

Duality Theory

Connecting Algebra and Topology via Logic

Lecture Notes

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Comments welcome!

Levin Hornischer

`Levin.Hornischer@lmu.de`

`https://levinhornischer.github.io/DualityTheory/`

Contents

Preface	1
1 Introduction: the key idea of duality	4
1.1 Intuitive examples of duality	4
1.1.1 Metaphysics: Properties vs objects	5
1.1.2 Semantics: Propositions vs possible worlds	7
1.1.3 Logic: formulas/syntax vs models/semantics	10
1.1.4 Further examples	12
1.2 Towards characterizing duality	13
1.3 Exercises	14
2 The algebraic side: Boolean algebras	17
2.1 Order theory	17
2.1.1 Objects: Partial orders	17
2.1.2 Morphisms: Monotone maps	21
2.2 Lattices	22
2.2.1 Objects: lattices	22
2.2.2 Morphisms: lattice homomorphisms	24
2.3 Distributive lattices and Boolean algebras	24
2.3.1 Boolean algebras	24
2.4 Exercises	24
3 The spatial side: topological spaces	30
4 Two sides of the same coin: Stone duality	31
5 Application: (Re)discovering semantics for modal logic	32
6 Generalization: Priestley duality	33
Bibliography	33
Index	36

Preface

This is the reader for the course “Duality Theory: Connecting Logic, Algebra, and Topology” given during the winter semester 2024/5 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. These lecture notes are updated as the course progresses. A website with all the course material is found at

<https://levinhornischer.github.io/DualityTheory/>.

Comments I’m happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just send an informal email to Levin.Hornischer@lmu.de.

Course description and objectives This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone’s duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (totally disconnected compact Hausdorff spaces). The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,
- open sets of a space vs. points of the space,
- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory, after a philosophical motivation. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications—especially to philosophical phenomena—that make use of dualities by combining the often opposing advantages of the two perspectives.

Prerequisites An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course **Philosophical Logic**. In that course, I stress two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. These approaches are often equivalent, which is a special case of duality.

Contents We start with an informal chapter describing the key idea of duality. The rest of the course is about developing this key idea precisely. We first precisely define the algebraic structure (Boolean algebras) and the topological structures (topological spaces), and then prove the duality result. The remainder of the course is about applying this result to modal logic (and sketching applications in computer science) and generalizing the result to Priestley duality.

Layout These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you'd rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end. Exercises are at the end of each chapter.

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Study material The main textbook that we use is by Gehrke and van Gool (2023). And informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press
- B. A. Davey and H. A. Priestley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press
- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press
- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag
- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer

- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser
- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory are

- P. T. Johnstone (1982). *Stone Spaces*. Cambridge studies in advanced mathematics 3. Cambridge: Cambridge University Press
- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press
- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: [10.1017/9781316543870](https://doi.org/10.1017/9781316543870)
- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press
- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser
- S. Abramsky and A. Jung (1994). “Domain Theory.” In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at <http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf> (last checked 24 January 2018). Oxford: Oxford University Press
- E. Orłowska et al. (2015). *Dualities for Structures of Applied Logic*. Studies in Logic 56. College Publications

Notation Throughout, ‘iff’ abbreviates ‘if and only if’.

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1 Introduction: the key idea of duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

Advice on how to read this chapter. Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the ‘rhyme and reason’ to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

Duality theory can be quite abstract. The advantage of this is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. So before attempting any general definition of duality, let us consider several examples (section 1.1). From those we can generalize an informal characterization of duality (section 1.2). This then hints at how duality theory is formalized mathematically and how it can be applied. Finally, in section 1.3 we list some exercises.

1.1 Intuitive examples of duality

We present several examples of duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).

To use the words of Abramsky (2023).

I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.

1.1.1 Metaphysics: Properties vs objects

When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon, and we will here also include merely possible objects like unicorns. Properties are, for example, being red, being higher than 300m, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Philosophically, it is difficult to make this talk of objects and properties precise (e.g., if we are too permissive about what counts as a property, Russell’s paradox creeps in). For now, let us just rely on our everyday intuitions about these concepts. Once we see where this will lead, the exercises at the end of this chapter will ask you to come back and scrutinize the concept of object and property at play (see exercise 1.c).

Let us write \mathcal{O} for the set of all possible objects and \mathcal{P} for the set of all properties. Crucially, observe that there is a certain dependency between \mathcal{O} and \mathcal{P} :

$(\mathcal{O} \rightarrow \overline{\mathcal{P}})$ Each object $x \in \mathcal{O}$ determines a set of properties $F_x \subseteq \mathcal{P}$ consisting of precisely those properties that x has.

(The bar in ‘ $\overline{\mathcal{P}}$ ’ indicates that we assign to each x a *set* of elements in \mathcal{P} rather than a single element of \mathcal{P} .) So we might wonder whether we can also go in the opposite direction ($\overline{\mathcal{P}} \rightarrow \mathcal{O}$)? Does a subset F of properties also determine an object, i.e., the unique object that has exactly the properties in F ? Actually, no: some sets of properties might not be satisfied by any object (e.g., $F = \{\text{being exactly 300m high, being exactly 200m high}\}$) or by more than one (e.g., $F = \{\text{being exactly 300m high}\}$).

But let us not give up too early. After all, the set F_x is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if F is a set of properties with these nice features, that then it determines a unique object.)

1. Assume $a, b \in \mathcal{P}$ are two properties such that having a implies having b ; we abbreviate this as $a \leq b$. For example,

$$a = \text{being higher than 300m} \leq \text{being higher than 200m} = b.$$

So if our object x has property a , then it also has property b , i.e., if $a \in F_x$, then $b \in F_x$. We may express this as: F_x is closed under implication.

If ‘being a property’ is a property, consider the property p of ‘not being a property’. Then p has p iff p does not have p , contradiction. But, arguably, ‘being a property’ is not an ‘everyday’ property.

Philosophers also call F_x the role of the individual x (McMichael 1983, p. 57).

Philosophers know phrases of the form ‘The F ’ (referring to the unique object satisfying F) as definite description. For their important role in philosophy, see e.g. Ludlow (2022).

Later we will say F_x is an upset. This sounds funny now, but by the end of the course, you will have said this so often that you won’t even notice.

