# Duality Theory

## Connecting Algebra and Topology via Logic

## Lecture Notes

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Comments welcome!

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# Contents

# Preface

This is the reader for the course "Duality Theory: Connecting Logic, Algebra, and Topology" given during the winter semester 2024/5 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. These lecture notes are updated as the course progresses. A website with all the course material is found at

https://levinhornischer.github.io/DualityTheory/.

**Comments** I'm happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just send an informal email to Levin.Hornischer@lmu.de.

**Course description and objectives** This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone's duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (totally disconnected compact Hausdorff spaces). The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,
- open sets of a space vs. points of the space,
- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory, after a philosophical motivation. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications—especially to philosophical phenomena—that make use of dualities by combining the often opposing advantages of the two perspectives.

**Prerequisites** An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course Philosophical Logic. In that course, I stress two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. These approaches are often equivalent, which is a special case of duality.

**Contents** We start with an informal chapter describing the key idea of duality. The rest of the course is about developing this key idea precisely. We first precisely define the algebraic structure (Boolean algebras) and the topological structures (topological spaces), and then prove the duality result. The remainder of the course is about applying this result to modal logic (and sketching applications in computer science) and generalizing the result to Priestley duality.

**Layout** These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you'd rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end. Exercises are at the end of each chapter.

*This is a margin note.*

This is a todo note

**Study material** The main textbook that we use is by Gehrke and van Gool (2023). And informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press
- B. A. Davey and H. A. Pristley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press
- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press
- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag
- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer

- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser
- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory are

- P. T. Johnstone (1982). *Stone Spaces*. Cambrdige studies in advanced mathematics 3. Cambridge: Cambridge University Press
- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press
- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: 10.1017/9781316543870
- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press
- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser
- S. Abramsky and A. Jung (1994). "Domain Theory." In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf (last checked 24 January 2018). Oxford: Oxford University Press
- E. Orłowska et al. (2015). *Dualities for Structures of Applied Logic*. Studies in Logic 56. College Publications

**Notation** Throughout, 'iff' abbreviates 'if and only if'.

**Acknowledgments** Many thanks to the participants of course—both past and present—as well to the colleagues whom I have discussed the material with. In particular, thank you for very helpful comments: Javier Belastegui Lazcano, Zimin Cheng, Aleksandar Nikolic, Ioannis Polychronopoulos, and Emils Zavelis.

# 1. Introduction: the key idea of duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

*For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).*

*Advice on how to read this chapter*. Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the 'rhyme and reason' to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

*To use the words of Abramsky (2023).*

Duality theory can be quite abstract. The advantage of this is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. So before attempting any general definition of duality, let us consider several examples (section 1.1). From those we can generalize an informal characterization of duality (section 1.2). This then hints at how duality theory is formalized mathematically and how it can be applied. Finally, in section 1.3 we list some exercises.

## 1.1. Intuitive examples of duality

We present several examples of duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

*I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.*

### 1.1.1. Metaphysics: Properties vs objects

When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon, and we will here also include merely possible objects like unicorns. Properties are, for example, being red, being higher than 300m, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Philosophically, it is difficult to make this talk of objects and properties precise (e.g., if we are too permissive about what counts as a property, Russell's paradox creeps in). For now, let us just rely on our everyday intuitions about these concepts. Once we see where this will lead, the exercises at the end of this chapter will ask you to come back and scrutinize the concept of object and property at play (see exercise 1.c).

*If 'being a property' is a property, consider the property $p$ of 'not being a property'. Then $p$ has $p$ iff $p$ does not have $p$, contradiction. But, arguably, 'being a property' is not an 'everyday' property.*

Let us write $\mathcal{O}$ for the set of all possible objects and $\mathcal{P}$ for the set of all properties. Crucially, observe that there is a certain dependency between $\mathcal{O}$ and $\mathcal{P}$:

**($\mathcal{O} \to \overline{\mathcal{P}}$)** Each object $x \in \mathcal{O}$ determines a set of properties $F_x \subseteq \mathcal{P}$ consisting of precisely those properties that $x$ has.

*Philosophers also call $F_x$ the* role *of the individual $x$ (McMichael 1983, p. 57).*

(The bar in '$\overline{\mathcal{P}}$' indicates that we assign to each $x$ a *set* of elements in $\mathcal{P}$ rather than a single element of $\mathcal{P}$.) So we might wonder whether we can also go in the opposite direction ($\overline{\mathcal{P}} \to \mathcal{O}$)? Does a subset $F$ of properties also determine an object, i.e., the unique object that has exactly the properties in $F$? Actually, no: some sets of properties might not be satisfied by any object (e.g., $F = \{$*being exactly* $300m$ *high*, *being exactly* $200m$ *high*$\}$) or by more than one (e.g., $F = \{$*being exactly* $300m$ *high*$\}$).

*Philosophers know phrases of the form 'The $F$' (referring to the unique object satisfying $F$) as definite description. For their important role in philosophy, see e.g. Ludlow (2022).*

But let us not give up too early. After all, the set $F_x$ is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if $F$ is a set of properties with these nice features, that then it determines a unique object.)

1. Assume $a, b \in \mathcal{P}$ are two properties such that having $a$ implies having $b$; we abbreviate this as $a \leqslant b$. For example,

$$a = \textit{being higher than } 300m \leqslant \textit{being higher than } 200m = b.$$

   So if our object $x$ has property $a$, then it also has property $b$, i.e., if $a \in F_x$, then $b \in F_x$. We may express this as: $F_x$ is closed under implication.

*Later we will say $F_x$ is an* upset*. This sounds funny now, but by the end of the course, you will have said this so often that you won't even notice.*

2. Assume $a, b \in \mathcal{P}$ are two properties. Note that then there is another property: namely, the property of having both property $a$ and property $b$. We denote this property $a \wedge b$. So $a \wedge b$ is again in $\mathcal{P}$ and we have $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$. Moreover, if our object $x$ has property $a$ and it has property $b$, then it has property $a \wedge b$, i.e., if $a, b \in F_x$, then $a \wedge b \in F_x$. We may express this as: $F_x$ is closed under conjunction.

3. Similarly, if $a, b \in \mathcal{P}$ are two properties, there also is the property of having either property $a$ or property $b$ (or both). We denote this property $a \vee b$. So $a \vee b$ is again in $\mathcal{P}$ and we have $a \leqslant a \vee b$ and $b \leqslant a \vee b$. Moreover, if our object $x$ has property $a \vee b$, then either it has property $a$ or it has property $b$, i.e., if $a \vee b \in F_x$, then either $a \in F_x$ or $b \in F_x$. Later, we express this as $F_x$ being prime.

*I will always read 'either $A$ or $B$' as inclusive-or (either only $A$ is the case, or only $B$ is the case, or both $A$ and $B$ are the case)*

*Cf. a number $p > 1$ is prime iff (that is Euclid's lemma), for all numbers $a$ and $b$, if $a \times b$ is divided by $p$, then either $a$ is divided by $p$ or $b$ is divided by $p$).*

4. Note that $\mathcal{P}$ also contains the trivial property like being identical to oneself. We denote this property $\top$. In particular, our object $x$ has it, i.e., $\top \in F_x$.

5. Similarly, note that $\mathcal{P}$ also contains the inconsistent property like not being identical to oneself. We denote this property $\perp$. In particular, our object $x$ does not have it, i.e., $\perp \notin F_x$.

Now, we can ask our question again: If $F$ is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in $F$? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

*Or is the list (1)–(5) not complete because we should also add a principle concerning negation: you can think about this in exercise 1.b.*

**($\overline{\mathcal{P}} \to \mathcal{O}$)** Each set of properties $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines an object $x \in \mathcal{O}$, namely, the unique object having exactly the properties in $F$.

The uniqueness part is close to Leibniz's principle about the identity of indiscernibles: if two objects $x$ and $x'$ have exactly the properties in $F$, they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description $F$ of an object, there in fact is a (possible) object that has these properties. This is why we consider the set $\mathcal{O}$ of all possible objects. The actual world need not be ontologically complete: $F$ might consistently describe a unicorn, even if this does not exist in the actual world.

*Actually, I don't know if a principle like this is considered in metaphysics: if you do, please let me know :-) Also see exercise 1.d asking for a comparison to formal concept analysis.*

We will see that this bidirectional determination ($\mathcal{O} \to \overline{\mathcal{P}}$) and ($\overline{\mathcal{P}} \to \mathcal{O}$) is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensionality principle: two properties $a$ and $b$ are identical if they apply to exactly the same possible objects (i.e., for all $x \in \mathcal{O}$, $x$ has $a$ iff $x$ has $b$). Each property $a$ determines a set of objects: namely, the set of those objects that have property $a$. This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

*Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements.*

*Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.*

### 1.1.2. Semantics: Propositions vs possible worlds

The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is 400$m$ high. So the proposition $a$ expressed by the sentence 'The Eiffel Tower is 330$m$ high' contains the actual world $x_0$ (i.e., $x_0 \in a$) but not the just described possible world $x_1$ (i.e., $x_1 \notin a$). Some common notation for the phrase 'world $x$ makes true proposition $a$' is $x \vDash a$; so possible world semantics analyses $\vDash$ as elementhood $\in$.

There is much debate in philosophy what the set $\mathcal{W}$ of possible worlds is (Menzel 2021) and what the set $\mathcal{P}$ of propositions is (McGrath and Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds ('worlds first, propositions later')? Or is it rather that worlds are maximally consistent sets of propositions ('propositions first, worlds later')? The latter goes by the name 'ersatzism' since full-blown possible worlds are substituted by something constructed out of linguistic entities—

and 'Ersatz' is German for substitute.

We won't enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

*World individuation* Possible worlds are individuated by the propositions they make true: if two possible worlds $x$ and $y$ make true exactly the same propositions (i.e., for every proposition $a$, we have $x \vDash a$ iff $y \vDash a$), then $x = y$.

*Cf. Leibniz's above principle about the identity of indiscernibles.*

*Proposition individuation* Propositions are individuated by the possible worlds at which they are true: if two propositions $a$ and $b$ are true at exactly the same possible worlds (i.e., for every possible world $x$, we have $x \vDash a$ iff $x \vDash b$), then $a = b$.

*A hyperintensional account of propositions would contest this; see Berto and Nolan (2021).*

And there is more. Just like properties, also the set of propositions has logical structure: If $a$ and $b$ are propositions, there also are the propositions $a \wedge b$ (conjunction), $a \vee b$ (disjunction), $\neg a$ (negation), $\top$ (logical truth), and $\perp$ (logical falsity). With this we can also express implications between propositions: proposition $a$ implies proposition $b$, written $a \leqslant b$, precisely if $a \wedge b = a$. The proposition expressed by 'I am in Munich' implies the proposition expressed by 'I am in Germany' because the sentence 'I am in Munich and I am in Germany' is equivalent to the sentence 'I am in Munich', i.e., they express identical propositions.

Thus, given a possible world $x \in \mathcal{W}$, we can again consider the set of propositions $F_x \subseteq \mathcal{P}$ that are true in $x$ (i.e., $F_x = \{a \in \mathcal{P} : x \vDash a\}$). And $F_x$ again satisfies the features (1)–(5) above: If $a \in F_x$, i.e., $x \vDash a$, and $a$ implies $b$, i.e., $a \leqslant b$, then $x \vDash b$, i.e., $b \in F_x$. If $a, b \in F_x$, then $x$ makes true both $a$ and $b$, so $a \wedge b \in F_x$. As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

*Metaphysical completeness* Each set of propositions $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines a possible world $x \in \mathcal{W}$, namely, the unique possible world making true exactly the propositions in $F$.

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions $F$ satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to $F$ without making it inconsistent (i.e., making it contain $\perp$). Ersatzism not only claims that then there is a world $x$ which makes true exactly the propositions in $F$, it even identifies this world $x$ with $F$. Hence both the existence and the uniqueness of $x$ follows.

