Duality Theory
Connecting Algebra and Topology via Logic

Lecture Notes

Draft from January 23, 2025.
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Comments welcome!

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Contents

Preface

This is the reader for the course "Duality Theory: Connecting Logic, Algebra, and Topology" given during the winter semester 2024/5 at *LMU Munich* as part of the *Master in Logic and Philosophy of Science*. These lecture notes are updated as the course progresses. A website with all the course material is found at

> https://levinhornischer.github.io/DualityTheory/.

Comments   I'm happy about any comments: spotting typos, finding mistakes, pointing out confusing parts, or simply questions triggered by the material. Just send an informal email to Levin.Hornischer@lmu.de.

Course description and objectives   This course is an introduction to duality theory, which is an exciting area of logic and neighboring subjects like math and computer science. The fundamental theorem is Stone's duality theorem stating that certain algebras (Boolean algebras) are in a precise sense equivalent to certain topological spaces (totally disconnected compact Hausdorff spaces). The underlying idea is that the two seemingly different perspectives—the algebraic one and the spatial one—are really two sides of the same coin:

- formulas/propositions vs. models/possible worlds,

- open sets of a space vs. points of the space,

- properties of a computational process vs. denotation of the computational process.

In terms of content, the focus of the course will be to introduce the mathematical theory, after a philosophical motivation. In terms of skills, the aim is to learn how to apply the tools of duality theory. We will illustrate this with applications—especially to philosophical phenomena—that make use of dualities by combining the often opposing advantages of the two perspectives.

Prerequisites   An introductory course in logic and some familiarity with mathematics (ideally, but not necessarily, having seen elementary concepts of topology and algebra), including the basics of writing mathematical proofs.

Apart from that, the course can be taken independently. But it also makes sense to take it as a follow-up course of the course Philosophical Logic. In that course, I stress two different approaches to giving semantics to various logics: the algebraic approach and the state-based approach. These approaches are often equivalent, which is a special case of duality.

Contents   We start with an informal chapter describing the key idea of duality. The rest of the course is about developing this key idea precisely. We first precisely define the algebraic structure (Boolean algebras) and the topological structures (topological spaces), and then prove the duality result. The remainder of the course is about applying this result to modal logic (and sketching applications in computer science) and generalizing the result to Priestley duality.

Layout   These notes are informal and partially still under construction. For example, there are margin notes to convey more casual comments that you'd rather find in a lecture but usually not in a book. Todo notes indicate, well, that something needs to be done. References are found at the end. Exercises are at the end of each chapter.

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Study material   The main textbook that we use is by Gehrke and van Gool (2023). And informal introduction to duality is provided by Gehrke (2009). Some further textbooks include:

- R. Balbes and P. Dwinger (1975). *Distributive lattices*. University of Missouri Press

- B. A. Davey and H. A. Pristley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press

- S. Vickers (1989). *Topology via Logic*. Cambridge: Cambridge University Press

- S. Givant and P. Halmos (2008). *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. New York: Springer-Verlag

- S. Givant (2014). Ed. by D. theories for Boolean algebras with operators. Springer

- G. Grätzer (2011). *Lattice Theory: Foundation*. Birkhäuser

- G. Grätzer (2003). *General Lattice Theory*. 2nd ed. Birkhäuser

Research monographs on duality theory are

- P. T. Johnstone (1982). *Stone Spaces*. Cambrdige studies in advanced mathematics 3. Cambridge: Cambridge University Press

- G. Gierz et al. (2003). *Continuous Lattices and Domains*. Cambridge: Cambridge University Press

- M. Dickmann et al. (2019). *Spectral Spaces*. New Mathematical Monographs. Cambridge University Press. DOI: 10.1017/9781316543870

- J. Goubault-Larrecq (2013). *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press

- J. Picado and A. Pultr (2012). *Frames and Locales*. Birkhäuser

- S. Abramsky and A. Jung (1994). "Domain Theory." In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky et al. Corrected and expanded version available at http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf (last checked 24 January 2018). Oxford: Oxford University Press

- E. Orłowska et al. (2015). *Dualities for Structures of Applied Logic*. Studies in Logic 56. College Publications

Notation   Throughout, 'iff' abbreviates 'if and only if'.

1. Introduction: the key idea of duality

Duality theory is a mathematical theory relating algebraic structures to geometric or spatial structures. It is a formal mathematical theory; but underlying it, is a deep philosophical idea. In this chapter, we describe this philosophical story—the key idea of duality—before developing the mathematical theory and its applications in the later chapters.

Advice on how to read this chapter. Duality theory can be confusing when one first hears about it. One has to keep track of many moving parts, making sure they all fit together. At least to me, reminding myself of the philosophical story helps: it provides the 'rhyme and reason' to the mathematics. So whenever you feel lost in the midst of the technical detail, you can come back to this philosophical story. It is a powerful and potentially unfamiliar idea, so give it some time to sink in and go through this conceptual motivation over and over again. Also, as you progress to the later, more technical chapters, be sure to come back to this introduction chapter to see how the intuitive ideas here are developed formally.

Duality theory can be quite abstract. The advantage of this is that it makes duality ubiquitous and widely applicable. But a disadvantage is that this makes it less accessible. So before attempting any general definition of duality, let us consider several examples (section 1.1). From those we can generalize an informal characterization of duality (section 1.2). This then hints at how duality theory is formalized mathematically and how it can be applied. Finally, in section 1.3 we list some exercises.

For other expositions of the philosophical idea behind duality, see, e.g., Abramsky (1991), Gehrke (2009), and Vickers (1989).

To use the words of Abramsky (2023).

1.1. Intuitive examples of duality

We present several examples of duality. We do so at a very informal and intuitive level, and we do not at all aim to be philosophically careful or mathematically precise. In fact, think of it as an *exercise* to revisit these examples once you know more about the formal development of duality theory—and see what more precise analysis you can provide.

I think this is a philosophically very fruitful exercise—or, better, research project. In particular, this makes for an excellent essay topic.

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1.1.1. Metaphysics: Properties vs objects

When we perceive and reason about the world, we naturally think in terms of there being various objects that have—or do not have—various properties. Objects are, for example, my laptop, the Eiffel Tower, or the Moon, and we will here also include merely possible objects like unicorns. Properties are, for example, being red, being higher than 300m, or being made of cheese. (We consider here only unary properties: i.e., those that apply to a single object, but not to multiple objects, like being taller than.) Philosophically, it is difficult to make this talk of objects and properties precise (e.g., if we are too permissive about what counts as a property, Russell's paradox creeps in). For now, let us just rely on our everyday intuitions about these concepts. Once we see where this will lead, the exercises at the end of this chapter will ask you to come back and scrutinize the concept of object and property at play (see exercise 1.c).

If 'being a property' is a property, consider the property p of 'not being a property'. Then p has p iff p does not have p, contradiction. But, arguably, 'being a property' is not an 'everyday' property.

Let us write $\mathcal{O}$ for the set of all possible objects and $\mathcal{P}$ for the set of all properties. Crucially, observe that there is a certain dependency between $\mathcal{O}$ and $\mathcal{P}$:

($\mathcal{O} \to \overline{\mathcal{P}}$) Each object $x \in \mathcal{O}$ determines a set of properties $F_x \subseteq \mathcal{P}$ consisting of precisely those properties that x has.

Philosophers also call F_x the role *of the individual x (McMichael 1983, p. 57).*

(The bar in '$\overline{\mathcal{P}}$' indicates that we assign to each x a *set* of elements in $\mathcal{P}$ rather than a single element of $\mathcal{P}$.) So we might wonder whether we can also go in the opposite direction ($\overline{\mathcal{P}} \to \mathcal{O}$)? Does a subset F of properties also determine an object, i.e., the unique object that has exactly the properties in F? Actually, no: some sets of properties might not be satisfied by any object (e.g., $F = \{being\ exactly\ 300m\ high, being\ exactly\ 200m\ high\}$) or by more than one (e.g., $F = \{being\ exactly\ 300m\ high\}$).

Philosophers know phrases of the form 'The F' (referring to the unique object satisfying F) as definite description. *For their important role in philosophy, see e.g. Ludlow (2022).*

But let us not give up too early. After all, the set F_x is not just *any* set of properties, but it has some nice features which we collect now. (And the hope is that if F is a set of properties with these nice features, that then it determines a unique object.)

1. Assume $a, b \in \mathcal{P}$ are two properties such that having a implies having b; we abbreviate this as $a \leqslant b$. For example,

   $a = being\ higher\ than\ 300m \leqslant being\ higher\ than\ 200m = b.$

   So if our object x has property a, then it also has property b, i.e., if $a \in F_x$, then $b \in F_x$. We may express this as: F_x is closed under implication.

Later we will say F_x is an upset. *This sounds funny now, but by the end of the course, you will have said this so often that you won't even notice.*

2. Assume $a, b \in \mathcal{P}$ are two properties. Note that then there is another property: namely, the property of having both property a and property b. We denote this property $a \wedge b$. So $a \wedge b$ is again in $\mathcal{P}$ and we have $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$. Moreover, if our object x has property a and it has property b, then it has property $a \wedge b$, i.e., if $a, b \in F_x$, then $a \wedge b \in F_x$. We may express this as: F_x is closed under conjunction.

3. Similarly, if $a, b \in \mathcal{P}$ are two properties, there also is the property of having either property a or property b (or both). We denote this property $a \vee b$. So $a \vee b$ is again in $\mathcal{P}$ and we have $a \leqslant a \vee b$ and $b \leqslant a \vee b$. Moreover, if our object x has property $a \vee b$, then either it has property a or it has property b, i.e., if $a \vee b \in F_x$, then either $a \in F_x$ or $b \in F_x$. Later, we express this as F_x being prime.

4. Note that $\mathcal{P}$ also contains the trivial property like being identical to oneself. We denote this property $\top$. In particular, our object x has it, i.e., $\top \in F_x$.

5. Similarly, note that $\mathcal{P}$ also contains the inconsistent property like not being identical to oneself. We denote this property $\bot$. In particular, our object x does not have it, i.e., $\bot \notin F_x$.

Now, we can ask our question again: If F is a set of properties with these features, does *it*—as opposed to any arbitrary set of properties—determine a unique object? In other words, is there exactly one object that has all the properties in F? It might be an attractive metaphysical (or, better, ontological) principle to answer yes and hold that:

($\overline{\mathcal{P}} \to \mathcal{O}$) Each set of properties $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines an object $x \in \mathcal{O}$, namely, the unique object having exactly the properties in F.

The uniqueness part is close to Leibniz's principle about the identity of indiscernibles: if two objects x and x' have exactly the properties in F, they are indiscernible, and hence are identical according to Leibniz. The existence part amounts to a certain *ontological completeness*: that for every consistent description F of an object, there in fact is a (possible) object that has these properties. This is why we consider the set $\mathcal{O}$ of all possible objects. The actual world need not be ontologically complete: F might consistently describe a unicorn, even if this does not exist in the actual world.

We will see that this bidirectional determination $(\mathcal{O} \to \overline{\mathcal{P}})$ and $(\overline{\mathcal{P}} \to \mathcal{O})$ is a hallmark of duality, here between objects and properties. We might also speak of mutual dependency, supervenience, or necessitation.

Moreover, we started our considerations from objects and considered their ontology; but we could also start from properties and wonder about their ontology. The analog of Leibniz's principle would be the extensionality principle: two properties a and b are identical if they apply to exactly the same possible objects (i.e., for all $x \in \mathcal{O}$, x has a iff x has b). Each property a determines a set of objects: namely, the set of those objects that have property a. This is known as the *extension* of the property. Analogously to before, we might also ask if every set of objects determines a property: namely, the property determined by having this set of objects as extension. Prima facie one would think that this should be the case, but we will see that duality provides a different answer: only some—and not all—sets of objects determine a property.

Cf. the extensionality principle in set theory which says that two sets are identical iff they have the same elements. Since we talk about all possible objects, not just the actual ones, some philosophers might rather call this the intension of the property, as it involves not just the actual world, but also objects from other possible worlds.

1.1.2. Semantics: Propositions vs possible worlds

The central question of philosophy of language is: What is the meaning of sentences? The meaning of a sentence is also called the *proposition* that the sentence expresses. The standard answer to this question, as far as there is one, is possible worlds semantics: The meaning of a sentence (i.e., the proposition it expresses) is the set of possible worlds in which the sentence is true. Here, a possible world is a consistent and complete description of how our world could have been. One example is the possible world which is just like our world but where the Eiffel Tower is 400m high. So the proposition a expressed by the sentence 'The Eiffel Tower is 330m high' contains the actual world x_0 (i.e., $x_0 \in a$) but not the just described possible world x_1 (i.e., $x_1 \notin a$). Some common notation for the phrase 'world x makes true proposition a' is $x \vDash a$; so possible world semantics analyses $\vDash$ as elementhood $\in$.

There is much debate in philosophy what the set $\mathcal{W}$ of possible worlds is (Menzel 2021) and what the set $\mathcal{P}$ of propositions is (McGrath and Frank 2023). Both are taken to exist in their own right and be important objects of study. But their nature is disputed. For example, is it really the case, as possible world semantics claims, that propositions are just sets of worlds ('worlds first, propositions later')? Or is it rather that worlds are maximally consistent sets of propositions ('propositions first, worlds later')? The latter goes by the name 'ersatzism' since full-blown possible worlds are substituted by something constructed out of linguistic entities—

and 'Ersatz' is German for substitute.

We won't enter this debate here. Instead, we observe again that there is a bidirectional determination between worlds and propositions. To start, a plausible principle to hold about worlds and propositions is the following. It is satisfied by possible worlds semantics, and, in fact, arguably its characteristic feature.

World individuation Possible worlds are individuated by the propositions they make true: if two possible worlds x and y make true exactly the same propositions (i.e., for every proposition a, we have $x \vDash a$ iff $y \vDash a$), then $x = y$.

Proposition individuation Propositions are individuated by the possible worlds at which they are true: if two propositions a and b are true at exactly the same possible worlds (i.e., for every possible world x, we have $x \vDash a$ iff $x \vDash b$), then $a = b$.

And there is more. Just like properties, also the set of propositions has logical structure: If a and b are propositions, there also are the propositions $a \wedge b$ (conjunction), $a \vee b$ (disjunction), $\neg a$ (negation), $\top$ (logical truth), and $\bot$ (logical falsity). With this we can also express implications between propositions: proposition a implies proposition b, written $a \leqslant b$, precisely if $a \wedge b = a$. The proposition expressed by 'I am in Munich' implies the proposition expressed by 'I am in Germany' because the sentence 'I am in Munich and I am in Germany' is equivalent to the sentence 'I am in Munich', i.e., they express identical propositions.

Thus, given a possible world $x \in \mathcal{W}$, we can again consider the set of propositions $F_x \subseteq \mathcal{P}$ that are true in x (i.e., $F_x = \{a \in \mathcal{P} : x \vDash a\}$). And F_x again satisfies the features (1)–(5) above: If $a \in F_x$, i.e., $x \vDash a$, and a implies b, i.e., $a \leqslant b$, then $x \vDash b$, i.e., $b \in F_x$. If $a, b \in F_x$, then x makes true both a and b, so $a \wedge b \in F_x$. As an exercise, go through the other cases as well.

Another plausible principle to hold about worlds and propositions is, again, that

Metaphysical completeness Each set of propositions $F \subseteq \mathcal{P}$ satisfying (1)–(5) determines a possible world $x \in \mathcal{W}$, namely, the unique possible world making true exactly the propositions in F.

Ersatzism, for example, endorses this principle; let us see why. We will later formally show that a set of propositions F satisfying (1)–(5) is maximally consistent: one cannot add a single more proposition to F without

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making it inconsistent (i.e., making it contain $\bot$). Ersatzism not only claims that then there is a world x which makes true exactly the propositions in F, it even identifies this world x with F. Hence both the existence and the uniqueness of x follows.

In other words, there is an exact match between possible worlds and sets of propositions satisfying (1)–(5). Formally, we say there is a bijective correspondence between the set $\mathcal{W}$ of possible worlds and the set $\overline{\mathcal{P}}$ of sets of propositions satisfying (1)–(5). (To anticipate terminology, these sets $F \in \overline{\mathcal{P}}$ will be called *prime filters* and $\overline{\mathcal{P}}$ will be called the *spectrum* of the algebra of propositions.)

$$\mathcal{W} \leftrightarrows \overline{\mathcal{P}}$$

$$x \mapsto F_x = \{a \in \mathcal{P} : x \vDash a\}$$

the x making true exactly the $a \in F \leftarrowtail F$

Let us verify that this really is a bijection: We have already checked that the function $f : \mathcal{W} \to \overline{\mathcal{P}}$ mapping x to F_x is well-defined. It is injective by the world individuation principle: if $x \neq y$, then there is a proposition a with $x \vDash a$ and $y \nvDash a$ (or vice versa), so $a \in F_x$ and $a \notin F_y$ (or vice versa), so $F_x \neq F_y$. It is surjective by metaphysical completeness: Given $F \in \overline{\mathcal{P}}$, let x be the unique world in $\mathcal{W}$ making true exactly the propositions in F. Then $F = F_x$ because: $a \in F$ iff $x \vDash a$ iff $a \in F_x$.

So far, we have looked at the relation between full-blown metaphysical worlds (the elements of $\mathcal{W}$) and their ersatz constructions as sets of propositions (the elements of $\overline{\mathcal{P}}$). But what about the other side: How do full-blown propositions (the elements of $\mathcal{P}$) relate to sets of worlds, i.e., their counterparts propagated by possible worlds semantics?

Every proposition $a \in \mathcal{P}$ determines the set of worlds $[\![a]\!] := \{x \in \mathcal{W} : x \vDash a\}$ where a is true. This is also known as the *truthset* of a. And we might again wonder whether we can also go in the opposite direction: whether every set of worlds also determines a proposition? This issue is actually not too much discussed in the philosophy of a language, and one often at least talks as if this is true. So let's see where this takes us. Let us write $\overline{\mathcal{W}}$ for the sets of worlds that determine propositions and $2^{\mathcal{W}}$ for the set of all sets of worlds. So our assumption for now is that $\overline{\mathcal{W}} = 2^{\mathcal{W}}$. Analogous to the previous case, we want to know if the function

$$[\![\cdot]\!] : \mathcal{P} \to 2^{\mathcal{W}}$$

$$a \mapsto [\![a]\!] = \{x \in \mathcal{W} : x \vDash a\}$$

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is a bijection. We are off to a good start: The function is injective by the proposition individuation principle: if $a \neq b$, there is a world x with $x \vDash a$ and $x \nvDash b$ (or vice versa), so $[\![a]\!] \neq [\![b]\!]$. In fact, it also preserves the logical structure: $[\![a \wedge b]\!] = [\![a]\!] \cap [\![b]\!]$, $[\![\bot]\!] = \emptyset$, etc. (Later we formalize this as $[\![\cdot]\!]$ being a Boolean algebra homomorphism.) However, the issue is surjectivity. (Above, this also required another assumption: metaphysical completeness.)

Here is one argument why $[\![\cdot]\!]$ is not surjective. Plausibly, since propositions are the meanings of sentences, every proposition is expressed by some sentence. But since there are only countably many sentences (they are generated by a 'finitistic' grammar), there hence only are countably many propositions. However, since there plausibly are infinitely many possible worlds (be it countably or uncountably many), the powerset 2^W of W is uncountable. So $\mathcal{P}$ and 2^W have different cardinalities, which means there cannot be a bijection between them, hence the already injective function $[\![\cdot]\!]$ cannot be surjective.

That is Cantor's diagonal argument.

So actually not any set of worlds determines a proposition, i.e., $\overline{W}$ is a proper subset of 2^W. The ingenious insight of Stone, who discovered Stone duality, was to realize how to precisely describe this special subset $\overline{W}$ of 2^W. The key idea is to realize that there is some additional structure on the set of worlds W that we have not seen so far: a topology. But this is something that needs more introduction, and we do this properly in chapter 3.

Also see exercise 1.e.

So we have a duality between worlds and propositions: even if we do not endorse a particular view about one side—like possible worlds semantics or ersatzism—, the duality still describes a bidirectional determination between the two. So accepting principles on one side translates to the other side, where we can use a very different set of intuitions to test the principles.