2. Assume $a, b \in \mathcal{P}$ are two properties. Note that then there is another property: namely, the property of having both property a and property b . We denote this property $a \wedge b$. So $a \wedge b$ is again in \mathcal{P} and we have $a \wedge b \leq a$ and $a \wedge b \leq b$. Moreover, if our object x has property a and it has property b , then it has property $a \wedge b$, i.e., if $a, b \in F_x$, then $a \wedge b \in F_x$. We may express this as: F_x is closed under conjunction.
3. Similarly, if $a, b \in \mathcal{P}$ are two properties, there also is the property of having either property a or property b (or both). We denote this property $a \vee b$. So $a \vee b$ is again in \mathcal{P} and we have $a \leq a \vee b$ and $b \leq a \vee b$. Moreover, if our object x has property $a \vee b$, then either it has property a or it has property b , i.e., if $a \vee b \in F_x$, then either $a \in F_x$ or $b \in F_x$. Later, we express this as F_x being prime.
4. Note that \mathcal{P} also contains the trivial property like being identical to oneself. We denote this property \top . In particular, our object x has it, i.e., $\top \in F_x$.
5. Similarly, note that \mathcal{P} also contains the inconsistent property like not being identical to oneself. We denote this property \perp . In particular, our object x does not have it, i.e., $\perp \notin F_x$.

Now, we can ask our question again: If F is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in F ? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

$(\overline{\mathcal{P}} \rightarrow \mathcal{O})$ Each set of properties $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines an object $x \in \mathcal{O}$, namely, the unique object having exactly the properties in F .

The uniqueness part is close to Leibniz's principle about the **identity of indiscernibles**: if two objects x and x' have exactly the properties in F , they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description F of an object, there in fact is a (possible) object that has these properties. This is why we consider the set \mathcal{O} of all possible objects. The actual world need not be ontologically complete: F might consistently describe a unicorn, even if this does not exist in the actual world.

I will always read 'either A or B' as inclusive-or (either only A is the case, or only B is the case, or both A and B are the case)

*Cf. a number $p > 1$ is prime iff (that is **Euclid's lemma**), for all numbers a and b , if $a \times b$ is divided by p , then either a is divided by p or b is divided by p).*

Or is the list (1)–(5) not complete because we should also add a principle concerning negation: you can think about this in exercise 1.b.

Actually, I don't know if a principle like this is considered in metaphysics: if you do, please let me know :-). Also see exercise 1.d asking for a comparison to formal concept analysis.

We will see that this bidirectional determination ($\mathcal{O} \rightarrow \overline{\mathcal{P}}$) and ($\overline{\mathcal{P}} \rightarrow \mathcal{O}$) is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensionality principle: two properties a and b are identical if they apply to exactly the same possible objects (i.e., for all $x \in \mathcal{O}$, x has a iff x has b). Each property a determines a set of objects: namely, the set of those objects that have property a . This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements.

Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.

1.1.2 Semantics: Propositions vs possible worlds

The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is 400m high. So the proposition a expressed by the sentence 'The Eiffel Tower is 330m high' contains the actual world x_0 (i.e., $x_0 \in a$) but not the just described possible world x_1 (i.e., $x_1 \notin a$). Some common notation for the phrase 'world x makes true proposition a ' is $x \models a$; so possible world semantics analyses \models as elementhood \in .

There is much debate in philosophy what the set \mathcal{W} of possible worlds is (Menzel 2021) and what the set \mathcal{P} of propositions is (McGrath and Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds ('worlds first, propositions later')? Or is it rather that worlds are maximally consistent sets of propositions ('propositions first, worlds later')? The latter goes by the name 'ersatzism' since full-blown possible worlds are substituted by something constructed out of linguistic entities—

and ‘Ersatz’ is German for substitute.

We won’t enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

World individuation Possible worlds are individuated by the propositions they make true: if two possible worlds x and y make true exactly the same propositions (i.e., for every proposition a , we have $x \models a$ iff $y \models a$), then $x = y$.

Cf. Leibniz’s above principle about the identity of indiscernibles.

Proposition individuation Propositions are individuated by the possible worlds at which they are true: if two propositions a and b are true at exactly the same possible worlds (i.e., for every possible world x , we have $x \models a$ iff $x \models b$), then $a = b$.

A hyperintensional account of propositions would contest this; see Berto and Nolan (2021).

And there is more. Just like properties, also the set of propositions has logical structure: If a and b are propositions, there also are the propositions $a \wedge b$ (conjunction), $a \vee b$ (disjunction), $\neg a$ (negation), \top (logical truth), and \perp (logical falsity). With this we can also express implications between propositions: proposition a implies proposition b , written $a \leq b$, precisely if $a \wedge b = a$. The proposition expressed by ‘I am in Munich’ implies the proposition expressed by ‘I am in Germany’ because the sentence ‘I am in Munich and I am in Germany’ is equivalent to the sentence ‘I am in Munich’, i.e., they express identical propositions.

Thus, given a possible world $x \in \mathcal{W}$, we can again consider the set of propositions $F_x \subseteq \mathcal{P}$ that are true in x (i.e., $F_x = \{a \in \mathcal{P} : x \models a\}$). And F_x again satisfies the features (1)–(5) above: If $a \in F_x$, i.e., $x \models a$, and a implies b , i.e., $a \leq b$, then $x \models b$, i.e., $b \in F_x$. If $a, b \in F_x$, then x makes true both a and b , so $a \wedge b \in F_x$. As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

Metaphysical completeness Each set of propositions $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines a possible world $x \in \mathcal{W}$, namely, the unique possible world making true exactly the propositions in F .

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions F satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to F without

making it inconsistent (i.e., making it contain \perp). Ersatzism not only claims that then there is a world x which makes true exactly the propositions in F , it even identifies this world x with F . Hence both the existence and the uniqueness of x follows.

