Duality Theory
Connecting Algebra and Topology via Logic

Lecture Notes

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Check the website for the latest version.
Comments welcome!

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Contents

Preface

This is the reader for the course "Duality Theory: Connecting Logic, Algebra, and Topology" given during the winter semester 2025/6 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. (Previous editions: winter 2023/4, winter 2024/5.) These lecture notes are updated as the course progresses. A website with all the course material is found at

> `https://levinhornischer.github.io/DualityTheory/`.

Comments I'm happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just email me at `Levin.Hornischer@lmu.de`.

Course description and objectives This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone's duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (zero-dimensional compact Hausdorff spaces). The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,
- open sets of a space vs. points of the space,
- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory, after a philosophical motivation. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications—especially to philosophical phenomena—that make use of dualities by combining the often opposing advantages of the two perspectives.

Prerequisites An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course Philosophical Logic. In that course, I stress two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. These approaches are often equivalent, which is a special case of duality.

Contents We start with an informal chapter describing the key idea of duality. Then we develop it formally. We first precisely define the algebraic structure (Boolean algebras) and the topological structures (topological spaces), and then prove the duality result. The remainder of the course is about applying this result to modal logic and generalizing the result to Priestley duality.

Both the mathematical theory and the computer science application of Stone duality are very well developed and accessible via many textbooks (see the study material below). However, the philosophical application still is underexplored and offers great potential. Hence, these notes aim to provide a philosophically motivated introduction to Stone duality.

Layout These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you'd rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end. Exercises are at the end of each chapter.

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Study material A great recent textbook is written by Gehrke and S. van Gool (2023). An informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press
- B. A. Davey and H. A. Pristley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press
- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press

- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag
- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer
- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser
- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory and related topics are

- P. T. Johnstone (1982). *Stone Spaces*. Cambrdige studies in advanced mathematics 3. Cambridge: Cambridge University Press
- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press
- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: 10.1017/9781316543870
- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press
- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser
- S. Abramsky and A. Jung (1994). "Domain Theory." In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf (last checked 28 August 2025). Oxford: Oxford University Press
- E. Orłowska et al. (2015). *Dualities for Structures of Applied Logic*. Studies in Logic 56. College Publications

Stone duality describes a *syntax–semantics duality* for propositional logic. This can be extended to first-order logic:

- M. Makkai (1987). "Stone duality for first order logic." In: *Advances in Mathematics* 65.2, pp. 97–170. DOI: https://doi.org/10.1016/0001-8708(87)90020-X. For a new and self-contained proof, see Lurie (2020). This is known as the Lurie–Makkai conceptual completeness theorem.
- S. Awodey and H. Forssell (2013). "First-order logical duality." In: *Annals of Pure and Applied Logic* 164.3, pp. 319–348. DOI: https://doi.org/10.1016/j.apal.2012.10.016.

This provides many connections to categorical logic (see, e.g. Pitts 2001) and topos theory (for a very quick introduction to topos theory with further references, see here). These are natural follow-up topics after a course on Stone duality. (Also see Caramello (2011) for a general topos-theoretic perspective on dualities.)

An influential computer science application of Stone duality—namely, to programming language semantics—is due to Abramsky (1991). There also are applications to automata theory: both applications are covered by Gehrke and S. van Gool (2023, ch. 7–8).

As mentioned, there aren't many philosophical applications of duality theory. But a great example is Holliday (2021). For further discussion, see section 1.4.

Notation Throughout, 'iff' abbreviates 'if and only if'.

Acknowledgments Many thanks to the participants of course—both past and present—as well to the colleagues whom I have discussed the material with. In particular, thank you for very helpful comments: Javier Belastegui Lazcano, Zimin Cheng, Aleksandar Nikolic, Ioannis Polychronopoulos, and Emils Zavelis.

1. Introduction: the key idea of Stone duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).

Advice on how to read this chapter. Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the 'rhyme and reason' to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

To use the words of Abramsky (2023).

Structure of this chapter. We start in section 1.1 by sketching the general idea of a duality, as it occurs in various science. Stone duality is a specific and precise example of this general idea. In section 1.2, we consider several instances of Stone duality, especially coming from philosophy. In section 1.3, we collect the different moving parts that play a role in Stone duality, to give a still informal but abstract description of Stone duality. This sketches how Stone duality is usually presented in the abstract. The advantage of the abstract formulation is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. Hence, we first start with concrete examples. Finally, in section 1.4, we provide some bibliographic background on duality in philosophy; and, in section 1.5, we list some exercises.

1.1. The general idea of duality

The general idea of a duality is that there are two, in a sense complementary perspectives on the same underlying phenomenon. Metaphorically speaking, there are two sides (the perspectives) of the same coin (the phenomenon). This occurs throughout the sciences. Here are three brief examples:

1. *Mathematics*: A fact that we grew accustomed to—but which is quite remarkable if one thinks about it—is that there is a close correspondence between *algebra* and *geometry*. Algebraic objects, like the polynomial $x^2 + y^2 = 1$, correspond to spatial objects, in this case to the unit circle. We can solve geometric problems, e.g., which ratios and angles can be constructed with a compass and straightedge, by translating them to the algebraic side and then using algebraic methods, in this case using Galois theory. The duality of algebra and geometry is at the heart of algebraic geometry and bears quite some similarities to Stone duality (Johnstone 1982; Mac Lane and Moerdijk 1992; Vickers 2007).

2. *Physics*: Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). On the one hand, a physical system is described by its *state space*, i.e., the collection of states that the system could be in. On the other hand, the system is also described by the *observations* we can make about it. We would hope that these two perspectives on the system are 'dual' to each other. That is to say the states should be determined by the observations that they give rise to; and the observations should be determined by the states that give rise to them. (For more, see, e.g., De Haro and Butterfield (2025), Rosenstock et al. (2015), Ruetsche (2011), and Wu and Weatherall (2023).)

3. *Computer science*: Computer programs are written in a programming language. Much like for sentences in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation* and is, roughly, the input-output function that it computes. But we can describe the program also by its *observable properties*, e.g., that given an even number as input, the program outputs a prime number. The far-reaching insight of Abramsky (1991) was that we again should have a duality between the denotational semantics and the program logic, i.e., the observable properties.

The purpose of this course is to add Stone duality as a specific and precise example of a duality (with many similarities to the above three examples). Stone duality itself is quite general, and its instances occur in logic, mathematics, computer science, and philosophy. In the next section, we develop several of these concrete instances.

1.2. Intuitive examples of Stone duality

We present several examples of Stone duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.

1.2.1. Metaphysics: Properties vs objects

When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon, and we will here also include merely possible objects like unicorns. Properties are, for example, being red, being higher than $300\mathrm{m}$, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Philosophically, it is difficult to make this talk of objects and properties precise (e.g., if we are too permissive about what counts as a property, Russell's paradox creeps in). For now, let us just rely on our everyday intuitions about these concepts. Once we see where this will lead, the exercises at the end of this chapter will ask you to come back and scrutinize the concept of object and property at play (see exercise 1.c).

If 'being a property' is a property, consider the property p of 'not being a property'. Then p has p iff p does not have p, contradiction. But, arguably, 'being a property' is not an 'everyday' property.

Let us write $\mathcal{O}$ for the set of all possible objects and $\mathcal{P}$ for the set of all properties. Crucially, observe that there is a certain dependency between $\mathcal{O}$ and $\mathcal{P}$:

($\mathcal{O} \to \overline{\mathcal{P}}$) Each object $x \in \mathcal{O}$ determines a set of properties $F_x \subseteq \mathcal{P}$ consisting of precisely those properties that x has.

Philosophers also call F_x the role *of the individual x (McMichael 1983, p. 57).*

(The bar in '$\overline{\mathcal{P}}$' indicates that we assign to each x a *set* of elements in $\mathcal{P}$ rather than a single element of $\mathcal{P}$.) So we might wonder whether we can also go in the opposite direction ($\overline{\mathcal{P}} \to \mathcal{O}$)? Does a subset F of properties also determine an object, i.e., the unique object that has exactly the properties in F? Actually, no: some sets of properties might not be satisfied by any object (e.g., $F = \{$*being exactly* $300\mathrm{m}$ *high*, *being exactly* $200\mathrm{m}$ *high*$\}$) or by more than one (e.g., $F = \{$*being exactly* $300\mathrm{m}$ *high*$\}$).

Philosophers know phrases of the form 'The F' (referring to the unique object satisfying F) as definite description. For their important role in philosophy, see e.g. Ludlow (2022).

But let us not give up too early. After all, the set F_x is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if F is a set of properties with these nice features, that then it determines a unique object.)

1. Assume $a, b \in \mathcal{P}$ are two properties such that having a implies having b; we abbreviate this as $a \leqslant b$. For example,

$$a = \textit{being higher than } 300\mathrm{m} \leqslant \textit{being higher than } 200\mathrm{m} = b.$$

   So if our object x has property a, then it also has property b, i.e., if $a \in F_x$, then $b \in F_x$. We may express this as: F_x is closed under implication.

Later we will say F_x is an upset*. This sounds funny now, but by the end of the course, you will have said this so often that you won't even notice.*

2. Assume $a, b \in \mathcal{P}$ are two properties. Note that then there is another property: namely, the property of having both property a and property b. We denote this property $a \wedge b$. So $a \wedge b$ is again in $\mathcal{P}$ and we have $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$. Moreover, if our object x has property a and it has property b, then it has property $a \wedge b$, i.e., if $a, b \in F_x$, then $a \wedge b \in F_x$. We may express this as: F_x is closed under conjunction.
3. Similarly, if $a, b \in \mathcal{P}$ are two properties, there also is the property of having either property a or property b (or both). We denote this property $a \vee b$. So $a \vee b$ is again in $\mathcal{P}$ and we have $a \leqslant a \vee b$ and $b \leqslant a \vee b$. Moreover, if our object x has property $a \vee b$, then either it has property a or it has property b, i.e., if $a \vee b \in F_x$, then either $a \in F_x$ or $b \in F_x$. Later, we express this as F_x being prime.

I will always read 'either A or B' as inclusive-or (either only A is the case, or only B is the case, or both A and B are the case)

Cf. a number $p > 1$ is prime iff (that is Euclid's lemma), for all numbers a and b, if $a \times b$ is divided by p, then either a is divided by p or b is divided by p).

4. Note that $\mathcal{P}$ also contains the trivial property like being identical to oneself. We denote this property $\top$. In particular, our object x has it, i.e., $\top \in F_x$.
5. Similarly, note that $\mathcal{P}$ also contains the inconsistent property like not being identical to oneself. We denote this property $\bot$. In particular, our object x does not have it, i.e., $\bot \notin F_x$.

Now, we can ask our question again: If F is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in F? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

($\overline{\mathcal{P}} \to \mathcal{O}$) Each set of properties $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines an object $x \in \mathcal{O}$, namely, the unique object having exactly the properties in F.

Or is the list (1)–(5) not complete because we should also add a principle concerning negation: you can think about this in exercise 1.b.

The uniqueness part is close to Leibniz's principle about the identity of indiscernibles: if two objects x and x' have exactly the properties in F,

Actually, I don't know if a principle like this is considered in metaphysics: if you do, please let me know :-) Also see exercise 1.d asking for a comparison to formal concept analysis.

they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description F of an object, there in fact is a (possible) object that has these properties. This is why we consider the set $\mathcal{O}$ of all possible objects. The actual world need not be ontologically complete: F might consistently describe a unicorn, even if this does not exist in the actual world.

We will see that this bidirectional determination ($\mathcal{O} \to \overline{\mathcal{P}}$) and ($\overline{\mathcal{P}} \to \mathcal{O}$) is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensionality principle: two properties a and b are identical if they apply to exactly the same possible objects (i.e., for all $x \in \mathcal{O}$, x has a iff x has b). Each property a determines a set of objects: namely, the set of those objects that have property a. This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements.

Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.

1.2.2. Semantics: Propositions vs possible worlds

The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is $400\mathrm{m}$ high. So the proposition a expressed by the sentence 'The Eiffel Tower is $330\mathrm{m}$ high' contains the actual world x_0 (i.e., $x_0 \in a$) but not the just described possible world x_1 (i.e., $x_1 \notin a$). Some common notation for the phrase 'world x makes true proposition a' is $x \vDash a$; so possible world semantics analyses $\vDash$ as elementhood $\in$.

There is much debate in philosophy what the set $\mathcal{W}$ of possible worlds is (Menzel 2021) and what the set $\mathcal{P}$ of propositions is (McGrath and

Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds ('worlds first, propositions later')? Or is it rather that worlds are maximally consistent sets of propositions ('propositions first, worlds later')? The latter goes by the name 'ersatzism' since full-blown possible worlds are substituted by something constructed out of linguistic entities—and 'Ersatz' is German for substitute.

We won't enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

World individuation Possible worlds are individuated by the propositions they make true: if two possible worlds x and y make true exactly the same propositions (i.e., for every proposition a, we have $x \vDash a$ iff $y \vDash a$), then $x = y$.

Cf. Leibniz's above principle about the identity of indiscernibles.

Proposition individuation Propositions are individuated by the possible worlds at which they are true: if two propositions a and b are true at exactly the same possible worlds (i.e., for every possible world x, we have $x \vDash a$ iff $x \vDash b$), then $a = b$.

A hyperintensional account of propositions would contest this; see Berto and Nolan (2021).

And there is more. Just like properties, also the set of propositions has logical structure: If a and b are propositions, there also are the propositions $a \wedge b$ (conjunction), $a \vee b$ (disjunction), $\neg a$ (negation), $\top$ (logical truth), and $\bot$ (logical falsity). With this we can also express implications between propositions: proposition a implies proposition b, written $a \leqslant b$, precisely if $a \wedge b = a$. The proposition expressed by 'I am in Munich' implies the proposition expressed by 'I am in Germany' because the sentence 'I am in Munich and I am in Germany' is equivalent to the sentence 'I am in Munich', i.e., they express identical propositions.

Thus, given a possible world $x \in \mathcal{W}$, we can again consider the set of propositions $F_x \subseteq \mathcal{P}$ that are true in x (i.e., $F_x = \{a \in \mathcal{P} : x \vDash a\}$). And F_x again satisfies the features (1)–(5) above: If $a \in F_x$, i.e., $x \vDash a$, and a implies b, i.e., $a \leqslant b$, then $x \vDash b$, i.e., $b \in F_x$. If $a, b \in F_x$, then x makes true both a and b, so $a \wedge b \in F_x$. As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

Metaphysical completeness Each set of propositions $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines a possible world $x \in \mathcal{W}$, namely, the unique possible world making true exactly the propositions in F.

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions F satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to F without making it inconsistent (i.e., making it contain $\bot$). Ersatzism not only claims that then there is a world x which makes true exactly the propositions in F, it even identifies this world x with F. Hence both the existence and the uniqueness of x follows.

This is assuming that the set of propositions forms what is known as a Boolean algebra.

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective correspondence between the set $\mathcal{W}$ of possible worlds and the set $\overline{\mathcal{P}}$ of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets $F \in \overline{\mathcal{P}}$ will be called *prime filters* and $\overline{\mathcal{P}}$ will be called the *spectrum* of the algebra of propositions.)

$$\begin{aligned} \mathcal{W} &\leftrightarrows \overline{\mathcal{P}} \\ x &\mapsto F_x = \{a \in \mathcal{P} : x \vDash a\} \\ \text{the } x \text{ making true exactly the } a \in F &\leftarrow\!\shortmid F \end{aligned}$$

Let us verify that this really is a bijection: We have already checked that the function $f : \mathcal{W} \to \overline{\mathcal{P}}$ mapping x to F_x is well-defined. It is injective by the world individuation principle: if $x \neq y$, then there is a proposition a with $x \vDash a$ and $y \nvDash a$ (or vice versa), so $a \in F_x$ and $a \notin F_y$ (or vice versa), so $F_x \neq F_y$. It is surjective by metaphysical completeness: Given $F \in \overline{\mathcal{P}}$, let x be the unique world in $\mathcal{W}$ making true exactly the propositions in F. Then $F = F_x$ because: $a \in F$ iff $x \vDash a$ iff $a \in F_x$.

A function $f : X \to Y$ is injective if $x \neq y$ implies $f(x) \neq f(y)$, it is surjective if for every $y \in Y$ there is $x \in X$ with $f(x) = y$, and it is bijective if it is both injective and surjective.

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of $\mathcal{W}$) and their ersatz constructions as sets of propositions (the elements of $\overline{\mathcal{P}}$). But what about the other side: How do full-blown propositions (the elements of $\mathcal{P}$) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition $a \in \mathcal{P}$ determines the set of worlds $[\![a]\!] := \{x \in \mathcal{W} : x \vDash a\}$ where a is true. This is also known as the *truthset* of a. And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write $\overline{\mathcal{W}}$ for the sets of worlds that determine propositions and $2^{\mathcal{W}}$ for the set of all sets of worlds. So our assumption for now is that $\overline{\mathcal{W}} = 2^{\mathcal{W}}$. Analogous to the previous case, we want to know if the function

If X is a set, the powerset of X is the set of all subsets of X and it is denoted 2^X or $\mathcal{P}(X)$.

$$\begin{aligned} [\![\cdot]\!] : \mathcal{P} &\to 2^{\mathcal{W}} \\ a &\mapsto [\![a]\!] = \{x \in \mathcal{W} : x \vDash a\} \end{aligned}$$

is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if $a \neq b$, there is a world x with $x \vDash a$ and $x \nvDash b$ (or vice versa), so $[\![a]\!] \neq [\![b]\!]$. In fact, it also preserves the logical structure: $[\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$, $[\![\bot]\!] = \emptyset$, etc. (Later we formalize this as $[\![\cdot]\!]$ being a Boolean algebra homomorphism.) However, the issue is surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why $[\![\cdot]\!]$ is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a 'finitistic' grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset $2^{\mathcal{W}}$ of $\mathcal{W}$ is uncountable. So $\mathcal{P}$ and $2^{\mathcal{W}}$ have different cardinalities, which means there cannot be a bijection between them, hence the already injective function $[\![\cdot]\!]$ cannot be surjective.

That is Cantor's diagonal argument.

So actually not any set of worlds determines a proposition, i.e., $\overline{\mathcal{W}}$ is a proper subset of $2^{\mathcal{W}}$. The ingenious insight of Stone, who discovered Stone duality, was to realize how to precisely describe this special subset $\overline{\mathcal{W}}$ of $2^{\mathcal{W}}$. The key idea is to realize that there is some additional structure on the set of worlds $\mathcal{W}$ that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in chapter 3.

Also see exercise 1.e.

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

1.2.3. Logic: formulas/syntax vs models/semantics

Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches—that are very different in spirit—actually are equivalent. This also is a form of duality. Let's explore this concretely.

Consider the language of classical propositional logic: sentences are formed from atomic sentences $p_0, p_1, \ldots$ using the connectives $\wedge, \vee, \neg$ and the constants $\perp$ and $\top$. And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence calculus for classical logic—whichever you prefer. It consists of various axioms and rules to define the relation $\Gamma \vdash \varphi$, i.e., when the sentence φ is derivable in the proof-system S using as axioms the sentences in the set Γ. This is the syntactic description of the logic.

The model-theoretic description of the logic defines the relation $\Gamma \vDash \varphi$, i.e., that the sentence φ is a logical consequence of the sentences in Γ. This is done as follows. A valuation is a function $\nu : \{p_0, p_1, \ldots\} \to \{0, 1\}$ that assigns each atomic sentence a truth-value, i.e., true (1) or false (0). This can be extended to all sentences: $\nu(\varphi \wedge \psi) = 1$ iff $\nu(\varphi) = 1$ and $\nu(\psi) = 1$; $\nu(\neg\varphi) = 1$ iff $\nu(\varphi) = 0$; $\nu(\perp) = 0$; etc. Then $\Gamma \vDash \varphi$ is defined as: for all valuations ν, if $\nu(\psi) = 1$ for all $\psi \in \Gamma$, then $\nu(\varphi) = 1$. Thus, logical consequence is truth-preservation.

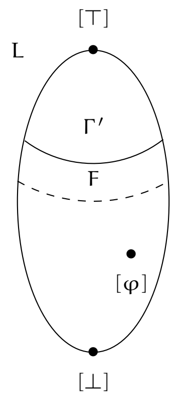
Now, the completeness theorem for classical propositional logic states that: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$. To be more precise, one often only calls the right-to-left implication 'completeness', and the left-to-right implication 'soundness'. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserve logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences φ and ψ equivalent, written $\varphi \equiv \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. An equivalence class of a sentence φ is the set of sentences that are equivalent to it: $[\varphi] := \{\psi : \varphi \equiv \psi\}$. Write L for the set of all equivalence classes. It also has logical structure: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$; $\neg[\varphi] = [\neg\varphi]$, etc. L is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each valuation ν determines a subset $F_\nu \subseteq L$: namely, those equivalence classes $[\varphi]$ with $\nu(\varphi) = 1$. Note again that F_ν has features (1)–(5): If

$[\varphi] \in F_v$ and $[\varphi] \leqslant [\psi]$ (i.e., $[\varphi \wedge \psi] = [\varphi]$), then $\varphi \vdash \psi$, so, by soundness, $\varphi \vDash \psi$, so, since $v(\varphi) = 1$, also $v(\psi) = 1$, so $[\psi] \in F_v$. If $[\varphi], [\psi] \in F_v$, then $v(\varphi) = 1$ and $v(\psi) = 1$, so $v(\varphi \wedge \psi) = 1$, so $[\varphi \wedge \psi] \in F_v$. Etc. Conversely, if $F \subseteq L$ satisfies (1)–(5), then v_F is a valuation mapping φ to 1 iff $[\varphi] \in F$. So, again, the set X of valuations is in bijective correspondence with the set $\overline{L}$ of subsets of L satisfying (1)–(5).

But how does completeness follow? For this, first note that subsets of L are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if $\Gamma \nvdash \varphi$, consider the deductive closure Γ' of Γ, i.e., the set of all sentences that can be derived from Γ, so also $\Gamma' \nvdash \varphi$. When we regard Γ' as a subset of L, this is, in formal terminology, a filter of L that does not intersect the ideal of all equivalence classes that imply $[\varphi]$. Now we use Stone's Prime Filter Theorem, which is at the heart of Stone duality and which we prove later on in the course. It says that we can extend this filter to a prime filter F which still does not intersect that ideal. Then v_F is a valuation that makes true all the premises in Γ but not the conclusion φ, hence $\Gamma \nvDash \varphi$, as desired.



1.3. Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will do this formally in the next chapters, but let's already give it an informal try here.

We had the following components in the examples:

- On the 'spatial' side, we have a set X, e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.
- On the 'algebraic' side, we have a set A, e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction ($\wedge$), disjunction ($\vee$), logical falsity ($\perp$), logical truth ($\top$), and possibly negation ($\neg$).
- We have a way to go from the spatial side to the algebraic side: each element of X is determined by a subsets of A with certain nice features, i.e., we have a bijective function $\epsilon : \overline{A} \to X$.
- We have a way to go from the algebraic side to the spatial side: each element from A can be assigned to a subset of X, so we have a bijective function $\eta : A \to \overline{X}$.

