the set of objects that are in B but not in A. If B is clear from context, we just write A^c.

A pair (aka ordered pair) is a list of two elements (a, b); here the order matters, so $(a, b) \neq (b, a)$. (We can define (a, b) as the set $\{\{a\}, \{a, b\}\}$.) More generally, an n-tuple is a list of n elements $(a_1, \ldots, a_n)$. Given n sets $A_1, \ldots, A_n$, their Cartesian product (written $A_1 \times \ldots \times A_n$) is the set of all n-tuples $(a_1, \ldots, a_n)$ such that, for all $i \in \{1, \ldots, n\}$, we have $a_i \in A_i$. More generally, the Cartesian product of a potentially infinite family $\{A_i : i \in I\}$ of sets is defined as the set $\prod_{i \in A} A_i$ of functions a that map each $i \in I$ to an element $a(i) \in A_i$. We often write such a function as $a = (a_i)_{i \in I}$; because in the above case where $I = \{1, \ldots, n\}$, we can think of a tuple $(a_1, \ldots, a_n)$ as a the function a mapping $i \in I$ to a_i. An n-ary relation between $A_1, \ldots, A_n$ is a subset of $A_1 \times \ldots \times A_n$. A 1-ary (resp., 2-ary, 3-ary) relation is also called a unary (resp., binary, ternary) relation. For a binary relation R, we usually write aRb instead of $(a, b) \in R$.

A function from a set A (its domain) to a set B (its codomain) is a binary relation f between A and B such that, for every $a \in A$, there is exactly one $b \in B$ such that afb. We then write $f : A \to B$ and $f(a) = b$ or, if f is clear

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from context, $a \mapsto b$. If $f : A \to B$ and $g : B \to C$ are functions, $g \circ f$ (g after f) denotes their composition, which maps $a \in A$ to $g(f(a)) \in C$. Given a set A, the identity function $\mathrm{id}_A : A \to A$ maps a to a. By an n-ary function on a set A we mean a function $f : A^n \to A$, where $A^n = A \times \dots \times A$ is the n-time Cartesian product of set A. Again, the first arities have special names: unary ($= 1$-ary), binary ($= 2$-ary), and ternary ($= 3$-ary). Sometimes it is convenient to take a 0-ary function to be a constant (i.e., an element or symbol which is fixed throughout).

An equivalence relation $\equiv$ on a set A is a binary relation on A such that

1. $\equiv$ is reflexive, i.e., for all $a \in A$, we have $a \equiv a$,

2. $\equiv$ is transitive, i.e., for all $a, b, c \in A$, if $a \equiv b$ and $b \equiv c$, then $a \equiv c$, and

3. $\equiv$ is symmetric, i.e., if $a \equiv b$, then $b \equiv a$.

If $a \in A$, then the $\equiv$-equivalence class of a is the set $[a]_\equiv := \{b \in A : a \equiv b\}$. An element $b \in [a]_\equiv$ is called a representative of $[a]_\equiv$. The quotient of A under $\equiv$ is defined as $A/\equiv := \{[a]_\equiv : a \in A\}$. The function $\pi : A \to A/\equiv$ defined by $\pi(a) := [a]_\equiv$ is called the projection of $\equiv$.

A.2. Category-theoretic terminology

To be written. For now, see, e.g., Leinster 2014, ch. 1.

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