Duality Theory
Connecting Logic, Algebra, and Topology

A Reader

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Comments welcome!

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Preface

This is the reader for the course "Duality Theory: Connecting Logic, Algebra, and Topology" given during the winter semester 2023/24 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. The reader is written as the course progresses. A website (or rather git repository) with all the course material is found at

> `https://github.com/LevinHornischer/DualityTheory`.

Comments I'm happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just send an informal email to `Levin.Hornischer@lmu.de`.

Course description and objectives This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone's duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (totally disconnected compact Hausdorff spaces). This has been extended in many ways. The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,
- open sets of a space vs. points of the space,
- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications that make use of dualities by combining the often opposing advantages of the two perspectives.

Prerequisites An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course "Philosophical Logic", which I thought in the summer semester 2023. In that course, I stressed two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. We've seen that these semantic approaches are often equivalent, and this is a special case of the more general phenomenon of duality.

Contents We start with an informal chapter describing the key idea of duality. The rest of the course is about developing this key idea precisely. For this, we follow the recent textbook Gehrke and van Gool 2023. We first precisely define the algebraic structure (lattices) and then topological structures (topological spaces), and we finally prove the duality result. The remainder of the course is about deepening this result and applying it in logic and computer science.

Layout These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you'd rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end.

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Study material The main textbook that we use is by Gehrke and van Gool (2023). And informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press
- B. A. Davey and H. A. Pristley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press
- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press
- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag
- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer

- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser
- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory are

- P. T. Johnstone (1982). *Stone Spaces*. Cambrdige studies in advanced mathematics 3. Cambridge: Cambridge University Press
- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press
- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: 10.1017/9781316543870
- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press
- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser
- S. Abramsky and A. Jung (1994). "Domain Theory." In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at `http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf` (last checked 24 January 2018). Oxford: Oxford University Press

Notation Throughout, 'iff' abbreviates 'if and only if'.

1 Introduction: the key idea of duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).

Advice on how to read this chapter Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, going in different directions, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the 'rhyme and reason' to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

To use the words of Abramsky (2023).

Duality theory can be quite abstract. The advantage of this is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. So before attempting any general definition of duality, let us consider several examples (section 1.1). From those we can generalize an informal characterization of duality (section 1.2). This then hints at how duality theory is formalized mathematically and how it can be applied. Finally, in section 1.3 we list some exercises.

1.1 Intuitive examples of duality

We present several examples of duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.

Metaphysics: Properties vs objects When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon. Properties are, for example, being red, being higher than $300\mathrm{m}$, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Let us write $\mathcal{O}$ for the set of all objects and $\mathcal{P}$ for the set of all properties. Crucially, observe that there is a certain dependency between $\mathcal{O}$ and $\mathcal{P}$:

($\mathcal{O} \to \overline{\mathcal{P}}$) Each object $x \in \mathcal{O}$ determines a set of properties $F_x \subseteq \mathcal{P}$ consisting of precisely those properties that x has.

(The bar in '$\overline{\mathcal{P}}$' indicates that we assign to each x a *set* of elements in $\mathcal{P}$ rather than a single element of $\mathcal{P}$.) So we might wonder whether we can also go in the opposite direction ($\overline{\mathcal{P}} \to \mathcal{O}$)? Does a subset F of properties also determine an object, i.e., the unique object that has exactly the properties in F? Actually, no: some sets of properties might not be satisfied by any object (e.g., $F = \{$*being exactly* $300\mathrm{m}$ *high*, *being exactly* $200\mathrm{m}$ *high*$\}$) or by more than one (e.g., $F = \{$*being exactly* $300\mathrm{m}$ *high*$\}$).

Philosophers know phrases of the form 'The F' (referring to the unique object satisfying F) as definite description. For their important role in philosophy, see e.g. Ludlow (2022).

But let us not give up too early. After all, the set F_x is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if F is a set of properties with these nice features, that then it determines a unique object.)

1. Assume $a, b \in \mathcal{P}$ are two properties such that having a implies having b; we abbreviate this as $a \leqslant b$. For example,

$$a = \textit{being higher than } 300\mathrm{m} \leqslant \textit{being higher than } 200\mathrm{m} = b.$$

   So if our object x has property a, then it also has property b, i.e., if $a \in F_x$, then $b \in F_x$. We may express this as: F_x is closed under implication.

Later we will say F_x is an upset. *This sounds funny now, but by the end of the course, you will have said this so often that you won't even notice.*

2. Assume $a, b \in \mathcal{P}$ are two properties. Note that then there is another property: namely, the property of having both property a and property b. We denote this property $a \wedge b$. So $a \wedge b$ is again in $\mathcal{P}$ and we have $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$. Moreover, if our object x has property a and it has property b, then it has property $a \wedge b$, i.e., if $a, b \in F_x$, then $a \wedge b \in F_x$. We may express this as: F_x is closed under conjunction.

3. Similarly, if $a, b \in \mathcal{P}$ are two properties, there also is the property of having either property a or property b (or both). We denote this property $a \vee b$. So $a \vee b$ is again in $\mathcal{P}$ and we have $a \leqslant a \vee b$ and $b \leqslant a \vee b$. Moreover, if our object x has property $a \vee b$, then either it has property a or it has property b, i.e., if $a \vee b \in F_x$, then either $a \in F_x$ or $b \in F_x$. Later, we express this as F_x being prime.

4. Note that $\mathcal{P}$ also contains the trivial property like being identical to oneself. We denote this property $\top$. In particular, our object x has it, i.e., $\top \in F_x$.

5. Similarly, note that $\mathcal{P}$ also contains the inconsistent property like not being identical to oneself. We denote this property $\perp$. In particular, our object x does not have it, i.e., $\perp \notin F_x$.

I will always read 'either A or B' as inclusive-or (either only A is the case, or only B is the case, or both A and B are the case) Cf. a number $p > 1$ is prime iff (that is Euclid's lemma), for all numbers a and b, if $a \times b$ is divided by p, then either a is divided by p or b is divided by p).

Now, we can ask our question again: If F is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in F? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

($\overline{\mathcal{P}} \to \mathcal{O}$) Each set of properties $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines an object $x \in \mathcal{O}$, namely, the unique object having exactly the properties in F.

Or is the list (1)–(5) not complete because we should also add a principle concerning negation: you can think about this in exercise 1.b.