What's 'algebraic' about this? Algebra is the study of rules for combining objects or symbols. In school, this means combining symbols like $ax^2 + bx + c$ *and studying when they equal another, like* 0*. Here it is about combining elements of* A *with the operations* $\wedge, \vee$*, etc.*

Finally, we want this translation manual to be *formulaic* in X and A: i.e., it should not depend on the idiosyncrasies of the specific X and A; rather, it should work for all X's and A's of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we want the above data for any X that is a candidate for the spatial side and for any A that is a candidate for the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates X, which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category consists of the algebraic candidates A and their relations. Then we will see that all the above components of the duality are succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

1.4. Bibliographic note: Duality in philosophy

Philosophy was conspicuously absent from the list in section 1.1 with instances of duality in the sciences. Compared to the impressive applications in mathematics and computer science, Stone duality has indeed found surprisingly few applications in philosophy. For example, Mormann (2013) writes:

> Stone coined the maxim "*You must always topologize*". He conceived of topology as a universal *method* or *perspective* . . . , i.e. all objects should be considered as topological ones. The topological was a kind of a general *a priori* form . . . Nevertheless, among philosophers his work has remained virtually unknown up to this day. (Mormann 2013, p. 428, emphasis original, footnotes omitted)

Fletcher and Lackey (2022) point out, in their history of how topological ideas entered analytic philosophy, that, in contrast to the last sentence of the quote, there are notable exceptions. Researchers working in the tradition of Stone viewed topology not as a generalization of geometry but rather as describing observable properties (Abramsky 1991; Smyth

1983; Vickers 1989), which is central to application 3 above. Kelly (1996) extends this to formal epistemology and philosophy of science, by analyzing which methods of inquiry reliably converge to the truth. Ensuing was a rich body of work connecting epistemic logic and topology (e.g., Baltag, Bezhanishvili, et al. (2013) and Baltag, Gierasimczuk, et al. (2019) and references therein), also including statistical approaches (e.g., Genin and Kelly (2017) and Huttegger (2022) and references therein). Moreover, going back to the work of Tarski and McKinsey, there is a rich interplay of logic—in particular, modal logic—and space (Aiello et al. 2007). To this, we may add recent uses of duality ideas, e.g., Holliday (2021), Wu and Weatherall (2023), Massas (2024), Randriamahazaka (2024, 2025), and Turner (2025). For more on the history of Stone duality, see Fletcher and Lackey (2022) and Johnstone (1982).

Nonetheless, it still seems fair to say that Stone duality is not widely used in philosophy. Hence, we added section 1.2 to showcase its potential in philosophy, in addition to the well-known applications in logic, mathematics, and computer science.

1.5. Exercises

Exercise 1.a. Complete the left-out details in the main text. For example, why, for a possible world x the set of propositions F_x really satisfied properties (1)–(5). Similarly for valuations v.

Exercise 1.b. Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If $a \in \mathcal{P}$ is a property, then there also is the property $\neg a$ of not having property a. It seems plausible to require that either a given object $x \in \mathcal{O}$ has a property or it does not. In other words, either $a \in F_x$ or $\neg a \in F_x$. Do you think this is plausible to require? What about vague properties? (Later we see that if we have a negation operator on our set of properties obeying the Boolean laws, then F being prime is equivalent to having the just mentioned negation property.)

Exercise 1.c. As promised in the first example (section 1.2.1), this exercise asks you to scrutinize the concepts of objects and properties. Here are two questions and what duality theory might respond. Philosophically evaluate these answers.

1. *Russell's paradox*: We mentioned the worry that if we are too permissive in our conception of properties, Russell's paradox might creep

in.

Response: For this paradox to occur, we would need the property a of being a property. However, such higher-order properties (i.e., properties of properties) are not considered in Stone duality, for the following reasons. First, the set $\mathcal{O}$ of objects is considered to be disjoint from the set $\mathcal{P}$ of properties. After all, the properties are 'isomorphic' to the (clopen) sets of objects, but sets of objects are not objects; similarly, objects are 'isomorphic' to sets of properties (that are prime filters), but sets of properties are not properties. Second, if we had the higher-order property a of being a property, we would have $\mathcal{P} \subseteq \mathcal{O}$. This is because the property a determines its extension $[\![a]\!] \subseteq \mathcal{O}$, though, by definition of a, $[\![a]\!] = \mathcal{P}$. But $\mathcal{P} \subseteq \mathcal{O}$ contradicts disjointness (since $\mathcal{P}$ is nonempty).

2. *Leibniz indiscernibility*: A trivial way to satisfy the principle of identity of indiscernibles is by acknowledging, for every object, the property of being x. This property is also known as the haecceity of the object x (see, e.g., Ladyman et al. 2012 for more on this). But, according to duality theory, are all haecceities really properties?

   Response: When we introduce topology, we will see that (1) the space of objects is compact and (2) the extensions of properties are clopen sets. The extension of the haecceity of object x is the singleton $\{x\}$. So, if the haecceity of every object is a property, then, by (2), all singletons are clopen, which, by (1), can only happen if there only are finitely many objects to start with. In other words, according to duality theory, if there are infinitely many objects, not all haecceities can be properties. (Also see exercise 1.f.)

Exercise 1.d (More of a research project than an exercise)**.** Consider to what extend the first example (objects vs properties) can be developed along the lines of formal concept analysis.

Exercise 1.e. Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

Exercise 1.f. For a logico-philosophical discussion of the principle of indiscernibility, see Ladyman et al. (2012). How does this inform the above philosophical discussion (section 1.2.1)? This paper is in the context of model theory, what does the above duality-theoretic perspective add? For a start, see exercise 1.c (2).

Exercise 1.g. Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states (computable theory of mind vs connectionism). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)? Or are these better seen as relations of supervenience rather than duality? What about the infamous Cartesian mind–body dualism between the physical and the mental world?

Exercise 1.h. Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

2. The algebraic side: Boolean algebras

This chapter introduces formally the algebraic side of duality, which, for us, will be Boolean algebras. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. The next chapter will deal with the other, spatial side of the duality. If you would like to refresh the standard set-theoretic terminology (which we use through-out the course), see appendix A.1.

2.1. Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The 'structure-preserving' maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

Even if we do not need category theory, we follow one of its keys lessons: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism).

For those interested in further reading on category theory, see, e.g., Leinster 2014, ch. 1. Eventually, I might add this into appendix A.2.

2.1.1. Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than. But the things could also be the dishes offered at your go-to lunch place with the sense of 'better' given by your preferences. Or, following the guiding intuition from chapter 1, the things can be propositions (or properties, etc.) ordered by implication: b is 'bigger' than a if a implies b. The formal definition goes as follows.

Definition 2.1. A *partial order* (or *partially ordered set*, or *poset*) is a pair $(P, \leqslant)$ where P is a (possibly empty) set and $\leqslant$ is a binary relation on P such that

A binary relation R on a set P is simply a subset of $P \times P = \{(a, b) : a, b \in P\}$. For $a, b \in P$, one writes aRb for $(a, b) \in R$.

1. *Reflexive*: For all $a \in P$, we have $a \leqslant a$.

2. *Transitive*: For all $a, b, c \in P$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

3. *Anti-symmetric*: For all $a, b \in P$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

If we do not require axiom 3, we speak of a *preorder*. We say $\leqslant$ is a (partial or pre-) order on P. If the order $\leqslant$ is clear from context, we often simply speak of the (partial or pre-) order P. We write $a < b$ if $a \leqslant b$ and $a \neq b$.

The name 'partial' is to indicate that not all elements need to be comparable: Formally, for $a, b \in P$, we say that a and b are *comparable*, if either $a \leqslant b$ or $b \leqslant a$; otherwise they are incomparable. If all elements are comparable, we say $(P, \leqslant)$ is *linear* (or *total*).

Formally, the example of the numbers is $(\mathbb{N}, \leqslant)$ where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ and, for $n, m \in \mathbb{N}$, the relation $n \leqslant m$ is defined as: n is smaller or equal to m (equivalently, there is $k \in \mathbb{N}$ such that $n + k = m$). Hence this is a linear order. In the example of your lunch place, if you have two dishes a and b that you find equally tasty—or, more precisely, none tastier than the other, i.e., a and b are incomparable—, then your preference order is only partial and not linear.

Check that this satisfies the axioms.

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder $(P, \leqslant)$ into a partial order $(\overline{P}, \overline{\leqslant})$ as follows. For $a, b \in P$, define $a \equiv b$ as $a \leqslant b$ and $b \leqslant a$. This is an equivalence relation. Equivalence classes are the sets $[a] := \{b \in P : a \equiv b\}$ for $a \in P$. The quotient of P under $\equiv$ is $\overline{P} := P/\equiv := \{[a] : a \in P\}$. Define $[a]\overline{\leqslant}[b]$ by $a \leqslant b$ (note that this is independent of the representatives a and b). This renders $(\overline{P}, \overline{\leqslant})$ a partial order. It is also called the *poset reflection* of P. Exercise 2.c makes formally precise in what sense it is the canonical or best possible poset approximating the preorder P.

See appendix A.1 for terminology around equivalence classes.

There is a nice visualization of partial orders. They are known as *Hasse diagrams*. An example is in figure 2.1. It depicts the partial order $(P, \leqslant)$ with $P = \{a, b, c, d\}$ and

$$\leqslant := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of P and the edges are the minimal information to recover the order:

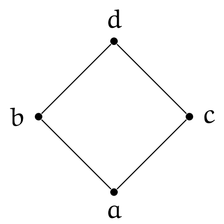


Figure 2.1.: The 'diamond' as an example of a partial order.

- if there is an edge between x and y and x is lower (on the page) than y, then $x \leqslant y$.
- we do not need to draw an edge from one node to itself because for all nodes x we have $x \leqslant x$.
- we do not need to draw edges that result from composing existing edges: for example, we have an edge from a to b and an edge from b to d, so we already know that $a \leqslant d$, hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order $(P, \leqslant)$ is as follows. For $a, b \in P$, we say that b *covers* a (short $a \lessdot b$) if $a < b$ and for all $c \in P$, if $a \leqslant c \leqslant b$, then $c = a$ or $c = b$. The elements of P are the nodes of the Hasse diagram, and an edge is drawn from node a to node b whenever b covers a. The direction of the edge is indicated by drawing b higher up in the diagram than a. So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

They can be confusing at first, but they really are worth learning. Make sure to draw little Hasse diagrams to illustrate the concepts and how they differ from each other (exercise 2.b).

Definition 2.2. Let $(P, \leqslant)$ be a partial order and $A \subseteq P$.

- An element $b \in P$ is a *lower bound* of A if, for all $a \in A$, we have $b \leqslant a$.
- An element $b \in P$ is an *upper bound* of A if, for all $a \in A$, we have $a \leqslant b$.
- An element $c \in P$ is an *infimum* or *greatest lower bound* of A if (1) c is a lower bound of A, and (2), for all lower bounds b of A, we have $b \leqslant c$.
- An element $c \in P$ is a *supremum* or *least upper bound* of A if (1) c is an upper bound of A, and (2), for all upper bounds b of A, we have $c \leqslant b$.

- An element $b \in P$ is a *least* or *bottom* or *minimum* element of P, if, for all $a \in P$, we have $b \leqslant a$ (i.e., b is the supremum of $A = \emptyset$).
- An element $b \in P$ is a *greatest* or *top* or *maximum* element of P, if, for all $a \in P$, we have $a \leqslant b$ (i.e., b is the infimum of $A = \emptyset$).
- An element $b \in P$ is *minimal* if, for all $a \in P$, if $a \leqslant b$, then $a = b$.
- An element $b \in P$ is *maximal* if, for all $a \in P$, if $b \leqslant a$, then $b = a$.
- An element $b \in P$ is *minimal in* A if (1) $b \in A$ and (2) for all $a \in A$, if $a \leqslant b$, then $a = b$.
- An element $b \in P$ is *maximal in* A if (1) $b \in A$ and (2) for all $a \in A$, if $b \leqslant a$, then $b = a$.
- A is an *upset* if for all $a, b \in P$, if $a \in A$ and $a \leqslant b$, then $b \in A$.
- A is a *downset* if for all $a, b \in P$, if $b \in A$ and $a \leqslant b$, then $a \in A$.
- A is *directed* (aka up-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. (Equivalently, all finite subsets of A have an upper bound in A.)
- A is *filtered* (aka filtering or down-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $c \leqslant a$ and $c \leqslant b$. (Equivalently, all finite subsets of A have a lower bound in A.)

(These notions also make sense in a preorder $(P, \leqslant)$, but if P is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted $\bigwedge A$, called the *meet* of A; and the supremum is denoted $\bigvee A$, called the *join* of A. If $A = \{a_1, \ldots, a_n\}$ is finite and nonempty, we write $\bigwedge A = a_1 \wedge \ldots \wedge a_n$ and $\bigvee A = a_1 \vee \ldots \vee a_n$. In particular, $\bigwedge\{a, b\} = a \wedge b$ and $\bigvee\{a, b\} = a \vee b$. The bottom element, if it exists, is denoted $\perp$ or 0; and the top element by $\top$ or 1. We write $\min(A)$ (resp. $\max(A)$) for the elements that are minimal (resp. maximal) in A. A *directed join* is the supremum of a directed set.

It is a good exercise to prove this.

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcop*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can 'turn them upside down' and get another preorder. Formally, if $(P, \leqslant)$ is a preorder, define the preorder $\leqslant'$ on P by $a \leqslant' b$ iff $b \leqslant a$. We write P^{op} for this preorder.

Verify that this again is a preorder (resp. partial order), and draw some Hasse diagram example to see that this really turns things upside down.

2.1.2. Morphisms: Monotone maps

What maps between partial orders should be considered to be 'structure preserving'? They should preserve the order structure, which yields the concept of a monotone map.

We consider the words 'map' and 'function' as synonymous.

Definition 2.3. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders and $f : P \to Q$ a function. We say f is

- *monotone* or *order preserving* if, for all $a, b \in P$, if $a \leqslant_P b$, then $f(a) \leqslant_Q f(b)$.

The converse notion is:

- *order reflecting* if, for all $a, b \in P$, if $f(a) \leqslant_Q f(b)$, then $a \leqslant_P b$.

Note that being order reflecting implies being injective. But injective monotone maps need not be order embeddings.

We call f an *order-embedding* if it is both order preserving and order reflecting. Finally, f is an *order-isomorphism* if it is monotone and it has a *monotone inverse*, i.e., there is a monotone function $g : Q \to P$ such that

- for all $a \in P$, we have $a = g(f(a))$, i.e., a is the g-inverse of $f(a)$ (in short, $\mathsf{id}_P = g \circ f$), and
- for all $b \in Q$, we have $f(g(b)) = b$, i.e., mapping the g-inverse of b along f yields b (in short, $f \circ g = \mathsf{id}_Q$).

Here id_X denotes the identity function on set X. And if $f : X \to Y$ and $g : Y \to Z$ are functions, $g \circ f$ (g after f) denotes their composition, which maps $x \in X$ to $g(f(x)) \in Z$.

We say two preorders are *isomorphic* if there is an order isomorphism between them.

We can consider two isomorphic preorders to be essentially identical (because any order-theoretic property that one has, the other has, too). For partial orders, the notion of isomorphism can be simplified. The above definition captures the general (category-theoretic) concept of an isomorphism, but in practice the following is often easier to check.

Proposition 2.4. *Let $f : P \to Q$ be a monotone function between posets. Then f is an isomorphism iff f is a surjective order-embedding.*

Proof. Exercise 2.d. $\square$

In exercise 2.e, we show how the notion of an isomorphism can be generalized to that of an adjunction. This provides yet another notion of morphism between posets. We do not need it here, but since it is a very useful (but also abstract) concept in the vicinity of the presented concepts, we include it as an exercise. Exercise 2.f shows how such adjunction naturally occur once one has a relation between two sets.

2.2. Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

2.2.1. Objects: lattices

The order-theoretic definition of a lattice goes as follows.

Definition 2.5 (Lattice, order-theoretic)**.** A *(bounded) lattice* is a partial order L in which every finite subset has a supremum and an infimum.

For example, the diamond of figure 2.1 is a lattice. To consider the example of the collection of propositions from chapter 1, we already said that it should be partially ordered by implication, and it makes sense to require that it is a lattice: Given two propositions a and b, there is, as we discussed, also the proposition $a \vee b$, and it is implied by both a and by b, and whenever a proposition c is implied by both a and b, then $a \vee b$ implies c—so $a \vee b$ is the supremum of $a \vee b$. This also works for a finite set of propositions, and also for $\wedge$ and the infimum.

Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.

As an exercise, prove this.

2. Often a lattice is defined as a partial order in which all binary suprema and infima exist (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exits (i.e., which have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word 'bounded' (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.

3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

Prove this. (Hint: think about the supremum of all lower bounds.)

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

Definition 2.6 (Lattice, algebraic)**.** A lattice is a tuple $(L, \vee, \wedge, \perp, \top)$ where $\vee$ (pronounced *join*) and $\wedge$ (pronounced *meet*) are binary operations on L (i.e., functions $L \times L \to L$), and $\perp$ (pronounced *bottom*) and $\top$ (pronounced *top*) are elements of L, such that the following axioms holds:

1. *commutative*: for all $a, b \in L$, we have $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.
2. *associative*: for all $a, b, c \in L$, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.
3. *idempotent*: for all $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$.
4. *absorption*: for all $a, b \in L$, we have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$
5. *neutrality*: for all $a \in L$, we have $\perp \vee a = a$ and $\top \wedge a = a$.

For example, if X is a set, then the powerset 2^X forms a lattice in this algebraic sense with union $\cup$ as join, intersection $\cap$ as meet, $\emptyset$ as bottom, and X as top. This also provides my mnemonic for remembering what 'join' and what 'meet' is. Think of X as a set of propositions, and let $a \in 2^X$ be the beliefs (opinions, values, etc.) that *A*lice holds, and let $b \in 2^X$ be the beliefs that *B*ob holds. Then the meet of a and b—i.e., $a \wedge b = a \cap b$—is where Alice and Bob can meet: the common (meeting) ground, the set of beliefs they agree on. And the join of a and b—i.e., $a \vee b = a \cup b$—is the result of joining Alice and Bob together: their joint beliefs, taking together all of their beliefs even if incoherent.

Though I'm happy to learn about a better one :-)

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.g asks you to prove it: that is a bit tedious, but quite instructive.

Theorem 2.7. *The algebraic and order-theoretic definitions of a lattice are equivalent in the following sense:*

1. *Given a lattice $(L, \vee, \wedge, \perp, \top)$ according to the algebraic definition, define $a \leqslant_L b$ as $a \wedge b = a$. Then $(L, \leqslant_L)$ is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by $\vee$ and $\wedge$.*

2. *Given a lattice* $(L, \leqslant)$ *according to the order-theoretic definition, define the binary operations* $\vee$ *and* $\wedge$ *as binary supremum and infimum, and take* $\bot$ *and* $\top$ *to be the least and greatest element of* L*. Then* $(L, \vee, \wedge, \bot, \top)$ *is a lattice according to the algebraic definition, with* $a \wedge b = a$ *iff* $a \leqslant b$ *iff* $a \vee b = b$.

From now on, we will often just speak of a lattice L and both use its order-theoretic definition (taking $\leqslant$ to be implicitly given) and its algebraic definitions (taking $\vee, \wedge, \bot, \top$ to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure $(L, \cdot, 1)$, where $\cdot$ is a commutative, associative, and idempotent binary operation on L, and 1 is a neutral element for the operation. The operation $\cdot$ can then either be seen as the binary infimum for the partial order defined by $a \leqslant b$ iff $a \cdot b = a$ (the join semilattice), or as the binary supremum for the opposite partial order defined by $a \leqslant b$ iff $a \cdot b = b$ (the meet semilattice).

2.2.2. Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

Definition 2.8. A function $f : L \to M$ between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all $a, b \in L$, we have $f(a \vee_L b) = f(a) \vee_M f(b)$
2. for all $a, b \in L$, we have $f(a \wedge_L b) = f(a) \wedge_M f(b)$
3. $f(\bot_L) = \bot_M$
4. $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*.

Prove this.

If a function $f : L \to M$ between lattices preserves $\bot$ and $\vee$, then it preserves all finite joins. This does, in general, *not* imply any preservation of arbitrary existing joins or preservation of infima. The analog statement is true for $\top$ and $\wedge$ and preservation of all finite meets.

Prove this.

2.2.3. Constructions: products, sublattices, homomorphic images, congruences

Whenever one has introduced a class of objects together and their structure-preserving maps, one also looks at the *constructions* one can perform: how to build new objects in the class from old ones. The typical ones are products, substructures, and quotients, and you might have seen this also for other structures, e.g. groups. (Here quotients will be given as homomorphic images or, equivalently, congruences.) Actually, in this course, they will not play a big part, but they will in more advanced texts on duality theory and are generally important to know. So it is enough if you just skim them.

Products. Given a family $(L_i)_{i \in I}$ of lattices, we can define a lattice $L = \prod_{i \in I} L_i$ on the Cartesian product where the operations are defined component-wise: e.g., for $a = (a_i)_i$ and $b = (b_i)_i$ in L, we define $a \leqslant_L b$ as $\forall i \in I : a_i \leqslant_{L_i} b_i$, and $(a \wedge b)_i = a_i \wedge b_i$ (similarly for $\vee$), and $(\bot_L)_i = \bot_{L_i}$ (similarly for $\top$). The projection maps $\pi_i : L \to L_i$, which map $a = (a_i)_i$ to a_i, is a surjective lattice homomorphism.

Recall (appendix A.1) that the Cartesian product of a family of sets is the set of functions a that map each $i \in I$ to an element $f(i) \in L_i$. We often write such a function as $a = (a_i)_{i \in I}$.

Sublattices. A sublattice of a lattice L is a subset L' of L that contains $\bot$ and $\top$ and that is closed under $\wedge$ and $\vee$ (i.e., if $a, b \in L'$, then $a \wedge b, a \vee b \in L'$). Then L' is a bounded lattice in its own right and the inclusion map $\iota : L' \to L$, which maps $a \in L'$ to $a \in L$, is a lattice embedding. If we do not require $\bot$ and $\top$ to be in L', we speak of an *unbounded sublattice*. And if we require L' to be closed under all suprema and infima, we call it a *complete sublattice*. If $f : L \to M$ is a lattice homomorphism, then the direct image $L' := f[L] = \{f(a) : a \in L\}$ is a sublattice of the lattice M.

Homomorphic images. A lattice L' is a *homomorphic image* of a lattice L if there is a surjective lattice homomorphism $f : L \to L'$.

Birkhoff's famous theorem in universal algebra says that a class of algebraic structures (like lattices) is closed under Homomorphic images, Subalgebras, and Products iff it is definable by equations (hence aka 'HSP theorem').

Congruences. A congruence on a lattice L is an equivalence relation ϑ on L that respects the lattice operations, i.e., for all $a, a', b, b' \in L$, if $a\vartheta a'$ and $b\vartheta b'$, then also $a \vee b \vartheta a' \vee b'$ and $a \wedge b \vartheta a' \wedge b'$. For an intuitive example, think of the elements of L as propositions and of ϑ as having the same subject matter. The quotient L/ϑ carries a unique lattice structure that turns the quotient map $p : L \to L/\vartheta$, which maps $a \in L$ to its equivalence class $[a]_\vartheta$ under ϑ, into a lattice homomorphism; concretely, this lattice structure is given by $[a]_\vartheta \vee [b]_\vartheta := [a \vee b]_\vartheta$ (similarly for $\wedge$) with bottom element $[\bot]_\vartheta$ (similarly for $\top$). Note how this is reminiscent of the Lindenbaum–Tarski algebra from the introduction (section 1.2.3).

The first isomorphism theorem for lattices. This says that any lattice homomorphism $f : L \to M$ can be factored as a surjective lattice homomorphism

The exciting thing about this is that lattice homomorphism can be very complicated, but this tells us that they can be broken down into two much simpler things: surjective lattice homomorphisms and injective lattice homomorphisms!

p followed by a lattice embedding e (i.e., $f = e \circ p$). These are given as follows. The *kernel* of f is the congruence relation

$$\ker f := \{(a, a') \in L \times L : f(a) = f(a')\}.$$

Choose $p : L \to L/\ker f$ (mapping a to $[a]$) and $e : L/\ker f \to M$ (mapping $[a]$ to $f(a)$). In particular, $L/\ker f$ is isomorphic to $f[L]$ (take $M := f[L]$, so e also is surjective); hence the homomorphic images of L are, up to isomorphism, the quotients of L.

Again, the quotients of L intuitively are much simpler: to determine them, we only have to look at L, while for homomorphic images we also need to consider other lattices M.

2.3. Distributive lattices and Boolean algebras

We get further subclasses of lattices by requiring that $\vee$ and $\wedge$ interact nicely, which is made precise as distributive lattices (section 2.3.1), and by additionally requiring that there is a sense of negation, which is made precise as Boolean algebras (section 2.3.2).