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective correspondence between the set \mathcal{W} of possible worlds and the set $\overline{\mathcal{P}}$ of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets $F \in \overline{\mathcal{P}}$ will be called *prime filters* and $\overline{\mathcal{P}}$ will be called the *spectrum* of the algebra of propositions.)

$$\mathcal{W} \xleftrightarrow{\sim} \overline{\mathcal{P}}$$

$$x \mapsto F_x = \{a \in \mathcal{P} : x \models a\}$$

the x making true exactly the $a \in F \leftrightarrow F$

Let us verify that this really is a bijection: We have already checked that the function $f : \mathcal{W} \rightarrow \overline{\mathcal{P}}$ mapping x to F_x is well-defined. It is injective by the world individuation principle: if $x \neq y$, then there is a proposition a with $x \models a$ and $y \not\models a$ (or vice versa), so $a \in F_x$ and $a \notin F_y$ (or vice versa), so $F_x \neq F_y$. It is surjective by metaphysical completeness: Given $F \in \overline{\mathcal{P}}$, let x be the unique world in \mathcal{W} making true exactly the propositions in F . Then $F = F_x$ because: $a \in F$ iff $x \models a$ iff $a \in F_x$.

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of \mathcal{W}) and their ersatz constructions as sets of propositions (the elements of $\overline{\mathcal{P}}$). But what about the other side: How do full-blown propositions (the elements of \mathcal{P}) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition $a \in \mathcal{P}$ determines the set of worlds $\llbracket a \rrbracket := \{x \in \mathcal{W} : x \models a\}$ where a is true. This is also known as the *truthset* of a . And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write $\overline{\mathcal{W}}$ for the sets of worlds that determine propositions and $2^{\mathcal{W}}$ for the set of all sets of worlds. So our assumption for now is that $\overline{\mathcal{W}} = 2^{\mathcal{W}}$. Analogous to the previous case, we want to know if the function

$$\llbracket \cdot \rrbracket : \mathcal{P} \rightarrow 2^{\mathcal{W}}$$

$$a \mapsto \llbracket a \rrbracket = \{x \in \mathcal{W} : x \models a\}$$

This is assuming that the set of propositions forms what is known as a Boolean algebra.

A function $f : X \rightarrow Y$ is injective if $x \neq y$ implies $f(x) \neq f(y)$, it is surjective if for every $y \in Y$ there is $x \in X$ with $f(x) = y$, and it is bijective if it is both injective and surjective.

If X is a set, the powerset of X is the set of all subsets of X and it is denoted 2^X or $\mathcal{P}(X)$.

is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if $a \neq b$, there is a world x with $x \models a$ and $x \not\models b$ (or vice versa), so $\llbracket a \rrbracket \neq \llbracket b \rrbracket$. In fact, it also preserves the logical structure: $\llbracket a \wedge b \rrbracket = \llbracket a \rrbracket \cap \llbracket b \rrbracket$, $\llbracket \perp \rrbracket = \emptyset$, etc. (Later we formalize this as $\llbracket \cdot \rrbracket$ being a Boolean algebra homomorphism.) However, the issue is surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why $\llbracket \cdot \rrbracket$ is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a ‘finitistic’ grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset $2^{\mathcal{W}}$ of \mathcal{W} is uncountable. So \mathcal{P} and $2^{\mathcal{W}}$ have different cardinalities, which means there cannot be a bijection between them, hence the already injective function $\llbracket \cdot \rrbracket$ cannot be surjective.

That is Cantor’s diagonal argument.

So actually not any set of worlds determines a proposition, i.e., $\overline{\mathcal{W}}$ is a proper subset of $2^{\mathcal{W}}$. The ingenious insight of Stone, who discovered Stone duality, was to realize how to precisely describe this special subset $\overline{\mathcal{W}}$ of $2^{\mathcal{W}}$. The key idea is to realize that there is some additional structure on the set of worlds \mathcal{W} that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in chapter 3.

Also see exercise 1.e.

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

1.1.3 Logic: formulas/syntax vs models/semantics

Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches—that are very different in spirit—actually are equivalent. This also is a form of duality. Let’s explore this concretely.

Consider the language of classical propositional logic: sentences are formed from atomic sentences p_0, p_1, \dots using the connectives \wedge, \vee, \neg and the constants \perp and \top . And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence

calculus for classical logic—whichever you prefer. It consists of various axioms and rules to define the relation $\Gamma \vdash \varphi$, i.e., when the sentence φ is derivable in the proof-system S using as axioms the sentences in the set Γ . This is the syntactic description of the logic.

The model-theoretic description of the logic defines the relation $\Gamma \models \varphi$, i.e., that the sentence φ is a logical consequence of the sentences in Γ . This is done as follows. A valuation is a function $v : \{p_0, p_1, \dots\} \rightarrow \{0, 1\}$ that assigns each atomic sentences a truth-value, i.e., true (1) or false (0). This can be extended to all sentences: $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$; $v(\neg\varphi) = 1$ iff $v(\varphi) = 0$; $v(\perp) = 0$; etc. Then $\Gamma \models \varphi$ is defined as: for all valuations v , if $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\varphi) = 1$. Thus, logical consequence is truth-preservation.

Now, the completeness theorem for classical propositional logic states that: $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$. To be more precise, one often only calls the right-to-left implication ‘completeness’, and the left-to-right implication ‘soundness’. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserves logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences φ and ψ equivalent, written $\varphi \equiv \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. An equivalence class of a sentence φ is the set of sentences that are equivalent to it: $[\varphi] := \{\psi : \varphi \equiv \psi\}$. Write L for the set of all equivalence classes. It also has logical structure: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$; $\neg[\varphi] = [\neg\varphi]$, etc. L is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each valuation v determines a subset $F_v \subseteq L$: namely, those equivalence classes $[\varphi]$ with $v(\varphi) = 1$. Note again that F_v has features (1)–(5): If $[\varphi] \in F_v$ and $[\varphi] \leq [\psi]$ (i.e., $[\varphi \wedge \psi] = [\varphi]$), then $\varphi \vdash \psi$, so, by soundness, $\varphi \models \psi$, so, since $v(\varphi) = 1$, also $v(\psi) = 1$, so $[\psi] \in F_v$. If $[\varphi], [\psi] \in F_v$, then $v(\varphi) = 1$ and $v(\psi) = 1$, so $v(\varphi \wedge \psi) = 1$, so $[\varphi \wedge \psi] \in F_v$. Etc. Conversely, if $F \subseteq L$ satisfies (1)–(5), then v_F is a valuation mapping φ to 1 iff $[\varphi] \in F$. So, again, the set X of valuations is in bijective correspondence with the set \bar{L} of subsets of L satisfying (1)–(5).