The uniqueness part is close to Leibniz's principle about the identity of indiscernibles: if two objects x and x' have exactly the properties in F, they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description F of an object, there in fact is a (possible) object that has these properties. For this, we should consider $\mathcal{O}$ to contain not only the objects in our world, but all possible objects. After all, the actual world need not be ontologically complete: F might consistently describe a unicorn, even if this does not exist in the actual world.

Actually, I don't know if a principle like this is considered in metaphysics: if you do, please let me know :-) Also see exercise 1.c asking for a comparison to formal concept analysis.

We will see that this bidirectional determination ($\mathcal{O} \to \overline{\mathcal{P}}$) and ($\overline{\mathcal{P}} \to \mathcal{O}$) is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensionality principle: two properties a and b are identical if they apply to exactly

Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements.

the same possible objects (i.e., for all $x \in \mathcal{O}$, x has a iff x has b). Each property a determines a set of objects: namely, the set of those objects that have property a. This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.

Semantics: Propositions vs possible worlds The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is 400m high. So the proposition a expressed by the sentence 'The Eiffel Tower is 330m high' contains the actual world x_0 (i.e., $x_0 \in a$) but not the just described possible world x_1 (i.e., $x_1 \notin a$). Some common notation for the phrase 'world x makes true proposition a' is $x \vDash a$; so possible world semantics analyses $\vDash$ as elementhood $\in$.

There is much debate in philosophy what the set $\mathcal{W}$ of possible worlds is (Menzel 2021) and what the set $\mathcal{P}$ of propositions is (McGrath and Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds ('worlds first, propositions later')? Or is it rather that worlds are maximally consistent sets of propositions ('propositions first, worlds later')? The latter goes by the name 'ersatzism' since full-blown possible worlds are substituted by something constructed out of linguistic entities—and 'Ersatz' is German for substitute.

We won't enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

World individuation Possible worlds are individuated by the propositions they make true: if two possible worlds x and y make true exactly the

Cf. Leibniz's above principle about the identity of indiscernibles.

same propositions (i.e., for every proposition a, we have $x \vDash a$ iff $y \vDash a$), then $x = y$.

Proposition individuation Propositions are individuated by the possible worlds at which they are true: if two propositions a and b are true at exactly the same possible worlds (i.e., for every possible world x, we have $x \vDash a$ iff $x \vDash b$), then $a = b$.

A hyperintensional account of propositions would contest this; see Berto and Nolan (2021).

And there is more. Just like properties, also the set of propositions has logical structure: If a and b are propositions, there also are the propositions $a \wedge b$ (conjunction), $a \vee b$ (disjunction), $\neg a$ (negation), $\top$ (logical truth), and $\bot$ (logical falsity). With this we can also express implications between propositions: proposition a implies proposition b, written $a \leqslant b$, precisely if $a \wedge b = a$. The proposition expressed by 'I am in Munich' implies the proposition expressed by 'I am in Germany' because the sentence 'I am in Munich and I am in Germany' is equivalent to the sentence 'I am in Munich', i.e., they express identical propositions.

Thus, given a possible world $x \in \mathcal{W}$, we can again consider the set of propositions $F_x \subseteq \mathcal{P}$ that are true in x (i.e., $F_x = \{a \in \mathcal{P} : x \vDash a\}$). And F_x again satisfies the features (1)–(5) above: If $a \in F_x$, i.e., $x \vDash a$, and a implies b, i.e., $a \leqslant b$, then $x \vDash b$, i.e., $b \in F_x$. If $a, b \in F_x$, then x makes true both a and b, so $a \wedge b \in F_x$. As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

Metaphysical completeness Each set of propositions $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines a possible world $x \in \mathcal{W}$, namely, the unique possible world making true exactly the propositions in F.

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions F satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to F without making it inconsistent (i.e., making it contain $\bot$). Ersatzism not only claims that then there is a world x which makes true exactly the propositions in F, it even identifies this world x with F. The metaphysic completeness claim only follows along with the existence claim, and the uniqueness of x follows from the world individuation principle above.

This is assuming that the set of propositions forms what is known as a Boolean algebra.

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective correspondence between the set $\mathcal{W}$ of possible worlds and the set $\overline{\mathcal{P}}$ of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets $F \in \overline{\mathcal{P}}$ will be called *prime filters* and $\overline{\mathcal{P}}$ will be called the *spectrum* of the algebra of propositions.)

$$\begin{aligned} \mathcal{W} &\leftrightarrows \overline{\mathcal{P}} \\ x &\mapsto F_x = \{a \in \mathcal{P} : x \vDash a\} \\ \text{the } x \text{ making true exactly the } a \in F &\leftarrow\!\shortmid F \end{aligned}$$

Let us verify that this really is a bijection: We have already checked that the function $f : \mathcal{W} \to \overline{\mathcal{P}}$ mapping x to F_x is well-defined. It is injective by the world individuation principle: if $x \neq y$, then there is a proposition a with $x \vDash a$ and $y \nvDash a$ (or vice versa), so $a \in F_x$ and $a \notin F_y$ (or vice versa), so $F_x \neq F_y$. It is surjective by metaphysical completeness: Given $F \in \overline{\mathcal{P}}$, let x be the unique world in $\mathcal{W}$ making true exactly the propositions in F. Then $F = F_x$ because: $a \in F$ iff $x \vDash a$ iff $a \in F_x$.

A function $f : X \to Y$ is injective if $x \neq y$ implies $f(x) \neq f(y)$, it is surjective if for every $y \in Y$ there is $x \in X$ with $f(x) = y$, and it is bijective if it is both injective and surjective.