2.3.1. Distributive lattices

The idea $\vee$ and $\wedge$ interact nicely is made precise as follows.

Definition 2.9. A lattice L is distributive if,

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{2.1}$$

Cf. distributivity from high school: $x \times (y + z) = (x \times y) + (x \times z)$

or, equivalently,

$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{2.2}$$

Proving the equivalence of 2.1 and 2.2 is exercise 2.h.

For example, the four diamond from figure 2.1 is distributive, as is any powerset 2^X, and also any chain $n = \{0, 1, \ldots, n-1\}$ with the usual ordering (see exercise 2.i). If we again consider the intuitive example of a collection of propositions from chapter 1, we already said that it is a lattice and we also expect it to be distributive because then 2.1 expresses a basic logical equivalence between propositions.

The equivalence of 2.1 and 2.2 implies that L is distributive iff L^{op} is distributive. So distributivity is a so-called *self-dual property*.

There also are strengthenings of the distributivity law. We mentioned one example here for context, but do not need it later. A *frame* is defined

In case you have heard of this: A frame is the same thing as a complete Heyting algebra, but their respective choice of morphisms differ.

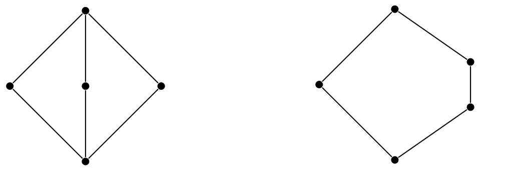

The diamond M_3 The pentagon N_5

Figure 2.2.: The forbidden substructures for distributivity.

as a complete lattice L satisfying the join infinite distributive law (JID)

$$\text{for any } a \in L \text{ and } B \subseteq L,\ a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b). \tag{2.3}$$

In a distributive lattice this, in general, only holds for all *finite* $B \subseteq L$.

A seemingly magic characterization of distributive lattices is the following.

Theorem 2.10 (The M_3–N_5 theorem)**.** *Let L be a lattice. Then L is distributive iff L does not contain an unbounded sublattice which is isomorphic to M_3 or N_5, depicted in figure 2.2.*

For a proof, see, e.g., Davey and Pristley (2002, 89 ff.).

2.3.2. Boolean algebras

So far, we have seen the order $\leqslant$ and the operations $\vee$ and $\wedge$ in a lattice, which act like implication, disjunction, and conjunction, respectively. So you might have wondered: what about negation? Especially since this also played a role in our motivating example of a collection of propositions from chapter 1: if we have a proposition a, we also have the proposition $\neg a$, and we expect $a \wedge \neg a$ to be a logical contradiction and $a \vee \neg a$ a logical truth. These ideas are made precise as follows.

Definition 2.11. Let L be a lattice and a an element of L. A *complement* of a is an element b of L such that $a \wedge b = \bot$ and $a \vee b = \top$. A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element a in a distributive lattice is unique, if it exists, and is denoted $\neg a$.

Prove this! Note that in non-distributive lattices, like M_3 and N_5 from figure 2.2, elements can have multiple complements.

For example, again the four diamond from figure 2.1 is a Boolean algebra, as is any powerset 2^X; but, for $n > 2$, the chain n is not a Boolean algebra (see exercise 2.i). Some further comments:

1. Usually, the negation is then taken into the signature: so a Boolean algebra is a tuple $(B, \wedge, \vee, \perp, \top, \neg)$ such that $(B, \wedge, \vee, \perp, \top)$ is a distributive lattice and $\neg : B \to B$ a unary function such that, for all $a \in B$, we have $a \wedge \neg a = \perp$ and $a \vee \neg a = \top$.

2. But if we have an additional operation around, shouldn't we require the morphisms to preserve it? Fortunately, they already do: If $f : B \to A$ is a lattice homomorphism between Boolean algebras, then, for all $a \in B$, we have $f(\neg a) = \neg f(a)$. We often still refer to them as *Boolean algebra homomorphisms* just to emphasize that we are dealing with Boolean algebras.

The fact that we can use the same morphisms is expressed in categorical terms as the category of Boolean algebras and Boolean algebra homomorphisms being a full *(as opposed to any) subcategory of the category of distributive lattices and lattice homomorphisms.*

3. However, with the notion of a sublattice we need to be more careful: A Boolean algebra may have many sublattices that themselves are not Boolean algebras; so by a (*Boolean*) *subalgebra* of a Boolean algebra B we mean a sublattice which is also closed under $\neg$.

4. If you like ring theory, a Boolean algebra can equivalently be defined as a commutative ring with unit in which all elements are idempotent, see exercise 2.j.

5. There is a best way to turn a distributive lattice L into a Boolean algebra B. This B is called the *Boolean envelope* or *free Boolean extension* of L. More precisely, this means that for every distributive lattice L there is a Boolean algebra B and an injective homomorphism $e : L \to B$ such that for any other lattice homomorphism $h : L \to A$ into a Boolean algebra A, there is a unique Boolean algebra homomorphism $\overline{h} : B \to A$ such that $\overline{h} \circ e = h$. As a diagram:

In categorical terms this means the category of Boolean algebras is a full reflective *subcategory of the category of distributive lattices.*

$$\begin{tikzcd} L \arrow[r, "e"] \arrow[dr, "h"'] & B \arrow[d, dashed, "\overline{h}"] \\ & A \end{tikzcd}$$

This can be shown using duality techniques (see, e.g., Gehrke and S. van Gool 2023, prop. 3.34 or Johnstone 1982, sec. 4.5).

2.4. Exercises

Exercise 2.a. Show that the following are partial orders and draw their Hasse diagrams:

- The chain $P = \{0, \ldots, n-1\}$ with the usual order. Draw it for, say, $n = 10$.

- The set $P = \{1, \dots, n\}$ with the order defined by $n \leqslant m$ iff n divides m (why is 0 excluded from the set?). Draw it for, say, $n = 10$.
- The powerset 2^X for some set X ordered by the subset relation, i.e., $A \leqslant B$ iff $A \subseteq B$. Draw it for, say, $X = \{4, 7\}$.

Exercise 2.b. Go through the partial order concepts defined in definition 2.2 and pick a few of them and draw (minimal) Hasse diagrams to show how they differ. For example, a maximal element that is not a greatest element; an upper bound that is not a greatest upper bound; or an upset that is not directed.

Exercise 2.c. Recall that for a preorder $(P, \leqslant)$, we have defined the poset reflection $(\overline{P}, \overline{\leqslant})$. This exercise makes precise in which sense this is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 1.1.5 in Gehrke and S. van Gool (2023), with small changes.

1. Prove that $\equiv$ is an equivalence relation.
2. Prove that the definition of $\overline{\leqslant}$ is independent of the representatives: If $a' \in [a]$ and $b' \in [b]$, then $a \leqslant b$ iff $a' \leqslant b'$.
3. Prove that $(\overline{P}, \overline{\leqslant})$ is indeed a partial order.
4. Prove that $\overline{\leqslant}$ is the smallest partial order on $\overline{P} = P/\equiv$ such that the quotient map $f : P \to P/\equiv$, which maps a to $[a]$, is order preserving: That is, if $\leqslant'$ is another such partial order on $P/\equiv$, then $\overline{\leqslant} \subseteq \leqslant'$.
5. Prove that, for any order preserving $g : P \to Q$ into a poset Q, there exists a unique order preserving $\overline{g} : P/\equiv \to Q$ such that $\overline{g} \circ f = g$. As a diagram:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "g"'] & P/\equiv \arrow[d, dashed, "\overline{g}"] \\ & Q \end{tikzcd}$$

The category-theoretic formulation of this fact is: the inclusion of the category of partial orders and monotone maps in the category of preorders and monotone maps has a left adjoint. Adjoint functors can be interpreted as formalizing the idea of finding a best possible approximation.

Think about how the last item formalizes the idea that $(\overline{P}, \overline{\leqslant})$ is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 2.d. Prove proposition 2.4.

The next two exercises introduce the notion of an order adjunction (this is a special case of the notion of an adjoint functor). The first states the general definition and the second a common situation how they occur.

Exercise 2.e (Order adjunction)**.** Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone maps. The pair (f, g) is called an *adjunction*, with f the *left* or *lower adjoint* and g the *right* or *upper adjoint*, if, for all $a \in P$ and $b \in Q$,

Exercise 1.1.8 in Gehrke and S. van Gool (2023).

$$f(a) \leqslant_Q b \text{ iff } a \leqslant_P g(b).$$

Note that f occurs on the left of '$\leqslant$' and g on the right.

We also write this as $f : P \leftrightarrows Q : g$. An adjunction between P^{op} and Q is called a *Galois connection* or *contravariant adjunction*.

1. Prove that (f, g) is an adjunction iff
   - for all $a \in P$, we have $a \leqslant_P g(f(a))$, i.e., the g-inverse of $f(a)$ is at least as good as a, and
   - for all $b \in Q$, we have $f(g(b)) \leqslant_Q b$, i.e., mapping the g-inverse of b along f approximates b.

Reflect on how an adjunction then generalizes *the notion of an isomorphism!*

For the rest of this exercise, assume that (f, g) is an adjunction.

Reflect on how this show that g is the best possible approximation to an inverse of f!

2. Prove that $f \circ g \circ f(a) \equiv f(a)$ and $g \circ f \circ g(b) \equiv g(b)$ for every $a \in P$ and $b \in Q$ (and $a \equiv b$ iff $a \leqslant b$ and $b \leqslant a$).
3. Conclude that, in particular, if P and Q are posets, then $fgf = f$ and $gfg = g$.
4. Prove that, if P is a poset, then for any $a \in P$, $gf(a)$ is the least element above a that lies in the image of g.
5. Formulate and prove a similar statement to the previous item about $fg(b)$, for $b \in Q$.
6. Prove that, for any subset $A \subseteq P$, if the supremum of A exists, then $f(\bigvee A) = \bigvee f(A)$ (where $f(A) = \{f(a) : a \in A\}$ is the image of A under f).
7. Prove that, for any subset $B \subseteq Q$, if the infimum of B exists, then $g(\bigwedge B) = \bigwedge g(B)$.

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

We will see that the converse holds for complete lattices. This is a special case of the Adjoint Functor Theorem.

Exercise 2.f (Galois connection from a relation)**.** Let $R \subseteq X \times Y$ be a relation between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define

$$u(a) := \{y \in Y : \forall x \in a. xRy\} \subseteq Y$$
$$l(b) := \{x \in X : \forall y \in b. xRy\} \subseteq X$$

Show that $l : \mathcal{P}(Y) \leftrightarrows \mathcal{P}(X) : u$ forms a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, i.e., for any $b \subseteq Y$ and $a \subseteq X$, we have $a \subseteq l(b)$ (i.e., $l(b) \subseteq^{op} a$) iff $b \subseteq u(a)$.

Here $\mathcal{P}(X)$ is the set of all subsets of the set X.

For those interested in further reading, here are three instances of this.

1. Maybe you know the name 'Galois' from the theory of fields in algebra. Then you know Galois theory as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation R between the set X of subfields of a given field and the set Y of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.

For an accessible introduction, take a look, e.g., at this or this video, or at these great lecture notes by Tom Leinster.

2. If X is a set and $R \subseteq X \times X$ is a preorder, then $u(a)$ is the set of upper bounds of $a \subseteq X$, and $l(b)$ is the set of lower bounds of $b \subseteq X$.

3. Consider a class of structures $\mathcal{C}$ (in, say, a first-order signature) and a class $\mathcal{F}$ of formulas (of this signature). Let $\models$ be the *interpretation* relation: For $M \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ means that structure M makes true formula φ. Then for a set of models a, $u(a)$ is the theory of a, i.e., the set of formulas that are true in all those models. And for a a theory $b \subseteq \mathcal{F}$, $l(b)$ is the class of models of b, i.e., the set of models which make true all the sentences in b.

Also recall the examples from section 1.2.

Exercise 2.g. Prove theorem 2.7.

Exercise 2.h. Prove the equivalence of the two ways of defining distributivity: 2.1 and 2.2.

Exercise 2.i. Show that the following are distributive lattices:

- The four diamond from figure 2.1.
- The powerset 2^X, for any set X.
- The chain $n = \{0, \ldots, n-1\}$ with the usual ordering, for any n.

Show that the first two also are Boolean algebras. Show that the last one is a Boolean algebra if $1 \leqslant n \leqslant 2$, and not if $n > 2$.

You might have had the suspicion that the join $\vee$ acts quite like addition $+$ and the meet $\wedge$ quite like the multiplication $\cdot$. If so, you might like the next exercise, which makes this precise.

Exercise 2.j. This exercise shows that Boolean algebras and Boolean rings are equivalent.

From Gehrke and S. van Gool 2023, ex. 1.2.13.

1. Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, i.e., a commutative ring with unit in which $a \cdot a = a$ for all $a \in B$. Define $a \leqslant b$ if $a \cdot b = a$. (We often write ab for $a \cdot b$.) Prove that $\leqslant$ is a distributive lattice order on B where

   - 1 is the greatest element and 0 is the least element,
   - meet is given by ab and join is given by $a + b + ab$, and
   - every element a has the complement $1 + a$ with respect to $\leqslant$.

   Hint: First show that $a + a = 0$ for all $a \in B$.

2. Conversely, let $(B, \wedge, \vee, \bot, \top, \neg)$ be a Boolean algebra. Define, for any $a, b \in B$,

$$a + b := (a \wedge \neg b) \vee (\neg a \wedge b) \qquad a \cdot b := a \wedge b$$
$$0 := \bot \qquad 1 := \top.$$

   The operation $+$ is known as symmetric difference.

   Prove that $(B, +, \cdot, 0, 1)$ is a Boolean ring.

3. Finally, show that the composition of these two assignments in either order yields the identity.

3. The spatial side: topological spaces

This chapter introduces formally the spatial side of duality, which, for us, will be certain topological spaces known as Stone spaces. This naturally structures this chapter: In section 3.1, we provide a general introduction to topological spaces. In section 3.2, we consider some further topological notions: separation axioms, compactness, and dimensionality. Finally, in section 3.3, we define Stone spaces and discuss one important example, namely the Cantor space. Then we have both the algebraic and the spatial side together, so we can prove the duality result in the next chapter.

3.1. Introduction to topological spaces

When we hear of 'space', we naturally think of the three-dimensional space we live in. And this indeed is an example of a topological space. It is the three-dimensional Euclidean space $\mathbb{R}^3$ whose points $x = (x_1, x_2, x_3)$ are described by the values on the x-axis, the y-axis, and the z-axis. From high-school, we also know what lines and planes are in this space, and what their geometry is.

E.g., the plane spanned by the x-axis and the y-axis is the set of points (x_1, x_2, x_3) with $x_3 = 0$.

But there also are other spaces. For example, the surface of a sphere. Its points are those (x_1, x_2, x_3) with $x_1^2 + x_2^2 + x_3^2 = 1$. But its geometry is different: for instance, the angles of a triangle add up to more than 180 degrees. Yet another space is the spacetime that we live in according to general relativity. Its points $x = (x_1, x_2, x_3, x_4)$ are four-dimensional—with three spatial and one temporal component—and its geometry is given by a metric tensor. And there are even wilder spaces, in which it might not even make sense to speak of a 'geometry' (e.g., angles between lines), but only of 'spatial' properties (e.g., continuous paths from one point to another).

Here we only refer to an intuitive difference between 'geometric' vs 'spatial' (or topological) properties: the latter are invariant under stretching and squishing the space, but the former are not. This is why topology is colloquially also described as rubber sheet geometry.

After much research, mathematicians—most notably Felix Hausdorff in 1914—came up with a general definition of a topological space that includes all these examples. When one first reads this rather abstract definition, one wonders how it possibly can cover all the relevant spatial concepts of the specific examples. But we see how, just from this parsimonious definition of a topological space, we can define many of the common spatial concepts. Again, we split this discussion into objects (topological

Mathematics provides many formal notions of space (e.g., Euclidean space, vector space, Hilbert space, probability space, Banach space, etc.). But topological spaces are a very general such notion.

spaces) and morphisms (continuous functions between spaces).

3.1.1. Objects: topological spaces

Without further ado, here is the abstract definition of a topological space.

Definition 3.1. A *topological space* is a pair (X, τ) where X is a nonempty set and τ is a collection of subsets of X such that

Some also allow the empty topological space.

1. $\emptyset$ and X are in τ
2. If $U, V \in \tau$, then $U \cap V \in \tau$
3. If $U_i \in \tau$ is a collection of sets indexed by a set I, then $\bigcup_{i \in I} U_i \in \tau$.

Equivalently: τ is closed under finite intersection and arbitrary union (which includes the empty intersection X and the empty union $\emptyset$). In particular, τ is a sublattice of 2^X.

We also call τ a topology on X. We call the elements of X *points*. The elements of τ are called *open sets* (or *opens*). Their complements, i.e., sets of the form $C = X \setminus U$ for $U \in \tau$, are called *closed sets*. A subset $K \subseteq X$ that is both open and closed (i.e., $K \in \tau$ and $K^c \in \tau$) is called *clopen*. We just speak of the topological space X if τ is clear from context. Then we write $\Omega(X)$ for the opens of X. The collection of closed (resp. clopen) subsets of X is denoted $\mathcal{C}(X)$ (resp. $\mathsf{Clp}(X)$).

Let's first see that this indeed generalizes our spatial intuitions about 'our' space:

Example 3.2. The three-dimensional space as a topological space: the underlying set is $X := \mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ and the opens are those subsets $U \subseteq \mathbb{R}^3$ that allow some 'wiggle-room', which is made precise as follows. Recall that the usual distance between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

So a subset $U \subseteq \mathbb{R}^3$ is defined to be open precisely if:

Exercise 3.a asks you to show that this then indeed is a topology.

1. for all $x \in \mathbb{R}^3$, if $x \in U$, then there is $\epsilon > 0$ such that for all $x' \in \mathbb{R}^3$ with $d(x, x') < \epsilon$, we have $x' \in U$.

This is called the *Euclidean topology* on $\mathbb{R}^3$. ⌟

Another, more abstract example are the two trivial topologies:

Example 3.3. For any nonempty set X, the set $\tau := 2^X$ is a topology on X. It is called the *discrete* topology. Also $\tau := \{\emptyset, X\}$ is a topology on X. It is called the *indiscrete* topology. ⌟

Next, we define some central concepts for a topological space X. They should give a sense of how many spatial concepts one can express with just talk of open sets.

Definition 3.4. Let (X, τ) be a topological space.

1. *Interior and closure.* If $S \subseteq X$ is a subset, there is a largest open set contained in S, which is called the *interior* of S:

$$\mathsf{Int}(S) := \bigcup \{U \in \tau : U \subseteq S\}.$$

Convince yourself that (a) this is an open set, (b) it is contained in S, and (c) it is the largest such set.

There also is a smallest closed set containing S, which is called the *closure* of S:

$$\mathsf{Cl}(S) := \bigcap \{C \in \mathcal{C}(X) : S \subseteq C\}.$$

Convince yourself that closed sets are closed under arbitrary intersection, so this is indeed a closed set.

2. *Boundary.* The *boundary* of a subset $S \subseteq X$ is defined as $\partial S := \mathsf{Cl}(S) \setminus \mathsf{Int}(S)$.

3. *Neighborhood.* A subset $S \subseteq X$ is a *neighborhood* of a point $x \in X$ if $x \in \mathsf{Int}(S)$. Accordingly, an *open neighborhood* of a point is an open set containing this point. (If it's clear we're talking about an open neighborhood, we might drop the adjective 'open'.)

4. *Dense.* A subset $S \subseteq X$ is *dense* (in X) if for all points $x \in X$ and open neighborhoods U of x, there is a point $s \in S$ with $s \in U$. So the points of X can be approximated arbitrarily closely by points in S. An equivalent formulation is: $\mathsf{Cl}(S) = X$.

For example, the (countable) set S of all points in $\mathbb{R}^3$ with rational coordinates is dense in $\mathbb{R}^3$.

5. *Convergence.* A sequence $(x_n)_{n \in \mathbb{N}}$ of points in X *converges* to a point $x \in X$ if for all open neighborhoods U of x, there is $N \geqslant 0$ such that, for all $n \geqslant N$, we have $x_n \in U$. We also say that x is the *limit* of the sequence (x_n).

6. *Generated topology.* A collection of subsets can naturally be turned into a topology: Any collection $\mathcal{S}$ of subsets of a nonempty set Y *generates* a topology $\langle \mathcal{S} \rangle$ on Y: namely, the smallest topology on Y that contains all subsets in $\mathcal{S}$. Concretely, $\langle \mathcal{S} \rangle$ is the set of arbitrary unions of finite intersections of elements of $\mathcal{S}$.

This exists because an arbitrary intersection of topologies on Y is again a topology on Y.

7. *Subbase.* The generating collection $\mathcal{S}$ can be simpler than the topology $\langle \mathcal{S} \rangle$ generated, so it allows for a more succinct description of the topology. Precisely: Given the topology τ on X, a collection $\mathcal{S}$ of

subsets of X is called a *subbase* of τ if $\tau = \langle \mathcal{S} \rangle$. So the opens of τ are arbitrary unions of finite intersection of subbasic elements.

8. *Base*. The nice subbases are those where we only need to consider arbitrary unions and can forget about the finite intersections. Precisely: A *base* for the topology τ is a collection $\mathcal{S} \subseteq \tau$ such that for every point $x \in X$ and every open neighborhood U of x, there is $V \in \mathcal{S}$ such that $x \in V \subseteq U$. Equivalently, a base is a collection of open subsets of X such that every open set is a union of elements from the base. Note that, in particular, a base is a subbase.

Exercise 3.5. Prove the four statements in the margins:

This also is the end-of-chapter exercise 3.b.

1. the existence of the interior,
2. the existence of the closure,
3. the rational points being dense in our three-dimensional space,
4. the existence of the generated topology.

Moreover, to illustrate the concept of a basis:

5. Show that a base for the Euclidean topology on $\mathbb{R}^3$ is given by the open balls $B_\epsilon(x) := \{y \in \mathbb{R}^3 : d(x, y) < \epsilon\}$ for $x \in \mathbb{R}^3$ and $\epsilon > 0$.
6. Can you improve the previous base by allowing only $x \in \mathbb{R}^3$ with rational coordinates and $\epsilon = \frac{1}{n}$ (for $n \in \mathbb{N}$)? Note that then you have found a countable base even though the space $\mathbb{R}^3$ has uncountably many points!

3.1.2. Morphisms: continuous functions

Now that we know what topological spaces are, what are the structure-preserving mappings between them? Again, there is a neat but abstract definition, which we state first and then link it to more familiar ideas in 'our' space.

Definition 3.6. Let X and Y be topological spaces and $f : X \to Y$ a function. We say f is *continuous* if, for all open subsets V of Y, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open subset of X.

Exercise 3.7. Maybe you have encountered the idea of a continuous function before but in the more concrete, so-called *epsilon–delta definition* of a continuous function $f : \mathbb{R} \to \mathbb{R}$. This definition says that $f : \mathbb{R} \to \mathbb{R}$ is continuous if

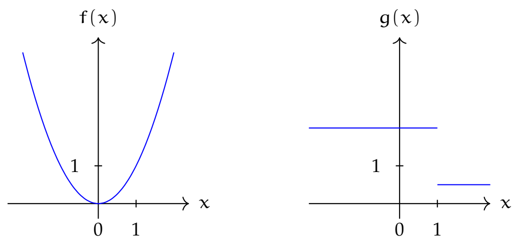


Figure 3.1.: A continuous function f (left) and a non-continuous function g (right).

1. For every $x \in \mathbb{R}$ and every $\epsilon > 0$, there is $\delta > 0$ such that, for all $y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

This captures the idea that, to draw the graph of the function, you do not have to lift your pen: If you want to continue drawing the graph a bit to the left or right of an argument x, the value outputted by the function will not 'jump away' but be close to the value at point x. To illustrate, consider the following two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := x^2 \qquad\qquad f(x) := \begin{cases} 2 & \text{if } x < 1 \\ 0.5 & \text{if } x \geqslant 1. \end{cases}$$

When drawing their graphs, as in figure 3.1, we can do this for f without lifting the pen, while for g we have to lift it at $x = 0$. And indeed, for $\epsilon := \frac{1}{4} > 0$, we cannot find the required $\delta > 0$.