But how does completeness follow? For this, first note that subsets of L are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if $\Gamma \not\vdash \varphi$, consider the deductive closure Γ' of Γ , i.e., the set of all sentences

that can be derived from Γ , so also $\Gamma' \not\models \varphi$. When we regard Γ' as a subset of L , this is, in formal terminology, a filter of L that does not intersect the ideal of all equivalence classes that imply $[\varphi]$. Now we use Stone's Prime Filter Theorem, which is at the heart of Stone duality and which we prove later on in the course. It says that we can extend this filter to a prime filter F which still does not intersect that ideal. Then v_F is a valuation that makes true all the premises in Γ but not the conclusion φ , hence $\Gamma \not\models \varphi$, as desired.

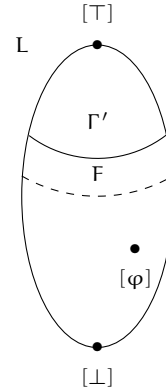
1.1.4 Further examples

Physics: observations vs states. Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). A physical system comes both with a *state space* X and an algebra \mathcal{A} of *observations* and these two again are dual in the sense that

- the states are determined by the observations that they give rise to,
- the observations are determined by the states that give rise to them.

The observations have logical structure: in a classical (as opposed to quantum) system, observing $A \wedge B$ means observing A and observing B , observing $A \vee B$ means observing A or observing B , etc. Each state x of the system determines a set of observations: namely, those that can be made if the system is in that state. Conversely, we can also start with the algebra of observations (they are empirically more accessible anyway) and postulate the states of the system as theoretical entities corresponding to certain subsets of observations.

Computation: observable properties vs denotations of programs. Computer programs are written in a programming language, and so, much like for sentences written in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation*. For example, the denotation of a program could be the (partial) function that it computes. Domain theory is the mathematical theory to systematically describe these meanings. There again also is a side that is dual to the side of meanings, and this was a crucial discovery in the development of domain theory by Abramsky (1991, p. 16). This is the side of *observable properties* of the computer programs. For example, it could be the property that, on input $x = 3$, the program halts and outputs $f(x) = 5$. Again, we would hope for a bidirectional determination in the sense that the meaning of a program is complete determined by its observable properties, and that these observable properties are determined by the denotations that have them.



Philosophy of science: observations vs theories.

to be added

Time: duration vs points.

to be added

1.2 Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will do this formally in the next chapters, but let's already give it an informal try here.

We had the following components in the examples:

- On the 'spatial' side, we have a set X , e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.
- On the 'algebraic' side, we have a set A , e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction (\wedge), disjunction (\vee), logical falsity (\perp), logical truth (\top), and possibly negation (\neg).
- We have a way to go from the spatial side to the algebraic side: each element of X is determined by a subsets of A with certain nice features, i.e., we have a bijective function $\epsilon : \overline{A} \rightarrow X$.
- We have a way to go from the algebraic side to the spatial side: each element from A can be assigned to a subset of X , so we have a bijective function $\eta : A \rightarrow \overline{X}$.

What's 'algebraic' about this? Algebra is the study of rules for combining objects or symbols. In school, this means combining symbols like $ax^2 + bx + c$ and studying when they equal another, like 0. Here it is about combining elements of A with the operations \wedge, \vee , etc.

Finally, we want this translation manual to be *formulaic* in X and A : i.e., it should not depend on the idiosyncrasies of the specific X and A ; rather, it should work for all X 's and A 's of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we want the above data for any X that is a candidate for the spatial side and for any A that is a candidate for the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates X , which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category consists of the algebraic candidates A and their relations. Then we will see

that all the above components of the duality is succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

1.3 Exercises

Exercise 1.a. Complete the left-out details in the main text. For example, why, for a possible world x the set of propositions F_x really satisfied properties (1)–(5). Similarly for valuations v .

Exercise 1.b. Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If $a \in \mathcal{P}$ is a property, then there also is the property $\neg a$ of not having property a . It seems plausible to require that either a given object $x \in \mathcal{O}$ has a property or it does not. In other words, either $a \in F_x$ or $\neg a \in F_x$. Do you think this is plausible to require? What about vague properties? (Later we see that this if if we have a negation operator on our set of properties obeying the Boolean laws, than F being prime is equivalent to having the just mentioned negation property.)

Exercise 1.c. As promised in the first example (section 1.1.1), this exercise asks you to scrutinize the concepts of objects and properties. Here are two questions and what duality theory might respond. Philosophically evaluate these answers.

1. *Russell's paradox*: We mentioned the worry that if we are too permissive in our conception of properties, Russell's paradox might creep in.

Response: For this paradox to occur, we would need the property a of being a property. However, such higher-order properties (i.e., properties of properties) are not considered in Stone duality, for the following reasons. First, the set \mathcal{O} of objects is considered to be disjoint from the set \mathcal{P} of properties. After all, the properties are 'isomorphic' to the (clopen) sets of objects, but sets of objects are not objects; similarly, objects are 'isomorphic' to sets of properties (that are prime filters), but sets of properties are not properties. Second, if we had the higher-order property a of being a property, we would have $\mathcal{P} \subseteq \mathcal{O}$. This is because the property a determines its extension

$\llbracket a \rrbracket \subseteq \mathcal{O}$, though, by definition of a , $\llbracket a \rrbracket = \mathcal{P}$. But $\mathcal{P} \subseteq \mathcal{O}$ contradicts disjointness (since \mathcal{P} is nonempty).

2. *Leibniz indiscernibility*: A trivial way to satisfy the principle of identity of indiscernibles is by acknowledging, for every object, the property of being x . This property is also known as the haecceity of the object x (see, e.g., Ladyman et al. 2012 for more on this). But, according to duality theory, are all haecceities really properties?

Response: When we introduce topology, we will see that (1) the space of objects is compact and (2) the extensions of properties are clopen sets. The extension of the haecceity of object x is the singleton $\{x\}$. So, if the haecceity of every object is a property, then, by (2), all singletons are clopen, which, by (1), can only happen if there only are finitely many objects to start with. In other words, according to duality theory, if there are infinitely many objects, not all haecceities can be properties. (Also see exercise 1.f.)