1.1.3. Logic: formulas/syntax vs models/semantics

Logic can be done both syntactically (aka proof-theoretically) or semantically (aka model-theoretically). The completeness theorem shows that the two approaches—that are very different in spirit—actually are equivalent. This also is a form of duality. Let's explore this concretely.

Consider the language of classical propositional logic: sentences are formed from atomic sentences $p_0, p_1, \ldots$ using the connectives $\wedge, \vee, \neg$ and the constants $\bot$ and $\top$. And consider a proof-system for classical logic: for example a Hilbert system, a natural deduction system, or a sequence

calculus for classical logic—whichever you prefer. It consists of various axioms and rules to define the relation $\Gamma \vdash \varphi$, i.e., when the sentence φ is derivable in the proof-system S using as axioms the sentences in the set Γ. This is the syntactic description of the logic.

The model-theoretic description of the logic defines the relation $\Gamma \vDash \varphi$, i.e., that the sentence φ is a logical consequence of the sentences in Γ. This is done as follows. A valuation is a function $v : \{p_0, p_1, \ldots\} \to \{0, 1\}$ that assigns each atomic sentences a truth-value, i.e., true (1) or false (0). This can be extended to all sentences: $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$; $v(\neg\varphi) = 1$ iff $v(\varphi) = 0$; $v(\bot) = 0$; etc. Then $\Gamma \vDash \varphi$ is defined as: for all valuations v, if $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\varphi) = 1$. Thus, logical consequence is truth-preservation.

Now, the completeness theorem for classical propositional logic states that: $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$. To be more precise, one often only calls the right-to-left implication 'completeness', and the left-to-right implication 'soundness'. However, soundness is easy to establish. (One just needs to check, roughly, that the finitely many axioms of the proof-system are indeed logical consequences, and that the finitely many rules of the system preserves logical consequences—so the proof-system will only ever produce logical consequences.) We take soundness for granted and want to show that completeness really is a duality result.

Let us start on the syntactic side. The proof-system naturally defines a notion of equivalence between sentences: we call two sentences φ and ψ equivalent, written $\varphi \equiv \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$. An equivalence class of a sentence φ is the set of sentences that are equivalent to it: $[\varphi] :=$ $\{\psi : \varphi \equiv \psi\}$. Write L for the set of all equivalence classes. It also has logical structure: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$; $\neg[\varphi] = [\neg\varphi]$, etc. L is also called the *Lindenbaum–Tarski algebra* of the logic.

Now, each valuation v determines a subset $F_v \subseteq L$: namely, those equivalence classes $[\varphi]$ with $v(\varphi) = 1$. Note again that F_v has features (1)–(5): If $[\varphi] \in F_v$ and $[\varphi] \leqslant [\psi]$ (i.e., $[\varphi \wedge \psi] = [\varphi]$), then $\varphi \vdash \psi$, so, by soundness, $\varphi \vDash \psi$, so, since $v(\varphi) = 1$, also $v(\psi) = 1$, so $[\psi] \in F_v$. If $[\varphi], [\psi] \in F_v$, then $v(\varphi) = 1$ and $v(\psi) = 1$, so $v(\varphi \wedge \psi) = 1$, so $[\varphi \wedge \psi] \in F_v$. Etc. Conversely, if $F \subseteq L$ satisfies (1)–(5), then v_F is a valuation mapping φ to 1 iff $[\varphi] \in F$. So, again, the set X of valuations is in bijective correspondence with the set $\overline{L}$ of subsets of L satisfying (1)–(5).

But how does completeness follow? For this, first note that subsets of L are *theories*, i.e., sets of sentences (modulo provable equivalence). Now, if $\Gamma \nvdash \varphi$, consider the deductive closure Γ' of Γ, i.e., the set of all sentences

that can be derived from Γ, so also Γ′ ⊬ φ. When we regard Γ′ as a subset of L, this is, in formal terminology, a filter of L that does not intersect the ideal of all equivalence classes that imply [φ]. Now we use Stone's Prime Filter Theorem, which is at the heart of Stone duality and which we prove later on in the course. It says that we can extend this filter to a prime filter F which still does not intersect that ideal. Then v_F is a valuation that makes true all the premises in Γ but not the conclusion φ, hence Γ ⊬ φ, as desired.

1.1.4. Further examples

Physics: observations vs states. Duality also is a central idea in physics (e.g. Strocchi 2008, p. 24). A physical system comes both with a *state space* X and an algebra $\mathcal{A}$ of *observations* and these two again are dual in the sense that

- the states are determined by the observations that they give rise to,

- the observations are determined by the states that give rise to them.

The observations have logical structure: in a classical (as opposed to quantum) system, observing $A \wedge B$ means observing A and observing B, observing $A \vee B$ means observing A or observing B, etc. Each state x of the system determines a set of observations: namely, those that can be made if the system is in that state. Conversely, we can also start with the algebra of observations (they are empirically more accessible anyway) and postulate the states of the system as theoretical entities corresponding to certain subsets of observations.

Also done in general relativity: e.g., in the substantivalist vs relationalist debate (Wu and Weatherall 2023)

Computation: observable properties vs denotations of programs. Computer programs are written in a programming language, and so, much like for sentences written in a natural language, we can ask what their meaning is. The meaning of a program is called its *denotation*. For example, the denotation of a program could be the (partial) function that it computes. Domain theory is the mathematical theory to systematically describe these meanings. There again also is a side that is dual to the side of meanings, and this was a crucial discovery in the development of domain theory by Abramsky (1991, p. 16). This is the side of *observable properties* of the computer programs. For example, it could be the property that, on input $x = 3$, the program halts and outputs $f(x) = 5$. Again, we would hope for a bidirectional determination in the sense that the meaning of a program is complete determined by its observable properties, and that these observable properties are determined by the denotations that have them.

Philosophy of science: observations vs theories. _____

Time: duration vs points. _____

1.2. Towards characterizing duality

By now, we have an interesting stock of examples involving duality. Now it is a matter of finding a concise way to systematically describe all the different components that are involved in a duality. We will do this formally in the next chapters, but let's already give it an informal try here.

We had the following components in the examples:

- On the 'spatial' side, we have a set X, e.g., of objects, possible worlds, models, states, or denotations. We hinted at the fact that this is not just a set, but actually a *space*, i.e., it also carries a topology.

- On the 'algebraic' side, we have a set A, e.g., of properties, propositions, sentences (modulo provable equivalence), observations, or observable properties. This set also has logical—or algebraic—structure: conjunction ($\wedge$), disjunction ($\vee$), logical falsity ($\bot$), logical truth ($\top$), and possibly negation ($\neg$).

- We have a way to go from the spatial side to the algebraic side: each element of X is determined by a subsets of A with certain nice features, i.e., we have a bijective function $\epsilon : \overline{A} \to X$.

- We have a way to go from the algebraic side to the spatial side: each element from A can be assigned to a subset of X, so we have a bijective function $\eta : A \to \overline{X}$.

Finally, we want this translation manual to be *formulaic* in X and A: i.e., it should not depend on the idiosyncrasies of the specific X and A; rather, it should work for all X's and A's of the same kind. This is because we do not always know the exact nature of the two sides (the objects, possible worlds, etc.; resp., the properties, propositions, etc.). So we want the above data for any X that is a candidate for the spatial side and for any A that is a candidate for the algebraic side.

Formally, the two sides are best represented as so-called *categories*. On the spatial side, the category consists of the spatial candidates X, which are called the *objects* of the category, and their relations, which are called the *morphisms* of the category. Similarly, on the algebraic side, the category consists of the algebraic candidates A and their relations. Then we will see

What's 'algebraic' about this? Algebra is the study of rules for combining objects or symbols. In school, this means combining symbols like $ax^2 + bx + c$ and studying when they equal another, like 0. Here it is about combining elements of A with the operations $\wedge, \vee,$ etc.

that all the above components of the duality is succinctly phrased as a *dual equivalence* between the spatial category and the algebraic category.

The key application of a duality is that it provides a precise back-and-forth translation between objects (or categories) of very different kinds. Thus, questions on one side translate to question on the other side where very different tools are available to solve the question.

1.3. Exercises

Exercise 1.a. Complete the left-out details in the main text. For example, why, for a possible world x the set of propositions F_x really satisfied properties (1)–(5). Similarly for valuations v.

Exercise 1.b. Right after the list of features (1)–(5), we asked in the margin if this list is lacking a principle concerning negation: If $a \in \mathcal{P}$ is a property, then there also is the property $\neg a$ of not having property a. It seems plausible to require that either a given object $x \in \mathcal{O}$ has a property or it does not. In other words, either $a \in F_x$ or $\neg a \in F_x$. Do you think this is plausible to require? What about vague properties? (Later we see that this if if we have a negation operator on our set of properties obeying the Boolean laws, than F being prime is equivalent to having the just mentioned negation property.)

Exercise 1.c. As promised in the first example (section 1.1.1), this exercise asks you to scrutinize the concepts of objects and properties. Here are two questions and what duality theory might respond. Philosophically evaluate these answers.

1. *Russell's paradox*: We mentioned the worry that if we are too permissive in our conception of properties, Russell's paradox might creep in.

   Response: For this paradox to occur, we would need the property a of being a property. However, such higher-order properties (i.e., properties of properties) are not considered in Stone duality, for the following reasons. First, the set $\mathcal{O}$ of objects is considered to be disjoint from the set $\mathcal{P}$ of properties. After all, the properties are 'isomorphic' to the (clopen) sets of objects, but sets of objects are not objects; similarly, objects are 'isomorphic' to sets of properties (that are prime filters), but sets of properties are not properties. Second, if we had the higher-order property a of being a property, we would have $\mathcal{P} \subseteq \mathcal{O}$. This is because the property a determines its extension

$[\![a]\!] \subseteq \mathcal{O}$, though, by definition of a, $[\![a]\!] = \mathcal{P}$. But $\mathcal{P} \subseteq \mathcal{O}$ contradicts disjointness (since $\mathcal{P}$ is nonempty).

2. *Leibniz indiscernibility*: A trivial way to satisfy the principle of identity of indiscernibles is by acknowledging, for every object, the property of being x. This property is also known as the haecceity of the object x (see, e.g., Ladyman et al. 2012 for more on this). But, according to duality theory, are all haecceities really properties?

    Response: When we introduce topology, we will see that (1) the space of objects is compact and (2) the extensions of properties are clopen sets. The extension of the haecceity of object x is the singleton $\{x\}$. So, if the haecceity of every object is a property, then, by (2), all singletons are clopen, which, by (1), can only happen if there only are finitely many objects to start with. In other words, according to duality theory, if there are infinitely many objects, not all haecceities can be properties. (Also see exercise 1.f.)

Exercise 1.d (More of a research project than an exercise). Consider to what extend the first example (objects vs properties) can be developed along the lines of formal concept analysis.

Exercise 1.e. Can you think of more structure on the set of possible worlds? For example, a relation of closeness (or comparative similarity) as in the semantics for counterfactuals? Note your ideas and come back to them once we later have learned about the topology that can be put on the set of possible worlds (as hinted at in the text above). Compare this topology to your ideas.

Exercise 1.f. For a logico-philosophical discussion of the principle of indiscernibility, see Ladyman et al. (2012). How does this inform the above philosophical discussion (section 1.1.1)? This paper is in the context of model theory, what does the above duality-theoretic perspective add? For a start, see exercise 1.c (2).

Exercise 1.g. Can you think of more examples where a duality is involved? In cognitive science: what about concepts vs. mental states (computable theory of mind vs connectionism). Or, related, in AI: or human-interpretable concepts (symbolic) vs. states of neural networks (subsymbolic)? Or are these better seen as relations of supervenience rather than duality? What about the infamous Cartesian duality between the physical and the mental world?

Exercise 1.h. Go through the discussed examples of duality again and think about where they should be made philosophically and/or mathematically more precise.

2. The algebraic side: Boolean algebras

This chapter introduces formally the algebraic side of duality, which, for us, will be Boolean algebras. They are particular partial orders. So, in section 2.1, we first recall order theory (which is very useful in general). Then, in section 2.2, we define lattices as particular partial orders, and we give an equivalent definition which is more algebraic (i.e., in terms of operations that satisfy equations). In section 2.3, we define when lattices are distributive and when they even are Boolean algebras. The next chapter will deal with the other, spatial side of the duality. If you would like to refresh the standard set-theoretic terminology (which we use through-out the course), see appendix A.1.

2.1. Order theory

The objects that order theory studies are known as partial orders. We define them in section 2.1.1. The 'structure-preserving' maps between partial orders are known as monotone maps. We define those, and variants thereof, in section 2.1.2.

Even if we do not need category theory, we follow one of its keys lessons: that one not only should specify the class of objects that one studies but also the class of appropriate maps—which are called morphisms—between them. These two data then constitute a category, provided some basic axioms are satisfied (that morphisms can be composed and that there is the identity morphism).

For those interested in further reading on category theory, see, e.g., Leinster 2014, ch. 1. Eventually, I might add this into appendix A.2.

2.1.1. Objects: Partial orders

Partial orders occur everywhere: when you have a bunch of things where it makes sense to say that some are bigger (better, higher, etc.) than others. The things could be numbers with the usual sense of being bigger than. But the things could also be the dishes offered at your go-to lunch place with the sense of 'better' given by your preferences. Or, following the guiding intuition from chapter 1, the things can be propositions (or properties, etc.) ordered by implication: b is 'bigger' than a if a implies b. The formal definition goes as follows.

Definition 2.1. A *partial order* (or *partially ordered set*, or *poset*) is a pair $(P, \leqslant)$ where P is a (possibly empty) set and $\leqslant$ is a binary relation on P such that

1. *Reflexive*: For all $a \in P$, we have $a \leqslant a$.

2. *Transitive*: For all $a, b, c \in P$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

3. *Anti-symmetric*: For all $a, b \in P$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

If we do not require axiom 3, we speak of a *preorder*. We say $\leqslant$ is a (partial or pre-) order on P. If the order $\leqslant$ is clear from context, we often simply speak of the (partial or pre-) order P. We write $a < b$ if $a \leqslant b$ and $a \neq b$.

The name 'partial' is to indicate that not all elements need to be comparable: Formally, for $a, b \in P$, we say that a and b are *comparable*, if either $a \leqslant b$ or $b \leqslant a$; otherwise they are incomparable. If all elements are comparable, we say $(P, \leqslant)$ is *linear* (or *total*).

Formally, the example of the numbers is $(\mathbb{N}, \leqslant)$ where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ and, for $n, m \in \mathbb{N}$, the relation $n \leqslant m$ is defined as: n is smaller or equal to m (equivalently, there is $k \in \mathbb{N}$ such that $n + k = m$). Hence this a linear order. In the example of your lunch place, if you have two dishes a and b that you find equally tasty—or, more precisely, none tastier than the other, i.e., a and b are incomparable—, then your preference order is only partial and not linear.

Every partial order in particular is a preorder, and in the other direction we can canonically turn a preorder $(P, \leqslant)$ into a partial order $(\overline{P}, \overline{\leqslant})$ as follows. For $a, b \in P$, define $a \equiv b$ as $a \leqslant b$ and $b \leqslant a$. This is an equivalence relation. Equivalence classes are the sets $[a] := \{b \in P : a \equiv b\}$ for $a \in P$. The quotient of P under $\equiv$ is $\overline{P} := P/\equiv := \{[a] : a \in P\}$. Define $[a]\overline{\leqslant}[b]$ by $a \leqslant b$ (note that this is independent of the representatives a and b). This renders $(\overline{P}, \overline{\leqslant})$ a partial order. It is also called the *poset reflection* of P. Exercise 2.c makes formally precise in what sense it is the canonical or best possible poset approximating the preorder P.

There is a nice visualization of partial orders. They are known as *Hasse diagrams*. An example is in figure 2.1. It depicts the partial order $(P, \leqslant)$ with $P = \{a, b, c, d\}$ and

$$\leqslant := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}.$$

This definition of the order is not particularly enlightening, but the diagram is. Its nodes are the elements of P and the edges are the minimal information to recover the order:
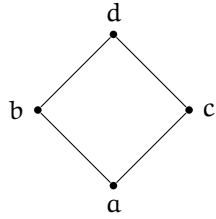
Figure 2.1.: The 'diamond' as an example of a partial order.

- if there is an edge between x and y and x is lower (on the page) than y, then $x \leqslant y$.

- we do not need to draw an edge from one node to itself because for all nodes x we have $x \leqslant x$.

- we do not need to draw edges that result from composing existing edges: for example, we have an edge from a to b and an edge from b to d, so we already know that $a \leqslant d$, hence we do not need to draw this.

More formally, the definition of a Hasse diagram of a partial order $(P, \leqslant)$ is as follows. For $a, b \in P$, we say that b *covers* a (short $a \lessdot b$) if $a < b$ and for all $c \in P$, if $a \leqslant c \leqslant b$, then $c = a$ or $c = b$. The elements of P are the nodes of the Hasse diagram, and an edge is drawn from node a to node b whenever b covers a. The direction of the edge is indicated by drawing b higher up in the diagram than a. So nodes on the same height are incomparable.

Next, some very useful concepts to talk about partial orders are the following.

Definition 2.2. Let $(P, \leqslant)$ be a partial order and $A \subseteq P$.

- An element $b \in P$ is a *lower bound* of A if, for all $a \in A$, we have $b \leqslant a$.

- An element $b \in P$ is an *upper bound* of A if, for all $a \in A$, we have $a \leqslant b$.

- An element $c \in P$ is an *infimum* or *greatest lower bound* of A if (1) c is a lower bound of A, and (2), for all lower bounds b of A, we have $b \leqslant c$.

- An element $c \in P$ is a *supremum* or *least upper bound* of A if (1) c is an upper bound of A, and (2), for all upper bounds b of A, we have $c \leqslant b$.

They can be confusing at first, but they really are worth learning. Make sure to draw little Hasse diagrams to illustrate the concepts and how they differ from each other (exercise 2.b).

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- An element $b \in P$ is a *least* or *bottom* or *minimum* element of P, if, for all $a \in P$, we have $b \leqslant a$ (i.e., b is the supremum of $A = \emptyset$).

- An element $b \in P$ is a *greatest* or *top* or *maximum* element of P, if, for all $a \in P$, we have $a \leqslant b$ (i.e., b is the infimum of $A = \emptyset$).

- An element $b \in P$ is *minimal* if, for all $a \in P$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal* if, for all $a \in P$, if $b \leqslant a$, then $b = a$.

- An element $b \in P$ is *minimal in A* if (1) $b \in A$ and (2) for all $a \in A$, if $a \leqslant b$, then $a = b$.

- An element $b \in P$ is *maximal in A* if (1) $b \in A$ and (2) for all $a \in A$, if $b \leqslant a$, then $b = a$.

- A is an *upset* if for all $a, b \in P$, if $a \in A$ and $a \leqslant b$, then $b \in A$.

- A is a *downset* if for all $a, b \in P$, if $b \in A$ and $a \leqslant b$, then $a \in A$.

- A is *directed* (aka up-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. (Equivalently, all finite subsets of A have an upper bound in A.)

- A is *filtered* (aka filtering or down-directed) if it is nonempty and for any $a, b \in A$, there is $c \in A$ with $c \leqslant a$ and $c \leqslant b$. (Equivalently, all finite subsets of A have a lower bound in A.)

(These notions also make sense in a preorder $(P, \leqslant)$, but if P is a partial order, then infimum and supremum are unique if they exist.) The infimum is denoted $\bigwedge A$, called the *meet* of A; and the supremum is denoted $\bigvee A$, called the *join* of A. If $A = \{a_1, \ldots, a_n\}$ is finite and nonempty, we write $\bigwedge A = a_1 \wedge \ldots \wedge a_n$ and $\bigvee A = a_1 \vee \ldots \vee a_n$. In particular, $\bigwedge\{a, b\} = a \wedge b$ and $\bigvee\{a, b\} = a \vee b$. The bottom element, if it exists, is denoted $\bot$ or 0; and the top element by $\top$ or 1. We write $\min(A)$ (resp. $\max(A)$) for the elements that are minimal (resp. maximal) in A. A *directed join* is the supremum of a directed set.

