Universal Analog Computation:
Fraïssé Limits of Dynamical Systems

LEVIN HORNISCHER

ABSTRACT. Analog computation is an alternative to digital computation, that has recently re-gained prominence, since it includes neural networks and neuromorphic computing. Further important examples are cellular automata and differential analyzers. While analog computers offer many advantages, they lack a notion of universality akin to universal digital computers. Since analog computers are best formalized as dynamical systems, we review scattered results on universal dynamical systems. We identify four senses of universality and connect to the two main general theories of computation: coalgebra and domain theory. For nondeterministic systems, we construct a universal system as a Fraïssé limit. It not only is universal in many of the identified senses, it also is unique in additionally being homogeneous. For deterministic systems, a universal system cannot exist, but we provide a simple method for constructing subclasses of deterministic systems with a universal and homogeneous system. This way, we introduce sofic proshifts: those systems that are limits of sofic shifts. In fact, their universal and homogeneous system even is a limit of shifts of finite type and has the shadowing property.

1. INTRODUCTION

Analog computation is an alternative to digital computation, with a long and rich history [18, 116]. Examples include *neural networks*, the drivers of the present surge of artificial intelligence. They do not manipulate discrete symbols according to a bespoke program as in digital computation. Rather, they operate on real numbers according to their internal parameters, called weights, which typically are learned from data in a continuous updating process.[1] More generally, *neuromorphic computing* develops computing systems which are inspired by the energy-efficient processing of biological brains. Other famous examples of analog computation are *cellular automata*, which distributively compute global behavior of cells via local update rules; or the *differential analyzer*, which computes by solving differential equations.

[1]As an abstract model of computation, neural networks are analog. But when they are (physically) implemented, they usually are still simulated on digital hardware. However, developing analog or 'neuromorphic' hardware for neural networks is an active field of research [77].

A major advantage that secured digital computation its dominant role is that it allows for universal computation: famously, the universal Turing machine can simulate all other digital machines. In analog computation, on the other hand, one typically has to construct one machine for each task. After all, the fairly uniform architecture of laptops stands in contrast to the many neural network architectures (as mandated, in a sense, by the No-Free-Lunch theorems). Notwithstanding, in this paper, we investigate when a universal analog computer can exists—and when not.

The plan is as follows. In section 2, we first recall that analog computers are best conceptualized as dynamical systems. Then we review the scattered literature on universal analog computation and universal dynamical systems, identifying four senses of universality. We also locate our results within existing work.

In section 3, we motivate and define the notion of time-discrete, space-continuous, possibly nondeterministic dynamical system that we use. Specifically, a *dynamical system* here is a pair (X, T) where X is a zero-dimensional and compact metrizable space and $T : X \rightrightarrows X$ is a closed-valued and upper-hemicontinuos multifunction. We also call (X, T) a *nondeterministic system*, and if T is a function, we speak of a *deterministic system*. We introduce embedding-factor pairs as the morphisms between dynamical systems. Equivalently, a morphism is a *factor* $f : (X, T) \to (Y, S)$, i.e., a surjective continuous function $f : X \to Y$ such that y' is an S-image of y iff there are x in X and a T-image x' of x such that $f(x) = y$ and $f(x') = y'$ (cf. bisimulations). For deterministic systems, this amounts to the usual notion of a factor, and this more general definition is also used for profinite graphs [63]. With this framework in place, we formally state our main results in section 4.

In section 5, we relate our dynamical systems to coalgebras and domain theory, thus contextualizing them within the two main general theories of computation. In particular, we show that our category of dynamical systems is equivalent to a category with a 'domain-theoretic spirit': the objects of this category are pairs (A, α) where A is an algebraic lattice (with an additional property) and $\alpha : A \to A$ is a Scott-continuous function (with an additional property), and the morphisms $(\epsilon, \pi) : (A, \alpha) \to (B, \beta)$ are embedding-projections pairs between A and B (with an additional property) such that $\pi \circ \beta \circ \epsilon \leqslant \alpha$ and $\alpha \circ \pi \leqslant \pi \circ \beta$.[2]

Sections 6–7 establish our universality result for nondeterministic systems. We use the category-theoretic version of Fraïssé limits due to Droste and Göbel [38], which works in so-called algebroidal categories. In section 6, prove a theorem with sufficient conditions for algebroidality and conclude that the category of all dynamical systems is algebroidal. In section 7, we then use the Fraïssé limit to prove the following universality result (generalizing one for undirected profinite graphs [20, 46, 63]):

Theorem (Theorem 4.2 below)**.** *There is a nondeterministic system* (U, T) *that is*

   (1) Universal: *For any system* (Y, S), *there is a factor* $f : (U, T) \to (Y, S)$ *(provided that S is nontrivial, i.e., there is $y \in Y$ with $S(y) \neq \emptyset$).*

---

[2]Note that these conditions are weaker than what is required for a coalgebra morphism between (A, α) and (B, β), where one would require $\alpha \circ \pi = \pi \circ \beta$, which implies $\pi \circ \beta \circ \epsilon = \alpha \circ \pi \circ \epsilon = \alpha$. In particular, to build universal systems, we cannot apply existence results of a final/terminal coalgebra [2, 97].

(2) Homogeneous: *If (Y, S) is a system over a finite set Y and $f, f' : (U, T) \to (Y, S)$ are two factors, then there is an isomorphism $\varphi : (U, T) \to (U, T)$ such that $f = f' \circ \varphi$.*

Moreover, (U, T) is unique up to isomorphism with these two properties.

In the remainder, we then turn to deterministic systems. Section 8 analyzes why universality fails for deterministic systems. The reason comes from the so-called category-theoretically finite objects. In an algebroidal category, there should only be countably many of those (up to isomorphism). Indeed, for nondeterministic systems, the category-theoretically finite objects simply are the systems with a finite state space. But for deterministic systems, we show that the category-theoretically finite objects are the shift spaces, and there are uncountably many of those systems.

However, the category of deterministic systems satisfies all other conditions for being algebroidal (i.e., it is semi-algebroidal). Thus, in section 9, we construct algebroidal subcategories. Any countable collection S of shift spaces induces, by taking all limits along countable inverse sequences, an algebroidal category $\mathsf{DS}^{\mathsf{ef}}(S)$ of deterministic systems. A natural choice for S is either the class SFT of shifts of finite type or the class SOF of sofic shifts. They are among the most studied classes of deterministic systems, and sofic shifts have close ties to automata theory. We call the systems in $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SFT})$ (resp., $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SOF})$) the ω-*proshifts of finite type* (resp., the *sofic ω-proshifts*). In section 10, we then show our second main universality result:

Theorem (Theorem 4.4 below). *There is a unique universal and homogeneous ω-proshift of finite type, and it also is the unique universal and homogeneous sofic ω-proshift.*

For brevity, we call this system the *universal proshift*. While it is not transitive, it does have the desirable shadowing property (which roughly says that an approximate orbit simulated step by step with rounding errors is close to a true orbit). We conclude in section 11 with questions for future research. An appendix contains outsourced proofs.

2. Background: Four senses of universal analog computation

With remarkable conceptual clarity, Turing [115] formalized digital computation by what is since known as Turing machines. Turing also constructed universal machines.[3] However, for analog computation, things are more complicated. There are many nonequivalent models of analog computation [18]. So what would even be a general enough definition of an analog computer? The answer is: they can all be regarded as dynamical systems [16, 18, 48, 105, 110]. (The converse—i.e., which dynamical systems are analog computers—is a deep philosophical issue beyond scope of this paper, but see, e.g., [43, 92, 93, 103, 108].) At any time, the analog computer is in some state, and when it computes, it updates its state: this is the dynamics. This includes—and hence generalizes—digital computation: the state of a digital computer is, roughly, its memory and CPU state, and the dynamics is given by the program that it runs.

---

[3]Via Gödel encoding, a specific Turing machine can be constructed which simulates, in a clear sense, any other Turing machine. Though, defining what it means, for an arbitrary Turing machine, to be universal is more subtle [30].

So different models of analog computation correspond to different classes of dynamical systems. But what would it mean to say that one analog computer—i.e., one dynamical system U—is universal (in its class)? We identify four answers in the scattered literature:

(1) *Turing universal*: U can simulate every Turing machine.
(2) *Approximation universal*: U can approximate any other dynamical system (in the class) up to arbitrary precision.
(3) *Embedding universal*: any other dynamical system (in the class) can be embedded into U.
(4) *Factor universal*: any other dynamical system (in the class) is a *factor* of U, i.e., an image under a dynamics-preserving function with domain U.

Concerning 1, von Neumann showed that there are Turing universal cellular automata (see [89] for discussion) and this also has been done for other models [90]. Moreover, recurrent neural networks with rational weights can simulate arbitrary Turing machines [104]. For continuous-time Hopfield networks, see [106], and for an overview, see [107]. Furthermore, Tao [113] provides a 'potential well' dynamical system such that "the halting of any Turing machine with a given input is equivalent to a certain bounded trajectory in this system entering a certain open set" (p. 1). Yet further dynamical systems that are Turing universal are provided by [23, 52, 74]; for an overview of 'fluid computers', see [22].

Concerning 2, the differential analyzer—a revolutionary analog computer first built in 1931—was mathematically analyzed to precisely compute the solutions of algebraic differential equations [94, 101]. Later, [96] provides "an analogue, for analog computers, of the Universal Turing Machine" (p. 345), in the form of a concrete algebraic differential equation that is approximation universal in the sense that, for every continuous function f on the real numbers and precision ϵ, there is a smooth solution y of that differential equation such that y is ϵ-close to f. Excitingly, [19] recently improved this to polynomial ordinary differential equations (which always have a unique and analytic solution). For neural networks, the universal approximation theorems [26, 59] similarly show that the functions realized by neural networks are dense in the set of all continuous functions (see [17] for a recent result connecting to computability theory).

Concerning 3, a general definition of a dynamical system is as a (Borel) action $\alpha : G \times X \to X$ of a (Polish) group G on a (Borel) space X. Here X is the state space of the system, G is time, and α the dynamics, so $\alpha(g, x) = g \cdot x$ is the state the system is in after time g when starting in state x. In this setting, there is an embedding universal system, i.e., a G-action β on a space U such that any other G-action α on a space X (Borel) embeds into β [69].

Concerning 4, viewing dynamical systems as G-*flows*, i.e., as continuous actions of a topological group G on a compact topological space X, it is well-known that, among the minimal G-flows (those where every orbit $\{g \cdot x : g \in G\}$ is dense in X) there is a factor universal one, which is even unique up to isomorphism (mentioned, e.g., in [70, p. 1]). Viewing dynamical systems as linear operators $T : X \to X$ on a Banach space X, there is a bounded such operator U such that every other such operator with norm 1 is a linear factor of it [29], which recently was generalized to many other categories by [7] (also see [76]). Coming back to cellular automata,

[55] asks if there are factor-universal cellular automata (noting the existence of universal effective $\mathbb{Z}$-dynamical systems [54]).

Concerning further related work, for a structurally similar discussion of universality of spin models, see [32], and for a category-theoretic framework for universality that includes spin models and Turing machines, see [50]. Moreover, our main tool, namely the Fraïssé-limit, has, beyond extensive use in model theory [58, ch. 7], also applications in the following fields: domain theory [38, 39], set theory [75], topos theory [21], graph theory [20, 46, 63], and dynamical systems theory [13, 79]—and see [84] for a more detailed overview. A stunning connection between these fields is provided by the KPT correspondence [70]: If A is the Fraïssé-limit of a Fraïssé-order class $\mathcal{K}$, then $\mathcal{K}$ has a combinatorial property known as the Ramsey property iff the automorphism group of A is extremely amenable, i.e., every continuous action of this group on a compact Hausdorff space has a fixed point. This can also be used to construct universal flows for automorphism groups of Fraïssé limits. For a continuation of this line of research, see, e.g., [3, 10, 11, 71, 82, 86, 87].

Moreover, related to the question of universality is the question of genericity: Given a space X, we can ask if there is a *generic* system on X, i.e., a dynamics $T : X \to X$ such that the conjugacy class of (X, T), i.e., the set of all systems (X, S) that are isomorphic to (X, T), forms a comeager subset of the space of all dynamics on X, appropriately topologized. A result of [73] states that this is true when X is the Cantor space and 'dynamics' means homeomorphism on X ([49] previously established density). This is further researched by [33, 34, 56, 57, 79].

Building on this rich body of work, our application of Fraïssé-limits is, to the best of our knowledge, new in constructing universal dynamical systems—motivated by the search for universal analog computation—in a way that unifies the case of nondeterministic systems and deterministic systems. Methodologically, we connect to the tools of theoretical computer science by using a category-theoretic version of the Fraïssé-limits and exploring links to domain theory and coalgebra.

The closest existing results in the nondeterministic case is the work cited above on the projectively universal homogeneous (undirected) graph [20, 46, 63] (as an instance of the category-theoretic Fraïssé limit [38, 75]). Indeed, our nondeterministic systems can be seen as directed graphs. Although "some ideas [for undirected graphs] can be easily adapted to the more general case of directed graphs" [46, p. 3], we still include proofs, since the directed setting is conceptually central to dynamical systems and we reuse some results for the deterministic systems.

The related work in the deterministic case is the literature cited above on KPT correspondence and genericity. But the focus is less on a universal system for actions of $\mathbb{N}$, but more on actions of automorphism groups of Fraïssé limits and understanding their universal minimal flow and their genericity. Moreover, we add category-theoretic tools to this investigation. For example, we corroborate the important role of shifts in dynamical systems theory by showing that they are the category-theoretically finite objects, and we define the category of, e.g., sofic ω-proshifts as an instance of an important category-theoretic construction (essentially that of a pro-category). Thus, we add to a growing body of literature of using category theory for dynamical systems [12, 80, 88, 99].

3. CATEGORIES OF DYNAMICAL SYSTEMS

We define the categories of dynamical systems that we will work with: subsection 3.1 defines the objects and subsection 3.2 the morphisms.

3.1. **Objects: Dynamical systems.** The dynamical systems that we consider (to represent analog computers) are time-discrete, state-continuous, and possibly nondeterministic dynamical systems. We first state the formal definition, and then we motivate this choice and provide examples.

Nondeterminism means that a state might not have a unique next state, but that there is a *set* of potential next states. Hence the dynamics is a multifunction (i.e., set-valued function). So we first recall the necessary definitions from set-valued analysis [5, ch. 17].

A *multifunction* $F : X \rightrightarrows Y$ is a function that maps each $x \in X$ to a subset $F(x) \subseteq Y$. We define

$$x \xrightarrow{F} y :\Leftrightarrow y \in F(x).$$

We call F *total* if each $F(x)$ is nonempty. If $A \subseteq X$, the *image* of A under F is $F[A] := \bigcup_{x \in A} F(x)$. Further, we call F *partition-injective* if $\{F(x) : x \in X\}$ is a partition of Y (i.e., for all $x \neq x'$ in X, we have $F(x) \cap F(x') = \emptyset$ and for all $y \in Y$, there is $x \in X$ with $y \in F(x)$). We identify a function $f : X \to Y$ with the multifunction $X \rightrightarrows Y$ that maps x to $\{f(x)\}$, and if a multifunction $F : X \rightrightarrows Y$ has only singletons as values, we identify it with the function $X \to Y$ mapping x to the single element of $F(x)$.

If X and Y are topological spaces, F is *closed-valued* if each $F(x)$ is a closed subset of Y. The appropriate notion of continuity for us is that F is *upper-hemicontinuous* if, for all $x \in X$ and open $V \subseteq Y$ with $F(x) \subseteq V$, there is an open $U \subseteq X$ with $x \in U$ and, for all $x' \in U$, we have $F(x') \subseteq V$. If F is a function, it is continuous in the usual sense iff it is upper-hemicontinuous. The *closed graph theorem* [5, thm. 17.11] says: If $F : X \rightrightarrows Y$ is a multifunction between two compact Hausdorff spaces, then F is upper-hemicontinuous and closed-valued iff the *graph* of F, i.e., the set $\mathrm{Gr}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$, is closed. A corollary is that then the image $F[A]$ of a closed set $A \subseteq X$ is closed.

Now we can define the notion of dynamical system that we will use.

Definition 3.1. By a *dynamical system* we mean a pair (X, T) where

- X is a zero-dimensional, second-countable, compact Hausdorff space.[4]
- $T : X \rightrightarrows X$ is a closed-valued and upper-hemicontinuous multifunction.[5]

We call a dynamical system (X, T):

- *nonempty* if X is nonempty.
- *nontrivial* if the graph of T is nonempty (i.e., there are $x, y \in X$ with $x \xrightarrow{T} y$),
- *total* if T is total (i.e., for every $x \in X$, there is $y \in X$ with $x \xrightarrow{T} y$),

---

[4]Recall that a topological space X is zero-dimensional if it has a basis of clopen (i.e., closed and open) sets. And it is second-countable if it has a countable basis. Note that X is a zero-dimensional, second-countable, compact Hausdorff space iff X is a compact and zero-dimensional Polish space iff X is a compact and zero-dimensional metrizable space iff X is a second-countable Stone space.

[5]Equivalently, by the closed graph theorem, the graph of F is closed.

- *deterministic* if T is a function (i.e., for every $x \in X$, there is exactly one $y \in X$ with $x \xrightarrow{T} y$).

Often, we just say 'system' instead of 'dynamical system'. We sometimes speak of a 'nondeterministic system' to stress that the system under consideration need not be deterministic.

(Note that a nontrivial dynamical system is nonempty; the converse holds for total systems, and hence also for deterministic systems, but it may fail for non-total systems since states may have no successor.)

To motivate this choice, topological dynamics is a well-established field studying dynamical systems as structures (X, T) where X is a Hausdorff space (typically compact and often also metrizable) and $T : X \to X$ is a continuous function [117]. (This was pioneered by Poincaré studying differential equations also in the absence of analytic solutions.) Studying zero-dimensional compact topological systems is an important subfield [36]. This is not just because it is the setting of symbolic dynamics [81] but also because classical results like the Jewett–Krieger theorem show that the systems in ergodic theory (i.e., dynamics on probability spaces that preserve the measure) have zero-dimensional topological systems as counterparts.

Moreover, as remark A.1 in appendix A.1 makes precise, we can also canonically turn the more general setting of a dynamics on a Polish space—like $\mathbb{R}^n$—into our setting. Specifically, if X is a Polish space and $T : X \to X$ is a continuous function, then there is a—in a sense best possible—way to extend this into a deterministic system (Y, S) in our sense. In particular, there is an injective function $\eta : X \to Y$ that is continuous with respect to a finer Polish topology on X and which is equivariant, i.e., $\eta \circ T = S \circ \eta$.

Here we generalize topological systems by also allowing nondeterministic dynamics. For simplicity, we still work with discrete time, i.e., T describes the *next* states. Future work should generalize our results to continuous time (as used in continuous time analog computation) and to group or monoid actions more generally.

To illustrate our definition, we consider, in the remainder of this section, some examples that highlight our computational perspective on dynamical systems.

Example 3.2 (Shifts). Symbolic dynamics is an important subfield of dynamical systems theory. It developed from studying physical dynamical systems by discretizing both space and time, but it also found applications in storing and transmitting digital data [81]. The objects of study in symbolic dynamics are shifts, which are defined as follows. Let A be a finite set, called *alphabet*. The (one-sided) *full shift* over A is the system (A^ω, σ) where A^ω carries the product topology of the discrete topology on A and σ is the shift map, i.e., $\sigma(x)_n = x_{n+1}$. A *shift* (aka *shift space*) is a closed, shift-invariant subset X of a full shift on some alphabet. Hence, if X is nonempty, (X, σ) is in $\mathsf{detSys}^{\mathsf{ef}}$.

The two most studied classes of shifts are the following. The *shifts of finite type* are those shifts X described by a finite list of forbidden words, i.e., X is the set of those sequences x over the alphabet A of X that do not contain a word w from a finite set F of words over A. Equivalently, shifts of finite type are those shifts whose sequences arise as the sequences of edges of infinite paths in a finite graph. The

sofic shifts are those shifts that are factors of shifts of finite type. Equivalently, sofic shifts are those shifts whose sequences arise as the sequences of labels of infinite paths in a finite graph whose edges are labeled. This makes sofic shifts analogous to regular languages in automata theory.

Example 3.3 (Cellular automata). A cellular automaton is a deterministic system $(\Sigma^{\mathbb{Z}^n}, G)$ in the above sense. Here Σ is the nonempty set of the finitely many states that a single cell can be in, so an element $c \in \Sigma^{\mathbb{Z}^n}$ describes the global configuration of the cellular automaton, i.e., the state of each cell in the n-dimensional grid $\mathbb{Z}^n$ of cells. The topology on $\Sigma^{\mathbb{Z}^n}$ is the product topology (where Σ has the discrete topology). The function G results from updating a global configuration according to the local updating rule. The local character renders G continuous.

For example, in dimension $n = 1$ with binary states $\Sigma = \{0, 1\}$, the well-known rule "110" says that a cell changes from 0 to 1 precisely if the cell to its right is 1 ("life spreads left'), and a cell changes from 1 to 0 precisely if both the cell to the left and to the right is 1 ("overpopulation"). Cook [24] shows that this simple rule yields a Turing universal cellular automaton.

Example 3.4 (Differential equations). Differential analyzers compute by solving (ordinary) differential equations—coming, e.g., from engineering [18, 116]. As usual [114, sec. 6.2], such solutions are, if globally defined, dynamical systems $\Phi : \mathbb{R} \times X \to X$ with $X \subseteq \mathbb{R}^n$ open; or in discrete time steps, $T := \Phi(1, \cdot) : X \to X$. As described by remark A.1, we can view this dynamics on the Polish space X in the larger, compact and zero-dimensional space Y, rendering it a deterministic system in our formal sense.

Example 3.5 (Training dynamics). One may view the training process of a neural network as computing weights for the neural networks with which it solves its task well. The training dynamics T takes the current weights w of the network and a batch of the training data, and then produces (using backpropagation) updated weights w'. Formally, the dynamics is given by stochastic gradient descent, so the choice of batch is not deterministic, hence there are several possible next states w'. Thus, the training dynamics $T : W \rightrightarrows W$ is a multifunction on the weight space W. Understanding this dynamics is an important problem in the theory of machine learning [65, 98].

Example 3.6 (Reservoir computing). Reservoir computing is one prominent approach to computing on non-digital hardware. The basic idea is that inputs are mapped into a high-dimensional state space of a nonlinear dynamical system—called the *reservoir*—and only a simple read-out function is trained to map from the high-dimensional state space to outputs [112]. This was initiated independently by [66] and [83], with a rich history since. For the purpose of this example, let us consider the first approach.[6] The reservoir is a recurrent neural network with weight matrix W whose state, at time $t \in \{0, 1, 2, \ldots\}$, is the vector $x(t)$ describing the activation value of each neuron. Additionally, there are input neurons whose state, at time t, is the vector $u(t)$ describing the value of each input neuron. (So the input can change over time, i.e., is sequential.) A matrix W^{in} maps input neuron

---

[6]In the case where there is no feedback from the output back to the reservoir.

$$x' \dashrightarrow^{\varphi} y' \qquad x' \dashrightarrow^{\varphi} y' \qquad x' \dashrightarrow^{\varphi} y' \qquad x' \dashrightarrow^{\varphi} y'$$

(A) Forth   (B) Back   (C) Factor   (D) Embedding
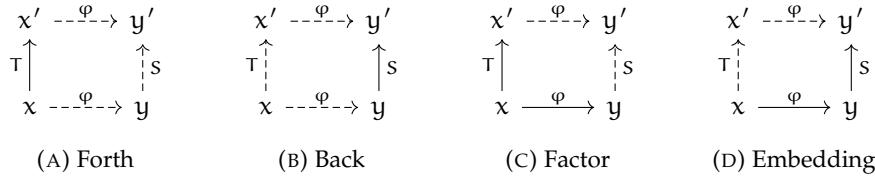
FIGURE 1. Nondeterministic equivariance: ways of requiring that the T-dynamics is matched, via φ, to the S-dynamics, and vice versa. Solid arrows are assumed relations, dashed arrows are relations required to exist.

states into states of the reservoir. The state of the reservoir at the next time step is computed by

$$(1) \qquad\qquad x(t+1) := f\big(W^{\text{in}}u(t+1) + Wx(t)\big),$$

where f is the function that applies to each neuron its activation function (e.g., a sigmoid). Moreover, there are output neurons whose activations, at time t, are given by the vector $y(t) = W^{\text{out}}x(t)$. Only the weight matrix W^{out} is updated during training. If the input sequence u is not fixed, we can view this as a nondeterministic system $T : X \rightrightarrows X$ on the state space X of the reservoir system, with $T(x)$ being the set of next states realized by some input vector as in 1.

3.2. **Morphisms: Embedding-factor pairs.** What should be the morphisms φ : $(X, T) \to (Y, S)$ between dynamical systems? For deterministic systems, these are the continuous functions $\varphi : X \to Y$ that are *equivariant*, i.e., the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ T\downarrow & & \downarrow S \\ X & \xrightarrow{\varphi} & Y \end{array}$$

How to generalize this to nondeterministic systems? Since we allow multifunctions for the dynamics, it is natural to also allow them for the morphisms. A natural way to generalize equivariance is to require that any step in the T-dynamics can be matched, via φ, to a step in the S-dynamics, and vice versa. For embeddings and factors we require an even closer match.

Definition 3.7. Let (X, T) and (Y, S) be two dynamical systems. A *system morphism* (or just *morphism*) $\varphi : (X, T) \to (Y, S)$ is a closed-valued and upper-hemicontinuous multifunction $\varphi : X \rightrightarrows Y$ such that

(1) for all $x, x' \in X$, if $x \xrightarrow{T} x'$, there is $y, y' \in Y$ such that $x \xrightarrow{\varphi} y$, $x' \xrightarrow{\varphi} y'$, and $y \xrightarrow{S} y'$ (figure 1a).