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of $\mathcal{W}$) and their ersatz constructions as sets of propositions (the elements of $\overline{\mathcal{P}}$). But what about the other side: How do full-blown propositions (the elements of $\mathcal{P}$) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition $a \in \mathcal{P}$ determines the set of worlds $[\![a]\!] := \{x \in \mathcal{W} : x \vDash a\}$ where a is true. This is also known as the *truthset* of a. And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write $\overline{\mathcal{W}}$ for the sets of worlds that determine propositions and $2^{\mathcal{W}}$ for the set of all sets of worlds. So our assumption for now is that $\overline{\mathcal{W}} = 2^{\mathcal{W}}$. Analogous to the previous case, we want to know if the function

If X is a set, the powerset of X is the set of all subsets of X and it is denoted 2^X or $\mathcal{P}(X)$.

$$\begin{aligned} [\![\cdot]\!] : \mathcal{P} &\to 2^{\mathcal{W}} \\ a &\mapsto [\![a]\!] = \{x \in \mathcal{W} : x \vDash a\} \end{aligned}$$

is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if $a \neq b$, there is a world x with $x \vDash a$ and $x \nvDash b$ (or vice versa), so $[\![a]\!] \neq [\![b]\!]$. In fact, it also preserves the logical structure: $[\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$, $[\![\bot]\!] = \emptyset$, etc. (Later we formalize this as $[\![\cdot]\!]$ being a Boolean algebra homomorphism.) However, the issue is surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why $[\![\cdot]\!]$ is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a 'finitistic' grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset $2^{\mathcal{W}}$ of $\mathcal{W}$ is uncountable. So $\mathcal{P}$ and $2^{\mathcal{W}}$ have different cardinalities, which means there cannot be a bijection between, hence the already injective function $[\![\cdot]\!]$ cannot be surjective.

That is Cantor's diagonal argument.

So actually not any set of worlds determines a proposition, i.e., $\overline{\mathcal{W}}$ is a proper subset of $2^{\mathcal{W}}$. The ingenious insight of Stone, who discovered the Stone duality, was to realize how to precisely describe this special subset $\overline{\mathcal{W}}$ of $2^{\mathcal{W}}$. The key idea is to realize that there is some additional structure on the set of worlds $\mathcal{W}$ that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in the formal chapters.

Also see exercise 1.d.

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

Logic: models vs formulas Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches, which are very different in spirit, actually are equivalent. This also is a form of duality. Let's explore this concretely.

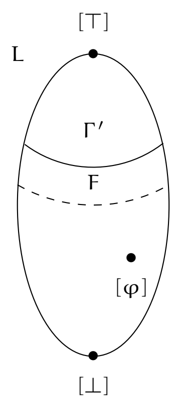
Consider the language of classical propositional logic: sentences are formed from atomic sentences $p_0, p_1, \dots$ using the connectives $\wedge, \vee, \neg$ and the constants $\bot$ and $\top$. And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence calculus for classical logic. It consists of various axioms and rules to define the relation $\Gamma \vdash \varphi$, i.e., when the sentence φ is derivable in the proof-system S using as axioms the sentences in the set Γ. This is the syntactic description of the logic.

The model-theoretic description of the logic defines the relation $\Gamma \vDash \varphi$, i.e., that the sentence φ is a logical consequence of the sentences in Γ. This is done as follows. A valuation is a function $\nu : \{p_0, p_1, \ldots\} \to \{0, 1\}$ that assigns each atomic sentences a truth-value, i.e., true (1) or false (0). This can be extended to all sentences: $\nu(\varphi \wedge \psi) = 1$ iff $\nu(\varphi) = 1$ and $\nu(\psi) = 1$; $\nu(\neg\varphi) = 1$ iff $\nu(\varphi) = 0$; $\nu(\bot) = 0$; etc. Then $\Gamma \vDash \varphi$ is defined as: for all valuations ν, if $\nu(\psi) = 1$ for all $\psi \in \Gamma$, then $\nu(\varphi) = 1$. Thus, logical consequence is truth-preservation.

Now, the completeness theorem for classical propositional logic states that: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$. To be more precise, one often only calls the right-to-left implication 'completeness', and the left-to-right implication 'soundness'. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserves logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences φ and ψ equivalent, written $\varphi \equiv \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. An equivalence class of a sentence φ is the set of sentences that are equivalent to it: $[\varphi] := \{\psi : \varphi \equiv \psi\}$. Write L for the set of all equivalence classes. It also has logical structure: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$; $\neg[\varphi] = [\neg\varphi]$, etc. L is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each a valuation ν determines a subset $F_\nu \subseteq L$: namely, those equivalence classes $[\varphi]$ with $\nu(\varphi) = 1$. Note again that F_ν has features (1)–(5): If $[\varphi] \in F_\nu$ and $[\varphi] \leqslant [\psi]$ (i.e., $[\varphi \wedge \psi] = [\varphi]$), then $\varphi \vdash \psi$, so, by soundness, $\varphi \vDash \psi$, so, since $\nu(\varphi) = 1$, also $\nu(\psi) = 1$, so $[\psi] \in F_\nu$. If $[\varphi], [\psi] \in F_\nu$, then $\nu(\varphi) = 1$ and $\nu(\psi) = 1$, so $\nu(\varphi \wedge \psi) = 1$, so $[\varphi \wedge \psi] \in F_\nu$. Etc. Conversely, if $F \subseteq L$ satisfies (1)–(5), then ν_F is a valuation mapping φ to 1 iff $[\varphi] \in F$. So, again, the set X of valuations is in bijective correspondence with the set $\overline{L}$ of subsets of L satisfying (1)–(5).



But how does completeness follow? For this, first note that subsets of L are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if $\Gamma \nvdash \varphi$, consider the deductive closure Γ' of Γ, i.e., the set of all sentences that can be derived from Γ, so also $\Gamma' \nvdash \varphi$. When we regard Γ' as a subset of L, this is, in formal terminology, a filter of L that does not intersect the ideal of all equivalence classes that imply $[\varphi]$. Now one only needs one formal result, namely Stone's Prime Filter Theorem (which we prove later on in the course), which says that we can extend this filter to a prime filter F which still does not intersect this ideal. Then ν_F is a valuation that makes true all the premises in Γ but not the conclusion φ, hence $\Gamma \not\models \varphi$, as desired.

Further examples in physics and computer science We sketch two further examples, one in physics and one in computer science.

Physics: states vs observations. Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). A physical system comes both with a *state space* X and an algebra $\mathcal{A}$ of *observations* and these two again are dual in the sense that

- the states are determined by the observations that they give rise to,
- the observations are determined by the states that give rise to them.

The observations have logical structure: in a classical (as opposed to quantum) system, observing $A \wedge B$ means observing A and observing B, observing $A \vee B$ means observing A or observing B, etc. Each state x of the system determines a set of observations: namely, those that can be made if the system is in that state. Conversely, we can also start with the algebra of observations (they are empirically more accessible anyway) and postulate the states of the system as theoretical entities corresponding to certain subsets of observations.