Verify this for yourself.

Now, a good exercise to build intuition is to show that this 'hands-on' definition of continuity is equivalent to—and hence generalized by—the abstract topological definition.

I admit, the pun is intended

For this, we have to define the standard topology on the real line $\mathbb{R}$. This is done just like in the three-dimensional case, except that the distance function now simplifies: Here, since $\mathbb{R}$ has just one dimension, $d(x, y) = \sqrt{(x - y)^2} = |x - y|$. So the opens of the real line are those subsets $U \subseteq \mathbb{R}$ such that, for all $x \in \mathbb{R}$, if $x \in U$, then there is $\epsilon > 0$ such that, for all $x' \in \mathbb{R}$ with $d(x, x') < \epsilon$, we have $x' \in U$.

In other words, for a given function $f : \mathbb{R} \to \mathbb{R}$, show that f satisfies 1 iff f is continuous according to definition 3.6 with the just defined topology on $\mathbb{R}$.

This also is the end-of-chapter exercise 3.c.

The continuous functions are for topological spaces what the monotone functions were for partial orders. But, like for partial orders, we sometimes also want to consider stronger properties of these structure-preserving maps—in particular, the notion of isomorphism.

Definition 3.8. A continuous function $f : X \to Y$ between topological spaces is

- *open* if, for all open $U \subseteq X$, the image $f[U] = \{f(x) : x \in U\}$ is an open subset of Y.
- *closed* if, for all closed $C \subseteq X$, the image $f[C] = \{f(x) : x \in C\}$ is a closed subset of Y.
- a *homeomorphism* (the topologists' name for isomorphism) , if f has a continuous inverse, i.e., f is a bijection and both f and f^{-1} are continuous. (Equivalently, as exercise 3.d shows, f is a continuous and open bijection; this is further equivalent to f being a continuous and closed bijection.)
- an *embedding*, f is injective and, for each open $U \subseteq X$, there is an open $V \subseteq Y$ such that $f[U] = f[X] \cap V$. Equivalently, the function $f : X \to f[X]$ is a homeomorphism when giving $f[X] \subseteq Y$ the subspace topology (whose opens are $V \cap f[X]$ for $V \subseteq Y$ open).

Note the additional 'e': it is not 'homomorphism' as with lattices.

This is the conceptual meaning of embedding: X is, up to homeomorphism, a subspace of Y.

Homeomorphisms are the isomorphisms of spaces: If there is a homeomorphism between spaces they are called homeomorphic and hence are topologically the same. The standard example is that a donut and a coffee mug are homeomorphic: you can obtain one from the other by squishing and squeezing, but—importantly—without breaking and tearing.

Hence the common joke that topologists cannot tell them apart.

3.1.3. Constructions: subspaces, products, quotients

Coming to constructions with topological spaces, we have the following.

Cf. the trinity of sublattice, product, and homomorphic images/quotient for lattices.

1. Subspace. Given a topological space (X, τ), any nonempty subset $Y \subseteq X$ can be naturally made into a topological space by equipping it with the *subspace topology*

$$\tau \restriction Y := \{U \cap Y : U \in \tau\}.$$

2. Product topology. If $(X_i)_{i \in I}$ is a collection of topological spaces indexed by a set I, the product space $\prod_{i \in I} X_i$ has as underlying set the Cartesian product of the sets X_i and its topology is generated by the subbase of sets of the form $\left\{x \in \prod_{i \in I} X_i : x_j \in V\right\}$ for $j \in I$ and $V \subseteq X_j$ open.

Equivalently, this is the smallest topology making continuous all the projections $\pi_i : \prod_I X_i \to X_i$ mapping x to its i-th component x_i.

3. Quotient space. If X is a topological space and $\equiv$ an equivalence relation on X, the quotient space has as underlying set $X/\equiv$ and the opens are those sets $U \subseteq X/\equiv$ such that $\{x \in X : [x]_\equiv \in U\}$ is open in X.

A construction specific to spaces is that we can take the *join* of two topologies that live on the same underlying set. This is made precise as follows.

1. If X is a nonempty set, then

$$\mathrm{Top}(X) := \left\{\tau \in 2^{2^X} : \tau \text{ is a topology}\right\}$$

is, when ordered under inclusion, a complete lattice.

2. Infima are given by intersections, and suprema are given by the topology generated by unions. The least element is the indiscrete topology, and the greatest element is the discrete topology.

3. In particular, if σ and τ are two topologies on X, then their *join* $\sigma \vee \tau$ is the topology generated by $\sigma \cup \tau = \{U \subseteq X : U \in \tau \text{ or } U \in \sigma\}$.

Some fun facts are that the Hausdorff topologies on X form an upset in $\mathrm{Top}(X)$, and the compact topologies form a downset. And compact-Hausdorff topologies are incomparable: If σ is a Hausdorff topology and τ a compact topology on the nonempty set X, then $\sigma \subseteq \tau$ implies $\sigma = \tau$.

A proof of this can be found in Gehrke and S. van Gool (2023, Prop. 2.10).

3.2. Important topological concepts

Now that we know topological spaces, we introduce some further topological concepts that are important in their own right and will figure in the definition of a Stone space.

3.2.1. Separation axioms

There is an important classification of topological spaces according to which so-called *separation axioms* they satisfy. There are many such axioms and they all are of the form that two distinct points can be—in various senses—separated by the topology. (Note that, despite them being called 'axioms', in general a topological space is not required to satisfy any of them.) The five main ones coming in increasing strength.

Definition 3.9. Let (X, τ) be a topological space.

1. X is T_0 (aka *Kolmogorov*) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that U contains exactly one of x and y.

2. X is T_1 (aka *Fréchet*) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that $x \in U$ and $y \notin U$.

3. X is T_2 (aka *Hausdorff*) if, for all $x \neq y$ in X, there are disjoint opens $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

4. X is T_3 (aka *regular*) if X is T_1 and, for all $x \in X$ and closed $C \subseteq X$ with $x \notin C$, there are disjoint opens $U, V \subseteq X$ with $x \in U$ and $C \subseteq V$.

5. X is T_4 (aka *normal*) if X is T_1 and, for all disjoint closed $C, D \subseteq X$, there are disjoint opens $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

Exercise 3.10. Show that these conditions indeed increase in strength, i.e., that we have the implications: $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$. Also check that the three-dimensional space $\mathbb{R}^3$ is normal and hence also satisfies the other separation axioms. Moreover, show that $(\mathbb{N}, \tau)$, where τ is the collection of upsets of $\mathbb{N}$ with the usual order, is a topological space that is T_0 but not T_1 (and hence also not T_2, T_3, or T_4).

That also is the end-of-chapter exercise 3.f.

3.2.2. Compactness

Another important concept is that of compactness. It formalizes the intuition that the space does not extend infinitely but has finite bounds.

Definition 3.11 (Compactness)**.** Let X be a topological space. If $S \subseteq X$ is a subset, an *open cover* $\mathcal{U}$ of S is a collection of open sets such that $S \subseteq \bigcup_{U \in \mathcal{U}} U$. A subset $S \subseteq X$ is *compact* if every open cover $\mathcal{U}$ of S contains a finite subcover, i.e., there is a finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of S. The space X is called *compact* if $S := X$ is compact.

For example, while the whole Euclidean space is not compact, closed boxes in it like the unit cube

$$[0, 1] \times [0, 1] \times [0, 1] = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leqslant x_1, x_2, x_3 \leqslant 1\right\}$$

are compact. Also any finite subset of a space is compact. Exercise 3.g asks you to establish this.

There also is a local version of compactness: A topological space X is *locally compact* if, for any open neighborhood U of any point $x \in X$, there is an open $V \subseteq X$ and compact $K \subseteq X$ such that $x \in V \subseteq K \subseteq U$. If X is Hausdorff, then compactness implies local compactness, but this is not true in general. And local compactness does also not imply compactness (the Euclidean space is locally compact but not compact).

The following exercise collects some very useful facts about compactness.

Exercise 3.12. Prove the following:

This is exercise 3.h.

1. A closed subset of a compact space is compact.
2. A compact subset of a Hausdorff space is closed. This need not be true without the Hausdorffness assumption.
3. The image of a compact subset under a continuous function is compact.
4. Conclude that a continuous function from a compact space to a Hausdorff space is closed.
5. Conclude (hint: exercise 3.d) that a continuous bijection between compact Hausdorff spaces is a homeomorphism.
6. A compact Hausdorff space is normal (hint: show it is regular first).

Some further and more advanced results are the following.

1. *Finite intersection property characterization*. Let X be a topological space and $S \subseteq X$ a subset. A collection $\mathcal{A}$ of closed sets has the *finite intersection property* with respect to S if for every finite subcollection $\mathcal{A}_0$, there is $x \in S$ such that $x \in \bigcap \mathcal{A}_0$. Then S is compact iff, for every collection $\mathcal{A}$ of closed sets with the finite intersection property with respect to S, there is $x \in S$ with $x \in \bigcap \mathcal{A}$. (The proof essentially is rewriting the open set definition by taking complements and using the de Morgan laws: see, e.g., here.)

   If $S = X$, we omit the 'with respect to S'.

2. *Alexander Subbase Theorem*. Let X be a topological space and $\mathcal{S}$ a subbase. If every cover $\mathcal{U} \subseteq \mathcal{S}$ of X has a finite subcover, then X is compact.

   The proof of this requires a non-constructive principle, i.e., a (strictly weaker) version of the axiom of choice. As this is an axiom of standard set theory, we assume this throughout in this course.

3. *Tychonoff's Theorem*. This says that the arbitrary product of compact spaces is again compact. (This is a corollary of the Alexander Subbase Theorem; see, e.g., here.)

3.2.3. Dimensionality

We say that the space we live in has three dimensions. But is there a way to reasonably define the dimension of a general topological space (X, τ) in such a way that the dimension of $\mathbb{R}^3$ considered as a topological space is indeed 3? It turns out there is; in fact there are several such reasonable definitions, though they may disagree on some topological space (X, τ) different from $\mathbb{R}^d$. The common ones are called small inductive dimension, large inductive dimension, and the Lebesgue covering dimension. For a full treatment, see Engelking (1989, ch. 7). Here we will only look at the small inductive dimension. In fact, eventually we will only consider zero-dimensional spaces. So we will only briefly discuss dimensionality to get the general motivation (definition 3.13) and then prove a simple characterization for the zero-dimensional spaces (proposition 3.14).

According to our intuitive notion of dimension, one-dimensional objects are things like a line or a circle, two-dimensional objects are things like a plane or a disk, and three-dimensional objects are things like 'our' space or also a sphere (say, a football). In the extreme case, we may also say that a single point has dimension 0. Moreover, the dimensions of these objects are connected as follows: if we consider the boundary of an object, we reduce the dimension by one. For example, the boundary—i.e., surface—of a sphere is two-dimensional (an ant walking on the football just experiences a two-dimensional world); and the boundary of a disk is just the circle.

The formal definition of the small inductive dimension bootstraps these intuitions: The simplest case is where the object is just the empty set. Its dimension is stipulated to be -1. Because then, if we consider the next more complicated object, namely a single point, then its is indeed 0: it must be one more than the dimension of its boundary, but its boundary is empty, hence has dimension -1, so the dimension of the point is one more, i.e., 0. Now we proceed inductively: Assume we have a yet more complicated object, but which is still not much more complicated in the sense that we can already determine the dimension of its boundary. Then the dimension of this object is one more than the dimension of its boundary.

The formal definition goes as follows (relativizing the above idea to each neighborhood of points of the considered object).

Definition 3.13. Given a regular topological space (X, τ) (which, only for this definition, we allow to also be empty) and an integer $n \geqslant 0$, say

- $\mathrm{ind}(X) = -1$ iff $X = \emptyset$

- $\operatorname{ind}(X) \leqslant n$ iff for every point $x \in X$ and open neighborhood V of x, there is an open $U \subseteq X$ with $x \in U \subseteq V$ and $\operatorname{ind}(\partial U) \leqslant n-1$.

Recall that $\partial S = \mathsf{Cl}(S) \setminus \mathsf{Int}(S)$ *is the boundary of the set* S.

- $\operatorname{ind}(X) = n$ iff $\operatorname{ind}(X) \leqslant n$ and $\operatorname{ind}(X) \nleqslant n-1$.

- $\operatorname{ind}(X) = \infty$ iff, for all n, $\operatorname{ind}(X) \nleqslant n$.

As mentioned, we will actually be interested in spaces with dimension zero. They have the following simple characterization. It is just an unpacking of definitions, so we leave the proof as exercise 3.i.

Proposition 3.14. *Let* (X, τ) *be a regular topological space. Then* X *is zero-dimensional (i.e.,* $\operatorname{ind}(X) = 0$*) iff* X *has a base consisting of clopen sets.*

Hence, from now on, we take the definition of *zero-dimensional* to be having a base of clopens.

It may seem that being zero-dimensional is quite different from our everyday intuitions about space. This is true to some extent, but the following example shows that zero-dimensional spaces are also not completely strange.

Exercise 3.15. Write $X := \mathbb{R} \setminus \mathbb{Q}$ for the set of *irrational* numbers and let τ be the subspace topology with respect to the usual topology on $\mathbb{R}$. Show that the space of irrational numbers (X, τ) is a zero-dimensional space. (Hint: show that $\{B_{\frac{1}{n}}(q) \cap X : n \in \mathbb{N}, q \in Q\}$ is a clopen base.)

This also is the end-of-chapter exercise 3.j.

3.3. Stone spaces

Now we can define Stone spaces (section 3.3.1). Then we will discuss an important example of a Stone space: namely, the Cantor space (section 3.3.2).

3.3.1. Definition

The definition of a Stone space is actually quite very simple.

Definition 3.16. A *Stone space* is a topological space that is compact, zero-dimensional, and Hausdorff.

Exercise 3.l provides some equivalent formulations of this definition.

Though you may wonder why it is exactly those spaces that we will consider. There are two answers. First, several important spaces—especially in descriptive sets theory—are Stone spaces. In the next subsection, we will consider the most important one: Cantor space. From this point of view, we consider Stone spaces simply because it makes sense to study the general properties of this important class of spaces.

The second answer is the insight of Stone that it is precisely those spaces that dually correspond to Boolean algebras under the constructions motivated in chapter 1. When we start with a Boolean algebra of, say, propositions and we consider the set of prime filters on it, we will see that we can naturally define a topology on this set. It is generated by calling those sets of prime filters open that make true a given proposition in the Boolean algebra. It turns out that this topology has all the defining features of a Stone space. Conversely, if we start with any topological space, the collection of its clopen subsets forms a Boolean algebra; but if this space actually was a Stone space, then this Boolean algebra is rich enough so we can recover the original space as the set of prime filters on this Boolean algebra.

So the second answer is a 'in hindsight' motivation of the definition: it characterizes a class of spaces that we obtain through a construction that we are interested in. But since we have not seen that yet, let's first explore the first answer: namely, example of Stone spaces that are important in their own right.

3.3.2. Example: Cantor space

The Cantor space plays an important role in descriptive set theory (Kechris 1995). One reason is that it is universal in the sense that every nonempty compact metrizable space is a continuous image of the Cantor space. In particular, every Stone space that is second-countable (i.e., has a countable base) is an image of the Cantor space. We will now define the Cantor space and show that it is a Stone space. We will not prove the just mentioned universality result (see Kechris 1995, p. 23), but it assures us that the Cantor space is an important example of a Stone space: it in a sense already contains all second-countable Stone spaces.

Second-countable Stone spaces are metrizable by Urysohn's metrization theorem.

The succinct definition of Cantor space is as follows; we will unpack it below.

Definition 3.17. Equip $2 := \{0, 1\}$ with the discrete topology. Define $2^{\mathbb{N}}$ to be the product space $\prod_{\mathbb{N}} 2$ of $\mathbb{N}$-many copies of 2. The space $2^{\mathbb{N}}$ is called the *Cantor space*.

The points of Cantor space are, by definition of the Cartesian product (appendix A.1), infinite binary sequences, i.e., functions $x : \mathbb{N} \to 2 = \{0, 1\}$. We also write these sequences as $x(0)x(1)x(2)\ldots$. For example, here are two points of Cantor space:

$$0101010101010101010\ldots \qquad 1110101001010111010\ldots.$$

The topology of Cantor space is concisely described via the product topology, but to build an intuition, it is useful to introduce *cylinder sets*. Given a binary string $\sigma = \sigma_0 \ldots \sigma_{n-1}$ (i.e., a finite sequence of 0's and 1's), the corresponding cylinder set is

$$[\sigma] := \left\{ x \in 2^{\mathbb{N}} : \forall k \in \{0, \ldots, n-1\}. x(k) = \sigma_k \right\},$$

so $[\sigma]$ contains all those binary sequences that have σ as initial segment.

Exercise 3.18. Show that the cylinder sets form a base for the topology of $2^{\mathbb{N}}$. (Hint: go back to the definition of the product topology in section 3.1.3.)

With this, it is quite straightforward to show that the Cantor space is a Stone space.

Exercise 3.19. Show the following for the Cantor space $2^{\mathbb{N}}$:

1. It is Hausdorff. (Hint: Conclude this from the previous exercise 3.18.)
2. It is zero-dimensional. (Hint: Show that the cylinder sets in fact are clopen.)
3. It is compact. This follows immediately from Tychonoff's theorem (since the finite space 2 is compact). But, as an exercise, also show this directly with the finite intersection property definition (section 3.2.2).

A few more comments on Cantor space. You might know it as the Cantor *set* from analysis: where you take the unit interval $[0, 1]$ and keep removing the middle thirds. More precisely, it consists of those numbers in the unit interval that have only 0's and 2's in their ternary expansion. One can show that this Cantor set is homeomorphic to the Cantor space $2^{\mathbb{N}}$. In general, a theorem of Brouwer characterizes the Cantor space as the, up to homeomorphism unique perfect, nonempty, compact, metrizable, zero-dimensional space (Kechris 1995, p. 35). Cantor space also is a natural setting for probability theory (as a standard Borel space) and also for algorithmic randomness (Downey and Hirschfeldt 2010).

A space is perfect if for every open neighborhood U of a point x, there is a point $y \in U$ with $y \neq x$.

3.4. Exercises

To get familiar with the abstract topological concepts from section 3.1.1, the next two exercises apply them to the usual three-dimensional space $\mathbb{R}^3$

(whose open sets are those with ‘wiggle-room’).

Exercise 3.a. 1. Prove that the collection τ of sets $U \subseteq \mathbb{R}^3$ with wiggle-room, as defined in example 3.2 (1), indeed forms a topology on $\mathbb{R}^3$.

2. A *closed interval* is of the form $[a, b] := \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ with $a, b \in \mathbb{R}$. An open interval is of the form $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ with $a, b \in \mathbb{R}$. (They are empty if $a \not\leqslant b$.) In three-dimensional space, a *rectangular cuboid* is of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ with $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$. (So a rectangular cuboid is just the 3D analogue of a rectangle; and a rectangle, in turn, is just the 2D analogue of an interval. For higher dimensions, one speaks of boxes or also hyperrectangles or k-cells.) Show that rectangular cuboids are closed in $\mathbb{R}^3$.

3. Show that the sequence of points $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n})_{n \geqslant 1}$ converges to $(0, 0, 0)$.

Exercise 3.b. Do exercise 3.5. (If you have done exercise 2.e, you can additionally show that the interior map $\mathsf{Int} : 2^X \to \Omega(X)$ is upper adjoint to the inclusion map $\iota : \Omega(X) \to 2^X$, and that the closure map $\mathsf{Cl} : 2^X \to \mathcal{C}(X)$ is lower adjoint to the inclusion $\iota' : \mathcal{C}(X) \to 2^X$ (Gehrke and S. van Gool 2023, ex. 2.1.3).)

Exercise 3.c. Do exercise 3.7.

(Gehrke and S. van Gool 2023, ex. 2.1.1 (c))

Exercise 3.d. Let $f : X \to Y$ be a continuous bijection. Show that the following are equivalent.

(Gehrke and S. van Gool 2023, ex. 2.1.4 (c))

1. f is a homeomorphism (i.e., its inverse is continuous)

2. f is open (i.e., maps open sets to open sets)

3. f is closed (i.e., maps closed sets to closed sets).

Exercise 3.e. Consider $X = \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Show that the standard ‘wiggle room’ topology on $X = \mathbb{R}^3$, as defined in example 3.2 (1), is the same topology as the product topology on $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ where each $\mathbb{R}$ is equipped with the standard topology.

Exercise 3.f. Do exercise 3.10.

Exercise 3.g. This exercise gets you acquainted with the concept of compactness via some examples.

1. Show that $\mathbb{R}^3$ is not compact but the unit cube is.

2. Show that any finite subset of any topological space is compact.

Exercise 3.h. Do exercise 3.12.

Exercise 3.i. Prove proposition 3.14.

Exercise 3.j. Do exercise 3.15.

Exercise 3.k. Do exercises 3.18 and 3.19 to establish that the Cantor space is a Stone space.

Exercise 3.l. Let X be a topological space. Prove that the following are equivalent.

1. X is a Stone space (in the sense of definition 3.16, i.e., compact, Hausdorff, and zero-dimensional)

2. X is compact, T_0, and zero-dimensional.

3. X is compact and *totally separated* (i.e., for any $x \neq y$ in X there are disjoint open sets U and V such that $x \in U$ and $y \in U$ *and* $U \cup V = X$).

4. Two sides of the same coin: Stone duality

In this chapter, we finally prove the Stone duality theorem that we have been working toward. It provides an exact correspondence between Boolean algebras on the one hand and Stone spaces on the other hand. In section 4.1, we first state the main idea of this correspondence informally. In section 4.2, we then formulate it precisely and, finally, prove it in section 4.3. In the next chapter, we discuss some applications.

4.1. Stone duality: The main idea

We already saw in the introduction (chapter 1) the conceptual idea of how algebras correspond to spaces. We now can state this more concretely, namely for Boolean algebras corresponding to Stone spaces. In this section, we do this informally—to have a succinct overview—, before doing it formally in the next section. We will use the 'possible worlds vs. propositions' example of a duality (as discussed in section 1.2.2), but you can also swap this to your favorite example.

From algebras to spaces Assume we have a Boolean algebra: say, the set of propositions, i.e., meanings of sentences. Then we can construct a space of possible worlds. We do so, by taking a possible world w to be a 'prime filter' on the algebra, i.e., a set of propositions that is appropriately closed under the operations of implications, conjunctions, disjunctions, logical truth, and logical falsity. What we could not see back than—but what we will see now—is that this collection of constructed worlds w carries a natural topology. It is generated by the sets $\{w : a \in w\}$, for each proposition a in the algebra. We will see that this defines a Stone space.

From spaces to algebras Conversely, if we start with the space of possible worlds, then we can construct a Boolean algebra of propositions. This is because, in line with possible worlds semantics, propositions simply are sets of possible worlds. Namely, a proposition is the set of those worlds where it is true. However, while propositions are sets of worlds, we do not consider every set of worlds to be a proposition. Rather, following the observation that the space of possible worlds naturally carries a topology, we only consider the 'clopen' sets of worlds to be a proposition. (Recall that a set is clopen if it is both closed and open in the topology.) We will see that the collection of clopen sets of worlds indeed forms a Boolean algebra.

Correspondence back and forth Now we have a way to map an algebra to a space and a way to map a space to an algebra. But if we want a (bijective) correspondence, we want that that these maps cancel each other out. If I start with an algebra, map it to a space, and—in turn—map that back to an algebra, I should get an algebra that is isomorphic to the one I started with. And we want the same for spaces (for spaces, though, we say 'homeomorphic' instead of 'isomorphic'). After all, if we take propositions as given and—following ersatzism—build substitute possible worlds (as prime filters), then the substitute propositions we get from those substitute worlds (as sets of substitute worlds) better be isomorphic to the initially given algebra of propositions. In the other direction, if we take possible worlds as given and—following possible worlds semantics—consider propositions to be sets of possible worlds, then the substitute worlds we can build from these propositions better be isomorphic to the initially given space of possible worlds.