(2) for all $y, y' \in Y$, if $y \xrightarrow{S} y'$, there is $x, x' \in X$ such that $x \xrightarrow{\varphi} y$, $x' \xrightarrow{\varphi} y'$, and $x \xrightarrow{T} x'$ (figure 1b).

A morphism φ is a *factor* if it is a surjective function with

(3) for all $x, x' \in X$ and $y \in Y$, if $x \xrightarrow{\varphi} y$ and $x \xrightarrow{T} x'$, there is $y' \in Y$ such that $y \xrightarrow{S} y'$ and $x' \xrightarrow{\varphi} y'$ (figure 1c).

A morphism φ is an *embedding* if it is total and partition-injective with

    (4) for all $x \in X$ and $y, y' \in Y$, if $x \xrightarrow{\varphi} y$ and $y \xrightarrow{S} y'$, there is $x' \in X$ such that $x \xrightarrow{T} x'$ and $x' \xrightarrow{\varphi} y'$ (figure 1d).

In the definition of a factor, clause 1 is superfluous; and, for an embedding, clause 2 is superfluous. Also, it is easily shown that $\varphi : (X, T) \to (Y, S)$ is a factor iff $\varphi : X \to Y$ is a continuous surjection such that, for all $y, y' \in Y$, we have

$$y \xrightarrow{S} y' \text{ iff } \exists x, x' \in X : \varphi(x) = y, \varphi(x') = y', \text{ and } x \xrightarrow{T} x',$$

which is the definition of a factor in [63]. Further, this notion of factor indeed generalizes that from deterministic systems: If (X, T) and (Y, S) are deterministic systems, then $\varphi : (X, T) \to (Y, S)$ is a factor iff $\varphi : X \to Y$ is a continuous and equivariant surjection. Moreover, our morphisms are closely related to the well-known notion of a *bisimulation*, which is a relation with both property 3 and 4.

Composition of morphisms is the usual composition of relations. (The proof is in appendix A.1.)

Proposition 3.8. *If $\varphi : (X, T) \to (Y, S)$ and $\psi : (Y, S) \to (Z, R)$ are morphisms (resp. factors or embeddings), then their composition*

$$\psi \circ \varphi : (X, T) \to (Z, R) \qquad\qquad x \mapsto \psi[\varphi(x)]$$

is again a morphism (resp. factor or embedding). The composition $\circ$ is associative and its unit is the identity function.

Our aim is to find a system (U, T) that is, at least, embedding and factor universal. So for any other system (Y, S), we want both an embedding $e : (Y, S) \to (U, T)$ and a factor $f : (U, T) \to (Y, S)$. Moreover, they should correspond to each other:

    (1) if one first embeds up and then factors down again, one arrives at the state one started with, and

    (2) if one first factors down and then embeds up, one might obtain several states (since the universal system is richer) but the state one started with should be among them.

We will then speak of an embedding-factor pair, in analogy to the embedding-projection pairs in domain theory [1].

Definition 3.9. An *embedding-factor pair* (*ef-pair* for short) from system (X, T) to system (Y, S) is a pair (e, f) of an embedding $e : (X, T) \to (Y, S)$ and a factor $f : (Y, S) \to (X, T)$ such that

    (1) for all $x \in X$, $f \circ e(x) = \{x\}$, and

    (2) for all $y \in Y$, $y \in e \circ f(y)$.

If $(e, f) : (X, T) \to (Y, S)$ and $(e', f') : (Y, S) \to (Z, R)$ are ef-pairs, their composition is $(e', f') \circ (e, f) := (e' \circ e, f \circ f') : (X, T) \to (Z, R)$, which again is an ef-pair.

Remark 3.10 (Simulation and abstraction)**.** The intuition behind an ef-pair $(e, f) :$ $(X, T) \to (Y, S)$ is that the factor $f : Y \to X$ *abstracts* the 'detailed' states y in Y into 'coarser' states x in X, and the embedding $e : X \to Y$ *simulates* states x in X by states $e(x)$ in Y (cf. [120]). Several states y (namely those $f^{-1}(x)$) can be identified under the abstraction into a single state x. And several states y (namely those in $e(x)$) can

simulate the state x: if there is a dynamic step $x \xrightarrow{T} x'$ in X, there are states y and y' simulating x and x', respectively, and $y \xrightarrow{S} y'$.

In an ef-pair, each of the two elements determines the other. This is analogous to embedding-projection pairs in domain theory forming an order adjunction. Thus, by considering ef-pairs, we, in sense, 'just' consider factors (or 'just' embeddings). However, like for embedding-projection pairs in domain theory, we will see that it is conceptually helpful to make both directions explicit (see remark 3.13 below). (The proof is in appendix A.1.)

Proposition 3.11. *Let* (X, T) *and* (Y, S) *be two systems.*

(1) If $f : (Y, S) \to (X, T)$ *is a factor, there is a unique embedding* $\underline{f} : (X, T) \to (Y, S)$ *such that* $(\underline{f}, f)$ *is an ef-pair. It is given by* $\underline{f}(x) := f^{-1}(x)$.

(2) If $e : (X, T) \to (Y, S)$ *is an embedding, there is a unique factor* $\overline{e} : (Y, S) \to (X, T)$ *such that* $(e, \overline{e})$ *is an ef-pair. It is given by* $\overline{e}(y) =$ *the* $x \in X$ *with* $y \in e(x)$.

Now we can define the categories of dynamical systems that we need here. We will focus on nontrivial systems. This is not just because trivial systems are uninteresting from a dynamical point of view. It is also because, by definition, no nontrivial system can factor onto a trivial one. So if we want to find a system that is both embedding and factor universal, we cannot, for that simple reason, include trivial systems.

Definition 3.12. Let $\mathsf{Sys}^{\mathsf{ef}}$ be the category whose objects are nontrivial dynamical systems (X, T) and whose morphisms are ef-pairs $(e, f) : (X, T) \to (Y, S)$. Should we need to talk about all systems, we write $\mathsf{allSys}^{\mathsf{ef}}$ for the supercategory consisting of all dynamical systems with ef-pairs. We are also interested in the full subcategories $\mathsf{totalSys}^{\mathsf{ef}}$ and $\mathsf{detSys}^{\mathsf{ef}}$ of $\mathsf{Sys}^{\mathsf{ef}}$ consisting of total and deterministic systems, respectively (which hence are nonempty).

We could also generalize away from ef-pairs and define, e.g., the category allSys of dynamical systems with system morphisms, but we do not need that here. Reassuringly, system isomorphism $(e, f) : (X, T) \to (Y, S)$ in the category-theoretic sense coincides with the usual sense, i.e., $e : X \to Y$ is a homeomorphism with inverse f such that $e \circ T = S \circ e$ (proposition A.3 in appendix A.1). The category $\mathsf{detSys}^{\mathsf{ef}}$ is a common setting for topological dynamics in dimension zero: the objects are pairs (X, T) with X a nonempty, compact, and zero-dimensional Polish space and $T : X \to X$ a continuous function, and the morphisms are factors (ignoring the corresponding embedding), i.e., continuous and equivariant surjections.

Remark 3.13 (Limit–colimit coincidence)**.** Because of proposition 3.11, we could, instead of ef-pairs, equivalently work with only embeddings or only factors. The latter is more natural from a dynamical systems point of view, but the former is more common for Fraïssé limits. With ef-pairs, we can capture both perspectives. In particular, in $\mathsf{allSys}^{\mathsf{ef}}$ and any of its full subcategories, a *colimit* via ef-pairs (or just the embeddings in the pairs) is the same thing as a *limit* of the factors in the pairs. This is, again, analogous to the *limit–colimit coincidence* in domain theory [1, sec. 3.3.2]. Thus, depending on one's preferred perspective, one can switch back and forth between the embedding/colimit or the factor/limit terminology.

4. Main results and proof technique

With this framework of dynamical systems in place, we can now state our main results (section 4.1). Then we describe the general proof technique, namely, Fraïssé limits (section 4.2).

4.1. **Statement of the main results.** To connect to existing theories of computation, section 5 establishes a categorical equivalence between our category $\mathsf{Sys}^{\mathsf{ef}}$ of systems and what we call the category dynAlg of dynamic algebraic lattices. But our two main results on the existence of universal systems are the following.

To state these results, recall that we are looking for a universal system (U, T), i.e., for any system (Y, S), we need an ef-pair $(e, f) : (Y, S) \to (U, T)$. Moreover, ideally the system (U, T) also is 'homogeneous' in the intuitive sense that 'every part of it looks the same', so it does not have any 'idiosyncrasies'. To formalize this, recall that a topological space X is homogeneous if, for any two points $x, y \in X$, there is a homeomorphism $h : X \to X$ such that $h(x) = y$.[7] Since points of X are essentially just continuous functions from the one-point space into X, we can rephrase this as follows: If Y is the one-point space and $f, g : Y \to X$ are continuous functions, then there is a homeomorphism $h : X \to X$ such that $h \circ g = f$. By demanding this for more spaces Y (e.g., finite spaces), we get a stronger homogeneity. Formally, then, we define these terms as follows.

Definition 4.1. Let C be a category and let D be a full subcategory. An object U in C is D-*universal*, if, for any object A in D, there is a morphism $f : A \to U$ (see figure 2a).[8] An object U in C is D-*homogeneous*, if, for any object A in D and morphisms $f, g : A \to U$, there is an isomorphism $h : U \to U$ such that $h \circ g = f$ (see figure 2b).

Regarding our existence result for nondeterministic systems, write $\mathsf{Sys}^{\mathsf{ef}}_{\mathsf{fin}}$ for the full subcategory of $\mathsf{Sys}^{\mathsf{ef}}$ consisting of those systems with a finite state space (i.e., those for which we want homogeneity). Then the result says:

Theorem 4.2 (Universal nondeterministic system). *There is an up to isomorphism unique system in $\mathsf{Sys}^{\mathsf{ef}}$ that is $\mathsf{Sys}^{\mathsf{ef}}$-universal and $\mathsf{Sys}^{\mathsf{ef}}_{\mathsf{fin}}$-homogeneous.*

Recall from the end of section 2 that this is a generalization of the result for profinite undirected graphs [20, 46, 63] (more details in remark 7.3 below). Section 7 discusses further properties of the universal nondeterministic system.

Regarding deterministic systems, there cannot be a universal system, as we discuss in section 8. The shifts are the basic building blocks of deterministic systems (i.e., every deterministic system is a limit of shifts). But the obstacle to universality is that there are uncountably many shifts. So this suggests a strategy to build universal systems for large subclasses of deterministic systems. Namely, choose a countable class S of shifts and consider the full subcategory $\mathsf{DS}^{\mathsf{ef}}(S)$ of $\mathsf{detSys}^{\mathsf{ef}}$ consisting of those systems that are limits of shifts in S. Recall remark 3.13

---

[7]Equivalently, the group G of hoemeomorphisms of X acts transitively on X.

[8]We follow [38, p. 138] in this terminology, but Kubiś [75, p. 1757] instead speaks of a 'cofinal' object, to avoid the category-theoretic connotation that a universal object is canonically determined by a universal property.

(A) Universal   (B) Homogeneous   (C) Joint embedding   (D) Amalgamation
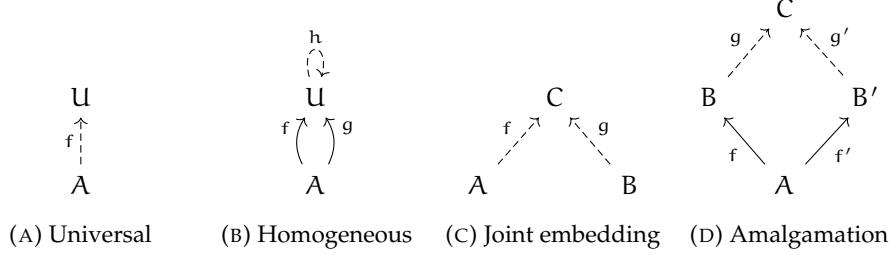
FIGURE 2. Definitions related to the category-theoretic Fraïssé theorem. Dashed arrows indicate the required existence of a morphism (however, unique existence is *not* required).

that a colimit along ef-pairs is the same thing as a limit along factors. So our formal definition is this:

Definition 4.3. If S is a collection of shifts, let $\mathsf{DS}^{\mathsf{ef}}(S)$ be the full subcategory of $\mathsf{detSys}^{\mathsf{ef}}$ consisting of those objects that are colimits of ω-chains of shifts in S. The objects of $\mathsf{DS}^{\mathsf{ef}}(S)$ we also call the ω-*proshifts* over S.

As mentioned in example 3.2, the most studied classes of shifts are the class SFT of shifts of finite type and the class SOF of sofic shifts. So they are natural choices for S. Surprisingly, they share a universal system, as our existence result for deterministic systems says:

Theorem 4.4 (Universal deterministic system). *There is a deterministic system* (U, T) *such that*

(1) (U, T) *is the up to isomorphism unique system in* $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SFT})$ *that is* $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SFT})$-*universal and* SFT-*homogeneous.*

(2) (U, T) *is the up to isomorphism unique system in* $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SOF})$ *that is* $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SOF})$-*universal and* SOF-*homogeneous.*

For brevity, we call (U, T) *the* universal proshift.

An important corollary of a result of [51] is that the universal proshift has the shadowing property (i.e., simulated orbits are close to a true orbit).

4.2. **Proof technique: Fraïssé limits.** To prove our universality results, we will use the Fraïssé theorem (aka Fraïssé–Jónsson theorem) from model theory in its category-theoretic generalization due to Droste and Göbel [38, thm. 1.1]. (This is further discussed and generalized by [75] and [21].)

We recall the terminology from [38] to state their version of the Fraïssé theorem. Let C be a category. We write $\omega = \{0, 1, 2, \ldots\}$. An ω-*chain* in C is a structure $(A_i, f_{i,i+1})_{i \in \omega}$ where each A_i is an object in C and each $f_{i,i+1} : A_i \to A_{i+1}$ is a morphism in C (we will drop the commas in the subscripts). For $i \leqslant j$ in ω, define $f_{ij} := f_{j-1j} \circ \ldots \circ f_{ii+1}$, with $f_{ij} := \mathrm{id}_{A_i}$ if $i = j$. So we have the diagram (A_i, f_{ij}) of shape ω in C. If it has a colimit in C, we denote it (A, f_i) with $f_i : A_i \to A$, and we also call it the colimit of the ω-chain (A_i, f_{ii+1}). An object B of C is *(category-theoretically) finite* if, for all ω-chains $(A_i, f_{ii+1})_{i \in \omega}$ with colimit (A, f_i), if $g : B \to A$

is a morphism, then there exists n such that, for all $i \geqslant n$, there is a unique morphism $h : B \to A_i$ such that $g = f_i \circ h$.

Definition 4.5. A category C is *semi-algebroidal* if

  (1) all morphisms in C are monic,
  (2) every object of C is a colimit of an ω-chain of finite objects in C, and
  (3) every ω-chain of finite objects in C has a colimit in C.

Finally, C is *algebroidal* if it is semi-algebroidal, it contains up to isomorphism only countably many finite objects, and between any two finite objects there are only countably many morphisms.

The term 'algebroidal' traces back, as [38, p. 138] note, to [109] (in domain theory) and [8] (in category theory). Indeed, [109, p. 780] see algebroidal categories as a generalization of ω-algebraic cpo's, which are a common kind of partial order studied in domain theory.[9] Thus, ω-algebraic cpo's—when viewed as a poset category[10]—provide a simple example of algebroidal categories. Other common examples also arise in domain theory: many categories of domains form algebroidal categories [38].

Definition 4.6. Let C be a category. Then C has the *joint embedding property* if for any objects A and B, there is an object C with morphisms $f : A \to C$ and $g : B \to C$ (see figure 2c). And C has the *amalgamation property* if for any objects A, B, and B' with morphisms $f : A \to B$ and $f' : A \to B'$, there is an object C with morphisms $g : B \to C$ and $g' : B' \to C$ such that $g \circ f = g' \circ f'$ (see figure 2d).

Now the category-theoretic Fraïssé theorem says:

Theorem 4.7 (Droste and Göbel [38])**.** *Let C be an algebroidal category and $\mathsf{C}_{\mathsf{fin}}$ the full subcategory of category-theoretically finite objects. Then there exists a C-universal and $\mathsf{C}_{\mathsf{fin}}$-homogeneous object U in C iff $\mathsf{C}_{\mathsf{fin}}$ has the joint embedding property and the amalgamation property. Moreover, if a C-universal and $\mathsf{C}_{\mathsf{fin}}$-homogeneous object exists in C, then it is unique up to isomorphism.*

Thus, our proof strategy, both for nondeterministic and deterministic systems, will be to show that the relevant category of systems (1) is algebroidal and (2) that the full subcategory of category-theoretically finite systems has the joint embedding property and the amalgamation property. We will achieve (1) by proving 'algebroidality theorems', i.e., theorems providing easily check-able sufficient conditions for a category of systems to be (semi-) algebroidal (theorems 6.2, 8.6, and 9.2).

---

[9]We will not discuss cpo's later, but we give here a definition for completeness and to highlight that it is a special case of an algebroidal category. A cpo is a partial order $(P, \leqslant)$ with a least element in which every increasing ω-chain has a least upper bound (i.e., 'colimit'). An element $b \in P$ is ω-compact (i.e., category-theoretically finite) if any increasing ω-chain (a_n) whose least upper bound is $\geqslant b$ contains an element a_n which is $\geqslant b$. Finally, P is ω-algebraic if it contains only countably many compact elements, and for any element $a \in P$, there is an increasing ω-chain of compact elements whose least upper bound is a. In section 5, we give domain-theoretic definitions in (the more common) terms of directed subsets rather than ω-chains, but these are equivalent if there are only countably many compact elements [1, sec. 2.2.4].

[10]The objects are the elements of the partial order $(P, \leqslant)$ and there is exactly one morphism from a to b if $a \leqslant b$, with no further morphisms.

5. CONNECTIONS TO COALGEBRA AND DOMAIN THEORY

To put things into perspective, we establish connections to the two main areas of research on computation from a general point of view: coalgebra and domain theory. In particular, we make precise the strong analogies to domain theory alluded to in section 3.

Coalgebra. The field of coalgebra studies the 'mathematics of computational dynamics' [64, p. vii]. From this perspective, we view our nondeterministic systems (X, T) as the coalgebra $T : X \to F(X)$ that maps each $x \in X$ to the (closed) set $T(x)$ of its successor states. To make this precise, we need to specify the endofunctor F. Since X is a second-countable Stone space, it is natural to work in that category. So we need to put a topology on the collection $F(X)$ of all closed subsets of X. Such topologies are well-studied as hyperspaces [14]. A standard one is the *Vietoris topology*: it is generated by (with $U \subseteq X$ open)

$$\Box U := \{A \in F(X) : A \subseteq U\} \qquad \Diamond U := \{A \in F(X) : A \cap U \neq \emptyset\}.$$

If X is a non-empty second-countable Stone space, so is $F(X)$ with this topology; and F indeed becomes an endofunctor on the category of second-countable Stone spaces by sending a morphism $f : X \to Y$ to the morphism $F(f) : F(X) \to F(Y)$ which maps a closed set A to its image $f[A]$ [78]. However, then T is an F-coalgebra only if it also is lower-hemicontinuous (and not only upper-hemicontinuous), and, further, coalgebra morphisms do not include all our system morphisms. We get a closer match, if we work only with the *upper Vietoris topology* (aka *miss topology*) which is generated only by the $\Box U$. The downside is that then $F(X)$ is only a spectral space, not necessarily a Stone space. This brings us to domain theory.

Domain theory. Edalat [40] studies dynamical systems via domain theory by considering, among others, the upper Vietoris topology on $F(X) \setminus \{\emptyset\}$ which (in our case where X is a second-countable Stone space) is an algebraic domain and $F(T)$ a Scott-continuous function (definitions below).[11] [60] provides another way of assigning a dynamical system (X, T) (viewed as a computational process) to a domain D with a continuous $[\![T]\!] : D \to D$ (viewed as the denotational semantics of that process: a program $[\![T]\!]$ of type D). This approach also incorporates probability measures on X. Here, we will make precise the analogy between, on the one hand, the dynamical systems that we study in this paper and, on the other hand, domain theory. To do so, we prove an equivalence between the category $\mathsf{Sys}^{\mathsf{ef}}$ and a 'domain-theoretic' category that we call dynAlg.

To state this equivalence, we recall some terminology [e.g. 47, sec. I-4]. Let $(P, \leqslant)$ be a partial order (aka poset). For a subset $S \subseteq P$, the least upper bound of S (resp., greatest lower bound), if it exists, is denoted $\bigvee S$ (resp., $\bigwedge S$). A subset $D \subseteq P$ is *directed* if it is nonempty and, for all $a, b \in D$, there is $c \in D$ with $a, b \leqslant c$. An element $x \in P$ is (order-theoretically) *compact* if for all directed subsets $D \subseteq P$, if $\bigvee D$ exists and $\bigvee D \geqslant x$, there is $d \in D$ with $d \geqslant x$. The set of compact elements of P is written $K(P)$. We call P *algebraic* if, for all $x \in P$, the set $\{c \in K(P) : c \leqslant x\}$ is directed and its least upper bound is x. We call P *directed complete* if every directed subset of P has a least upper bound. An *algebraic domain* is a directed complete

---

[11]Moreover, the algebraic domain has only countably many compact elements (definition below) and is bounded-complete (any two elements with an upper bound have a least upper bound).

partial order that also is algebraic. An *algebraic lattice* is an algebraic domain that also is a lattice (i.e., each finite subset has a least upper bound and a greatest lower bound). A function $f : A \to B$ between directed complete posets is *Scott continuous* if it preserves directed least upper bounds, i.e., for all directed $D \subseteq A$, we have $f(\bigvee D) = \bigvee f[D]$. The category Alg of algebraic lattices with Scott continuous functions is a category of domains that was considered early on, when Scott [100] used it for an "Outline of a Mathematical Theory of Computation".[12]

Here, we are interested in the following subcategory of Alg. Call an element c of an algebraic lattice A a *co-atom* if c is not the greatest element of A, but for all $y \in A$, if $c \leqslant y$, then either $y = c$ or y is the greatest element of A.[13] For an element $x \in A$, write $\text{CoAt}(x)$ for the set of all co-atoms c in A with $c \geqslant x$. A function $f : A \to B$ between algebraic lattices is *co-atomic* if, for all $x \in A$, we have $f(x) = \bigwedge f[\text{CoAt}(x)]$. Now, we define the subcategory bAlg^c of Alg as follows: The objects are the algebraic lattices whose compact elements form a countable sublattice that is a Boolean algebra (i.e., distributive and complemented), and the morphisms are the Scott-continuous and co-atomic functions.[14]

Then our domain-theoretic characterization of the category Sys^{ef} of dynamical systems goes as follows. For a second-countable Stone space X, we (still) write $F(X)$ for the set of closed subsets of X ordered by reverse inclusion, and if $f : X \rightrightarrows Y$ is a closed-valued and upper-hemicontinuous multifunction between Stone spaces, then $F(f) : F(X) \to F(Y)$ maps A to the image $f[A] = \bigcup\{f(x) : x \in A\}$.

Theorem 5.1. *The category* Sys^{ef} *is equivalent to the following category* dynAlg *of 'dynamic algebraic lattices':*

- *Objects: Pairs* (A, α) *where* A *is an object in* bAlg^c *and* $\alpha : A \to A$ *is a morphism in* bAlg^c.
- *Morphisms: Pairs* $(\epsilon, \pi) : (A, \alpha) \to (B, \beta)$ *of morphisms* $\epsilon : A \to B$ *and* $\pi : B \to A$ *in* bAlg^c *such that*
  (1) π *preserves co-atoms (if* b *is a co-atom in* B*, then* $\pi(b)$ *is a co-atom in* A*)*
  (2) $\pi \circ \epsilon = \text{id}_A$
  (3) $\epsilon \circ \pi \leqslant \text{id}_B$
  (4) $\pi \circ \beta \circ \epsilon \leqslant \alpha$
  (5) $\alpha \circ \pi \leqslant \pi \circ \beta$

via the functor that sends the system (X, T) *to* $(F(X), F(T))$ *and that sends the ef-pair* $(e, f) : (X, T) \to (Y, S)$ *to* $(F(e), F(f))$.

---

[12]Later, other categories of domains were considered in which the partial orders need not have a top element—e.g., algebraic domains. But for us, the top element has an intuitive interpretation: If X is a second-countable Stone space, then $F(X)$, the set of closed subsets of X ordered by reverse inclusion, is an algebraic lattice. For a dynamics T on X, the bottom element X and the top element $\emptyset$ play two dual roles, when considered as possible sets of successor states: The bottom element X is one extreme of being nondeterministic (every state can be a successor), and the top element $\emptyset$ is the other extreme (no state can be a successor).

[13]The order-theoretically dual notion is that of an *atom*: an element $a \in A$ that is not the least element but for all $y \in A$, if $y \leqslant a$, then either $y = a$ or y is the least element.

[14]Algebraic lattices whose compact elements form a sublattice are known as arithmetic lattices [47, def. I-4.7]. So we might call the objects of bAlg^c 'Boolean-arithmetic lattices'.