*This is assuming that the set of propositions forms what is known as a Boolean algebra.*

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective correspondence between the set $\mathcal{W}$ of possible worlds and the set $\overline{\mathcal{P}}$ of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets $F \in \overline{\mathcal{P}}$ will be called *prime filters* and $\overline{\mathcal{P}}$ will be called the *spectrum* of the algebra of propositions.)

$$
\begin{aligned}
\mathcal{W} &\leftrightarrows \overline{\mathcal{P}} \\
x &\mapsto F_x = \{a \in \mathcal{P} : x \vDash a\} \\
\text{the } x \text{ making true exactly the } a \in F &\leftarrow\!\shortmid F
\end{aligned}
$$

Let us verify that this really is a bijection: We have already checked that the function $f : \mathcal{W} \to \overline{\mathcal{P}}$ mapping $x$ to $F_x$ is well-defined. It is injective by the world individuation principle: if $x \neq y$, then there is a proposition $a$ with $x \vDash a$ and $y \nvDash a$ (or vice versa), so $a \in F_x$ and $a \notin F_y$ (or vice versa), so $F_x \neq F_y$. It is surjective by metaphysical completeness: Given $F \in \overline{\mathcal{P}}$, let $x$ be the unique world in $\mathcal{W}$ making true exactly the propositions in $F$. Then $F = F_x$ because: $a \in F$ iff $x \vDash a$ iff $a \in F_x$.

*A function $f : X \to Y$ is injective if $x \neq y$ implies $f(x) \neq f(y)$, it is surjective if for every $y \in Y$ there is $x \in X$ with $f(x) = y$, and it is bijective if it is both injective and surjective.*

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of $\mathcal{W}$) and their ersatz constructions as sets of propositions (the elements of $\overline{\mathcal{P}}$). But what about the other side: How do full-blown propositions (the elements of $\mathcal{P}$) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition $a \in \mathcal{P}$ determines the set of worlds $[\![a]\!] := \{x \in \mathcal{W} : x \vDash a\}$ where $a$ is true. This is also known as the *truthset* of $a$. And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write $\overline{\mathcal{W}}$ for the sets of worlds that determine propositions and $2^{\mathcal{W}}$ for the set of all sets of worlds. So our assumption for now is that $\overline{\mathcal{W}} = 2^{\mathcal{W}}$. Analogous to the previous case, we want to know if the function

*If $X$ is a set, the powerset of $X$ is the set of all subsets of $X$ and it is denoted $2^X$ or $\mathcal{P}(X)$.*

$$
\begin{aligned}
[\![\cdot]\!] &: \mathcal{P} \to 2^{\mathcal{W}} \\
a &\mapsto [\![a]\!] = \{x \in \mathcal{W} : x \vDash a\}
\end{aligned}
$$

is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if $a \neq b$, there is a world $x$ with $x \vDash a$ and $x \nvDash b$ (or vice versa), so $[\![a]\!] \neq [\![b]\!]$. In fact, it also preserves the logical structure: $[\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$, $[\![\bot]\!] = \emptyset$, etc. (Later we formalize this as $[\![\cdot]\!]$ being a Boolean algebra homomorphism.) However, the issue is surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why $[\![\cdot]\!]$ is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a 'finitistic' grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset $2^{\mathcal{W}}$ of $\mathcal{W}$ is uncountable. So $\mathcal{P}$ and $2^{\mathcal{W}}$ have different cardinalities, which means there cannot be a bijection between them, hence the already injective function $[\![\cdot]\!]$ cannot be surjective.

*That is Cantor's diagonal argument.*

So actually not any set of worlds determines a proposition, i.e., $\overline{\mathcal{W}}$ is a proper subset of $2^{\mathcal{W}}$. The ingenious insight of Stone, who discovered Stone duality, was to realize how to precisely describe this special subset $\overline{\mathcal{W}}$ of $2^{\mathcal{W}}$. The key idea is to realize that there is some additional structure on the set of worlds $\mathcal{W}$ that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in chapter 3.

*Also see exercise 1.e.*

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

### 1.1.3. Logic: formulas/syntax vs models/semantics

Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches—that are very different in spirit—actually are equivalent. This also is a form of duality. Let's explore this concretely.

Consider the language of classical propositional logic: sentences are formed from atomic sentences $p_0, p_1, \ldots$ using the connectives $\wedge, \vee, \neg$ and the constants $\bot$ and $\top$. And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence calculus for classical logic—whichever you prefer. It consists of various axioms and rules to define the relation $\Gamma \vdash \varphi$, i.e., when the sentence $\varphi$ is derivable in the proof-system S using as axioms the sentences in the set $\Gamma$. This is the syntactic description of the logic.

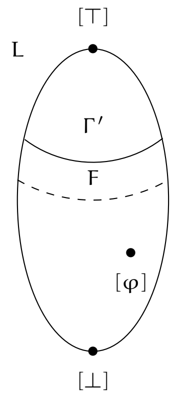
The model-theoretic description of the logic defines the relation $\Gamma \vDash \varphi$, i.e., that the sentence $\varphi$ is a logical consequence of the sentences in $\Gamma$. This is done as follows. A valuation is a function $\nu : \{p_0, p_1, \ldots\} \to \{0, 1\}$ that assigns each atomic sentences a truth-value, i.e., true (1) or false (0). This can be extended to all sentences: $\nu(\varphi \wedge \psi) = 1$ iff $\nu(\varphi) = 1$ and $\nu(\psi) = 1$; $\nu(\neg\varphi) = 1$ iff $\nu(\varphi) = 0$; $\nu(\bot) = 0$; etc. Then $\Gamma \vDash \varphi$ is defined as: for all valuations $\nu$, if $\nu(\psi) = 1$ for all $\psi \in \Gamma$, then $\nu(\varphi) = 1$. Thus, logical consequence is truth-preservation.

Now, the completeness theorem for classical propositional logic states that: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$. To be more precise, one often only calls the right-to-left implication 'completeness', and the left-to-right implication 'soundness'. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserves logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences $\varphi$ and $\psi$ equivalent, written $\varphi \equiv \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. An equivalence class of a sentence $\varphi$ is the set of sentences that are equivalent to it: $[\varphi] := \{\psi : \varphi \equiv \psi\}$. Write L for the set of all equivalence classes. It also has logical structure: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$; $\neg[\varphi] = [\neg\varphi]$, etc. L is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each valuation $\nu$ determines a subset $F_\nu \subseteq L$: namely, those equivalence classes $[\varphi]$ with $\nu(\varphi) = 1$. Note again that $F_\nu$ has features (1)–(5): If $[\varphi] \in F_\nu$ and $[\varphi] \leqslant [\psi]$ (i.e., $[\varphi \wedge \psi] = [\varphi]$), then $\varphi \vdash \psi$, so, by soundness, $\varphi \vDash \psi$, so, since $\nu(\varphi) = 1$, also $\nu(\psi) = 1$, so $[\psi] \in F_\nu$. If $[\varphi], [\psi] \in F_\nu$, then $\nu(\varphi) = 1$ and $\nu(\psi) = 1$, so $\nu(\varphi \wedge \psi) = 1$, so $[\varphi \wedge \psi] \in F_\nu$. Etc. Conversely, if $F \subseteq L$ satisfies (1)–(5), then $\nu_F$ is a valuation mapping $\varphi$ to 1 iff $[\varphi] \in F$. So, again, the set X of valuations is in bijective correspondence with the set $\overline{L}$ of subsets of L satisfying (1)–(5).

But how does completeness follow? For this, first note that subsets of L are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if $\Gamma \nvdash \varphi$, consider the deductive closure $\Gamma'$ of $\Gamma$, i.e., the set of all sentences that can be derived from $\Gamma$, so also $\Gamma' \nvdash \varphi$. When we regard $\Gamma'$ as a subset of $L$, this is, in formal terminology, a filter of $L$ that does not intersect the ideal of all equivalence classes that imply $[\varphi]$. Now we use Stone's Prime Filter Theorem, which is at the heart of Stone duality and which we prove later on in the course. It says that we can extend this filter to a prime filter $F$ which still does not intersect that ideal. Then $v_F$ is a valuation that makes true all the premises in $\Gamma$ but not the conclusion $\varphi$, hence $\Gamma \nvDash \varphi$, as desired.




### 1.1.4. Further examples

*Physics: observations vs states*. Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). A physical system comes both with a *state space* $X$ and an algebra $\mathcal{A}$ of *observations* and these two again are dual in the sense that

- the states are determined by the observations that they give rise to,
- the observations are determined by the states that give rise to them.

The observations have logical structure: in a classical (as opposed to quantum) system, observing $A \wedge B$ means observing $A$ and observing $B$, observing $A \vee B$ means observing $A$ or observing $B$, etc. Each state $x$ of the system determines a set of observations: namely, those that can be made if the system is in that state. Conversely, we can also start with the algebra of observations (they are empirically more accessible anyway) and postulate the states of the system as theoretical entities corresponding to certain subsets of observations.

Also done in general relativity: e.g., in the substantivalist vs relationalist debate (Wu and Weatherall 2023)

*Computation: observable properties vs denotations of programs*. Computer programs are written in a programming language, and so, much like for sentences written in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation*. For example, the denotation of a program could be the (partial) function that it computes. Domain theory is the mathematical theory to systematically describe these meanings. There again also is a side that is dual to the side of meanings, and this was a crucial discovery in the development of domain theory by Abramsky (1991, p. 16). This is the side of *observable properties* of the computer programs. For example, it could be the property that, on input $x = 3$, the program halts and outputs $f(x) = 5$. Again, we would hope for a bidirectional determination in the sense that the meaning of a program is complete determined by its observable properties, and that these observable properties are determined by the denotations that have them.

*Philosophy of science: observations vs theories.* to be added

*Time: duration vs points.* to be added

## 1.2. Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will do this formally in the next chapters, but let's already give it an informal try here.

We had the following components in the examples:

- On the 'spatial' side, we have a set $X$, e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.

- On the 'algebraic' side, we have a set $A$, e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction ($\bigwedge$), disjunction ($\bigvee$), logical falsity ($\bot$), logical truth ($\top$), and possibly negation ($\neg$).

- We have a way to go from the spatial side to the algebraic side: each element of $X$ is determined by a subsets of $A$ with certain nice features, i.e., we have a bijective function $\epsilon : \overline{A} \to X$.

- We have a way to go from the algebraic side to the spatial side: each element from $A$ can be assigned to a subset of $X$, so we have a bijective function $\eta : A \to \overline{X}$.

*What's 'algebraic' about this? Algebra is the study of rules for combining objects or symbols. In school, this means combining symbols like* $ax^2 + bx + c$ *and studying when they equal another, like* $0$*. Here it is about combining elements of* $A$ *with the operations* $\bigwedge, \bigvee$*, etc.*

Finally, we want this translation manual to be *formulaic* in $X$ and $A$: i.e., it should not depend on the idiosyncrasies of the specific $X$ and $A$; rather, it should work for all $X$'s and $A$'s of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we want the above data for any $X$ that is a candidate for the spatial side and for any $A$ that is a candidate for the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates $X$, which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category consists of the algebraic candidates $A$ and their relations. Then we will see that all the above components of the duality is succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

## 1.3. Exercises

**Exercise 1.a.** Complete the left-out details in the main text. For example, why, for a possible world $x$ the set of propositions $F_x$ really satisfied properties (1)–(5). Similarly for valuations $v$.

**Exercise 1.b.** Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If $a \in \mathcal{P}$ is a property, then there also is the property $\neg a$ of not having property $a$. It seems plausible to require that either a given object $x \in \mathcal{O}$ has a property or it does not. In other words, either $a \in F_x$ or $\neg a \in F_x$. Do you think this is plausible to require? What about vague properties? (Later we see that this if if we have a negation operator on our set of properties obeying the Boolean laws, than $F$ being prime is equivalent to having the just mentioned negation property.)

**Exercise 1.c.** As promised in the first example (section 1.1.1), this exercise asks you to scrutinize the concepts of objects and properties. Here are two questions and what duality theory might respond. Philosophically evaluate these answers.

1. *Russell's paradox*: We mentioned the worry that if we are too permissive in our conception of properties, Russell's paradox might creep in.

   *Response*: For this paradox to occur, we would need the property $a$ of being a property. However, such higher-order properties (i.e., properties of properties) are not considered in Stone duality, for the following reasons. First, the set $\mathcal{O}$ of objects is considered to be disjoint from the set $\mathcal{P}$ of properties. After all, the properties are 'isomorphic' to the (clopen) sets of objects, but sets of objects are not objects; similarly, objects are 'isomorphic' to sets of properties (that are prime filters), but sets of properties are not properties. Second, if we had the higher-order property $a$ of being a property, we would have $\mathcal{P} \subseteq \mathcal{O}$. This is because the property $a$ determines its extension

$[\![a]\!] \subseteq \mathcal{O}$, though, by definition of $a$, $[\![a]\!] = \mathcal{P}$. But $\mathcal{P} \subseteq \mathcal{O}$ contradicts disjointness (since $\mathcal{P}$ is nonempty).