It is a good exercise to prove this.

Partial orders where various suprema and infima exist get special names. For example, *lattices* (which we study in the next section) are partial orders where all finite subsets have an infimum and a supremum; *complete lattices* are partial orders where all subsets have an infimum and a supremum; *directed-complete partial orders* (*dcop*'s) are partial orders where all directed subsets have a supremum.

Finally, one useful operation on preorders is that we can 'turn them upside down' and get another preorder. Formally, if $(P, \leqslant)$ is a preorder, define the preorder $\leqslant'$ on P by $a \leqslant' b$ iff $b \leqslant a$. We write P^{op} for this preorder.

2.1.2. Morphisms: Monotone maps

What maps between partial orders should be considered to be 'structure preserving'? They should preserve the order structure, which yields the concept of a monotone map.

Definition 2.3. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders and $f : P \to Q$ a function. We say f is

- *monotone* or *order preserving* if, for all $a, b \in P$, if $a \leqslant_P b$, then $f(a) \leqslant_Q f(b)$.

The converse notion is:

- *order reflecting* if, for all $a, b \in P$, if $f(a) \leqslant_Q f(b)$, then $a \leqslant_P b$.

We call f an *order-embedding* if it is both order preserving and order reflecting. Finally, f is an *order-isomorphism* if it is monotone and it has a *monotone inverse*, i.e., there is a monotone function $g : Q \to P$ such that

- for all $a \in P$, we have $a = g(f(a))$, i.e., a is the g-inverse of $f(a)$ (in short, $\mathrm{id}_P = g \circ f$), and

- for all $b \in Q$, we have $f(g(b)) = b$, i.e., mapping the g-inverse of b along f yields b (in short, $f \circ g = \mathrm{id}_Q$).

We say two preorders are *isomorphic* if there is an order isomorphism between them.

We can consider two isomorphic preorders to be essentially identical (because any order-theoretic property that one has, the other has, too). For partial orders, the notion of isomorphism can be simplified. The above definition captures the general (category-theoretic) concept of an isomorphism, but in practice the following is often easier to check.

Proposition 2.4. *Let $f : P \to Q$ be a monotone function between posets. Then f is an isomorphism iff f is a surjective order-embedding.*

Proof. Exercise 2.d. $\qquad\square$

In exercise 2.e, we show how the notion of an isomorphism can be generalized to that of an adjunction. This provides yet another notion of morphism between posets. We do not need it here, but since it is a very useful (but also abstract) concept in the vicinity of the presented concepts, we include it as an exercise. Exercise 2.f shows how such adjunction naturally occur once one has a relation between two sets.

2.2. Lattices

In this section, we define lattices as particular partial orders (and provide an equivalent algebraic definition), we define the appropriate morphisms between lattices, and we discuss some basic constructions with lattices.

2.2.1. Objects: lattices

The order-theoretic definition of a lattice goes as follows.

Definition 2.5 (Lattice, order-theoretic). A *(bounded) lattice* is a partial order L in which every finite subset has a supremum and an infimum.

For example, the diamond of figure 2.1 is a lattice. To consider the example of the collection of propositions from chapter 1, we already said that it should be partially ordered by implication, and it makes sense to require that it is a lattice: Given two propositions a and b, there is, as we discussed, also the proposition $a \vee b$, and it is implied by both a and by b, and whenever a proposition c is implied by both a and b, then $a \vee b$ implies c—so $a \vee b$ is the supremum of $a \vee b$. This also works for a finite set of propositions, and also for $\wedge$ and the infimum.

Some comments:

1. In fact, it is enough that the empty set and all two-element sets have suprema and infima.

   As an exercise, prove this.

2. Often a lattice is defined as a partial order in which all binary suprema and infima exist (i.e., those of two-element sets), and a bounded lattice is a lattice where also the supremum and infimum of the empty set exits (i.e., which a have a least and a greatest element). Here we assume all lattices to be bounded, because this is more convenient for duality theory. Hence we drop the word 'bounded' (unless we want to stress this assumption). A non necessarily bounded lattice can always be bounded by adding a new top and bottom element.

3. A complete lattice is a partial order in which all subsets have suprema and infima. In fact, for this it is enough that every subset has a supremum.

Alternatively, lattices are also defined algebraically (i.e., in terms of operations satisfying certain equations). Interestingly, these two definitions are equivalent, as we will show afterward.

Definition 2.6 (Lattice, algebraic). A lattice is a tuple $(L, \vee, \wedge, \perp, \top)$ where $\vee$ (pronounced *join*) and $\wedge$ (pronounced *meet*) are binary operations on L (i.e., functions $L \times L \to L$), and $\perp$ (pronounced *bottom*) and $\top$ (pronounced *top*) are elements of L, such that the following axioms holds:

1. *commutative*: for all $a, b \in L$, we have $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

2. *associative*: for all $a, b, c \in L$, we have $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

3. *idempotent*: for all $a \in L$, we have $a \vee a = a$ and $a \wedge a = a$.

4. *absorption*: for all $a, b \in L$, we have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

5. *neutrality*: for all $a \in L$, we have $\perp \vee a = a$ and $\top \wedge a = a$.

For example, if X is a set, then the powerset 2^X forms a lattice in this algebraic sense with union $\cup$ as join, intersection $\cap$ as meet, $\emptyset$ as bottom, and X as top. This also provides my mnemonic for remembering what 'join' and what 'meet' is. Think of X as a set of propositions, and let $a \in 2^X$ be the beliefs (opinions, values, etc.) that *A*lice holds, and let $b \in 2^X$ be the beliefs that *B*ob holds. Then the meet of a and b—i.e., $a \wedge b = a \cap b$—is where Alice and Bob can meet: the common (meeting) ground, the set of beliefs they agree on. And the join of a and b—i.e., $a \vee b = a \cup b$—is the result of joining Alice and Bob together: their joint beliefs, taking together all of their beliefs even if incoherent.

The equivalence of the two definitions is made precise in the following theorem. Exercise 2.g asks you to prove it: that is a bit tedious, but quite instructive.

Theorem 2.7. *The algebraic and order-theoretic definitions of a lattice are equivalent in the following sense:*

1. *Given a lattice $(L, \vee, \wedge, \perp, \top)$ according to the algebraic definition, define $a \leqslant_L b$ as $a \wedge b = a$. Then $(L, \leqslant_L)$ is a partial order which is a lattice according to the order-theoretic definition, with binary suprema and infima being given by $\vee$ and $\wedge$.*

2. *Given a lattice* $(L, \leqslant)$ *according to the order-theoretic definition, define the binary operations* $\vee$ *and* $\wedge$ *as binary supremum and infimum, and take* $\bot$ *and* $\top$ *to be the least and greatest element of* L. *Then* $(L, \vee, \wedge, \bot, \top)$ *is a lattice according to the algebraic definition, with* $a \wedge b = a$ *iff* $a \leqslant b$ *iff* $a \vee b = b$.

From now on, we will often just speak of a lattice L and both use its order-theoretic definition (taking $\leqslant$ to be implicitly given) and its algebraic definitions (taking $\vee, \wedge, \bot, \top$ to be implicitly given).

Finally, in some situations we might only have one of the two binary operations: then we speak of a semilattice. Formally, a *semilattice* is a structure $(L, \cdot, 1)$, where $\cdot$ is a commutative, associative, and idempotent binary operation on L, and 1 is a neutral element for the operation. The operation $\cdot$ can then either be seen as the binary infimum for the partial order defined by $a \leqslant b$ iff $a \cdot b = a$ (the join semilattice), or as the binary supremum for the opposite partial order defined by $a \leqslant b$ iff $a \cdot b = b$ (the meet semilattice).

2.2.2. Morphisms: lattice homomorphisms

The appropriate structure preserving map between lattices is the following:

Definition 2.8. A function $f : L \to M$ between lattices is a lattice homomorphism if it preserves all the lattice operations, i.e.,

1. for all $a, b \in L$, we have $f(a \vee_L b) = f(a) \vee_M f(b)$

2. for all $a, b \in L$, we have $f(a \wedge_L b) = f(a) \wedge_M f(b)$

3. $f(\bot_L) = \bot_M$

4. $f(\top_L) = \top_M$

Note that lattice homomorphisms are always order preserving, and injective lattice homomorphisms are order-embeddings. An injective lattice homomorphism is called a *lattice embedding*. Bijective lattice homomorphisms are order-isomorphisms and are called *lattice isomorphisms*. *Prove this.*

If a function $f : L \to M$ between lattices preserves $\bot$ and $\vee$, then it preserves all finite joins. This does, in general, *not* imply any preservation *Prove this.* of arbitrary existing joins or preservation of infima. The analog statement is true for $\top$ and $\wedge$ and preservation of all finite meets.

2.2.3. Constructions: products, sublattices, homomorphic images, congruences

Whenever one has introduced a class of objects together and their structure-preserving maps, one also looks at the *constructions* one can perform: how to build new objects in the class from old ones. The typical ones are products, substructures, and quotients, and you might have seen this also for other structures, e.g. groups. (Here quotients will be given as homomorphic images or, equivalently, congruences.) Actually, in this course, they will not play a big part, but they will in more advanced texts on duality theory and are generally important to know. So it is enough if you just skim them.

Products. Given a family $(L_i)_{i \in I}$ of lattices, we can define a lattice $L = \prod_{i \in I} L_i$ on the Cartesian product where the operations are defined component-wise: e.g., for $a = (a_i)_i$ and $b = (b_i)_i$ in L, we define $a \leqslant_L b$ as $\forall i \in I : a_i \leqslant_{L_i} b_i$, and $(a \wedge b)_i = a_i \wedge b_i$ (similarly for $\vee$), and $(\perp_L)_i = \perp_{L_i}$ (similarly for $\top$). The projection maps $\pi_i : L \to L_i$, which map $a = (a_i)_i$ to a_i, is a surjective lattice homomorphism.

Sublattices. A sublattice of a lattice L is a subset L' of L that contains $\perp$ and $\top$ and that is closed under $\wedge$ and $\vee$ (i.e., if $a, b \in L'$, then $a \wedge b, a \vee b \in L'$). Then L' is a bounded lattice in its own right and the inclusion map $\iota : L' \to L$, which maps $a \in L'$ to $a \in L$, is a lattice embedding. If we do not require $\perp$ and $\top$ to be in L', we speak of an *unbounded sublattice*. And if we require L' to be closed under all suprema and infima, we call it a *complete sublattice*. If $f : L \to M$ is a lattice homomorphism, then the direct image $L' := f[L] = \{f(a) : a \in L\}$ is a sublattice of the lattice M.

Homomorphic images. A lattice L' is a *homomorphic image* of a lattice L if there is a surjective lattice homomorphism $f : L \to L'$.

Congruences. A congruence on a lattice L is an equivalence relation ϑ on L that respects the lattice operations, i.e., for all $a, a', b, b' \in L$, if $a \vartheta a'$ and $b \vartheta b'$, then also $a \vee b \vartheta a' \vee b'$ and $a \wedge b \vartheta a' \wedge b'$. For an intuitive example, think of the elements of L as propositions and of ϑ as having the same subject matter. The quotient L/ϑ carries a unique lattice structure that turns the quotient map $p : L \to L/\vartheta$, which maps $a \in L$ to its equivalence class $[a]_\vartheta$ under ϑ, into a lattice homomorphism; concretely, this lattice structure is given by $[a]_\vartheta \vee [b]_\vartheta := [a \vee b]_\vartheta$ (similarly for $\wedge$) with bottom element $[\perp]_\vartheta$ (similarly for $\top$). Note how this is reminiscent of the Lindenbaum–Tarski algebra from the introduction (section 1.1.3).

The first isomorphism theorem for lattices. This says that any lattice homomorphism $f : L \to M$ can be factored as a surjective lattice homomorphism

p followed by a lattice embedding e (i.e., $f = e \circ p$). These are given as follows. The *kernel* of f is the congruence relation

$$\ker f := \{(a, a') \in L \times L : f(a) = f(a')\}.$$

Choose $p : L \to L/\ker f$ (mapping a to $[a]$) and $e : L/\ker f \to M$ (mapping $[a]$ to $f(a)$). In particular, $L/\ker f$ is isomorphic to $f[L]$ (take $M := f[L]$, so e also is surjective); hence the homomorphic images of L are, up to isomorphism, the quotients of L.

2.3. Distributive lattices and Boolean algebras

We get further subclasses of lattices by requiring that $\vee$ and $\wedge$ interact nicely, which is made precise as distributive lattices (section 2.3.1), and by additionally requiring that there is a sense of negation, which is made precise as Boolean algebras (section 2.3.2).

2.3.1. Distributive lattices

The idea $\vee$ and $\wedge$ interact nicely is made precise as follows.

Definition 2.9. A lattice L is distributive if,

$$\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{2.1}$$

or, equivalently,

$$\forall a, b, c \in L : a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{2.2}$$

For example, the four diamond from figure 2.1 is distributive, as is any powerset 2^X, and also any chain $n = \{0, 1, \ldots, n-1\}$ with the usual ordering (see exercise 2.i). If we again consider the intuitive example of a collection of propositions from chapter 1, we already said that it is a lattice and we also expect it to be distributive because then 2.1 expresses a basic logical equivalence between propositions.

The equivalence of 2.1 and 2.2 implies that L is distributive iff L^{op} is distributive. So distributivity is a so-called *self-dual property*.

There also are strengthenings of the distributivity law. We mentioned one example here for context, but do not need it later. A *frame* is defined

The diamond M_3 The pentagon N_5
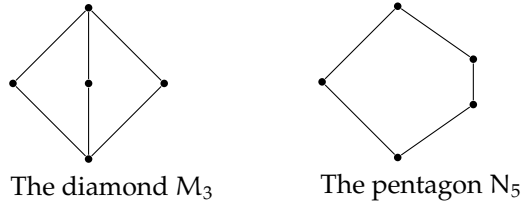
Figure 2.2.: The forbidden substructures for distributivity.

as a complete lattice L satisfying the join infinite distributive law (JID)

$$\text{for any } a \in L \text{ and } B \subseteq L, \ a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b). \tag{2.3}$$

In a distributive lattice this, in general, only holds for all *finite* $B \subseteq L$.

A seemingly magic characterization of distributive lattices is the following.

Theorem 2.10 (The M_3–N_5 theorem)**.** *Let L be a lattice. Then L is distributive iff L does not contain an unbounded sublattice which is isomorphic to M_3 or N_5, depicted in figure 2.2.*

2.3.2. Boolean algebras

So far, we have seen the order $\leqslant$ and the operations $\vee$ and $\wedge$ in a lattice, which act like implication, disjunction, and conjunction, respectively. So you might have wondered: what about negation? Especially since this also played a role in our motivating example of a collection of propositions from chapter 1: if we have a proposition a, we also have the proposition $\neg a$, and we expect $a \wedge \neg a$ to be a logical contradiction and $a \vee \neg a$ a logical truth. These ideas are made precise as follows.

Definition 2.11. Let L be a lattice and a an element of L. A *complement* of a is an element b of L such that $a \wedge b = \bot$ and $a \vee b = \top$. A *Boolean algebra* is a distributive lattice in which every element has a complement. The complement of an element a in a distributive lattice is unique, if it is exist, an denoted $\neg a$.

For example, again the four diamond from figure 2.1 is a Boolean algebra, as is any powerset 2^X; but, for $n > 2$, the chain n is not a Boolean algebra (see exercise 2.i). Some further comments:

1. Usually, the negation is then taken into the signature: so a Boolean algebra is a tuple $(B, \wedge, \vee, \bot, \top, \neg)$ such that $(B, \wedge, \vee, \bot, \top)$ is a distributive lattice and $\neg : B \to B$ a unary function such that, for all $a \in B$, we have $a \wedge \neg a = \bot$ and $a \vee \neg a = \top$.

2. But if we have an additional operation around, shouldn't we require the morphisms to preserve it? Fortunately, they already do: If $f : B \to A$ is a lattice homomorphism between Boolean algebras, then, for all $a \in B$, we have $f(\neg a) = \neg f(a)$. We often still refer to them as *Boolean algebra homomorphisms* just to emphasize that we are dealing with Boolean algebras.

   The fact that we can use the same morphisms is expressed in categorical terms as the category of Boolean algebras and Boolean algebra homomorphisms being a full *(as opposed to any) subcategory of the category of distributive lattices and lattice homomorphisms.*

3. However, with the notion of a sublattice we need to be more careful: A Boolean algebra may have many sublattices that themselves are not Boolean algebras; so by a (*Boolean*) *subalgebra* of a Boolean algebra B we mean a sublattice which is also closed under $\neg$.

4. If you like ring theory, a Boolean algebra can equivalently be defined as a commutative ring with unit in which all elements are idempotent, see exercise 2.j.

5. There is a best way to turn a distributive lattice L into a Boolean algebra B. This B is called the *Boolean envelope* or *free Boolean extension* of L. More precisely, this means that for every distributive lattice L there is a Boolean algebra B and an injective homomorphism $e : L \to B$ such that for any other lattice homomorphism $h : L \to A$ into a Boolean algebra A, there is a unique Boolean algebra homomorphism $\overline{h} : B \to A$ such that $\overline{h} \circ e = h$. As a diagram:

   In categorical terms this means the category of Boolean algebras is a full reflective subcategory of the category of distributive lattices.

   $$
   \begin{array}{ccc}
   L & \xrightarrow{\ e\ } & B \\
    & \searrow{\scriptstyle h} & \Big\downarrow{\scriptstyle \overline{h}} \\
    & & A
   \end{array}
   $$

   This can be shown using duality techniques (see, e.g., Gehrke and van Gool 2023, prop. 3.34 or Johnstone 1982, sec. 4.5).

2.4. Exercises

Exercise 2.a. Show that the following are partial orders and draw their Hasse diagrams:

- The chain $P = \{0, \ldots, n-1\}$ with the usual order. Draw it for, say, $n = 10$.

- The set $P = \{1, \dots, n\}$ with the order defined by $n \leqslant m$ iff n divides m (why is 0 excluded from the set?). Draw it for, say, $n = 10$.

- The powerset 2^X for some set X ordered by the subset relation, i.e., $A \leqslant B$ iff $A \subseteq B$. Draw it for, say, $X = \{4, 7\}$.

Exercise 2.b. Go through the partial order concepts defined in definition 2.2 and pick a few of them and draw (minimal) Hasse diagrams to show how they differ. For example, a maximal element that is not a greatest element; an upper bound that is not a greatest upper bound; or an upset that is not directed.

Exercise 2.c. Recall that for a preorder $(P, \leqslant)$, we have defined the poset reflection $(\overline{P}, \overline{\leqslant})$. This exercise makes precise in which sense this is the best possible poset approximating the preorder $(P, \leqslant)$.

1. Prove that $\equiv$ is an equivalence relation.

2. Prove that the definition of $\overline{\leqslant}$ is independent of the representatives: If $a' \in [a]$ and $b' \in [b]$, then $a \leqslant b$ iff $a' \leqslant b'$.

3. Prove that $(\overline{P}, \overline{\leqslant})$ is indeed a partial order.

4. Prove that $\overline{\leqslant}$ is the smallest partial order on $\overline{P} = P/\equiv$ such that the quotient map $f : P \to P/\equiv$, which maps a to $[a]$, is order preserving: That is, if $\leqslant'$ is another such partial order on $P/\equiv$, then $\overline{\leqslant} \subseteq \leqslant'$.

5. Prove that, for any order preserving $g : P \to Q$ into a poset Q, there exists a unique order preserving $\overline{g} : P/\equiv \to Q$ such that $\overline{g} \circ f = g$. As a diagram:

$$P \xrightarrow{\ f\ } P/\equiv$$
$$g \searrow \quad \downarrow \overline{g}$$
$$Q$$

Think about how the last item formalizes the idea that $(\overline{P}, \overline{\leqslant})$ is the best possible poset approximating the preorder $(P, \leqslant)$.

Exercise 2.d. Prove proposition 2.4.

The next two exercises introduce the notion of an order adjunction (this is a special case of the notion of an adjoint functor). The first states the general definition and the second a common situation how they occur.