To not digress from establishing the universality results for $\mathsf{Sys}^{\mathrm{ef}}$, we move the proof to appendix A.2. Of course, the universality result then also holds for dynAlg. Hence the result can be seen within the tradition of existence results concerning universal domains [38], albeit now for the new category of dynamic algebraic lattices. To connect back to coalgebras, note that the objects of dynAlg are coalgebras for the identity functor on $\mathsf{bAlg}^{\mathrm{c}}$. However, the morphisms are *not* coalgebra morphisms. In this case, a coalgebra morphism $\varphi : (A, \alpha) \to (B, \beta)$ would be a morphism $\varphi : A \to B$ in $\mathsf{bAlg}^{\mathrm{c}}$ such that $\varphi \circ \alpha = \beta \circ \varphi$, yet here we do not have equality, but only the inequality expressed in 5, which, together with 4, forms a weaker equivariance condition. On the other hand, the morphisms in dynAlg are also stronger in the sense that they are not just a 'one-directional' morphism in $\mathsf{bAlg}^{\mathrm{c}}$ but they rather are 'bidirectional': They are embedding-projection pairs, as demanded by 2 and 3, which play an important role in Alg. Thus, this corroborates the analogy to ef-pairs, which we mentioned in section 3.2.

6. Algebroidal categories of nondeterministic systems

We now work toward the universality result for nondeterministic systems. As described in the proof strategy (section 4.2), we start with an algebroidality theorem: sufficient conditions for when a category of (nondeterministic) systems is algebroidal. We state the theorem in section 6.1 and prove it in section 6.2. In the next section, we use this to prove the existence of a universal nondeterministic system.

6.1. **Algebroidality theorem for nondeterministic systems.** To state the theorem, let us first recall some terminology from topology. By a *countable inverse* (or *projective*) *system* of topological spaces we mean the data of a countable directed partial order $(I, \leqslant)$, a family $(X_i)_{i \in I}$ of topological spaces, and, for each $i \leqslant j$ in I, a surjective map $f_{ij} : X_j \to X_i$ such that f_{ii} is the identity function on X_i and, for $i \leqslant j \leqslant k$ in I, we have $f_{ij} \circ f_{jk} = f_{ik}$. The category-theoretic limit (X, f_i), also called *inverse* or *projective limit*, is given as

$$(2) \qquad X := \left\{ \langle x_i : i \in I \rangle \in \prod_{i \in I} X_i : \forall i, j \in I \text{ with } i \leqslant j . f_{ij}(x_j) = x_i \right\},$$

endowed with the subspace topology of the product topology, and $f_i(\langle x_i : i \in I \rangle) := x_i$ are the canonical projection maps. One can show that the f_i are surjective, as is any mediating morphism from another cone. A standard result is (e.g., [28, prop. 3.1]):

Lemma 6.1. *A topological space is a countable projective limit of finite discrete spaces iff it is a compact, zero-dimensional Polish space. Because a countable directed partial order has a cofinal chain, the index set of the projective system can be chosen to be $I = \omega$.*

Now we can state the sufficient condition for algebroidality of a category of systems, i.e., a full subcategory of the category of all dynamical systems $\mathsf{allSys}^{\mathrm{ef}}$.

Theorem 6.2. *Let S^{ef} be a full subcategory of $\mathsf{allSys}^{\mathrm{ef}}$ such that*

(1) *If* $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i\in\omega}$ *is an ω-chain in* S^{ef}*, then* (X, T) *is again in* S^{ef}*, where X is defined as in equation 2 with canonical projections f_i and*

$$T(x) := \big\{ x' \in X : \forall i \in \omega. f_i(x) \xrightarrow{T_i} f_i(x') \big\}.$$

(2) *If (X, T) is in* S^{ef} *and $f : X \to Y$ is a continuous surjection into a finite discrete space Y, then (Y, S) is in* S^{ef} *with $S := f \circ T \circ f^{-1}$, i.e.,*

$$S(y) = \big\{ y' \in Y : \exists x, x' \in X. y = f(x) \text{ and } x \xrightarrow{T} x' \text{ and } f(x') = y' \big\}.$$

Then S^{ef} *is algebroidal and its category-theoretically finite objects are precisely those systems in* S^{ef} *that have a finite state space.*

We will prove this result in section 6.2. We end this subsection by checking to which categories of systems the result applies (theorem 6.3 below) and to which it does not (example 6.4 below).

Theorem 6.3. *The following full subcategories of* $\mathsf{allSys}^{\mathsf{ef}}$ *satisfy the conditions 1 and 2 of theorem 6.2.*

(1) $\mathsf{allSys}^{\mathsf{ef}}$ *itself (i.e., the full subcategory of all systems in* $\mathsf{allSys}^{\mathsf{ef}}$*).*
(2) $\mathsf{Sys}^{\mathsf{ef}}$ *consisting of all nontrivial systems in* $\mathsf{allSys}^{\mathsf{ef}}$*.*
(3) $\mathsf{totalSys}^{\mathsf{ef}}$ *consisting of all total and nontrivial systems in* $\mathsf{allSys}^{\mathsf{ef}}$*.*

Proof. Ad (1). Concerning condition 1, let $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i\in\omega}$ be an ω-chain in $\mathsf{allSys}^{\mathsf{ef}}$. We need to show that (X, T) is a dynamical system, i.e., that (a) X is a compact, zero-dimensional Polish space and (b) T is a closed-valued, upper-hemicontinuous multifunction. Regarding (a), X is, via the f_i, a projective limit of a countable projective diagram of compact, zero-dimensional Polish spaces, and these spaces are closed under such limits. Regarding (b), we show that the graph of T is closed. First, note that, since f_i is continuous, its graph is closed, and so is the graph of f_i^{-1}. By assumption, the graph of T_i is closed, so the graph of $f^{-1} \circ T_i \circ f_i$ is closed qua composition of multifunctions with closed graphs. Hence $\mathrm{Gr}(T) = \bigcap_{i\in\omega} f^{-1} \circ T_i \circ f_i$ is closed qua intersection of closed sets.

Concerning condition 2, let (X, T) be a dynamical system and $f : X \to Y$ a continuous surjection into a finite discrete space Y. We need to show that (Y, S) is a dynamical system. Indeed, qua finite discrete space, Y is compact zero-dimensional Polish and the multifunction S trivially has a closed graph.

Ad (2). Concerning condition 1, let $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i\in\omega}$ be an ω-chain in $\mathsf{Sys}^{\mathsf{ef}}$. From (1), we know that (X, T) is a dynamical system, so it remains to show that it is nontrivial. Indeed, since X_0 is nontrivial, let $x_0 \xrightarrow{T_0} x_0'$. Since $f_{01} : X_1 \to X_0$ is a factor, there is $x_1 \xrightarrow{T_1} x_1'$ with $f_{01}(x_1) = x_0$ and $f_{01}(x_1') = x_0'$. Since $f_{12} : X_2 \to X_1$ is a factor, there is $x_2 \xrightarrow{T_2} x_2'$ with $f_{12}(x_2) = x_1$ and $f_{12}(x_2') = x_1'$. We continue like this to build sequences $x = \langle x_i : i \in \omega \rangle$ and $x' = \langle x_i' : i \in \omega \rangle$ with $x_i \xrightarrow{T_i} x_i'$ and $f_{ii+1}(x_{i+1}) = x_i$ and $f_{ii+1}(x_{i+1}') = x_i'$. So, by definition, $x, x' \in X$ and $x \xrightarrow{T} x'$.

Concerning condition 2, let (X, T) be a nontrivial dynamical system and $f : X \to Y$ a continuous surjection into a finite discrete space Y. From (1), we know that (Y, S) is a dynamical system, and it again is nontrivial: if $x \xrightarrow{T} x'$, then $f(x) \xrightarrow{S} f(x')$ by definition of S.

Ad (3). Concerning condition 1, let $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$ be an ω-chain in $\mathsf{totalSys}^{\mathsf{ef}}$. From (2), we know that (X, T) is a nontrivial dynamical system, so it remains to show that T is total. Let $x \in X$ and find $x' \in T(x)$. Consider $F_i := f_i^{-1}(T_i(f_i(x)))$, which is a nonempty closed set. If $i \leqslant j$, then $F_i \supseteq F_j$: if $x' \in F_j$, then $f_j(x') \in T_j(f_j(x))$, so, since f_{ij} is a factor and hence $f_{ij} \circ T_j \subseteq T_i \circ f_{ij}$, we have

$$f_i(x') = f_{ij}\big(f_j(x')\big) \in f_{ij}\big[T_j(f_j(x))\big] \subseteq T_i\big[f_{ij}(f_j(x))\big] = T_i\big[f_i(x)\big],$$

hence $x' \in F_i$. By compactness, there is $x' \in \bigcap_i F_i = T(x)$.

Concerning condition 2, let (X, T) be a total and nontrivial dynamical system and $f : X \to Y$ a continuous surjection into a finite discrete space Y. From (2), we know that (Y, S) is a nontrivial dynamical system, so it remains to show that it is total. Let $y \in Y$ and find y' with $y' \in S(y)$. Since f is surjective, there is $x \in X$ with $f(x) = y$. Since T is total, there is $x' \in T(x)$. Define $y' := f(x')$. Then, by definition of S, $y' \in S(y)$, as needed. $\qquad\square$

The list in theorem 6.3 begs the question: can deterministic systems be included? No, as the next example shows: they do not satisfy the conclusion of theorem 6.2 (they violate part (2) of the sufficient condition).

Example 6.4. The subcategory $\mathsf{detSys}^{\mathsf{ef}}$ of deterministic systems cannot be algebroidal with category-theoretically finite objects being the systems with a finite state space. This is because not every deterministic system is a colimit along ef-pairs of finite deterministic systems. In fact, the full shift $(2^\omega, \sigma)$ does not have any nontrivial finite factor. (An elementary proof is in appendix A.3.) Hence there are not enough factors to form a diagram with the full shift being the colimit along ef-pairs.

However, we will see, in section 8, that the category $\mathsf{detSys}^{\mathsf{ef}}$ *is* semi-algebroidal, but its category-theoretically finite objects are very different: they are the shifts. Since there are uncountably many such shifts, $\mathsf{detSys}^{\mathsf{ef}}$ is only semi-algebroidal and not algebroidal. This indicates that the notion of category-theoretical finiteness depends quite subtly on the surrounding category.

6.2. **Proof of the algebroidality theorem.** In this section, we prove theorem 6.2 in a sequence of propositions. (We will use some of these again when considering deterministic systems.)

Regarding part (1) of being algebroidal, we have the following.

Proposition 6.5. *Let S^{ef} be a full subcategory of allSys. Then all morphisms in S^{ef} are monic.*

Proof. Let $(e, f) : (X, T) \to (Y, S)$ and $(e_1, f_1), (e_2, f_2) : (Z, R) \to (X, T)$ be morphisms in S^{ef} such that $(e, f) \circ (e_1, f_1) = (e, f) \circ (e_2, f_2)$. We show that $(e_1, f_1) = (e_2, f_2)$. By definition, $f_1 \circ f = f_2 \circ f$. Since f is a surjective function, $f_1 = f_2$. Since e_1 (resp. e_2) is the unique embedding corresponding to f_1 (resp. f_2), also $e_1 = e_2$. $\qquad\square$

We show, more than part (3) requires, that any ω-chain has a colimit.

Proposition 6.6. *Let S^{ef} be a full subcategory of allSys with property (1). Then every ω-chain $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$ in S^{ef} has a colimit in S^{ef}, namely $\big((X, T), (e_i, f_i)\big)$*

with (X, T) and f_i as in condition 1. In particular, in any (up to unique isomorphism unique) colimit $\big((X, T), (e_i, f_i)\big)$, we have, for all $x, x' \in X$:

    (1) $x = x'$ iff $\forall i \in \omega : f_i(x) = f_i(x')$.
    (2) $x' \in T(x)$ iff $\forall i \in \omega : f_i(x') \in T_i(f_i(x))$.

Proof. By property 1, (X, T) is an object in S^{ef}. We first show that $f_i : (X, T) \to (X_i, T_i)$ is a factor. Then, writing $e_i := \underline{f_i}$, we have the ef-pairs $(e_i, f_i) : (X_i, T_i) \to (X, T)$, which commute with the (e_{ij}, f_{ij}). In particular, we have that $\big((X, T), (e_i, f_i)\big)$ is a cocone.

To show that f_i is a factor, note that it already is a surjective continuous function. Concerning equivariance 3, if $x \xrightarrow{f_i} f_i(x)$ and $x \xrightarrow{T} x'$, then, by definition, $f_i(x') \in T_i(f_i(x))$, so $x' \xrightarrow{f_i} f_i(x')$ and $f_i(x) \xrightarrow{T_i} f_i(x')$. For equivariance 2, Assume $x_i \xrightarrow{T_i} x_i'$. For $k < i$, define $x_k := f_{ki}(x_i) \in X_k$ and $x_k' := f_{ki}(x_i') \in X_k$, so, since, f_{ki} is a factor, $x_k \xrightarrow{T_k} x_k'$. For $i + 1$, there is, since f_{ii+1} is a factor, some $x_{i+1} \xrightarrow{T_{i+1}} x_{i+1}'$ such that $f_{ii+1}(x_{i+1}) = x_i$ and $f_{ii+1}(x_{i+1}') = x_i'$. Similarly, for $i + 2$, and so on. So we get sequences $x = (x_j)$ and $x' = (x_j')$ with $x_j \xrightarrow{T_j} x_j'$ and $f_{jj+1}(x_{j+1}) = x_j$ and $f_{jj+1}(x_{j+1}') = x_j'$. So, by definition, $x, x' \in X$ and $x \xrightarrow{T} x'$ with $x \xrightarrow{f_i} x_i$ and $x' \xrightarrow{f_i} x_i'$.

Finally, we show that $\big((X, T), (e_i, f_i)\big)$ is limiting. Let $\big((Y, S), (g_i, h_i)\big)$ be a cocone in S^{ef} to $((X_i, T_i), (e_{ij}, f_{ij}))$. In particular, (Y, h_i) is a cone to (X_i, f_{ij}), so there is a unique mediating $u : Y \to X$, which is a continuous surjective function that is, as usual, given by $u(y) := \langle h_i(y) : i \in \omega \rangle$. So it suffices to show that it is a factor.

Concerning equivariance 3, assume $y \xrightarrow{u} u(y) =: x$ and $y \xrightarrow{S} y'$. We need to show $u(y) \xrightarrow{T} u(y') =: x'$. Given $i \in \omega$, we have, since h_i is a factor, $h_i(y') \in T_i(h_i(y))$. Since $h_i = f_i \circ u$, we have $h_i(y) = f_i \circ u(y) = f_i(x)$ and similarly $h_i(y') = f_i(x')$. Hence $f_i(x') \in T_i(f_i(x))$, as needed.

Concerning equivariance 2, assume $x \xrightarrow{T} x'$, and find $y, y' \in Y$ with $u(y) = x$, $u(y') = x'$, and $y \xrightarrow{S} y'$. We make a compactness argument: Define

$$F_i := \big\{ (y, y') \in Y \times Y : h_i(y) = f_i(x) \text{ and } h_i(y') = f_i(x') \text{ and } y \xrightarrow{S} y' \big\}.$$

It is straightforward to check that F_i is nonempty and closed, and if $i \leqslant j$, then $F_i \supseteq F_j$. So, by compactness, there is $y, y' \in Y$ with $(y, y') \in \bigcap_i F_i$, hence $y \xrightarrow{S} y'$ and, for all $i \in \omega$, $f_i(u(y)) = h_i(y) = f_i(x)$ and $f_i(u(y')) = h_i(y') = f_i(x')$, so $u(y) = x$ and $u(y') = x'$, as needed. $\square$

The remaining part (2) of being semi-algebroidal we show in two steps. First, we show it when understanding 'finite' in the set-theoretic sense, and then we show that this is equivalent to the category-theoretic sense.

We start, though, with a lemma, which is a version of [63, lem. 2.1]. For completeness, we add a proof in appendix A.3.

Lemma 6.7. *Let* $f : (X, T) \to (Y, S)$ *and* $g : (X, T) \to (Z, R)$ *be factors of systems in* allSys *and let* $h : Y \to Z$ *be a function with* $h \circ f = g$:

$$(X, T)$$

$$f \qquad\qquad g$$

$$(Y, S) \xrightarrow{\quad h \quad} (Z, R)$$

Then $h : (Y, S) \to (Z, R)$ *already is a factor.*

Proposition 6.8. *Let* S^{ef} *be a full subcategory of* allSys *with property* (2). *Then every object of* S^{ef} *is a colimit of an* ω-*chain of dynamical systems in* S^{ef} *that have finite state spaces.*

Proof. Let (X, T) be an object in S^{ef}. Since X is a compact zero-dimensional Polish space, there is, by lemma 6.1, an inverse sequence $(X_i, f_{ij})_{i \in \omega}$ of finite discrete spaces whose limit is (X, f_i). For $i \in \omega$, define $T_i : X_i \rightrightarrows X_i$ as $f_i \circ T \circ f_i^{-1}$, so, by condition 2, (X_i, T_i) is in S^{ef}.

We first show that each $f_i : X \to X_i$ is a factor. By construction, it is a continuous surjection. Concerning equivariance 3, if $x \xrightarrow{f_i} x_i$ and $x \xrightarrow{T} x'$, then take $x_i' := f_i(x')$, so, since $x \in f_i^{-1}(x_i)$ and $x' \in T(x)$, we have $x_i \xrightarrow{T_i} x_i'$ and $x' \xrightarrow{f_i} x_i'$. Concerning equivariance 2, if $x_i \xrightarrow{T_i} x_i'$, then, by definition, there is $x, x' \in X$ with $x \in f_i^{-1}(x_i)$ and $x' \in T(x)$ and $x_i' = f_i(x')$, as needed.

By construction, $f_{ij} : X_j \to X_i$ is a function with $f_{ij} \circ f_j = f_i$. So, by lemma 6.7, also f_{ij} is a factor. Hence $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$, with $e_{ii+1} := \underline{f_{ii+1}}$, is an ω-chain in S^{ef} where each X_i is finite. Moreover, $\big((X, T), (e_i, f_i)\big)$, with $e_i := \underline{f_i}$, is a cocone to the ω-chain.

So it remains to show that the cocone is limiting: If $\big((Y, S), (g_i, h_i)\big)$ is another cocone, then (Y, h_i) is a cone of (X_i, f_{ij}), so there is a unique continuous surjection $u : Y \to X$ commuting with the f_i. Hence it suffices to show that u is a factor from (Y, S) to (X, T).

Concerning equivariance 2, assume $x \xrightarrow{T} x'$, and find $y, y' \in Y$ with $u(y) = x$, $u(y') = x'$, and $y \xrightarrow{S} y'$. Consider

$$F_i := \big\{ (y, y') \in Y \times Y : h_i(y) = f_i(x) \text{ and } h_i(y') = f_i(x') \text{ and } y \xrightarrow{S} y' \big\}.$$

It is readily checked that F_i is closed and nonempty, and if $i \leqslant j$, then $F_i \supseteq F_j$. Hence, by compactness, there is $(y, y') \in \bigcap_i F_i$, so $y \xrightarrow{S} y'$ and, for all $i \in \omega$, $f_i(u(y)) = h_i(y) = f_i(x)$ and $f_i(u(y')) = h_i(y') = f_i(x')$, so, since (X, f_i) is the projective limit, $u(y) = x$ and $u(y') = x'$, as needed.

Concerning equivariance 3, assume $y \xrightarrow{u} x$ and $y \xrightarrow{S} y'$, and find $x' \in X$ with $x \xrightarrow{T} x'$ and $y' \xrightarrow{u} x'$. Consider

$$F_i := \big\{ (x, x') \in X : x \xrightarrow{T} x' \text{ and } f_i(x) = h_i(y) \text{ and } f_i(x') = h_i(y') \big\}.$$

Again, it is readily checked that F_i is closed and nonempty, and if $i \leqslant j$, then $F_i \supseteq F_j$. Hence, by compactness, there is $(x, x') \in \bigcap_i F_i$. So $x \xrightarrow{T} x'$. Moreover, $x = u(y)$

because, for any i, we have $f_i(x) = h_i(y) = f_i(u(y))$. Similarly, $x' = u(y')$. So $u(y) \xrightarrow{T} x'$ and $y' \xrightarrow{u} x'$, as needed. $\qquad\square$

To show the equivalence of the two senses of 'finite', we need two more lemmas. The first is a useful fact about algebroidal categories.

Lemma 6.9. *Let* C *be a category in which all morphisms are monic. Let* B *be a category-theoretically finite object of* C. *If* (B, f_i) *is the colimit of an ω-chain* $(A_i, f_{ii+1})_{i \in \omega}$ *in* C, *then some* $f_i : A_i \to B$ *is an isomorphism.*

Proof. By using $g = \mathrm{id}_B$ in the definition of being category-theoretically finite, there is, in particular, some i and a unique morphism $h : B \to A_i$ such that $g = f_i \circ h$. We claim that h is the inverse of $f_i : A_i \to B$. Since $\mathrm{id}_B = g = f_i \circ h$, it remains to show $h \circ f_i = \mathrm{id}_{A_i}$. Indeed, note that $f_i \circ \mathrm{id}_{A_i} = \mathrm{id}_B \circ f_i = f_i \circ h \circ f_i$. Since f_i is monic, this implies $\mathrm{id}_{A_i} = h \circ f_i$, as needed. $\qquad\square$

The second lemma is a standard fact about the interplay between partitions and projections in inverse limits. For completeness, we add a proof in appendix A.3.

Lemma 6.10. *Let* S^{ef} *be a full subcategory of* allSys. *Let* $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$ *be an ω-chain in* S^{ef}. *Let* $\big((X, T), (e_i, f_i)\big)$ *be a colimit in* S^{ef}. *Assume* $\mathcal{C} = \{C_1, \ldots, C_n\}$ *is a finite partition of* X *consisting of closed sets. Then there is* $i \in \omega$ *such that* X/f_i *(i.e., the quotient under the equivalence relation* $x \equiv_i x'$ *iff* $f_i(x) = f_i(x')$*) refines* $\mathcal{C}$.[15]

Proposition 6.11. *Let* S^{ef} *be a full subcategory of* allSys *with property (2). Let* (X, T) *be in* S^{ef}. *Then* (X, T) *is category-theoretically finite in* S^{ef} *iff* X *is a finite set.*

Proof. ($\Rightarrow$) Assume (X, T) is category-theoretically finite. By proposition 6.8, (X, T) is the colimit in S^{ef} of an ω-chain $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)$ of systems with finite state spaces. By lemma 6.9, (X, T) is isomorphic to some (X_i, T_i), so, like X_i, also X is a finite set.

($\Leftarrow$) Assume X is finite. Let $((Y_i, S_i), (e_{ii+1}, f_{ii+1}))_i$ be an ω-chain with colimit $((Y, S), (e_i, f_i))$ in S^{ef} and $(g, h) : (X, T) \to (Y, S)$ a morphism. We need to find $n \in \omega$ such that, for all $i \geqslant n$, there is a unique morphism $(g', h') : (X, T) \to (Y_i, S_i)$ such that $(g, h) = (e_i, f_i) \circ (g', h')$.

By lemma 6.10, there is $n \in \omega$ such that $\mathcal{D} := Y/f_n$ refines the closed and finite partition $\mathcal{C} := \{g(x) : x \in X\}$ of Y. We claim:

(∗) For each $y_n \in Y_n$, there is a unique $x \in X$ such that $f_n^{-1}(y_n) \subseteq g(x)$. We write $u(y_n) := x$.

Indeed, regarding existence, given $y_n \in Y_n$, since $\{f_n^{-1}(y_n) : y_n \in Y_n\}$ refines $\mathcal{C}$, there is $x \in X$ with $f_n^{-1}(y_n) \subseteq g(x)$. Regarding uniqueness, if $x, x' \in X$ are such that $f_n^{-1}(y_n) \subseteq g(x)$ and $f_n^{-1}(y_n) \subseteq g(x')$, then, by surjectivity of f_n, we have $\emptyset \neq f_n^{-1}(y_n) \subseteq g(x) \cap g(x')$, so, since g is partition-injective, $x = x'$.

By (∗), we have the function $u : Y_n \to X$. For the functions $f_n : Y \to Y_n$ and $h : Y \to X$, we have $h = u \circ f_n$: Given $y \in Y$, we need to show that $x := h(y)$ is such that $f_n^{-1}(f_n(y)) \subseteq g(x)$. Since $\mathcal{D}$ refines $\mathcal{C}$, it suffices to show $f_n^{-1}(f_n(y)) \cap g(x) \neq \emptyset$.

---

[15]If $\mathcal{C}$ and $\mathcal{D}$ are two partitions of an underlying set X, we say that $\mathcal{D}$ *refines* $\mathcal{C}$ if, for every $D \in \mathcal{D}$, there is $C \in \mathcal{C}$ such that $D \subseteq C$.

Indeed, we trivially have $y \in f_n^{-1}(f_n(y))$ and, qua ef-pair, we have $y \in g \circ h(y) = g(x)$.

By lemma 6.7, since f_n and h are factors, also u is. This finishes the proof: Given $i \geqslant n$, consider $(g', h') := (e_{ni}, f_{ni}) \circ (\underline{u}, u) : (X, T) \to (Y_i, S_i)$. Then $(e_i, f_i) \circ (g', h') = (e_n, f_n) \circ (\underline{u}, u) = (g, h)$. And (g', h') is unique with this property since (e_i, f_i) is monic (proposition 6.5). $\qquad\square$

Now we know that a full subcategory S^{ef} of allSys with property 1 and 2 is semi-algebroidal. And it immediately is algebroidal:

Proposition 6.12. *Let S^{ef} be a full subcategory of allSys with property (2). Then S^{ef} contains up to isomorphism only countable many category-theoretically finite objects, and between any two such objects there are only countably many morphisms.*

Proof. Immediate, since the category-theoretically finite objects in S^{ef} are the systems with a finite state space. $\qquad\square$

7. The universal and homogeneous nondeterministic system

Now we can show the universality result for nondeterministic systems (subsection 7.1). Then we comment on the nature of the universal system (subsection 7.2).