2. *Leibniz indiscernibility*: A trivial way to satisfy the principle of identity of indiscernibles is by acknowledging, for every object, the property of being $x$. This property is also known as the haecceity of the object $x$ (see, e.g., Ladyman et al. 2012 for more on this). But, according to duality theory, are all haecceities really properties?

   *Response*: When we introduce topology, we will see that (1) the space of objects is compact and (2) the extensions of properties are clopen sets. The extension of the haecceity of object $x$ is the singleton $\{x\}$. So, if the haecceity of every object is a property, then, by (2), all singletons are clopen, which, by (1), can only happen if there only are finitely many objects to start with. In other words, according to duality theory, if there are infinitely many objects, not all haecceities can be properties. (Also see exercise 1.f.)

**Exercise 1.d** (More of a research project than an exercise)**.** Consider to what extend the first example (objects vs properties) can be developed along the lines of formal concept analysis.

**Exercise 1.e.** Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

**Exercise 1.f.** For a logico-philosophical discussion of the principle of indiscernibility, see Ladyman et al. (2012). How does this inform the above philosophical discussion (section 1.1.1)? This paper is in the context of model theory, what does the above duality-theoretic perspective add? For a start, see exercise 1.c (2).

**Exercise 1.g.** Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states (computable theory of mind vs connectionism). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)? Or are these better seen as relations of supervenience rather than duality? What about the infamous Cartesian duality between the physical and the mental world?

**Exercise 1.h.** Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

# 2. The algebraic side: Boolean algebras

This chapter introduces formally the algebraic side of duality, which, for us, will be Boolean algebras. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. The next chapter will deal with the other, spatial side of the duality. If you would like to refresh the standard set-theoretic terminology (which we use through-out the course), see appendix A.1.

## 2.1. Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The 'structure-preserving' maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

Even if we do not need category theory, we follow one of its keys lessons: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism).

For those interested in further reading on category theory, see, e.g., Leinster 2014, ch. 1. Eventually, I might add this into appendix A.2.

### 2.1.1. Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than. But the things could also be the dishes offered at your go-to lunch place with the sense of 'better' given by your preferences. Or, following the guiding intuition from chapter 1, the things can be propositions (or properties, etc.) ordered by implication: $b$ is 'bigger' than $a$ if $a$ implies $b$. The formal definition goes as follows.

**Definition 2.1.** A *partial order* (or *partially ordered set*, or *poset*) is a pair $(P, \leqslant)$ where $P$ is a (possibly empty) set and $\leqslant$ is a binary relation on $P$ such that

*A binary relation $R$ on a set $P$ is simply a subset of $P \times P = \{(a, b) : a, b \in P\}$. For $a, b \in P$, one writes $aRb$ for $(a, b) \in R$.*

1. *Reflexive*: For all $a \in P$, we have $a \leqslant a$.

2. *Transitive*: For all $a, b, c \in P$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

3. *Anti-symmetric*: For all $a, b \in P$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

If we do not require axiom 3, we speak of a *preorder*. We say $\leqslant$ is a (partial or pre-) order on $P$. If the order $\leqslant$ is clear from context, we often simply speak of the (partial or pre-) order $P$. We write $a < b$ if $a \leqslant b$ and $a \neq b$.

The name 'partial' is to indicate that not all elements need to be comparable: Formally, for $a, b \in P$, we say that $a$ and $b$ are *comparable*, if either $a \leqslant b$ or $b \leqslant a$; otherwise they are incomparable. If all elements are comparable, we say $(P, \leqslant)$ is *linear* (or *total*).

Formally, the example of the numbers is $(\mathbb{N}, \leqslant)$ where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ and, for $n, m \in \mathbb{N}$, the relation $n \leqslant m$ is defined as: $n$ is smaller or equal to $m$ (equivalently, there is $k \in \mathbb{N}$ such that $n + k = m$). Hence this a linear order. In the example of your lunch place, if you have two dishes $a$ and $b$ that you find equally tasty—or, more precisely, none tastier than the other, i.e., $a$ and $b$ are incomparable—, then your preference order is only partial and not linear.

*Check that this satisfies the axioms.*

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder $(P, \leqslant)$ into a partial order $(\overline{P}, \overline{\leqslant})$ as follows. For $a, b \in P$, define $a \equiv b$ as $a \leqslant b$ and $b \leqslant a$. This is an equivalence relation. Equivalence classes are the sets $[a] := \{b \in P : a \equiv b\}$ for $a \in P$. The quotient of $P$ under $\equiv$ is $\overline{P} := P/\equiv := \{[a] : a \in P\}$. Define $[a]\overline{\leqslant}[b]$ by $a \leqslant b$ (note that this is independent of the representatives $a$ and $b$). This renders $(\overline{P}, \overline{\leqslant})$ a partial order. It is also called the *poset reflection* of $P$. Exercise 2.c makes formally precise in what sense it is the canonical or best possible poset approximating the preorder $P$.

*See appendix A.1 for terminology around equivalence classes.*

There is a nice visualization of partial orders. They are known as *Hasse diagrams*. An example is in figure 2.1. It depicts the partial order $(P, \leqslant)$ with $P = \{a, b, c, d\}$ and

$$\leqslant := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of $P$ and the edges are the minimal information to recover the order:

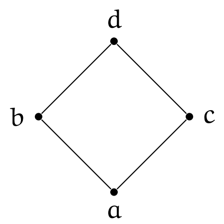



Figure 2.1.: The 'diamond' as an example of a partial order.

- if there is an edge between $x$ and $y$ and $x$ is lower (on the page) than $y$, then $x \leqslant y$.
- we do not need to draw an edge from one node to itself because for all nodes $x$ we have $x \leqslant x$.
- we do not need to draw edges that result from composing existing edges: for example, we have an edge from $a$ to $b$ and an edge from $b$ to $d$, so we already know that $a \leqslant d$, hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order $(P, \leqslant)$ is as follows. For $a, b \in P$, we say that $b$ *covers* $a$ (short $a \lessdot b$) if $a < b$ and for all $c \in P$, if $a \leqslant c \leqslant b$, then $c = a$ or $c = b$. The elements of $P$ are the nodes of the Hasse diagram, and an edge is drawn from node $a$ to node $b$ whenever $b$ covers $a$. The direction of the edge is indicated by drawing $b$ higher up in the diagram than $a$. So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

*They can be confusing at first, but they really are worth learning. Make sure to draw little Hasse diagrams to illustrate the concepts and how they differ from each other (exercise 2.b).*

**Definition 2.2.** Let $(P, \leqslant)$ be a partial order and $A \subseteq P$.

- An element $b \in P$ is a *lower bound* of $A$ if, for all $a \in A$, we have $b \leqslant a$.
- An element $b \in P$ is an *upper bound* of $A$ if, for all $a \in A$, we have $a \leqslant b$.
- An element $c \in P$ is an *infimum* or *greatest lower bound* of $A$ if (1) $c$ is a lower bound of $A$, and (2), for all lower bounds $b$ of $A$, we have $b \leqslant c$.
- An element $c \in P$ is a *supremum* or *least upper bound* of $A$ if (1) $c$ is an upper bound of $A$, and (2), for all upper bounds $b$ of $A$, we have $c \leqslant b$.

- An element $b \in P$ is a *least* or *bottom* or *minimum* element of $P$, if, for all $a \in P$, we have $b \leqslant a$ (i.e., $b$ is the supremum of $A = \emptyset$).

- An element $b \in P$ is a *greatest* or *top* or *maximum* element of $P$, if, for all $a \in P$, we have $a \leqslant b$ (i.e., $b$ is the infimum of $A = \emptyset$).

- An element $b \in P$ is *minimal* if, for all $a \in P$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal* if, for all $a \in P$, if $b \leqslant a$, then $b = a$.

- An element $b \in P$ is *minimal in* $A$ if (1) $b \in A$ and (2) for all $a \in A$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal in* $A$ if (1) $b \in A$ and (2) for all $a \in A$, if $b \leqslant a$, then $b = a$.

- $A$ is an *upset* if for all $a, b \in P$, if $a \in A$ and $a \leqslant b$, then $b \in A$.

- $A$ is a *downset* if for all $a, b \in P$, if $b \in A$ and $a \leqslant b$, then $a \in A$.

- $A$ is *directed* (aka up-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. (Equivalently, all finite subsets of $A$ have an upper bound in $A$.)

- $A$ is *filtered* (aka filtering or down-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $c \leqslant a$ and $c \leqslant b$. (Equivalently, all finite subsets of $A$ have a lower bound in $A$.)

(These notions also make sense in a preorder $(P, \leqslant)$, but if $P$ is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted $\bigwedge A$, called the *meet* of $A$; and the supremum is denoted $\bigvee A$, called the *join* of $A$. If $A = \{a_1, \dots, a_n\}$ is finite and nonempty, we write $\bigwedge A = a_1 \wedge \dots \wedge a_n$ and $\bigvee A = a_1 \vee \dots \vee a_n$. In particular, $\bigwedge\{a, b\} = a \wedge b$ and $\bigvee\{a, b\} = a \vee b$. The bottom element, if it exists, is denoted $\bot$ or 0; and the top element by $\top$ or 1. We write $\min(A)$ (resp. $\max(A)$) for the elements that are minimal (resp. maximal) in $A$. A *directed join* is the supremum of a directed set.

*It is a good exercise to prove this.*

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcop*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can 'turn them upside down' and get another preorder. Formally, if $(P, \leqslant)$ is a preorder, define the preorder $\leqslant'$ on $P$ by $a \leqslant' b$ iff $b \leqslant a$. We write $P^{\mathrm{op}}$ for this preorder.

*Verify that this again is a preorder (resp. partial order), and draw some Hasse diagram example to see that this really turns things upside down.*

### 2.1.2. Morphisms: Monotone maps

*We consider the words 'map' and 'function' as synonymous.*

What maps between partial orders should be considered to be 'structure preserving'? They should preserve the order structure, which yields the concept of a monotone map.

**Definition 2.3.** Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders and $f : P \to Q$ a function. We say $f$ is

- *monotone* or *order preserving* if, for all $a, b \in P$, if $a \leqslant_P b$, then $f(a) \leqslant_Q f(b)$.

The converse notion is:

- *order reflecting* if, for all $a, b \in P$, if $f(a) \leqslant_Q f(b)$, then $a \leqslant_P b$.

*Note that being order reflecting implies being injective. But injective monotone maps need not be order embeddings.*

We call $f$ an *order-embedding* if it is both order preserving and order reflecting. Finally, $f$ is an *order-isomorphism* if it is monotone and it has a *monotone inverse*, i.e., there is a monotone function $g : Q \to P$ such that

- for all $a \in P$, we have $a = g(f(a))$, i.e., $a$ is the $g$-inverse of $f(a)$ (in short, $\mathsf{id}_P = g \circ f$), and
- for all $b \in Q$, we have $f(g(b)) = b$, i.e., mapping the $g$-inverse of $b$ along $f$ yields $b$ (in short, $f \circ g = \mathsf{id}_Q$).

*Here $\mathsf{id}_X$ denotes the identity function on set $X$. And if $f : X \to Y$ and $g : Y \to Z$ are functions, $g \circ f$ ($g$ after $f$) denotes their composition, which maps $x \in X$ to $g(f(x)) \in Z$.*

We say two preorders are *isomorphic* if there is an order isomorphism between them.

We can consider two isomorphic preorders to be essentially identical (because any order-theoretic property that one has, the other has, too). For partial orders, the notion of isomorphism can be simplified. The above definition captures the general (category-theoretic) concept of an isomorphism, but in practice the following is often easier to check.

**Proposition 2.4.** *Let $f : P \to Q$ be a monotone function between posets. Then $f$ is an isomorphism iff $f$ is a surjective order-embedding.*

*Proof.* Exercise 2.d. $\square$

In exercise 2.e, we show how the notion of an isomorphism can be generalized to that of an adjunction. This provides yet another notion of morphism between posets. We do not need it here, but since it is a very useful (but also abstract) concept in the vicinity of the presented concepts, we include it as an exercise. Exercise 2.f shows how such adjunction naturally occur once one has a relation between two sets.

## 2.2. Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

### 2.2.1. Objects: lattices

The order-theoretic definition of a lattice goes as follows.

**Definition 2.5** (Lattice, order-theoretic)**.** A *(bounded) lattice* is a partial order $L$ in which every finite subset has a supremum and an infimum.