Exercise 2.e (Order adjunction). Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be two preorders, and let $f : P \to Q$ and $g : Q \to P$ be monotone maps. The pair (f, g) is called an *adjunction*, with f the *left* or *lower adjoint* and g the *right* or *upper adjoint*, if, for all $a \in P$ and $b \in Q$,

$$f(a) \leqslant_Q b \text{ iff } a \leqslant_P g(b).$$

We also write this as $l : P \leftrightarrows Q : u$. An adjunction between P^{op} and Q is called a *Galois connection* or *contravariant adjunction*.

1. Prove that (f, g) is an adjunction iff
   - for all $a \in P$, we have $a \leqslant_P g(f(a))$, i.e., the g-inverse of $f(a)$ is at least as good as a, and
   - for all $b \in Q$, we have $f(g(b)) \leqslant_Q b$, i.e., mapping the g-inverse of b along f approximates b.

For the rest of this exercise, assume that (f, g) is an adjunction.

2. Prove that $f \circ g \circ f(a) \equiv f(a)$ and $g \circ f \circ g(b) \equiv g(b)$ for every $a \in P$ and $b \in Q$ (and $a \equiv b$ iff $a \leqslant b$ and $b \leqslant a$).

3. Conclude that, in particular, if P and Q are posets, then $fgf = f$ and $gfg = g$.

4. Prove that, if P is a poset, then for any $a \in P$, $gf(a)$ is the least element above a that lies in the image of g.

5. Formulate and prove a similar statement to the previous item about $fg(b)$, for $b \in Q$.

6. Prove that, for any subset $A \subseteq P$, if the supremum of A exists, then $f(\bigvee A) = \bigvee f(A)$ (where $f(A) = \{f(a) : a \in A\}$ is the image of A under f).

7. Prove that, for any subset $B \subseteq Q$, if the infimum of B exists, then $g(\bigwedge B) = \bigwedge g(B)$.

In words, the last two items say that *lower adjoints preserve existing suprema* and *upper adjoints preserve existing infima*.

Exercise 2.f (Galois connection from a relation). Let $R \subseteq X \times Y$ be a relation between two sets. For any $a \subseteq X$ and $b \subseteq Y$, define

$$u(a) := \{y \in Y : \forall x \in a. xRy\} \subseteq Y$$
$$l(b) := \{x \in X : \forall y \in b. xRy\} \subseteq X$$

Show that $l : \mathcal{P}(Y) \leftrightarrows \mathcal{P}(X) : u$ forms a Galois connection between the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, i.e., for any $b \subseteq Y$ and $a \subseteq X$, we have $a \subseteq l(b)$ (i.e., $l(b) \subseteq^{op} a$) iff $b \subseteq u(a)$.

For those interested in further reading, here are three instances of this.

1. Maybe you know the name 'Galois' from the theory of fields in algebra. Then you know Galois theory as relating fields to groups (and showing why quintic equations cannot be solved). This connection arises via the above lemma from the relation R between the set X of subfields of a given field and the set Y of automorphisms of this field, which relates a subfield to the automorphisms which are the identity on this subfield.

2. If X is a set and $R \subseteq X \times X$ is a preorder, then $u(a)$ is the set of upper bounds of $a \subseteq X$, and $l(b)$ is the set of lower bounds of $b \subseteq X$.

3. Consider a class of structures $\mathcal{C}$ (in, say, a first-order signature) and a class $\mathcal{F}$ of formulas (of this signature). Let $\vDash$ be the *interpretation* relation: For $M \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ means that structure M makes true formula φ. Then for a set of models a, $u(a)$ is the theory of a, i.e., the set of formulas that are true in all those models. And for a a theory $b \subseteq \mathcal{F}$, $l(b)$ is the class of models of b, i.e., the set of models which make true all the sentences in b.

Exercise 2.g. Prove theorem 2.7.

Exercise 2.h. Prove the equivalence of the two ways of defining distributivity: 2.1 and 2.2.

Exercise 2.i. Show that the following are distributive lattices:

- The four diamond from figure 2.1.

- The powerset 2^X, for any set X.

- The chain $n = \{0, \ldots, n-1\}$ with the usual ordering, for any n.

Show that the first two also are Boolean algebras. Show that the last one is a Boolean algebra if $1 \leqslant n \leqslant 2$, and not if $n > 2$.

You might have had the suspicion that the join $\vee$ acts quite like addition $+$ and the meet $\wedge$ quite like the multiplication $\cdot$. If so, you might like the next exercise, which makes this precise.

Exercise 2.j. This exercise shows that Boolean algebras and Boolean rings are equivalent.

1. Let $(B, +, \cdot, 0, 1)$ be a Boolean ring, i.e., a commutative ring with unit in which $a \cdot a = a$ for all $a \in B$. Define $a \leqslant b$ if $a \cdot b = a$. (We often write ab for $a \cdot b$.) Prove that $\leqslant$ is a distributive lattice order on B where

   - 1 is the greatest element and 0 is the least element,

   - meet is given by ab and join is given by $a + b + ab$, and

   - every element a of has the complement $1 + a$ with respect to $\leqslant$.

   Hint: First show that $a + a = 0$ for all $a \in B$.

2. Conversely, let $(B, \wedge, \vee, \bot, \top, \neg)$ be a Boolean algebra. Define, for any $a, b \in B$,

   $$a + b := (a \wedge \neg b) \vee (\neg a \wedge b) \qquad a \cdot b := a \wedge b$$

   $$0 := \bot \qquad\qquad\qquad\qquad 1 := \top.$$

   The operation $+$ is known as symmetric difference.

   Prove that $(B, +, \cdot, 0, 1)$ is a Boolean ring.

3. Finally, show that the composition of these two assignments in either order yields the identity.

3. The spatial side: topological spaces

This chapter introduces formally the spatial side of duality, which, for us, will be certain topological spaces known as Stone spaces. This naturally structures this chapter: In section 3.1, we provide a general introduction to topological spaces. In section 3.2, we consider some further topological notions: separation axioms, compactness, and dimensionality. Finally, in section 3.3, we define Stone spaces and discuss one important example, namely the Cantor space. Then we have both the algebraic and the spatial side together, so we can prove the duality result in the next chapter.

3.1. Introduction to topological spaces

When we hear of 'space', we naturally think of the three-dimensional space we live in. And this indeed is an example of a topological space. It is the three-dimensional Euclidean space $\mathbb{R}^3$ whose points $x = (x_1, x_2, x_3)$ are described by the values on the x-axis, the y-axis, and the z-axis. From high-school, we also know what lines and planes are in this space, and what their geometry is.

E.g., the plane spanned by the x-axis and the y-axis is the set of points (x_1, x_2, x_3) with $x_3 = 0$.

But there also are other spaces. For example, the surface of a sphere. Its points are those (x_1, x_2, x_3) with $x_1^2 + x_2^2 + x_3^2 = 1$. But its geometry is different: for instance, the angles of a triangle add up to more than 180 degrees. Yet another space is the spacetime that we live in according to general relativity. Its points $x = (x_1, x_2, x_3, x_4)$ are four-dimensional—with three spatial and one temporal component—and its geometry is given by a metric tensor. And there are even wilder spaces, in which it might not even make sense to speak of a 'geometry' (e.g., angles between lines), but only of 'spatial' properties (e.g., continuous paths from one point to another).

Here we only refer to an intuitive difference between 'geometric' vs 'spatial' (or topological) properties: the latter are invariant under stretching and squishing the space, but the former are not. This is why topology is colloquially also described as rubber sheet geometry.

After much research, mathematicians—most notably Felix Hausdorff in 1914—came up with a general definition of a topological space that includes all these examples. When one first reads this rather abstract definition, one wonders how it possibly can cover all the relevant spatial concepts of the specific examples. But we see how, just from this parsimonious definition of a topological space, we can define many of the common spatial concepts. Again, we split this discussion into objects (topological

Mathematics provides many formal notions of space (e.g., Euclidean space, vector space, Hilbert space, probability space, Banach space, etc.). But topological spaces are a very general such notion.

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spaces) and morphisms (continuous functions between spaces).

3.1.1. Objects: topological spaces

Without further ado, here is the abstract definition of a topological space.

Definition 3.1. A *topological space* is a pair (X, τ) where X is a nonempty set and τ is a collection of subsets of X such that

1. $\emptyset$ and X are in τ

2. If $U, V \in \tau$, then $U \cap V \in \tau$

3. If $U_i \in \tau$ is a collection of sets indexed by a set I, then $\bigcup_{i \in I} U_i \in \tau$.

Equivalently: τ is closed under finite intersection and arbitrary union (which includes the empty intersection X and the empty union $\emptyset$). In particular, τ is a sublattice of 2^X.

We also call τ a topology on X. We call the elements of X *points*. The elements of τ are called *open sets* (or *opens*). Their complements, i.e., sets of the form $C = X \setminus U$ for $U \in \tau$, are called *closed sets*. A subset $K \subseteq X$ that is both open and closed (i.e., $K \in \tau$ and $K^c \in \tau$) is called *clopen*. We just speak of the topological space X if τ is clear from context. Then we write $\Omega(X)$ for the opens of X. The collection of closed (resp. clopen) subsets of X is denoted $\mathcal{C}(X)$ (resp. $\mathsf{Clp}(X)$).

Let's first see that this indeed generalizes our spatial intuitions about 'our' space:

Example 3.2. The three-dimensional space as a topological space: the underlying set is $X := \mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ and the opens are those subsets $U \subseteq \mathbb{R}^3$ that allow some 'wiggle-room', which is made precise as follows. Recall that the usual distance between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

So a subset $U \subseteq \mathbb{R}^3$ is defined to be open precisely if:

Exercise 3.a asks you to show that this then indeed is a topology.

1. for all $x \in \mathbb{R}^3$, if $x \in U$, then there is $\epsilon > 0$ such that for all $x' \in \mathbb{R}^3$ with $d(x, x') < \epsilon$, we have $x' \in U$.

This is called the *Euclidean topology* on $\mathbb{R}^3$. ⌟

Another, more abstract example are the two trivial topologies:

Example 3.3. For any nonempty set X, the set $\tau := 2^X$ is a topology on X. It is called the *discrete* topology. Also $\tau := \{\emptyset, X\}$ is a topology on X. It is called the *indiscrete* topology. ⌟

Next, we define some central concepts for a topological space X. They should give a sense of how many spatial concepts one can express with just talk of open sets.

Definition 3.4. Let (X, τ) be a topological space.

1. *Interior and closure.* If $S \subseteq X$ is a subset, there is a largest open set contained in S, which is called the *interior* of S:

$$\mathsf{Int}(S) := \bigcup \{U \in \tau : U \subseteq S\}.$$

Convince yourself that (a) this is an open set, (b) it is contained in S, and (c) it is the largest such set.

There also is a smallest closed set containing S, which is called the *closure* of S:

$$\mathsf{Cl}(S) := \bigcap \{C \in \mathcal{C}(X) : S \subseteq C\}.$$

Convince yourself that closed sets are closed under arbitrary intersection, so this is indeed a closed set.

2. *Boundary.* The *boundary* of a subset $S \subseteq X$ is defined as $\partial S := \mathsf{Cl}(S) \setminus \mathsf{Int}(S)$.

3. *Neighborhood.* A subset $S \subseteq X$ is a *neighborhood* of a point $x \in X$ if $x \in \mathsf{Int}(S)$. Accordingly, an *open neighborhood* of a point is an open set containing this point. (If it's clear we're talking about an open neighborhood, we might drop the adjective 'open'.)

4. *Dense.* A subset $S \subseteq X$ is *dense* (in X) if for all points $x \in X$ and open neighborhoods U of x, there is a point $s \in S$ with $s \in U$. So the points of X can be approximated arbitrarily closely by points in S. An equivalent formulation is: $\mathsf{Cl}(S) = X$.

For example, the (countable) set S of all points in $\mathbb{R}^3$ with rational coordinates is dense in $\mathbb{R}^3$.

5. *Convergence.* A sequence $(x_n)_{n \in \mathbb{N}}$ of points in X *converges* to a point $x \in X$ if for all open neighborhoods U of x, there is $N \geqslant 0$ such that, for all $n \geqslant N$, we have $x_n \in U$. We also say that x is the *limit* of the sequence (x_n).

6. *Generated topology.* A collection of subsets can naturally be turned into a topology: Any collection $\mathcal{S}$ of subsets of a nonempty set Y *generates* a topology $\langle \mathcal{S} \rangle$ on Y: namely, the smallest topology on Y that contains all subsets in $\mathcal{S}$. Concretely, $\langle \mathcal{S} \rangle$ is the set of arbitrary unions of finite intersections of elements of $\mathcal{S}$.

This exists because an arbitrary intersection of topologies on Y is again a topology on Y.

7. *Subbase.* The generating collection $\mathcal{S}$ can be simpler than the topology $\langle \mathcal{S} \rangle$ generated, so it allows for a more succinct description of the topology. Precisely: Given the topology τ on X, a collection $\mathcal{S}$ of

subsets of X is called a *subbase* of τ if $\tau = \langle S \rangle$. So the opens of τ are arbitrary unions of finite intersection of subbasic elements.

8. *Base*. The nice subbases are those where we only need to consider arbitrary unions and can forget about the finite intersections. Precisely: A *base* for the topology τ is a collection $S \subseteq \tau$ such that for every point $x \in X$ and every open neighborhood U of x, there is $V \in S$ such that $x \in V \subseteq U$. Equivalently, a base is a collection of open subsets of X such that every open set is a union of of elements from the base. Note that, in particular, a base is a subbase.

Exercise 3.5. Prove the four statements in the margins:

1. the existence of the interior,

2. the existence of the closure,

3. the rational points being dense in our three-dimensional space,

4. the existence of the generated topology.

Moreover, to illustrate the concept of a basis:

5. Show that a base for the Euclidean topology on $\mathbb{R}^3$ is given by the open balls $B_\epsilon(x) := \{y \in \mathbb{R}^3 : d(x,y) < \epsilon\}$ for $x \in \mathbb{R}^3$ and $\epsilon > 0$.

6. Can you improve the previous base by allowing only $x \in \mathbb{R}^3$ with rational coordinates and $\epsilon = \frac{1}{n}$ (for $n \in \mathbb{N}$)? Note that then you have found a countable base even though the space $\mathbb{R}^3$ has uncountably many points!

3.1.2. Morphisms: continuous functions

Now that we know what topological spaces are, what are the structure-preserving mappings between them? Again, there is a neat but abstract definition, which we state first and then link it to more familiar ideas in 'our' space.

Definition 3.6. Let X and Y be topological spaces and $f : X \to Y$ a function. We say f is *continuous* if, for all open subsets V of Y, the preimage $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is an open subset of X.

Exercise 3.7. Maybe you have encountered the idea of a continuous function before but in the more concrete, so-called *epsilon–delta definition* of a continuous function $f : \mathbb{R} \to \mathbb{R}$. This definition says that $f : \mathbb{R} \to \mathbb{R}$ is continuous if
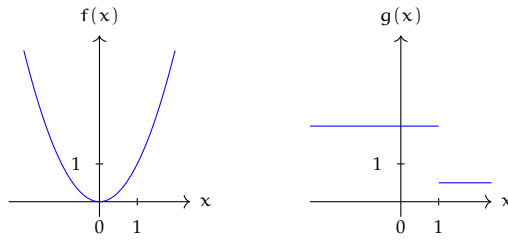
Figure 3.1.: A continuous function f (left) and a non-continuous function g (right).

1. For every $x \in \mathbb{R}$ and every $\epsilon > 0$, there is $\delta > 0$ such that, for all $y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

This captures the idea that, to draw the graph of the function, you do not have to lift your pen: If you want to continue drawing the graph a bit to the left or right of an argument x, the value outputted by the function will not 'jump away' but be close to the value at point x. To illustrate, consider the following two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := x^2 \qquad\qquad f(x) := \begin{cases} 2 & \text{if } x < 1 \\ 0.5 & \text{if } x \geqslant 1. \end{cases}$$

When drawing their graphs, as in figure 3.1, we can do this for f without lifting the pen, while for g we have to lift it at $x = 0$. And indeed, for $\epsilon := \frac{1}{4} > 0$, we cannot find the required $\delta > 0$.

Verify this for yourself.

Now, a good exercise to build intuition is to show that this 'hands-on' definition of continuity is equivalent to—and hence generalized by—the abstract topological definition.

I admit, the pun is intended

For this, we have to define the standard topology on the real line $\mathbb{R}$. This is done just like in the three-dimensional case, except that the distance function now simplifies: Here, since $\mathbb{R}$ has just one dimension, $d(x, y) = \sqrt{(x - y)^2} = |x - y|$. So the opens of the real line are those subsets $U \subseteq \mathbb{R}$ such that, for all $x \in \mathbb{R}$, if $x \in U$, then there is $\epsilon > 0$ such that, for all $x' \in \mathbb{R}$ with $d(x, x') < \epsilon$, we have $x' \in U$.

In other words, for a given function $f : \mathbb{R} \to \mathbb{R}$, show that f satisfies 1 iff f is continuous according to definition 3.6 with the just defined topology on $\mathbb{R}$.

This also is the end-of-chapter exercise 3.c.

The continuous functions are for topological spaces what the monotone functions were for partial orders. But, like for partial orders, we sometimes

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also want to consider stronger properties of these structure-preserving maps—in particular, the notion of isomorphism.

Definition 3.8. A continuous function $f : X \to Y$ between topological spaces is

- *open* if, for all open $U \subseteq X$, the image $f[U] = \{f(x) : x \in U\}$ is an open subset of Y.

- *closed* if, for all closed $C \subseteq X$, the image $f[C] = \{f(x) : x \in C\}$ is a closed subset of Y.

- a *homeomorphism* (the topologists' name for isomorphism) , if f has a continuous inverse, i.e., f is a bijection and both f and f^{-1} are continuous. (Equivalently, as exercise 3.d shows, f is a continuous and open bijection; this is further equivalent to f being a continuous and closed bijection.)

  Note the additional 'e': it is not 'homomorphism' as with lattices.

- an *embedding*, f is injective and, for each open $U \subseteq X$, there is an open $V \subseteq Y$ such that $f[U] = f[X] \cap V$. Equivalently, the function $f : X \to f[X]$ is a homeomorphism when giving $f[X] \subseteq Y$ the subspace topology (whose opens are $V \cap f[X]$ for $V \subseteq Y$ open).

  This is the conceptual meaning of embedding: X is, up to homeomorphism, a subspace of Y.

Homeomorphisms are the isomorphisms of spaces: If there is a homeomorphism between spaces they are called homeomorphic and hence are topologically the same. The standard example is that a donut and a coffee mug are homeomorphic: you can obtain one from the other by squishing and squeezing, but—importantly—without breaking and tearing.

Hence the common joke that topologists cannot tell them apart.

3.1.3. Constructions: subspaces, products, quotients

Coming to constructions with topological spaces, we have the following.

Cf. the trinity of sublattice, product, and homomorphic images/quotient for lattices.

1. Subspace. Given a topological space (X, τ), any nonempty subset $Y \subseteq X$ can be naturally made into a topological space by equipping it with the *subspace topology*

$$\tau \restriction Y := \{U \cap Y : U \in \tau\}.$$

2. Product topology. If $(X_i)_{i \in I}$ is a collection of topological spaces indexed by a set I, the product space $\prod_{i \in I} X_i$ has as underlying set the Cartesian product of the sets X_i and its topology is generated by the subbase of sets of the form $\{x \in \prod_{i \in I} X_i : x_j \in V\}$ for $j \in I$ and $V \subseteq X_j$ open.

Equivalently, this is the smallest topology making continuous all the projections $\pi_i : \prod_I X_i \to X_i$ mapping x to its i-th component x_i.

3. Quotient space. If X is a topological space and $\equiv$ an equivalence relation on X, the quotient space has as underlying set $X/\equiv$ and the opens are those sets $U \subseteq X/\equiv$ such that $\{x \in X : [x]_\equiv \in U\}$ is open in X.

A construction specific to spaces is that we can take the *join* of two topologies that live on the same underlying set. This is made precise as follows.

1. If X is a nonempty set, then

$$\mathrm{Top}(X) := \left\{\tau \in 2^{2^X} : \tau \text{ is a topology}\right\}$$

is, when ordered under inclusion, a complete lattice.