7.1. **Proof of theorem 4.2.** Having established the algebroidality of $\mathsf{Sys}^{\mathrm{ef}}$, it only remains, according to our proof strategy (section 4.2), to establish the joint embedding property and the amalgamation property of $\mathsf{Sys}^{\mathrm{ef}}_{\mathrm{fin}}$. (To recall, $\mathsf{Sys}^{\mathrm{ef}}_{\mathrm{fin}}$ is the full subcategory of $\mathsf{Sys}^{\mathrm{ef}}$ consisting of the category-theoretically finite objects in $\mathsf{Sys}^{\mathrm{ef}}$.) Then theorem 4.2 follows from the Fraïssé theorem. Fortunately, here this is straightforward.

We start with the amalgamation property. We can use the usual pullback (or fiber product) construction for sets.

Proposition 7.1. $\mathsf{Sys}^{\mathrm{ef}}_{\mathrm{fin}}$ *has the amalgamation property.*

Proof. Let (X, T), (Y_0, S_0), and (Y_1, S_1) be in $\mathsf{Sys}^{\mathrm{ef}}_{\mathrm{fin}}$ and let $(e_0, f_0) : (X, T) \to (Y_0, S_0)$ and $(e_1, f_1) : (X, T) \to (Y_1, S_1)$ be morphisms in $\mathsf{Sys}^{\mathrm{ef}}_{\mathrm{fin}}$. Consider

$$Z := \big\{ (y_0, y_1) \in Y_0 \times Y_1 : f_0(y_0) = f_1(y_1) \big\},$$

Define $R : Z \rightrightarrows Z$ by

$$R(y_0, y_1) := \big\{ (y_0', y_1') \in Z : y_0' \in S_0(y_0) \text{ and } y_1' \in S_1(y_1) \big\}.$$

So (Z, R) is a finite dynamical system, and it is nontrivial since (X, T) is nontrivial and f_0 and f_1 are factors. It is readily checked that the projections $\pi_0 : Z \to Y_0$ and $\pi_1 : Z \to Y_1$ to the first and second component, respectively, are factors. Then $(\underline{\pi_0}, \pi_0) : (Y_0, S_0) \to (Z, R)$ and $(\underline{\pi_1}, \pi_1) : (Y_1, S_1) \to (Z, R)$ are morphisms in $\mathsf{Sys}^{\mathrm{ef}}_{\mathrm{fin}}$ and, by construction, $f_0 \circ \pi_0 = f_1 \circ \pi_1$, so $(\underline{\pi_0}, \pi_0) \circ (e_0, f_0) = (\underline{\pi_1}, \pi_1) \circ (e_1, f_1)$, as needed.[16] $\qquad\square$

---

[16]This proof does not carry over to total systems: The dynamics R is nontrivial, but it need not be total, even if the systems to be amalgamated are. In section 8.2, we show that in fact no other construction can work, because the total systems do not have a universal system.

Regarding the joint embedding property, it is well-known that this is implied by the amalgamation property if there is an initial object.

Proposition 7.2. $\mathsf{Sys}^{\mathsf{ef}}$ *has an initial object. Since* $\mathsf{Sys}^{\mathsf{ef}}_{\mathsf{fin}}$ *has the amalgamation property, this implies that it has the joint embedding property.*

Proof. The initial object is the system (X, T) consisting of one state $*$ with a self-loop, i.e., $T(*) = \{*\}$. Given any nontrivial system (Y, S), the unique ef-pair $(e, f) : (X, T) \to (Y, S)$ is given by f mapping every $y \in Y$ to $*$, which is readily checked to be a factor. $\square$

This finishes the proof of theorem 4.2. As noted, the theorem generalizes the result for profinite undirected graphs. More precisely:

Remark 7.3. As mentioned in section 2, from a dynamical systems perspective, we consider *directed* edges $x \xrightarrow{T} x'$, rather than *undirected* (i.e., reflexive and symmetric) edges. But, since theorem 6.2 applies to any subcategory of $\mathsf{Sys}^{\mathsf{ef}}$, we can, as mentioned in section 2, also recover the existence result of the projectively universal homogeneous (undirected) graph [20, 46, 63]. Specifically, call a system (X, T) in $\mathsf{Sys}^{\mathsf{ef}}$ a *graph* if, for all $x, x' \in X$, we have $x \xrightarrow{T} x$ (reflexivity) and, if $x \xrightarrow{T} x'$, then $x' \xrightarrow{T} x$ (symmetry). Let G^{ef} be the full subcategory of $\mathsf{Sys}^{\mathsf{ef}}$ consisting of those systems that are graphs. Then conditions 1 and 2 of theorem 6.2 are readily checked, so G^{ef} is algebroidal and the full subcategory $\mathsf{G}^{\mathsf{ef}}_{\mathsf{fin}}$ of category-theoretically finite objects consists of precisely the finite graphs. The proofs of propositions 7.1 and 7.2 still work and yield the amalgamation property and the joint embedding property for $\mathsf{G}^{\mathsf{ef}}_{\mathsf{fin}}$. Hence the Fraïssé limit theorem yields that G^{ef} has a G^{ef}-universal and $\mathsf{G}^{\mathsf{ef}}_{\mathsf{fin}}$-homogeneous system.

7.2. **The nature of the universal system.** We end this section with some comments about the nature of the universal nondeterministic system (U, T): its symmetry, universality, complexity, and its status as an analog computer.

Symmetry. The homogeneity property, that makes (U, T) unique, is quite remarkable. In section 4.1, we motivated it as saying that 'every part of (U, T) looks the same'. Another way of saying this is that (U, T) is 'maximally symmetric': whenever we have two ways of embedding a finite dynamical system into the universal system, there is a symmetry (i.e. isomorphism) of the universal system that transforms one embedding into the other. Thus, the embedding of each finite system into the universal one is, in a sense, unique up to isomorphism.

Universality. In which of the senses that we have identified in section 2 is (U, T) universal?

(4) By construction, (U, T) is factor universal: every dynamical system is a factor of (U, T).

(3) By construction, (U, T) also is embedding universal, provided that 'embedding' is understood as a multi-valued function. This sense of 'embedding' comes natural in a multi-valued setting, but it is not the usual 'injective function' idea of an embedding.

(1) Since every Turing machine can be seen as a dynamical system, it is a factor of and embeds into the universal system. Such an ef-pair can be seen as a simulation (see remark 3.10). Thus, in this sense, (U, T) is Turing universal.

(2) Whether (U, T) is approximation universal depends on the understanding of 'approximation'. Simulation understood as an ef-pair is, in a sense, best possible approximation, so in this sense (U, T) is approximation universal. But a sense of a universal function approximator (as with neural networks or differential equations) is not available in our general setting here, as there are no associated real-valued functions that can approximate a given function.

Complexity. One would expect the universal system to be a very 'complicated' or 'complex' object since it contains every other system. However, it actually is surprisingly simple, as we will show now: Its state space is a Cantor space (i.e., isomorphic to 2^ω), and its dynamics has no complicated long-term behavior because the orbits consist of at most two states. The latter, surprising fact is known for the universal profinite graph, proven by Camerlo [20, prop. 12] (also discussed by [46]). However, in our systems setting, a much simpler proof is available.

First, the claim about the state space [cf. 46, p. 13].

Proposition 7.4. *The state space of the universal system (U, T) from theorem 4.2 is a Cantor space (i.e., isomorphic to 2^ω).*

Proof. By Brouwer's theorem [e.g. 72, thm. 7.4], a space is a Cantor space if it is a nonempty, compact, zero-dimensional, metrizable space without isolated points (i.e., no points whose singleton is open). Since (U, T) is in $\mathsf{Sys}^{\mathsf{ef}}$, it hence remains to show that U has no isolated points.

Toward a contradiction, assume $x_0 \in U$ is isolated. Consider the two-element discrete space $Y = \{y_0, y_1\}$ and the function $f : U \to Y$ which maps x_0 to y_0 and any $x \in U \setminus \{x_0\}$ to y_1—so f is a continuous function. Define the dynamics S on Y by: $y \xrightarrow{S} y'$ iff there is $x, x' \in U$ with $f(x) = y$, $f(x') = y'$, and $x \xrightarrow{T} x'$. So $f : (U, T) \to (Y, S)$ is a factor. Now define the system (Z, R) by 'doubling' y_0: let $Z := \{y_0, z_0, y_1\}$ be a three-element discrete space and let $R(z) := S(z)$ if $z \in \{y_0, y_1\}$ and $R(z_0) := \emptyset$. Define $g : Z \to Y$ by mapping $y_0 \mapsto y_0$, $z_0 \mapsto y_0$, and $y_1 \mapsto y_1$. So $g : (Z, R) \to (Y, S)$ is a factor. By universality, there is a factor $h : (U, T) \to (Z, R)$, and, by homogeneity, we can choose h such that $g \circ h = f$.[17] Hence, since $g(h(x_0)) = f(x_0) = y_0$ and $g^{-1}(y_0) = \{y_0, z_0\}$, we must have either $h(x_0) = y_0$ or $h(x_0) = z_0$. Let's consider the first case (the other is analogous). By surjectivity, there is $x \in U$ with $h(x) = z_0$. But then $x \in U \setminus \{x_0\}$, so $y_1 = f(x) = g(h(x)) = g(z_0) = y_0$, contradiction. $\qquad\square$

Now, the claim about the dynamics. Let us first fix some terminology. Let (X, T) be a system. An *orbit* in (X, T) is a sequence $(x_n)_{n=0}^{\kappa}$ in X with $\kappa \leqslant \omega$ and $x_n \xrightarrow{T} x_{n+1}$ for all $n < \kappa$. So the orbit is finite if $\kappa < \omega$ or infinite if $\kappa = \omega$. If $x \xrightarrow{T} y$, we call y a *successor state* of state x. We call a state $x \in X$ a *dead end* (or *sink*) if x has no successor states. A *loop* in (X, T) is a finite orbit $(x_0, \ldots, x_\kappa)$ with $x_0 = x_\kappa$ and $\kappa \geqslant 1$. It is a *self-loop* if $\kappa = 1$.

---

[17]This is the idea of (U, T) being a 'saturated object', which we will define and use again in the proof of proposition 10.4.

Proposition 7.5. *Orbits in the universal system* (U, T) *have at most* 2 *states. In other words, for every state* $x \in U$, *either* x *is a dead end or all its successor states are dead ends. In particular,* (U, T) *does not have any loops (also no self-loops).*

Proof. Toward a contradiction, assume there was an orbit with three states: $x_0 \xrightarrow{T} x_1 \xrightarrow{T} x_2$. Consider the system (Y, S) with $Y = \{y_0, y_1\}$ and $S(y_0) = \{y_1\}$ and $S(y_1) = \emptyset$. It is a non-trivial system and hence is in $\mathsf{Sys}^{\mathsf{ef}}$. So there is a factor $f : (U, T) \to (Y, S)$. We cannot have $f(x_0) = y_1$: Otherwise, since f is a factor and $x_0 \xrightarrow{T} x_1$, there must be $y' \in Y$ with $f(x_1) = y'$ and $y_1 \xrightarrow{S} y'$, but y_1 is a dead end. Hence $f(x_0) = y_0$. Similarly, we cannot have $f(x_1) = y_1$. Hence $f(x_1) = y_0$. But then, since $x_0 \xrightarrow{T} x_1$ and f is a factor, we have $f(x_0) \xrightarrow{S} f(x_1)$, and hence $y_0 \xrightarrow{S} y_0$, contradiction. $\square$

Analog computer? In section 2, we said that every analog computer can be seen as a dynamical system. But we also noted that the converse is a difficult philosophical question. Accordingly, it is out of scope of this paper to discuss whether the universal system (U, T) is an analog computer, and hence is a universal analog computer. We only note that the simple nature of (U, T) seems to intuitively speak in favor of (U, T) being an analog computer—if it were an extremely intricate and complicated system, it would seem less plausible to view it as a 'computing machine'. We pick this up again as an interesting open question in section 11.

8. UNIVERSALITY FOR DETERMINISTIC SYSTEMS?

We have established the existence of a universal nondeterministic system, which even is unique in also being homogenous. Now, for the remainder of this paper, we consider the special case of deterministic systems. After all, they are more prominently researched than nondeterministic systems. One might expect similar universality results, since there are known ways to transform a nondeterministic system into a deterministic one, which we review in section 8.1. However, the connection is not close enough: in section 8.2, we see that there cannot be a universal deterministic system. But we are very close to universality: in section 8.3, we show that the category $\mathsf{detSys}^{\mathsf{ef}}$ still is semi-algebroidal, with category-theoretically finite objects being shifts, which have the joint embedding property and the amalgamation property. So all that is missing for universality is the countability of the category-theoretically finite objects. In the next section, we hence consider how to get subcategories of $\mathsf{detSys}^{\mathsf{ef}}$ with universal systems.

8.1. **Deterministic vs nondeterministic systems.** There are two standard ways to transform a nondeterministic system into a deterministic one. Conceptually, these two ways are reminiscent of a right and a left adjoint to the inclusion $\mathsf{Inc} : \mathsf{detSys}^{\mathsf{ef}} \hookrightarrow \mathsf{Sys}^{\mathsf{ef}}$, but they do *not* formally satisfy all the requirements for an adjunction.

The first way (cf. 'right adjoint' to Inc) is reminiscent of how the powerset functor is right adjoint to the inclusion of the category of sets and functions in the category of sets and relations. Given a nondeterministic system (X, T), we already saw in section 5 that we can consider the 'deterministic system' $(F(X), F(T))$, where $F(X)$ is the set of closed subsets of X and $F(T) : F(X) \to F(X)$ maps A to $T[A]$. But the

problem was how to topologize $F(X)$: if we use the Vietoris topology, $F(X)$ is again a second-countable Stone space but $F(T)$ need not be continuous; if we use the upper Vietoris topology, $F(T)$ is continuous but then X in general only is a spectral space. Moreover, the natural choice for a universal morphism $\epsilon_{(X,T)} : (F(X), F(T)) \to (X, T)$ is a multifunction e mapping $A \in F(X)$ to $A \subseteq X$ or, in the other direction, a function f mapping $x \in X$ to $\{x\} \in F(X)$. However, these do not form an ef-pair (e.g., f is not surjective).

The second way (cf. 'left adjoint' to Inc) is reminiscent of the unraveling functor for labeled transition systems (or Kripke frames).[18] Given a nondeterministic system (X, T), define $\overline{X}$ to be the set of all infinite orbits. To recall, an infinite orbit—which, for brevity, we call *path*—is a sequence $\overline{x} = (x_n)_{n \in \omega}$ in X such that, for all $n \in \omega$, we have $x_n \xrightarrow{T} x_{n+1}$. Hence $\overline{X}$ is a subset of $\prod_\omega X$ and we equip it with the subset topology. Define $\sigma(\overline{x})$ as the shift of $\overline{x}$, i.e., the sequence $(x_{n+1})_{n \in \omega}$, which again is a path. Write $\mathsf{Path}(X, T) := (\overline{X}, \sigma)$. Moreover, the natural choice for a universal morphism $\eta_{(X,T)} : (X, T) \to \mathsf{Path}(X, T)$ is the ef-pair whose factor $f : \overline{X} \to X$ projects to the first component, i.e., $f((x_n)) := x_0$. The next proposition shows that, this is all well-defined for total nondeterministic systems. (The straightforward proof is in appendix A.4.)

Proposition 8.1. *Let* (X, T) *be a system in* $\mathsf{Sys}^{\mathsf{ef}}$. *Then*

(1) $\mathsf{Path}(X, T)$ *is in* $\mathsf{detSys}^{\mathsf{ef}}$ *iff* (X, T) *has an infinite path (which, in particular, is the case if* (X, T) *is total).*
(2) *The projection to the first component* $\pi_0 : \mathsf{Path}(X, T) \to (X, T)$ *is a factor iff* (X, T) *is total.*

However, as we will see in corollary 8.4 below, neither Path nor any other functor can be a left adjoint to the inclusion $\mathsf{Inc} : \mathsf{detSys}^{\mathsf{ef}} \hookrightarrow \mathsf{Sys}^{\mathsf{ef}}$.

8.2. **No universal deterministic system.** Despite the close connections between deterministic systems and nondeterministic systems, the former cannot have a universal system, in contrast to the latter.

Proposition 8.2. *The category* $\mathsf{detSys}^{\mathsf{ef}}$ *does not have a universal system.*

Proof. The claim follows from two results by Darji and Matheron [29]. For the purposes of this proof, let us call (X, T) a compact (metric) system if X is a compact (metric) space and $T : X \to X$ a continuous function. A Cantor system is a pair $(2^\omega, S)$ where 2^ω is the Cantor space and $S : 2^\omega \to 2^\omega$ a continuous function.

(1) [29, Corollary 4.2]: If (X, T) is a compact metric system, then there is a Cantor system $(2^\omega, S)$ and a factor $f : (2^\omega, S) \to (X, T)$. (This extends the universality of Cantor space from spaces to systems.)
(2) [29, Remark 4.6]: Assume Z is a compact system that is universal for compact metric systems, i.e., if (X, T) is a compact metric system, there is a factor $f : (Z, R) \to (X, T)$. Then Z cannot be metrizable. (In fact, (Z, R) cannot even factor onto all rotations of the circle.)

---

[18]See, e.g., [120, p. 30] or [15, p. 220].

Toward a contradiction, assume (Z, R) is universal in $\mathsf{detSys}^{\mathsf{ef}}$. Then (Z, R) is a compact system, and it is universal for compact metric systems (not just zero-dimensional ones): If (X, T) is a compact metric system, then, by 1, there is a Cantor system $(2^\omega, S)$ and a factor $f : (2^\omega, S) \to (X, T)$. Hence $(2^\omega, S)$ is in $\mathsf{detSys}^{\mathsf{ef}}$, so, by universality, there is a factor $g : (Z, R) \to (2^\omega, S)$. In sum, there is a factor $f \circ g : (Z, R) \to (X, T)$, as needed. However, then 2 implies that Z is not metrizable, in contradiction to Z being a Polish space. $\square$

The nonexistence of a universal system in $\mathsf{detSys}^{\mathsf{ef}}$ can also be extended to some categories of nondeterministic systems, namely $\mathsf{totalSys}^{\mathsf{ef}}$. In theorem 6.2, we have seen that this category is algebroidal. But, as we see now, $\mathsf{totalSys}^{\mathsf{ef}}$ cannot have a universal system. In particular, the finite total nondeterministic systems also cannot have the amalgamation property.

Corollary 8.3. *The category* $\mathsf{totalSys}^{\mathsf{ef}}$ *does not have a universal system.*

Proof. Assume (Z, R) were universal in $\mathsf{totalSys}^{\mathsf{ef}}$. By proposition 8.1, we can define the deterministic system $\mathsf{Path}(Z, R)$ and have a factor $\pi_0 : \mathsf{Path}(Z, R) \to (Z, R)$. So $\mathsf{Path}(Z, R)$ is in $\mathsf{detSys}^{\mathsf{ef}}$ and it is universal in there: if (X, T) is a deterministic system, it is, in particular, in $\mathsf{totalSys}^{\mathsf{ef}}$, so, by assumption, there is a factor $f : (Z, R) \to (X, T)$, so there indeed is a factor $f \circ \pi_0 : \mathsf{Path}(Z, R) \to (X, T)$. This contradicts proposition 8.2. $\square$

Another corollary is that, despite the close connections observed in section 8.1, the deterministic systems do not form a reflective subcategory of the nondeterministic systems. In other words:

Corollary 8.4. *The inclusion* $\mathsf{Inc} : \mathsf{detSys}^{\mathsf{ef}} \hookrightarrow \mathsf{Sys}^{\mathsf{ef}}$ *does not have a left adjoint.*

Proof. Toward a contradiction, assume there is a left adjoint (aka reflector) $R : \mathsf{Sys}^{\mathsf{ef}} \to \mathsf{detSys}^{\mathsf{ef}}$ with unit η. Let (U, T) be the universal system in $\mathsf{Sys}^{\mathsf{ef}}$ (from theorem 4.2). We claim that $R(U, T)$ is universal in $\mathsf{detSys}^{\mathsf{ef}}$, hence contradicting proposition 8.2. Indeed, let (Y, S) be in $\mathsf{detSys}^{\mathsf{ef}}$. We need to find an ef-pair $(Y, S) \to R(U, T)$. By universality of (U, T), there is an ef-pair $(e, f) : (Y, S) \to (U, T)$. Hence, qua left adjoint, there is a unique ef-pair $R(e, f) = (g, h)$ making the following diagram commute:

$$
\begin{array}{ccc}
(Y, S) & \xrightarrow{\eta_{(Y,S)}} & R(Y, S) \\
{\scriptstyle (e,f)}\Big\downarrow & & \Big\downarrow{\scriptstyle (g,h)} \\
(U, T) & \xrightarrow{\eta_{(U,T)}} & R(U, T)
\end{array}
$$

Hence $(g, h) \circ \eta_{(Y,S)} : (Y, S) \to R(U, T)$ is the desired ef-pair. $\square$

8.3. **The category of deterministic systems still is semi-algebroidal.** In order to understand the nonexistence result from the preceding subsection, we now analyze where exactly the Fraïssé-limit method—that worked for nondeterministic systems—breaks down for deterministic systems. As announced, we will find that $\mathsf{detSys}^{\mathsf{ef}}$ still is semi-algebroidal, and the category-theoretically finite objects are precisely the shifts. (We already saw in example 6.4 that the category-theoretically finite objects cannot be those systems with finite state spaces: with those, $\mathsf{detSys}^{\mathsf{ef}}$

cannot be semi-algebroidal.) Moreover, a later result (lemma 10.2) will show that the category-theoretically finite objects of $\mathsf{detSys}^{\mathsf{ef}}$ also have the joint embedding property and the amalgamation property. So the only reason why the Fraïssé-limit method does not yield a universal system for $\mathsf{detSys}^{\mathsf{ef}}$ is that the countability requirement is violated: there are uncountably many shifts.

We start by recalling some terminology from dynamical systems theory [81]. A *finite topological generator* of a system (X, T) in $\mathsf{detSys}^{\mathsf{ef}}$ is a finite clopen partition $\mathcal{C}$ of X (i.e., $\mathcal{C}$ has finitely many cells and they are clopen) such that $\{T^{-n}(C) : C \in \mathcal{C}, n \in \omega\}$ generates the topology of X.[19] If (X, T) is in $\mathsf{detSys}^{\mathsf{ef}}$ and $\mathcal{C}$ is a finite clopen partition of X, the $\mathcal{C}$-*name* of $x \in X$ is the sequence (C_n) of $\mathcal{C}$-partition cells where, for each n, C_n is the cell with $T^n(x) \in C_n$. Then the set Y of all $\mathcal{C}$-names is a shift over alphabet $\mathcal{C}$ and the function $f : X \to Y$ that maps x to its $\mathcal{C}$-name is a factor. We call Y the $\mathcal{C}$-*shift* and f the $\mathcal{C}$-*factor*.

A well-known characterization of finite topological generators is the following [53]. To state it, recall a classic result of De Groot [31]: An ultrametric on a set X is a metric d on X which, instead of the usual triangle inequality, satisfies the strong triangle inequality $d(x, z) \leqslant \max(d(x, y), d(y, z))$. If X is a Hausdorff space with a countable base (U_n) of clopens, one can define a topology-generating ultrametric on X by $d(x, y) = \frac{1}{n}$ where n is the first n such that U_n separates x and y (i.e., $x \in U_n$ and $y \notin U_n$ or vice versa). In particular, if (X, T) is in $\mathsf{detSys}^{\mathsf{ef}}$, there is an ultrametric on X that generates the topology.

Proposition 8.5 (Hedlund's theorem [53]). *Let (X, T) be in $\mathsf{detSys}^{\mathsf{ef}}$. Let d be an ultrametric that generates the topology on X. The following are equivalent:*

(1) (X, T) has a finite topological generator.
(2) (X, T) is isomorphic to a shift.
(3) (X, T) is expansive, *i.e., there is $\epsilon > 0$ such that, for all $x \neq y$ in X, there is n such that $d(T^n(x), T^n(y)) \geqslant \epsilon$.*

With this terminology, we can formulate an analogue to theorem 6.2 for deterministic systems.

Theorem 8.6. *Let $\mathsf{DS}^{\mathsf{ef}}$ be a full subcategory of $\mathsf{detSys}^{\mathsf{ef}}$ such that*

(1) If $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$ is an ω-chain in $\mathsf{DS}^{\mathsf{ef}}$, then (X, T) is again in $\mathsf{DS}^{\mathsf{ef}}$, where X is defined as in equation 2 with canonical projections f_i and $T(\langle x_i : i \in \omega \rangle) := \langle T_i(x_i) : i \in \omega \rangle$.
(2) If (X, T) is in $\mathsf{DS}^{\mathsf{ef}}$, then it is the colimit of an ω-chain of shifts in $\mathsf{DS}^{\mathsf{ef}}$.

Then $\mathsf{DS}^{\mathsf{ef}}$ is semi-algebroidal and its category-theoretically finite objects are precisely those systems in $\mathsf{DS}^{\mathsf{ef}}$ that are isomorphic to a shift.

Before proving this, we note a sufficient condition for requirement 2 (proof below).

Lemma 8.7. *Let $\mathsf{DS}^{\mathsf{ef}}$ be a full subcategory of $\mathsf{detSys}^{\mathsf{ef}}$ such that: If (X, T) is in $\mathsf{DS}^{\mathsf{ef}}$ and $\mathcal{C}$ is a finite clopen partition of X, then the $\mathcal{C}$-shift is again in $\mathsf{DS}^{\mathsf{ef}}$. Then $\mathsf{DS}^{\mathsf{ef}}$ satisfies requirement 2 of theorem 8.6.*

---

[19]Recall that a collection $\mathcal{F}$ of subsets of a topological space X *generates* the topology τ of X if τ is the smallest topology containing all subsets of $\mathcal{F}$.