For example, the diamond of figure 2.1 is a lattice. To consider the example of the collection of propositions from chapter 1, we already said that it should be partially ordered by implication, and it makes sense to require that it is a lattice: Given two propositions $a$ and $b$, there is, as we discussed, also the proposition $a \vee b$, and it is implied by both $a$ and by $b$, and whenever a proposition $c$ is implied by both $a$ and $b$, then $a \vee b$ implies $c$—so $a \vee b$ is the supremum of $a \vee b$. This also works for a finite set of propositions, and also for $\wedge$ and the infimum.

Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.

   *As an exercise, prove this.*

2. Often a lattice is defined as a partial order in which all binary suprema and infima exist (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exits (i.e., which a have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word 'bounded' (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.

3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

*Prove this. (Hint: think about the supremum of all lower bounds.)*

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

**Definition 2.6** (Lattice, algebraic)**.** A lattice is a tuple $(L, \vee, \wedge, \perp, \top)$ where $\vee$ (pronounced *join*) and $\wedge$ (pronounced *meet*) are binary operations on $L$ (i.e., functions $L \times L \to L$), and $\perp$ (pronounced *bottom*) and $\top$ (pronounced *top*) are elements of $L$, such that the following axioms holds:

1. *commutative*: for all $a, b \in L$, we have $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.
2. *associative*: for all $a, b, c \in L$, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.
3. *idempotent*: for all $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$.
4. *absorption*: for all $a, b \in L$, we have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$
5. *neutrality*: for all $a \in L$, we have $\perp \vee a = a$ and $\top \wedge a = a$.

For example, if $X$ is a set, then the powerset $2^X$ forms a lattice in this algebraic sense with union $\cup$ as join, intersection $\cap$ as meet, $\emptyset$ as bottom, and $X$ as top. This also provides my mnemonic for remembering what 'join' and what 'meet' is. Think of $X$ as a set of propositions, and let $a \in 2^X$ be the beliefs (opinions, values, etc.) that *A*lice holds, and let $b \in 2^X$ be the beliefs that *B*ob holds. Then the meet of $a$ and $b$—i.e., $a \wedge b = a \cap b$—is where Alice and Bob can meet: the common (meeting) ground, the set of beliefs they agree on. And the join of $a$ and $b$—i.e., $a \vee b = a \cup b$—is the result of joining Alice and Bob together: their joint beliefs, taking together all of their beliefs even if incoherent.

*Though I'm happy to learn about a better one :-)*

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.g asks you to prove it: that is a bit tedious, but quite instructive.

**Theorem 2.7.** *The algebraic and order-theoretic definitions of a lattice are equivalent in the following sense:*

1. *Given a lattice $(L, \vee, \wedge, \perp, \top)$ according to the algebraic definition, define $a \leqslant_L b$ as $a \wedge b = a$. Then $(L, \leqslant_L)$ is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by $\vee$ and $\wedge$.*

2. *Given a lattice* $(L, \leqslant)$ *according to the order-theoretic definition, define the binary operations* $\vee$ *and* $\wedge$ *as binary supremum and infimum, and take* $\bot$ *and* $\top$ *to be the least and greatest element of* $L$*. Then* $(L, \vee, \wedge, \bot, \top)$ *is a lattice according to the algebraic definition, with* $a \wedge b = a$ *iff* $a \leqslant b$ *iff* $a \vee b = b$.

From now on, we will often just speak of a lattice $L$ and both use its order-theoretic definition (taking $\leqslant$ to be implicitly given) and its algebraic definitions (taking $\vee, \wedge, \bot, \top$ to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure $(L, \cdot, 1)$, where $\cdot$ is a commutative, associative, and idempotent binary operation on $L$, and $1$ is a neutral element for the operation. The operation $\cdot$ can then either be seen as the binary infimum for the partial order defined by $a \leqslant b$ iff $a \cdot b = a$ (the join semilattice), or as the binary supremum for the opposite partial order defined by $a \leqslant b$ iff $a \cdot b = b$ (the meet semilattice).

#### 2.2.2. Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

**Definition 2.8.** A function $f : L \to M$ between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all $a, b \in L$, we have $f(a \vee_L b) = f(a) \vee_M f(b)$
2. for all $a, b \in L$, we have $f(a \wedge_L b) = f(a) \wedge_M f(b)$
3. $f(\bot_L) = \bot_M$
4. $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*.

*Prove this.*

If a function $f : L \to M$ between lattices preserves $\bot$ and $\vee$, then it preserves all finite joins. This does, in general, *not* imply any preservation of arbitrary existing joins or preservation of infima. The analog statement is true for $\top$ and $\wedge$ and preservation of all finite meets.

*Prove this.*

### 2.2.3. Constructions: products, sublattices, homomorphic images, congruences

Whenever one has introduced a class of objects together and their structure-preserving maps, one also looks at the *constructions* one can perform: how to build new objects in the class from old ones. The typical ones are products, substructures, and quotients, and you might have seen this also for other structures, e.g. groups. (Here quotients will be given as homomorphic images or, equivalently, congruences.) Actually, in this course, they will not play a big part, but they will in more advanced texts on duality theory and are generally important to know. So it is enough if you just skim them.

*Products*. Given a family $(L_i)_{i \in I}$ of lattices, we can define a lattice $L = \prod_{i \in I} L_i$ on the Cartesian product where the operations are defined component-wise: e.g., for $a = (a_i)_i$ and $b = (b_i)_i$ in L, we define $a \leqslant_L b$ as $\forall i \in I : a_i \leqslant_{L_i} b_i$, and $(a \wedge b)_i = a_i \wedge b_i$ (similarly for $\vee$), and $(\bot_L)_i = \bot_{L_i}$ (similarly for $\top$). The projection maps $\pi_i : L \to L_i$, which map $a = (a_i)_i$ to $a_i$, is a surjective lattice homomorphism.

*Recall (appendix A.1) that the Cartesian product of a family of sets is the set of functions* $a$ *that map each* $i \in I$ *to an element* $f(i) \in L_i$*. We often write such a function as* $a = (a_i)_{i \in I}$.

*Sublattices*. A sublattice of a lattice L is a subset $L'$ of L that contains $\bot$ and $\top$ and that is closed under $\wedge$ and $\vee$ (i.e., if $a, b \in L'$, then $a \wedge b, a \vee b \in L'$). Then $L'$ is a bounded lattice in its own right and the inclusion map $\iota : L' \to L$, which maps $a \in L'$ to $a \in L$, is a lattice embedding. If we do not require $\bot$ and $\top$ to be in $L'$, we speak of an *unbounded sublattice*. And if we require $L'$ to be closed under all suprema and infima, we call it a *complete sublattice*. If $f : L \to M$ is a lattice homomorphism, then the direct image $L' := f[L] = \{f(a) : a \in L\}$ is a sublattice of the lattice M.

*Homomorphic images*. A lattice $L'$ is a *homomorphic image* of a lattice L if there is a surjective lattice homomorphism $f : L \to L'$.

*Birkhoff's famous theorem in universal algebra says that a class of algebraic structures (like lattices) is closed under Homomorphic images, Subalgebras, and Products iff it is definable by equations (hence aka 'HSP theorem').*

*Congruences*. A congruence on a lattice L is an equivalence relation $\vartheta$ on L that respects the lattice operations, i.e., for all $a, a', b, b' \in L$, if $a \vartheta a'$ and $b \vartheta b'$, then also $a \vee b \vartheta a' \vee b'$ and $a \wedge b \vartheta a' \wedge b'$. For an intuitive example, think of the elements of L as propositions and of $\vartheta$ as having the same subject matter. The quotient $L/\vartheta$ carries a unique lattice structure that turns the quotient map $p : L \to L/\vartheta$, which maps $a \in L$ to its equivalence class $[a]_\vartheta$ under $\vartheta$, into a lattice homomorphism; concretely, this lattice structure is given by $[a]_\vartheta \vee [b]_\vartheta := [a \vee b]_\vartheta$ (similarly for $\wedge$) with bottom element $[\bot]_\vartheta$ (similarly for $\top$). Note how this is reminiscent of the Lindenbaum–Tarski algebra from the introduction (section 1.1.3).

*The first isomorphism theorem for lattices*. This says that any lattice homomorphism $f : L \to M$ can be factored as a surjective lattice homomorphism

*The exciting thing about this is that lattice homomorphism can be very complicated, but this tells us that they can be broken down into two much simpler things: surjective lattice homomorphisms and injective lattice homomorphisms!*

$p$ followed by a lattice embedding $e$ (i.e., $f = e \circ p$). These are given as follows. The *kernel* of $f$ is the congruence relation

$$\ker f := \{(a, a') \in L \times L : f(a) = f(a')\}.$$

Choose $p : L \to L/\ker f$ (mapping $a$ to $[a]$) and $e : L/\ker f \to M$ (mapping $[a]$ to $f(a)$). In particular, $L/\ker f$ is isomorphic to $f[L]$ (take $M := f[L]$, so $e$ also is surjective); hence the homomorphic images of $L$ are, up to isomorphism, the quotients of $L$.

*Again, the quotients of* $L$ *intuitively are much simpler: to determine them, we only have to look at* $L$*, while for homomorphic images we also need to consider other lattices* $M$*.*

## 2.3. Distributive lattices and Boolean algebras

We get further subclasses of lattices by requiring that $\vee$ and $\wedge$ interact nicely, which is made precise as distributive lattices (section 2.3.1), and by additionally requiring that there is a sense of negation, which is made precise as Boolean algebras (section 2.3.2).

### 2.3.1. Distributive lattices

The idea $\vee$ and $\wedge$ interact nicely is made precise as follows.

**Definition 2.9.** A lattice $L$ is distributive if,

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{2.1}$$

*Cf. distributivity from high school:* $x \times (y + z) = (x \times y) + (x \times z)$

or, equivalently,

$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{2.2}$$

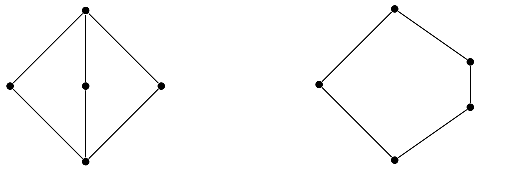
*Proving the equivalence of 2.1 and 2.2 is exercise 2.h.*

For example, the four diamond from figure 2.1 is distributive, as is any powerset $2^X$, and also any chain $n = \{0, 1, \ldots, n-1\}$ with the usual ordering (see exercise 2.i). If we again consider the intuitive example of a collection of propositions from chapter 1, we already said that it is a lattice and we also expect it to be distributive because then 2.1 expresses a basic logical equivalence between propositions.

The equivalence of 2.1 and 2.2 implies that $L$ is distributive iff $L^{\text{op}}$ is distributive. So distributivity is a so-called *self-dual property*.

There also are strengthenings of the distributivity law. We mentioned one example here for context, but do not need it later. A *frame* is defined

*In case you have heard of this: A frame is the same thing as a complete Heyting algebra, but their respective choice of morphisms differ.*



The diamond $M_3$ The pentagon $N_5$

Figure 2.2.: The forbidden substructures for distributivity.

as a complete lattice $L$ satisfying the join infinite distributive law (JID)

$$\text{for any } a \in L \text{ and } B \subseteq L,\ a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b). \tag{2.3}$$

In a distributive lattice this, in general, only holds for all *finite* $B \subseteq L$.

A seemingly magic characterization of distributive lattices is the following.

**Theorem 2.10** (The $M_3$–$N_5$ theorem)**.** *Let $L$ be a lattice. Then $L$ is distributive iff $L$ does not contain an unbounded sublattice which is isomorphic to $M_3$ or $N_5$, depicted in figure 2.2.*

*For a proof, see, e.g., Davey and Pristley (2002, 89 ff.).*

#### 2.3.2. Boolean algebras

So far, we have seen the order $\leqslant$ and the operations $\vee$ and $\wedge$ in a lattice, which act like implication, disjunction, and conjunction, respectively. So you might have wondered: what about negation? Especially since this also played a role in our motivating example of a collection of propositions from chapter 1: if we have a proposition $a$, we also have the proposition $\neg a$, and we expect $a \wedge \neg a$ to be a logical contradiction and $a \vee \neg a$ a logical truth. These ideas are made precise as follows.

**Definition 2.11.** Let $L$ be a lattice and $a$ an element of $L$. A *complement* of $a$ is an element $b$ of $L$ such that $a \wedge b = \bot$ and $a \vee b = \top$. A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element $a$ in a distributive lattice is unique, if it is exist, an denoted $\neg a$.