Duality Now we have a correspondence between spaces and algebras. But why do we speak of a 'duality'? The reason is this. The correspondence that we have so far is between the *objects* in the category of spaces on the one hand and the category of algebras on the other hand. But we would also like to have a correspondence between the *morphisms* of the categories, i.e., the structure-preserving maps between the objects. So a continuous map between Stone spaces should correspond uniquely to a Boolean algebra homomorphism between the Boolean algebras corresponding to the spaces. We speak of a (categorical) 'equivalence' if a morphism $f : X \to Y$ of Stone spaces corresponds uniquely to a morphism $g : A \to B$, where A and B are the Boolean algebras corresponding to X and Y, respectively. However, we will see that we do not have this; but something close. What we have to do is to reverse the direction of the arrow g: the morphism f uniquely corresponds to some $g : B \to A$. This phenomenon occurs often enough to get a specific name: one then speaks of a *dual equivalence* between the categories.

4.2. Stone duality: The precise statement

In this section, formally state the Stone duality in a series of propositions. So the Stone duality formally is the conjunction of propositions 4.3–4.10 below. Category-theoretically this is swiftly expressed as: The category of Boolean algebras and Boolean algebra homomorphisms is dually equivalent to the category of Stone spaces and continuous functions.

4.2.1. From Boolean algebras to Stone spaces: Prime filters

Let's recall again the idea of how to recover a space from a Boolean algebra. If we have a Boolean algebra L of propositions, we can recover the possible worlds as 'maximally decided' subsets F of L, i.e., those which have the expected closure conditions for implication ($\leqslant$), conjunction ($\wedge$), disjunction ($\vee$), logical truth ($\top$), and logical falsity ($\perp$). We have already called these subsets prime filters:

By 'recover' we mean 'retrieve' or 'regain': Assume we have the Boolean algebra but lost the space, then how can we deduce how the space must have been like only using the information provided by the Boolean algebra?

Definition 4.1. Let L be a lattice. (So, in particular, this definition applies if L is a Boolean algebra.) A subset F of L is a filter if it is a nonempty upset that is closed under meet. It is proper, if $F \neq L$ (equivalently, $\perp \notin F$). A *prime filter* is a proper filter such that, for all $a, b \in L$, if $a \vee b \in F$, then either $a \in F$ or $b \in F$. We write $\mathsf{PrFilt}(L)$ for the set of all prime filters of L. If needed, we write $\mathsf{Filt}(L)$ for the set of all filters of L.

Being nonempty is, for an upset, equivalent to $\top \in F$.

The order-dual notion of a filter is sometimes also useful and is called an *ideal*: Concretely, these are nonempty downsets $I \subseteq L$ closed under join (if $a, b \in I$, then $a \vee b \in I$). An ideal is *proper* if $I \neq L$ and *prime* if, additionally, $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

The notion of an ideal is also important in the theory of rings which is central in commutative algebra. They are used to build the spectra *of rings, which provide useful topological tools to understand a ring. They form spectral spaces—hence the name. To see how lattice ideals relate to ring ideals, see exercise 3.1.9 of Gehrke and S. van Gool 2023.*

Here we will always work with filters and rarely mention ideals (only if we need to also talk about duals of filters). But, again, ultimately this is a convention and many textbooks primarily use ideals.

An equivalent way to define a prime filter F of a lattice L is by requiring that its characteristic function

$$
\begin{aligned}
\chi_F &: L \to \mathbf{2} \\
a &\mapsto \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}
\end{aligned}
$$

is a lattice homomorphism. This brings out maybe more clearly the intuition that a prime filter decides, for every proposition in L, whether it is true according to it. Although these two characterizations (subsets vs homomorphisms) are equivalent, it often is conceptually useful to consider both. Exercise 4.a asks you to prove this equivalence and a third one that prime filters are those subsets whose complements are prime ideals.

So far, what we said about prime filters works in any lattice. But when we restrict us to Boolean algebras, a striking fact is the following characterization of prime filters. The proof is a recommended exercise (exercise 4.b).

Proposition 4.2. *Let A be a Boolean algebra, and let $F \subseteq A$ be a filter. Then the following are equivalent.*

1. F *is a prime filter.*
2. F *is a* maximal *filter, i.e.,* F *is proper and for any proper filter* G *with* $F \subseteq G$*, we have* $F = G$*.*
3. F *is an* ultrafilter*, i.e.,* F *is proper and for any* $a \in A$*, either* $a \in F$ *or* $\neg a \in F$*.*

The philosophical meaning of this is that the usual, well-motivated notion of a prime filter is equivalent to other common notions of a 'model' in classical logic (and Boolean algebras are the algebraic version of classical logic). Prime filters require the models to respect conjunction and disjunction. Ultrafilters require models to respect conjunction and negation. Maximal filters require models to be maximally consistent. All three are common ways of specifying a classical model or possible world (or whatever the philosophical interpretation of the points in the space).

Cf. section 2.3.2: being a lattice homomorphism (respecting $\wedge$ and $\vee$) already implies being a Boolean algebra homomorphism (i.e., also respecting $\neg$).

With this understanding of prime filters, let's move to understanding the structure of the set of all prime filters on a Boolean algebra L: because this is how we recover the space of possible worlds. What we couldn't yet see so far is that this space really is a space in the sense of topology! So we need to say what the open sets are on $\mathsf{PrFilt}(L)$. But what's a natural choice? The general intuition for the open sets is that they represent different degrees of closeness or similarity: For example, in our space, the open ball $B_\epsilon(x)$ around a point x represents closeness to degree ϵ. What would be basic degrees of similarity for prime filters? The idea is that two prime filters can be close to degree $a \in L$ by agreeing on the property $a \in L$, i.e., both contain a or both do not contain a. So we declare open, for each $a \in L$, the sets

$$\{F \in \mathsf{PrFilt}(L) : a \in F\}$$

This indeed produces a Stone space as the next propositions shows.

Proposition 4.3. *Let* L *be a Boolean algebra. Equip* $\mathsf{X} := \mathsf{PrFilt}(\mathsf{L})$ *with the topology generated by* $\mathcal{S} := \{\widehat{a} : a \in \mathsf{L}\}$*, where*

$$\widehat{a} := \{\mathsf{F} \in \mathsf{X} : a \in \mathsf{F}\}.$$

Then X *with this topology is a Stone space, which we also denoted* $\mathsf{St}(\mathsf{L})$*. Moreover,* $\mathcal{S} = \mathsf{Clp}(\mathsf{X})$*.*

In this and the next section, we often keep writing $\mathsf{PrFilt}(\mathsf{L})$ *for* $\mathsf{St}(\mathsf{L})$ *to remind us that we take prime filters.*

Example 4.4. Consider the diamond Boolean algebra with two incomparable elements a and b between the bottom element $\bot$ and the top element $\top$ (cf. figure 2.1). What are its prime filters? To compute them by brute force, let's first list all its upsets (since every prime filter in particular is an upset):

$$\emptyset, \{\top\}, \{a, \top\}, \{b, \top\}, \{a, b, \top\}, \{\bot, a, b, \top\}.$$

Which of those are filters? Not the empty set, because filters are required to be nonempty. And also note the second-last set, because it is not closed under $\wedge$. But the remaining sets are filters. The last one is, by definition, not proper. Which of the remaining proper filters are prime? The filter $\{\top\}$ is not prime, because $a \vee b = \top$, but neither a nor b are in the filter. But the two filters

$$\mathsf{F}_1 := \{a, \top\} \qquad\qquad \mathsf{F}_2 := \{b, \top\}$$

are prime. Also not that they are maximal and ultra. So they form the points of the dual space $\mathsf{X} = \{\mathsf{F}_1, \mathsf{F}_2\}$.

What is the topology of the space X? We in particular have the open sets:

$$\begin{aligned}
\widehat{\bot} &= \{\mathsf{F} \in \mathsf{X} : \bot \in \mathsf{F}\} = \emptyset \\
\widehat{a} &= \{\mathsf{F} \in \mathsf{X} : a \in \mathsf{F}\} = \{\mathsf{F}_1\} \\
\widehat{b} &= \{\mathsf{F} \in \mathsf{X} : b \in \mathsf{F}\} = \{\mathsf{F}_2\} \\
\widehat{\top} &= \{\mathsf{F} \in \mathsf{X} : \top \in \mathsf{F}\} = \{\mathsf{F}_1, \mathsf{F}_2\}
\end{aligned}$$

These in fact already are all the subsets of the space X, to the topology is discrete in this case (later we will see that this is no coincidence). $\lrcorner$

4.2.2. From Stone spaces to Boolean algebras: Clopens

Now going in the other direction, the idea was that if we have a space of possible worlds, then propositions can be identified with their intensions: i.e., the set of possible worlds where the proposition is true. So intensions are subsets of the space. But they are not just *any* subset. What comes to light via the topology on the space, a plausible requirement is that intensions are clopen sets.

We already saw this above for the recovered possible worlds: If we have a Boolean algebra L, then for each proposition $a \in L$, its intension in the recovered space X is $\widehat{a} = \{F \in X : a \in F\}$, and by construction this $\widehat{a}$ is a clopen set. And it stands to reason that this should also be the case if we start with a space of possible worlds X (if you believe in 'proper' possible worlds, that aren't just constructed out of propositions). Indeed, it seems plausible to require that the intension of a proposition a is closed under similarity: if world x makes true proposition a and world y is very similar to x, then also y makes true a. If similarity is spelled out topologically, this requirement naturally is formalized as the intension of a being clopen:

This also makes sense for the 'object vs. property' example: there the principle of tolerance says that if object x has property a and object y is very similar to x, then also y has property a.[1]

- if x is in the intension, there is a degree of similarity (i.e., an open set) such that all objects similar to x by at least this degree also are in the intension (hence the intension is open), and
- if x is not in the intension, there is a degree of similarity (i.e., an open set) such that all objects similar to x by at least this degree also are not in the extension (hence the intension is closed).[2]

So the propositions are those subsets of the space of possible worlds that are clopen—to respect the similarities between possible worlds that is formalized in the toplogy. Fortunately, the clopens form a Boolean algebra.

Proposition 4.5. *Let (X, τ) be a Stone space. Let $L := \mathsf{Clp}(X)$ be the set of clopen subsets of X ordered by inclusion. Then $(L, \cap, \cup, \emptyset, X)$ is a Boolean algebra, with negation given by the set-theoretic complement.*

[1] Actually, more precisely this is the much less problematic *margin for error principle* (see, e.g., Williamson (1994, sec. 8.3) or Égré (2015, sec. 2)): If object x has property a, then there is some degree of similarity ϵ such that any object y that is ϵ-similar to x also has property a. Also see Belastegui Lazcano (2020).

[2] This may be even more compelling when formulated with a metric: if x is in a, then there is an $\epsilon > 0$ such that all ϵ-close y are also in a; similarly for the complement a^c. Such a metric exists if the space X is metrizable, i.e., there is a metric d on X such that the open balls of the metric generate the topology of X. By the Urysohn metrization theorem, a compact Hausdorff space (which X is) is metrizable iff it is second-countable (i.e., has a countable base). Hence the metric formulation works provided there are countably many propositions.

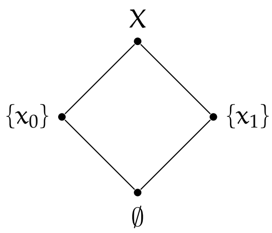


Figure 4.1.: The Boolean algebra of clopens $\mathsf{Clp}(X)$ of the two-element discrete space $X = \{x_0, x_1\}$.

Example 4.6. Let's consider the two element space $X = \{x_0, x_1\}$ with the discrete topology. Up to renaming the points (i.e., up to homeomorphism), this is the Stone space we ended up with in example 4.4. What is the Boolean algebra of clopens $\mathsf{Clp}(X)$?

Let first list all the possible subsets: $\emptyset, \{x_0\}, \{x_1\}, X$. Since the topology is discrete, all of them are clopen: any set is open, so also the complement of any set is open, so any set is both open and closed. If we order these sets by inclusion, we get the Boolean algebra in figure 4.1.

As we will see next, it is no coincidence that this Boolean algebra is isomorphic to the diamond Boolean algebra we started out with in example 4.4. ⌟

4.2.3. Boolean algebras are isomorphic to their double-duals

If we start with an algebra of propositions and construct worlds as prime filters, then we would expect that the algebra of intensions of the propositions with respect to the constructed worlds is isomorphic to the original algebra of propositions. Concretely, if we are given a proposition a in a Boolean algebra L of propositions, we can consider its intension $\widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}$ with respect to the constructed worlds—and we expect a to play the same role as $\widehat{a}$. In other words, we expect that this map $a \mapsto \widehat{a}$ in fact is an isomorphism. Fortunately, this is the case.

Proposition 4.7. *Let L be a Boolean algebra. Then the following is a well-defined lattice isomorphism between L and its double-dual:*

$$\begin{aligned} \widehat{\cdot} : L &\to \mathsf{Clp}(\mathsf{PrFilt}(L)) \\ a &\mapsto \widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}. \end{aligned}$$

4.2.4. Spaces are isomorphic to their double-duals

Similarly, if we start with a space of possible worlds, we also expect that it is homeomorphic to the space of worlds constructed as prime filter over intensions formed with the original possible worlds. In section 1.2.2, we discussed the motivation behind this:

- Each world x determines the set F_x of propositions/intensions that are true at x—and this is a prime filter.
- Each set F of propositions that is a prime filer should determine a possible world x: namely, the unique possible world making true exactly the propositions in F.

This indeed yields an isomorphism between a space and its double-dual:

Proposition 4.8. *Let (X, τ) be a Stone space. Then the following is a well-defined homeomorphism:*

$$\begin{aligned} \beta : X &\to \mathsf{PrFilt}(\mathsf{Clp}(X)) \\ x &\mapsto \{a \in \mathsf{Clp}(X) : x \in a\} \end{aligned}$$

the x *with* $x \in a$ *for all* $a \in F \leftarrow F$.

4.2.5. Also including morphisms

Now, we have the correspondence between the objects (spaces and algebras), but we also want a correspondence between the morphisms (continuous maps and Boolean algebra homomorphisms). Again, we first see how to move in both directions (proposition 4.9), and then we check that these moves cancel each other out (proposition 4.10).

Let's start by moving from space to algebras. If X and Y are Stone spaces, their dual Boolean algebras are $\mathsf{Clp}(X)$ and $\mathsf{Clp}(Y)$. If we have a continuous function $f : X \to Y$, how do we get a corresponding morphism between those dual Boolean algebras? We already mentioned that we corresponding morphism will go in the opposite direction: $\mathsf{Clp}(Y) \to \mathsf{Clp}(X)$. The reason is preimage of a function is much better behaved than the direct image: it preserves more set-theoretic operations like intersection, unions, etc.—see exercise 4.c. So rather than trying to map a clopen subset $A \subseteq X$ to $f[A] \subseteq B$, we map a clopen subset $B \subseteq Y$ to $f^{-1}(B) \subseteq X$. Since f is continuous, $f^{-1}(B)$ is indeed clopen. The result below says that this indeed is a Boolean algebra homomorphism.

This is a trick worth remembering!

Now, let's consider moving from algebras to spaces. If L and M are algebras, their dual Stone spaces are $\mathsf{PrFilt}(L)$ and $\mathsf{PrFilt}(M)$. If we have a

Boolean algebra homomorphism $h : L \to M$, how do we get a corresponding continuous function between the dual spaces? We again expect to go in the opposite direction: $\mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$. If G is a prime filter on M, it is, in particular, a subset of M, so we again map it to $h^{-1}(F) \subseteq L$. Exercise 4.e asks you to verify that this is indeed a prime filter. Formally, the result then is the following.

I told you so: this is a trick to remember :-)

Proposition 4.9.

1. *If* $f : X \to Y$ *is a continuous function between Stone spaces, then*

$$\begin{aligned} \mathsf{Clp}(f) : \mathsf{Clp}(Y) &\to \mathsf{Clp}(X) \\ B &\mapsto f^{-1}(B) \end{aligned}$$

   is a Boolean algebra homomorphism.

2. *If* $h : L \to M$ *is a Boolean algebra homomorphism, then*

$$\begin{aligned} \mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) &\to \mathsf{PrFilt}(L) \\ G &\mapsto h^{-1}(G) \end{aligned}$$

   is a continuous function.

So now we have extended Clp and PrFilt from objects to also including morphisms: they not only map Stone spaces to Boolean algebras and vice versa, but also continuous functions to Boolean algebra homomorphisms and vice versa. Exercise 4.f asks you to show that, in doing so, they also respect the composition of morphisms, i.e., $\mathsf{Clp}(g \circ f) = \mathsf{Clp}(g) \circ \mathsf{Clp}(f)$ and $\mathsf{PrFilt}(g \circ f) = \mathsf{PrFilt}(g) \circ \mathsf{PrFilt}(f)$. In category-theoretic terminology we say: they are functors.

To get a correspondence, we need to check that also on morphisms PrFilt and Clp cancel each other out. But what should that mean precisely? Let's start with the spatial side. Like with the objects, we want to say that if we start on the spatial side with $f : X \to Y$, and map it to the algebraic side to get $\mathsf{Clp}(f)$, and then map it back again to $\mathsf{PrFilt}(\mathsf{Clp}(f))$, we should get something that is isomorphic to f. But what should being 'identical up to isomorphism' mean for functions? The answer is: the functions f and $\mathsf{PrFilt}(\mathsf{Clp}(f))$ should behave the same under the isomorphisms that we already have for the objects, namely, $\beta_X : X \to \mathsf{PrFilt}(\mathsf{Clp}(X))$ and $\beta_Y : Y \to \mathsf{PrFilt}(\mathsf{Clp}(Y))$. In other words, first applying f and then the isomorphism (i.e., β_Y) should be the same as first applying the isomorphism (i.e., β_X) and then $\mathsf{PrFilt}(\mathsf{Clp}(f))$. One says that the following diagram should commute:

$$
\begin{tikzcd}
X \arrow[r, "\beta_X"] \arrow[d, "f"] & \mathsf{PrFilt}(\mathsf{Clp}(X)) \arrow[d, "\mathsf{PrFilt}(\mathsf{Clp}(f))"] \\
Y \arrow[r, "\beta_Y"] & \mathsf{PrFilt}(\mathsf{Clp}(Y))
\end{tikzcd}
\tag{4.1}
$$

To say that a diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "g'"] \\
C \arrow[r, "f'"] & D
\end{tikzcd}
$$

commutes is to say that $g' \circ f = f' \circ g$.

We also want the same one the algebraic side: If we start on the algebraic side with $h : L \to M$, and map it to the spatial side side to get $\mathsf{PrFilt}(h)$, and then map it back again to $\mathsf{Clp}(\mathsf{PrFilt}(h))$, we should get something that is 'isomorphic' to h. Again, this means that the functions h and $\mathsf{Clp}(\mathsf{PrFilt}(h))$ should behave the same under the isomorphisms that we already have for the objects, namely, $\widehat{\cdot}_L : L \to \mathsf{Clp}(\mathsf{PrFilt}(L))$ and $\widehat{\cdot}_M : M \to \mathsf{Clp}(\mathsf{PrFilt}(M))$. In other words, the following diagram should commute:

$$
\begin{tikzcd}
L \arrow[r, "\widehat{\cdot}_L"] \arrow[d, "h"] & \mathsf{Clp}(\mathsf{PrFilt}(L)) \arrow[d, "\mathsf{Clp}(\mathsf{PrFilt}(h))"] \\
M \arrow[r, "\widehat{\cdot}_M"] & \mathsf{Clp}(\mathsf{PrFilt}(M))
\end{tikzcd}
\tag{4.2}
$$

Hence, to finalize the correspondence also between the morphisms, we claim that the above two diagrams commute.

Proposition 4.10. *The diagrams 4.1 and 4.2 commute.*

This concludes the statement of the Stone duality: as mentioned, it is the conjunction of propositions 4.3–4.10. Category-theorist would express this swiftly by saying:

Theorem 4.11 (Stone duality)**.** *The functors* PrFilt *and* Clp *form a dual equivalence between, on the one side, the category* BA *of Boolean algebras with Boolean algebra homomorphisms and, on the other side, the category* Stone *of Stone spaces with continuous functions.*

4.3. Stone duality: The proof

In this section, we provide the proofs for the Stone duality, i.e., propositions 4.3–4.10 above. We have one subsection per proposition plus a first subsection with two methods for constructing filters that we will need for the proofs.

4.3.1. Constructing filters

Given a Boolean algebra L of propositions, possible worlds are (re-) constructed as prime filters. So, given a set of propositions $A \subseteq L$, it would be very useful to construct prime filters F that contain all the propositions in A, i.e., $A \subseteq F$. Because this means that we were able to construct a possible world F that makes true all the desired propositions A.

Surely this is not always possible, for example if A contains two inconsistent propositions. But in this subsection we provide two results that show when this is possible. They correspond to two stages of the construction. The first result first extends A to a filter of L, and the second result then says when we can further extend this filter to a prime filter.

Proposition 4.12. *Let L be a lattice.*

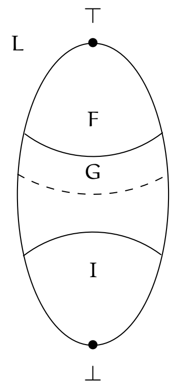
Proving this is exercise 4.g.

1. *For any subset $A \subseteq L$, there is a $\subseteq$-smallest filter F that contains A. It is called the filter* generated by A *and denoted $\langle A \rangle_{\mathrm{filt}}$.*
2. *Concretely, this filter is given as*
$$\langle A \rangle_{\mathrm{filt}} = \left\{ a \in L : \text{ there is finite } A' \subseteq A \text{ such that } \bigwedge A' \leqslant a \right\}.$$
3. *If $F \subseteq L$ is a filter and we want to extend it by an element $b \in L$, this is concretely given as*
$$\langle F \cup \{b\} \rangle_{\mathrm{filt}} = \left\{ a \in L : \text{ there is } f \in F \text{ such that } f \wedge b \leqslant a \right\}.$$

Order-dual results hold for ideals.

Theorem 4.13 (Stone's Prime Filter Extension Theorem)**.** *Let L be a distributive lattice. If F is a filter and I and ideal in L such that $F \cap I = \emptyset$, then there is a prime filter G in L such that $F \subseteq G$ and $G \cap I = \emptyset$.*

As a picture (cf. section 1.2.3):



The formulation using ideals makes this theorem more general: By choosing $I := \{\perp\}$, it says that any filter F not containing the inconsistent property $\perp$ can be extended to a prime filters. So the obstruction we mentioned before—that A is inconsistent—really is the only obstruction to extend A to a prime filter. But with the more general formulation in terms of ideal we can also, for example, take a property $b \notin \langle A \rangle_{\mathrm{filt}}$ (i.e., b is not above a finite meet of properties in A) and, by choosing $I := {\downarrow} b$, construct a prime filter extension G that does not contain b.

Proof. We go for a typical Zorn's lemma argument: Order the potential candidates for a solution so that any maximal element is an actual solution. Here the candidates are the filters extending F that don't intersect I:

$$\mathcal{P} := \{G \in \mathsf{Filt}(L) : F \subseteq G \text{ and } G \cap I = \emptyset\}$$

which we order by inclusion. To apply Zorn's lemma, we need to check that $\mathcal{P}$ is nonempty—which it is, since $F \in \mathcal{P}$—and that any nonempty chain $\mathcal{C}$ in $\mathcal{P}$ has an upper bound: this is the case since $\bigcup_{G \in \mathcal{C}} G$ is a filter that belongs to $\mathcal{P}$ (verify this as an exercise). So Zorn's lemma applies and says that $\mathcal{P}$ has a maximal element G.

Zorn's lemma is equivalent to the axiom of choice and says: A (nonempty) partially ordered set containing upper bounds for every (nonempty) chain must have a maximal element.