Exercise 1.d (More of a research project than an exercise). Consider to what extent the first example (objects vs properties) can be developed along the lines of **formal concept analysis**.

Exercise 1.e. Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

Exercise 1.f. For a logico-philosophical discussion of the principle of indiscernibility, see Ladyman et al. (2012). How does this inform the above philosophical discussion (section 1.1.1)? This paper is in the context of model theory, what does the above duality-theoretic perspective add? For a start, see exercise 1.c (2).

Exercise 1.g. Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states (computable theory of mind vs **connectionism**). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)? Or are these better seen as relations of supervenience rather than duality? What about the infamous Cartesian duality between the physical and the mental world?

Exercise 1.h. Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

2 The algebraic side: Boolean algebras

This chapter introduces formally the algebraic side of duality, which, for us, will be Boolean algebras. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. The next chapter will deal with the other, spatial side of the duality.

2.1 Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The ‘structure-preserving’ maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

Even if we do not need or formally introduce category, we follow one of its key lessons: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism).

For those interested in further reading on category theory, see, e.g., Leinster 2014, ch. 1. To be self-contained, I might eventually add an appendix on this.

2.1.1 Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than. But the things could also be the dishes offered at your go-to lunch place with the sense of ‘better’ given by your preferences. The formal definition goes as follows.

Definition 2.1. A *partial order* (or *partially ordered set*, or *poset*) is a pair (P, \leq) where P is a (possibly empty) set and \leq is a binary relation on P such that

A binary relation R on a set P is simply a subset of $P \times P = \{(a, b) : a, b \in P\}$. For $a, b \in P$, one writes aRb for $(a, b) \in R$.

1. *Reflexive*: For all $a \in P$, we have $a \leq a$.
2. *Transitive*: For all $a, b, c \in P$, if $a \leq b$ and $b \leq c$, then $a \leq c$.
3. *Anti-symmetric*: For all $a, b \in P$, if $a \leq b$ and $b \leq a$, then $a = b$.

If we do not require axiom 3, we speak of a *preorder*. We say \leq is a (partial or pre-) order on P . If the order \leq is clear from context, we often simply speak of the (partial or pre-) order P . We write $a < b$ if $a \leq b$ and $a \neq b$.

The name ‘partial’ is to indicate that not all elements need to be comparable: Formally, for $a, b \in P$, we say that a and b are *comparable*, if either $a \leq b$ or $b \leq a$; otherwise they are *incomparable*. If all elements are comparable, we say (P, \leq) is *linear* (or *total*).

Formally, the example of the numbers is (\mathbb{N}, \leq) where \mathbb{N} is the set $\{0, 1, 2, \dots\}$ and, for $n, m \in \mathbb{N}$, the relation $n \leq m$ is defined as: n is smaller or equal to m (equivalently, there is $k \in \mathbb{N}$ such that $n + k = m$). Hence this is a linear order. In the example of your lunch place, if you have two dishes a and b that you find equally tasty—or, more precisely, none tastier than the other, i.e., a and b are incomparable—, then your preference order is only partial and not linear.

Check that this satisfies the axioms.

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder (P, \leq) into a partial order $(\bar{P}, \bar{\leq})$ as follows. For $a, b \in P$, define $a \equiv b$ as $a \leq b$ and $b \leq a$. This is an equivalence relation (reflexive, symmetric, and transitive). Equivalence classes are the sets $[a] := \{b \in P : a \equiv b\}$ for $a \in P$. The quotient of P under \equiv is $\bar{P} := P / \equiv := \{[a] : a \in P\}$. Define $[a] \bar{\leq} [b]$ by $a \leq b$ (note that this is independent of the representatives a and b). This renders $(\bar{P}, \bar{\leq})$ a partial order. It is also called the *poset reflection* of P . Exercise 2.b makes formally precise in what sense it is the canonical or best possible poset approximating the preorder P .

There is a nice visualization of partial orders. They are known as *Hasse diagrams*. An example is in figure 2.1. It depicts the partial order (P, \leq) with $P = \{a, b, c, d\}$ and

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of P and the edges are the minimal information to recover the order:

- if there is an edge between x and y and x is lower (on the page) than y , then $x \leq y$.

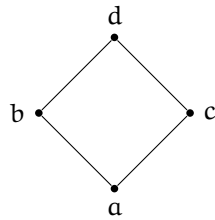


Figure 2.1: The ‘diamond’ as an example of a partial order.

- we do not need to draw an edge from one node to itself because for all nodes x we have $x \leq x$.
- we do not need to draw edges that result from composing existing edges: for example, we have an edge from a to b and an edge from b to d , so we already know that $a \leq d$, hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order (P, \leq) is as follows. For $a, b \in P$, we say that b *covers* a (short $a < b$) if $a < b$ and for all $c \in P$, if $a \leq c \leq b$, then $c = a$ or $c = b$. The elements of P are the nodes of the Hasse diagram, and an edge is drawn from node a to node b whenever b covers a . The direction of the edge is indicated by drawing b higher up in the diagram than a . So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

Definition 2.2. Let (P, \leq) be a partial order and $A \subseteq P$.

- An element $b \in P$ is a *lower bound* of A if, for all $a \in A$, we have $b \leq a$.
- An element $b \in P$ is an *upper bound* of A if, for all $a \in A$, we have $a \leq b$.
- An element $c \in P$ is an *infimum* or *greatest lower bound* of A if (1) c is a lower bound of A , and (2), for all lower bounds b of A , we have $b \leq c$.
- An element $c \in P$ is a *supremum* or *least upper bound* of A if (1) c is an upper bound of A , and (2), for all upper bounds b of A , we have $c \leq b$.

They can be confusing at first, but they really are worth learning. Make sure to draw little Hasse diagrams to illustrate the concepts and how they differ from each other (exercise 2.a).