*Prove this! Note that in non-distributive lattices, like $M_3$ and $N_5$ from figure 2.2, elements can have multiple complements.*

For example, again the four diamond from figure 2.1 is a Boolean algebra, as is any powerset $2^X$; but, for $n > 2$, the chain $n$ is not a Boolean algebra (see exercise 2.i). Some further comments:

1. Usually, the negation is then taken into the signature: so a Boolean algebra is a tuple $(B, \wedge, \vee, \perp, \top, \neg)$ such that $(B, \wedge, \vee, \perp, \top)$ is a distributive lattice and $\neg : B \to B$ a unary function such that, for all $a \in B$, we have $a \wedge \neg a = \perp$ and $a \vee \neg a = \top$.

2. But if we have an additional operation around, shouldn't we require the morphisms to preserve it? Fortunately, they already do: If $f : B \to A$ is a lattice homomorphism between Boolean algebras, then, for all $a \in B$, we have $f(\neg a) = \neg f(a)$. We often still refer to them as *Boolean algebra homomorphisms* just to emphasize that we are dealing with Boolean algebras.

*The fact that we can use the same morphisms is expressed in categorical terms as the category of Boolean algebras and Boolean algebra homomorphisms being a* full *(as opposed to any) subcategory of the category of distributive lattices and lattice homomorphisms.*

3. However, with the notion of a sublattice we need to be more careful: A Boolean algebra may have many sublattices that themselves are not Boolean algebras; so by a (*Boolean*) *subalgebra* of a Boolean algebra $B$ we mean a sublattice which is also closed under $\neg$.

4. If you like ring theory, a Boolean algebra can equivalently be defined as a commutative ring with unit in which all elements are idempotent, see exercise 2.j.

5. There is a best way to turn a distributive lattice $L$ into a Boolean algebra $B$. This $B$ is called the *Boolean envelope* or *free Boolean extension* of $L$. More precisely, this means that for every distributive lattice $L$ there is a Boolean algebra $B$ and an injective homomorphism $e : L \to B$ such that for any other lattice homomorphism $h : L \to A$ into a Boolean algebra $A$, there is a unique Boolean algebra homomorphism $\overline{h} : B \to A$ such that $\overline{h} \circ e = h$. As a diagram:

$$\begin{tikzcd} L \arrow[r, "e"] \arrow[dr, "h"'] & B \arrow[d, dashed, "\overline{h}"] \\ & A \end{tikzcd}$$

   This will be a corollary from Stone duality theorem.

*In categorical terms this means the category of Boolean algebras is a full* reflective *subcategory of the category of distributive lattices.*

## 2.4. Exercises

**Exercise 2.a.** Show that the following are partial orders and draw their Hasse diagrams:

- The chain $P = \{0, \dots, n-1\}$ with the usual order. Draw it for, say, $n = 10$.

- The set $P = \{1, \dots, n\}$ with the order defined by $n \leqslant m$ iff $n$ divides $m$ (why is 0 excluded from the set?). Draw it for, say, $n = 10$.
- The powerset $2^X$ for some set $X$ ordered by the subset relation, i.e., $A \leqslant B$ iff $A \subseteq B$. Draw it for, say, $X = \{4, 7\}$.

**Exercise 2.b.** Go through the partial order concepts defined in definition 2.2 and pick a few of them and draw (minimal) Hasse diagrams to show how they differ. For example, a maximal element that is not a greatest element; an upper bound that is not a greatest upper bound; or an upset that is not directed.

**Exercise 2.c.** Recall that for a preorder $(P, \leqslant)$, we have defined the poset reflection $(\overline{P}, \overline{\leqslant})$. This exercise makes precise in which sense this is the best possible poset approximating the preorder $(P, \leqslant)$.

*Exercise 1.1.5 in Gehrke and van Gool (2023), with small changes.*

1. Prove that $\equiv$ is an equivalence relation.
2. Prove that the definition of $\overline{\leqslant}$ is independent of the representatives: If $a' \in [a]$ and $b' \in [b]$, then $a \leqslant b$ iff $a' \leqslant b'$.
3. Prove that $(\overline{P}, \overline{\leqslant})$ is indeed a partial order.
4. Prove that $\overline{\leqslant}$ is the smallest partial order on $\overline{P} = P/\equiv$ such that the quotient map $f : P \to P/\equiv$, which maps $a$ to $[a]$, is order preserving: That is, if $\leqslant'$ is another such partial order on $P/\equiv$, then $\overline{\leqslant} \subseteq \leqslant'$.
5. Prove that, for any order preserving $g : P \to Q$ into a poset $Q$, there exists a unique order preserving $\overline{g} : P/\equiv \to Q$ such that $\overline{g} \circ f = g$. As a diagram:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "g"'] & P/\equiv \arrow[d, dashed, "\overline{g}"] \\ & Q \end{tikzcd}$$

*The category-theoretic formulation of this fact is: the inclusion of the category of partial orders and monotone maps in the category of preorders and monotone maps has a left adjoint. Adjoint functors can be interpreted as formalizing the idea of finding a best possible approximation.*

Think about how the last item formalizes the idea that $(\overline{P}, \overline{\leqslant})$ is the best possible poset approximating the preorder $(P, \leqslant)$.

**Exercise 2.d.** Prove proposition 2.4.

The next two exercises introduce the notion of an order adjunction (this is a special case of the notion of an adjoint functor). The first states the general definition and the second a common situation how they occur.

**Exercise 2.e** (Order adjunction)**.** Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone maps. The pair $(f, g)$ is called an *adjunction*, with $f$ the *left* or *lower adjoint* and $g$ the *right* or *upper adjoint*, if, for all $a \in P$ and $b \in Q$,

*Exercise 1.1.8 in Gehrke and van Gool (2023).*

$$f(a) \leqslant_Q b \text{ iff } a \leqslant_P g(b).$$

*Note that $f$ occurs on the left of '$\leqslant$' and $g$ on the right.*

We also write this as $l : P \leftrightarrows Q : u$. An adjunction between $P^{op}$ and $Q$ is called a *Galois connection* or *contravariant adjunction*.

1. Prove that $(f, g)$ is an adjunction iff
   - for all $a \in P$, we have $a \leqslant_P g(f(a))$, i.e., the $g$-inverse of $f(a)$ is at least as good as $a$, and
   - for all $b \in Q$, we have $f(g(b)) \leqslant_Q b$, i.e., mapping the $g$-inverse of $b$ along $f$ approximates $b$.

*Reflect on how an adjunction then* generalizes *the notion of an isomorphism!*

For the rest of this exercise, assume that $(f, g)$ is an adjunction.

*Reflect on how this show that $g$ is the best possible approximation to an inverse of $f$!*

2. Prove that $f \circ g \circ f(a) \equiv f(a)$ and $g \circ f \circ g(b) \equiv g(b)$ for every $a \in P$ and $b \in Q$ (and $a \equiv b$ iff $a \leqslant b$ and $b \leqslant a$).
3. Conclude that, in particular, if $P$ and $Q$ are posets, then $fgf = f$ and $gfg = g$.
4. Prove that, if $P$ is a poset, then for any $a \in P$, $gf(a)$ is the least element above $a$ that lies in the image of $g$.
5. Formulate and prove a similar statement to the previous item about $fg(b)$, for $b \in Q$.
6. Prove that, for any subset $A \subseteq P$, if the supremum of $A$ exists, then $f(\bigvee A) = \bigvee f(A)$ (where $f(A) = \{f(a) : a \in A\}$ is the image of $A$ under $f$).
7. Prove that, for any subset $B \subseteq Q$, if the infimum of $B$ exists, then $g(\bigwedge B) = \bigwedge g(B)$.

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

*We will see that the converse holds for complete lattices. This is a special case of the Adjoint Functor Theorem.*

**Exercise 2.f** (Galois connection from a relation)**.** Let $R \subseteq X \times Y$ be a relation between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define

$$u(a) := \{y \in Y : \forall x \in a. xRy\} \subseteq Y$$
$$l(b) := \{x \in X : \forall y \in b. xRy\} \subseteq X$$

Show that $l : \mathcal{P}(Y) \leftrightarrows \mathcal{P}(X) : u$ forms a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, i.e., for any $b \subseteq Y$ and $a \subseteq X$, we have $a \subseteq l(b)$ (i.e., $l(b) \subseteq^{op} a$) iff $b \subseteq u(a)$.

*Here $\mathcal{P}(X)$ is the set of all subsets of the set $X$.*

For those interested in further reading, here are three instances of this.

1. Maybe you know the name 'Galois' from the theory of fields in algebra. Then you know Galois theory as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation $R$ between the set $X$ of subfields of a given field and the set $Y$ of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.

*For an accessible introduction, take a look, e.g., at this or this video, or at these great lecture notes by Tom Leinster.*

2. If $X$ is a set and $R \subseteq X \times X$ is a preorder, then $u(a)$ is the set of upper bounds of $a \subseteq X$, and $l(b)$ is the set of lower bounds of $b \subseteq X$.

3. Consider a class of structures $\mathcal{C}$ (in, say, a first-order signature) and a class $\mathcal{F}$ of formulas (of this signature). Let $\vDash$ be the *interpretation* relation: For $M \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ means that structure $M$ makes true formula $\varphi$. Then for a set of models $a$, $u(a)$ is the theory of $a$, i.e., the set of formulas that are true in all those models. And for a a theory $b \subseteq \mathcal{F}$, $l(b)$ is the class of models of $b$, i.e., the set of models which make true all the sentences in $b$.

*Also recall the examples from section 1.1.*

**Exercise 2.g.** Prove theorem 2.7.

**Exercise 2.h.** Prove the equivalence of the two ways of defining distributivity: 2.1 and 2.2.

**Exercise 2.i.** Show that the following are distributive lattices:

- The four diamond from figure 2.1.
- The powerset $2^X$, for any set $X$.
- The chain $n = \{0, \dots, n-1\}$ with the usual ordering, for any $n$.

Show that the first two also are Boolean algebras. Show that the last one is a Boolean algebra if $1 \leqslant n \leqslant 2$, and not if $n > 2$.

You might have had the suspicion that the join $\vee$ acts quite like addition $+$ and the meet $\wedge$ quite like the multiplication $\cdot$. If so, you might like the next exercise, which makes this precise.

**Exercise 2.j.** This exercise shows that Boolean algebras and Boolean rings are equivalent.

*From Gehrke and van Gool 2023, ex. 1.2.13.*

1. Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, i.e., a commutative ring with unit in which $a \cdot a = a$ for all $a \in B$. Define $a \leqslant b$ if $a \cdot b = a$. (We often write $ab$ for $a \cdot b$.) Prove that $\leqslant$ is a distributive lattice order on $B$ where
   - 1 is the greatest element and 0 is the least element,
   - meet is given by $ab$ and join is given by $a + b + ab$, and
   - every element $a$ of has the complement $1 + a$ with respect to $\leqslant$.

   *Hint*: First show that $a + a = 0$ for all $a \in B$.

2. Conversely, let $(B, \wedge, \vee, \perp, \top, \neg)$ be a Boolean algebra. Define, for any $a, b \in B$,

$$a + b := (a \wedge \neg b) \vee (\neg a \wedge b) \qquad a \cdot b := a \wedge b$$
$$0 := \perp \qquad 1 := \top.$$

*The operation + is known as* symmetric difference.

   Prove that $(B, +, \cdot, 0, 1)$ is a Boolean ring.

3. Finally, show that the composition of these two assignments in either order yields the identity.

The following is a fact that we will later use a lot.

**Exercise 2.k.** Let $f : X \to Y$ be a function between two sets $X$ and $Y$. Show that the function from $2^Y$ to $2^X$ defined by

$$B \mapsto f^{-1}(B)$$

is a Boolean algebra homomorphism.

# 3. The spatial side: topological spaces

This chapter introduces formally the spatial side of duality, which, for us, will be certain topological spaces known as Stone spaces. This naturally structures this chapter: In section 3.1, we provide a general introduction to topological spaces. In section 3.2, we consider some further topological notions: separation axioms, compactness, and dimensionality. Finally, in section 3.3, we define Stone spaces and discuss one important example, namely the Cantor space. Then we have both the algebraic and the spatial side together, so we can prove the duality result in the next chapter.

## 3.1. Introduction to topological spaces

When we hear of 'space', we naturally think of the three-dimensional space we live in. And this indeed is an example of a topological space. It is the three-dimensional Euclidean space $\mathbb{R}^3$ whose points $x = (x_1, x_2, x_3)$ are described by the values on the $x$-axis, the $y$-axis, and the $z$-axis. From high-school, we also know what lines and planes are in this space, and what their geometry is.