This sufficient condition is clearly satisfied for $DS^{ef} = detSys^{ef}$, so we in particular get the announced result of this subsection:

Corollary 8.8. *The category of deterministic systems $detSys^{ef}$ is semi-algebroidal and its category-theoretically finite objects are precisely the deterministic systems that are isomorphic to a shift.*

In the remainder of this subsection, we prove these results.

Proof of lemma 8.7. Let (X, T) be in DS^{ef} and show that it is the colimit of an ω-chain of shifts in DS^{ef}. Let $(\mathcal{C}_i)$ be a refining sequence of finite clopen partitions of X generating the topology. Define $f_i : (X, T) \to (X_i, T_i)$ as the $\mathcal{C}_i$-factor. By assumption, (X_i, T_i) is in DS^{ef}. Define $f_{ij} : X_j \to X_i$ by mapping a sequence (D_n) of $\mathcal{C}_j$-partition cells to the sequence (C_n) of $\mathcal{C}_i$-partition cells with $D_n \subseteq C_n$. Hence $f_{ii} = id$ and $f_{ij} = f_{ii+1} \circ \ldots \circ f_{j-1j}$, and it is also easily checked that $f_i = f_{ij} \circ f_j$. We show that (X, T) with the f_i is the limit of the ω-chain $\big((X_i, T_i), f_{ii+1}\big)$.

Note that $\big((X_i, T_i), f_{ii+1}\big)$ is indeed an ω-chain and $\big((X, T), f_i\big)$ a cone, because the $f_i : (X, T) \to (X_i, T_i)$ are factors and, since $f_i = f_{ij} \circ f_j$, lemma 6.7 implies that also f_{ij} is a factor. So we need to check that $\big((X, T), f_i\big)$ is limiting. If $\big((Y, S), g_i\big)$ is another cone, define the mediating morphism $u : Y \to X$ by mapping y to the x in $\bigcap_i F_i^y$ with $F_i^y := f_i^{-1}(\{g_i(y)\})$. We now show that this is well-defined and has the required properties.

Well-defined: It is straightforward to check that the F_i^y's for a descending sequence of nonempty closed sets, so, by compactness, $\bigcap_i F_i^y$ is indeed nonempty. Concerning the uniqueness of x, if $x \neq x'$ were in $\bigcap_i F_i^y$, there is, since $(\mathcal{C}_i)$ generates the topology, some i and $C \in \mathcal{C}_i$ such that $x \in C$ and $x' \notin C$. But then $f_i(x) = g_i(y) = f_i(x')$, so $C = f_i(x)_0 = f_i(x')_0 \ni x'$, contradiction.

Surjective: Let $x \in X$ and find $y \in Y$ with $u(y) = x$. Again it is straightforward to check that $G_i := g_i^{-1}(\{f_i(x)\})$ forms a descending sequence of nonempty closed sets, so, by compactness, there is $y \in \bigcap_i G_i$ and, by construction, $x \in \bigcap_i F_i^y$, so $u(y) = x$.

Commuting: Let $y \in Y$ and $i \in \omega$ and show $f_i(u(y)) = g_i(y)$. By definition, $u(y) \in \bigcap_i F_i^y$, so $f_i(u(y)) = g_i(y)$.

Equivariant: Let $y \in Y$ and show $u(S(y)) = T(u(y))$. So let $i \in \omega$ and show $T(u(y)) \in F_i^{S(y)}$. Indeed, we have

$$f_i(T(u(y))) = T_i(f_i(u(y))) = T_i(g_i(y)) = g_i(S(y)).$$

Unique: If $u' : (Y, S) \to (X, T)$ is another factor with $f_i \circ u' = g_i$ and $y \in Y$, we claim $u'(y) = u(y)$. Indeed, for $i \in \omega$, we have $f_i(u'(y)) = g_i(y)$, so $u'(y) \in F_i^y$. Since $u(y)$ is the unique element in $\bigcap_i F_i^y$, the claim follows. □

Lemma 8.9. *Let DS^{ef} be a full subcategory of $detSys^{ef}$. If (X, T) has a finite topological generator, then (X, T) is category-theoretically finite in DS^{ef}. If DS^{ef} satisfies requirement 2 of theorem 8.6, then the converse holds, too.*

Proof. ($\Rightarrow$) Assume $\mathcal{C}$ is a finite topological generator of (X, T). Moreover, let $\big((Y_i, S_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$ be an ω-chain in DS^{ef} with colimit (Y, S) and let $(g, h) : (X, T) \to (Y, S)$ be an ef-pair. We need to find $n \in \omega$ such that, for all $i \geqslant n$, there is a unique morphism $(g', h') : (X, T) \to (Y_i, S_i)$ such that $(g, h) = (e_i, f_i) \circ (g', h')$.

By lemma 6.10, there is $n \in \omega$ such that $\mathcal{D} := Y/f_n$ refines the closed and finite partition $\{h^{-1}(C) : C \in \mathcal{C}\}$. So for each $y_n \in Y_n$, there is a unique $C = C(y_n) \in \mathcal{C}$ such that $f_n^{-1}(y_n) \subseteq h^{-1}(C)$. Now we claim:

(a) For any $y \in Y$, $y_n \in Y_n$ and $x \in X$, if $f_n(y) = y_n$ and $x = h(y)$, then, for all k, the cell $C(S_n^k(y_n))$ is the one containing $T^k(x)$.

(b) For each $y_n \in Y_n$, there is a unique $x \in X$ such that, for all k, we have $T^k(x) \in C(S_n^k(y_n))$.

Indeed, to show (a), let k be given and let $C \in \mathcal{C}$ be such that $T^k(x) \in C$. We need to show that $C = C(S_n^k(y_n))$, i.e., that $f_n^{-1}(S_n^k(y_n)) \subseteq h^{-1}(C)$. By the refinement of the partitions, it suffices to show that $f_n^{-1}(S_n^k(y_n)) \cap h^{-1}(C) \neq \emptyset$. For this, note that $f_n(S^k(y)) = S_n^k(f_n(y)) = S_n^k(y_n)$ and $h(S^k(y)) = T^k(h(y)) = T^k(x) \in C$, so $S^k(y) \in f_n^{-1}(S_n^k(y_n)) \cap h^{-1}(C)$, as needed. To show (b), uniqueness follows since $\mathcal{C}$ is generator: if x and x' have the described property, they have the same $\mathcal{C}$-name, so they must be identical. For existence, there is, since f_n is surjective, some $y \in Y$ with $f_n(y) = y_n$. Set $x := h(y) \in X$. Then (a) implies $T^k(x) \in C(S_n^k(y_n))$ for all k.

Now, by (b), there is a function $u : Y_n \to X$ that maps y_n to the unique $x \in X$ mentioned in (b). For the functions $f_n : Y \to Y_n$ and $h : Y \to X$, note that $u \circ f_n = h$: Indeed, for $y \in Y$, we have, by (a) and writing $y_n := f_n(y)$ and $x := h(y)$, that $T^k(x) \in C(S_n^k(y_n))$ for all k, i.e., $u(y_n) = x$, so $h(y) = u(f_n(y))$. Hence, by lemma 6.7, $u : Y_n \to X$ is a factor. This finishes the proof: Given $i \geqslant n$, consider $(g', h') := (e_{ni}, f_{ni}) \circ (u, u) : (X, T) \to (Y_i, S_i)$. Then $(e_i, f_i) \circ (g', h') = (e_n, f_n) \circ (u, u) = (g, h)$. And (g', h') is unique with this property since (e_i, f_i) is monic.

($\Leftarrow$) Now assume that $\mathsf{DS}^{\mathsf{ef}}$ satisfies requirement 2 and let (X, T) be category-theoretically finite. So (X, T) is a colimit of an ω-chain $((X_i, T_i), (e_{ii+1}, f_{ii+1}))$ of shifts in $\mathsf{DS}^{\mathsf{ef}}$. By lemma 6.9, (X, T) is isomorphic to some (X_i, T_i), which, qua shift, has a finite topological generator (proposition 8.5). □

Proof of theorem 8.6. Let $\mathsf{DS}^{\mathsf{ef}}$ be a full subcategory of $\mathsf{detSys}^{\mathsf{ef}}$ with properties 1 and 2. By lemma 8.9, the category-theoretically finite objects of $\mathsf{DS}^{\mathsf{ef}}$ are precisely those which are isomorphic to a shift. It remains to show semi-algebroidality. First, all morphisms in $\mathsf{DS}^{\mathsf{ef}}$ are monic by proposition 6.5. Second, if (X, T) is in $\mathsf{DS}^{\mathsf{ef}}$, then, by 2, it is a colimit of an ω-chain of shifts in $\mathsf{DS}^{\mathsf{ef}}$, i.e., of category-theoretically finite objects of $\mathsf{DS}^{\mathsf{ef}}$. Third, if $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$ is an ω-chain of (finite) objects in $\mathsf{DS}^{\mathsf{ef}}$, then (X, T), as described as in requirement 1, is in $\mathsf{DS}^{\mathsf{ef}}$, and by proposition 6.6, it is the colimit of the chain in $\mathsf{DS}^{\mathsf{ef}}$. □

9. ALGEBROIDAL CATEGORIES OF DETERMINISTIC SYSTEMS

We have seen that the category $\mathsf{detSys}^{\mathsf{ef}}$ fails to be algebroidal because its collection of category-theoretically finite objects—namely, the set of all shifts—is uncountable. As a result, the Fraïssé-limit method cannot be applied, and indeed there cannot be a universal deterministic system. So a natural question is whether we still get algebroidal *sub*categories of deterministic systems that then, further, also have universal and homogeneous systems.

As described in section 4.1, our strategy is to choose a countable class S of shifts and consider the full subcategory $\mathsf{DS}^{\mathsf{ef}}(S)$ of $\mathsf{detSys}^{\mathsf{ef}}$ consisting of those systems

that are limits of shifts in S. We defined $\mathsf{DS}^{\mathsf{ef}}(S)$ in definition 4.3 and called its objects ω-proshifts over S. Thus, intuitively, the shifts in S are our basic building blocks with which we can build more complex systems, namely the ω-proshifts of over S.

Remark 9.1. Such constructions of categories are common in mathematics, because of their desirable properties. For example, in topology, Stone spaces (aka profinite spaces) are the limits of finite discrete spaces; similarly for profinite groups and profinite graphs. In domain theory, bifinite domains are the colimits along embedding-projection pairs of finite partial orders with a least element. The general idea is that of an Ind-completion (resp., Pro-completion), i.e., the 'closure' of a category under filtered colimits (resp., limits) [67, ch. 6]. Given remark 3.13, we can use Ind- and Pro-completions interchangeably here. However, one difference in our definition is that we only consider ω-chains and not any filtered diagram. We stress this by adding the prefix 'ω' to 'proshift'.

In this section, we show that this strategy indeed yields algebroidal subcategories of deterministic systems:

Theorem 9.2. *Let S be a countable collection of shifts. Then $\mathsf{DS}^{\mathsf{ef}}(S)$ is an algebroidal category and its category-theoretically finite objects are, up to isomorphism, exactly those that are in S. Moreover, $\mathsf{DS}^{\mathsf{ef}}(S)$ contains a $\mathsf{DS}^{\mathsf{ef}}(S)$-universal and S-homogeneous deterministic system (U, T) iff*

(1) if $X_0, X_1 \in S$, there is $Y \in S$ and there are factors $f_0 : Y \to X_0$ and $f_1 : Y \to X_1$.

(2) if $X, Y_0, Y_1 \in S$ and $f_0 : Y_0 \to X$ and $f_1 : Y_1 \to X$ are factors, then there is $Z \in S$ and there are factors $g_0 : Z \to Y_0$ and $g_1 : Z \to Y_1$ such that $f_0 \circ g_0 = f_1 \circ g_1$.

We prove the theorem in the remainder of this section. In the next section, we then apply it to the choice of S as the class of shifts of finite type and the class of sofic shifts, respectively.

For the proof, we need the Curtis-Lyndon-Hedlund Theorem (see, e.g., [81, thm. 6.2.9]) in the following version. To fix some terminology: Let X be a shift over alphabet A. Write $B_n(X)$ for the set of all words $w \in A^n$ that occur in some $x \in X$ (i.e., there is k such that $w = (x(k), \ldots, x(k + n - 1))$). For a word $w \in A^n$, we call $C_w(X) := \{x \in X : w = (x(0), \ldots, x(n - 1))\}$ the *cylinder set* of w in X.

Lemma 9.3 (Curtis-Lyndon-Hedlund Theorem). *Let X and Y be shifts over alphabets A and B, respectively. Let $f : X \to Y$ be a continuous function that commutes with the shifts (i.e., $f \circ \sigma = \sigma \circ f$). Then there is $N \in \omega$ such that f is an N-sliding block code, i.e., there is a function $\varphi : B_N(X) \to B$ (aka block map) such that, for all $x \in X$,*

$$f(x)(k) = \varphi\big((x(k), \ldots, x(k + N - 1))\big) \qquad (k \in \omega).$$

Moreover, f also is an n-sliding block code for all $n \geqslant N$.

For completeness, we add a proof of this in appendix A.5. Next, we need the simple but important observation that generating new systems with shifts does *not* add any new shifts.

Lemma 9.4. *Let S be a collection of shifts and let X be a shift. Then X is in $\mathsf{DS}^{\mathsf{ef}}(S)$ iff X is isomorphic to a system in S.*

Proof. The right-to-left implication is trivial. For the other direction, assume X is a colimit of an ω-chain of shifts X_i in S. By lemma 8.9, since X is a shift—and hence has a finite topological generator—X is category-theoretically finite in $\mathsf{DS}^{\mathsf{ef}}(S)$. Since all morphisms in $\mathsf{DS}^{\mathsf{ef}}(S)$ are monic (proposition 6.5), lemma 6.9 implies that X hence is isomorphic to some X_i, which is in S. $\qquad\square$

Finally, we need a basic category-theoretic fact: that $\mathsf{DS}^{\mathsf{ef}}(S)$ is closed under taking colimits of ω-chains. In other words, a colimit of systems which are colimits of shifts is itself a colimit of shifts. That is well-known for Ind-completions [67, ch. 6]. A sketch of an elementary proof is in appendix A.5.

Lemma 9.5. *Let C be a category. Let A be a colimit of an ω-chain with objects A_i. Assume each A_i is, in turn, a colimit of an ω-chain with objects A_i^j, such that each A_i^j is category-theoretically finite. Then A also is a colimit of an ω-chain with objects that are among the A_i^j's.*

Now we are ready to prove theorem 9.2.

Proof of theorem 9.2. We want to apply theorem 8.6, so we need to show that $\mathsf{DS}^{\mathsf{ef}}(S)$ satisfies 1 and 2. The former follows from lemma 9.5 and the latter is satisfied by construction.

Now theorem 8.6 entails that $\mathsf{DS}^{\mathsf{ef}}(S)$ is a semi-algebroidal category and its category-theoretic finite objects are precisely those systems in $\mathsf{DS}^{\mathsf{ef}}(S)$ that are shifts. By lemma 9.4, these, in turn, are, up to isomorphism, exactly the elements of S, as needed.

To show that $\mathsf{DS}^{\mathsf{ef}}(S)$ is algebroidal, we need to show that (a) it contains, up to isomorphism, only countably many finite objects, and (b) between any two finite objects there are only countably many morphisms. Now (a) follows from the assumption that S is countable and the just proven fact that, up to isomorphism, S contains all the finite objects. Moreover, (b) follows from lemma 9.3: If X and Y are two finite objects, they are isomorphic to shifts, and between shifts there are only countably many factors, because each such factor is given by a block map, which is a finite object.

Finally, the necessary and sufficient conditions for the existence of a universal and homogeneous system simply is theorem 4.7. $\qquad\square$

10. THE UNIVERSAL PROSHIFT

Theorem 9.2 provides a general strategy to build algebroidal categories of deterministic systems: first, choose a countable class of shifts, and then close it under (co-) limits along ω-chains. As mentioned in example 3.2, the two most studied classes of shifts are the class SFT of shifts of finite type and the class SOF of sofic shifts [81, ch. 2 and 3, respectively]. Indeed, both SFT and SOF are countable (as they are characterized by finite graphs). In this section, we apply our strategy to those classes of shifts and obtain our second universality result, namely theorem 4.2 stating the existence of the universal proshift, i.e., the deterministic system that is the universal and homogeneous system both in $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SFT})$ and in $\mathsf{DS}^{\mathsf{ef}}(\mathsf{SOF})$. The proof is in subsection 10.1. Then subsection 10.2 investigates the universal proshift and avenues for future work.

10.1. **Proof of theorem 4.4.** First, let us give a name to $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ and $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$.

Definition 10.1. We call $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ the category of *ω-proshifts of finite type*, and we call $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$ the category of *sofic ω-proshifts*.[20]

From theorem 9.2, we know that $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ and $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$ are algebroidal. The next lemma is the key to show the joint embedding property and the amalgamation property.

Lemma 10.2. *Let* X, Y_0, Y_1 *be shifts over alphabets* A, B_0, B_1, *respectively, and let* $f_0 : Y_0 \to X$ *and* $f_1 : Y_1 \to X$ *be factors. For* $y_0 \in Y_0$ *and* $y_1 \in Y_1$, *let* $\langle y_0, y_1 \rangle$ *be the sequence whose n-th element is* $(y_0(n), y_1(n))$, *so* $\langle y_0, y_1 \rangle \in (B_0 \times B_1)^\omega$. *Write*

$$Y_0 \times_{f_0, f_1} Y_1 := \{ \langle y_0, y_1 \rangle : y_0 \in Y_0, y_1 \in Y_1, f_0(y_0) = f_1(y_1) \}.$$

Then we have the following:

(1) $Y_0 \times_{f_0, f_1} Y_1$ *is a shift over the alphabet* $B_0 \times B_1$.
(2) *If* Y_0 *and* Y_1 *are shifts of finite type, so is* $Y_0 \times_{f_0, f_1} Y_1$.
(3) *For* $i \in \{0, 1\}$, *the function* $g_i : Y_0 \times_{f_0, f_1} Y_1 \to Y_i$, *which maps* $\langle y_0, y_1 \rangle$ *to* y_i, *is a factor.*

Proof. Write $Z := Y_0 \times_{f_0, f_1} Y_1$. For $w_0 \in B_0^n$ and $w_1 \in B_1^n$, we also write $\langle w_0, w_1 \rangle$ for the word in $(B_0 \times B_1)^n$ whose k-th element is $(w_0(k), w_1(k))$. For $i \in \{0, 1\}$, let F_i be a set of B_i-words such that Y_i is the set of B_i-sequences that avoid F_i.[21] Define:

$$\overline{F}_0 := \bigcup_{w_0 \in F_0} \{ \langle w_0, w_1 \rangle : w_1 \in B_1^{|w_0|} \} \qquad \overline{F}_1 := \bigcup_{w_1 \in F_1} \{ \langle w_0, w_1 \rangle : w_0 \in B_0^{|w_1|} \}.\text{[22]}$$

By lemma 9.3, f_0 and f_1 are N-sliding block codes for some sufficiently big N. Let $\varphi_0 : B_N(Y_0) \to A$ and $\varphi_1 : B_N(Y_1) \to A$ be their respective block maps. For $(a, a') \in A \times A$ with $a \neq a'$, define

$$\overline{F}_{(a,a')} := \{ \langle w_0, w_1 \rangle : w_0 \in \varphi_0^{-1}(a), w_1 \in \varphi_1^{-1}(a') \} \subseteq (B_0 \times B_1)^N.$$

Let $\overline{F}$ be the union of all the $\overline{F}_{(a,a')}$ for $(a, a') \in A \times A$ with $a \neq a'$. Now it is straightforward to show

$$Z = \{ z \in (B_0 \times B_1)^\omega : z \text{ avoids } \overline{F}_0 \cup \overline{F}_1 \cup \overline{F} \}.$$

So 1 follows: Z is a shift qua set of sequence defined by avoiding a set of words.

Regarding 2, if Y_0 and Y_1 are shifts of finite type, then F_0 and F_1 can be chosen to be finite, so also $\overline{F}_0 \cup \overline{F}_1 \cup \overline{F}$ is finite, rendering Z a shift of finite type, as needed.

Regarding 3, we show the claim for $i = 0$ (the case $i = 1$ is symmetric). It is straightforward to check that g_0 is continuous and surjective, and, for equivariance, we have

$$g_0\big(\sigma(\langle y_0, y_1 \rangle)\big) = g_0\big(\langle \sigma(y_0), \sigma(y_1) \rangle\big) = \sigma(y_0) = \sigma\big(g_0(\langle y_0, y_1 \rangle)\big),$$

so $g_0 \circ \sigma = \sigma \circ g_0$, as needed. $\square$

---

[20]One might also call them 'shifts of profinite type' and 'prosofic shifts', respectively, but this would falsely suggests that they are shifts (example 10.5 describes a proshift which is not a shift).

[21]It is a basic fact that shifts can equivalently be defined via a set F of avoided words [81, p. 5 and 27]. (The special case where F is finite yields the shifts of finite type.)

[22]Here $|w|$ denotes the length of word w.

Proposition 10.3. *The following collections of shifts satisfy both the joint embedding property 1 and the amalgamation property 2 of theorem 9.2:*

(1) the collection SHI *of all shifts,*
(2) the collection SFT *of all shifts of finite type,*
(3) the collection SOF *of all sofic shifts.*

Moreover, SFT *and* SOF *are countable.*

Proof. Both SFT and SOF are countable because they can be represented by finite graphs and finite labeled graphs, respectively. All three collections contain the trivial shift (i.e., the full shift over the one-element alphabet), which is a factor of any shift. Hence the joint-embedding property follows from the amalgamation property, and it remains to show the latter. For SHI and SFT, this immediately follows from lemma 10.2. So let us show it for SOF.

Let $X, Y_0, Y_1 \in$ SOF with factors $f_0 : Y_0 \to X$ and $f_1 : Y_1 \to X$. Since sofic shifts are factors of shifts of finite type, there are $Z_0, Z_1 \in$ SFT with factors $h_0 : Z_0 \to Y_0$ and $h_1 : Z_1 \to Y_1$. Now we can apply lemma 10.2 to $f_0 \circ h_0 : Z_0 \to X$ and $f_1 \circ h_1 : Z_1 \to X$. So we have $Z := Z_0 \times_{f_0 \circ h_0, f_1 \circ h_1} Z_1 \in$ SFT with factors $\pi_0 : Z \to Z_0$ and $\pi_1 : Z \to Z_1$ such that $(f_0 \circ h_0) \circ \pi_0 = (f_1 \circ h_1) \circ \pi_1$. In particular, Z is in SOF (since SFT $\subseteq$ SOF) and we have factors $g_0 := h_0 \circ \pi_0 : Z \to Y_0$ and $g_1 := h_1 \circ \pi_1 : Z \to Y_1$ with $f_0 \circ g_0 = f_1 \circ g_1$, as needed. $\square$

Hence both $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ and $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$ have a universal and homogeneous object. To establish theorem 4.2 it remains to show that these two objects are isomorphic:
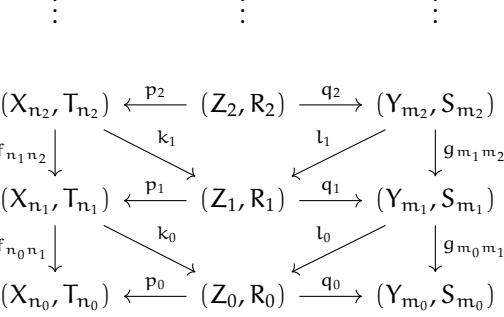
Proposition 10.4. *Let* (X, T) *be the universal and homogeneous ω-proshift of finite type and let* (Y, S) *be the universal and homogeneous sofic ω-proshift. Then* (X, T) *and* (Y, S) *are isomorphic.*

Proof. For the proof we use the concept of a saturated object (from [38]). Let C be a semi-algebroidal category and let D be a full subcategory. An object U in C is D-*saturated*, if, for any objects A and B in D and morphisms $f : A \to U$ and $g : A \to B$ in C, there is a morphism $h : B \to U$ in C such that $h \circ g = f$. It is easy to show that an object U that is D-universal and D-homogeneous is D-saturated. In particular, (X, T) is SFT-saturated and (Y, S) is SOF-saturated.

Let $(X_i, T_i), f_{ii+1})_{i \in \omega}$ be an ω-chain of shifts in SFT with colimit $((X, T), f_i)_{i \in \omega}$. Let $(Y_i, S_i), g_{ii+1})_{i \in \omega}$ be an ω-chain of shifts in SOF with colimit $((Y, S), g_i)_{i \in \omega}$. We construct the diagram in figure 3 inductively as follows.

Base step. Set $n_0 := 0$ and $m_0 := 0$. By the joint embedding property of SOF, there is a sofic shift (Z_0, R_0) with factors $p_0 : (Z_0, R_0) \to (X_{n_0}, T_{n_0})$ and $q_0 : (Z_0, R_0) \to (Y_{m_0}, S_{m_0})$. Since sofic shifts are factors of shifts of finite type, we can assume that (Z_0, R_0) in fact is a shift of finite type.