- An element $b \in P$ is a *least* or *bottom* or *minimum* element of P , if, for all $a \in P$, we have $b \leq a$ (i.e., b is the supremum of $A = \emptyset$).
- An element $b \in P$ is a *greatest* or *top* or *maximum* element of P , if, for all $a \in P$, we have $a \leq b$ (i.e., b is the infimum of $A = \emptyset$).
- An element $b \in P$ is *minimal* if, for all $a \in P$, if $a \leq b$, then $a = b$.
- An element $b \in P$ is *maximal* if, for all $a \in P$, if $b \leq a$, then $b = a$.
- An element $b \in P$ is *minimal in A* if (1) $b \in A$ and (2) for all $a \in A$, if $a \leq b$, then $a = b$.
- An element $b \in P$ is *maximal in A* if (1) $b \in A$ and (2) for all $a \in A$, if $b \leq a$, then $b = a$.
- A is an *upset* if for all $a, b \in P$, if $a \in A$ and $a \leq b$, then $b \in A$.
- A is a *downset* if for all $a, b \in P$, if $b \in A$ and $a \leq b$, then $a \in A$.
- A is *directed* (aka up-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $a \leq c$ and $b \leq c$. (Equivalently, all finite subsets of A have an upper bound in A .)
- A is *filtered* (aka filtering or down-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $c \leq a$ and $c \leq b$. (Equivalently, all finite subsets of A have a lower bound in A .)

(These notions also make sense in a preorder (P, \leq) , but if P is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted $\bigwedge A$, called the *meet* of A ; and the supremum is denoted $\bigvee A$, called the *join* of A . If $A = \{a_1, \dots, a_n\}$ is finite and nonempty, we write $\bigwedge A = a_1 \wedge \dots \wedge a_n$ and $\bigvee A = a_1 \vee \dots \vee a_n$. In particular, $\bigwedge \{a, b\} = a \wedge b$ and $\bigvee \{a, b\} = a \vee b$. The bottom element, if it exists, is denoted \perp or 0 ; and the top element by \top or 1 . We write $\min(A)$ (resp. $\max(A)$) for the elements that are minimal (resp. maximal) in A . A *directed join* is the supremum of a directed set.

It is a good exercise to prove this.

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcpo*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can ‘turn them upside down’ and get another preorder. Formally, if (P, \leq) is a preorder, define the preorder \leq' on P by $a \leq' b$ iff $b \leq a$. We write P^{op} for this preorder.

Verify that this again is a preorder (resp. partial order), and draw some Hasse diagram example to see that this really turns things upside down.

2.1.2 Morphisms: Monotone maps

What maps between partial orders should be considered to be ‘structure preserving’? They should preserve the order structure, which yields the concept of a monotone map.

We consider the words ‘map’ and ‘function’ as synonymous.

Definition 2.3. Let (P, \leq_P) and (Q, \leq_Q) be two preorders and $f : P \rightarrow Q$ a function. We say f is

- *monotone* or *order preserving* if, for all $a, b \in P$, if $a \leq_P b$, then $f(a) \leq_Q f(b)$.

The converse notion is:

- *order reflecting* if, for all $a, b \in P$, if $f(a) \leq_Q f(b)$, then $a \leq_P b$.

We call f an *order-embedding* if it is both order preserving and order reflecting. Finally, f is an *order-isomorphism* if it is monotone and it has a *monotone inverse*, i.e., there is a monotone function $g : Q \rightarrow P$ such that

- for all $a \in P$, we have $a = g(f(a))$, i.e., a is the g -inverse of $f(a)$ (in short, $\text{id}_P = g \circ f$), and
- for all $b \in Q$, we have $f(g(b)) = b$, i.e., mapping the g -inverse of b along f yields b (in short, $f \circ g = \text{id}_Q$).

We say two preorders are *isomorphic* if there is an order isomorphism between them.

We can consider two isomorphic preorders to be essentially identical (because any order-theoretic property that one has, the other has, too). For partial orders, the notion of isomorphism can be simplified. The above definition captures the general (category-theoretic) concept of an isomorphism, but in practice the following is often easier to check.

Proposition 2.4. Let $f : P \rightarrow Q$ be a monotone function between posets. Then f is an isomorphism iff f is a surjective order-embedding.

Proof. Exercise 2.c. □

Order-embeddings between posets are injective, but the converse fails (i.e., there are injective order preserving maps between posets which are not order-embeddings. Here id_X denotes the identity function on set X . And if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions, $g \circ f$ (g after f) denotes their composition, which maps $x \in X$ to $g(f(x)) \in Z$.

In exercise 2.d, we show how the notion of an isomorphism can be generalized to that of an adjunction. This provides yet another notion of morphism between posets. We do not need it here, but since it is a very useful (but also abstract) concept in the vicinity of the presented concepts, we include it as an exercise. Exercise 2.e shows how such adjunction naturally occur once one has a relation between two sets.

2.2 Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

2.2.1 Objects: lattices

The order-theoretic definition of a lattice goes as follows.

Definition 2.5 (Lattice, order-theoretic). A (bounded) lattice is a partial order L in which every finite subset has a supremum and an infimum.

For example, the diamond of figure 2.1 is a lattice. Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.
2. Often a lattice is defined as a partial order in which all binary suprema and infima exist (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exists (i.e., which have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word 'bounded' (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.
3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

As an exercise, prove this.

Prove this. (Hint: think about the supremum of all lower bounds.)

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

Definition 2.6 (Lattice, algebraic). A lattice is a tuple $(L, \vee, \wedge, \perp, \top)$ where \vee (pronounced *join*) and \wedge (pronounced *meet*) are binary operations on L (i.e., functions $L \times L \rightarrow L$), and \perp (pronounced *bottom*) and \top (pronounced *top*) are elements of L , such that the following axioms holds:

1. *commutative*: for all $a, b \in L$, we have $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.
2. *associative*: for all $a, b, c \in L$, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.
3. *idempotent*: for all $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$.
4. *absorption*: for all $a, b \in L$, we have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.
5. *neutrality*: for all $a \in L$, we have $\perp \vee a = a$ and $\top \wedge a = a$.

For example, if X is a set, then the powerset 2^X forms a lattice in this algebraic sense with union \cup as join, intersection \cap as meet, \emptyset as bottom, and X as top. This also provides my mnemonic for remembering what ‘join’ and what ‘meet’ is. Think of X as a set of propositions, and let $a \in 2^X$ be the beliefs (opinions, values, etc.) that Alice holds, and let $b \in 2^X$ be the beliefs that Bob holds. Then the meet of a and b —i.e., $a \wedge b = a \cap b$ —is where Alice and Bob can meet: the common (meeting) ground, the set of beliefs they agree on. And the join of a and b —i.e., $a \vee b = a \cup b$ —is the result of joining Alice and Bob together: their joint beliefs, taking together all of their beliefs even if incoherent.