Computation: denotations of programs vs observable properties. Computer programs are written in a programming language, and so, much like for sentences written in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation*. For example, the denotation of a program could be the (partial) function that it computes. Domain theory is the mathematical theory to systematically describe these meanings. There again also is a side that is dual to the side of meanings, and this was a crucial discovery in the development of domain theory by Abramsky (1991, p. 16). This is the side of *observable properties* of the computer programs. For example, it could be the property that, on input $x = 3$, the program halts and outputs $f(x) = 5$. Again, we would hope for a bidirectional determination in the sense that the meaning of a program is complete determined by its observable properties, and that these observable properties are determined by the denotations that have them.

1.2 Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will work toward doing this formally for a good part of the course. But let's already give it an informal try here.

We had the following components in the examples:

- On the one side, we have a set X, e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.

- On the other side, we have a set A, e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction ($\wedge$), disjunction ($\vee$), logical falsity ($\bot$), logical truth ($\top$), and possibly negation ($\neg$).

- And we have a way to go from the spatial side to the algebraic side, and we also have a way to go in to other direction. In particular, we have:
  - A canonical way to determine from subsets of A with certain nice features an element from X, i.e., a function $\epsilon : \overline{A} \to X$.
  - A canonical way to assign to each element from A a subset of X, i.e., a function $\eta : A \to \overline{X}$.

Finally, we want to translation manual to be *formulaic* in X and A: i.e., it should not depend on the idiosyncrasies of the specific X and A; rather, it should work for all X's and A's of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we do not want the above data for specific X and A. Rather, we want it to hold for any X that is a candidate set for the spatial side, and for any A that is a candidate for the algebraic side. And hence we also want the ways of going back and forth between the two sides to respect the relations between these candidates for the spatial side and the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates X, which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category consists of the algebraic candidates A and their relations. Then we will see that all the above components of the duality is succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

1.3 Exercises

Exercise 1.a. Complete the left-out details in the main text. For example, why, for a possible world x the set of propositions F_x really satisfied properties (1)–(5). Similarly for valuations v.

Exercise 1.b. Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If $a \in \mathcal{P}$ is a property, then there also is the property $\neg a$ of not having property a. It seems plausible to require that either a given object $x \in \mathcal{O}$ has a property or it does not. In other words, either $a \in F_x$ or $\neg a \in F_x$. Do you think this is plausible to require? What about vague properties? (Later we see that this if if we have a negation operator on our set of properties obeying the Boolean laws, than F being prime is equivalent to having the just mentioned negation property.)

Exercise 1.c (More of a research project than an exercise)**.** Consider to what extend the first example (objects vs properties) can be developed along the lines of formal concept analysis.

Exercise 1.d. Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

Exercise 1.e. Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states (computable theory of mind vs connectionism). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)?

Exercise 1.f. Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

2 The algebraic side: distributive lattices

This chapter introduces formally the algebraic side of duality, which, for us, will be distributive lattices. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. And we end with section 2.4, where we already establish a duality between finite sets (resp. finite partial orders) on the one hand and finite Boolean algebras (resp. finite distributive lattices) on the other hand. This will provide a good idea of the more general case of Stone (resp. Priestley) duality. The main missing ingredient for the general case is topology, which will be the topic of the next chapter.

2.1 Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The 'structure-preserving' maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

We follow one of the keys lessons of category theory: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism). We will introduce basic notions from category theory later when we need them. For now we only foreshadow it with the 'objects' and 'morphisms' distinction.

Probably we add an extra chapter on it. If so, link to it here.

2.1.1 Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than.

But the things could also be the dishes offered at your go-to lunch place with the sense of 'better' given by your preferences. The formal definition goes as follows.

Definition 2.1. A *partial order* (or *partially ordered set*, or *poset*) is a pair $(P, \leqslant)$ where P is a (possibly empty) set and $\leqslant$ is a binary relation on P such that

A binary relation R *on a set* P *is simply a subset of* $P \times P = \{(a, b) : a, b \in P\}$*. For* $a, b \in P$*, one writes* aRb *for* $(a, b) \in R$.

1. *Reflexive*: For all $a \in P$, we have $a \leqslant a$.
2. *Transitive*: For all $a, b, c \in P$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.
3. *Anti-symmetric*: For all $a, b \in P$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

If we do not require axiom 3, we speak of a *preorder*. We say $\leqslant$ is a (partial or pre-) order on P. If the order $\leqslant$ is clear from context, we often simply speak of the (partial or pre-) order P.

The name 'partial' is to indicate that not all elements need to be comparable: Formally, for $a, b \in P$, we say that a and b are *comparable*, if either $a \leqslant b$ or $b \leqslant a$; otherwise they are incomparable. If all elements are comparable, we say $(P, \leqslant)$ is a *linear* (or *total*).

Formally, the example of the numbers is $(\mathbb{N}, \leqslant)$ where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ and, for $n, m \in \mathbb{N}$, the relation $n \leqslant m$ is defined as: n is smaller or equal to m (equivalently, there is $k \in \mathbb{N}$ such that $n + k = m$). Hence this a linear order. In the example of you lunch place, if you have two dishes a and b that you find equally tasty—or, more precisely, none tastier than the other, i.e., a and b are incomparable—, then your preference order is only partial and not linear.

Check that this satisfies the axioms.

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder $(P, \leqslant)$ into a partial order $(\overline{P}, \overline{\leqslant})$ as follows. For $a, b \in P$, define $a \equiv b$ as $a \leqslant b$ and $b \leqslant a$. This is an equivalence relation (reflexive, symmetric, and transitive). Equivalence classes are the sets $[a] := \{b \in P : a \equiv b\}$ for $a \in P$. The quotient of P under $\equiv$ is $\overline{P} := P/\equiv := \{[a] : a \in P\}$. Define $[a]\overline{\leqslant}[b]$ by $a \leqslant b$ (note that this is independent of the representatives a and b). This renders $(\overline{P}, \overline{\leqslant})$ a partial order. It is also called the *poset reflection* of P. Exercise 2.a makes formally precise in what sense it is the canonical or best possible poset approximating the preorder P.