Inductive step. Since (X, T) is SFT-saturated, there is a factor $u : (X, T) \to (Z_0, R_0)$ with $p_0 \circ u = f_{n_0}$. Since the shifts in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ are category-theoretically finite, there in particular is $n_1 > n_0$ and a factor $k_0 : (X_{n_1}, T_{n_1}) \to (Z_0, R_0)$ such that $k_0 \circ f_{n_1} = u$. Note that $f_{n_0 n_1} \circ f_{n_1} = f_{n_0} = p_0 \circ u = p_0 \circ k_0 \circ f_{n_1}$, so, since factors are epic (and ef-pairs monic), $f_{n_0 n_1} = p_0 \circ k_0$. Similarly, we get $m_1 > m_0$ and a factor $l_0 : (Y_{m_1}, S_{m_1}) \to (Z_0, R_0)$ such that $g_{m_0 m_1} = q_0 \circ l_0$. Now, by the amalgamation property of SOF, there is a sofic shift (Z_1, R_1) with factors $p_1 : (Z_1, R_1) \to (X_{n_1}, T_{n_1})$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$(X_{n_2}, T_{n_2}) \xleftarrow{\ p_2\ } (Z_2, R_2) \xrightarrow{\ q_2\ } (Y_{m_2}, S_{m_2})$$

$$f_{n_1 n_2} \downarrow \quad \searrow^{k_1} \qquad\qquad \swarrow^{l_1} \qquad \downarrow g_{m_1 m_2}$$

$$(X_{n_1}, T_{n_1}) \xleftarrow{\ p_1\ } (Z_1, R_1) \xrightarrow{\ q_1\ } (Y_{m_1}, S_{m_1})$$

$$f_{n_0 n_1} \downarrow \quad \searrow^{k_0} \qquad\qquad \swarrow^{l_0} \qquad \downarrow g_{m_0 m_1}$$

$$(X_{n_0}, T_{n_0}) \xleftarrow{\ p_0\ } (Z_0, R_0) \xrightarrow{\ q_0\ } (Y_{m_0}, S_{m_0})$$

FIGURE 3. Diagram to prove that (X, T) and (Y, S) are isomorphic.

and $q_1 : (Z_1, R_1) \to (Y_{m_1}, S_{m_1})$ such that $k_0 \circ p_1 = l_0 \circ q_1$. Again, we can assume that (Z_1, R_1) in fact is a shift of finite type. Now we proceed with the next layers of the diagram.

Now, $\big((X, T), q_i \circ k_i \circ f_{n_{i+1}}\big)_i$ is a cone to the ω-chain $\big((Y_{m_i}, S_{m_i}), g_{m_i m_{i+1}}\big)_i$, whose limit is $\big((Y, S), g_{m_i}\big)_i$. So there is a factor $u : (X, T) \to (Y, S)$ such that $g_{m_i} \circ u = q_i \circ k_i \circ f_{n_{i+1}}$. Similarly, $\big((Y, S), p_i \circ l_i \circ g_{m_{i+1}}\big)_i$ is a cone to the ω-chain $\big((X_{n_i}, T_{n_i}), f_{n_i n_{i+1}}\big)_i$, whose limit is $\big((X, T), f_{n_i}\big)_i$. So there is a factor $v : (Y, S) \to (X, T)$ such that $f_{n_i} \circ v = p_i \circ l_i \circ g_{m_{i+1}}$.

We claim that $v \circ u = \mathrm{id}_X$ and $u \circ v = \mathrm{id}_Y$, yielding the desired isomorphism. Indeed, some diagram-chasing shows, for any i, that $f_{n_i} \circ v \circ u = f_{n_i}$ and $g_{m_i} \circ u \circ v = g_{m_i}$. □

10.2. **The nature of the universal proshift.** We explore the categories $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ and $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$ and their shared universal and homogeneous object. Then we sketch avenues for further investigation, especially in connection to automata theory.

Examples. To better understand the two categories of proshifts, let's consider some examples. Recall that we can think of the collection S of shifts as the basic building blocks with which we build the ω-proshifts in $\mathsf{DS}^{\mathrm{ef}}(S)$. Lemma 9.4 told us that when building more complex systems, we do not generate any further shifts. In particular, a sofic shift that is not a shift of finite type—like the even shift [81, p. 67]—is in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$ but not in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$. Similarly, a shift that is not sofic—like the context-free shift [81, p. 74]—is neither in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$ nor in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$. However, what are examples of systems which are not basic but complex, i.e., which are not shifts but are in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ and hence also in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$? Arguably the simplest such example is this:

Example 10.5. Let $X := 2^\omega$ be Cantor space and $T := \mathrm{id}_X$ the identity function, i.e., the trivial dynamics. We claim that (X, T) is in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ but it is not in SFT, in fact, (X, T) is not isomorphic to a shift at all. The latter follows from Hedlund's theorem, since the identity function is not expansive. To show the former, we need to show that (X, T) is a limit of shifts of finite type. Recall that the Cantor space 2^ω is the limit of the inverse sequence $(2^i, \rho_{ij})$, where 2^i is the discrete space consisting of all binary strings of length i with $\rho_{ij} : 2^j \to 2^i$ and $\rho_i : 2^\omega \to 2^i$ being the

restriction functions (mapping a sequence x to its restriction $x \upharpoonright i$). We regard 2^i as a dynamical system with the identity function id as dynamics, so the restriction maps ρ_{ij} and ρ_i trivially are factors. Moreover, each $(2^i, \mathrm{id})$ is isomorphic to a shift of finite type X_i: The alphabet is 2_i and X_i consists of all the constant sequences (so the forbidden words are those $w = ab$ where a and b are distinct letters in 2^i). The isomorphism $\bar{\cdot} : (2^i, \mathrm{id}) \to (X_i, \sigma)$ maps a string $a \in 2^i$ to the constant sequence $\bar{a} := (a_i)$ with $a_i := a$ for all $i \in \omega$. Hence (X, T) is a limit of the shifts of finite type X_i.[23]

Another example comes from odometers. They play an important role in dynamical systems theory (for an overview, see [35]) and have connections to cellular automata [25].

Example 10.6. We show that every odometer is in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ and hence also in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$. An *odometer* (aka adding machine or group of s-adic integers) is defined, in the most concrete way, as an inverse limit of finite cyclic groups, as follows. Let $s = (s_i)_{i \in \omega}$ be a sequence of positive integers such that s_i divides s_{i+1}. Write C_n for the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$. Let $\pi_{ii+1} : C_{s_{i+1}} \to C_{s_i}$ be the function mapping k to $k \bmod s_i$ (a surjective group homomorphism). The s-odometer G_s then is the inverse limit of $(C_{s_i}, \pi_{ii+1})_{i \in \omega}$. So an element x of G_s is a sequence $(x_i)_{i \in \omega}$ with $x_i \in C_{s_i}$ and $x_i \equiv x_{i+1} \bmod s_i$. Addition is defined coordinatewise, as addition modulo s_i. In particular, we can consider the '+1 dynamics' $+_1 : G_s \to G_s$ which maps $(x_i)_{i \in \omega}$ to $(x_i + 1 \bmod s_i)_{i \in \omega}$. Then $(G_s, +_1)$ is a deterministic dynamical system in our sense.

In particular, we can define the sequence $(s_i)_{i \in \omega}$ by $s_i := (i + 1)!$. The resulting odometer $(G_s, +_1)$ is universal among the odometers (i.e., every other odometer is a factor of it). Moreover, [56] shows that the isomorphism class of $(G_s, +_1)$ is generic in the class of transitive and invertible dynamics on compact metric spaces.

To see that each odometer $(G_s, +_1)$ is in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$, first note that each cyclic group C_n, with the +1 dynamics $a \mapsto a + 1 \bmod n$, can be regarded as a shift of finite type. Indeed, let $n := \{0, \ldots, n-1\}$ be the alphabet and let X_n be the shift over n defined by avoiding the words $w = ab$ where $b \not\equiv a + 1 \bmod n$. Then X_n contains the sequences

$$
\begin{array}{rccccccccccc}
\bar{0} & := & 0 & 1 & 2 & \ldots & n-2 & n-1 & 0 & 1 & 2 & \ldots \\
\bar{1} & := & 1 & 2 & 3 & \ldots & n-1 & 0 & 1 & 2 & 3 & \ldots \\
& \vdots & & & & & & & & & & \\
\overline{n-1} & := & n-1 & 0 & 1 & \ldots & n-3 & n-2 & n-1 & 0 & 1 & \ldots
\end{array}
$$

with the discrete topology, and $\sigma(\bar{k}) = \overline{k+1 \bmod n}$. So $\bar{\cdot}$ is an isomorphism between $(C_n, +_1)$ and (X_n, σ). Now, since G_s is the limit of the C_{s_i}'s, $(G_s, +_1)$ is the limit of the ω-chain of shifts of finite type X_{s_i}.

Properties. Regarding the 'complexity' of the universal proshift, we can make similar arguments as for the universal nondeterministic system in section 7.2. Knowing that the universal proshift must have as factor every sofic shift tells us

---

[23]Note, though, that the set of homeomorphisms on Cantor space that are isomorphic to subshifts of finite type is dense in the set of all homeomorphisms on Cantor space with the topology of uniform convergence [102].

quite a lot about it. For example, it does not have any dense orbits and, indeed, it is not topologically transitive.[24]

Proposition 10.7. *The universal proshift is not topologically transitive.*

Proof. Recall, from example 10.5, that $(2, \mathrm{id})$, the two-element system with trivial dynamics, is a shift of finite type. Clearly, it is not transitive. By universality, it is a factor of the universal proshift. Because topological transitivity is preserved along factors, the universal proshift hence also cannot be transitive.   □

This is noted, e.g., by [57, p. 1128] and [34, p. 2] for the generic system on Cantor space. Similarly to the nondeterministic case, this renders the universal system less interesting from a dynamical perspective. However, from a computational perspective, not having dense orbits actually is welcome: a system that transverses all of its state space does not look like a system that implements a goal-oriented algorithm.

Another important concept in dynamical systems theory is the shadowing property [91]. Intuitively, it says that when we numerically simulate an orbit starting in x_0 and computing, at each step x_n, the next x_{n+1} with some error from the true $T(x_n)$, then there is a true orbit close to the simulated orbit. Clearly, such a stability is again welcome from a computational perspective. Formally, the property is defined on a compact metric space X with metric d and continuous function $T : X \to X$. (But it can also be defined purely topologically.) A sequence (x_i) in X is a δ-*pseudo-orbit* if $d(x_{i+1}, T(x_i)) < \delta$ for all $i \in \omega$. A point $x \in X$ ϵ-*shadows* a sequence (x_i) in X if $d(x_i, T^i(x)) < \epsilon$ for all $i \in \omega$. Then (X, T) has the *shadowing property* if, for all $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo-orbit is ϵ-shadowed by some point.

A result of [51] is that a system (X, T), where X is a compact totally disconnected space and $T : X \to X$ continuous, has the shadowing property iff (X, T) is the inverse limit of a directed system satisfying the Mittag-Leffler condition and consisting of shifts of finite type. If the bonding maps of the inverse system are surjective, i.e., factors in our sense, then the Mittag-Leffler condition is satisfied. Hence:

Corollary 10.8 (of [51])**.** *All ω-proshifts of finite type, i.e., the systems in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$, have the shadowing property. In particular, the universal proshift has the shadowing property.*

It is also known that a shift has the shadowing property iff it is a shift of finite type [118]. So, for example, a sofic shift which is not a shift of finite type does not have the shadowing property. In particular, some systems in $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$ fail to have the shadowing property, despite having a universal system with the shadowing property.

Future work. There is much more to be investigated in the categories $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SFT})$ and $\mathsf{DS}^{\mathrm{ef}}(\mathsf{SOF})$. So we now sketch four avenues for future work.

First, one way to better understand an object is to understand its automorphism group. Thus, picking up again the KPT correspondence discussed in section 2,

---

[24]Recall that a system (X, T), with X a topological space and $T : X \to X$ continuous, is *topologically transitive*, if for every two nonempty open sets $U, V \subseteq X$, there is an integer $n \geqslant 0$ such that $f^n(U) \cap V \neq \emptyset$.

one can relate dynamical properties of the automorphism group of our universal systems to combinatorial properties of their category-theoretically finite objects. This way, one could investigate, e.g., if the automorphism group is extremely amenable (and hence uniquely ergodic) or if it admits ample generics [71].

Second, the latter question about genericity is particularly interesting (see again [33, 34, 56, 57, 73, 79]). It also connects to questions about computability, as [57] shows: There are effective actions of $\mathbb{Z}^1$ on Cantor space with dense conjugacy class. However, for dimensions $d \geqslant 2$, there are no dense effective actions of $\mathbb{Z}^d$ on Cantor space, while there are non-effective ones. Other computability-theoretic aspects of symbolic dynamics are studied in [42], and in [9] also inverse limits of shifts of finite type occur. In this vein, one might also ask for a concrete description of the universal proshift, not just an existence results (cf. [4]).

Third, one can connect to the construction of compacta [61, 62, 85] using inverse limits along multifunctions, especially through their study via shift maps [68].

Fourth, given the close connection between sofic shifts and automata, it is natural to investigate if there is a similarly fruitful connection between our sofic ω-proshifts and profinite automata [95]. Here duality-theoretic tools might also be useful (see [44] and [45, ch. 8]): they consider the Boolean algebra of regular languages (i.e., those languages that are recognized by some automaton) over a given alphabet and with certain residuation operations, and they show that dual to this Boolean algebra is the free profinite monoid. For further connections between profinite semigroups and shift spaces, see [6].

11. CONCLUSION

We end with a brief summary and some direction for future work.

Summary. Motivated by the search for a universal analog computer, we considered possibly nondeterministic systems and identified four senses of universality. We provided an equivalent domain-theoretic description of the systems, and we used the Fraïssé-limit method to show the existence of a universal and homogeneous nondeterministic system. The subclass of deterministic systems cannot have such a universal system, but we introduced ω-proshifts and showed that both the ω-proshifts of finite type and the sofic ω-proshifts have a shared universal and homogeneous system, which has the shadowing property.

Future work. We already mentioned open questions on ω-proshifts: e.g., exploring their automorphism groups, their genericity, their computability, or their connections to compacta. Particularly exciting is to extend the connection between sofic shifts and automata to the profinite—including links to duality theory.

Further questions include, as mentioned, generalizing our results from discrete time to other monoid actions (in particular, see [34, 71]). Quite generally, it is worth better understanding the structure of the different categories of systems—and their corresponding categories of domains (also see [119]).

Moreover, it is pertinent to clarifying the connection between our existence result on universal analog computation (which includes 'learning systems' like neural networks) to results like the No-Free-Lunch theorems stating the non-existence of a universal learner (see [111] for a study of universal prediction methods).

Finally, at a philosophical level, one should clarify the implication of universality and genericity results on the question—which we had to leave open—whether a dynamical system counts as an analog computer. We have mentioned the density and genericity results about actions on the Cantor space, which are within a longer tradition in ergodic theory on understanding which properties a 'typical' dynamical system has (see, e.g., [49] or [56] for a short history). Beyond deterministic systems and closer to nondeterministic systems, Erdős and Rényi [41] probabilistically constructed a countable graph that, with probability 1, yields Fraïssé's universal homogenous graph; [39] provides a related probabilistic construction for universal domains in domain theory; and [46, sec. 3.1] provide a concrete model of the projectively universal profinite graph. So one may try to make precise the question whether being an analog computer (whatever this means exactly) is a typical property. Does the fact that the generic automorphism on Cantor space is effective have any bearing on this question? What about the fact that this fails for higher dimensions? Note that algorithmic randomness also studies such questions of 'typicality' [37].

APPENDIX A. PROOFS

A.1. **Proofs from section 3.**

Remark A.1 (Zero-dimensionalizing and compactifying). Starting with a 'Polish system' (X, T), i.e., where X is a Polish space and $T : X \to X$ is a continuous function, we want to construct a—in a sense best possible—system (Y, S), with Y a compact and zero-dimensional Polish space and $S : Y \to Y$ continuous, together with an injective and equivariant function $\eta : X \to Y$ that is continuous with respect to a finer Polish topology on X.

We use a result from Dahlqvist, Danos, and Garnier [27] on how to zero-dimensionalize and compactify Polish spaces. To state this precisely, let us introduce their terminology: A *based Polish space* is a pair $(X, \mathcal{B})$ where X is a Polish space and $\mathcal{B}$ is a countable base for the topology of X. A *base-preserving map* f from $(X, \mathcal{B})$ to $(Y, \mathcal{C})$ is a function $f : X \to Y$ such that $f^{-1}(C) \in \mathcal{B}$ for all $C \in \mathcal{C}$ (so f is continuous). A compact zero-dimensional Polish space X can canonically be regarded as a based Polish space by taking the collection $\mathrm{Clp}(X)$ of all clopen subsets of X as the base. Next, [27] consider the following categories:

- Pol_{cz}: compact zero-dimensional Polish spaces with continuous maps.
- $\mathsf{Pol}_z^\flat$: based Polish spaces $(X, \mathcal{B})$, where X is zero-dimensional and $\mathcal{B}$ is a Boolean algebra, with base-preserving maps.
- $\mathsf{Pol}^\flat$: based Polish spaces with base-preserving maps.

Finally, [27] show that (1) Pol_{cz} is a reflective subcategory of $\mathsf{Pol}_z^\flat$ with reflector c and (2) $\mathsf{Pol}_z^\flat$, in turn, is a coreflective subcategory of $\mathsf{Pol}^\flat$ with coreflector z. For (1), given a based space $(X, \mathcal{B})$ in $\mathsf{Pol}_z^\flat$, the compactification $c(X, \mathcal{B})$ is the compact zero-dimensional space Y obtained as the limit of the diagram of finite spaces P, where P is a finite partition of X with partition cells in $\mathcal{B}$. The unit $\eta_{(X, \mathcal{B})} : (X, \mathcal{B}) \to Y$ is the canonical embedding of X into its compactification Y. For (2), given a based space $(X, \mathcal{B})$ in $\mathsf{Pol}^\flat$, the zero-dimensionalization $z(X, \mathcal{B})$ is the based space $(X, \overline{\mathcal{B}})$ on the same underlying set X but whose topology is generated by the base $\overline{\mathcal{B}}$, which is the

Boolean algebra generated by $\mathcal{B}$. The counit $\epsilon_{(X,\mathcal{B})} : z(X,\mathcal{B}) \to (X,\mathcal{B})$ simply is the identity function on X.

Now, we use this result to turn a Polish system (X,T) into a deterministic system in our sense:

- *Step 0.* Pick a countable base $\mathcal{B}$ of X. For example, if $X = \mathbb{R}^n$, pick the open balls centered on rational vectors with rational radii. Next close $\mathcal{B}$ under the dynamics, i.e., consider $\mathcal{B}_T := \{T^{-k}(B) : B \in \mathcal{B}, k \in \omega\}$. Then $(X,\mathcal{B}_T)$ is a based Polish space. (This step is unnecessary if one thinks that Polish systems should make their choice of dynamically closed base explicit.)
- *Step 1.* Take the zero-dimensionalization $z(X,\mathcal{B}_T) = (X, \overline{\mathcal{B}_T})$. Note that, by construction, $T : z(X,\mathcal{B}_T) \to (X,\mathcal{B}_T)$ is base-preserving. So, by the adjunction, it lifts to a dynamics $T : z(X,\mathcal{B}_T) \to z(X,\mathcal{B}_T)$.
- *Step 2.* Compactify $z(X,\mathcal{B}_T)$ into $Y := c(z(X,\mathcal{B}_T))$, with the embedding $\eta_X := \eta_{z(X,\mathcal{B}_T)} : X \to Y$. Consider the map $\eta_X \circ T : X \to Y$, i.e., viewing the result of the dynamics T in the larger, compactified space Y. By the adjunction, there is a unique commuting map $c(T)$:

$$
\begin{array}{ccc}
z(X,\mathcal{B}_T) & \xrightarrow{\eta_X} & c(z(X,\mathcal{B}_T)) \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle c(T)} \\
z(X,\mathcal{B}_T) & \xrightarrow{\eta_X} & c(z(X,\mathcal{B}_T))
\end{array}
$$

So $S := c(T) : Y \to Y$ extends the dynamics T on X.

In sum, we have the larger system (Y,S) and an injective function $\eta_X : X \to Y$, which is equivariant by the above diagram and continuous with respect to the finer Polish topology on X generated by $\overline{\mathcal{B}_T}$. The fact that c and z are adjoint functors provides a sense in which this (Y,S) is optimal.[25]

Lemma A.2. *Let* $F : X \rightrightarrows Y$ *and* $G : Y \rightrightarrows Z$ *be closed-valued and upper-hemicontinuous multifunctions between compact Hausdorff spaces. Then* $G \circ F : X \rightrightarrows Z$ *is again such a function. If both* F *and* G *are total, so is* $G \circ F$.

Proof. Closed-valued: For $x \in X$, $\varphi(x)$ is closed, so the image $\psi[\varphi(x)]$ is closed by the corollary to the closed graph theorem. Upper-hemicontinuous: The composition of upper-hemicontinuous multifunctions is again upper-hemicontinuous [5, thm. 17.23]. If F and G are total, then, for $x \in X$, $\varphi(x)$ is nonempty, so $\psi \circ \varphi(x) = \psi[\varphi(x)] \neq \emptyset$. $\qquad\square$

Proof of proposition 3.8. Assume $\varphi : (X,T) \to (Y,S)$ and $\psi : (Y,S) \to (Z,R)$ are morphisms. By lemma A.2, $\psi \circ \varphi$ is again closed-valued and upper-hemicontinuous. So it remains to check the equivariance conditions.

Concerning 1, if $x \xrightarrow{T} x'$, there is $y, y' \in Y$ with $x \xrightarrow{\varphi} y$, $x' \xrightarrow{\varphi} y'$, and $y \xrightarrow{S} y'$, so there is $z, z' \in Z$ with $y \xrightarrow{\psi} z$, $y' \xrightarrow{\psi} z'$, and $z \xrightarrow{R} z'$, hence also $x \xrightarrow{\psi \circ \varphi} z$, $x' \xrightarrow{\psi \circ \varphi} z'$, and $z \xrightarrow{R} z'$, as needed.

---

[25]Similar ideas are used in [60, sec. 5.3].

Concerning 2, if $z \xrightarrow{R} z'$, there is $y, y' \in Y$ with $y \xrightarrow{\psi} z$, $y' \xrightarrow{\psi} z'$, and $y \xrightarrow{S} y'$, so there is $x, x' \in X$ with $x \xrightarrow{\varphi} y$, $x' \xrightarrow{\varphi} y'$, and $x \xrightarrow{T} x'$, hence also $x \xrightarrow{\psi \circ \varphi} z$, $x' \xrightarrow{\psi \circ \varphi} z'$, and $x \xrightarrow{T} x'$, as needed.

Now assume that φ and ψ are factors. By the above, $\psi \circ \varphi$ is a morphism, and, qua composition of continuous surjections, it again is a continuous surjection, so it remains to show 3. So assume $x \xrightarrow{\psi \circ \varphi} z$ and $x \xrightarrow{T} x'$. So there is y with $x \xrightarrow{\varphi} y$ and $y \xrightarrow{\psi} z$. Since φ satisfies 3, there is y' with $y \xrightarrow{S} y'$ and $x' \xrightarrow{\varphi} y'$. Since ψ satisfies 3, there is z' with $z \xrightarrow{R} z'$ and $y' \xrightarrow{\psi} z'$. Hence also $x' \xrightarrow{\psi \circ \varphi} z'$, as needed.

Now assume that φ and ψ are embeddings. By the above, $\psi \circ \varphi$ is a morphism and total, hence it remains to show that it is partition-injective and satisfies 4. For the former, we need to show that $\{\psi \circ \varphi(x) : x \in X\}$ partitions Z. Indeed, if $x \neq x'$, then it is readily checked that $\psi \circ \varphi(x) \cap \psi \circ \varphi(x') = \emptyset$; and if $z \in Z$, then there is $y \in Y$ such that $z \in \psi(y)$, and hence there is $x \in X$ such that $y \in \varphi(x)$, so $z \in \varphi \circ \psi(x)$. For 4, assume $x \xrightarrow{\psi \circ \varphi} z$ and $z \xrightarrow{R} z'$. So there is y with $x \xrightarrow{\varphi} y$ and $y \xrightarrow{\psi} z$. Since ψ satisfies 4, there is y' with $y \xrightarrow{S} y'$ and $y' \xrightarrow{\psi} z'$. Since φ satisfies 4, there is x' with $x \xrightarrow{T} x'$ and $x' \xrightarrow{\varphi} y'$. Hence also $x' \xrightarrow{\psi \circ \varphi} z'$, as needed.

Finally, it is well-known that relation composition, which is given by $\circ$, is associative with unit the identity function. □

Proof of proposition 3.11. Ad 1. For uniqueness, if $\underline{f}$ and $\underline{f}'$ are two embeddings $(X, T) \to (Y, S)$ such that $(\underline{f}, f)$ and $(\underline{f}', f)$ are ef-pairs, then, for $x \in X$, we have $\underline{f}(x) = \underline{f}[f \circ \underline{f}'(x)] = \underline{f} \circ f[\underline{f}'(x)] = \underline{f}'(x)$.

For existence, we show that the multifunction $e : X \rightrightarrows Y$ given by $e(x) := f^{-1}(x)$ is such that (e, f) is an ef-pair.

First, we show that e is an embedding. Since f is surjective, each $e(x)$ is nonempty, so e is total, and since the graph of f is closed, so is the graph of $e = f^{-1}$. By construction, $\{e(x) : x \in X\}$ partitions Y. Regarding equivarience 1, assume $x \xrightarrow{T} x'$. Since f satisfies 2, there is $y, y' \in Y$ such that $y \xrightarrow{f} x$, $y' \xrightarrow{f} x'$, and $y \xrightarrow{S} y'$, hence also $x \xrightarrow{e} y$, $x' \xrightarrow{e} y'$, as needed. Regarding 4, if $x \xrightarrow{e} y$ and $y \xrightarrow{S} y'$, then $y \xrightarrow{f} x$, so, since f satisfies 3, there is x' with $x \xrightarrow{T} x'$ and $y' \xrightarrow{f} x'$, so also $x' \xrightarrow{e} y'$, as needed.