2. Infima are given by intersections, and suprema are given by the topology generated by unions. The least element is the indiscrete topology, and the greatest element is the discrete topology.

3. In particular, if σ and τ are two topologies on X, then their *join* $\sigma \vee \tau$ is the topology generated by $\sigma \cup \tau = \{U \subseteq X : U \in \tau \text{ or } U \in \sigma\}$.

Some fun facts are that the Hausdorff topologies on X form an upset in $\mathrm{Top}(X)$, and the compact topologies form a downset. And compact-Hausdorff topologies are incomparable: If σ is a Hausdorff topology and τ a compact topology on the nonempty set X, then $\sigma \subseteq \tau$ implies $\sigma = \tau$.

A proof of this can be found in Gehrke and van Gool (2023, Prop. 2.10).

3.2. Important topological concepts

Now that we know topological spaces, we introduce some further topological concepts that are important in their own right and will figure in the definition of a Stone space.

3.2.1. Separation axioms

There is an important classification of topological spaces according to which so-called *separation axioms* they satisfy. There are many such axioms and they all are of the form that two distinct points can be—in various senses—separated by the topology. (Note that, despite them being called 'axioms', in general a topological space is not required to satisfy any of them.) The five main ones coming in increasing strength.

Definition 3.9. Let (X, τ) be a topological space.

1. X is T_0 (aka *Kolmogorov*) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that U contains exactly one of x and y.

2. X is T_1 (aka *Fréchet*) if, for all $x \neq y$ in X, there is an open $U \subseteq X$ such that $x \in U$ and $y \notin U$.

3. X is T_2 (aka *Hausdorff*) if, for all $x \neq y$ in X, there are disjoint opens $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

4. X is T_3 (aka *regular*) if X is T_1 and, for all $x \in X$ and closed $C \subseteq X$ with $x \notin C$, there are disjoint opens $U, V \subseteq X$ with $x \in U$ and $C \subseteq V$.

5. X is T_4 (aka *normal*) if X is T_1 and, for all disjoint closed $C, D \subseteq X$, there are disjoint opens $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

Exercise 3.10. Show that these conditions indeed increase in strength, i.e., that we have the implications: $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$. Also check that the three-dimensional space $\mathbb{R}^3$ is normal and hence also satisfies the other separation axioms. Moreover, show that $(\mathbb{N}, \tau)$, where τ is the collection of upsets of $\mathbb{N}$ with the usual order, is a topological space that is T_0 but not T_1 (and hence also not T_2, T_3, or T_4).

3.2.2. Compactness

Another important concept is that of compactness. It formalizes the intuition that the space does not extend infinitely but has finite bounds.

Definition 3.11 (Compactness). Let X be a topological space. If $S \subseteq X$ is a subset, an *open cover* $\mathcal{U}$ of S is a collection of open sets such that $S \subseteq \bigcup_{U \in \mathcal{U}} U$. A subset $S \subseteq X$ is *compact* if every open cover $\mathcal{U}$ of S contains a finite subcover, i.e., there is a finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of S. The space X is called *compact* if $S := X$ is compact.

For example, while the whole Euclidean space is not compact, closed boxes in it like the unit cube

$$[0, 1] \times [0, 1] \times [0, 1] = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leqslant x_1, x_2, x_3 \leqslant 1\big\}$$

are compact. Also any finite subset of a space is compact. Exercise 3.f asks you to establish this.

There also is a local version of compactness: A topological space X is *locally compact* if, for any open neighborhood U of any point $x \in X$, there is an open $V \subseteq X$ and compact $K \subseteq X$ such that $x \in V \subseteq K \subseteq U$. If X is Hausdorff, then compactness implies local compactness, but this is not

true in general. And local compactness does also not imply compactness (the Euclidean space is locally compact but not compact).

The following exercise collects some very useful facts about compactness.

Exercise 3.12. Prove the following:

1. A closed subset of a compact space is compact.

2. A compact subset of a Hausdorff space is closed. This need not be true without the Hausdorffness assumption.

3. The image of a compact subset under a continuous function is compact.

4. Conclude that a continuous function from a compact space to a Hausdorff space is closed.

5. Conclude (hint: exercise 3.d) that a continuous bijection between compact Hausdorff spaces is a homeomorphism.

6. A compact Hausdorff space is normal (hint: show it is regular first).

Some further and more advanced results are the following.

1. *Finite intersection property characterization.* Let X be a topological space and $S \subseteq X$ a subset. A collection $\mathcal{A}$ of closed sets has the *finite intersection property* with respect to S if for every finite subcollection $\mathcal{A}_0$, there is $x \in S$ such that $x \in \bigcap \mathcal{A}_0$. Then S is compact iff, for every collection $\mathcal{A}$ of closed sets with the finite intersection property with respect to S, there is $x \in S$ with $x \in \bigcap \mathcal{A}$. (The proof essentially is rewriting the open set definition by taking complements and using the de Morgan laws: see, e.g., here.)

If $S = X$, we omit the 'with respect to S'.

2. *Alexander Subbase Theorem.* Let X be a topological space and $\mathcal{S}$ a subbase. If every cover $\mathcal{U} \subseteq \mathcal{S}$ of X has a finite subcover, then X is compact.

The proof of this requires a non-constructive principle, i.e., a (strictly weaker) version of the axiom of choice. As this is an axiom of standard set theory, we assume this throughout in this course.

3. *Tychonoff's Theorem.* This says that the arbitrary product of compact spaces is again compact. (This is a corollary of the Alexander Subbase Theorem; see, e.g., here.)

3.2.3. Dimensionality

We say that the space we live in has three dimensions. But is there a way to reasonably define the dimension of a general topological space (X, τ) in such a way that the dimension of $\mathbb{R}^3$ considered as a topological space is indeed 3? It turns out there is; in fact there are several such reasonable definitions, though they may disagree on some topological space (X, τ) different from $\mathbb{R}^d$. The common ones are called small inductive dimension, large inductive dimension, and the Lebesgue covering dimension. For a full treatment, see Engelking (1989, ch. 7). Here we will only look at the small inductive dimension. In fact, eventually we will only consider zero-dimensional spaces. So we will only briefly discuss dimensionality to get the general motivation (definition 3.13) and then prove a simple characterization for the zero-dimensional spaces (proposition 3.14).

According to our intuitive notion of dimension, one-dimensional objects are things like a line or a circle, two-dimensional objects are things like a plane or a disk, and three-dimensional objects are things like 'our' space or also a sphere (say, a football). In the extreme case, we may also say that a single point has dimension 0. Moreover, the dimensions of these objects are connected as follows: if we consider the boundary of an object, we reduce the dimension by one. For example, the boundary—i.e., surface—of a sphere is two-dimensional (an ant walking on the football just experiences a two-dimensional world); and the boundary of a disk is just the circle.

The formal definition of the small inductive dimension bootstraps these intuitions: The simplest case is where the object is just the empty set. Its dimension is stipulated to be -1. Because then, if we consider the next more complicated object, namely a single point, then its is indeed 0: it must be one more than the dimension of its boundary, but its boundary is empty, hence has dimension -1, so the dimension of the point is one more, i.e., 0. Now we proceed inductively: Assume we have a yet more complicated object, but which is still not much more complicated in the sense that we can already determine the dimension of its boundary. Then the dimension of this object is one more than the dimension of its boundary.

The formal definition goes as follows (relativizing the above idea to each neighborhood of points of the considered object).

Definition 3.13. Given a regular topological space (X, τ) (which, only for this definition, we allow to also be empty) and an integer $n \geqslant 0$, say

- $\mathrm{ind}(X) = -1$ iff $X = \emptyset$

- $\mathrm{ind}(X) \leqslant n$ iff for every point $x \in X$ and open neighborhood V of x, there is an open $U \subseteq X$ with $x \in U \subseteq V$ and $\mathrm{ind}(\partial U) \leqslant n - 1$.

- $\mathrm{ind}(X) = n$ iff $\mathrm{ind}(X) \leqslant n$ and $\mathrm{ind}(X) \not\leqslant n - 1$.

- $\mathrm{ind}(X) = \infty$ iff, for all n, $\mathrm{ind}(X) \not\leqslant n$.

As mentioned, we will actually be interested in spaces with dimension zero. They have the following simple characterization. It is just an unpacking of definitions, so we leave the proof as exercise 3.h.

Proposition 3.14. *Let (X, τ) be a regular topological space. Then X is zero-dimensional (i.e., $\mathrm{ind}(X) = 0$) iff X has a base consisting of clopen sets.*

Hence, from now on, we take the definition of *zero-dimensional* to be having a base of clopens.

It may seem that being zero-dimensional is quite different from our everyday intuitions about space. This is true to some extent, but the following example shows that zero-dimensional spaces are also not completely strange.

Exercise 3.15. Write $X := \mathbb{R} \setminus \mathbb{Q}$ for the set of *irrational* numbers and let τ be the subspace topology with respect to the usual topology on $\mathbb{R}$. Show that the space of irrational numbers (X, τ) is a zero-dimensional space. (Hint: show that $\{B_{\frac{1}{n}}(q) \cap X : n \in \mathbb{N}, q \in \mathbb{Q}\}$ is a clopen base.)

3.3. Stone spaces

Now we can define Stone spaces (section 3.3.1). Then we will discuss an important example of a Stone space: namely, the Cantor space (section 3.3.2).

3.3.1. Definition

The definition of a Stone space is actually quite very simple.

Definition 3.16. A *Stone space* is a topological space that is compact, zero-dimensional, and Hausdorff.

Though you may wonder why it is exactly those spaces that we will consider. There are two answers. First, several important spaces—especially in descriptive sets theory—are Stone spaces. In the next subsection, we will consider the most important one: Cantor space. From this point of

view, we consider Stone spaces simply because it makes sense to study the general properties of this important class of spaces.

The second answer is the insight of Stone that it is precisely those spaces that dually correspond to Boolean algebras under the constructions motivated in chapter 1. When we start with a Boolean algebra of, say, propositions and we consider the set of prime filters on it, we will see that we can naturally define a topology on this set. It is generated by calling those sets of prime filters open that make true a given proposition in the Boolean algebra. It turns out that this topology has all the defining features of a Stone space. Conversely, if we start with any topological space, the collection of its clopen subsets forms a Boolean algebra; but if this space actually was a Stone space, then this Boolean algebra is rich enough so we can recover the original space as the set of prime filters on this Boolean algebra.

So the second answer is a 'in hindsight' motivation of the definition: it characterizes a class of spaces that we obtain through a construction that we are interested in. But since we have not seen that yet, let's first explore the first answer: namely, example of Stone spaces that are important in their own right.

3.3.2. Example: Cantor space

The Cantor space plays an important role in descriptive set theory (Kechris 1995). One reason is that it is universal in the sense that every nonempty compact metrizable space is a continuous image of the Cantor space. In particular, every Stone space that is second-countable (i.e., has a countable base) is an image of the Cantor space. We will now define the Cantor space and show that it is a Stone space. We will not prove the just mentioned universality result (see Kechris 1995, p. 23), but it assures us that the Cantor space is an important example of a Stone space: it in a sense already contains all second-countable Stone spaces.

Second-countable Stone spaces are metrizable by Urysohn's metrization theorem.

The succinct definition of Cantor space is as follows; we will unpack it below.

Definition 3.17. Equip $2 := \{0, 1\}$ with the discrete topology. Define $2^{\mathbb{N}}$ to be the product space $\prod_{\mathbb{N}} 2$ of $\mathbb{N}$-many copies of 2. The space $2^{\mathbb{N}}$ is called the *Cantor space*.

The points of Cantor space are, by definition of the Cartesian product (appendix A.1), infinite binary sequences, i.e., functions $x : \mathbb{N} \to 2 = \{0, 1\}$. We also write these sequences as $x(0)x(1)x(2)\dots$. For example, here are

two points of Cantor space:

$$0101010101010101010\ldots \qquad 1110101001010111010\ldots.$$

The topology of Cantor space is concisely described via the product topology, but to build an intuition, it is useful to introduce *cylinder sets*. Given a binary string $\sigma = \sigma_0 \ldots \sigma_{n-1}$ (i.e., a finite sequence of 0's and 1's), the corresponding cylinder set is

$$[\sigma] := \big\{x \in 2^{\mathbb{N}} : \forall k \in \{0, \ldots, n-1\}.x(k) = \sigma_k\big\},$$

so $[\sigma]$ contains all those binary sequences that have σ as initial segment.

Exercise 3.18. Show that the cylinder sets form a base for the topology of $2^{\mathbb{N}}$. (Hint: go back to the definition of the product topology in section 3.1.3.)

With this, it is quite straightforward to show that the Cantor space is a Stone space.

Exercise 3.19. Show the following for the Cantor space $2^{\mathbb{N}}$:

1. It is Hausdorff. (Hint: Conclude this from the previous exercise 3.18.)

2. It is zero-dimensional. (Hint: Show that the cylinder sets in fact are clopen.)

3. It is compact. This follows immediately from Tychonoff's theorem (since the finite space 2 is compact). But, as an exercise, also show this directly with the finite intersection property definition (section 3.2.2).

A few more comments on Cantor space. You might know it as the Cantor *set* from analysis: where you take the unit interval $[0, 1]$ and keep removing the middle thirds. More precisely, it consists of those numbers in the unit interval that have only 0's and 2's in their ternary expansion. One can show that this Cantor set is homeomorphic to the Cantor space $2^{\mathbb{N}}$. Similarly, the irrational numbers in the unit interval, i.e., $[0, 1] \setminus \mathbb{Q}$ also are homeomorphic to the Cantor space. These examples are instances of a theorem of Brouwer which characterizes the Cantor space as the, up to homeomorphism unique perfect, nonempty, compact, metrizable, zero-dimensional space (Kechris 1995, p. 35). Cantor space also is a natural setting for probability theory (as a standard Borel space) and also for algorithmic randomness (Downey and Hirschfeldt 2010).

Cf. exercise 3.15

A space is perfect if for every open neighborhood U of a point x, there is a point $y \in U$ with $y \neq x$.

3.4. Exercises

To get familiar with the abstract topological concepts from section 3.1.1, the next two exercises apply them to the usual three-dimensional space $\mathbb{R}^3$ (whose open sets are those with 'wiggle-room').

Exercise 3.a.   1. Prove that the collection τ of sets $U \subseteq \mathbb{R}^3$ with wiggle-room, as defined in example 3.2 (1), indeed forms a topology on $\mathbb{R}^3$.

2. A *closed interval* is of the form $[a, b] := \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ with $a, b \in \mathbb{R}$. An open interval is of the form $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ with $a, b \in \mathbb{R}$. (They are empty if $a \not\leqslant b$.) In three-dimensional space, a *rectangular cuboid* is of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ with $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$. (So a rectangular cuboid is just the 3D analogue of a rectangle; and a rectangle, in turn, is just the 2D analogue of an interval. For higher dimensions, one speaks of boxes or also hyperrectangles or k-cells.) Show that rectangular cuboids are closed in $\mathbb{R}^3$.

3. Show that the sequence of points $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n})_{n \geqslant 1}$ converges to $(0, 0, 0)$.

Exercise 3.b. Do exercise 3.5. (If you have done exercise 2.e, you can additionally show that the interior map $\mathsf{Int} : 2^X \to \Omega(X)$ is upper adjoint to the inclusion map $\iota : \Omega(X) \to 2^X$, and that the closure map $\mathsf{Cl} : 2^X \to \mathcal{C}(X)$ is lower adjoint to the inclusion $\iota' : \mathcal{C}(X) \to 2^X$ (Gehrke and van Gool 2023, ex. 2.1.3).)

Exercise 3.c. Do exercise 3.7.

(Gehrke and van Gool 2023, ex. 2.1.1 (c))

Exercise 3.d. Let $f : X \to Y$ be a continuous bijection. Show that the following are equivalent.

(Gehrke and van Gool 2023, ex. 2.1.4 (c))

1. f is a homeomorphism (i.e., its inverse is continuous)

2. f is open (i.e., maps open sets to open sets)

3. f is closed (i.e., maps closed sets to closed sets).

Exercise 3.e. Do exercise 3.10.

Exercise 3.f. This exercise gets you acquainted with the concept of compactness via some examples.

1. Show that $\mathbb{R}^3$ is not compact but the unit cube is.

2. Show that any finite subset of any topological space is compact.

Exercise 3.g. Do exercise 3.12.

Exercise 3.h. Prove proposition 3.14.

Exercise 3.i. Do exercise 3.15.

Exercise 3.j. Do exercises 3.18 and 3.19 to establish that the Cantor space is a Stone space.

Exercise 3.k. Let X be a topological space. Prove that the following are equivalent.

1. X is a Stone space (in the sense of definition 3.16, i.e., compact, Hausdorff, and zero-dimensional)

2. X is compact, T_0, and zero-dimensional.

3. X is compact and *totally separated* (i.e., for any $x \neq y$ in X there are disjoint open sets U and V such that $x \in U$ and $y \in U$ *and* $U \cup V = X$).

4. Two sides of the same coin: Stone duality

In this chapter, we finally prove the Stone duality theorem that we have been working toward. It provides an exact correspondence between Boolean algebras on the one hand and Stone spaces on the other hand. In section 4.1, we first state the main idea of this correspondence informally. In section 4.2, we then formulate it precisely and, finally, prove it in section 4.3. In the next chapter, we discuss some applications.

4.1. Stone duality: The main idea

We already saw in the introduction (chapter 1) the conceptual idea of how algebras correspond to spaces. We now can state this more concretely, namely for Boolean algebras corresponding to Stone spaces. In this section, we do this informally—to have a succinct overview—, before doing it formally in the next section. We will use the 'possible worlds vs. propositions' example of a duality (as discussed in section ssec: example duality semantics), but you can also swap this to your favorite example.

From algebras to spaces   Assume we have a Boolean algebra: say, the set of propositions, i.e., meanings of sentences. Then we can construct a space of possible worlds. We do so, by taking a possible world to be a 'prime filter' on the algebra, i.e., a set of propositions that is appropriately closed under the operations of implications, conjunctions, disjunctions, logical truth, and logical falsity. What we could not see back than—but what we will see now—is that this collection of constructed worlds w carries a natural topology. It is generated by the sets $\{w : a \in w\}$, for each proposition a in the algebra. We will see that this defines a Stone space.

From spaces to algebras   Conversely, if we start with the space of possible worlds, then we can construct a Boolean algebra of propositions. This is because, in line with possible worlds semantics, propositions simply are sets of possible worlds. Namely, a proposition is the set of those worlds where it is true. However, while propositions are sets of worlds, we do not consider every set of worlds to be a proposition. Rather, following the observation that the space of possible worlds naturally carries a topology,

we only consider the 'clopen' sets of worlds to be a proposition. (Recall that a set is clopen if it is both closed and open in the topology.) We will see that the collection of clopen sets of worlds indeed forms a Boolean algebra.

Correspondence back and forth Now we a way to map an algebra to a space and a way to map a space to an algebra. But if we want a (bijective) correspondence, we want that that these maps cancel each other out. If I start with an algebra, map it to a space, and—in turn—map that back to an algebra, I should get an algebra that is isomorphic to the one I started with. And we want the same for spaces (for spaces, though, we say 'homeomorphic' instead of 'isomorphic'). After all, if we take propositions as given and—following ersatzism—build substitute possible worlds (as prime filters), then the substitute propositions we get from those substitute worlds (as sets of substitute worlds) better be isomorphic to the initially given algebra of propositions. In the other direction, if we take possible worlds as given and—following possible worlds semantics—consider propositions to be sets of possible worlds, then the substitute worlds we can build from these propositions better be isomorphic to the initially given space of possible worlds.

Duality Now we have a correspondence between spaces and algebras. But why do we speak of a 'duality'? The reason is this. The correspondence that we have so far is between the *objects* in the category of spaces on the one hand and the category of algebras on the other hand. But we would also like to have a correspondence between the *morphisms* of the categories, i.e., the structure-preserving maps between the objects. So a continuous map between Stone spaces should correspond uniquely to a Boolean algebra homomorphism between the Boolean algebras corresponding to the spaces. We speak of a (categorical) 'equivalence' if a morphism f : X → Y of Stone spaces corresponds uniquely to a morphism g : A → B, where A and B are the Boolean algebras corresponding to X and Y, respectively. However, we will see that we do not have this; but something close. What we have to do is to reverse the direction of the arrow g: the morphism f uniquely corresponds to some g : B → A. This phenomenon occurs often enough to get a specific name: one then speaks of a *dual equivalence* between the categories.