There is a nice visualization of partial orders. They are known as Hasse diagrams. An example is in figure 2.1. It depicts the partial order $(P, \leqslant)$

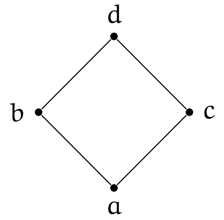


Figure 2.1: The 'diamond' as an example of a partial order.

with $P = \{a, b, c, d\}$ and

$$\leqslant := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of P and the edges are the minimal information to recover the order:

- if there is an edge between x and y and x is lower (on the page) than y, then $x \leqslant y$.
- we do not need to draw an edge from one node to itself because for all nodes x we have $x \leqslant x$.
- we do not need to draw edges that result from composing existing edges: for example, we have an edge from a to b and an edge from b to d, so we already know that $a \leqslant d$, hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order $(P, \leqslant)$ is as follows. For $a, b \in P$, we say that b *covers* a (short $a \lessdot b$) if $a \leqslant b$ and for all $c \in P$, if $a \leqslant c \leqslant b$, then $c = a$ or $c = b$. The elements of P are the nodes of the Hasse diagram, and an edge is drawn from node a to node b whenever b covers a. The direction of the edge is indicated by drawing b higher up in the diagram than a. So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

They can be confusing at first, but they really are worth learning. Make sure to draw little Hasse diagrams to illustrate the concepts and how they differ from each other.

Definition 2.2. Let $(P, \leqslant)$ be a partial order and $A \subseteq P$.

- An element $b \in P$ is a *lower bound* of A if, for all $a \in A$, we have $b \leqslant a$.

- An element $b \in P$ is an *upper bound* of A if, for all $a \in A$, we have $a \leqslant b$.

- An element $c \in P$ is an *infimum* or *greatest lower bound* of A if (1) c is a lower bound of A, and (2), for all lower bounds b of A, we have $b \leqslant c$.

- An element $c \in P$ is a *supremum* or *least upper bound* of A if (1) c is an upper bound of A, and (2), for all upper bounds b of A, we have $c \leqslant b$.

- An element $b \in P$ is a *least* or *bottom* or *minimum* element of P, if, for all $a \in P$, we have $b \leqslant a$ (i.e., b is the supremum of $A = \emptyset$).

- An element $b \in P$ is a *greatest* or *top* or *maximum* element of P, if, for all $a \in P$, we have $a \leqslant b$ (i.e., b is the infimum of $A = \emptyset$).

- An element $b \in P$ is *minimal* if, for all $a \in P$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal* if, for all $a \in P$, if $b \leqslant a$, then $b = a$.

- An element $b \in P$ is *minimal in* A if (1) $b \in A$ and (2) for all $a \in A$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal in* A if (1) $b \in A$ and (2) for all $a \in A$, if $b \leqslant a$, then $b = a$.

- A is *directed* (aka up-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. (Equivalently, all finite subsets of A have an upper bound in A.)

- A is *filtered* (aka filtering or down-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $c \leqslant a$ and $c \leqslant b$. (Equivalently, all finite subsets of A have a lower bound in A.)

(These notions also make sense in a preorder $(P, \leqslant)$, but if P is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted $\bigwedge A$, called the *meet* of A; and the supremum is denoted $\bigvee A$, called the *join* of A. If $A = \{a_1, \ldots, a_n\}$ is finite and nonempty, we write $\bigwedge A = a_1 \wedge \ldots \wedge a_n$ and $\bigvee A = a_1 \vee \ldots \vee a_n$. In particular, $\bigwedge\{a, b\} = a \wedge b$ and $\bigvee\{a, b\} = a \vee b$. The bottom element, if it exists, is denoted $\perp$ or 0; and the top element by $\top$ or 1. We write $\min(A)$ (resp. $\max(A)$) for the elements that are minimal (resp. maximal) in A. A *directed join* is the supremum of a directed set.

It is a good exercise to prove this.

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcop*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can 'turn them upside down' and get another preorder. Formally, if $(P, \leqslant)$ is a preorder, define the preorder $\leqslant'$ on P by $a \leqslant' b$ iff $b \leqslant a$. We write P^{op} for this preorder.

Verify that this again is a preorder (resp. partial order), and draw some Hasse diagram example to see that this really turns things upside down.

2.1.2 Morphisms: Monotone maps

We consider the words 'map' and 'function' as synonymous.

What maps between partial orders should be considered to be 'structure preserving'? Surely they should preserve the order structure. This yields the concept of a monotone map, and is the standard choice. But there also are other ones, which we mention as well.

Definition 2.3. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders and $f : P \to Q$ a function. We say f is

- *monotone* or *order preserving* if, for all $a, b \in P$, if $a \leqslant_P b$, then $f(a) \leqslant_Q f(b)$.
- *order reflecting* if, for all $a, b \in P$, if $f(a) \leqslant_Q f(b)$, then $a \leqslant_P b$.
- an *order-embedding* if f is both order preserving and order reflecting.
- an *order-isomorphism* if f is monotone with a monotone inverse (further comments below).

Order-embeddings between posets are injective, but the converse fails (i.e., there are injective order preserving maps between posets which are not order-embeddings.

Verify this.

If P and Q are posets, an equivalent condition for f being an order-isomorphism is that f is a surjective order-embedding. In practice, this is often easier to check, although the definition via a monotone inverse better captures the (category-theoretic) concept of an isomorphism. In full, the latter says: A monotone function $f : P \to Q$ between two preorders is an order-isomorphism if there is a monotone function $g : Q \to P$ such that

- for all $a \in P$, we have $a = g(f(a))$, i.e., a is the g-inverse of $f(a)$ (in short, $\mathsf{id}_P = g \circ f$), and
- for all $b \in Q$, we have $f(g(b)) = b$, i.e., mapping the g-inverse of b along f yields b (in short, $f \circ g = \mathsf{id}_Q$).

Here id_X denotes the identity function on set X. And if $f : X \to Y$ and $g : Y \to Z$ are functions, $g \circ f$ (g after f) denotes their composition, which maps $x \in X$ to $g(f(x)) \in Z$.

If two preorders are isomorphic (i.e., there is an order isomorphism between them), we can consider them to be essentially identical. This is difficult to achieve, so it makes sense to look for a *generalization* of the concept of an isomorphism. The key idea is to still require a monotone function $g : Q \to P$ in the other direction, but it need not be the *true* inverse but only the best possible *approximation* to an inverse:

- for all $a \in P$, we have $a \leqslant_P g(f(a))$, i.e., the g-inverse of $f(a)$ is at least as good as a, and
- for all $b \in Q$, we have $f(g(b)) \leqslant_Q b$, i.e., mapping the g-inverse of b along f approximates b.

Exercise 2.b shows why this approximation then really is best possible; and it also provides the following equivalent definition.

This is an advanced concept. Give yourself the time to let it sink in by coming back to it over and over again.