Second, we show properties 1 and 2: For $x \in X$, since f is surjective, $f \circ e(x) = f(f^{-1}(x)) = \{x\}$. For $y \in Y$, trivially $y \in f^{-1}(f(y)) = e \circ f(y)$.

Ad 2. For uniqueness, if $\overline{e}$ and $\overline{e}'$ are two factors $(Y, S) \to (X, T)$ such that $(e, \overline{e})$ and $(e, \overline{e}')$ are ef-pairs, then, for $y \in Y$, we have $y \in e \circ \overline{e}(y)$ and $y \in e \circ \overline{e}'(y)$, so there is $x \in \overline{e}(y)$ and $x' \in \overline{e}'(y)$ with $y \in e(x)$ and $y \in e(x')$. Since e is partition-injective, $x = x'$. So, since $\overline{e}$ and $\overline{e}'$ are functions, $\overline{e}(y) = x = x' = \overline{e}'(y)$.

For existence, we show that the (multi)function $f : Y \rightrightarrows X$, given by $f(y) :=$ the x with $y \in e(x)$, is such that (e, f) is an ef-pair. Note that, since e is partition-injective, f is indeed a function. Also note that, for any $A \subseteq X$, we have $f^{-1}(A) = e[A]$: for $y \in Y$, we have $y \in f^{-1}(A)$ iff $f(y) \in A$ iff the x with $y \in e(x)$ is in A iff there is $x \in A$ with $y \in e(x)$ iff $y \in e[A]$.

First, we show that f is a factor. To show that f is surjective, let $x \in X$. Since e is total, pick $y \in e(x)$. Then $f(y) = x$. Also, f is continuous: if $A \subseteq X$ is closed, then $f^{-1}(A) = e[A]$ is closed, since, by the corollary to the closed graph theorem, the

image of a closed set is closed. Regarding equivarience 2, if $x \xrightarrow{T} x'$, then, since e satisfies 1, there is $y, y' \in Y$ with $x \xrightarrow{e} y$, $x' \xrightarrow{e} y'$, and $y \xrightarrow{S} y'$, so also $y \xrightarrow{f} x$ and $y' \xrightarrow{f} x'$, as needed. For 3, if $y \xrightarrow{f} x$ and $y \xrightarrow{S} y'$, then $x \xrightarrow{e} y$, so, since e is an embedding, there is x' with $x \xrightarrow{T} x'$ and $x' \xrightarrow{e} y'$, so also $y' \xrightarrow{f} x'$, as needed.

Second, we show properties 1 and 2: For $x \in X$, since f is surjective, $f \circ e(x) = f[f^{-1}(x)] = \{x\}$. For $y \in Y$, we have, by definition, that $f(y)$ is the $x \in X$ with $y \in e(x)$, so $y \in e(x) = e \circ f(y)$. □

Proposition A.3. *Let* $(e, f) : (X, T) \to (Y, S)$ *be an ef-pair. The following are equivalent.*

  (1) $(e, f) : (X, T) \to (Y, S)$ *is an isomorphism in* $\mathsf{allSys}^{\mathsf{ef}}$
  (2) $e : X \to Y$ *is a homeomorphism with inverse* $f : Y \to X$ *such that* $e \circ T = S \circ e$
      (and hence also $f \circ S = T \circ f$).

Proof. (1)$\Rightarrow$(2). Assume $(e, f) : (X, T) \to (Y, S)$ is an isomorphism, with inverse $(e', f') : (Y, s) \to (X, T)$. We show that e is a continuous bijective function $X \to Y$ such that $e \circ T = S \circ e$ and $e^{-1} = f$. (Recall that bijective continuous functions on compact Hausdorff spaces are homeomorphisms; and if $e \circ T = S \circ e$, then by 'multiplying' e^{-1} to the left and to the right, $e^{-1}eTe^{-1} = e^{-1}See^{-1}$, so $Te^{-1} = e^{-1}S$.)

First, e is a function, because if $y \neq y'$ were in $e(x)$, then both $e'(y)$ and $e'(y')$ are, by totality, nonempty subsets of $e' \circ e(x) = \{x\}$, so $e'(y) = \{x\} = e'(y')$, contradicting that e' is partition-injective. And e is bijective: if $x \neq x'$, then $e(x) \cap e(x') = \emptyset$, and if $y \in Y$, then it is in some $e(x)$, so $e(x) = y$. Further, as a upper-hemicontinuous multifunction, e is continuous. And $e^{-1} = f$ because, given $y \in Y$, we need to show that $x := f(y)$ is such that $e(x) = y$; indeed, since f is the factor corresponding to e, we have $y \in e(f(y))$, and since e is a function, we actually have $y = e(f(y)) = e(x)$.

Finally, we check, for $x \in X$, that $e \circ T(x) = S \circ e(x)$. ($\subseteq$) If $y' \in e \circ T(x)$, there is $x' \in X$ with $x \xrightarrow{T} x'$ and $x' \xrightarrow{e} y'$, so, since e is a morphism, there is $y_0, y_1 \in Y$ such that $x \xrightarrow{e} y_0$, $x' \xrightarrow{e} y_1$, and $y_0 \xrightarrow{S} y_1$, so, since e is a function, $y' = y_1 \in S \circ e(x)$. ($\supseteq$) If $y' \in S \circ e(x)$, there is $y \in Y$ with $x \xrightarrow{e} y$ and $y \xrightarrow{S} y'$, so, since e is an embedding, there is $x' \in X$ such that $x \xrightarrow{T} x'$ and $x' \xrightarrow{e} y'$, so $y' \in e \circ T(x)$.

(2)$\Rightarrow$(1). Assume $e : X \to Y$ is a homeomorphism with inverse f and $e \circ T = S \circ e$ (so, as noted, also $T \circ f = f \circ S$). We show that $(e, f) : (X, T) \to (Y, S)$ is an isomorphism in $\mathsf{allSys}^{\mathsf{ef}}$. It suffices to show that both (e, f) and (f, e) are ef-pairs: Then $(f, e) \circ (e, f) = (f \circ e, f \circ e) = (\mathrm{id}_X, \mathrm{id}_X) = \mathrm{id}_{(X,T)}$ and similarly $(e, f) \circ (f, e) = \mathrm{id}_{(Y,S)}$.

Indeed, qua continuous functions, both e and f are closed-valued and upper-hemicontinuous multifunctions. We show that e is both an embedding and a factor: Qua bijective continuous function, it is both a surjective function and a total partition-injective multifunction. It suffices to show equivariance 3 and 4 (which implies 1 and 2). For 3, if $x \xrightarrow{e} y$ and $x \xrightarrow{T} x'$, then $y' := e(x') \in e \circ T(x) = S \circ e(x)$, so there is y_0 such that $x \xrightarrow{e} y_0$ and $y_0 \xrightarrow{S} y'$, hence, since e is a function $y_0 = y$, as needed. For 4, if $x \xrightarrow{e} y$ and $y \xrightarrow{S} y'$, then $y' \in S \circ e(x) = e \circ T(x)$, so there is x' with $x \xrightarrow{T} x'$ and $x' \xrightarrow{e} y'$, as needed. Similarly, we show that f is both an embedding and a factor (now using $T \circ f = f \circ S$). Moreover, $f \circ e = \mathrm{id}_X$ and $e \circ f = \mathrm{id}_Y$, so both (e, f) and (f, e) are ef-pairs. □

A.2. **Proofs from section 5.** Here we prove theorem 5.1. We call a poset P *co-atomic* if each element is the greatest lower bound of its co-atoms, i.e., for all $x \in A$, we have $x = \bigwedge \mathrm{CoAt}(x)$.[26]

The next two propositions are fairly standard (cf. e.g. [40]), but, for completeness, we still add proofs.

Proposition A.4. *Let X be a compact and zero-dimensional Polish space. Then* $\mathsf{F}(X)$, *i.e., the set of closed subsets of X ordered by reverse inclusion, is an algebraic lattice. Its compact elements (in the order-theoretic sense) are the clopen subsets of X. Moreover,* $\mathsf{F}(X)$ *is co-atomic, with co-atoms being the singletons* $\{x\}$.

Proof. Least upper bounds in $\mathsf{F}(X)$ are given by intersections, so $\mathsf{F}(X)$ is a complete lattice. (The greatest lower bounds hence are closures of unions.) The greatest element is $\emptyset$, hence, since singletons are closed, each $\{x\} \in \mathsf{F}(X)$ is a co-atom, and every co-atom is a issingleton. If $A \in \mathsf{F}(X)$, then $\mathrm{CoAt}(A) := \{\{a\} : a \in A\}$, so $A = \bigcup \mathrm{CoAt}(A)$, and since A is already closed, it is the greatest lower bound of $\mathrm{CoAt}(A)$.

It remains to show that $\mathsf{F}(X)$ is algebraic, with the set $\mathsf{K}(\mathsf{F}(X))$ of compact elements being the set $\mathrm{Clp}(X)$ of clopen subsets of X. We start by showing, for $A \in \mathsf{F}(X)$, that $\mathcal{F} := \{B \in \mathrm{Clp}(X) : B \leqslant A\}$ is directed with $\bigcap \mathcal{F} = A$.

Indeed, $\mathcal{F}$ is nonempty: If $A = X$, then it is clopen, so $A \in \mathcal{F}$. If $A \neq X$, let $x \in X$ with $x \notin A$. Since X is normal and A closed, there are disjoint opens U and V with $x \in U$ and $A \subseteq V$. Since the clopens form a base, there in particular is a clopen B with $x \in B \subseteq U$ and $A \subseteq V \subseteq U^c \subseteq B^c$. So $B^c \in \mathcal{F}$. Moreover, $\mathcal{F}$ is closed under intersection, i.e., least upper bounds, hence directed. Finally, by construction, $A \subseteq \bigcap \mathcal{F}$. For the other inclusion, if $x \in X \setminus A$, there is, as above, a clopen B such that $x \in B$ and $A \subseteq B^c$, so $B^c \in \mathcal{F}$, hence $x \notin \bigcap \mathcal{F}$.

Hence it remains to show $\mathsf{K}(\mathsf{F}(X)) = \mathrm{Clp}(X)$. If $A \in \mathsf{K}(\mathsf{F}(X))$, then A is clopen: Indeed, we already know that $\mathcal{F} := \{B \in \mathrm{Clp}(X) : B \leqslant A\}$ is directed with $\bigcap \mathcal{F} \subseteq A$, so there is $B \in \mathcal{F}$ with $B \subseteq A$, so $B \in \mathrm{Clp}(X)$ with $B \supseteq A$ and $B \subseteq A$, i.e., $A = B$ is clopen. For the other direction, assume A is clopen and show that A is order-theoretically compact. So let $\mathcal{D} \subseteq \mathsf{F}(X)$ be directed with $\bigcap \mathcal{D} \subseteq A$. We need to find $B \in \mathcal{D}$ with $B \subseteq A$. Note that $\{B^c : B \in \mathcal{D}\}$ is an open cover of A^c: if $x \notin A$, then $x \notin \bigcap \mathcal{D}$, so $x \notin B$ for some $B \in \mathcal{D}$. Now, since A^c is a closed subset of a compact space, there is a finite subcover, say, $\{B_1^c, \ldots, B_n^c\}$. By directedness, let $B \in \mathcal{D}$ with $B \subseteq B_k$ for $k = 1, \ldots, n$. Then $B \subseteq A$: If $x \notin A$, then $x \in B_k^c$ for some k, so $x \notin B$. $\square$

Proposition A.5. *If* $T : X \rightrightarrows Y$ *is an upper-hemicontinuous and closed-valued multi-function between compact and zero-dimensional Polish spaces, then* $\mathsf{F}(T) : \mathsf{F}(X) \to \mathsf{F}(Y)$ *mapping A to* $T[A]$ *is a Scott-continuous and co-atomic function.*

Proof. Note that $\mathsf{F}(T)$ is well-defined: by the corollary to the closed graph theorem, $T[A]$ is a closed subset of Y. It is monotone: if $A \supseteq B$, then $T[A] \supseteq T[B]$. And $\mathsf{F}(T)$ is co-atomic: For $A \in \mathsf{F}(X)$, by definition $\mathsf{F}(T)(A) = T[A] = \bigcup\{T(x) : x \in A\}$, and, since $\mathrm{CoAt}(A) = \{\{x\} : x \in A\}$, this is further identical to $\bigcup \mathsf{F}(T)[\mathrm{CoAt}(A)]$. So, since $\mathsf{F}(T)(A)$ already is closed, we have $\mathsf{F}(T)(A) = \bigwedge \mathsf{F}(T)[\mathrm{CoAt}(A)]$.

---

[26]Note that a function between co-atomic partial orders that preserves arbitrary infima is co-atomic, so co-atomicity can be seen as a weakening of infima preservation (which, for monotone functions on complete lattices, is equivalent to having a lower adjoint).

To show Scott-continuity, let $\mathcal{F} \subseteq \mathsf{F}(X)$ be directed and show

$$\mathsf{T}\Big[\bigcap \mathcal{F}\Big] = \bigcap_{A \in \mathcal{F}} \mathsf{T}[A].$$

($\subseteq$) For $A \in \mathcal{F}$, we have $\bigcap \mathcal{F} \subseteq A$, so, by monotonicity, $\mathsf{T}\big[\bigcap \mathcal{F}\big] \subseteq \mathsf{T}[A]$. Hence $\mathsf{T}\big[\bigcap \mathcal{F}\big] \subseteq \bigcap_{A \in \mathcal{F}} \mathsf{T}[A]$.

($\supseteq$) Assume $y \in \bigcap_{A \in \mathcal{F}} \mathsf{T}[A]$. Then, for each $A \in \mathcal{F}$, there is $x_A \in A$ with $y \in \mathsf{T}(x_A)$. So the set $C_A := \{x \in X : x \in A \text{ and } y \in \mathsf{T}(x)\}$ is nonempty. It also is closed: it is the intersection of A and $\{x \in X : y \in \mathsf{T}(x)\}$, and the latter also is closed, since the graph of T is closed. Moreover, if $A \supseteq B$, then $C_A \supseteq C_B$. Hence $\{C_A : A \in \mathcal{F}\}$ is a collection of closed subsets of X with the finite intersection property. By compactness of X, the intersection of the C_A is nonempty, so let $x \in \bigcap_{A \in \mathcal{F}} C_A$. Then $x \in \bigcap \mathcal{F}$ and $y \in \mathsf{T}(x)$ (note that there is some $A \in \mathcal{F}$), so $y \in \mathsf{T}\big[\bigcap \mathcal{F}\big]$, as needed. $\square$

Write czPol^m for the category of compact and zero-dimensional Polish spaces together with upper-hemicontinuous and closed-valued multifunctions. Recall that bAlg^c is the category of algebraic lattices whose compact elements form a countable sublattice that is a Boolean algebra together with Scott-continuous and co-atomic functions.

Proposition A.6. $\mathsf{F} : \mathsf{czPol}^m \to \mathsf{bAlg}^c$ *is a functor establishing a categorical equivalence.*

In particular, this shows that the algebraic lattices in bAlg^c are co-atomic.

Proof. By proposition A.4, if X is a compact and zero-dimensional Polish space, then $\mathsf{F}(X)$ is indeed an algebraic lattice whose compact element are $\mathsf{Clp}(X)$, which is a countable sublattice and a Boolean algebra. By proposition A.5, F sends upper-hemicontinuous and closed-valued multifunctions to Scott-continuous and co-atomic functions. And F preserves composition: $\mathsf{F}(S \circ \mathsf{T})(A) = S \circ \mathsf{T}[A] = S[\mathsf{T}[A]] = \mathsf{F}(S)(\mathsf{T}[A]) = \mathsf{F}(S) \circ \mathsf{F}(\mathsf{T})(A)$.

Hence F is a functor. We show that it is (1) essentially surjective, (2) faithful, and (3) full.

Ad (1). Let L be an algebraic lattice whose set $K(L)$ of compact elements is a countable sublattice of L and a Boolean algebra. Let X be the Stone space corresponding to $K(L)$ under Stone duality. Then X is a compact and zero-dimensional Polish space and $\mathsf{Clp}(X)$ is isomorphic to $K(L)$. Hence also the algebraic lattice $\mathsf{F}(X)$ (whose compact elements are $\mathsf{Clp}(X)$) is isomorphic to the algebraic lattice L (whose compact elements are $K(L)$).

Ad (2). If $S, \mathsf{T} : X \rightrightarrows Y$ are two upper-hemicontinuous and closed-valued multifunctions between compact and zero-dimensional Polish spaces with $S \neq \mathsf{T}$, then there is $x \in X$ with $S(x) \neq \mathsf{T}(x)$, hence $\mathsf{F}(S)(\{x\}) \neq \mathsf{F}(\mathsf{T})(\{x\})$, so $\mathsf{F}(S) \neq \mathsf{F}(\mathsf{T})$.

Ad (3). Let X and Y be compact and zero-dimensional Polish spaces and let $f : \mathsf{F}(X) \to \mathsf{F}(Y)$ be a Scott-continuous and co-atomic function. Define the multifunction $\mathsf{T} : X \rightrightarrows Y$ by $\mathsf{T}(x) := f(\{x\})$. We show that T is upper-hemicontinuous and closed-valued with $\mathsf{F}(\mathsf{T}) = f$.

By construction, $\mathsf{T}(x) \in \mathsf{F}(Y)$, so T is closed-valued. For upper-hemicontinuity, let $x \in X$ and $V \subseteq Y$ open with $\mathsf{T}(x) \subseteq V$. We need to find an open $U \subseteq X$ with $x \in U$ and $\mathsf{T}[U] \subseteq V$.

We first note that it is enough to show this for clopen V: For a (merely) open V, it is, by zero-dimensionality, the union of clopen V_i, which hence form an open cover of the closed set $T(x)$, so there is a finite subcover $V_{i_1}, \ldots, V_{i_n}$ of $T(x)$. Hence, for the clopen $V^* := V_{i_1} \cup \ldots \cup V_{i_n}$, we have $T(x) \subseteq V^* \subseteq V$. Hence, if the claim holds for clopens, there is an open $U \subseteq X$ with $x \in U$ and $T[U] \subseteq V^* \subseteq V$, as needed.

So assume V is clopen. So $V \in F(Y)$ is (order-theoretically) compact. By Scott-continuity of f,

$$\bigcap \{f(C) : x \in C \in \mathrm{Clp}(X)\} = f\big(\bigcap \{C \in \mathrm{Clp}(X) : \{x\} \subseteq C\}\big) = f(\{x\}) = T(x) \subseteq V.$$

Since V is compact and $\leqslant$ is reverse inclusion, there hence is $U \in \mathrm{Clp}(X)$ with $x \in U$ and $f(U) \geqslant V$. Hence, for $x' \in U$, we have, by monotonicity of f, that $T(x') = f(\{x'\}) \geqslant f(U) \geqslant V$, so $T(x') \subseteq V$, as needed.

To show $F(T) = f$, let $A \in F(X)$ and show $F(T)(A) = f(A)$. Since f is co-atomic, we have

$$f(A) = \bigwedge f[\mathrm{CoAt}(A)] = \mathrm{Cl}\bigcup\{T(x) : x \in A\} = \mathrm{Cl}T[A].$$

Since we already know that T is upper-hemicontinuous and closed-valued, the corollary to the closed graph theorem implies that $T[A]$ is closed, so $f(A) = \mathrm{Cl}T[A] = T[A]$, as needed.   □

Finally, we prove theorem 5.1: The category $\mathsf{Sys}^{\mathrm{ef}}$ is equivalent to the category dynAlg via the functor G sending (X, T) to $(F(X), F(T))$ and sending $(e, f) : (X, T) \to (Y, S)$ to $(F(e), F(f))$.

Proof of theorem 5.1. First note that dynAlg is indeed a category: Composition of $(\epsilon, \pi) : (A, \alpha) \to (B, \beta)$ and $(\epsilon', \pi') : (B, \beta) \to (C, \gamma)$ is given by $(\epsilon' \circ \epsilon, \pi \circ \pi')$, which again satisfies properties 1–5. The identity morphism is $\mathrm{id}_{(A,\alpha)} = (\mathrm{id}_A, \mathrm{id}_A)$ and clearly satisfies properties 1–5.

Next we show that G is a well-defined functor. If (X, T) is in $\mathsf{Sys}^{\mathrm{ef}}$, then, by proposition A.6, $G(X, T) = (F(X), F(T))$ is in dynAlg. So assume $(e, f) : (X, T) \to (Y, S)$ is an ef-pair and show that $G(e, f) = (F(e), F(f))$ has properties 1–5. (1) Recall that the co-atoms of $F(Y)$ are singletons, and $F(f)(\{y\}) = f(y)$ is again a singleton since f is a function (i.e., multifunction with singleton values). (2) Show $F(f) \circ F(e) = \mathrm{id}_{F(X)}$. Indeed, for $A \in F(X)$, we have, since $f \circ e(x) = \{x\}$, that $F(f) \circ F(e)(A) = f \circ e[A] = A$. (3) Show $F(e) \circ F(f) \leqslant \mathrm{id}_{F(Y)}$. Indeed, for $B \in F(Y)$, we have, since $y \in e \circ f(y)$, that $F(e) \circ F(f)(B) = e \circ f(B) \supseteq B$. (4) Show $F(f)F(S)F(e) \leqslant F(T)$. It suffices to show, for $x \in X$, that $F(f)F(S)F(e)(x) \supseteq F(T)(x)$. So let $x' \in X$ with $x' \in F(T)(x) = T(x)$ and show $x' \in F(f)F(S)F(e)(x) = f \circ S \circ e(x)$. Since $f : (Y, S) \to (X, T)$ is a factor, $x \xrightarrow{T} x'$ implies that there are $y, y' \in Y$ such that $y \xrightarrow{f} x$, $y' \xrightarrow{f} x'$, and $y \xrightarrow{S} y'$. Since $y \in e \circ f(y) = e(x)$, we have $x \xrightarrow{e} y$. So $x \xrightarrow{e} y \xrightarrow{S} y' \xrightarrow{f} x'$, so $x' \in f \circ S \circ e(x)$, as needed. (5) Show $F(T)F(f) \leqslant F(f)F(S)$. We show, for $y \in Y$, that $f \circ S(y) \subseteq T \circ f(y)$. If $x' \in f \circ S(y)$, there is $y' \in Y$ with $y \xrightarrow{S} y' \xrightarrow{f} x'$. Write $x := f(y)$. Since f is a factor, there is $x'' \in X$ with $y' \xrightarrow{f} x''$ and $x \xrightarrow{T} x''$. Since f is a function, $x'' = f(y') = x'$. Hence $y \xrightarrow{f} x \xrightarrow{T} x'$, so $x' \in T \circ f(y)$. Also, it is straightforward that G preserves composition.

It remains to show that G is essentially surjective, full, and faithful.

Essentially surjective: Since, by proposition A.6, F is an equivalence, given (A, α) in dynAlg, there is X in czPol$^{\mathrm{m}}$ with an isomorphism $\iota : F(X) \to A$. Moreover, for the morphism

$$F(X) \xrightarrow{\iota} A \xrightarrow{\alpha} A \xrightarrow{\iota^{-1}} F(X)$$

there is $T : X \rightrightarrows X$ in czPol$^{\mathrm{m}}$ with $F(T) = \iota^{-1} \circ \alpha \circ \iota$. Hence $G(X, T) = (F(X), F(T))$ is isomorphic to (A, α) via the isomorphism (ι, ι^{-1}) in dynAlg, whose inverse is (ι^{-1}, ι).

Faithful: If $(e, f), (e', f') : (X, T) \to (Y, S)$ are ef-pairs with $(e, f) \neq (e', f')$, then $e \neq e'$ (and also $f \neq f'$, since one determines the other), so, since F is faithful, $F(e) \neq F(e')$, so $(F(e), F(f)) \neq (F(e'), F(f'))$.

Full: If $(\epsilon, \pi) : G(X, T) \to G(Y, S)$ is a morphism, then $\epsilon : F(X) \to F(Y)$ and $\pi : F(Y) \to F(X)$ are morphisms in bAlg$^{\mathrm{c}}$, so, since F is full, there are upper-hemicontinuous and closed-valued multifunctions $e : X \rightrightarrows Y$ and $f : Y \rightrightarrows X$ such that $F(e) = \epsilon$ and $F(f) = \pi$. It suffices to show that $(e, f) : (X, T) \to (Y, S)$ is an ef-pair.

We first show that f is a surjective function. It is a function, since, by 1, $f(y) = F(f)(\{y\}) = \pi(\{y\})$ is a co-atom, i.e., a singleton. To show that f is surjective, note that, given $x \in X$, the set $\epsilon(\{x\})$ is nonempty: otherwise we get the contradiction

$$\{x\} = \pi \circ \epsilon(\{x\}) = \pi(\emptyset) = F(f)(\emptyset) = f[\emptyset] = \emptyset.$$

So let $y \in \epsilon(\{x\})$. Hence $\{y\} \subseteq \epsilon(\{x\})$ and, by monotonicity, $f(y) = \pi(\{y\}) \subseteq \pi \circ \epsilon(\{x\}) = \{x\}$. So the singleton $f(y)$ is identical to $\{x\}$. Hence, by viewing f as a function rather than a multifunction with singleton values, we have $f(y) = x$, as needed.

Next we note that, for all $x \in X$, we have $e(x) = f^{-1}(x)$. Indeed, if $y \in e(x) = \epsilon(\{x\})$, then, as above, $f(y) = x$. And if $f(y) = x$, or, when rather viewing f as a multifunction, $f(y) = \{x\}$, then $\{y\} \subseteq \epsilon(\pi(\{y\}) = \epsilon(f(y)) = \epsilon(\{x\}) = e(x)$, so $y \in e(x)$, as needed.