4.2. Stone duality: The precise statement

In this section, formally state the Stone duality in a series of propositions. So the Stone duality formally is the conjunction of propositions 4.3–4.10 below. Category-theoretically this is swiftly expressed as: The category of Boolean algebras and Boolean algebra homomorphisms is dually equivalent to the category of Stone spaces and continuous functions.

4.2.1. From Boolean algebras to Stone spaces: Prime filters

Let's recall again the idea of how to recover a space from a Boolean algebra. If we have a Boolean algebra L of propositions, we can recover the possible worlds as 'maximally decided' subsets F of L, i.e., those which have the expected closure conditions for implication ($\leqslant$), conjunction ($\wedge$), disjunction ($\vee$), logical truth ($\top$), and logical falsity ($\bot$). We have already called these subsets prime filers:

Definition 4.1. Let L be a lattice. (So, in particular, this definition applies if L is a Boolean algebra.) A subset F of L is a filter if it is a nonempty upset that is closed under meet. It is proper, if $F \neq L$ (equivalently, $\bot \notin F$). A *prime filter* is a proper filter such that, for all $a, b \in L$, if $a \vee b \in F$, then either $a \in F$ or $b \in F$. We write $\mathsf{PrFilt}(L)$ for the set of all prime filters of L. If needed, we write $\mathsf{Filt}(L)$ for the set of all filters of L.

The order-dual notion of a filter is sometimes also useful and is called an *ideal*: Concretely, these are nonempty downsets $I \subseteq L$ closed under join (if $a, b \in I$, then $a \vee b \in I$). An ideal is *proper* if $I \neq L$ and *prime* if, additionally, $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Here we will always work with filters and rarely mention ideals (only if we need to also talk about duals of filters). But, again, ultimately this is a convention and many textbooks primarily use ideals.

An equivalent way to define a prime filter F of a lattice L is by requiring that its characteristic function

$$\chi_F : L \to \mathbf{2}$$
$$a \mapsto \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$$

is a lattice homomorphism. This brings out maybe more clearly the intuition that a prime filter decides, for every proposition in L, whether it is true according to it. Although these two characterizations (subsets vs

By 'recover' we mean 'retrieve' or 'regain': Assume we have the Boolean algebra but lost the space, then how can we deduce how the space must have been like only using the information provided by the Boolean algebra?

Being nonempty is, for an upset, equivalent to $\top \in F$.

The notion of an ideal is also important in the theory of rings which is central in commutative algebra. They are used to build the spectra *of rings, which provide useful topological tools to understand a ring. They form spectral spaces—hence the name. To see how lattice ideals relate to ring ideals, see exercise 3.1.9 of Gehrke and van Gool 2023.*

homomorphisms) are equivalent, it often is conceptually useful to consider both. Exercise 4.a asks you to prove this equivalence and a third one that prime filters are those subsets whose complements are prime ideals.

So far, what we said about prime filters works in any lattice. But when we restrict us to Boolean algebras, a striking fact is the following characterization of prime filters. The proof is a recommended exercise (exercise 4.b).

Proposition 4.2. *Let* A *be a Boolean algebra, and let* $F \subseteq A$ *be a filter. Then the following are equivalent.*

1. F *is a prime filter.*

2. F *is a* maximal *filter, i.e.,* F *is proper and for any proper filter* G *with* $F \subseteq G$, *we have* $F = G$.

3. F *is an* ultrafilter, *i.e.,* F *is proper and for any* $a \in A$, *either* $a \in F$ *or* $\neg a \in F$.

The philosophical meaning of this is that the usual, well-motivated notion of a prime filter is equivalent to other common notions of a 'model' in classical logic (and Boolean algebras are the algebraic version of classical logic). Prime filters require the models to respect conjunction and disjunction. Ultrafilters require models to respect conjunction and negation. Maximal filters require models to be maximally consistent. All three are common ways of specifying a classical model or possible world (or whatever the philosophical interpretation of the points in the space).

Cf. section 2.3.2: being a lattice homomorphism (respecting $\wedge$ and $\vee$) already implies being a Boolean algebra homomorphism (i.e., also respecting $\neg$).

With this understanding of prime filters, let's move to understanding the structure of the set of all prime filters on a Boolean algebra L: because this is how we recover the space of possible worlds. What we couldn't yet see so far is that this space really is a space in the sense of topology! So we need to say what the open sets are on $\mathsf{PrFilt}(L)$. But what's a natural choice? The general intuition for the open sets is that they represent different degrees of closeness or similarity: For example, in our space, the open ball $B_\epsilon(x)$ around a point x represents closeness to degree ϵ. What would be basic degrees of similarity for prime filters? The idea is that two prime filters can be close to degree $a \in L$ by agreeing on the property $a \in L$, i.e., both contain a or both do not contain a. So we declare open, for each $a \in L$, the sets

$$\{F \in \mathsf{PrFilt}(L) : a \in F\}$$

This indeed produces a Stone space as the next propositions shows.

Proposition 4.3. *Let* L *be a Boolean algebra. Equip* $X := \mathsf{PrFilt}(L)$ *with the topology generated by* $\mathcal{S} := \{\widehat{a} : a \in L\}$*, where*

$$\widehat{a} := \{F \in X : a \in F\}.$$

Then X *with this topology is a Stone space, which we also denoted* $\mathsf{St}(L)$*. Moreover,* $\mathcal{S} = \mathsf{Clp}(X)$*.*

Example 4.4. Consider the diamond Boolean algebra with two incomparable elements a and b between the bottom element $\bot$ and the top element $\top$ (cf. figure 2.1). What are its prime filters? To compute them by brute force, let's first list all its upsets (since every prime filter in particular is an upset):

$$\emptyset, \{\top\}, \{a, \top\}, \{b, \top\}, \{a, b, \top\}, \{\bot, a, b, \top\}.$$

Which of those are filters? Not the empty set, because filters are required to be nonempty. And also not the second-last set, because it is not closed under $\wedge$. But the remaining sets are filters. The last one is, by definition, not proper. Which of the remaining proper filters are prime? The filter $\{\top\}$ is not prime, because $a \vee b = \top$, but neither a nor b are in the filter. But the two filters

$$F_1 := \{a, \top\} \qquad\qquad F_2 := \{b, \top\}$$

are prime. Also not that they are maximal and ultra. So they form the points of the dual space $X = \{F_1, F_2\}$.

What is the topology of the space X? We in particular have the open sets:

$$\widehat{\bot} = \{F \in X : \bot \in F\} = \emptyset$$
$$\widehat{a} = \{F \in X : a \in F\} = \{F_1\}$$
$$\widehat{b} = \{F \in X : b \in F\} = \{F_2\}$$
$$\widehat{\top} = \{F \in X : \top \in F\} = \{F_1, F_2\}$$

These in fact already are all the subsets of the space X, to the topology is discrete in this case (later we will see that this is no coincidence). $\lrcorner$

4.2.2. From Stone spaces to Boolean algebras: Clopens

Now going in the other direction, the idea was that if we have a space of possible worlds, then propositions can be identified with their intensions: i.e., the set of possible worlds where the proposition is true. So intensions are subsets of the space. But they are not just *any* subset. What comes to light via the topology on the space, a plausible requirement is that intensions are clopen sets.

We already saw this above for the recovered possible worlds: If we have a Boolean algebra L, then for each proposition $a \in L$, its intension in the recovered space X is $\hat{a} = \{F \in X : a \in F\}$, and by construction this $\hat{a}$ is a clopen set. And it stands to reason that this should also be the case if we start with a space of possible worlds X (if you believe in 'proper' possible worlds, that aren't just constructed out of propositions). Indeed, it seems plausible to require that the intension of a proposition a is closed under similarity: if world x makes true proposition a and world y is very similar to x, then also y makes true a. If similarity is spelled out topologically, this requirement naturally is formalized as the intension of a being clopen:

- if x is in the intension, there is a degree of similarity (i.e., an open set) such that all objects similar to x by at least this degree also are in the intension (hence the intension is open), and

- if x is not in the intension, there is a degree of similarity (i.e., an open set) such that all objects similar to x by at least this degree also are not in the extension (hence the intension is closed).[2]

This also makes sense for the 'object vs. property' example: there the principle of tolerance says that if object x has property a and object y is very similar to x, then also y has property a.[1]

So the propositions are those subsets of the space of possible worlds that are clopen—to respect the similarities between possible worlds that is formalized in the toplogy. Fortunately, the clopens form a Boolean algebra.

Proposition 4.5. *Let (X, τ) be a Stone space. Let $L := \mathsf{Clp}(X)$ be the set of clopen subsets of X ordered by inclusion. Then $(L, \cap, \cup, \emptyset, X)$ is a Boolean algebra, with negation given by the set-theoretic complement.*

---

[1]Actually, more precisely this is the much less problematic *margin for error principle* (see, e.g., Williamson (1994, sec. 8.3) or Égré (2015, sec. 2)): If object x has property a, then there is some degree of similarity ϵ such that any object y that is ϵ-similar to x also has property a. Also see Belastegui Lazcano (2020).

[2]This may be even more compelling when formulated with a metric: if x is in a, then there is an $\epsilon > 0$ such that all ϵ-close y are also in a; similarly for the complement a^c. Such a metric exists if the space X is metrizable, i.e., there is a metric d on X such that the open balls of the metric generate the topology of X. By the Urysohn metrization theorem, a compact Hausdorff space (which X is) is metrizable iff it is second-countable (i.e., has a countable base). Hence the metric formulation works provided there are countably many propositions.
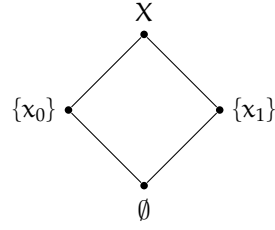
Figure 4.1.: The Boolean algebra of clopens $\mathsf{Clp}(X)$ of the two-element discrete space $X = \{x_0, x_1\}$.

Example 4.6. Let's consider the two element space $X = \{x_0, x_1\}$ with the discrete topology. Up to renaming the points (i.e., up to homeomorphism), this is the Stone space we ended up with in example 4.4. What is the Boolean algebra of clopens $\mathsf{Clp}(X)$?

Let first list all the possible subsets: $\emptyset, \{x_0\}, \{x_1\}, X$. Since the topology is discrete, all of them are clopen: any set is open, so also the complement of any set is open, so any set is both open and closed. If we order these sets by inclusion, we get the Boolean algebra in figure 4.1.

As we will see next, it is no coincidence that this Boolean algebra is isomorphic to the diamond Boolean algebra we started out with in example 4.4. ⌟

4.2.3. Boolean algebras are isomorphic to their double-duals

If we start with an algebra of propositions and construct worlds as prime filters, then we would expect that the algebra of intensions of the propositions with respect to the constructed worlds is isomorphic to the original algebra of propositions. Concretely, if we are given a proposition a in a Boolean algebra L of propositions, we can consider its intension $\widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}$ with respect to the constructed worlds—and we expect a to play the same role as $\widehat{a}$. In other words, we expect that this map $a \mapsto \widehat{a}$ in fact is an isomorphism. Fortunately, this is the case.

Proposition 4.7. *Let L be a Boolean algebra. Then the following is a well-defined lattice isomorphism between L and its double-dual:*

$$\widehat{\cdot} : L \to \mathsf{Clp}(\mathsf{PrFilt}(L))$$
$$a \mapsto \widehat{a} = \{F \in \mathsf{PrFilt}(L) : a \in F\}.$$

4.2.4. Spaces are isomorphic to their double-duals

Similarly, if we start with a space of possible worlds, we also expect that it is homeomorphic to the space of worlds constructed as prime filter over intensions formed with the original possible worlds. In section 1.1.2, we discussed the motivation behind this:

- Each world x determines the set F_x of propositions/intensions that are true at x—and this is a prime filter.

- Each set F of propositions that is a prime filer should determine a possible world x: namely, the unique possible world making true exactly the propositions in F.

This indeed yields an isomorphism between a space and its double-dual:

Proposition 4.8. (X, τ) *be a Stone space. Then the following is a well-defined homeomorphism:*

$$\beta : X \to \mathsf{PrFilt}(\mathsf{Clp}(X))$$
$$x \mapsto \{a \in \mathsf{Clp}(X) : x \in a\}$$

the x with $x \in a$ for all $a \in F \hookleftarrow F$.

4.2.5. Also including morphisms

Now, we have the correspondence between the objects (spaces and algebras), but we also want a correspondence between the morphisms (continuous maps and Boolean algebra homomorphisms). Again, we first see how to move in both directions (proposition 4.9), and then we check that these moves cancel each other out (proposition 4.10).

Let's start by moving from space to algebras. If X and Y are Stone spaces, their dual Boolean algebras are $\mathsf{Clp}(X)$ and $\mathsf{Clp}(Y)$. If we have a continuous function $f : X \to Y$, how do we get a corresponding morphism between those dual Boolean algebras? We already mentioned that we corresponding morphism will go in the opposite direction: $\mathsf{Clp}(Y) \to \mathsf{Clp}(X)$. The reason is preimage of a function is much better behaved than the direct image: it preserves more set-theoretic operations like intersection, unions, etc.—see exercise 4.c. So rather than trying to map a clopen subset $A \subseteq X$ to $f[A] \subseteq B$, we map a clopen subset $B \subseteq Y$ to $f^{-1}(B) \subseteq X$. Since f is continuous, $f^{-1}(B)$ is indeed clopen. The result below says that this indeed is a Boolean algebra homomorphism.

This is a trick worth remembering!

Now, let's consider moving from algebras to spaces. If L and M are algebras, their dual Stone spaces are $\mathsf{PrFilt}(L)$ and $\mathsf{PrFilt}(M)$. If we have a

Boolean algebra homomorphism $h : L \to M$, how do we get a corresponding continuous function between the dual spaces? We again expect to go in the opposite direction: $\mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$. If G is a prime filter on M, it is, in particular, a subset of M, so we again map it to $h^{-1}(F) \subseteq L$. Exercise 4.e asks you to verify that this is indeed a prime filter. Formally, the result then is the following.

Proposition 4.9.

1. If $f : X \to Y$ *is a continuous function between Stone spaces, then*

$$\mathsf{Clp}(f) : \mathsf{Clp}(Y) \to \mathsf{Clp}(X)$$
$$B \mapsto f^{-1}(B)$$

   is a Boolean algebra homomorphism.

2. If $h : L \to M$ *is a Boolean algebra homomorphism, then*

$$\mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$$
$$G \mapsto h^{-1}(G)$$

   is a continuous function.

So now we have extended Clp and PrFilt from objects to also including morphisms: they not only map Stone spaces to Boolean algebras and vice versa, but also continuous functions to Boolean algebra homomorphisms and vice versa. Exercise 4.f asks you to show that, in doing so, they also respect the composition of morphisms, i.e., $\mathsf{Clp}(g \circ f) = \mathsf{Clp}(g) \circ \mathsf{Clp}(f)$ and $\mathsf{PrFilt}(g \circ f) = \mathsf{PrFilt}(g) \circ \mathsf{PrFilt}(f)$. In category-theoretic terminology we say: they are functors.

To get a correspondence, we need to check that also on morphisms PrFilt and Clp cancel each other out. But what should that mean precisely? Let's start with the spatial side. Like with the objects, we want to say that if we start on the spatial side with $f : X \to Y$, and map it to the algebraic side to get $\mathsf{Clp}(f)$, and then map it back again to $\mathsf{PrFilt}(\mathsf{Clp}(f))$, we should get something that is isomorphic to f. But what should being 'identical up to isomorphism' mean for functions? The answer is: the functions f and $\mathsf{PrFilt}(\mathsf{Clp}(f))$ should behave the same under the isomorphisms that we already have for the objects, namely, $\beta_X : X \to \mathsf{PrFilt}(\mathsf{Clp}(X))$ and $\beta_Y : Y \to \mathsf{PrFilt}(\mathsf{Clp}(Y))$. In other words, first applying f and then the isomorphism (i.e., β_Y) should be the same as first applying the isomorphism (i.e., β_X) and then $\mathsf{PrFilt}(\mathsf{Clp}(f))$. One says that the following diagram

should commute:

$$X \xrightarrow{\beta_X} \mathsf{PrFilt}(\mathsf{Clp}(X))$$
$$\downarrow f \qquad\qquad \downarrow \mathsf{PrFilt}(\mathsf{Clp}(f))$$
$$Y \xrightarrow{\beta_Y} \mathsf{PrFilt}(\mathsf{Clp}(Y))$$

(4.1)

$$A \xrightarrow{f} B$$
$$\downarrow g \qquad \downarrow g'$$
$$C \xrightarrow{f'} D$$

commutes is to say that
$g' \circ f = f' \circ g$.

We also want the same one the algebraic side: If we start on the algebraic side with $h : L \to M$, and map it to the spatial side side to get $\mathsf{PrFilt}(h)$, and then map it back again to $\mathsf{Clp}(\mathsf{PrFilt}(h))$, we should get something that is 'isomorphic' to h. Again, this means that the functions h and $\mathsf{Clp}(\mathsf{PrFilt}(h))$ should behave the same under the isomorphisms that we already have for the objects, namely, $\widehat{\cdot}_L : L \to \mathsf{Clp}(\mathsf{PrFilt}(L))$ and $\widehat{\cdot}_M : M \to \mathsf{Clp}(\mathsf{PrFilt}(M))$. In other words, the following diagram should commute:

$$L \xrightarrow{\widehat{\cdot}_L} \mathsf{Clp}(\mathsf{PrFilt}(L))$$
$$\downarrow h \qquad\qquad \downarrow \mathsf{Clp}(\mathsf{PrFilt}(h))$$
$$M \xrightarrow{\widehat{\cdot}_M} \mathsf{Clp}(\mathsf{PrFilt}(M))$$

(4.2)

Hence, to finalize the correspondence also between the morphisms, we claim that the above two diagrams commute.

Proposition 4.10. *The diagrams 4.1 and 4.2 commute.*

This concludes the statement of the Stone duality: as mentioned, it is the conjunction of propositions 4.3–4.10. Category-theorist would express this swiftly by saying:

Theorem 4.11 (Stone duality)**.** *The functors* PrFilt *and* Clp *form a dual equivalence between, on the one side, the category* BA *of Boolean algebras with Boolean algebra homomorphisms and, on the other side, the category* Stone *of Stone spaces with continuous functions.*

4.3. Stone duality: The proof

In this section, we provide the proofs for the Stone duality, i.e., propositions 4.3–4.10 above. We have one subsection per proposition plus a first subsection with two methods for constructing filters that we will need for the proofs.

4.3.1. Constructing filters

Given a Boolean algebra L of propositions, possible worlds are (re-) constructed as prime filters. So, given a set of propositions $A \subseteq L$, it would be very useful to construct prime filters F that contain all the propositions in A, i.e., $A \subseteq F$. Because this means that we were able to construct a possible world F that makes true all the desired propositions A.

Surely this is not always possible, for example if A contains two inconsistent propositions. But in this subsection we provide two results that show when this is possible. They correspond to two stages of the construction. The first result first extends A to a filter of L, and the second result then says when we can further extend this filter to a prime filter.

Proposition 4.12. *Let L be a lattice.*

1. *For any subset $A \subseteq L$, there is a $\subseteq$-smallest filter F that contains A. It is called the filter generated by A and denoted $\langle A \rangle_{\text{filt}}$.*

2. *Concretely, this filter is given as*

$$\langle A \rangle_{\text{filt}} = \left\{ a \in L : \text{ there is finite } A' \subseteq A \text{ such that } \bigwedge A' \leqslant a \right\}.$$

3. *If $F \subseteq L$ is a filter and we want to extend it by an element $b \in L$, this is concretely given as*

$$\langle F \cup \{b\} \rangle_{\text{filt}} = \left\{ a \in L : \text{ there is } f \in F \text{ such that } f \wedge b \leqslant a \right\}.$$

Order-dual results hold for ideals.