It remains to check that the maximality of G implies that it is prime. So assume $a \vee b \in G$ and show that either $a \in G$ or $b \in G$. The idea is to try and add a and b to G and conclude from the maximality of G that at least one of them must already be in G.

If L is a Boolean algbera and $I = \{\bot\}$, we are done already: Then G being maximal in $\mathcal{P}$ implies that G is a maximal filter, and hence also a prime filter.

So consider the filter G_a (resp., G_b) generated by $G \cup \{a\}$ (resp. $G \cup \{b\}$). Recall from propositions 4.12 that it contains precisely those $c \in L$ for which there is $g \in G$ such that $g \wedge a \leqslant c$ (resp. $g \wedge b \leqslant c$). So G_a and G_b are still filters extending F and we claim that either $G_a \cap I = \emptyset$ or $G_b \cap I = \emptyset$. This implies $G \subseteq G_a, G_b \in \mathcal{P}$, so, since G is maximal, either $G = G_a$ or $G = G_b$, hence either $a \in G$ or $b \in G$, as needed.

Indeed, if there were $c \in G_a \cap I$ and $d \in G_b \cap I$, then $c \vee d$ is in I (since I is an ideal and hence closed under $\vee$) and both in G_a and in G_b (qua upsets). The latter implies that there is g_a and g_b in G such that $g_a \wedge a \leqslant c \vee d$ and $g_b \wedge b \leqslant c \vee d$. In particular, $g := g_a \wedge g_b \in G$ is such that $g \wedge a \leqslant c \vee d$ and $g \wedge b \leqslant c \vee d$. Hence, by distributivity,

$$g \wedge (a \vee b) = (g \wedge a) \vee (g \wedge b) \leqslant c \vee d.$$

Since $g \in G$ and $a \vee b \in G$, we have that the left-hand-side of the inequality is in G, and hence, qua upset, also the right-hand-side $c \vee d$ is in G. But this element also is in I, so $G \cap I \neq \emptyset$, contradiction. $\square$

4.3.2. Proof of proposition 4.3

Let L be a Boolean algebra. Equip $X := \mathsf{PrFilt}(L)$ with the topology generated by $\mathcal{S} := \{\widehat{a} : a \in L\}$ with $\widehat{a} := \{F \in X : a \in F\}$. We need to show that X is zero-dimensional, Hausdorff, and compact, which we do in turn, and we finally show that $\mathcal{S} = \mathsf{Clp}(X)$.

Zero-dimensional. Note that $\mathcal{S}$ is a base for the topology because[3] it is closed under finite intersections: if $\widehat{a_1}, \ldots, \widehat{a_n} \in \mathcal{S}$, then, since prime filters are upsets closed under meet,

$$\widehat{a_1} \cap \ldots \cap \widehat{a_n} = a_1 \widehat{\wedge \ldots \wedge} \, a_n \in \mathcal{S}.$$

Moreover, each $\widehat{a}$ in $\mathcal{S}$ is clopen: By the ultrafilter characterization of prime filters in a Boolean algebra (proposition 4.2), we have

$$\widehat{a}^c = \{F \in \mathsf{PrFilt}(L) : a \notin F\} = \{F \in \mathsf{PrFilt}(L) : \neg a \in F\} = \widehat{\neg a},$$

so not just $\widehat{a}$ but also its complement is open, so $\widehat{a}$ is clopen. Hence $\mathcal{S}$ is a base of clopens for X, so X is zero-dimensional.

Hausdorff. If $F \neq G$ are two prime filters of L, there is, qua subsets of L, some $a \in L$ with $a \in F$ and $a \notin G$, or vice versa, but this other case will be analogous. Then $F \in \widehat{a}$ and $G \notin \widehat{a}$, so, as seen above, $G \in \widehat{\neg a}$. Hence $\widehat{a}$ and $\widehat{\neg a}$ are two disjoint open sets separating F and G.

Compactness. We show that the compactness of X really is just the compactness theorem from logic. So we first formulate this logical compactness theorem (step 1), then show that it implies the topological compactness of X (step 2), and finally prove the logical compactness theorem (step 3).

Step 1. A theory (in logic) is simply a set of sentences, so by an L-theory T we here mean a subset of L. Since models are prime filters, we say that an L-theory T has a model if there is a prime filter F of L such that $T \subseteq F$ (i.e., F makes true every sentence in T). Then the logical compactness theorem says:

(∗) If T is an L-theory such that every finite subset T' of T has a model, then T has a model.

Step 2. Let's show the topological compactness of X. We use the finite intersection formulation of compactness. So let $\mathcal{F}$ be a collection of closed subsets of X with the finite intersection property, and show that $\bigcap \mathcal{F} \neq \emptyset$. Since $\mathcal{S}$ is a base of clopens, we can assume that $\mathcal{F} \subseteq \mathcal{S}$.[4] So $\mathcal{F}$ is of the form $\{\widehat{a} : a \in T\}$ for some $T \subseteq L$. The finite intersection property implies

[3]A subbase $\mathcal{S}$ of a topological space (X, τ) that is closed under finite intersection is a base: if U is an open subset, it is a union of finite intersection of elements from $\mathcal{S}$, so, by closure under finite intersection, it also is a union of elements from $\mathcal{S}$, so $\mathcal{S}$ is a base.

[4]Every closed set is an intersection of sets from $\mathcal{S}$. Write $\mathcal{F}'$ for the collection of sets from $\mathcal{S}$ that occur in these intersections. Then $\bigcap \mathcal{F}' = \bigcap \mathcal{F}$ and $\mathcal{F}'$ still has the finite intersection property: if $A_1, \ldots, A_n$ are sets from $\mathcal{F}'$, they occur in intersections for some sets $C_1, \ldots, C_n$ in $\mathcal{F}$, so $A_1 \cap \ldots \cap A_n \supseteq C_1 \cap \ldots \cap C_n \neq \emptyset$, as needed. Thus, if the claim holds for $\mathcal{F}' \subseteq \mathcal{S}$, we can conclude it for general $\mathcal{F}$.

that every finite subset T' of T has a model: Write $T' = \{a_1, \ldots, a_n\}$, so $\widehat{a_1} \cap \ldots \cap \widehat{a_n}$ is nonempty, so it contains a prime filter F, so each a_i is in F, so $T' \subseteq F$, as needed. Hence $(*)$ implies that also T has a model, so there is a prime filter F containing every $a \in T$. So, for every $a \in T$, $F \in \widehat{a}$, i.e., $F \in \bigcap \mathcal{F}$, as needed.

Step 3. Finally, we prove $(*)$. So let $T \subseteq L$ such that every finite subset has a model. To show that T has a model, we need to show that T extends to a prime filter. So we use the two theorems for this. So, first, let F be the filter generated by T, so, by proposition 4.12, $F = \{a \in L : \exists T' \subseteq T. \bigwedge T' \leqslant a\}$. Second, writing $I = \{\bot\}$, note that we have $F \cap I = \emptyset$, because otherwise $\bot \in F$, so there is a finite $T' \subseteq T$ with $\bigwedge T' \leqslant \bot$, but then T' cannot have a model (otherwise there is a prime filter containing T', but hence also contains $\bigwedge T' = \bot$, contradiction). So Stone's Prime Filter Extension Theorem applies and yields a prime filter G such that $T \subseteq F \subseteq G$, as needed.

Characterizing the clopens. To show $\mathcal{S} = \mathsf{Clp}(X)$, it remains to show the $\supseteq$ inclusion. If U is a clopen set, it is, since $\mathcal{S}$ is a base, a union of elements $\widehat{a_i}$ in $\mathcal{S}$. Since U also is a closed subset of a compact space, it is compact. So the open cover $(\widehat{a_i})$ has a finite subcover, so, for some $a_1, \ldots, a_n$, we have $U = \widehat{a_1} \cup \ldots \cup \widehat{a_n}$. Since prime filters are upset and prime, we further have

$$U = \widehat{a_1} \cup \ldots \cup \widehat{a_n} = a_1 \widehat{\vee \ldots \vee} a_n \in \mathcal{S},$$

as needed.

4.3.3. Proof of proposition 4.5

Let (X, τ) be a Stone space. Let $L := \mathsf{Clp}(X)$ be the set of clopens of X ordered by inclusion. We have to show that L is a Boolean algebra with the usual set-theoretic operations.

We first show that L is a sublattice of the powerset lattice 2^X, which implies that it is a distributive lattice. Indeed, the empty set $\emptyset$ and the whole set X are clopen. And if A and B are clopen, so is $A \cap B$ and $A \cup B$, because both open and closed sets are closed under finite intersection and finite union.

Finally, we show that the set-theoretic complement $A^c := X \setminus A$ is a (lattice-theoretic) complement. Indeed, if A is clopen, so is A^c, and $A \cap A^c = \emptyset$ and $A \cup A^c = X$.

4.3.4. Proof of proposition 4.7

Let L be a Boolean algebra. We want to show that

$$\begin{aligned} \widehat{\cdot}: L &\to \mathsf{Clp}(\mathsf{PrFilt}(L)) \\ a &\mapsto \widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}. \end{aligned}$$

is a well-defined lattice isomorphism. We also write $X := \mathsf{PrFilt}(L)$.

The function is well-defined since $\widehat{a}$ is, by construction, a clopen subsets of $\mathsf{PrFilt}(L)$. We first show that it is a lattice homomorphism:

It maps $\perp$ to $\widehat{\mathrm{bot}} = \emptyset$ since prime filters are proper, and it maps $\top$ to $\widehat{\top} = X$ since prime filters are nonempty. Moreover, $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$ since prime filters are closed under $\wedge$. And $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$ since prime filters are prime.

The function is injective: if $a \neq b$, we show $\widehat{a} \neq \widehat{b}$. By assumption, either $a \not\leq b$ or $b \not\leq a$. Without loss of generality, assume the former. Then $F := {\uparrow}a$ and $I := {\downarrow}b$ are a filter and ideal of L, respectively, with $F \cap I = \emptyset$. By Stone's Prime Filter Extension Theorem, there is a prime filter G in L such that $F \subseteq G$ and $G \cap I = \emptyset$. So $a \in F \subseteq G$ and $b \notin G$ (otherwise $b \in G \cap I$). So $G \in \widehat{a}$ but $G \notin \widehat{b}$, as needed.

Finally, the function is surjective, because, by proposition 4.3, $\mathsf{Clp}(X) = \{\widehat{a} : a \in L\}$.

4.3.5. Proof of proposition 4.8

Let (X, τ) be a Stone space. We want to show that

$$\begin{aligned} \beta : X &\to \mathsf{PrFilt}(\mathsf{Clp}(X)) \\ x &\mapsto \{a \in \mathsf{Clp}(X) : x \in a\} \\ \text{the } x \text{ with } x \in a \text{ for all } a \in F &\leftarrow\!\shortmid F \end{aligned}$$

is a well-defined order homeomorphism. We also write $L := \mathsf{ClpD}(X)$.

We first observe that β is well-defined: if $x \in X$, then $\{a \in \mathsf{Clp}(X) : x \in a\}$ is indeed a prime filter on the Boolean algebra $L = \mathsf{Clp}(X)$. Indeed, this is a nonempty (since $x \in X$) upset closed under intersection, doesn't contain $\emptyset$ and if $x \in a \cup b$, then either $x \in a$ or $x \in b$.

So, since we deal with compact Hausdorff spaces, it suffices to show that β is a continuous bijection (recall exercise 3.12).

Continuous. Given a basic open set $\widehat{a}$ of $\mathsf{PrFilt}(L)$, we need to show that

$\beta^{-1}(\widehat{a}) \subseteq X$ is open. Indeed,

$$\beta^{-1}(\widehat{a}) = \{x \in X : \beta(x) \in \widehat{a}\} = \{x \in X : a \in \beta(x)\}$$
$$= \{x \in X : x \in a\} = a,$$

and a is a clopen subset of X, hence open.

Injective. If $x \neq y$ in X, there are, by Hausdorffness, disjoint open sets U and V with $x \in U$ and $y \in V$. Since X has a clopen base, U is a union of clopens, so x is in one of them, say $a \in \mathsf{Clp}(X)$. Hence $x \in a$ and $y \notin a$. So $a \in \beta(x)$ and $a \notin \beta(y)$. Hence $\beta(x) \neq \beta(y)$.

Surjective. If $F \subseteq \mathsf{Clp}(X)$ is a prime filter, we find $x \in X$ such that $x \in \bigcap_{a \in F} a$. Because then we have $F = \beta(x)$: if $a \in F$, then, by construction of F, we have $x \in a$, so $a \in \beta(x)$; and if $a \notin F$, then, since F is an ultrafilter, $a^c = \neg a \in F$, so $x \in a^c$, so $x \notin a$, so $a \notin \beta(x)$.

Finding such an x screams for compactness: We have a collection F of closed subsets of X, and we need to show that its intersection is nonempty. Since X is compact, this reduces to showing that F has the finite intersection property. So let $a_1, \ldots, a_n \in F$. Since F is a filter, also $a_1 \cap \ldots \cap a_n \in F$, and since F is prime, this cannot be the empty set, as needed.

4.3.6. Proof of propositions 4.9 and 4.10

Concerning proposition 4.9, if $f : X \to Y$ is a continuous function between Stone spaces, we have already argued in section 4.2.5 that $\mathsf{Clp}(f) : \mathsf{Clp}(Y) \to \mathsf{Clp}(X)$ is well defined, and it is a good exercise to show that it is a Boolean algebra homomorphism (see exercise 4.c). If $h : L \to M$ is a Boolean algebra homomorphism, we also mentioned that exercise 4.e shows that $\mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$ is well-defined. So it remains to show that it is continuous. Indeed, if $\widehat{a}$ is a basic open set of $\mathsf{PrFilt}(L)$ with $a \in L$, we need to show that $\mathsf{PrFilt}(h)^{-1}(\widehat{a}) \subseteq \mathsf{PrFilt}(M)$ is open. Indeed,

$$\begin{aligned}
\mathsf{PrFilt}(h)^{-1}(\widehat{a}) &= \{F \in \mathsf{PrFilt}(M) : \mathsf{PrFilt}(h)(F) \in \widehat{a}\} \\
&= \{F \in \mathsf{PrFilt}(M) : a \in h^{-1}(F)\} \\
&= \{F \in \mathsf{PrFilt}(M) : h(a) \in F\} \\
&= \widehat{h(a)},
\end{aligned}$$

which is (cl)open.

Concerning proposition 4.10, we need to show that diagrams 4.1 and 4.2 commute.

Let's start with the first one. Let $f : X \to Y$ be a continuous function between Stone spaces. We need to show $\beta_Y \circ f = \mathsf{PrFilt}(\mathsf{Clp}(f)) \circ \beta_X$. Let $x \in X$. On the left-hand-side, by definition, $\beta_Y(f(x)) = \{b \in \mathsf{Clp}(Y) : f(x) \in b\}$. On the right-hand-side, write $F := \beta_X(x) = \{a \in \mathsf{Clp}(X) : x \in a\}$ and $h := \mathsf{Clp}(f) : \mathsf{Clp}(Y) \to \mathsf{Clp}(X)$, so

$$\begin{aligned}
\mathsf{PrFilt}(\mathsf{Clp}(f)) \circ \beta_X(x) = \mathsf{PrFilt}(h)(F) &= h^{-1}(F) \\
&= \{b \in \mathsf{Clp}(Y) : h(b) \in F\} \\
&= \{b \in \mathsf{Clp}(Y) : f^{-1}(b) \in F\} \\
&= \{b \in \mathsf{Clp}(Y) : x \in f^{-1}(b)\} \\
&= \{b \in \mathsf{Clp}(Y) : f(x) \in b\} \\
&= \beta_Y \circ f(x).
\end{aligned}$$

Now for the second diagram. Let $h : L \to M$ be a Boolean algebra homomorphism. We need to show $\widehat{\cdot}_M \circ h = \mathsf{Clp}(\mathsf{PrFilt}(h)) \circ \widehat{\cdot}_L$. Let $a \in L$. On the left-hand-side, by definition, $\widehat{\cdot}_M(h(a)) = \{G \in \mathsf{PrFilt}(M) : h(a) \in G\}$. On the right-hand-side, write $f := \mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$, so

$$\begin{aligned}
\mathsf{Clp}(\mathsf{PrFilt}(h)) \circ \widehat{\cdot}_L(a) = \mathsf{Clp}(f)(\widehat{a}) &= f^{-1}(\widehat{a}) \\
&= \{G \in \mathsf{PrFilt}(M) : f(G) \in \widehat{a}\} \\
&= \{G \in \mathsf{PrFilt}(M) : h^{-1}(G) \in \widehat{a}\} \\
&= \{G \in \mathsf{PrFilt}(M) : a \in h^{-1}(G)\} \\
&= \{G \in \mathsf{PrFilt}(M) : h(a) \in G\} \\
&= \widehat{\cdot}_M \circ h(a).
\end{aligned}$$

This completes the proof of Stone duality.

4.4. Exercises

Exercise 4.a. Let L be a lattice and $F \subseteq L$. Show that the following are equivalent.

1. The set F is a prime filter.
2. The complement $I := L \setminus F$ is a prime ideal.
3. The characteristic function $\chi_F : L \to \mathbf{2}$ is a lattice homomorphism.

Exercise 4.b. Prove the characterization of prime filters in Boolean algebras stated in proposition 4.2.

Exercise 4.c. Let $f : X \to Y$ be a function between two sets X and Y.

1. The preimage is well-behaved: For $B, B' \subseteq Y$, show that
   - $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$
   - $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$
   - $f^{-1}(B^c) = f^{-1}(B)^c$
   - $f^{-1}(\emptyset) = \emptyset$
   - $f^{-1}(Y) = X$

   So $f^{-1} : 2^Y \to 2^X$ is a Boolean algebra homomorphism.

2. The direct image is not so well behaved: Now consider the direct image function $2^X \to 2^Y$ given by $f[A] := \{f(x) : x \in A\}$. Which of the above preservation properties does it still have? Give a proof for those properties it still has and give a counterexample for those it does not have anymore.

Exercise 4.d. Show that, when restricting Stone duality to finite Boolean algebras and their homomorphisms, they simply correspond to finite sets and functions between them. Conclude that if B is a finite Boolean algebra, it has 2^n elements, for some $n \in \mathbb{N}$. Note that, on the other hands, there are finite distributive lattices of any finite cardinality (since chains are distributive lattices).

Exercise 4.e. Let $h : L \to M$ be a lattice homomorphism and $F \subseteq M$ a prime filter. Then $h^{-1}(F) \subseteq L$ is a prime filter.

Exercise 4.f.

1. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between Stone spaces. Show that $\mathsf{Clp}(g \circ f) = \mathsf{Clp}(f) \circ \mathsf{Clp}(g)$.

2. Let $f : L \to M$ and $g : M \to N$ be Boolean algebra homomorphisms. Show that $\mathsf{PrFilt}(g \circ f) = \mathsf{PrFilt}(f) \circ \mathsf{PrFilt}(g)$.

Exercise 4.g. Prove proposition 4.12.

This is exercise 3.1.13 of Gehrke and S. van Gool (2023).

5. Applications

There are a plethora of applications of Stone duality (and generalizations thereof that we will see in the next chapter). For example, take a look at chapters 4–8 of Gehrke and S. van Gool (2023). While many of these applications are in theoretical computer science (e.g., programming semantics or automata theory), we will here pick two applications in philosophy.

In section 5.1, we discuss an application to modal logic: We add a less understood logical concept—here, necessity—to our logical concepts, and we get a better understanding of it by studying its behavior on the dual side of spaces.

In section 5.2, we discuss an application to the philosophical quest of understanding the nature of propositions.

5.1. (Re)discovering the semantics of 'necessity'

Necessity is an important philosophical concept. In section 5.1.1, we see that it is, like the Boolean operators, a function on the set of all propositions. However, unlike the Boolean operators, it is not clear what its truth-conditions are. In section 5.1.2, we apply Stone duality, to move this question from propositions to possible worlds, and find an answer there. In section 5.1.3, we see that this is exactly the famous answer Kripke gave: the Kripke semantics for the necessity operator.

Also see Goldblatt 2000.

5.1.1. The 'necessity' operator

Assume we have a set L of propositions with the usual Boolean connectives $\wedge, \vee, \neg$; i.e., L is a Boolean algebra. Its elements are the meanings of declarative sentences. The connectives represent the special logical role played by conjunction, disjunction, and negation of sentences. For example, if sentence φ expresses proposition a and the sentence ψ expresses proposition b, then the sentence 'φ and ψ' expresses the proposition $a \wedge b$. Since the two sentences 'φ and ψ' and 'ψ and φ' have the same meaning, even if they are syntactically distinct, we have $a \wedge b = b \wedge a$, and similarly for the other laws of a Boolean algebra.

Analogously for $\vee$ and $\neg$.

As philosophers, we're not quite happy yet with this choice of logical connectives since it is not yet expressive enough. For philosophical discussions, it is, e.g., also important whether a sentence is necessarily or just accidentally true. For instance, the sentence 'My laptop is black' is true, but only accidentally—i.e., not necessarily—because it also could have been silver. So we would say 'It is not necessary that my laptop is black'. There might be other philosophical concepts that we also would like to express—like belief, knowledge, eternity, essence, etc.—but for now, let's focus on necessity.

So, if φ is a sentence expressing proposition a, we also consider the sentence 'Necessarily, φ'—customarily written as $\Box\varphi$—which expresses the proposition that it is necessarily the case that a. Just like the Boolean logical connectives translate to functions on the set L of propositions, also our new necessity connective translates to a function

$$\Box : L \to L, \quad a \mapsto \Box a.$$

As with negation, we typically write $\Box a$ instead of $\Box(a)$.

This already tells us a little bit about 'necessity': namely, the type of thing it is (a function that maps propositions to propositions). But we don't know yet what it really means: we don't know its identity.This is the situation that philosophers also were in before Kripke and others (in the late 1950s).

5.1.2. The search for truth-conditions of 'necessity'

You might already know the Kripke semantics for modal logic, which is (among others) the logic of the necessity connective. If you do, then forget it again for now. If you don't, even better—because we want to rediscover it using the tools of duality theory. Before the advent of Kripke semantics, philosophers—like us now—struggled to understand the necessity connective.

For the usual Boolean connectives, they could say what their meaning is: for example, they could point to their truth-tables. In more fancy terminology, they had compositional truth-conditions for the Boolean connectives: the sentence $\varphi \vee \psi$ is true (in some situation) iff either φ is true (in that situation) or ψ is true (in that situation). However, for 'necessity', they could only provide some plausible reasoning principles, like the following.

1. It is necessary that φ and ψ ($\Box(\varphi \wedge \psi)$) if and only if it is necessary that φ and it is necessary that ψ ($\Box\varphi \wedge \Box\psi$).

2. If φ is a logical truth, then 'it is necessary that φ' also is a logical truth.

So the situation was as if philosophers could point to reasoning rules for disjunction like 'If φ, then $\varphi \vee \psi$', but not to the truth-table or truth-conditions of $\vee$.

Now, how can duality theory help to get such an understanding of the meaning of 'necessity'? The trick is—as always when it comes to applications of duality theory—to move to the dual side and hope for clearer intuitions there. So let's do this.

All we know so far is that $\Box$ is some (so far unknown) function from L to L which—by stating the principles 1 and 2 above more formally—satisfies:

1. For all $a, b \in L$: $\Box(a \wedge b) = \Box a \wedge \Box b$.

2. $\Box \top = \top$.

How can we translate this function $\Box : L \to L$ to the dual side involving the Stone space $\mathsf{PrFilt}(L)$ of prime filters on L? (We described this in section 4.2.) We already know that we can think of these prime filters as (Ersatz) possible worlds: they are maximally consistent sets of sentences. And we know that a function $h : L \to M$ on Boolean algebras that in fact is a lattice homomorphism translates to a continuous function $\mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$ by mapping the ultrafilter G on M to the ultrafilter $h^{-1}(G) = \{a \in L : h(a) \in G\}$.