Though I’m happy to learn about a better one :-)

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.f asks you to prove it: that is a bit tedious, but quite instructive.

Theorem 2.7. 1. *Given a lattice $(L, \vee, \wedge, \perp, \top)$ according to the algebraic definition, define $a \leq_L b$ as $a \wedge b = a$. Then (L, \leq_L) is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by \vee and \wedge .*

2. *Given a lattice (L, \leq) according to the order-theoretic definition, define the binary operations \vee and \wedge as binary supremum and infimum, and take \perp and \top to be the least and greatest element of L . Then $(L, \vee, \wedge, \perp, \top)$ is a lattice according to the algebraic definition, with $a \wedge b = a$ iff $a \leq b$ iff $a \vee b = b$.*

From now on, we will often just speak of a lattice L and both use its order-theoretic definition (taking \leq to be implicitly given) and its algebraic definitions (taking $\vee, \wedge, \perp, \top$ to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure $(L, \cdot, 1)$, where \cdot is a commutative, associative, and idempotent binary operation on L , and 1 is a neutral element for the operation. The operation \cdot can then either be seen as the binary infimum for the partial order defined by $a \leq b$ iff $a \cdot b = a$ (the join semilattice), or as the binary supremum for the opposite partial order defined by $a \leq b$ iff $a \cdot b = b$ (the meet semilattice).

2.2.2 Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

Definition 2.8. A function $f : L \rightarrow M$ between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all $a, b \in L$, we have $f(a \vee_L b) = f(a) \vee_M f(b)$
2. for all $a, b \in L$, we have $f(a \wedge_L b) = f(a) \wedge_M f(b)$
3. $f(\perp_L) = \perp_M$
4. $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*.

Prove this.

If a function $f : L \rightarrow M$ between lattices preserves \perp and \vee , then it preserves all finite joins. This does, in general, *not* imply any preservation of arbitrary existing joins or preservation of infima. The analog statement is true for \top and \wedge and preservation of all finite meets.

Prove this.

2.3 Distributive lattices and Boolean algebras

2.3.1 Boolean algebras

2.4 Exercises

Exercise 2.a. Go through the partial order concepts defined in definition 2.2 and pick a few of them and draw (minimal) Hasse diagrams to show how they differ. For example, a maximal element that is not a greatest element; an upper bound that is not a greatest upper bound; or an upset that is not directed.

Exercise 2.b. Recall that for a preorder (P, \leq) , we have defined the poset reflection $(\bar{P}, \bar{\leq})$. This exercise makes precise in which sense this is the best possible poset approximating the preorder (P, \leq) .

Exercise 1.1.5 in Gehrke and van Gool (2023), with small changes.

1. Prove that \equiv is an equivalence relation.
2. Prove that the definition of $\bar{\leq}$ is independent of the representatives:
If $a' \in [a]$ and $b' \in [b]$, then $a \leq b$ iff $a' \leq b'$.
3. Prove that $(\bar{P}, \bar{\leq})$ is indeed a partial order.
4. Prove that $\bar{\leq}$ is the smallest partial order on $\bar{P} = P / \equiv$ such that the quotient map $f : P \rightarrow P / \equiv$, which maps a to $[a]$, is order preserving: That is, if \leq' is another such partial order on P / \equiv , then $\bar{\leq} \subseteq \leq'$.
5. Prove that, for any order preserving $g : P \rightarrow Q$ into a poset Q , there exists a unique order preserving $\bar{g} : P / \equiv \rightarrow Q$ such that $\bar{g} \circ f = g$. As a diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & P / \equiv \\ & \searrow g & \downarrow \bar{g} \\ & & Q \end{array}$$

The category-theoretic formulation of this fact is: the inclusion of the category of partial orders and monotone maps in the category of preorders and monotone maps has a left adjoint. *Adjoint functors* can be interpreted as formalizing the idea of finding a best possible approximation.

Think about how the last item formalizes the idea that $(\bar{P}, \bar{\leq})$ is the best possible poset approximating the preorder (P, \leq) .

Exercise 2.c. Prove proposition 2.4.

The next two exercises introduce the notion of an order adjunction (this is a special case of the notion of an adjoint functor). The first states the general definition and the second a common situation how they occur.

Exercise 2.d (Order adjunction). Let (P, \leq_P) and (Q, \leq_Q) be two preorders, and let $f : P \rightarrow Q$ and $g : Q \rightarrow P$ be monotone maps. The pair (f, g) is called an *adjunction*, with f the *left* or *lower adjoint* and g the *right* or *upper adjoint*, if, for all $a \in P$ and $b \in Q$,

$$f(a) \leq_Q b \text{ iff } a \leq_P g(b).$$

Exercise 1.1.8 in Gehrke and van Gool (2023).

Note that f occurs on the left of ' \leq ' and g on the right.

We also write this as $l : P \rightleftarrows Q : u$. An adjunction between P^{op} and Q is called a *Galois connection* or *contravariant adjunction*.

1. Prove that (f, g) is an adjunction iff

Reflect on how an adjunction then generalizes the notion of an isomorphism!

- for all $a \in P$, we have $a \leq_P g(f(a))$, i.e., the g -inverse of $f(a)$ is at least as good as a , and
- for all $b \in Q$, we have $f(g(b)) \leq_Q b$, i.e., mapping the g -inverse of b along f approximates b .

For the rest of this exercise, assume that (f, g) is an adjunction.

2. Prove that $f \circ g \circ f(a) \equiv f(a)$ and $g \circ f \circ g(b) \equiv g(b)$ for every $a \in P$ and $b \in Q$ (and $a \equiv b$ iff $a \leq b$ and $b \leq a$).
3. Conclude that, in particular, if P and Q are posets, then $fgf = f$ and $gfg = g$.
4. Prove that, if P is a poset, then for any $a \in P$, $gf(a)$ is the least element above a that lies in the image of g .
5. Formulate and prove a similar statement to the previous item about $fg(b)$, for $b \in Q$.
6. Prove that, for any subset $A \subseteq P$, if the supremum of A exists, then $f(\bigvee A) = \bigvee f(A)$ (where $f(A) = \{f(a) : a \in A\}$ is the image of A under f).
7. Prove that, for any subset $B \subseteq Q$, if the infimum of B exists, then $g(\bigwedge B) = \bigwedge g(B)$.