*E.g., the plane spanned by the $x$-axis and the $y$-axis is the set of points $(x_1, x_2, x_3)$ with $x_3 = 0$.*

But there also are other spaces. For example, the surface of a sphere. Its points are those $(x_1, x_2, x_3)$ with $x_1^2 + x_2^2 + x_3^2 = 1$. But its geometry is different: for instance, the angles of a triangle add up to more than 180 degrees. Yet another space is the spacetime that we live in according to general relativity. Its points $x = (x_1, x_2, x_3, x_4)$ are four-dimensional—with three spatial and one temporal component—and its geometry is given by a metric tensor. And there are even wilder spaces, in which it might not even make sense to speak of a 'geometry' (e.g., angles between lines), but only of 'spatial' properties (e.g., continuous paths from one point to another).

*Here we only refer to an intuitive difference between 'geometric' vs 'spatial' (or topological) properties: the latter are invariant under stretching and squishing the space, but the former are not. This is why topology is colloquially also described as* rubber sheet geometry.

After much research, mathematicians—most notably Felix Hausdorff in 1914—came up with a general definition of a topological space that includes all these examples. When one first reads this rather abstract definition, one wonders how it possibly can cover all the relevant spatial concepts of the specific examples. But we see how, just from this parsimonious definition of a topological space, we can define many of the common spatial concepts. Again, we split this discussion into objects (topological

*Mathematics provides many formal notions of space (e.g., Euclidean space, vector space, Hilbert space, probability space, Banach space, etc.). But topological spaces are a very general such notion.*

spaces) and morphisms (continuous functions between spaces).

### 3.1.1. Objects: topological spaces

Without further ado, here is the abstract definition of a topological space.

**Definition 3.1.** A *topological space* is a pair $(X, \tau)$ where $X$ is a nonempty set and $\tau$ is a collection of subsets of $X$ such that

*Some also allow the empty topological space.*

1. $\emptyset$ and $X$ are in $\tau$
2. If $U, V \in \tau$, then $U \cap V \in \tau$
3. If $U_i \in \tau$ is a collection of sets indexed by a set $I$, then $\bigcup_{i \in I} U_i \in \tau$.

*Equivalently: $\tau$ is closed under finite intersection and arbitrary union (which includes the empty intersection $X$ and the empty union $\emptyset$). In particular, $\tau$ is a sublattice of $2^X$.*

We also call $\tau$ a topology on $X$. We call the elements of $X$ *points*. The elements of $\tau$ are called *open sets* (or *opens*). Their complements, i.e., sets of the form $C = X \setminus U$ for $U \in \tau$, are called *closed sets*. A subset $K \subseteq X$ that is both open and closed (i.e., $K \in \tau$ and $K^c \in \tau$) is called *clopen*. We just speak of the topological space $X$ if $\tau$ is clear from context. Then we write $\Omega(X)$ for the opens of $X$. The collection of closed (resp. clopen) subsets of $X$ is denoted $\mathcal{C}(X)$ (resp. $\mathsf{Clp}(X)$).

Let's first see that this indeed generalizes our spatial intuitions about 'our' space:

**Example 3.2.** The three-dimensional space as a topological space: the underlying set is $X := \mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ and the opens are those subsets $U \subseteq \mathbb{R}^3$ that allow some 'wiggle-room', which is made precise as follows. Recall that the usual distance between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

So a subset $U \subseteq \mathbb{R}^3$ is defined to be open precisely if:

*Exercise 3.a asks you to show that this then indeed is a topology.*

1. for all $x \in \mathbb{R}^3$, if $x \in U$, then there is $\epsilon > 0$ such that for all $x' \in \mathbb{R}^3$ with $d(x, x') < \epsilon$, we have $x' \in U$.

This is called the *Euclidean topology* on $\mathbb{R}^3$. $\lrcorner$

Another, more abstract example are the two trivial topologies:

**Example 3.3.** For any nonempty set $X$, the set $\tau := 2^X$ is a topology on $X$. It is called the *discrete* topology. Also $\tau := \{\emptyset, X\}$ is a topology on $X$. It is called the *indiscrete* topology. $\lrcorner$

Next, we define some central concepts for a topological space X. They should give a sense of how many spatial concepts one can express with just talk of open sets.

**Definition 3.4.** Let $(X, \tau)$ be a topological space.

1. *Interior and closure*. If $S \subseteq X$ is a subset, there is a largest open set contained in S, which is called the *interior* of S:

   $$\mathsf{Int}(S) := \bigcup \{U \in \tau : U \subseteq S\}.$$

   *Convince yourself that (a) this is an open set, (b) it is contained in S, and (c) it is the largest such set.*

   There also is a smallest closed set containing S, which is called the *closure* of S:

   $$\mathsf{Cl}(S) := \bigcap \{C \in \mathcal{C}(X) : S \subseteq C\}.$$

   *Convince yourself that closed sets are closed under arbitrary intersection, so this is indeed a closed set.*

2. *Boundary*. The *boundary* of a subset $S \subseteq X$ is defined as $\partial S := \mathsf{Cl}(S) \setminus \mathsf{Int}(S)$.

3. *Neighborhood*. A subset $S \subseteq X$ is a *neighborhood* of a point $x \in X$ if $x \in \mathsf{Int}(S)$. Accordingly, an *open neighborhood* of a point is an open set containing this point. (If it's clear we're talking about an open neighborhood, we might drop the adjective 'open'.)

4. *Dense*. A subset $S \subseteq X$ is *dense* (in X) if for all points $x \in X$ and open neighborhoods U of $x$, there is a point $s \in S$ with $s \in U$. So the points of X can be approximated arbitrarily closely by points in S. An equivalent formulation is: $\mathsf{Cl}(S) = X$.

   *For example, the (countable) set S of all points in $\mathbb{R}^3$ with rational coordinates is dense in $\mathbb{R}^3$.*

5. *Convergence*. A sequence $(x_n)_{n \in \mathbb{N}}$ of points in X *converges* to a point $x \in X$ if for all open neighborhoods U of $x$, there is $N \geqslant 0$ such that, for all $n \geqslant N$, we have $x_n \in U$. We also say that $x$ is the *limit* of the sequence $(x_n)$.

6. *Generated topology*. A collection of subsets can naturally be turned into a topology: Any collection $\mathcal{S}$ of subsets of a nonempty set Y *generates* a topology $\langle \mathcal{S} \rangle$ on Y: namely, the smallest topology on Y that contains all subsets in $\mathcal{S}$. Concretely, $\langle \mathcal{S} \rangle$ is the set of arbitrary unions of finite intersections of elements of $\mathcal{S}$.

   *This exists because an arbitrary intersection of topologies on Y is again a topology on Y.*

7. *Subbase*. The generating collection $\mathcal{S}$ can be simpler than the topology $\langle \mathcal{S} \rangle$ generated, so it allows for a more succinct description of the topology. Precisely: Given the topology $\tau$ on X, a collection $\mathcal{S}$ of

subsets of $X$ is called a *subbase* of $\tau$ if $\tau = \langle \mathcal{S} \rangle$. So the opens of $\tau$ are arbitrary unions of finite intersection of subbasic elements.

8. *Base*. The nice subbases are those where we only need to consider arbitrary unions and can forget about the finite intersections. Precisely: A *base* for the topology $\tau$ is a collection $\mathcal{S} \subseteq \tau$ such that for every point $x \in X$ and every open neighborhood $U$ of $x$, there is $V \in \mathcal{S}$ such that $x \in V \subseteq U$. Equivalently, a base is a collection of open subsets of $X$ such that every open set is a union of of elements from the base. Note that, in particular, a base is a subbase.

**Exercise 3.5.** Prove the four statements in the margins:

*This also is the end-of-chapter exercise 3.b.*

1. the existence of the interior,
2. the existence of the closure,
3. the rational points being dense in our three-dimensional space,
4. the existence of the generated topology.

Moreover, to illustrate the concept of a basis:

5. Show that a base for the Euclidean topology on $\mathbb{R}^3$ is given by the open balls $B_\epsilon(x) := \{y \in \mathbb{R}^3 : d(x, y) < \epsilon\}$ for $x \in \mathbb{R}^3$ and $\epsilon > 0$.
6. Can you improve the previous base by allowing only $x \in \mathbb{R}^3$ with rational coordinates and $\epsilon = \frac{1}{n}$ (for $n \in \mathbb{N}$)? Note that then you have found a countable base even though the space $\mathbb{R}^3$ has uncountably many points!

### 3.1.2. Morphisms: continuous functions

Now that we know what topological spaces are, what are the structure-preserving mappings between them? Again, there is a neat but abstract definition, which we state first and then link it to more familiar ideas in 'our' space.

**Definition 3.6.** Let $X$ and $Y$ be topological spaces and $f : X \to Y$ a function. We say $f$ is *continuous* if, for all open subsets $V$ of $Y$, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open subset of $X$.

**Exercise 3.7.** Maybe you have encountered the idea of a continuous function before but in the more concrete, so-called *epsilon–delta definition* of a continuous function $f : \mathbb{R} \to \mathbb{R}$. This definition says that $f : \mathbb{R} \to \mathbb{R}$ is continuous if

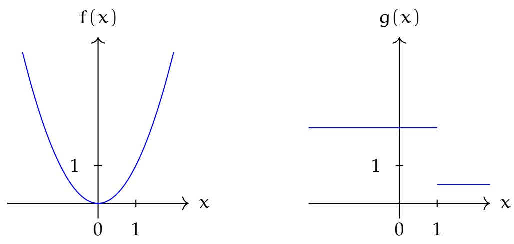



Figure 3.1.: A continuous function f (left) and a non-continuous function g (right).

1. For every $x \in \mathbb{R}$ and every $\epsilon > 0$, there is $\delta > 0$ such that, for all $y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

This captures the idea that, to draw the graph of the function, you do not have to lift your pen: If you want to continue drawing the graph a bit to the left or right of an argument $x$, the value outputted by the function will not 'jump away' but be close to the value at point $x$. To illustrate, consider the following two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := x^2 \qquad\qquad f(x) := \begin{cases} 2 & \text{if } x < 1 \\ 0.5 & \text{if } x \geqslant 1. \end{cases}$$

When drawing their graphs, as in figure 3.1, we can do this for $f$ without lifting the pen, while for $g$ we have to lift it at $x = 0$. And indeed, for $\epsilon := \frac{1}{4} > 0$, we cannot find the required $\delta > 0$.

*Verify this for yourself.*

Now, a good exercise to build intuition is to show that this 'hands-on' definition of continuity is equivalent to—and hence generalized by—the abstract topological definition.

*I admit, the pun is intended*

For this, we have to define the standard topology on the real line $\mathbb{R}$. This is done just like in the three-dimensional case, except that the distance function now simplifies: Here, since $\mathbb{R}$ has just one dimension, $d(x, y) = \sqrt{(x - y)^2} = |x - y|$. So the opens of the real line are those subsets $U \subseteq \mathbb{R}$ such that, for all $x \in \mathbb{R}$, if $x \in U$, then there is $\epsilon > 0$ such that, for all $x' \in \mathbb{R}$ with $d(x, x') < \epsilon$, we have $x' \in U$.

In other words, for a given function $f : \mathbb{R} \to \mathbb{R}$, show that $f$ satisfies 1 iff $f$ is continuous according to definition 3.6 with the just defined topology on $\mathbb{R}$.

*This also is the end-of-chapter exercise 3.c.*

The continuous functions are for topological spaces what the monotone functions were for partial orders. But, like for partial orders, we sometimes also want to consider stronger properties of these structure-preserving maps—in particular, the notion of isomorphism.

**Definition 3.8.** A continuous function $f : X \to Y$ between topological spaces is

- *open* if, for all open $U \subseteq X$, the image $f[U] = \{f(x) : x \in U\}$ is an open subset of $Y$.
- *closed* if, for all closed $C \subseteq X$, the image $f[C] = \{f(x) : x \in C\}$ is a closed subset of $Y$.
- a *homeomorphism* (the topologists' name for isomorphism) , if $f$ has a continuous inverse, i.e., $f$ is a bijection and both $f$ and $f^{-1}$ are continuous. (Equivalently, as exercise 3.d shows, $f$ is a continuous and open bijection; this is further equivalent to $f$ being a continuous and closed bijection.)
- an *embedding*, $f$ is injective and, for each open $U \subseteq X$, there is an open $V \subseteq Y$ such that $f[U] = f[X] \cap V$. Equivalently, the function $f : X \to f[X]$ is a homeomorphism when giving $f[X] \subseteq Y$ the subspace topology (whose opens are $V \cap f[X]$ for $V \subseteq Y$ open).