Theorem 4.13 (Stone's Prime Filter Extension Theorem). *Let L be a distributive lattice. If F is a filter and I and ideal in L such that $F \cap I = \emptyset$, then there is a prime filter G in L such that $F \subseteq G$ and $G \cap I = \emptyset$.*

The formulation using ideals makes this theorem more general: By choosing $I := \{\bot\}$, it says that any filter F not containing the inconsistent property $\bot$ can be extended to a prime filters. So the obstruction we mentioned before—that A is inconsistent—really is the only obstruction to extend A to a prime filter. But with the more general formulation in terms of ideal we can also, for example, take a property $b \notin \langle A \rangle_{\text{filt}}$ (i.e., b is not above a finite meet of properties in A) and, by choosing $I := \downarrow b$, construct a prime filter extension G that does not contain b.
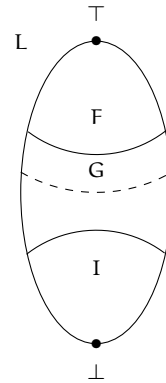
Proof. We go for a typical Zorn's lemma argument: Order the potential candidates for a solution so that any maximal element is an actual solution. Here the candidates are the filters extending F that don't intersect I:

$$\mathcal{P} := \big\{ G \in \mathsf{Filt}(L) : F \subseteq G \text{ and } G \cap I = \emptyset \big\}$$

which we order by inclusion. To apply Zorn's lemma, we need to check that $\mathcal{P}$ is nonempty—which it is, since $F \in \mathcal{P}$—and that any chain $\mathcal{C}$ in $\mathcal{P}$ has a maximal element: this is the case since $\bigcup_{G \in \mathcal{C}} G$ is a filter that belongs to $\mathcal{P}$ (verify this as an exercise). So Zorn's lemma applies and says that $\mathcal{P}$ has a maximal element G.

It remains to check that the maximality of G implies that it is prime. So assume $a \vee b \in G$ and show that either $a \in G$ or $b \in G$. The idea is to try and add a and b to G and conclude from the maximality of G that at least one of them must already be in G.

So consider the filter G_a (resp., G_b) generated by $G \cup \{a\}$ (resp. $G \cup \{b\}$). Recall from propositions 4.12 that it contains precisely those $c \in L$ for which there is $g \in G$ such that $g \wedge a \leqslant c$ (resp. $g \wedge b \leqslant c$). So G_a and G_b are still filters extending F and we claim that either $G_a \cap I = \emptyset$ or $G_b \cap I = \emptyset$. This implies $G \subseteq G_a, G_b \in \mathcal{P}$, so, since G is maximal, either $G = G_a$ or $G = G_b$, hence either $a \in G$ or $b \in G$, as needed.

Indeed, if there were $c \in G_a \cap I$ and $d \in G_b \cap I$, then $c \vee d$ is in I (since I is an ideal and hence closed under $\vee$) and both in G_a and in G_b (qua upsets). The latter implies that there is g_a and g_b in G such that $g_a \wedge a \leqslant c \vee d$ and $g_b \wedge b \leqslant c \vee d$. In particular, $g := g_a \wedge g_b \in G$ is such that $g \wedge a \leqslant c \vee d$ and $g \wedge b \leqslant c \vee d$. Hence, by distributivity,

$$g \wedge (a \vee b) = (g \wedge a) \vee (g \wedge b) \leqslant c \vee d.$$

Since $g \in G$ and $a \vee b \in G$, we have that the left-hand-side of the inequality is in G, and hence, qua upset, also the right-hand-side $c \vee d$ is in G. But this element also is in I, so $G \cap I \neq \emptyset$, contradiction. $\square$

4.3.2. Proof of proposition 4.3

Let L be a Boolean algebra. Equip $X := \mathsf{PrFilt}(L)$ with the topology generated by $\mathcal{S} := \{\widehat{a} : a \in L\}$ with $\widehat{a} := \{F \in X : a \in F\}$. We need to show that X is zero-dimensional, Hausdorff, and compact, which we do in turn, and we finally show that $\mathcal{S} = \mathsf{Clp}(X)$.

Zero-dimensional. Note that $\mathcal{S}$ is a base for the topology because[3] it is closed under finite intersections: if $\widehat{a_1}, \ldots, \widehat{a_n} \in \mathcal{S}$, then, since prime filters are upsets closed under meet,

$$\widehat{a_1} \cap \ldots \cap \widehat{a_n} = \widehat{a_1 \wedge \ldots \wedge a_n} \in \mathcal{S}.$$

Moreover, each $\widehat{a}$ in $\mathcal{S}$ is clopen: By the ultrafilter characterization of prime filters in a Boolean algebra (proposition 4.2), we have

$$\widehat{a}^c = \big\{ F \in \mathsf{PrFilt}(L) : a \notin F \big\} = \big\{ F \in \mathsf{PrFilt}(L) : \neg a \in F \big\} = \widehat{\neg a},$$

so not just $\widehat{a}$ but also its complement is open, so $\widehat{a}$ is clopen. Hence $\mathcal{S}$ is a base of clopens for X, so X is zero-dimensional.

Hausdorff. If $F \neq G$ are two prime filters of L, there is, qua subsets of L, some $a \in L$ with $a \in F$ and $a \notin G$, or vice versa, but this other case will be analogous. Then $F \in \widehat{a}$ and $G \notin \widehat{a}$, so, as seen above, $G \in \widehat{\neg a}$. Hence $\widehat{a}$ and $\widehat{\neg a}$ are two disjoint open sets separating F and G.

Compactness. We show that the compactness of X really is just the compactness theorem from logic. So we first formulate this logical compactness theorem (step 1), then show that it implies the topological compactness of X (step 2), and finally prove the logical compactness theorem (step 3).

Step 1. A theory (in logic) is simply a set of sentences, so by an L-theory T we here mean a subset of L. Since models are prime filters, we say that an L-theory T has a model if there is a prime filter F of L such that $T \subseteq F$ (i.e., F makes true every sentence in T). Then the logical compactness theorem says:

($*$) If T is an L-theory such that every finite subset T' of T has a model, then T has a model.

Step 2. Let's show the topological compactness of X. We use the finite intersection formulation of compactness. So let $\mathcal{F}$ be a collection of closed subsets of X with the finite intersection property, and show that $\bigcap \mathcal{F} \neq \emptyset$. Since $\mathcal{S}$ is a base of clopens, we can assume that $\mathcal{F} \subseteq \mathcal{S}$.[4] So $\mathcal{F}$ is of the form $\{\widehat{a} : a \in T\}$ for some $T \subseteq L$. The finite intersection property implies

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[3] A subbase $\mathcal{S}$ of a topological space (X, τ) that is closed under finite intersection is a base: if U is an open subset, it is a union of finite intersection of elements from $\mathcal{S}$, so, by closure under finite intersection, it also is a union of elements from $\mathcal{S}$, so $\mathcal{S}$ is a base.

[4] Every closed set is an intersection of sets from $\mathcal{S}$. Write $\mathcal{F}'$ for the collection of sets from $\mathcal{S}$ that occur in these intersections. Then $\bigcap \mathcal{F}' = \bigcap \mathcal{F}$ and $\mathcal{F}'$ still has the finite intersection property: if $A_1, \ldots, A_n$ are sets from $\mathcal{F}'$, they occur in intersections for some sets $C_1, \ldots, C_n$ in $\mathcal{F}$, so $A_1 \cap \ldots \cap A_n \supseteq C_1 \cap \ldots \cap C_n \neq \emptyset$, as needed. Thus, if the claim holds for $\mathcal{F}' \subseteq \mathcal{S}$, we can conclude it for general $\mathcal{F}$.

that every finite subset T' of T has a model: Write $T' = \{a_1, \ldots, a_n\}$, so $\widehat{a_1} \cap \ldots \cap \widehat{a_n}$ is nonempty, so it contains a prime filter F, so each a_i is in F, so $T' \subseteq F$, as needed. Hence $(*)$ implies that also T has a model, so there is a prime filter F containing every $a \in T$. So, for every $a \in T$, $F \in \widehat{a}$, i.e., $F \in \bigcap \mathcal{F}$, as needed.

Step 3. Finally, we prove $(*)$. So let $T \subseteq L$ such that every finite subset has a model. To show that T has a model, we need to show that T extends to a prime filter. So we use the two theorems for this. So, first, let F be the filter generated by T, so, by proposition 4.12, $F = \{a \in L : \exists T' \subseteq T. \bigwedge T' \leqslant a\}$. Second, writing $I = \{\bot\}$, note that we have $F \cap I = \emptyset$, because otherwise $\bot \in F$, so there is a finite $T' \subseteq T$ with $\bigwedge T' \leqslant \bot$, but then T' cannot have a model (otherwise there is a prime filter containing T', but hence also contains $\bigwedge T' = \bot$, contradiction). So Stone's Prime Filter Extension Theorem applies and yields a prime filter G such that $T \subseteq F \subseteq G$, as needed.

Characterizing the clopens. To show $\mathcal{S} = \mathsf{Clp}(X)$, it remains to show the $\supseteq$ inclusion. If U is a clopen set, it is, since $\mathcal{S}$ is a base, a union of elements $\widehat{a_i}$ in $\mathcal{S}$. Since U also is a closed subset of a compact space, it is compact. So the open cover $(\widehat{a_i})$ has a finite subcover, so, for some $a_1, \ldots, a_n$, we have $U = \widehat{a_1} \cup \ldots \cup \widehat{a_n}$. Since prime filters are upset and prime, we further have

$$U = \widehat{a_1} \cup \ldots \cup \widehat{a_n} = \widehat{a_1 \vee \ldots \vee a_n} \in \mathcal{S},$$

as needed.

4.3.3. Proof of proposition 4.5

Let (X, τ) be a Stone space. Let $L := \mathsf{Clp}(X)$ be the set of clopens of X ordered by inclusion. We have to show that L is a Boolean algebra with the usual set-theoretic operations.

We first show that L is a sublattice of the powerset lattice 2^X, which implies that it is a distributive lattice. Indeed, the empty set $\emptyset$ and the whole set X are clopen. And if A and B are clopen, so is $A \cap B$ and $A \cup B$, because both open and closed sets are closed under finite intersection and finite union.

Finally, we show that the set-theoretic complement $A^c := X \setminus A$ is a (lattice-theoretic) complement. Indeed, if A is clopen, so is A^c, and $A \cap A^c = \emptyset$ and $A \cup A^c = X$.

4.3.4. Proof of proposition 4.7

Let L be a Boolean algebra. We want to show that

$$\widehat{\cdot} : \mathsf{L} \to \mathsf{Clp}(\mathsf{PrFilt}(\mathsf{L}))$$
$$a \mapsto \widehat{a} = \{\mathsf{F} \in \mathsf{PrFilt}(\mathsf{L}) : a \in \mathsf{F}\}.$$

is a well-defined lattice isomorphism. We also write $X := \mathsf{PrFilt}(\mathsf{L})$.

The function is well-defined since $\widehat{a}$ is, by construction, a clopen subsets of $\mathsf{PrFilt}(\mathsf{L})$. We first show that it is a lattice homomorphism:

It maps $\bot$ to $\widehat{\mathsf{bot}} = \emptyset$ since prime filters are proper, and it maps $\top$ to $\widehat{\top} = X$ since prime filters are nonempty. Moreover, $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$ since prime filters are closed under $\wedge$. And $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$ since prime filters are prime.

The function is injective: if $a \neq b$, we show $\widehat{a} \neq \widehat{b}$. By assumption, either $a \not\leq b$ or $b \not\leq a$. Without loss of generality, assume the former. Then $\mathsf{F} := \uparrow a$ and $\mathsf{I} := \downarrow b$ are a filter and ideal of L, respectively, with $\mathsf{F} \cap \mathsf{I} = \emptyset$. By Stone's Prime Filter Extension Theorem, there is a prime filter G in L such that $\mathsf{F} \subseteq \mathsf{G}$ and $\mathsf{G} \cap \mathsf{I} = \emptyset$. So $a \in \mathsf{F} \subseteq \mathsf{G}$ and $b \notin \mathsf{G}$ (otherwise $b \in \mathsf{G} \cap \mathsf{I}$). So $\mathsf{G} \in \widehat{a}$ but $\mathsf{G} \notin \widehat{b}$, as needed.

Finally, the function is surjective, because, by proposition 4.3, $\mathsf{Clp}(X) = \{\widehat{a} : a \in \mathsf{L}\}$.

4.3.5. Proof of proposition 4.8

Let (X, τ) be a Stone space. We want to show that

$$\beta : X \to \mathsf{PrFilt}(\mathsf{Clp}(X))$$
$$x \mapsto \{a \in \mathsf{Clp}(X) : x \in a\}$$
$$\text{the } x \text{ with } x \in a \text{ for all } a \in \mathsf{F} \hookleftarrow \mathsf{F}$$

is a well-defined order homeomorphism. We also write $\mathsf{L} := \mathsf{ClpD}(X)$.

We first observe that β is well-defined: if $x \in X$, then $\{a \in \mathsf{Clp}(X) : x \in a\}$ is indeed a prime filter on the Boolean algebra $\mathsf{L} = \mathsf{Clp}(X)$. Indeed, this is a nonempty (since $x \in X$) upset closed under intersection, doesn't contain $\emptyset$ and if $x \in a \cup b$, then either $x \in a$ or $x \in b$.

So, since we deal with compact Hausdorff spaces, it suffices to show that β is a continuous bijection (recall exercise 3.12).

Continuous. Given a basic open set $\widehat{a}$ of $\mathsf{PrFilt}(\mathsf{L})$, we need to show that

$\beta^{-1}(\widehat{a}) \subseteq X$ is open. Indeed,

$$\beta^{-1}(\widehat{a}) = \{x \in X : \beta(x) \in \widehat{a}\} = \{x \in X : a \in \beta(x)\}$$
$$= \{x \in X : x \in a\} = a,$$

and a is a clopen subset of X, hence open.

Injective. If $x \neq y$ in X, there are, by Hausdorffness, disjoint open sets U and V with $x \in U$ and $y \in V$. Since X has a clopen base, U is a union of clopens, so x is in one of them, say $a \in \mathsf{Clp}(X)$. Hence $x \in a$ and $y \notin a$. So $a \in \beta(x)$ and $a \notin \beta(y)$. Hence $\beta(x) \neq \beta(y)$.

Surjective. If $F \subseteq \mathsf{Clp}(X)$ is a prime filter, we find $x \in X$ such that $x \in \bigcap_{a \in F} a$. Because then we have $F = \beta(x)$: if $a \in F$, then, by construction of F, we have $x \in a$, so $a \in \beta(x)$; and if $a \notin F$, then, since F is an ultrafilter, $a^c = \neg a \in F$, so $x \in a^c$, so $x \notin a$, so $a \notin \beta(x)$.

Finding such an x screams for compactness: We have a collection F of closed subsets of X, and we need to show that its intersection is nonempty. Since X is compact, this reduces to showing that F has the finite intersection property. So let $a_1, \ldots, a_n \in F$. Since F is a filter, also $a_1 \cap \ldots \cap a_n \in F$, and since F is prime, this cannot be the empty set, as needed.

4.3.6. Proof of propositions 4.9 and 4.10

Concerning proposition 4.9, if $f : X \to Y$ is a continuous function between Stone spaces, we have already argued in section 4.2.5 that $\mathsf{Clp}(f) : \mathsf{Clp}(Y) \to \mathsf{Clp}(X)$ is well defined, and it is a good exercise to show that it is a Boolean algebra homomorphism (see exercise 4.c). If $h : L \to M$ is a Boolean algebra homomorphism, we also mentioned that exercise 4.e shows that $\mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$ is well-defined. So it remains to show that it is continuous. Indeed, if $\widehat{a}$ is a basic open set of $\mathsf{PrFilt}(L)$ with $a \in L$, we need to show that $\mathsf{PrFilt}(h)^{-1}(\widehat{a}) \subseteq \mathsf{PrFilt}(M)$ is open. Indeed,

$$\mathsf{PrFilt}(h)^{-1}(\widehat{a}) = \{F \in \mathsf{PrFilt}(M) : \mathsf{PrFilt}(h)(F) \in \widehat{a}\}$$
$$= \{F \in \mathsf{PrFilt}(M) : a \in h^{-1}(F)\}$$
$$= \{F \in \mathsf{PrFilt}(M) : h(a) \in F\}$$
$$= \widehat{h(a)},$$

which is (cl)open.

Concerning proposition 4.10, we need to show that diagrams 4.1 and 4.2 commute.

Let's start with the first one. Let $f : X \to Y$ be a continuous function between Stone spaces. We need to show $\beta_Y \circ f = \mathsf{PrFilt}(\mathsf{Clp}(f)) \circ \beta_X$. Let $x \in X$. On the left-hand-side, by definition, $\beta_Y(f(x)) = \{b \in \mathsf{Clp}(Y) : f(x) \in b\}$. On the right-hand-side, write $F := \beta_X(x) = \{a \in \mathsf{Clp}(X) : x \in a\}$ and $h := \mathsf{Clp}(f) : \mathsf{Clp}(Y) \to \mathsf{Clp}(X)$, so

$$
\begin{aligned}
\mathsf{PrFilt}(\mathsf{Clp}(f)) \circ \beta_X(x) = \mathsf{PrFilt}(h)(F) &= h^{-1}(F) \\
&= \{b \in \mathsf{Clp}(Y) : h(b) \in F\} \\
&= \{b \in \mathsf{Clp}(Y) : f^{-1}(b) \in F\} \\
&= \{b \in \mathsf{Clp}(Y) : x \in f^{-1}(b)\} \\
&= \{b \in \mathsf{Clp}(Y) : f(x) \in b\} \\
&= \beta_Y \circ f(x).
\end{aligned}
$$

Now for the second diagram. Let $h : L \to M$ be a Boolean algebra homomorphism. We need to show $\widehat{\cdot}_M \circ h = \mathsf{Clp}(\mathsf{PrFilt}(h)) \circ \widehat{\cdot}_L$. Let $a \in L$. On the left-hand-side, by definition, $\widehat{\cdot}_M(h(a)) = \{G \in \mathsf{PrFilt}(M) : h(a) \in G\}$. On the right-hand-side, write $f := \mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$, so

$$
\begin{aligned}
\mathsf{Clp}(\mathsf{PrFilt}(h)) \circ \widehat{\cdot}_L(a) = \mathsf{Clp}(f)(\widehat{a}) &= f^{-1}(\widehat{a}) \\
&= \{G \in \mathsf{PrFilt}(M) : f(G) \in \widehat{a}\} \\
&= \{G \in \mathsf{PrFilt}(M) : h^{-1}(G) \in \widehat{a}\} \\
&= \{G \in \mathsf{PrFilt}(M) : a \in h^{-1}(G)\} \\
&= \{G \in \mathsf{PrFilt}(M) : h(a) \in G\} \\
&= \widehat{\cdot}_M \circ h(a).
\end{aligned}
$$

This completes the proof of Stone duality.

4.4. Exercises

Exercise 4.a. Let L be a lattice and $F \subseteq L$. Show that the following are equivalent.

1. The set F is a prime filter.

2. The complement $I := L \setminus F$ is a prime ideal.

3. The characteristic function $\chi_F : L \to \mathbf{2}$ is a lattice homomorphism.

Exercise 4.b. Prove the characterization of prime filters in Boolean algebras stated in proposition 4.2.

Exercise 4.c. Let $f : X \to Y$ be a function between two sets X and Y.

1. The preimage is well-behaved: For $B, B' \subseteq Y$, show that

   - $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$

   - $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$

   - $f^{-1}(B^c) = f^{-1}(B)^c$

   - $f^{-1}(\emptyset) = \emptyset$

   - $f^{-1}(Y) = X$

   So $f^{-1} : 2^Y \to 2^X$ is a Boolean algebra homomorphism.

2. The direct image is not so well behaved: Now consider the direct image function $2^X \to 2^Y$ given by $f[A] := \{f(x) : x \in A\}$. Which of the above preservation properties does it still have? Give a proof for those properties it still has and give a counterexample for those it does not have anymore.