Unfortunately, our $\Box$ is something more general than a lattice homomorphism: properties 1 and 2 only state preservation of $\top$ and $\wedge$, but not of $\bot$ and $\vee$. As a result, we cannot guarantee anymore that, if $F \in \mathsf{PrFilt}(L)$, also $\Box^{-1}(F)$ is again a prime filter. In fact, preservation of $\top$ and $\wedge$ only yields that this is a filter, but not necessarily a prime filter. So, if $\Box^{-1}(F)$ is not a prime filter, the next best thing is to consider the *set* of all prime filters that extend this filter:

This is a good exercise: In fact, if you have done exercise 4.e, you did even more.

$$\varphi(F) := \{G \in \mathsf{PrFilt}(L) : G \supseteq \Box^{-1}(F)\}.$$

Hence:

- For a Boolean algebra homomorphism $h : L \to M$, the dual map is the function $\mathsf{PrFilt}(h)$ that maps each prime filter $F \in \mathsf{PrFilt}(L)$ to a single prime filter, namely $h^{-1}(F)$.

- The more general case of a function $\Box : L \to L$ that only preserves $\top$ and $\wedge$ yields the more general dual morphism that is a *multifunction*

$\varphi : \mathsf{PrFilt}(L) \rightrightarrows \mathsf{PrFilt}(L)$ that maps each F to a set of prime filters $\varphi(F)$, namely the set of prime filter extending $\Box^{-1}(F)$.

Multifunctions are also known as set-valued functions.

The graph of a multifunction is a relation: so we write $F \mathrel{R} G$ for $G \in \varphi(F)$. So

$$F \mathrel{R} G \Leftrightarrow G \supseteq \Box^{-1}(F) \Leftrightarrow \forall a \in L : \Box a \in F \Rightarrow a \in G. \tag{5.1}$$

So our function $\Box : L \to L$ on the algebraic side becomes, when translated to the spatial side, the multifunction φ or, equivalently, its graph R. But this doesn't yet give us truth-conditions for the necessity operator. For that we need to say what it means that $\Box a$ is true at the possible world F, i.e., $\Box a \in F$? So, for $F \in \mathsf{PrFilt}(L)$, we need to fill in the question marks in

$$\Box a \in F \Leftrightarrow \ ???.$$

Actually, can get a necessary condition from the definition of R: If $\Box a \in F$, then, if $F \mathrel{R} G$, we have $\forall a \in L : \Box a \in F \Rightarrow a \in G$, so, in particular, $a \in G$. So a first try is to see if this condition actually also is necessary. And we're in luck:

That's the punchline of this section. Do you recognize this? (To be revealed below the theorem.)

Theorem 5.1. *In the preceding notation, we have for all $a \in L$ and prime filters $F \in \mathsf{PrFilt}(L)$*

$$\Box a \in F \Leftrightarrow \forall G \in \mathsf{PrFilt}(L) : F \mathrel{R} G \Rightarrow a \in G.$$

Before proving this, here is the punchline: if you have seen Kripke semantics, this is *exactly* the truth-condition for the necessity connective:

- $\Box a$ is true at a possible world F iff for all R-accessible worlds G, we have that a is true at G.

So we have a way of relating the truth of the complex proposition $\Box a$ to the truth of its constituent a: hence this truth-condition is compositional. Recall, by the properties of prime filters, we also have compositional truth-conditions for the Boolean connectives.

- $a \wedge b$ is true at a possible world F (i.e., $a \wedge b \in F$) iff both a is true at F (i.e., $a \in F$) and b is true at F (i.e., $b \in F$).
- $a \vee b$ is true at a possible world F (i.e., $a \vee b \in F$) iff either a is true at F (i.e., $a \in F$) or b is true at F (i.e., $b \in F$).
- $\neg a$ is true at a possible world F (i.e., $\neg a \in F$) iff a is not true at F (i.e., $a \notin F$).

So we have truth-conditions for all the connectives!

But now, let's prove the theorem, i.e., show that we have indeed found the truth-conditions for $\Box a$.

Proof. ($\Rightarrow$) This was the necessary condition which we already saw to hold.

($\Leftarrow$) Contrapositively, assume $\Box a \notin F$. We want to find an ultrafilter $G \in \mathsf{PrFilt}(L)$ with $F\ R\ G$ but $a \notin G$. So of course we turn to the Prime Filter Extension Theorem (theorem 4.13). We already noted that $\Box^{-1}(F)$ is a filter on L and, by assumption, $a \notin \Box^{-1}(F)$, hence, for the ideal $I := \downarrow a$, we have $\Box^{-1}(F) \cap I = \emptyset$. Hence, by the Extension Theorem, there is a prime filter G on L such that $G \supseteq \Box^{-1}(F)$ and $G \cap I = \emptyset$. The former means, by definition, that $F\ R\ G$, and the latter implies $a \notin G$, as needed. $\Box$

5.1.3. Kripke semantics

Now, with that hindsight, the usual Kripke semantics for modal logic almost seems obvious: We use the language $\mathcal{L}$ whose sentences are built from the atomic sentences in the set $\mathrm{At} = \{p_0, p_1, \ldots\}$ using $\wedge, \vee, \neg, \Box$. A *Kripke model* M is a triple (W, R, V) where

- W is a set of worlds. (So far, this was the set of prime filters $\mathsf{PrFilt}(L)$.)
- $R \subseteq W \times W$ is a binary relation. (So far, this was the one defined in equation 5.1.)
- V is a function that assigns each possible world $x \in W$ to a function V_x that assigns each atomic sentence a truth-value in $\{0, 1\}$, i.e., $V_x : \mathrm{At} \to \{0, 1\}$. (So far, V was given by the map $F \mapsto \chi_F$ which maps a prime filter to its characteristic function, which in turn assigns truth values not only to 'atomic' propositions but all propositions.)

The valuation of atomic sentences is extended to all sentences: We recursively define when a sentence φ is true at a world x, written $x \vDash \varphi$, by

- $x \vDash p$ iff $V_x(p) = 1$
- $x \vDash \varphi \wedge \psi$ iff $x \vDash \varphi$ and $x \vDash \psi$
- $x \vDash \varphi \vee \psi$ iff $x \vDash \varphi$ or $x \vDash \psi$
- $x \vDash \neg\varphi$ iff $x \nvDash \varphi$
- $x \vDash \Box\varphi$ iff for all $y \in W$, if xRy, then $y \vDash \varphi$.

What the Kripke semantics omits is the topological structure on the set of possible worlds that we still have on our topological approach. This additional topological information amounts to the following: In the usual Stone duality, Boolean algebra homomorphisms correspond to continuous functions on the dual spaces. Now, we've generalized Boolean algebra homomorphisms to functions that preserve $\wedge$ and $\top$, and on the dual side they correspond to the relations defined in 5.1. The additional topological properties of these relations—i.e., the appropriate generalization of continuity—is that they are *Boolean compatible*: If X and Y are Stone spaces, a relation $R \subseteq X \times Y$ is Boolean compatible if (1) for all $x \in X$, the set $\{y : xRy\} \subseteq Y$ is closed and (2) for all clopen $U \subseteq Y$, the set $\{x : \exists y \in U.xRy\} \subseteq X$ is clopen. Then the functions $f : B \to A$ from a Boolean algebra B to a Boolean algebra A which preserve $\wedge$ and $\top$ are in one-to-one correspondence with the Boolean compatible relations $R \subseteq X \times Y$ on the dual spaces X of A and Y of B(Gehrke and S. van Gool 2023, cor. 4.43). If we have a Stone space X with Boolean compatible relation R, we can find the dual Boolean algebra $B = \mathsf{Clp}(X)$ with $\Box U := \{x \in X : \forall x' \in X.xRx' \Rightarrow x' \in U\}$ (Gehrke and S. van Gool 2023, cor. 4.51).

Note the swap of direction, as usual for dualities

5.2. The ontology of propositions

We said that propositions are the meanings of declarative sentences. But we did not care very much what these meanings *are*, we just cared about what they *do*, i.e., how they interact with each other via implication, conjunction, etc. However, the ontology of propositions—i.e., what they are—is an important topic in the philosophy of language. Can Stone duality offer some insight as well? Yes!

Using Stone duality, we will show that the purely *functional* assumption that—whatever propositions are—they should form a Boolean algebra implies the *ontological* claim that propositions are intensions, i.e., functions from possible worlds to truth-values.

Famously, Frege distinguished between two aspects of the meaning of a sentence (and other linguistic expressions): its reference and its sense. The reference of a sentence is simply its truth-value, while the sense of a sentence is, intuitively speaking, the way the sentence determines its reference. Carnap made explicated these notions as extension and intension, respectively. The extension of the sentence is 0 or 1, depending on whether or not it is true or false in the actual world. The intension is a function that maps a possible world to 0 or 1 depending on whether or not the sentence is true or false in that possible world. This gave rise to the idea that, ontologically speaking, a proposition is simply a function from the set of possible worlds to the set $2 = \{0, 1\}$.

This is because Stone duality offers a representation theorem: Every Boolean algebra L is isomorphic to $\mathsf{Clp}(\mathsf{PrFilt}(L))$, which, in turn, is a sub-Boolean algebra of the powerset Boolean algebra $\mathcal{P}(\mathsf{PrFilt}(L))$. The following theorem expresses essentially the same idea in different terms, making this embedding more explicit. adjust text

Theorem 5.2. *Every Boolean algebra L is, up to isomorphism, a sub-Boolean algebra of powers of the two-element Boolean algebra $\mathbf{2}$, namely, of $\mathbf{2}^{\mathsf{PrFilt}(L)}$.*

Proof. We define the required Boolean algebra embedding $e : L \to \mathbf{2}^{\mathsf{PrFilt}(L)}$ by mapping $a \in L$ to the function

$$\begin{aligned} e(a) : \mathsf{PrFilt}(L) &\to \mathbf{2} \\ F &\mapsto \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Exercise 5.b asks you to show that this indeed is a Boolean algebra embedding (i.e., an injective Booleam algebra homomorphism). □

5.3. Exercises

Exercise 5.a. Tell the story that we've told in section 5.1 but now not for the 'propositions vs. possible worlds' duality, but for the 'properties vs. objects' duality. Here the new operator is Δ read as 'definitely' instead of $\Box$ read as 'necessarily'. This operator is used a lot in theories of vagueness (e.g. Williamson 1999). If p is the property of being red, then Δp is the property of definitely being red. Which semantic of 'definitely' do you get on the dual side? How does it compare to existing semantics? What does the topological perspective add? Is it philosophically plausible?

Exercise 5.b. Show that the function $e : L \to \mathbf{2}^{\mathsf{PrFilt}(L)}$ defined in the proof of theorem 5.2 is indeed a Boolean algebra embedding.

6. Generalizations

The world of classical logic—with Boolean algebras and their corresponding Stone spaces—is a beautiful one. But reality—or, rather, the domain where we would like to apply this theory—might not be. In the last chapter, we have seen that, in some cases, we can *extend* the framework to include further interesting concepts, like necessity. But, in other cases, it might be that not all laws of classical logic are satisfied. For example, we might deal with vague predicates and take them to sometimes be neither true nor false (so $\neg p \vee p$ may fail), or we might write a computer program and only allow constructive reasoning (so $\neg\neg p \to p$ may fail). If we want to apply Stone duality in these cases, too, we need to *generalize* the framework.

We discuss these examples in more detail in section 6.1.

To generalize Stone duality, we need to do the following:

1. On the algebraic side, generalize from Boolean algebras to a larger class of algebras.
2. On the spatial side, generalize from Stone spaces to a larger class of spaces.
3. Ensure that the constructions PrFilt and Clp with which we can go between Boolean algebras and Stone spaces also extend to these larger classes of algebras and spaces.

But how should we find such a generalization? A good place to start is to look at the intended application and see which Boolean laws may fail there. This will indicate which generalization of Boolean algebras to consider. We do so in section 6.1 and argue that distributive lattices are a good generalization (i.e., dropping negation). (Though, we note that other choices are possible, too.) The next step is to then investigate what the spatial side will look like when applying the prime filter construction PrFilt and see if the resulting class of spaces can be inherently defined. We do so in section 6.2 and actually find two equivalent answers: one known as spectral spaces and one known as Priestley spaces. The former was also provided by Stone and is sometimes also referred to as *Stone duality*. The latter was later provided by Priestley and is called *Priestley duality*.

The most general version of Stone duality being that between locales and frames, see here.

The overall picture that we get—i.e., the summary of this chapter—is shown in figure 6.1.

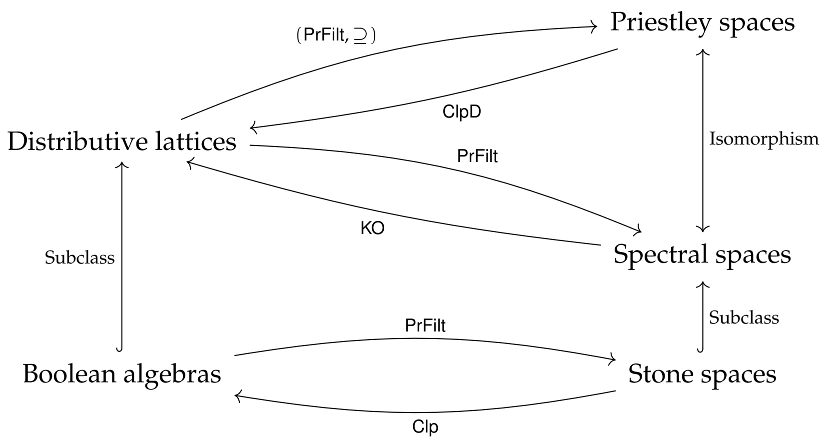


Figure 6.1.: Generalization of Stone duality.

6.1. Generalization on the algebraic side: distributive lattices

Let's look at some domains of applications that may take us outside the world of classical logic.

1. *Vague predicates*: We might consider propositions involving vague predicates. There are many theories of vagueness, but, on a straightforward one, we might take some such propositions to be neither true nor false; e.g., the proposition that this shirt is red, when referring to, say, a dark-orange shirt. Hence the proposition $a \vee \neg a$ would fail to be a logical truth, i.e., is not the proposition 1. So the condition on negation in Boolean algebras that $a \vee \neg a = 1$ is violated.

2. *Constructive reasoning*: When we write a computer program, we might only allow constructive reasoning. So from the fact that we cannot show that a claim is false, we cannot conclude that it is true. Hence $\neg\neg p \to p$ fails to be the logically true proposition. But again this violates the equality $\neg\neg p \to p = 1$ holding in Boolean algebras.

Quick exercise: show that $\neg\neg p \to p = 1$ indeed holds in Boolean algebras.

See the introduction of Vickers (1989) for a nice exposition of these ideas.

3. *Verifiable propositions*: As scientist we might not be interested in the set of all propositions but only in the subset of verifiable propositions, i.e., those propositions that, if true, can be verified by an experiment. For example, 'This weighs exactly 5 kg' is not verifiable because, if true of some object, we will never be able to verify it, because any scale that we use will have some margin of error. On the other hand, 'This weighs strictly more than 5 kg' is verifiable, because, if

true of some object, it weighs $\epsilon > 0$ more than 5 kg, so if we use a scale whose margin of error is less than ϵ, we can verify it. Now the subset of verifiable propositions is a closed under conjunction and disjunction and also includes the logically true and the logically false proposition. So it is a distributive lattice (assuming the set of all propositions to be a Boolean algebra). But it is not closed under negation. For example, 'This does not weigh strictly more than 5 kg' is not verifiable: it is true of an object that weighs exactly 5 kg, but we cannot verify the proposition, since, no matter the scale, its margin of error will also include the possibility that the object weighs $5 + \epsilon$ kg and thus does not make the proposition true.

Why?

In all these examples, we still have a distributive lattice, because the required equations concerning conjunction and disjunction are still logical laws in the intended application. However, the problem is with negation. Either there still is some operation of negation—like in the first two examples—but it doesn't satisfy all the Boolean laws, or it doesn't even make sense to speak of negation anymore, like in the last example.

Hence a natural generalization on the algebraic side is to move from Boolean algebras to distributive lattices. This is what we will explore here.

But of course other choices would be possible. For example, we might be less radical not drop negation all together but keep a weaker negation operator. In the first example, we might weaken the requirements on negation to $\neg(a \wedge b) = \neg a \vee \neg b$, $\neg\neg a = a$, and $a \wedge \neg a \leqslant b \vee \neg b$, yielding what is known as a Kleene logic algebra (not to be confused with Kleene algebras in regular languages). In the second example, we might use Heyting algebras since they are the algebras for intuitionistic logic, i.e., the logic of constructive reasoning. They are distributive lattice with an additional implication operator $\to$ which can be used to define a notion of implication: $\neg a := a \to \bot$. This, for example does not satisfy double negation elimination (i.e., some Heyting algebras have elements a with $\neg\neg a \nleqslant a$).

On the other hand, we might also be more radical and also drop the distributivity axiom $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. This is more difficult to motivate, but here is a try. Say we have a light sensor which measures the wavelength of visible light. So its possible readings are in the interval $X = [380, 750]$, reaching from violet light (380 nm) to red light (750 nm). We think of X as a set of possible worlds. So a proposition is the set of worlds where the proposition is true. For example, the proposition 'The registered light is green' is the interval $[500, 565]$ (the wavelengths we

For more on the visible light spectrum, see here.

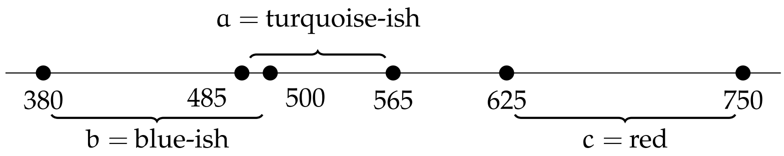


Figure 6.2.: The visible light spectrum

perceive as green). Now, a conjecture of conceptual space theory is that all 'natural' concepts are convex subsets (Gärdenfors 2004). Since we consider for simplicity here just the real line (and not some higher-dimensional $\mathbb{R}^n$), the convex subsets simply are the intervals. Hence we only allow intervals as propositions. The intersection of two intervals is again an interval (we also allow the empty interval), so we can interpret conjunction ($\wedge$). But the union of two intervals need not be an interval. So we interpret disjunction ($\vee$) as the smallest interval containing the union of the two intervals. (In the general $\mathbb{R}^n$, we would take the convex closure.)

This is an instance of the general strategy to give semantics to non-distributive logics via closure operators (Restall 2000, ch. 12).

Now consider the following propositions, visualized in figure 6.2

- $a = [485, 565]$, i.e., the light is turquoise-ish (cyan or green).
- $b = [380, 500]$, i.e., the light is blue-ish (violet, blue, or cyan).
- $c = [625, 750]$, i.e., the light is red.

Then the smallest interval containing $b \cup c$ actually is the whole set $X = [380, 750]$, so $b \vee c = X$. Hence, on the one hand,

$$a \wedge (b \vee c) = a \cap X = a = [485, 565],$$

but, on the other hand, the smallest interval containing and interval I and the empty set $\emptyset$ is simply I, so

$$(a \wedge b) \vee (a \wedge c) = (a \cap b) \vee (a \cap c) = [485, 500] \vee \emptyset = [485, 500],$$

so the distributive law is violated. Hence, if we want to include this application, then we should generalize to just lattices.[5]

[5] Of course, one can continue and ask if we should be even more radical and drop some lattice axioms. One might argue for dropping the absorption axiom of lattices, i.e., $a \vee (a \wedge b) = a$, on the grounds that the meaning of the sentence 'This coffee is delicious or this coffee is delicious and poisoned' is not the same as the meaning of the sentence 'This coffee is delicious. This is a phenomenon that goes under the name of hyperintensionality.

Whether a duality theory exist for these other choices depends. For example, Esakia duality provides a duality between Heyting algebras and what is known as Esakia spaces. There also is a duality theory for lattices.[6] But duality theory without distributivity becomes much more difficult.

6.2. Generalization on the spatial side: spectral spaces and Priestley spaces

As just said, we will explore the generalization of Stone duality where we generalize, on the algebraic side, from Boolean algebras to distributive lattices. But how should the dual spaces look like, that then generalize Stone spaces? The strategy is to apply the prime filter construction and characterize the resulting spaces. We first do this verbatim, and as a result we get the spectral spaces (section 6.2.1). Then we do it in a slightly different, but equally motivated way and get the Priestley spaces (section 6.2.2).

6.2.1. Spectral spaces

For the entire subsection, let L be a distributive lattice (and not necessarily a Boolean algebra). The notion of a prime filter is still defined. However, it need not coincide anymore with the notion of a maximal filter or an ultrafilter. Ultrafilters are not even defined anymore in the absence of complements, and maximal filter are more special than prime filters. (Maximal filters are prime, but prime filters need not be maximal, as we will see in the example below.) This is also why we took prime filters as the basic notion and stated it as a theorem that, for Boolean algebras, they coincide with maximal filters and ultrafilters.

Hence, for a distributive lattice, we can still consider the set $X := \mathsf{PrFilt}(L)$ of all prime filters on L. And, also just like with Stone spaces, we can still define the topology on it generated by the sets of the form

$$\widehat{a} := \{F \in X : a \in F\} \qquad \text{(for } a \in L\text{).}$$

However, in general, this cannot be a Stone space anymore, since Stone spaces correspond exactly to Boolean algebras under this PrFilt construction. So let's go through the three defining properties (Hausdorff, compact, zero-dimensional) and see what is different.

First, Hausdorffness fails, as the following example shows.

[6] See, e.g., Urquhart (1978), Gehrke and S. J. van Gool (2014), Goldblatt (2019), Bezhanishvili et al. (2022), or Hartonas (2023).

Example 6.1. Consider the simplest distributive lattice that is not a Boolean algebra, namely the three-element chain L, which we write as $0 < a < 1$. Its prime filters are $F_1 = \{1\}$ and $F_2 = \{a, 1\}$. (Note that only the second one is maximal.) The topology on $X = \mathsf{PrFilt}(L)$ is generated by the sets $\widehat{0} = \emptyset$, $\widehat{a} = \{F_2\}$, and $\widehat{1} = X$. These sets already form a topology because they are closed under union and intersection (there are only finitely many sets, so arbitrary union and finite union amounts to the same). But then X cannot be Hausdorff: we would need two disjoint open sets one of which contains F_1 and the other F_2. But the only open set that contains F_1 is X and this cannot be disjoint from an open set containing F_2. ⌟

However, $\mathsf{PrFilt}(L)$ still is T_0, though: If $F \neq G$, there is, by definition, some $a \in L$ with $a \in F$ and $a \notin G$, or vice versa. Hence $F \in \widehat{a}$ and $G \notin \widehat{a}$, or vice versa, so $\widehat{a}$ is an open set that contains one but not the other point.

Second, the good news is t for distinct points we do indeed find hat one can still show that the space X is compact: the main tools for that—the (prime) filter extension theorems from section 4.3.1—hold for distributive lattices in general. (We won't do the proof here, but see exercise 6.a.)

Third, zero-dimensionality followed in the Boolean case from two observation: (1) Since $\widehat{a} \cap \widehat{b} = \widehat{a \wedge b}$, the collection of $\widehat{a}$'s is closed under intersection and hence a base, and (2) the $\widehat{a}$ are clopen, since they not only are open by definition but also closed since $\widehat{a}^c = \widehat{\neg a}$. Part (1) still works since filters are upsets and closed under $\wedge$. But part (2) does not work due to the absence of a negation operator. And indeed, the above example shows that the $\widehat{a}$ there is not closed: its complement is $\{F_1\}$ and this is not open (the only open sets are $\emptyset, \{F_2\}, X$).

Still we would expect the $\widehat{a}$'s to play as important a role as the clopens did for Sone spaces. After all, the $\widehat{a}$'s are the intension of the proposition a over the reconstructed set of possible worlds $\mathsf{PrFilt}(L)$. So what makes them special? Are they something close to being closed? Yes, and that's the key insight to spectral spaces! Recall that in a compact Hausdorff space, being closed is the same thing as being compact. This is not true for our X now, since it is not Hausdorff anymore. But it suggests that we replace 'closed-and-open' by 'compact-and-open'.