Definition 2.4. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone functions. The pair (f, g) is called an *adjunction*, with f the *left* or *lower adjoint* and g the *right* or *upper adjoint*, if, for all $a \in P$ and $b \in Q$,

$$f(a) \leqslant_Q b \text{ iff } a \leqslant_P g(b).$$

Note that f occurs on the left of '$\leqslant$' and g on the right.

We also write this as $l : P \leftrightarrows Q : u$. An adjunction between P^{op} and Q is called a *Galois connection* or *contravariant adjunction*.

It is best to illustrate this abstract concept with examples. An important template of how Galois connections arise is the following (which includes the instance coining them).

Lemma 2.5. *Let $R \subseteq X \times Y$ be a relation between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define*

$$u(a) := \{y \in Y : \forall x \in a.xRy\} \subseteq Y$$
$$l(b) := \{x \in X : \forall y \in b.xRy\} \subseteq X$$

Then $l : \mathcal{P}(Y) \leftrightarrows \mathcal{P}(X) : u$ forms a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, i.e., for any $b \subseteq Y$ and $a \subseteq X$, we have $a \subseteq l(b)$ (i.e., $l(b) \subseteq^{op} a$) iff $b \subseteq u(a)$.

Here $\mathcal{P}(X)$ is the set of all subsets of the set X.

Proof. ($\Rightarrow$). Assume $a \subseteq l(b)$. To show $b \subseteq u(a)$, let $y \in b$ and show $y \in u(a)$. So let $x \in a$ and show xRy. By the assumption, $x \in l(b)$, so for our $y \in b$ we have xRy.

($\Leftarrow$). Assume $b \subseteq u(a)$. To show $a \subseteq l(b)$, let $x \in a$ and show $x \in l(b)$. So let $y \in b$ and show xRy. By the assumption, $y \in u(a)$, so for our $x \in a$ we have xRy. $\square$

Here are three instances of this lemma.

1. Maybe you know the name 'Galois' from the theory of fields in algebra. Then you know Galois theory as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation R between the set X of subfields of a given field and the set Y of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.

But you don't need to know this for the course. If you'd like an accessible introduction, have a look, e.g., at this or this video, or at these great lecture notes by Tom Leinster.

2. If X is a set and $R \subseteq X \times X$ is a preorder, then $u(a)$ is the set of upper bounds of $a \subseteq X$, and $l(b)$ is the set of lower bounds of $b \subseteq X$.

3. Consider a class of structures $\mathcal{C}$ (in, say, a first-order signature) and a class $\mathcal{F}$ of formulas (of this signature). Let $\vDash$ be the *interpretation* relation: For $M \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ means that structure M makes true formula φ. Then for a set of models a, $u(a)$ is the theory of a, i.e., the set of formulas that are true in all those models. And for a a theory $b \subseteq \mathcal{F}$, $l(b)$ is the class of models of b, i.e., the set of models which make true all the sentences in b.

Also recall the examples from section 1.1.

2.2 Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

2.2.1 Objects: lattices

The order-theoretic definition of a lattice goes as follows.

Definition 2.6 (Lattice, order-theoretic)**.** A *(bounded) lattice* is a partial order L in which every finite subset has a supremum and an infimum.

Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.

As an exercise, prove this.

2. Often a lattice is defined as a partial order in which all binary suprema and infima exists (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exits (i.e., which a have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word 'bounded' (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.

3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

Prove this. (Hint: think about the supremum of all lower bounds.)

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

Definition 2.7 (Lattice, algebraic)**.** A lattice is a tuple $(L, \vee, \wedge, \perp, \top)$ where $\vee$ and $\wedge$ are binary operations on L (i.e., functions $L \times L \to L$), and $\perp$ and $\top$ are elements of L, such that the following axioms holds:

1. *commutative*: for all $a, b \in L$, we have $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.
2. *associative*: for all $a, b, c \in L$, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.
3. *idempotent*: for all $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$.
4. *absorption*: for all $a, b \in L$, we have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$
5. *neutrality*: for all $a \in L$, we have $\perp \vee a = a$ and $\top \wedge a = a$.

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.c asks you to prove it: that is a bit tedious, but quite instructive.

Theorem 2.8. 1. *Given a lattice $(L, \vee, \wedge, \perp, \top)$ according to the algebraic definition, define $a \leqslant_L b$ as $a \wedge b = a$. Then $(L, \leqslant_L)$ is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by $\vee$ and $\wedge$.*

2. *Given a lattice $(L, \leqslant)$ according to the order-theoretic definition, define the binary operations $\vee$ and $\wedge$ as binary supremum and infimum, and take $\perp$ and $\top$ to be the least and greatest element of L. Then $(L, \vee, \wedge, \perp, \top)$ is a*

lattice according to the algebraic definition, with $a \wedge b = a$ *iff* $a \leqslant b$ *iff* $a \vee b = b$.

From now on, we will often just speak of a lattice L and both use its order-theoretic definition (taking $\leqslant$ to be implicitly given) and its algebraic definitions (taking $\vee, \wedge, \perp, \top$ to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure $(L, \cdot, 1)$, where $\cdot$ is a commutative, associative, and idempotent binary operation on L, and 1 is a neutral element for the operation. The operation $\cdot$ can then either be seen as the binary infimum for the partial order defined by $a \leqslant b$ iff $a \cdot b = a$ (the join semilattice), or as the binary supremum for the opposite partial order defined by $a \leqslant b$ iff $a \cdot b = b$ (the meet semilattice).

2.2.2 Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

Definition 2.9. A function $f : L \to M$ between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all $a, b \in L$, we have $f(a \vee_L b) = f(a) \vee_M f(b)$
2. for all $a, b \in L$, we have $f(a \wedge_L b) = f(a) \wedge_M f(b)$
3. $f(\perp_L) = \perp_M$
4. $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*.

Prove this.

If a function $f : L \to M$ between lattices preserves $\perp$ and $\vee$, then it preserves all finite joins. This does, in general, *not* imply any preservation of arbitrary existing joins or preservation of infima. The analog statement is true for $\top$ and $\wedge$ and preservation of all finite meets.

Prove this.

2.2.3 Constructions: products, sublattices, homomorphic images, congruences

We introduce several common constructions on lattices. They are common algebraic operations that you might have seen already in other contexts (e.g., group theory); and, in any case, they are worth knowing as they come up quite often.