*Note the additional 'e': it is not 'homomorphism' as with lattices.*

*This is the conceptual meaning of embedding: $X$ is, up to homeomorphism, a subspace of $Y$.*

Homeomorphisms are the isomorphisms of spaces: If there is a homeomorphism between spaces they are called homeomorphic and hence are topologically the same. The standard example is that a donut and a coffee mug are homeomorphic: you can obtain one from the other by squishing and squeezing, but—importantly—without breaking and tearing.

*Hence the common joke that topologists cannot tell them apart.*

### 3.1.3. Constructions: subspaces, products, quotients

Coming to constructions with topological spaces, we have the following.

*Cf. the trinity of sublattice, product, and homomorphic images/quotient for lattices.*

1. Subspace. Given a topological space $(X, \tau)$, any nonempty subset $Y \subseteq X$ can be naturally made into a topological space by equipping it with the *subspace topology*

$$\tau \restriction Y := \{U \cap Y : U \in \tau\}.$$

2. Product topology. If $(X_i)_{i \in I}$ is a collection of topological spaces indexed by a set $I$, the product space $\prod_{i \in I} X_i$ has as underlying set the Cartesian product of the sets $X_i$ and its topology is generated by the subbase of sets of the form $\left\{x \in \prod_{i \in I} X_i : x_j \in V\right\}$ for $j \in I$ and $V \subseteq X_j$ open.

*Equivalently, this is the smallest topology making continuous all the projections $\pi_i : \prod_I X_i \to X_i$ mapping $x$ to its $i$-th component $x_i$.*

3. Quotient space. If $X$ is a topological space and $\equiv$ an equivalence relation on $X$, the quotient space has as underlying set $X/\equiv$ and the opens are those sets $U \subseteq X/\equiv$ such that $\{x \in X : [x]_\equiv \in U\}$ is open in $X$.

A construction specific to spaces is that we can take the *join* of two topologies that live on the same underlying set. This is made precise as follows.

1. If $X$ is a nonempty set, then

$$\mathrm{Top}(X) := \left\{\tau \in 2^{2^X} : \tau \text{ is a topology}\right\}$$

is, when ordered under inclusion, a complete lattice.

2. Infima are given by intersections, and suprema are given by the topology generated by unions. The least element is the indiscrete topology, and the greatest element is the discrete topology.

3. In particular, if $\sigma$ and $\tau$ are two topologies on $X$, then their *join* $\sigma \vee \tau$ is the topology generated by $\sigma \cup \tau = \{U \subseteq X : U \in \tau \text{ or } U \in \sigma\}$.

Some fun facts are that the Hausdorff topologies on $X$ form an upset in $\mathrm{Top}(X)$, and the compact topologies form a downset. And compact-Hausdorff topologies are incomparable: If $\sigma$ is a Hausdorff topology and $\tau$ a compact topology on the nonempty set $X$, then $\sigma \subseteq \tau$ implies $\sigma = \tau$.

*A proof of this can be found in Gehrke and van Gool (2023, Prop. 2.10).*

## 3.2. Important topological concepts

Now that we know topological spaces, we introduce some further topological concepts that are important in their own right and will figure in the definition of a Stone space.

### 3.2.1. Separation axioms

There is an important classification of topological spaces according to which so-called *separation axioms* they satisfy. There are many such axioms and they all are of the form that two distinct points can be—in various senses—separated by the topology. (Note that, despite them being called 'axioms', in general a topological space is not required to satisfy any of them.) The five main ones coming in increasing strength.

**Definition 3.9.** Let $(X, \tau)$ be a topological space.

1. $X$ is $T_0$ (aka *Kolmogorov*) if, for all $x \neq y$ in $X$, there is an open $U \subseteq X$ such that $U$ contains exactly one of $x$ and $y$.

2. $X$ is $T_1$ (aka *Fréchet*) if, for all $x \neq y$ in $X$, there is an open $U \subseteq X$ such that $x \in U$ and $y \notin U$.

3. $X$ is $T_2$ (aka *Hausdorff*) if, for all $x \neq y$ in $X$, there are disjoint opens $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

4. $X$ is $T_3$ (aka *regular*) if $X$ is $T_1$ and, for all $x \in X$ and closed $C \subseteq X$ with $x \notin C$, there are disjoint opens $U, V \subseteq X$ with $x \in U$ and $C \subseteq V$.

5. $X$ is $T_4$ (aka *normal*) if $X$ is $T_1$ and, for all disjoint closed $C, D \subseteq X$, there are disjoint opens $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

**Exercise 3.10.** Show that these conditions indeed increase in strength, i.e., that we have the implications: $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$. Also check that the three-dimensional space $\mathbb{R}^3$ is normal and hence also satisfies the other separation axioms. Moreover, show that $(\mathbb{N}, \tau)$, where $\tau$ is the collection of upsets of $\mathbb{N}$ with the usual order, is a topological space that is $T_0$ but not $T_1$ (and hence also not $T_2$, $T_3$, or $T_4$).

*That also is the end-of-chapter exercise 3.e.*

### 3.2.2. Compactness

Another important concept is that of compactness. It formalizes the intuition that the space does not extend infinitely but has finite bounds.

**Definition 3.11** (Compactness)**.** Let $X$ be a topological space. If $S \subseteq X$ is a subset, an *open cover* $\mathcal{U}$ of $S$ is a collection of open sets such that $S \subseteq \bigcup_{U \in \mathcal{U}} U$. A subset $S \subseteq X$ is *compact* if every open cover $\mathcal{U}$ of $S$ contains a finite subcover, i.e., there is a finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of $S$. The space $X$ is called *compact* if $S := X$ is compact.

For example, while the whole Euclidean space is not compact, closed boxes in it like the unit cube

$$[0,1] \times [0,1] \times [0,1] = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leqslant x_1, x_2, x_3 \leqslant 1\right\}$$

are compact. Also any finite subset of a space is compact. Exercise 3.f asks you to establish this.

There also is a local version of compactness: A topological space $X$ is *locally compact* if, for any open neighborhood $U$ of any point $x \in X$, there is an open $V \subseteq X$ and compact $K \subseteq X$ such that $x \in V \subseteq K \subseteq U$. If $X$ is Hausdorff, then compactness implies local compactness, but this is not true in general. And local compactness does also not imply compactness (the Euclidean space is locally compact but not compact).

The following exercise collects some very useful facts about compactness.

**Exercise 3.12.** Prove the following:

*This is exercise 3.g.*

1. A closed subset of a compact space is compact.

2. A compact subset of a Hausdorff space is closed. This need not be true without the Hausdorffness assumption.

3. The image of a compact subset under a continuous function is compact.

4. Conclude that a continuous function from a compact space to a Hausdorff space is closed.

5. Conclude (hint: exercise 3.d) that a continuous bijection between compact Hausdorff spaces is a homeomorphism.

6. A compact Hausdorff space is normal (hint: show it is regular first).

Some further and more advanced results are the following.

1. *Finite intersection property characterization.* Let $X$ be a topological space and $S \subseteq X$ a subset. A collection $\mathcal{A}$ of closed sets has the *finite intersection property* with respect to $S$ if for every finite subcollection $\mathcal{A}_0$, there is $x \in S$ such that $x \in \bigcap \mathcal{A}_0$. Then $S$ is compact iff, for every collection $\mathcal{A}$ of closed sets with the finite intersection property with respect to $S$, there is $x \in S$ with $x \in \bigcap \mathcal{A}$. (The proof essentially is rewriting the open set definition by taking complements and using the de Morgan laws: see, e.g., here.)

   *If $S = X$, we omit the 'with respect to $S$'.*

2. *Alexander Subbase Theorem.* Let $X$ be a topological space and $\mathcal{S}$ a subbase. If every cover $\mathcal{U} \subseteq \mathcal{S}$ of $X$ has a finite subcover, then $X$ is compact.

   *The proof of this requires a non-constructive principle, i.e., a (strictly weaker) version of the axiom of choice. As this is an axiom of standard set theory, we assume this throughout in this course.*

3. *Tychonoff's Theorem.* This says that the arbitrary product of compact spaces is again compact. (This is a corollary of the Alexander Subbase Theorem; see, e.g., here.)

### 3.2.3. Dimensionality

We say that the space we live in has three dimensions. But is there a way to reasonably define the dimension of a general topological space $(X, \tau)$ in such a way that the dimension of $\mathbb{R}^3$ considered as a topological space is indeed 3? It turns out there is; in fact there are several such reasonable definitions, though they may disagree on some topological space $(X, \tau)$ different from $\mathbb{R}^d$. The common ones are called small inductive dimension, large inductive dimension, and the Lebesgue covering dimension. For a full treatment, see Engelking (1989, ch. 7). Here we will only look at the small inductive dimension. In fact, eventually we will only consider zero-dimensional spaces. So we will only briefly discuss dimensionality to get the general motivation (definition 3.13) and then prove a simple characterization for the zero-dimensional spaces (proposition 3.14).

According to our intuitive notion of dimension, one-dimensional objects are things like a line or a circle, two-dimensional objects are things like a plane or a disk, and three-dimensional objects are things like 'our' space or also a sphere (say, a football). In the extreme case, we may also say that a single point has dimension 0. Moreover, the dimensions of these objects are connected as follows: if we consider the boundary of an object, we reduce the dimension by one. For example, the boundary—i.e., surface—of a sphere is two-dimensional (an ant walking on the football just experiences a two-dimensional world); and the boundary of a disk is just the circle.

The formal definition of the small inductive dimension bootstraps these intuitions: The simplest case is where the object is just the empty set. Its dimension is stipulated to be $-1$. Because then, if we consider the next more complicated object, namely a single point, then its is indeed 0: it must be one more than the dimension of its boundary, but its boundary is empty, hence has dimension $-1$, so the dimension of the point is one more, i.e., 0. Now we proceed inductively: Assume we have a yet more complicated object, but which is still not much more complicated in the sense that we can already determine the dimension of its boundary. Then the dimension of this object is one more than the dimension of its boundary.

The formal definition goes as follows (relativizing the above idea to each neighborhood of points of the considered object).

**Definition 3.13.** Given a regular topological space $(X, \tau)$ (which, only for this definition, we allow to also be empty) and an integer $n \geqslant 0$, say

- $\mathrm{ind}(X) = -1$ iff $X = \emptyset$

- $\operatorname{ind}(X) \leqslant n$ iff for every point $x \in X$ and open neighborhood $V$ of $x$, there is an open $U \subseteq X$ with $x \in U \subseteq V$ and $\operatorname{ind}(\partial U) \leqslant n-1$.
- $\operatorname{ind}(X) = n$ iff $\operatorname{ind}(X) \leqslant n$ and $\operatorname{ind}(X) \nleqslant n-1$.
- $\operatorname{ind}(X) = \infty$ iff, for all $n$, $\operatorname{ind}(X) \nleqslant n$.

*Recall that* $\partial S = \mathsf{Cl}(S) \setminus \mathsf{Int}(S)$ *is the boundary of the set* $S$.

As mentioned, we will actually be interested in spaces with dimension zero. They have the following simple characterization. It is just an unpacking of definitions, so we leave the proof as exercise 3.h.

**Proposition 3.14.** *Let* $(X, \tau)$ *be a regular topological space. Then* $X$ *is zero-dimensional (i.e.,* $\operatorname{ind}(X) = 0$*) iff* $X$ *has a base consisting of clopen sets.*

Hence, from now on, we take the definition of *zero-dimensional* to be having a base of clopens.

It may seem that being zero-dimensional is quite different from our everyday intuitions about space. This is true to some extent, but the following example shows that zero-dimensional spaces are also not completely strange.

**Exercise 3.15.** Write $X := \mathbb{R} \setminus \mathbb{Q}$ for the set of *irrational* numbers and let $\tau$ be the subspace topology with respect to the usual topology on $\mathbb{R}$. Show that the space of irrational numbers $(X, \tau)$ is a zero-dimensional space. (Hint: show that $\{B_{\frac{1}{n}}(q) \cap X : n \in \mathbb{N}, q \in Q\}$ is a clopen base.)

*This also is the end-of-chapter exercise 3.i.*

## 3.3. Stone spaces

Now we can define Stone spaces (section 3.3.1). Then we will discuss an important example of a Stone space: namely, the Cantor space (section 3.3.2).