The usual abbreviation is only 'compact-open', unfortunately not as fun as 'clopen'.

In fact, requiring propositions to be compact-and-open makes philosophical sense, too—so we might have required it from the start. As before, being open means that, if a possible world makes true the proposition a, there is some range of similarity around that possible world (i.e., some open set containing the possible world), such that all worlds within that range already make true a. Being compact means that the proposition a is 'finitely falsifiable': Say a set of propositions T falsifies a if, for any possible world, if no proposition of T is true in that world, also a is not true in that world. The compactness of a then means that, for any set of propositions T that falsifies a, there already is a finite subset $T' \subseteq T$ that falsifies a.

Indeed, for the above sets $\widehat{a}$, i.e., the intension of the proposition a over the reconstructed worlds $X = \mathsf{PrFilt}(L)$, one can show that they are indeed compact, again using the (prime) filter extension theorems (see exercise 6.a).

So, if we write $\mathsf{KO}(X)$ for the set of compact-and-open subsets of $X = \mathsf{PrFilt}(L)$, we have $\{\widehat{a} : a \in L\} \subseteq \mathsf{KO}(X)$. And we also have the converse inclusion: If $U \subseteq X$ is compact and open, then, since U is open and the $\widehat{a}$'s form a base, $U = \bigcup_{i \in I} \widehat{a_i}$ for some $a_i \in L$. Since U is compact, there is a finite subcover $U = \widehat{a_1} \cup \ldots \cup \widehat{a_n}$. Since prime filters are upsets and prime, we hence have, for $a := a_1 \vee \ldots \vee a_n \in L$ that $U + \widehat{a}$, as needed.

Hence, $\mathsf{KO}(X)$ is a base of X and closed under intersection. Thus, we have found four of the five conditions of an intrinsic definition of a spectral space. The fifth one is a more technical condition known as being sober. One finds it when trying to prove that a spectral space should be homeomorphic to the prime filters of the distributive lattice of its compact-opens.

Cf. Gehrke and S. van Gool (2023, prop. 6.17).

Definition 6.2. Let (X, τ) be a topological space. Write $\mathsf{KO}(X)$ for the collection of subsets that are both compact and open (compact-open for short). We call X a *spectral space* if

An excellent book on spectral spaces is Dickmann et al. (2019). Spectral spaces also arise precisely as the spectra of commutative rings (with the Zariski topology), so they are important in algebraic topology.

1. X is T_0
2. X is compact.
3. $\mathsf{KO}(X)$ is a base of X
4. $\mathsf{KO}(X)$ is closed under intersection.
5. X is *sober*, i.e., for every nonempty closed subset $C \subseteq X$, if C is irreducible (i.e., cannot be written as the union of two closed proper subsets), then there is a point $x \in X$ such that C is the topological closure of $\{x\}$.

One also says that x is a generic point of C. If the space is T_0, generic points, if they exist, are unique.

Exercise 6.b asks to show that the fifth condition is indeed also satisfied for $\mathsf{PrFilt}(L)$. With the above discussion, we hence know that $\mathsf{PrFilt}(L)$ is indeed a spectral space.

Finally, what should be the structure preserving maps between spectral spaces? Surely, like for any topological spaces, such functions should be continuous. But since the compact-opens play such an important role, it is natural to require that preimages of compact-opens should again be compact-open. In the case of clopen sets, this is already implied by continuity: the preimage of a clopen set is clopen. But, when closed and compact comes apart, this is no longer true. So we need to require it explicitly. The resulting functions are called spectral.

Definition 6.3. A function $f : X \to Y$ between spectral spaces is called *spectral* if preimages of compact-open sets are compact-open.

As a consequence, if $f : X \to Y$ is a spectral map, $f^{-1} : \mathsf{KO}(Y) \to \mathsf{KO}(X)$ again is a lattice homomorphism, analogous to the Stone case.

6.2.2. Priestley spaces

The insight behind Priestley spaces is twofold. First, the set of prime filters of a distributive lattice has a natural order: namely, ordering filters by extension. (As mentioned, in the Boolean case, all prime filters are maximal, so this order is trivial, but not so in the general distributive case.) Second, by making this order explicit, we can use a topology which still is a Stone space and hence much simpler than the topology of a spectral space. Let's see how this is done.

If L is a distributive lattice, again consider the set of all prime filters $X := \mathsf{PrFilt}(L)$. Order it by reverse inclusion: for $F, G \in X$, define $F \leqslant G$ as $F \supseteq G$. So G is more *general* than F and, equivalently, F is more *special* than G: everything that G makes true, also F makes true, but F might make more true than G does.

We also could have chosen inclusion as order. Some textbooks also do so, but here we follow Gehrke and S. van Gool (2023). Ultimately, it just matters that we stick to one choice.

We can still think of $\mathsf{PrFilt}(L)$ as the set of possible worlds. Prime filters are closed under implication, they make true the logical truth and they do not make true the logical falsity (i.e., are consistent), they make a conjunction true precisely if they make true both conjuncts, and they make a disjunction true precisely if they make at least one disjunct true. But, in the absence of Boolean negation, this is not equivalent to the conception of a possible world as a maximally consistent set of propositions (since prime filters need not be maximal). As a result, we have a non-trivial ordering on possible worlds by 'specificity', 'determinateness', or 'informativeness'. This squares well with the idea of moving from a Boolean algebra to merely a distributive lattice in order to incorporate vague predicates (see section 6.1). Then, like in many theories of vagueness, one also has a 'determinateness' order on the possible situations making true vague sentences. However, here the 'situations' are not partial in the sense that sometimes do not give truth-values to sentences. Rather, they are 'classical

To think more about this, see exercise 6.e.

possible worlds' in the sense of having the usual classical truth-conditions (i.e., are prime filters), but they can be ordered non-trivially by specificity.

When it comes to the topology on $X = \mathsf{PrFilt}(L)$, the intuition was that we declare all $\widehat{a} := \{F \in X : a \in F\}$ to be open, since they represent the 'degree of similarity' of being similar with respect to making true proposition a. In the Boolean case, this implies that we also have the 'degree of similarity' $\widehat{\neg a} = \{F \in X : a \notin F\}$ of being similar with respect to not making true a. We do not get this for free in the general case of a distributive lattice, as we saw in the spectral spaces. But we might also just *stipulate* this for our notion of similarity that is formalized with our choice of topology. This amount to implicitly adding a negation operator and also require similarity with respect to propositions using it.

Hence, we consider the topology generated by the sets of the form, for $a \in L$,

$$\widehat{a} := \{F \in X : a \in F\} \qquad\qquad \widehat{a}^c = \{F \in X : a \notin F\}.$$

Let's call this topology τ. So, we have the structure $(X, \tau, \leqslant)$. Let's again consider an example.

Example 6.4. In example 6.1, we have considered the spectral space dual to the 3-element chain L (the smallest distributive lattice that is not a Boolean algebra). How does its Priestley space look like? The only prime filters again are $F_1 = \{1\}$ and $F_2 = \{a, 1\}$, so $X := \mathsf{PrFilt}(L) = \{F_1, F_2\}$, with $F_2 \supseteq F_1$, i.e., $F_2 \leqslant F_1$. The topology is generated by:

$$\begin{aligned}
\widehat{0} &= \emptyset & \qquad\qquad \widehat{0}^c &= X \\
\widehat{a} &= \{F_2\} & \widehat{a}^c &= \{F_1\} \\
\widehat{1} &= X & \widehat{1}^c &= \emptyset.
\end{aligned}$$

These are already all subsets of X, so it is the discrete topology. ⌟

Next, let's explore the general properties of $(X, \tau, \leqslant) = (\mathsf{PrFilt}(L), \tau, \supseteq)$. As before, let's start with its separation properties. If $F \neq G$ are two distinct points in X, we now have, using the order, that either $F \nleqslant G$ or $G \nleqslant F$. Say, $F \nleqslant G$, i.e., $F \not\supseteq G$. So there is $a \in L$ with $a \in G$ but $a \notin F$. Hence $\widehat{a}$ is an open set with $G \in \widehat{a}$ and $F \notin \widehat{a}$. Moreover, by definition, $\widehat{a}$ is clopen (since also $\widehat{a}^c$ is declared to be open). Considering the order, $\widehat{a}$ also is a downset: if $G' \leqslant G \in \widehat{a}$, then $a \in G \subseteq G'$, so also $G' \in \widehat{a}$. In sum, if $F \nleqslant G$, then there is a clopen downset $\widehat{a} \subseteq X$ such that $G \in \widehat{a}$ and $F \notin \widehat{a}$.

What about compactness? Again, using the filter extension theorems (and also the Alexander subbase theorem), one can show that the topology on $\mathsf{PrFilt}(L)$ generated by the $\widehat{a}$'s together with their complements is indeed compact (see exercise 6.c).

Thus, we already have found the definition of a Priestley space.

Definition 6.5. A *Priestley space* is a triple $(X, \tau, \leqslant)$ such that

1. (X, τ) is a compact topological space,
2. $(X, \leqslant)$ is a partial order, and
3. it is *totally order-disconnected* (TOD), i.e., for all $x \not\leqslant y$ in X, there is a clopen downset $U \subseteq X$ such that $y \in U$ and $x \notin U$.

As usual, we often leave the topology τ implicit and just write $(X, \leqslant)$.

By exercise 3.l, if $(X, \leqslant)$ is a Priestley space, then X is a Stone space (since it is compact and totally separated). So Priestley spaces really are just Stone spaces together with a partial order (a partial order that interacts nicely with the topology, as mandated by TOD). The role of clopens for Stone space is now the role of clopen downsets for Priestley spaces.

The structure-preserving maps between Priestley spaces are as one would expect: they preserve the topology and the order.

Definition 6.6. A *Priestley space morphism* $f : (X, \leqslant) \to (X', \leqslant')$ is a continuous function $f : X \to X'$ that also is monotone (i.e., for all $x, y \in X$, if $x \leqslant y$, then $f(x) \leqslant' f(y)$).

As a result, the preimage of a clopen downset under a Priestley space morphism is again a clopen downset.

6.2.3. Moving between spectral and Priestley spaces

Spectral spaces and Priesley spaces share some similarities, but also are different in other respect. We now note, without proof, how to exactly translate one into the other.

Some preliminary terminology: Let (X, τ) be a topological space. The *specialization order* $\leqslant_\tau$ on X is defined by

$$x \leqslant_\tau y :\Leftrightarrow \forall U \subseteq X \text{ open:} x \in U \Rightarrow y \in U.$$

A subset $A \subseteq X$ is *saturated* if it is an upset in the specialization order (equivalently, it is an intersection of open sets). In a compact Hausdorff space, a set if compact and saturated iff it is closed. Thus, a useful generalization of closed sets in compact non-Hausdorff spaces is the set of compact and saturated subsets (like the compact-opens are a good generalization of the clopens). The *patch topology* τ^p on X is the topology generated by

$$\tau \cup \{K^c : K \subseteq X \text{ is compact and saturated}\}.$$

Finally, if $(X, \tau, \leqslant)$ is a Priestley space, define

$$\tau^{\downarrow} := \{U \in \tau : U \text{ is a } \leqslant\text{-downset}\}.$$

Theorem 6.7. *The assignments*

$$\begin{aligned} (X, \tau, \leqslant) &\mapsto (X, \tau^{\downarrow}) \\ (X, \rho^p, \geqslant_\rho) &\leftarrow\!\shortmid (X, \rho) \end{aligned}$$

form a bijective correspondence between Priestley spaces and spectral spaces.

See Gehrke and S. van Gool (2023, Thm. 6.4).

In fact, this is an isomorphism of categories which, on morphisms, simply is the identity.

6.3. The Stone and the Priestley duality

Finally, we state the Stone duality and Priestley duality for distributive lattices, i.e., the two equivalent generalizations of the Stone duality for Boolean algebras. We don't do the full proofs, but we already covered the crucial bits in our motivation for spectral spaces and Priestley spaces, respectively. You may take it as an exercise to fill in some of the remaining gaps.

6.3.1. The Stone duality for distributive lattices

The Stone duality translates as follows between distributive lattices and spectral spaces:

- If L is a distributive lattice, the dual space is $\mathsf{PrFilt}(L)$ with the topology generated by the sets $\widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}$, which indeed is a spectral space.

  If $h : L \to M$ is a lattice homomorphism, then $\mathsf{PrFilt}(f) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$ maps a prime filter G of M to the prime filter $h^{-1}(G) = \{a \in L : h(a) \in G\}$. This is indeed a well-defined spectral map.

  Recall exercise 4.e.

- If (X, τ) is a spectral space, the dual distributive lattice is $\mathsf{KO}(X)$, the set of compact-open subsets of X ordered by inclusion.

If $f : X \to Y$ is a spectral map between spectral spaces, then $\mathsf{KO}(f) : \mathsf{KO}(Y) \to \mathsf{KO}(X)$ maps a compact-open set B of Y to the preimage $f^{-1}(B)$. This is a lattice homomorphism.

Now one can show that these again form a duality: every distributive lattice L is isomorphic to its double-dual $\mathsf{KO}(\mathsf{PrFilt}(L))$ and every spectral space X is homeomorphic to its double dual $\mathsf{PrFilt}(\mathsf{KO}(X))$; and for all lattice homomorphisms $h : \mathsf{KO}(Y) \to \mathsf{KO}(X)$ there is a unique spectral map $f : X \to Y$ such that $h = \mathsf{KO}(f)$. Again, formulated in category-theoretic language this means:

Theorem 6.8 (Stone duality for distributive lattices)**.** *The functors* PrFilt *and* KO *form a dual equivalence between, on the one side, the category* DL *of distributive lattices with lattice homomorphisms and, on the other side, the category* $\mathsf{Spectral}$ *of spectral spaces with spectral maps.*

6.3.2. The Priestley duality

The Priestley duality translates as follows between distributive lattices and Priestley spaces:

- If L is a distributive lattice, the dual Priestley space is $(\mathsf{PrFilt}(L), \supseteq)$ with the topology generated by the sets $\widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}$ and $\widehat{a}^c$. This indeed is a Priestley space.

  If $h : L \to M$ is a lattice homomorphism, then $\mathsf{PrFilt}(f) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$ maps a prime filter G of M to the prime filter $h^{-1}(G) = \{a \in L : h(a) \in G\}$. This is indeed a well-defined Priestley space morphism.

- If $(X, \tau, \leqslant)$ is a Priestley space, the dual distributive lattice is the set $\mathsf{ClpD}(X)$ of clopen downsets of X, with the usual set-theoretic operations as lattice operations. This indeed is a distributive lattice.

  If $f : X \to Y$ is a Priestley space morphism, then $\mathsf{ClpD}(f) : \mathsf{ClpD}(Y) \to \mathsf{ClpD}(X)$ maps a clopen downset B of Y to the preimage $f^{-1}(B)$. This is a lattice homomorphism.

Now one can show that these again form a duality: every distributive lattice L is isomorphic to its double-dual $\mathsf{ClpD}(\mathsf{PrFilt}(L), \supseteq)$ and every Priestley space $(X, \leqslant)$ is isomorphic to its double dual $(\mathsf{PrFilt}(\mathsf{ClpD}(X)), \supseteq)$; and for all lattice homomorphisms $h : \mathsf{ClpD}(Y) \to \mathsf{ClpD}(X)$ there is a unique Priestley space morphism $f : X \to Y$ such that $h = \mathsf{ClpD}(f)$. Again, formulated in category-theoretic language this means:

Theorem 6.9 (Priestley duality)**.** *The functors* $(\mathsf{PrFilt}, \supseteq)$ *and* ClpD *form a dual equivalence between, on the one side, the category* DL *of distributive lattices with lattice homomorphisms and, on the other side, the category* $\mathsf{Priestley}$ *of Priestley spaces with Priestley space morphisms.*

If we now look back at figure 6.1, we have indeed established all promised connections.

6.4. Exercises

Exercise 6.a. Let L be a distributive lattice. Consider $X := \mathsf{PrFilt}(L)$ with the topology generated by the $\widehat{a} := \{F \in X : a \in F\}$ for $a \in L$. Show that each $\widehat{a}$ is compact. In particular, $X = \widehat{1}$ is compact.

Hint: When proving compactness of $\mathsf{PrFilt}(L)$ in the Boolean case (section 4.3.2), we used the compactness theorem, which essentially says, writing $\models$ for $\subseteq$, that $\bigcap_{b \in T} \widehat{b} \models \widehat{a}$ implies $\bigcap_{b \in T'} \widehat{b} \models \widehat{a}$ for some finite $T' \subseteq T$. Here prove and use the 'dual' saying $\widehat{a} \models \bigcup_{b \in T} \widehat{b}$ implies $\widehat{a} \models \bigcup_{b \in T'} \widehat{b}$ for some finite $T' \subseteq T$. Note that this precisely the 'finite falsifiability' condition discussed in section 6.2.1: that $\bigcap_{b \in T} \widehat{b}^c \models \widehat{a}^c$ implies $\bigcap_{b \in T'} \widehat{b}^c \models \widehat{a}^c$ for some finite $T' \subseteq T$.

Exercise 6.b. Let L be a distributive lattice. Consider $X := \mathsf{PrFilt}(L)$ with the topology generated by the $\widehat{a} := \{F \in X : a \in F\}$ for $a \in L$. Show that X is sober, i.e., every nonempty, irreducible, and closed subset $C \subseteq X$ is the closure of a singleton.

Hint: Use irreducibility to show that $F := \{a \in L : C \cap \widehat{a} \neq \emptyset\}$ is a prime filter. To show $C = \mathsf{Cl}(\{F\})$, use and prove that, for $D \subseteq X$ closed, we have $C \subseteq D$ iff $F \in D$.

Exercise 6.c. Let L be a distributive lattice. Consider $X := \mathsf{PrFilt}(L)$ with the topology generated by the $\widehat{a} := \{F \in X : a \in F\}$ and $\widehat{a}^c$ for $a \in L$. Show that X, with this topology, is compact.

Hint: Use the Alexander Subbase Theorem and, again, find a way to apply the idea of the logical compactness theorem.

Exercise 6.d. Consider the set $\omega + 1 = \{0, 1, 2, \ldots, \infty\}$ with the order $\leqslant$ where the natural numbers are ordered in the usual way and ∞ is bigger than all natural numbers. This is depicted in figure 6.3. As a topology on $\omega + 1$, consider the so-called (Alexandroff) one-point compactification of the natural numbers $\mathbb{N}$ equipped with the discrete topology. Concretely,

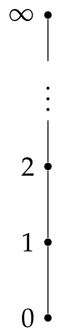
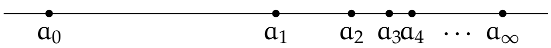


Figure 6.3.: The order $\omega + 1$.

this means the open sets of $\omega + 1$ are given by

$$\tau := \{A : A \subseteq \mathbb{N}\} \cup \{(\mathbb{N} \setminus F) \cup \{\infty\} : F \subseteq \mathbb{N} \text{ finite}\}.$$

More visually, you can also think of it as the subspace of the real line consisting of those points $a_n = 1 - \frac{1}{n+1}$ (for $n = 0, 1, \ldots$) together with their limit point $a_\infty = 1$ (with their inherited order).



For this exercise, you can assume that τ is indeed a topology (i.e., you don't have to prove this).

1. Show that $(\omega + 1, \tau, \leqslant)$ is a Priestley space.

Next, also consider the topology

$$\tau' := \{U \in \tau : U \text{ is an } \leqslant\text{-downset}\}.$$

2. Show that $(\omega + 1, \tau')$ is a spectral space.

Exercise 6.e. Think about how you could use Priestley spaces as a theory of vagueness, as initiated in section 6.2.2.

Also cf. exercise 5.a.

A. Appendix

A.1. Set-theoretic terminology

See, e.g., Priest (2008, sec. 0.1).

We use standard set-theoretic terminology as it is common in mathematics. A set is a collection of objects. We write $a \in A$ to say that object a is in (or is an element of, or is a member of) the set A. If $a_1, \ldots, a_n$ are objects, we write $\{a_1, \ldots, a_n\}$ for the set of these objects. Sets do not count 'order' and 'multiplicities', so $\{1, 0, 2, 2\} = \{0, 1, 2\}$. A set with just one element is called a singleton, and if a is an object, $\{a\}$ is the singleton of a (note $\{a\} \neq a$). The set without any elements is called the empty set and is denoted $\emptyset$.

If A and B are sets, we say A is a subset of B (written $A \subseteq B$) if every element of A is an element of B. So the empty set trivially is a subset of any set. And two sets A and B are identical iff $A \subseteq B$ and $B \subseteq A$. If A and B are sets, then the union of A and B (written $A \cup B$) is the set containing exactly those objects that either are in A or in B (or both). The intersection of A and B (written $A \cap B$) is the set containing exactly those objects that are both in A and in B. The complement of a set A relative to a set B (written $B \setminus A$) is the set of objects that are in B but not in A. If B is clear from context, we just write A^c.

A pair (aka ordered pair) is a list of two elements (a, b); here the order matters, so $(a, b) \neq (b, a)$. (We can define (a, b) as the set $\{\{a\}, \{a, b\}\}$.) More generally, an n-tuple is a list of n elements $(a_1, \ldots, a_n)$. Given n sets $A_1, \ldots, A_n$, their Cartesian product (written $A_1 \times \ldots \times A_n$) is the set of all n-tuples $(a_1, \ldots, a_n)$ such that, for all $i \in \{1, \ldots, n\}$, we have $a_i \in A_i$. More generally, the Cartesian product of a potentially infinite family $\{A_i : i \in I\}$ of sets is defined as the set $\prod_{i \in A} A_i$ of functions a that map each $i \in I$ to an element $a(i) \in A_i$. We often write such a function as $a = (a_i)_{i \in I}$; because in the above case where $I = \{1, \ldots, n\}$, we can think of a tuple $(a_1, \ldots, a_n)$ as a the function a mapping $i \in I$ to a_i. An n-ary relation between $A_1, \ldots, A_n$ is a subset of $A_1 \times \ldots \times A_n$. A 1-ary (resp., 2-ary, 3-ary) relation is also called a unary (resp., binary, ternary) relation. For a binary relation R, we usually write aRb instead of $(a, b) \in R$.

A function from a set A (its domain) to a set B (its codomain) is a binary relation f between A and B such that, for every $a \in A$, there is exactly one $b \in B$ such that afb. We then write $f : A \to B$ and $f(a) = b$ or, if f is clear from context, $a \mapsto b$. If $f : A \to B$ and $g : B \to C$ are functions, $g \circ f$ (g after f) denotes their composition, which maps $a \in A$ to $g(f(a)) \in C$. Given a set A, the identity function $\mathsf{id}_A : A \to A$ maps a to a. By an n-ary function on a set A we mean a function $f : A^n \to A$, where $A^n = A \times \ldots \times A$ is the n-time Cartesian product of set A. Again, the first arities have special names: unary (= 1-ary), binary (= 2-ary), and ternary (= 3-ary). Sometimes it is convenient to take a 0-ary function to be a constant (i.e., an element or symbol which is fixed throughout).

An equivalence relation $\equiv$ on a set A is a binary relation on A such that

1. $\equiv$ is reflexive, i.e., for all $a \in A$, we have $a \equiv a$,
2. $\equiv$ is transitive, i.e., for all $a, b, c \in A$, if $a \equiv b$ and $b \equiv c$, then $a \equiv c$, and
3. $\equiv$ is symmetric, i.e., if $a \equiv b$, then $b \equiv a$.

If $a \in A$, then the $\equiv$-equivalence class of a is the set $[a]_\equiv := \{b \in A : a \equiv b\}$. An element $b \in [a]_\equiv$ is called a representative of $[a]_\equiv$. The quotient of A under $\equiv$ is defined as $A/\equiv := \{[a]_\equiv : a \in A\}$. The function $\pi : A \to A/\equiv$ defined by $\pi(a) := [a]_\equiv$ is called the projection of $\equiv$.

A.2. Category-theoretic terminology

To be written. For now, see, e.g., Leinster 2014, ch. 1.

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