Products. Given a family $(L_i)_{i\in I}$ of lattices, we can define a lattice $L = \prod_{i\in I}$ on the Cartesian product where the operations are define componentwise: e.g., for $a = (a_i)_i$ and $b = (b_i)_i$ in L, we define $a \leqslant_L b$ as $\forall i \in I : a_i \leqslant_{L_i} b$, and $(a \wedge b)_i = a_i \wedge b_i$ (similarly for $\vee$), and $(\bot_L)_i = \bot_{L_i}$ (similarly for $\top$). The projection maps $\pi_i : L \to L_i$, which map $a = (a_i)_i$ to a_i, is a surjective lattice homomorphism.

Recall that the Cartesian product of a family of sets is the set of functions a *that map each* $i \in I$ *to an element* $f(i) \in L_i$. *We often write such a function as* $a = (a_i)_{i\in I}$.

Sublattices. A sublattice of a lattice L is a subset L' of L that contains $\bot$ and $\top$ and that is closed under $\wedge$ and $\vee$ (i.e., if $a, b \in L'$, then $a \wedge b, a \vee b \in L'$). Then L' is a bounded lattice in its own right and the inclusion map $\iota : L' \to L$, which maps $a \in L'$ to $a \in L$, is a lattice embedding. If we do not require $\bot$ and $\top$ to be in L', we speak of an *unbounded sublattice*. And if we require L' to be closed under all suprema and infima, we call it a *complete sublattice*. If $f : L \to M$ is a lattice homomorphism, then the direct image $L' := f[L] = \{f(a) : a \in L\}$ is a sublattice of the lattice M.

Homomorphic images. A lattice L' is a *homomorphic image* of a lattice L if there is a surjective lattice homomorphism $f : L \to L'$.

Birkhoff's famous theorem in universal algebra says that a class of algebraic structures (like lattices) is closed under Homomorphic images, Subalgebras, and Products iff it is definable by equations (hence aka 'HSP theorem').

Congruences. A congruence on a lattice L is an equivalence relation ϑ on L that respects the lattice operations, i.e., for all $a, a', b, b' \in L$, if $a\vartheta a'$ and $b\vartheta b'$, then also $a \vee b\vartheta a' \vee b'$ and $a \wedge b\vartheta a' \wedge b'$. The quotient L/ϑ carries a unique lattice structure that turns the quotient map $p : L \to L/\vartheta$, which maps $a \in L$ to its equivalence class $[a]_\vartheta$ under ϑ; this is given by $[a]_\vartheta \vee [b]_\vartheta := [a \vee b]_\vartheta$ (similarly for $\wedge$) with bottom element $[\bot]_\vartheta$ (similarly for $\top$).

The first isomorphism theorem for lattices. This says that any lattice homomorphism $f : L \to M$ can be factored as a surjective lattice homomorphism p followed by a lattice embedding e (i.e., $f = e \circ p$). These are given as follows. The *kernel* of f is the congruence relation

$$\ker f := \{(a, a') \in L \times L : f(a) = f(a')\}.$$

Choose $p : L \to L/\ker f$ (mapping a to $[a]$) and $e : L/\ker f \to M$ (mapping $[a]$ to $f(a)$). In particular, $L/\ker f$ is isomorphic to $f[L]$ (take $M := f[L]$, so e also is surjective); hence the homomorphic images of L are, up to isomorphism, the quotients of L.

2.3 Distributive lattices and Boolean algebras

We get further subclasses of lattices by requiring that $\vee$ and $\wedge$ interact nicely, which is made precise as distributive lattices (section 2.3.1), and by additionally requiring that there is a sense of negation, which is made precise as Boolean algebras (section 2.3.2).

2.3.1 Distributive lattices

The idea $\vee$ and $\wedge$ interact nicely is made precise as follows.

Definition 2.10. A lattice L is distributive if,

Cf. distributivity from high school: $x \times (y + z) = (x \times y) + (x \times z)$

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{2.1}$$

or, equivalently,

Proving the equivalence of 2.1 and 2.2 is a good exercise!

$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{2.2}$$

The equivalence of 2.1 and 2.2 implies that L is distributive iff L^{op} is distributive. So distributivity is a so-called self-dual property. Moreover, homomorphic images, sublattices, and products of distributive lattices are again distributive. (This follows from the 'HSP theorem' and the fact that distributive lattices are defined equationally.)

Again important special cases are as follows: A *frame* is a complete lattice L satisfying the join infinite distributive law (JID)

In case you have heard of this: A frame is the same thing as a complete Heyting algebra, but their respective choice of morphisms differ.

$$\text{for any } a \in L \text{ and } B \subseteq L,\ a \wedge \bigvee A = \bigvee_{b \in B} (a \wedge b). \tag{2.3}$$

In a distributive lattice this, in general, only holds for all *finite* $B \subseteq L$.

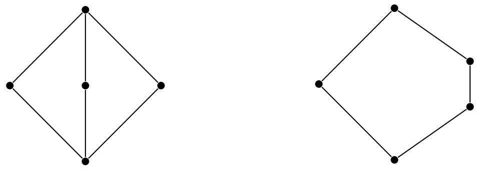
A seemingly magic characterization of distributive lattices is the following.

Theorem 2.11 (The M_3–N_5 theorem)**.** *Let L be a lattice. Then L is distributive iff L does not contain an unbounded sublattice which is isomorphic to M_3 or N_5, depicted in figure 2.3.1.*

For a proof, see, e.g., Davey and Pristley (2002, 89 ff.).

2.3.2 Boolean algebras

So far, we have seen the order $\leqslant$ and the operations $\vee$ and $\wedge$ in a lattice, which act like implication, disjunction, and conjunction, respectively. So you might have wondered: what about negation? Especially since this



The diamond M_3 The pentagon N_5

Figure 2.2: The forbidden substructures for distributivity.

also played a role in our motivating introduction (chapter 1). The (or, more precisely, a) idea of negation is made precise as follows.

Definition 2.12. Let L be a lattice and a an element of L. A *complement* of a is an element b of L such that $a \wedge b = \bot$ and $a \vee b = \top$. A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element a in a distributive lattice is unique, if it is exist, an denoted $\neg a$.

1. Usually, the negation is then taken into the signature: so a Boolean algebra is a tuple $(B, \wedge, \vee, \bot, \top, \neg)$ such that $(B, \wedge, \vee, \bot, \top)$ is a distributive lattice and $\neg : B \to B$ a unary function such that, for all $a \in B$, we have $a \wedge \neg a = \bot$ and $a \vee \neg a = \top$.