### 3.3.1. Definition

The definition of a Stone space is actually quite very simple.

**Definition 3.16.** A *Stone space* is a topological space that is compact, zero-dimensional, and Hausdorff.

Though you may wonder why it is exactly those spaces that we will consider. There are two answers. First, several important spaces—especially in descriptive sets theory—are Stone spaces. In the next subsection, we will consider the most important one: Cantor space. From this point of view, we consider Stone spaces simply because it makes sense to study the general properties of this important class of spaces.

The second answer is the insight of Stone that it is precisely those spaces that dually correspond to Boolean algebras under the constructions motivated in chapter 1. When we start with a Boolean algebra of, say, propositions and we consider the set of prime filters on it, we will see that we can naturally define a topology on this set. It is generated by calling those sets of prime filters open that make true a given proposition in the Boolean algebra. It turns out that this topology has all the defining features of a Stone space. Conversely, if we start with any topological space, the collection of its clopen subsets forms a Boolean algebra; but if this space actually was a Stone space, then this Boolean algebra is rich enough so we can recover the original space as the set of prime filters on this Boolean algebra.

So the second answer is is a 'hindsight' motivation of the definition: it characterizes a class of spaces that we obtain through a construction that we are interested in. But since we have not seen that yet, let's first explore the first answer: namely, example of Stone spaces that are important in their own right.

### 3.3.2. Example: Cantor space

The Cantor space plays an important role in descriptive set theory (Kechris 1995). One reason is that it is universal in the sense that every nonempty compact metrizable space is a continuous image of the Cantor space. In particular, every Stone space that is second-countable (i.e., has a countable base) is an image of the Cantor space. We will now define the Cantor space and show that it is a Stone space. We will not prove the just mentioned universality result (see Kechris 1995, p. 23), but it assures us that the Cantor space is an important example of a Stone space: it in a sense already contains all second-countable Stone spaces.

*Second-countable Stone spaces are metrizable by Urysohn's metrization theorem.*

The succinct definition of Cantor space is as follows; we will unpack it below.

**Definition 3.17.** Equip $2 := \{0, 1\}$ with the discrete topology. Define $2^{\mathbb{N}}$ to be the product space $\prod_{\mathbb{N}} 2$ of $\mathbb{N}$-many copies of 2. The space $2^{\mathbb{N}}$ is called the *Cantor space*.

The points of Cantor space are, by definition of the Cartesian product (appendix A.1), infinite binary sequences, i.e., functions $x : \mathbb{N} \to 2 = \{0, 1\}$. We also write these sequences as $x(0)x(1)x(2)\ldots$. For example, here are two points of Cantor space:

$$0101010101010101010\ldots \qquad 1110101001010111010\ldots.$$

The topology of Cantor space is concisely described via the product topology, but to build an intuition, it is useful to introduce *cylinder sets*. Given a binary string $\sigma = \sigma_0 \ldots \sigma_{n-1}$ (i.e., a finite sequence of 0's and 1's), the corresponding cylinder set is

$$[\sigma] := \left\{x \in 2^{\mathbb{N}} : \forall k \in \{0, \ldots, n-1\}. x(k) = \sigma_k\right\},$$

so $[\sigma]$ contains all those binary sequences that have $\sigma$ as initial segment.

**Exercise 3.18.** Show that the cylinder sets form a base for the topology of $2^{\mathbb{N}}$. (Hint: go back to the definition of the product topology in section 3.1.3.)

With this, it is quite straightforward to show that the Cantor space is a Stone space.

**Exercise 3.19.** Show the following for the Cantor space $2^{\mathbb{N}}$:

1. It is Hausdorff. (Hint: Conclude this from the previous exercise 3.18.)
2. It is zero-dimensional. (Hint: Show that the cylinder sets in fact are clopen.)
3. It is compact. This follows immediately from Tychonoff's theorem (since the finite space 2 is compact). But, as an exercise, also show this directly with the finite intersection property definition (section 3.2.2).

A few more comments on Cantor space. You might know it as the Cantor *set* from analysis: where you take the unit interval $[0,1]$ and keep removing the middle thirds. More precisely, it consists of those numbers in the unit interval that have only 0's and 2's in their ternary expansion. One can show that this Cantor set is homeomorphic to the Cantor space $2^{\mathbb{N}}$. Similarly, the irrational numbers in the unit interval, i.e., $[0,1] \setminus \mathbb{Q}$ also are homeomorphic to the Cantor space. These examples are instances of a theorem of Brouwer which characterizes the Cantor space as the, up to homeomorphism unique perfect, nonempty, compact, metrizable, zero-dimensional space (Kechris 1995, p. 35). Cantor space also is a natural setting for probability theory (as a standard Borel space) and also for algorithmic randomness (Downey and Hirschfeldt 2010).

*Cf. exercise 3.15*

*A space is perfect if for every open neighborhood $U$ of a point $x$, there is a point $y \in U$ with $y \neq x$.*

## 3.4. Exercises

To get familiar with the abstract topological concepts from section 3.1.1, the next two exercises apply them to the usual three-dimensional space $\mathbb{R}^3$ (whose open sets are those with 'wiggle-room').

**Exercise 3.a.**
1. Prove that the collection $\tau$ of sets $U \subseteq \mathbb{R}^3$ with wiggle-room, as defined in example 3.2 (1), indeed forms a topology on $\mathbb{R}^3$.
2. A *closed interval* is of the form $[a, b] := \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ with $a, b \in \mathbb{R}$. An open interval is of the form $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ with $a, b \in \mathbb{R}$. (They are empty if $a \not\leqslant b$.) In three-dimensional space, a *rectangular cuboid* is of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ with $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$. (So a rectangular cuboid is just the 3D analogue of a rectangle; and a rectangle, in turn, is just the 2D analogue of an interval. For higher dimensions, one speaks of boxes or also hyperrectangles or $k$-cells.) Show that rectangular cuboids are closed in $\mathbb{R}^3$.
3. Show that the sequence of points $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n})_{n \geqslant 1}$ converges to $(0, 0, 0)$.

**Exercise 3.b.** Do exercise 3.5. (If you have done exercise 2.e, you can additionally show that the interior map $\mathsf{Int} : 2^X \to \Omega(X)$ is upper adjoint to the inclusion map $\iota : \Omega(X) \to 2^X$, and that the closure map $\mathsf{Cl} : 2^X \to \mathcal{C}(X)$ is lower adjoint to the inclusion $\iota' : \mathcal{C}(X) \to 2^X$ (Gehrke and van Gool 2023, ex. 2.1.3).)

**Exercise 3.c.** Do exercise 3.7.

*(Gehrke and van Gool 2023, ex. 2.1.1 (c))*

**Exercise 3.d.** Let $f : X \to Y$ be a continuous bijection. Show that the following are equivalent.

*(Gehrke and van Gool 2023, ex. 2.1.4 (c))*

1. $f$ is a homeomorphism (i.e., its inverse is continuous)
2. $f$ is open (i.e., maps open sets to open sets)
3. $f$ is closed (i.e., maps closed sets to closed sets).

**Exercise 3.e.** Do exercise 3.10.

**Exercise 3.f.** This exercise gets you acquainted with the concept of compactness via some examples.

1. Show that $\mathbb{R}^3$ is not compact but the unit cube is.

2. Show that any finite subset of any topological space is compact.

**Exercise 3.g.** Do exercise 3.12.

**Exercise 3.h.** Prove proposition 3.14.

**Exercise 3.i.** Do exercise 3.15.

**Exercise 3.j.** Do exercises 3.18 and 3.19 to establish that the Cantor space is a Stone space.

# 4. Two sides of the same coin: Stone duality

# 5. Application: (Re)discovering semantics for modal logic

# 6. Generalization: Priestley duality

# A. Appendix

## A.1. Set-theoretic terminology

We use standard set-theoretic terminology as it is common in mathematics. A set is a collection of objects. We write $a \in A$ to say that object $a$ is in (or is an element of, or is a member of) the set $A$. If $a_1, \ldots, a_n$ are objects, we write $\{a_1, \ldots, a_n\}$ for the set of these objects. Sets do not count 'order' and 'multiplicities', so $\{1, 0, 2, 2\} = \{0, 1, 2\}$. A set with just one element is called a singleton, and if $a$ is an object, $\{a\}$ is the singleton of $a$ (note $\{a\} \neq a$). The set without any elements is called the empty set and is denoted $\emptyset$.

*See, e.g., Priest (2008, sec. 0.1).*

If $A$ and $B$ are sets, we say $A$ is a subset of $B$ (written $A \subseteq B$) if every element of $A$ is an element of $B$. So the empty set trivially is a subset of any set. And two sets $A$ and $B$ are identical iff $A \subseteq B$ and $B \subseteq A$. If $A$ and $B$ are sets, then the union of $A$ and $B$ (written $A \cup B$) is the set containing exactly those objects that either are in $A$ or in $B$ (or both). The intersection of $A$ and $B$ (written $A \cap B$) is the set containing exactly those objects that are both in $A$ and in $B$. The complement of a set $A$ relative to a set $B$ (written $B \setminus A$) is the set of objects that are in $B$ but not in $A$. If $B$ is clear from context, we just write $A^c$.

A pair (aka ordered pair) is a list of two elements $(a, b)$; here the order matters, so $(a, b) \neq (b, a)$. (We can define $(a, b)$ as the set $\{\{a\}, \{a, b\}\}$.) More generally, an $n$-tuple is a list of $n$ elements $(a_1, \ldots, a_n)$. Given $n$ sets $A_1, \ldots, A_n$, their Cartesian product (written $A_1 \times \ldots \times A_n$) is the set of all $n$-tuples $(a_1, \ldots, a_n)$ such that, for all $i \in \{1, \ldots, n\}$, we have $a_i \in A_i$. More generally, the Cartesian product of a potentially infinite family $\{A_i : i \in I\}$ of sets is defined as the set $\prod_{i \in A} A_i$ of functions $a$ that map each $i \in I$ to an element $a(i) \in A_i$. We often write such a function as $a = (a_i)_{i \in I}$; because in the above case where $I = \{1, \ldots, n\}$, we can think of a tuple $(a_1, \ldots, a_n)$ as a the function $a$ mapping $i \in I$ to $a_i$. An $n$-ary relation between $A_1, \ldots, A_n$ is a subset of $A_1 \times \ldots \times A_n$. A 1-ary (resp., 2-ary, 3-ary) relation is also called a unary (resp., binary, ternary) relation. For a binary relation $R$, we usually write $aRb$ instead of $(a, b) \in R$.

A function from a set $A$ (its domain) to a set $B$ (its codomain) is a binary relation $f$ between $A$ and $B$ such that, for every $a \in A$, there is exactly one $b \in B$ such that $afb$. We then write $f : A \to B$ and $f(a) = b$ or, if $f$ is clear from context, $a \mapsto b$. If $f : A \to B$ and $g : B \to C$ are functions, $g \circ f$ ($g$ after $f$) denotes their composition, which maps $a \in A$ to $g(f(a)) \in C$. Given a set $A$, the identity function $\mathsf{id}_A : A \to A$ maps $a$ to $a$. By an $n$-ary function on a set $A$ we mean a function $f : A^n \to A$, where $A^n = A \times \ldots \times A$ is the $n$-time Cartesian product of set $A$. Again, the first arities have special names: unary (= 1-ary), binary (= 2-ary), and ternary (= 3-ary). Sometimes it is convenient to take a 0-ary function to be a constant (i.e., an element or symbol which is fixed throughout).

An equivalence relation $\equiv$ on a set $A$ is a binary relation on $A$ such that

1. $\equiv$ is reflexive, i.e., for all $a \in A$, we have $a \equiv a$,
2. $\equiv$ is transitive, i.e., for all $a, b, c \in A$, if $a \equiv b$ and $b \equiv c$, then $a \equiv c$, and
3. $\equiv$ is symmetric, i.e., if $a \equiv b$, then $b \equiv a$.

If $a \in A$, then the $\equiv$-equivalence class of $a$ is the set $[a]_\equiv := \{b \in A : a \equiv b\}$. An element $b \in [a]_\equiv$ is called a representative of $[a]_\equiv$. The quotient of $A$ under $\equiv$ is defined as $A/\equiv := \{[a]_\equiv : a \in A\}$. The function $\pi : A \to A/\equiv$ defined by $\pi(a) := [a]_\equiv$ is called the projection of $\equiv$.

## A.2. Category-theoretic terminology

To be written. For now, see, e.g., Leinster 2014, ch. 1.

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