Exercise 4.d. Show that, when restricting Stone duality to finite Boolean algebras and their homomorphisms, they simply correspond to finite sets and functions between them. Conclude that if B is a finite Boolean algebra, it has 2^n elements, for some $n \in \mathbb{N}$. Note that, on the other hands, there are finite distributive lattices of any finite cardinality (since chains are distributive lattices).

Exercise 4.e. Let $h : L \to M$ be a lattice homomorphism and $F \subseteq M$ a prime filter. Then $h^{-1}(F) \subseteq L$ is a prime filter.

Exercise 4.f.

1. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions between Stone spaces. Show that $\mathsf{Clp}(g \circ f) = \mathsf{Clp}(g) \circ \mathsf{Clp}(f)$.

2. Let $f : L \to M$ and $g : M \to N$ be Boolean algebra homomorphisms. Show that $\mathsf{PrFilt}(g \circ f) = \mathsf{PrFilt}(g) \circ \mathsf{PrFilt}(f)$.

Exercise 4.g. Prove proposition 4.12.

This is exercise 3.1.13 of Gehrke and van Gool (2023).

5. Application: (Re)discovering Kripke semantics

There are a plethora of applications of Stone duality (and generalizations thereof that we will see in the next chapter). For example, take a look at chapters 4–8 of Gehrke and van Gool (2023). While many of these applications are in theoretical computer science (e.g., programming semantics or automata theory), we will here pick two applications in philosophy.

In section 5.1, we discuss an application to modal logic: We add a less understood logical concept—here, necessity—to our logical concepts, and we get a better understanding of it by studying its behavior on the dual side of spaces.

In section 5.2, we discuss an application to the philosophical quest of understanding the nature of propositions.

5.1. (Re)discovering the semantics of 'necessity'

Necessity is an important philosophical concept. In section 5.1.1, we see that it is, like the Boolean operators, a function on the set of all propositions. However, unlike the Boolean operators, it is not clear what its truth-conditions are. In section 5.1.2, we apply Stone duality, to move this question from propositions to possible worlds, and find an answer there. In section 5.1.3, we see that this is exactly the famous answer Kripke gave: the Kripke semantics for the necessity operator.

5.1.1. The 'necessity' operator

Assume we have a set L of propositions with the usual Boolean connectives $\wedge, \vee, \neg$; i.e., L is a Boolean algebra. Its elements are the meanings of declarative sentences. The connectives represent the special logical role played by conjunction, disjunction, and negation of sentences. For example, if sentence φ expresses proposition a and the sentence ψ expresses proposition b, then the sentence 'φ and ψ' expresses the proposition $a \wedge b$. Since the two sentences 'φ and ψ' and 'ψ and φ' have the same meaning, even if they are syntactically distinct, we have $a \wedge b = b \wedge a$, and similarly for the other laws of a Boolean algebra.

Analogously for $\vee$ and $\neg$.

As philosophers, we're not quite happy yet with this choice of logical connectives since it is not yet expressive enough. For philosophical discussions, it is, e.g., also important whether a sentence is necessarily or just accidentally true. For instance, the sentence 'My laptop is black' is true, but only accidentally—i.e., not necessarily—because it also could have been silver. So we would say 'It is not necessary that my laptop is black'. There might be other philosophical concepts that we also would like to express—like belief, knowledge, eternity, essence, etc.—but for now, let's focus on necessity.

So, if φ is a sentence expressing proposition a, we also consider the sentence 'Necessarily, φ'—customarily written as $\square\varphi$—which expresses the proposition that it is necessarily the case that a. Just like the Boolean logical connectives translate to functions on the set L of propositions, also our new necessity connective translates to a function

$$\square : L \to L, \quad a \mapsto \square a.$$

As with negation, we typically write $\square a$ instead of $\square(a)$.

This already tells us a little bit about 'necessity': namely, the type of thing it is (a function that maps propositions to propositions). But we don't know yet what it really means: we don't know its identity. This is the situation that philosophers also were in before Kripke and others (in the late 1950s).

5.1.2. The search for truth-conditions of 'necessity'

You might already know the Kripke semantics for modal logic, which is (among others) the logic of the necessity connective. If you do, then forget it again for now. If you don't, even better—because we want to rediscover it using the tools of duality theory. Before the advent of Kripke semantics, philosophers—like us now—struggled to understand the necessity connective.

For the usual Boolean connectives, they could say what their meaning is: for example, they could point to their truth-tables. In more fancy terminology, they had compositional truth-conditions for the Boolean connectives: the sentence $\varphi \vee \psi$ is true (in some situation) iff either φ is true (in that situation) or ψ is true (in that situation). However, for 'necessity', they could only provide some plausible reasoning principles, like the following.

1. It is necessary that φ and ψ ($\square(\varphi \wedge \psi)$) if and only if it is necessary that φ and it is necessary that ψ ($\square\varphi \wedge \square\psi$).

2. If φ is a logical truth, then 'it is necessary that φ' also is a logical truth.

So the situation was as if philosophers could point to reasoning rules for disjunction like 'If φ, then φ ∨ ψ', but not to the truth-table or truth-conditions of ∨.

Now, how can duality theory help to get such an understanding of the meaning of 'necessity'? The trick is—as always when it comes to applications of duality theory—to move to the dual side and hope for clearer intuitions there. So let's do this.

All we know so far is that □ is some (so far unknown) function from L to L which—by stating the principles 1 and 2 above more formally—satisfies:

1. For all $a, b \in L$: $\Box(a \land b) = \Box a \land \Box b$.

2. $\Box\top = \top$.

How can we translate this function $\Box : L \to L$ to the dual side involving the Stone space $\mathsf{PrFilt}(L)$ of prime filters on L? (We described this in section 4.2.) We already know that we can think of these prime filters as (Ersatz) possible worlds: they are maximally consistent sets of sentences. And we know that a function $h : L \to M$ on Boolean algebras that in fact is a lattice homomorphism translates to a continuous function $\mathsf{PrFilt}(h) : \mathsf{PrFilt}(M) \to \mathsf{PrFilt}(L)$ by mapping the ultrafilter G on M to the ultrafilter $h^{-1}(G) = \{a \in L : h(a) \in G\}$.

Unfortunately, our □ is something more general than a lattice homomorphism: properties 1 and 2 only state preservation of ⊤ and ∧, but not of ⊥ and ∨. As a result, we cannot guarantee anymore that, if $F \in \mathsf{PrFilt}(L)$, also $\Box^{-1}(F)$ is again a prime filter. In fact, preservation of ⊤ and ∧ only yields that this is a filter, but not necessarily a prime filter. So, if $\Box^{-1}(F)$ is not a prime filter, the next best thing is to consider the *set* of all prime filters that extend this filter:

This is a good exercise: In fact, if you have done exercise 4.e, you did even more.

$$\varphi(F) := \{G \in \mathsf{PrFilt}(L) : G \supseteq \Box^{-1}(F)\}.$$

Hence:

- For a Boolean algebra homomorphism $h : L \to M$, the dual morphism is the function $\mathsf{PrFilt}(h)$ that maps each prime filter $F \in \mathsf{PrFilt}(L)$ to a single prime filter, namely $h^{-1}(F)$.

- The more general case of a function $\Box : L \to L$ that only preserves ⊤ and ∧ yields the more general dual morphism that is a *multifunction*

$\varphi : \mathsf{PrFilt}(\mathsf{L}) \rightrightarrows \mathsf{PrFilt}(\mathsf{L})$ that maps each F to a set of prime filters $\varphi(\mathsf{F})$, namely the set of prime filter extending $\square^{-1}(\mathsf{F})$.

The graph of a multifunction is a relation: so we write $\mathsf{F}\ \mathsf{R}\ \mathsf{G}$ for $\mathsf{G} \in \varphi(\mathsf{F})$. So

$$\mathsf{F}\ \mathsf{R}\ \mathsf{G} \Leftrightarrow \mathsf{G} \supseteq \square^{-1}(\mathsf{F}) \Leftrightarrow \forall a \in \mathsf{L} : \square a \in \mathsf{F} \Rightarrow a \in \mathsf{G}. \tag{5.1}$$

So our function $\square : \mathsf{L} \to \mathsf{L}$ on the algebraic side becomes, when translated to the spatial side, the multifunction φ or, equivalently, its graph R. But this doesn't yet give us truth-conditions for the necessity operator. For that we need to say what it means that $\square a$ is true at the possible world F, i.e., $\square a \in \mathsf{F}$? So, for $\mathsf{F} \in \mathsf{PrFilt}(\mathsf{L})$, we need to fill in the question marks in

$$\square a \in \mathsf{F} \Leftrightarrow\ \text{???}.$$

Actually, can get a necessary condition from the definition of R: If $\square a \in \mathsf{F}$, then, if $\mathsf{F}\ \mathsf{R}\ \mathsf{G}$, we have $\forall a \in \mathsf{L} : \square a \in \mathsf{F} \Rightarrow a \in \mathsf{G}$, so, in particular, $a \in \mathsf{G}$. So a first try is to see if this condition actually also is necessary. And we're in luck:

Theorem 5.1. *In the preceding notation, we have for all* $a \in \mathsf{L}$ *and prime filters* $\mathsf{F} \in \mathsf{PrFilt}(\mathsf{L})$

$$\square a \in \mathsf{F} \Leftrightarrow \forall \mathsf{G} \in \mathsf{PrFilt}(\mathsf{L}) : \mathsf{F}\ \mathsf{R}\ \mathsf{G} \Rightarrow a \in \mathsf{G}.$$

Before proving this, here is the punchline: if you have seen Kripke semantics, this is *exactly* the truth-condition for the necessity connective:

- $\square a$ is true at a possible world F iff for all R-accessible worlds G, we have that a is true at G.

So we have a way of relating the truth of the complex proposition $\square a$ to the truth of its constituent a: hence this truth-condition is compositional. Recall, by the properties of prime filters, we also have compositional truth-conditions for the Boolean connectives.

- $a \wedge b$ is true at a possible world F (i.e., $a \wedge b \in \mathsf{F}$) iff both a is true at F (i.e., $a \in \mathsf{F}$) and b is true at F (i.e., $b \in \mathsf{F}$).

- $a \vee b$ is true at a possible world F (i.e., $a \vee b \in \mathsf{F}$) iff either a is true at F (i.e., $a \in \mathsf{F}$) or b is true at F (i.e., $b \in \mathsf{F}$).

- $\neg a$ is true at a possible world F (i.e., $\neg a \in \mathsf{F}$) iff a is not true at F (i.e., $a \notin \mathsf{F}$).

So we have truth-conditions for all the connectives!

But now, let's prove the theorem, i.e., show that we have indeed found the truth-conditions for $\Box a$.

Proof. ($\Rightarrow$) This was the necessary condition which we already saw to hold.

($\Leftarrow$) Contrapositively, assume $\Box a \notin F$. We want to find an ultrafilter $G \in \mathsf{PrFilt}(L)$ with $F\,R\,G$ but $a \notin G$. So of course we turn to the Prime Filter Extension Theorem (theorem 4.13). We already noted that $\Box^{-1}(F)$ is a filter on L and, by assumption, $a \notin \Box^{-1}(F)$, hence, for the ideal $I := \downarrow a$, we have $\Box^{-1}(F) \cap I = \emptyset$. Hence, by the Extension Theorem, there is a prime filter G on L such that $G \supseteq \Box^{-1}(F)$ and $G \cap I = \emptyset$. The former means, by definition, that $F\,R\,G$, and the latter implies $a \notin G$, as needed. $\qquad\square$

5.1.3. Kripke semantics

Now, with that hindsight, the usual Kripke semantics for modal logic almost seems obvious: We use the language $\mathcal{L}$ whose sentences are built from the atomic sentences in the set $\mathsf{At} = \{p_0, p_1, \ldots\}$ using $\wedge, \vee, \neg, \Box$. A *Kripke model* M is a triple (W, R, V) where

- W is a set of worlds. (So far, this was the set of prime filters $\mathsf{PrFilt}(L)$.)

- $R \subseteq W \times W$ is a binary relation. (So far, this was the one defined in equation 5.1.)

- V is a function that assigns each possible world $x \in W$ to a function V_x that assigns each atomic sentence a truth-value in $\{0, 1\}$, i.e., $V_x : \mathsf{At} \to \{0, 1\}$. (So far, V was given by the map $F \mapsto \chi_F$ which maps a prime filter to its characteristic function, which in turn assigns truth values not only to 'atomic' propositions but all propositions.)

The valuation of atomic sentences is extended to all sentences: We recursively define when a sentence φ is true at a world x, written $x \vDash \varphi$, by

- $x \vDash p$ iff $V_x(p) = 1$

- $x \vDash \varphi \wedge \psi$ iff $x \vDash \varphi$ and $x \vDash \psi$

- $x \vDash \varphi \vee \psi$ iff $x \vDash \varphi$ or $x \vDash \psi$

- $x \vDash \neg\varphi$ iff $x \nvDash \varphi$

- $x \vDash \Box\varphi$ iff for all $y \in W$, if xRy, then $y \vDash \varphi$.

What the Kripke semantics omits is the topological structure on the set of possible worlds that we still have on our topological approach. This additional topological information amounts to the following: In the usual Stone duality, Boolean algebra homomorphisms correspond to continuous functions on the dual spaces. Now, we've generalized Boolean algebra homomorphisms to functions that preserve $\wedge$ and $\top$, and on the dual side they correspond to the relations defined in 5.1. The additional topological properties of these relations—i.e., the appropriate generalization of continuity—is that they are *Boolean compatible*: If X and Y are Stone spaces, a relation $R \subseteq X \times Y$ is Boolean compatible if (1) for all $x \in X$, the set $\{y : xRy\} \subseteq Y$ is closed and (2) for all clopen $U \subseteq Y$, the set $\{x : \exists y \in U.xRy\} \subseteq X$ is clopen. Then the functions $f : B \to A$ from a Boolean algebra B to a Boolean algebra A which preserve $\wedge$ and $\top$ are in one-to-one correspondence with the Boolean compatible relations $R \subseteq X \times Y$ on the dual spaces X of A and Y of B(Gehrke and van Gool 2023, cor. 4.43). If we have a Stone space X with Boolean compatible relation R, we can find the dual Boolean algebra $B = \mathsf{Clp}(X)$ with $\Box U := \{x \in X : \forall x' \in X.xRx' \Rightarrow x' \in U\}$ (Gehrke and van Gool 2023, cor. 4.51).

Note the swap of direction, as usual for dualities

5.2. Characterizing propositions

Surrounding story to be written.

Theorem 5.2. *Every Boolean algebra L is, up to isomorphism, a sub-Boolean algebra of powers of the two-element Boolean algebra $\mathbf{2}$, namely, of $\mathbf{2}^{\mathsf{PrFilt}(L)}$.*

Proof. We define the required Boolean algebra embedding $e : L \to \mathbf{2}^{\mathsf{PrFilt}(L)}$ by mapping $a \in L$ to the function

$$e(a) : \mathsf{PrFilt}(L) \to \mathbf{2}$$

$$F \mapsto \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{otherwise} \end{cases}$$

Exercise 5.b asks you to show that this indeed is a Boolean algebra embedding (i.e., an injective Booleam algebra homomorphism). $\qquad\square$

5.3. Exercises

Exercise 5.a. Tell the story that we've told in section 5.1 but now not for the 'propositions vs. possible worlds' duality, but for the 'properties vs. objects' duality. Here the new operator is Δ read as 'definitely' instead of $\Box$ read

as 'necessarily'. This operator is used a lot in theories of vagueness (e.g. Williamson 1999). If p is the property of being red, then Δp is the property of definitely being red. Which semantic of 'definitely' do you get on the dual side? How does it compare to existing semantics? What does the topological perspective add? Is it philosophically plausible?

Exercise 5.b. Show that the function $e : L \to \mathbf{2}^{\mathsf{PrFilt}(L)}$ defined in the proof of theorem 5.2 is indeed a Boolean algebra embedding.

6. Generalization: Priestley duality

A. Appendix

A.1. Set-theoretic terminology

We use standard set-theoretic terminology as it is common in mathematics. A set is a collection of objects. We write $a \in A$ to say that object a is in (or is an element of, or is a member of) the set A. If $a_1, \ldots, a_n$ are objects, we write $\{a_1, \ldots, a_n\}$ for the set of these objects. Sets do not count 'order' and 'multiplicities', so $\{1, 0, 2, 2\} = \{0, 1, 2\}$. A set with just one element is called a singleton, and if a is an object, $\{a\}$ is the singleton of a (note $\{a\} \neq a$). The set without any elements is called the empty set and is denoted $\emptyset$.

If A and B are sets, we say A is a subset of B (written $A \subseteq B$) if every element of A is an element of B. So the empty set trivially is a subset of any set. And two sets A and B are identical iff $A \subseteq B$ and $B \subseteq A$. If A and B are sets, then the union of A and B (written $A \cup B$) is the set containing exactly those objects that either are in A or in B (or both). The intersection of A and B (written $A \cap B$) is the set containing exactly those objects that are both in A and in B. The complement of a set A relative to a set B (written $B \setminus A$) is the set of objects that are in B but not in A. If B is clear from context, we just write A^c.

A pair (aka ordered pair) is a list of two elements (a, b); here the order matters, so $(a, b) \neq (b, a)$. (We can define (a, b) as the set $\{\{a\}, \{a, b\}\}$.) More generally, an n-tuple is a list of n elements $(a_1, \ldots, a_n)$. Given n sets $A_1, \ldots, A_n$, their Cartesian product (written $A_1 \times \ldots \times A_n$) is the set of all n-tuples $(a_1, \ldots, a_n)$ such that, for all $i \in \{1, \ldots, n\}$, we have $a_i \in A_i$. More generally, the Cartesian product of a potentially infinite family $\{A_i : i \in I\}$ of sets is defined as the set $\prod_{i \in A} A_i$ of functions a that map each $i \in I$ to an element $a(i) \in A_i$. We often write such a function as $a = (a_i)_{i \in I}$; because in the above case where $I = \{1, \ldots, n\}$, we can think of a tuple $(a_1, \ldots, a_n)$ as a the function a mapping $i \in I$ to a_i. An n-ary relation between $A_1, \ldots, A_n$ is a subset of $A_1 \times \ldots \times A_n$. A 1-ary (resp., 2-ary, 3-ary) relation is also called a unary (resp., binary, ternary) relation. For a binary relation R, we usually write aRb instead of $(a, b) \in R$.

A function from a set A (its domain) to a set B (its codomain) is a binary relation f between A and B such that, for every $a \in A$, there is exactly one $b \in B$ such that afb. We then write $f : A \rightarrow B$ and $f(a) = b$ or, if f is clear

from context, $a \mapsto b$. If $f : A \to B$ and $g : B \to C$ are functions, $g \circ f$ (g after f) denotes their composition, which maps $a \in A$ to $g(f(a)) \in C$. Given a set A, the identity function $\mathrm{id}_A : A \to A$ maps a to a. By an n-ary function on a set A we mean a function $f : A^n \to A$, where $A^n = A \times \ldots \times A$ is the n-time Cartesian product of set A. Again, the first arities have special names: unary ($= 1$-ary), binary ($= 2$-ary), and ternary ($= 3$-ary). Sometimes it is convenient to take a 0-ary function to be a constant (i.e., an element or symbol which is fixed throughout).

An equivalence relation $\equiv$ on a set A is a binary relation on A such that

1. $\equiv$ is reflexive, i.e., for all $a \in A$, we have $a \equiv a$,

2. $\equiv$ is transitive, i.e., for all $a, b, c \in A$, if $a \equiv b$ and $b \equiv c$, then $a \equiv c$, and

3. $\equiv$ is symmetric, i.e., if $a \equiv b$, then $b \equiv a$.

If $a \in A$, then the $\equiv$-equivalence class of a is the set $[a]_\equiv := \{b \in A : a \equiv b\}$. An element $b \in [a]_\equiv$ is called a representative of $[a]_\equiv$. The quotient of A under $\equiv$ is defined as $A/\equiv := \{[a]_\equiv : a \in A\}$. The function $\pi : A \to A/\equiv$ defined by $\pi(a) := [a]_\equiv$ is called the projection of $\equiv$.

A.2. Category-theoretic terminology

To be written. For now, see, e.g., Leinster 2014, ch. 1.

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