2. But if we have an additional operation around, shouldn't we require the morphisms to preserve it? Fortunately, they already do: If $f : B \to A$ is a lattice homomorphism between Boolean algebras, then, for all $a \in B$, we have $f(\neg a) = \neg f(a)$. We often still refer to them as *Boolean algebra homomorphisms* just to emphasize that we are dealing with Boolean algebras.

The fact that we can use the same morphisms is expressed in categorical terms as the category of Boolean algebras and Boolean algebra homomorphisms being a full *(as opposed to any) subcategory of the category of distributive lattices and lattice homomorphisms.*

3. However, with the notion of a sublattice we need to be more careful: A Boolean algebra may have many sublattices that themselves are not Boolean algebras; so by a (*Boolean*) *subalgebra* of a Boolean algebra B we mean a sublattice which is also closed under $\neg$.

4. If you like ring theory, a Boolean algebra can equivalently be defined as a commutative ring with unit in which all elements are idempotent, see exercise 2.d.

5. There is a best way to turn a distributive lattice L into a Boolean algebra B. This B is called the *Boolean envelope* or *free Boolean extension* of L. More precisely, this means that for every distributive lattice L there is a Boolean algebra B and an injective homomorphism $e : L \to$

In categorical terms this means the category of Boolean algebras is a full reflective *subcategory of the category of distributive lattices.*

B such that for any other lattice homomorphism $h : L \to A$ into a Boolean algebra A, there is a unique Boolean algebra homomorphism $\overline{h} : B \to A$ such that $\overline{h} \circ e = h$. As a diagram:

$$\begin{tikzcd} L \arrow[r, "e"] \arrow[dr, "h"'] & B \arrow[d, dashed, "\overline{h}"] \\ & A \end{tikzcd}$$

We later will prove this theorem using duality theory.

2.4 Duality for finite distributive lattices and finite Boolean algebras

In this section, we prove a 'baby version' of the duality result that we are working toward. Of course, the baby version will follow from the full version, but we prove it now already mostly for pedagogical reasons (1) to already reap some benefits of the build-up of theory so far and (2) to already get used to how a duality theorem looks like.

2.5 Exercises

Exercise 2.a. Recall that for a preorder $(P, \leqslant)$, we have defined the poset reflection $(\overline{P}, \overline{\leqslant})$. This exercise makes precise in which sense this is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 1.1.5 in Gehrke and van Gool (2023), with small changes.

1. Prove that $\equiv$ is an equivalence relation.

2. Prove that the definition of $\overline{\leqslant}$ is independent of the representatives: If $a' \in [a]$ and $b' \in [b]$, then $a \leqslant b$ iff $a' \leqslant b'$.

3. Prove that $(\overline{P}, \overline{\leqslant})$ is indeed a partial order.

4. Prove that $\overline{\leqslant}$ is the smallest partial order on $\overline{P} = P/\equiv$ such that the quotient map $f : P \to P/\equiv$, which maps a to $[a]$, is order preserving: That is, if $\leqslant'$ is another such partial order on $P/\equiv$, then $\overline{\leqslant} \subseteq \leqslant'$.

5. Prove that, for any order preserving $g : P \to Q$ into a poset Q, there exists a unique order preserving $\overline{g} : P/\equiv \to Q$ such that $\overline{g} \circ f = g$. As

The category-theoretic formulation of this fact is: the inclusion of the category of partial orders and monotone maps in the category of preorders and monotone maps has a left adjoint. Adjoint functors can be interpreted as formalizing the idea of finding a best possible approximation.

a diagram:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "g"'] & P/\equiv \arrow[d, dashed, "\overline{g}"] \\ & Q \end{tikzcd}$$

Think about how the last item formalizes the idea that $(\overline{P}, \overline{\leqslant})$ is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 2.b. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone maps.

Exercise 1.1.8 in Gehrke and van Gool (2023).

1. Prove that (f, g) is an adjunction iff for all $a \in P$ we have $a \leqslant_P g(f(a))$ and for all $b \in Q$ we have $f(g(b)) \leqslant b$.

For the rest of this exercise, assume that (f, g) is an adjunction.

2. Prove that $f \circ g \circ f(a) \equiv f(a)$ and $g \circ f \circ g(b) \equiv g(b)$ for every $a \in P$ and $b \in Q$ (and $a \equiv b$ iff $a \leqslant b$ and $b \leqslant a$).

3. Conclude that, in particular, if P and Q are posets, then $fgf = f$ and $gfg = g$.

4. Prove that, if P is a poset, then for any $a \in P$, $gf(a)$ is the least element above a that lies in the image of g.

5. Formulate and prove a similar statement to the previous item about $fg(b)$, for $b \in Q$.

6. Prove that, for any subset $AsubseteqP$, if the supremum of A exists, then $f(\bigvee A) = \bigvee f(A)$ (where $f(A) = \{f(a) : a \in A\}$ is the image of A under f).

7. Prove that, for any subset $B \subseteq Q$, if the infimum of B exists, then $g(\bigwedge B) = \bigwedge g(B)$.

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

We will see that the converse holds for complete lattices. This is a special case of the Adjoint Functor Theorem.

Exercise 2.c. Prove theorem 2.8.

Exercise 2.d. This exercise shows that Boolean algebras and Boolean rings are equivalent.

From Gehrke and van Gool 2023, ex. 1.2.13.

Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, i.e., a commutative ring with unit in which $a^2 = a$ for all $a \in B$. Define $a \leqslant b$ if $a \cdot b = a$. Prove that

$\leqslant$ is a distributive lattice order on B, and that every element of B has a complement with respect to $\leqslant$. *Hint*: First show that $a + a = 0$ for all $a \in B$.

Conversely, let $(B, \wedge, \vee, \perp, \top, \neg)$ be a Boolean algebra. Define, for any $a, b \in B$,

$$a + b := (a \wedge \neg b) \vee (\neg a \wedge b) \qquad a \cdot b := a \wedge b$$
$$0 := \perp \qquad 1 := \top.$$

The operation + is known as symmetric difference.

Prove that $(B, +, \cdot, 0, 1)$ is a Boolean ring.

Finally, show that the composition of these two assignments in either order yields the identity.

3 The spatial side: topological spaces

4 Two sides of the same coin: Priestley and Stone duality

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