Reflect on how this shows that g is the best possible approximation to an inverse of f !

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

Exercise 2.e (Galois connection from a relation). Let $R \subseteq X \times Y$ be a relation between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define

$$u(a) := \{y \in Y : \forall x \in a. xRy\} \subseteq Y$$

$$l(b) := \{x \in X : \forall y \in b. xRy\} \subseteq X$$

Show that $l : \mathcal{P}(Y) \rightleftarrows \mathcal{P}(X) : u$ forms a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, i.e., for any $b \subseteq Y$ and $a \subseteq X$, we have $a \subseteq l(b)$ (i.e., $l(b) \subseteq^{\text{op}} a$) iff $b \subseteq u(a)$.

For those interested in further reading, here are three instances of this.

1. Maybe you know the name ‘Galois’ from the theory of fields in algebra. Then you know **Galois theory** as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation R between the set X of

*We will see that the converse holds for complete lattices. This is a special case of the **Adjoint Functor Theorem**.*

Here $\mathcal{P}(X)$ is the set of all subsets of the set X .

For an accessible introduction, take a look, e.g., at [this](#) or [this](#) video, or at [these](#) great lecture notes by Tom Leinster.

subfields of a given field and the set Y of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.

2. If X is a set and $R \subseteq X \times X$ is a preorder, then $u(a)$ is the set of upper bounds of $a \subseteq X$, and $l(b)$ is the set of lower bounds of $b \subseteq X$.
3. Consider a class of structures \mathcal{C} (in, say, a first-order signature) and a class \mathcal{F} of formulas (of this signature). Let \models be the *interpretation* relation: For $M \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ means that structure M makes true formula φ . Then for a set of models \mathfrak{a} , $u(\mathfrak{a})$ is the theory of \mathfrak{a} , i.e., the set of formulas that are true in all those models. And for a theory $b \subseteq \mathcal{F}$, $l(b)$ is the class of models of b , i.e., the set of models which make true all the sentences in b .

Also recall the examples from section 1.1.

Exercise 2.f. Prove theorem 2.7.

Exercise 2.g. This exercise shows that Boolean algebras and Boolean rings are equivalent.

From Gehrke and van Gool 2023, ex. 1.2.13.

1. Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, i.e., a commutative ring with unit in which $a \cdot a = a$ for all $a \in B$. Define $a \leq b$ if $a \cdot b = a$. (We often write ab for $a \cdot b$.) Prove that \leq is a distributive lattice order on B where
 - 1 is the greatest element and 0 is the least element,
 - meet is given by ab and join is given by $a + b + ab$, and
 - every element a of has the complement $1 + a$ with respect to \leq .

Hint: First show that $a + a = 0$ for all $a \in B$.

2. Conversely, let $(B, \wedge, \vee, \perp, \top, \neg)$ be a Boolean algebra. Define, for any $a, b \in B$,

$$\begin{aligned} a + b &:= (a \wedge \neg b) \vee (\neg a \wedge b) & a \cdot b &:= a \wedge b \\ 0 &:= \perp & 1 &:= \top. \end{aligned}$$

The operation $+$ is known as symmetric difference.

Prove that $(B, +, \cdot, 0, 1)$ is a Boolean ring.

3. Finally, show that the composition of these two assignments in either order yields the identity.

Exercise 2.h. To provide examples and non-examples about the notion of complement:

1. Show that the n -chain $\mathbf{n} = \{0, 1, \dots, n-1\}$ is a distributive lattice.
2. Show that, for $n = 2$, the two-element lattice $\mathbf{2} = \{0, 1\}$ is a Boolean algebra.
3. But for $n \geq 3$, \mathbf{n} is not a Boolean algebra. (Complements don't always exist.)
4. Show that the distributive lattices M_3 and N_5 from figure ?? are not Boolean algebras. (Complements exists but are not unique.)
5. Show that the complement of an element of a distributive lattice is unique if it exists.

Exercise 2.i. This exercise explores the difference and similarities of join-prime and join-irreducible elements. Let L be a lattice. Recall that an element $a \in L$ is

- join-prime if $a \neq \perp$ and for all $b, c \in L$, if $a \leq b \vee c$, then either $a \leq b$ or $a \leq c$.
- join-irreducible if $a \neq \perp$ and for all $b, c \in L$, if $a = b \vee c$, then either $a = b$ or $a = c$.

Now, establish the following relationships between these concepts.

1. Show that a is join-prime iff for all finite $S \subseteq L$, if $a \leq \bigvee S$, then there is $s \in S$ such that $a \leq s$.
2. Convince yourself that you can analogously show that a is join-irreducible iff for all finite $S \subseteq L$, if $a = \bigvee S$, then there is $s \in S$ such that $a = s$.
3. Show that if a is join-prime, then a is join-irreducible.
4. Show that the converse holds if L is distributive.
5. Provide an example of a lattice L which has a join-irreducible element a which is not join-prime.

Exercise 2.j. Let X be a poset and $\mathcal{D}(X)$ the set of its downsets. Show that $\mathcal{D}(X)$ is a sublattice of 2^X . In particular, intersections and unions of downsets are again downsets.

Exercise 2.k. Let $f : X \rightarrow Y$ be a function between two sets X and Y . Show that the function from 2^Y to 2^X defined by

$$B \mapsto f^{-1}(B)$$

is a Boolean algebra homomorphism. Show that if f additionally is a monotone function between two posets X and Y , then this mapping restricts to a lattice homomorphism $\mathcal{D}(Y) \rightarrow \mathcal{D}(X)$.

Exercise 2.1. In the context of proposition ??, show that if $f, f' : X \rightarrow Y$ are such that $\mathcal{D}(f) = \mathcal{D}(f')$, then $f = f'$.

3 The spatial side: topological spaces

4 Two sides of the same coin: Stone duality

5 Application: (Re)discovering semantics for modal logic

6 Generalization: Priestley duality

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