Now we show that $f : Y \to X$ is a factor. Concerning equivariance 2, if $x \xrightarrow{T} x'$, then

$$x' \in T(x) = F(T)(\{x\}) \subseteq \pi \circ F(S) \circ \epsilon(\{x\}) = \pi \circ F(S)(e(x)) = f[S[f^{-1}(x)]]$$

hence there is $y, y' \in Y$ with $x \xrightarrow{f^{-1}} y \xrightarrow{S} y' \xrightarrow{f} x'$. In other words, $y \xrightarrow{f} x$, $y' \xrightarrow{f} x'$, and $y \xrightarrow{S} y'$, as needed. Concerning equivariance 3, if $y \xrightarrow{S} y'$ and $y \xrightarrow{f} x$, write $x' := f(y')$. Then

$$x' \in f[S(y)] = \pi \circ F(S)(\{y\}) \subseteq F(T) \circ \pi(\{y\}) = T[f(y)],$$

so there is $x_0 \in X$ with $y \xrightarrow{f} x_0 \xrightarrow{T} x'$. Since f is a function, $x_0 = x$. In sum, we have $x' \in X$ with $x \xrightarrow{T} x'$ and $y' \xrightarrow{f} x'$, as needed.

Finally, to show that (e, f) is an ef-pair, we know, by proposition 3.11, that there is a unique embedding $\underline{f} : (X, T) \to (Y, S)$ such that $(\underline{f}, f)$ is an ef-pair and, for all x, $\underline{f}(x) = f^{-1}(x) = e(x)$. $\qquad\square$

A.3. **Proofs from section 6.**

Proof of the claim in example 6.4. Assume there was a factor $f : (2^\omega, \sigma) \to (Y, S)$ with $S : Y \to Y$ a function on a finite discrete space Y with at least two elements. Then $\{f^{-1}(y) : y \in Y\}$ is a clopen partition of 2^ω, so there is k such that any two sequences $x, x' \in 2^\omega$ that agree on their first k elements are in the same partition cell, i.e., $f(x) = f(x')$.

Our strategy is to find $y \neq y'$ in Y and $j \geqslant k$ such that $S^j(y) \neq y'$. Then we get the desired contradiction as follows: Since f is surjective, there are $x, x' \in 2^\omega$ with $f(x) = y$ and $f(x') = y'$. Consider the concatenation $x'' := (x \restriction j)x' \in 2^\omega$. Then, by agreement on the first k elements, $f(x'') = f(x) = y$. By construction, $\sigma^j(x'') = x'$. So, by equivariance, $S^j(y) = S^j(f(x'')) = f(\sigma^j(x'')) = f(x') = y'$, contradiction.

If S is the identity function on Y, this is easy: since Y has at least two elements, let $y \neq y'$ and pick $j := k$, so $S^j(y) = y \neq y'$. So we assume that there is $y \in Y$, with $S(y) \neq y$. Consider the orbit $y_n := S^n(y)$. Since Y is finite, there is a first time n such that $y_n = y_i$ for some $i < n$. If $i = 0$, the orbit (y_n) is of the form $y_0(y_1 \ldots y_{n-1}y_0)^\omega$ with $y_1 \neq y_0$. If $i > 0$, the orbit (y_n) is of the form $y_0 \ldots y_{i-1}(y_i \ldots y_{n-1})^\omega$ with $y_i \neq y_0$ (otherwise $y_0 = y_i = y_n$ with $0 < i < n$, so n would not be minimal). In either case, the orbit (y_n) is of the form $\rho_0(\rho_1)^\omega$ with ρ_0 and ρ_1 being finite paths in (Y, S). The first state of ρ_0 is $y_0 = y$. Let y' be the first state of ρ_1 (either y_1 or y_i). Then, for $j := k|\rho_1| \geqslant k$, we have $S^j(y') = y' \neq y$, as needed. $\qquad\square$

Proof of lemma 6.7. Let $f : (X, T) \to (Y, S)$ and $g : (X, T) \to (Z, R)$ be factors of systems in allSys and let $h : Y \to Z$ be a function with $h \circ f = g$. We show that h is a factor.

First note that, for $C \subseteq Z$, we have $h^{-1}(C) = f[g^{-1}(C)]$: Indeed, for $y \in Y$, if $y \in h^{-1}(C)$, then, by surjectivity, let $x \in X$ with $f(x) = y$, so $g(x) = h(f(x)) = h(y) \in C$, so $y \in f[g^{-1}(C)]$. Conversely, if $y = f(x)$ for some $x \in g^{-1}(C)$, then $h(y) = h(f(x)) = g(x) \in C$, so $y \in h^{-1}(C)$.

Now, h is continuous: If $C \subseteq Z$ is closed, then $h^{-1}(C) = f[g^{-1}(C)]$ is an image of a closed set and hence closed, since h is closed qua continuous function between compact Hausdorff spaces. Also, h is surjective: if $z \in Z$, let, by surjectivity, $x \in X$ with $g(x) = z$, then, for $y := f(x) \in Y$, we have $h(y) = h(f(x)) = g(x) = z$.

Concerning equivariance 2, if $z, z' \in Z$ with $z \xrightarrow{R} z'$, there is $x, x' \in X$ with $g(x) = z$, $g(x') = z'$, and $x \xrightarrow{T} x'$, so, for $y := f(x)$ and $y' := f(x')$ in Y we have $y \xrightarrow{S} y'$ and $h(y) = h(f(x)) = g(x) = z$ and, similarly, $h(y') = z'$, as needed.

Concerning equivariance 3, if $y, y' \in Y$ and $z \in Z$ with $h(y) = z$ and $y \xrightarrow{S} y'$, consider $z' := h(y')$. We need to show $z \xrightarrow{R} z'$. Since f is a factor, there is $x, x' \in X$ with $f(x) = y$, $f(x') = y'$, and $x \xrightarrow{T} x'$. Hence $g(x) \xrightarrow{R} g(x')$, and we have $g(x) = h(f(x)) = h(y)$ and, similarly, $g(x') = h(y')$, so $z = h(y) \xrightarrow{R} h(y') = z'$. $\qquad\square$

Proof of lemma 6.10. Recall that $\big((X_i, T_i), (e_{ii+1}, f_{ii+1})\big)_{i \in \omega}$ is an ω-chain in S^{ef} and $\big((X, T), (e_i, f_i)\big)$ is a colimit. Assume $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a finite partition of X consisting of closed sets. Assume for contradiction, that there is no $i \in \omega$ such that X/f_i refines $\mathcal{C}$. Then, for every $i \in \omega$, there is $x, x' \in X$ with $f_i(x) = f_i(x')$ but

$x \in C_k$ and $x' \in C_l$ for $k \neq l$, so the following subset of $X \times X$ is nonempty:

$$F_i := \big\{(x, x') \in X \times X : f_i(x) = f_i(x')\big\} \cup \bigcup_{k \neq l \text{ in } \{1,\dots,n\}} C_k \times C_l.$$

This set is closed qua finite union of closed sets (the equivalence relation $\equiv$ of a continuous function f into a Hausdorff space is closed). Moreover, if $i \leqslant j$ in ω, then $F_i \supseteq F_j$ (if $f_j(x) = f_j(x')$, then $f_i(x) = f_{ij} \circ f_j(x) = f_{ij} \circ f_j(x') = f_i(x')$). Since $X \times X$ is compact, there is $(x, x') \in \bigcap_{i \in \omega} F_i$. So there is $k \neq l$ with $x \in C_k$ and $x' \in C_l$, so $x \neq x'$. However, for all $i \in \omega$, we also have $f_i(x) = f_i(x')$, which implies $x = x'$ (proposition 6.6). □

A.4. **Proofs from section 8.**

Proof of proposition 8.1. Ad 1. It is straightforward to show that $\overline{X}$ is a closed subset of $\prod_\omega X$. So $\overline{X}$ again is a zero-dimensional compact Polish space. It also is immediate that $\sigma : \overline{X} \to \overline{X}$ is a continuous function. Hence, per definition 3.12, $\mathsf{Path}(X, T) = (\overline{X}, \sigma)$ is in $\mathsf{detSys}^{\mathsf{ef}}$ iff $\overline{X}$ is nonempty, as needed.

Ad 2. First, let us assume that (X, T) is total. Write $p := \pi_0 : \overline{X} \to X$ for the projection to the first component, and show that p is a factor. Indeed, p is a continuous: if $U \subseteq X$ is open, then $p^{-1}(U) = \pi_0^{-1}(U) \cap \overline{X}$, which is open. And p is surjective: If $x \in X$, then, since T is total, we can continue x to some path $\overline{x} \in \overline{X}$, so $p(\overline{x}) = x$. Concerning equivariance 2, if $x \xrightarrow{T} x'$, continue this, since T is total, into a path $\overline{x}$:

$$x \xrightarrow{T} x' \xrightarrow{T} x_2 \xrightarrow{T} x_3 \xrightarrow{T} \dots,$$

then $\overline{x} \xrightarrow{P} x$, $\sigma(\overline{x}) \xrightarrow{P} x'$, and $\overline{x} \xrightarrow{\sigma} \sigma(\overline{x})$. Concerning equivariance 3, if $\overline{x} \xrightarrow{\sigma} \sigma(\overline{x})$ and $\overline{x} \xrightarrow{P} x$, then, for $x' := p(\sigma(\overline{x}))$, we have, since $\overline{x}$ is a path, that $x \xrightarrow{T} x'$, as needed.

Conversely, assume that $p : \overline{X} \to X$ is a factor, and show that (X, T) is total. Given $x \in X$, there is, since p is surjective, some path $(x_n) \in \overline{X}$ with $p\big((x_n)\big) = x$. Hence $x = x_0 \xrightarrow{T} x_1$, i.e., x has a successor state, as needed. □

A.5. **Proofs from section 9.**

Proof of lemma 9.3. We can assume that X and Y are nonempty: Otherwise, if $X = \emptyset$, then $f = \emptyset$ and we can choose $N := 0$ and $\varphi = \emptyset : B_0(X) \to B$. If $Y = \emptyset$, then, since f is a function, also $X = \emptyset$, and the claim, as just seen, holds.

Since f is continuous and $\{C_b(Y) : b \in B\}$ is a clopen partition of Y, also $\{f^{-1}(C_b(Y)) : b \in B\}$ is a clopen partition of X. (Since X and Y are nonempty, these partitions contain at least one cell.) So there is N, such that this partition is refined by the partition $\{C_w(X) : w \in A^N\}$. So we can define $\varphi : B_N(X) \to B$ by mapping $w \in B_N(X)$ to the unique $b \in B$ such that $C_w(X) \subseteq f^{-1}(C_b(Y))$. Finally, let $x \in X$ and $k \geqslant 0$ and show $f(x)(k) = \varphi(x(k), \dots, x(k + N - 1))$. By equivariance,

$$b := f(x)(k) = \sigma^k(f(x))(0) = f(\sigma^k(x))(0) \in B.$$

So $x' := \sigma^k(x) \in f^{-1}(C_b(Y))$. Write $w := (x'(0), \dots, x'(N-1)) = (x(k), \dots, x'(k + N - 1)) \in B_N(X)$. We have $x' \in C_w(X) \cap f^{-1}(C_b(Y))$, hence, qua refinement, $C_w(X) \subseteq f^{-1}(C_b(Y))$, so $\varphi(w) = b$. Hence $f(x)(k) = b = \varphi(w) = \varphi(x(k), \dots, x(k + N - 1))$, as needed.

$$A_0 \xrightarrow{\ f_{01}\ } A_1 \xrightarrow{\ f_{12}\ } A_2 \longrightarrow \cdots \qquad\qquad A$$
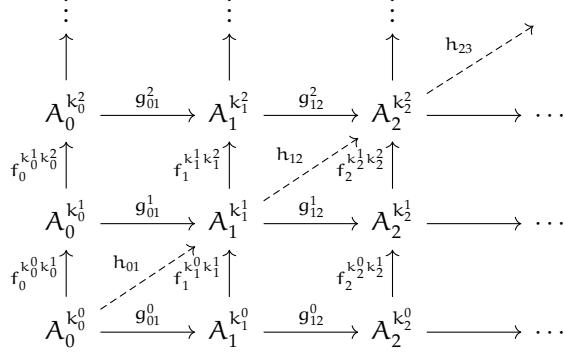
FIGURE 4. Finding a diagonal ω-chain.

For the 'moreover' part, note that, for $n \geqslant N$, the partition $\{C_w(X) : w \in A^n\}$ still refines $\{C_w(X) : w \in A^N\}$, which was all that is needed for the above proof. □

Proof of lemma 9.5. To fix notation, let $(A_i, f_{ii+1})_{i\in\omega}$ be an ω-chain with colimit $(A, f_i)_{i\in\omega}$. For each $i \in \omega$, let $(A_i^j, f_i^{jj+1})_{j\in\omega}$ be an ω-chain with colimit $(A_i, f_i^j)_{j\in\omega}$ such that each A_i^j is category-theoretically finite in C. For each $i \in \omega$, we construct a strictly increasing sequence $(k_i^j)_{j\in\omega}$ and morphisms $g_{ii+1}^j : A_i^{k_i^j} \to A_{i+1}^{k_{i+1}^j}$ as follows (see figure 4).

For $i = 0$, we set $k_i^j := j$. For $i = 1$, we build $k_1^0 < k_1^1 < k_1^2 < \ldots$ inductively. Set $k_1^{-1} := 0$ and assume k_1^{j-1} is defined, Note that we have the ω-chain $(A_1^j, f_1^{jj+1})_{j\in\omega}$ with colimit A_1, and we have the morphism $f_{01} \circ f_0^{k_0^j} : A_0^{k_0^j} \to A_1$. Since $A_0^{k_0^j}$ is category-theoretically finite, there hence is k_1^j and a morphism $g_{01}^j : A_0^{k_0^j} \to A_1^{k_1^j}$ such that $f_{01} \circ f_0^{k_0^j} = f_1^{k_1^j} \circ g_{01}^j$ and

(∗) For each $l \geqslant k_1^j$, there is a unique morphism $u : A_0^{k_0^j} \to A_1^l$ such that
     $f_{01} \circ f_0^{k_0^j} = f_1^l \circ u$.

Now we continue similarly for $i = 2, 3, \ldots$, and we get the diagram in figure 4.

It is straightforward to show that this diagram commutes. So we can define the 'diagonal' ω-chain $(A_m^{k_m^m}, h_{mm+1})_{m\in\omega}$, where

$$h_{mm+1} := f_{m+1}^{k_{m+1}^m k_{m+1}^{m+1}} \circ g_{mm+1}^m = g_{mm+1}^{m+1} \circ f_m^{k_m^m k_m^{m+1}}.$$

Now it is a bit tedious but straightforward to show that A also is a colimit of this ω-chain, via the natural choice for the projections: $h_m := f_m \circ f_m^{k_m^m}$. □

REFERENCES

[1] S. Abramsky and A. Jung. "Domain Theory". In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum. http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf. Oxford: Oxford University Press, 1994.

[2] J. Adámek, S. Milius, and L. S. Moss. "Fixed points of functors". In: *Journal of Logical and Algebraic Methods in Programming* 95 (02/2018), pp. 41–81. DOI: 10.1016/j.jlamp.2017.11.003.

[3] E. Akin. "Conjugacy in the Cantor set automorphism group". In: *Ergodic Theory, Dynamical Systems, and the Continuing Influence of John C. Oxtoby*. Contemporary Mathematics, vol. 678. Providence, RI: American Mathematical Society, 2016, pp. 1–42. DOI: 10.1090/conm/678/13639.

[4] E. Akin, E. Glasner, and B. Weiss. "Generically there is but one self homeomorphism of the Cantor set". In: *Transactions of the American Mathematical Society* 360.07 (02/2008), pp. 3613–3631. DOI: 10.1090/S0002-9947-08-04450-4.

[5] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. 3rd ed. Berlin: Springer-Verlag, 2006.

[6] J. Almeida, A. Costa, and H. Goulet-Ouellet. *Profinite approach to S-adic shift spaces I: Saturating directive sequences*. 2025. arXiv: 2509.00991 [math.DS]. URL: https://arxiv.org/abs/2509.00991.

[7] M. Balcerzak and T. Kania. "The existence of UFO implies projectively universal morphisms". In: *Proc. Amer. Math. Soc.* 151.9 (2023), pp. 3737–3742.

[8] B. Banaschewski and H. Herrlich. "Subcategories defined by implications". In: *Houston Journal of Mathematics* 2 (1976), pp. 149–171. URL: https://www.math.uh.edu/~hjm/vol02-2.html.

[9] S. Barbieri and L. Poirier. *A remark on inverse limits of effective subshifts*. 2025. arXiv: 2506.00183 [math.DS]. URL: https://arxiv.org/abs/2506.00183.

[10] A. Bartoš, T. Bice, K. Dasilva Barbosa, and W. Kubiś. "The weak Ramsey property and extreme amenability". In: *Forum of Mathematics, Sigma* 12 (2024), e96. DOI: 10.1017/fms.2024.64.

[11] D. Bartošová. "Universal Minimal Flows of Groups of Automorphisms of Uncountable Structures". In: *Canadian Mathematical Bulletin* 56.4 (12/2013), pp. 709–722. DOI: 10.4153/CMB-2012-023-4.

[12] M. Behrisch, S. Kerkhoff, R. Pöschel, F. M. Schneider, and S. Siegmund. "Dynamical Systems in Categories". In: *Applied Categorical Structures* 25 (2017), pp. 29–57. DOI: https://doi.org/10.1007/s10485-015-9409-8.

[13] N. C. Bernardes Jr. and U. B. Darji. "Graph theoretic structure of maps of the Cantor space". In: *Advances in Mathematics* 231.3–4 (10/2012), pp. 1655–1680. DOI: 10.1016/j.aim.2012.05.024.

[14] G. Bezhanishvili, J. Harding, and P. J. Morandi. *Remarks on Hyperspaces for Priestley Spaces*. 2022. DOI: 10.48550/arXiv.2207.03604. arXiv: 2207.03604.

[15] P. Blackburn, M. Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge: Cambridge University Press, 2001.

[16] O. Bournez, N. Dershowitz, and P. Néron. "Axiomatizing Analog Algorithms". In: *CiE 2016*. Ed. by A. Beckmann, L. Bienvenu, and N. Jonoska. Vol. 9709. LNCS. Cham: Springer, 2016, pp. 215–224.

[17] O. Bournez, J. Cohen, and A. Wurm. "A Universal Uniform Approximation Theorem for Neural Networks". In: *50th International Symposium on Mathematical Foundations of Computer Science (MFCS 2025)*. Ed. by P. Gawrychowski, F. Mazowiecki, and M. Skrzypczak. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025, 29:1–29:20. DOI: `10.4230/LIPIcs.MFCS.2025.29`.

[18] O. Bournez and A. Pouly. "A Survey on Analog Models of Computation". In: *Handbook of Computability and Complexity in Analysis*. Ed. by V. Brattka and P. Hertling. Cham: Springer, 2021, pp. 173–226.

[19] O. Bournez and A. Pouly. "A Universal Ordinary Differential Equation". In: *Logical Methods in Computer Science* 16.1 (2020).

[20] R. Camerlo. "Characterising quotients of projective Fraïssé limits". In: *Topology and its Applications* 157.12 (2010), pp. 1980–1989. DOI: `https://doi.org/10.1016/j.topol.2010.04.012`.

[21] O. Caramello. "Fraïssé's construction from a topos-theoretic perspective". In: *Logica Universalis* 8 (2014), pp. 261–281. DOI: `https://doi.org/10.1007/s11787-014-0104-6`.

[22] R. Cardona, E. Miranda, and D. Peralta-Salas. *Towards a Fluid computer*. 2024. arXiv: `2405.20999 [math.DS]`. URL: `https://arxiv.org/abs/2405.20999`.

[23] R. Cardona, E. Miranda, D. Peralta-Salas, and F. Presas. "Constructing Turing complete Euler flows in dimension 3". In: *Proceedings of the National Academy of Sciences* 118.19 (05/2021). DOI: `https://doi.org/10.1073/pnas.2026818118`.

[24] M. Cook. "Universality in Elementary Cellular Automata". In: *Complex Systems* 15.1 (03/2004), pp. 1–40. DOI: `10.25088/ComplexSystems.15.1.1`.

[25] E. M. Coven, M. Pivato, and R. Yassawi. "Prevalence of odometers in cellular automata". In: *Proceedings of the American Mathematical Society* 135.3 (09/2006), pp. 815–821. DOI: `10.1090/S0002-9939-06-08754-5`.

[26] G. Cybenko. "Approximations by superpositions of sigmoidal functions". In: *Mathematics of Control, Signals, and Systems* 2.4 (1989), pp. 303–314.

[27] F. Dahlqvist, V. Danos, and I. Garnier. "Giry and the Machine". In: *Electronic Notes in Theoretical Computer Science* 325 (2016). The Thirty-second Conference on the Mathematical Foundations of Programming Semantics (MFPS XXXII), pp. 85–110. DOI: `https://doi.org/10.1016/j.entcs.2016.09.033`.

[28] V. Danos and I. Garnier. "Dirichlet is Natural". In: *Electronic Notes in Theoretical Computer Science* 319 (2015), pp. 137–164.

[29] U. Darji and É. Matheron. "Some universality results for dynamical systems". In: *Proc. Amer. Math. Soc.* 145.1 (2017), pp. 251–265.

[30] M. Davis. "The definition of universal Turing machine". In: *Proc. Amer. Math. Soc.* 8 (1957), pp. 1125–1126.

[31] J. De Groot. "Non-archimedean metrics in topology". In: *Proceedings of the American Mathematical Society* 7.5 (1956), pp. 948–953. DOI: `https://doi.org/10.1090/S0002-9939-1956-0080905-8`.

[32] G. De las Cuevas and T. S. Cubitt. "Simple universal models capture all classical spin physics". In: *Science* 351.6278 (03/2016), pp. 1180–1183. DOI: `https://doi.org/10.1126/science.aab3326`.

[33] M. Doucha. *Strong topological Rokhlin property, shadowing, and symbolic dynamics of countable groups*. To appear in JEMS. 2024. arXiv: `2211.08145 [math.DS]`. URL: `https://arxiv.org/abs/2211.08145`.

[34] M. Doucha, J. Melleray, and T. Tsankov. *Dense and comeager conjugacy classes in zero-dimensional dynamics*. 2025. arXiv: `2507.05474 [math.DS]`. URL: `https://arxiv.org/abs/2507.05474`.

[35] T. Downarowicz. "Survey of odometers and Toeplitz flows". In: *Algebraic and Topological Dynamics*. Ed. by S. Kolyada, Y. Manin, and T. Ward. Vol. 385. Contemporary Mathematics. Providence, RI: American Mathematical Society, 2005, pp. 7–37. DOI: `10.1090/conm/385`.

[36] T. Downarowicz and O. Karpel. "Dynamics in dimension zero: A survey". In: (2016). DOI: `10.48550/arXiv.1610.02727`. arXiv: `1610.02727`.

[37] R. G. Downey and D. R. Hirschfeldt. *Algorithmic Randomness and Complexity*. New York: Springer, 2010.

[38] M. Droste and R. Göbel. "Universal domains and the amalgamation property". In: *Mathematical Structures in Computer Science* 3.2 (1993), pp. 137–159.

[39] M. Droste and D. Kuske. "Almost Every Domain is Universal". In: *Electronic Notes in Theoretical Computer Science* 173 (2007), pp. 103–119.

[40] A. Edalat. "Dynamical Systems, Measures, and Fractals via Domain Theory". In: *Information and Computation* 120.1 (1995), pp. 32–48.

[41] P. Erdős and A. Rényi. "Asymmetric graphs". In: *Acta Math. Acad. Sci. Hungar* 14.3 (1963), pp. 295–315.

[42] S. Galatolo, M. Hoyrup, and C. Rojas. "Effective symbolic dynamics, random points, statistical behavior, complexity and entropy". In: *Information and Computation* 208.1 (2010), pp. 23–41. DOI: `https://doi.org/10.1016/j.ic.2009.05.001`.

[43] R. Gandy. "Church's Thesis and Principles for Mechanisms". In: *The Kleene Symposium*. Ed. by J. Barwise, H. J. Keisler, and K. Kunen. Amsterdam: North-Holland Publishing Company, 1980, pp. 123–148.

[44] M. Gehrke. "Stone duality, topological algebra, and recognition". In: *Journal of Pure and Applied Algebra* 220.7 (07/2016), pp. 2711–2747. DOI: `http://dx.doi.org/10.1016/j.jpaa.2015.12.007`.

[45] M. Gehrke and S. van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*. Cambridge: Cambridge University Press, 02/2024. DOI: `https://doi.org/10.1017/9781009349680`.

[46] S. Geschke, S. Głąb, and W. Kubiś. *Evolutions of finite graphs*. 2022. arXiv: `2209.14626 [math.CO]`. URL: `https://arxiv.org/abs/2209.14626`.

[47] G. Gierz, K. H. Hoffman, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*. Cambridge: Cambridge University Press, 2003.

[48] M. Giunti. *Computation, Dynamics, and Cognition*. New York, Oxford: Oxford University Press, 1997.

[49] E. Glasner and B. Weiss. "The topological Rohlin property and topological entropy". In: *American Journal of Mathematics* 123.6 (12/2001), pp. 1055–1070. DOI: 10.1353/ajm.2001.0039.

[50] T. Gonda, T. Reinhart, S. Stengele, and G. D. l. Coves. "A Framework for Universality in Physics, Computer Science, and Beyond". In: *Compositionality* Volume 6 (2024) (08/2024). DOI: 10.46298/compositionality-6-3. URL: http://dx.doi.org/10.46298/compositionality-6-3.

[51] C. Good and J. Meddaugh. "Shifts of finite type as fundamental objects in the theory of shadowing". In: *Inventiones mathematicae* 220.3 (12/2020), pp. 715–736. DOI: https://doi.org/10.1007/s00222-019-00936-8.

[52] D. S. Graça, M. L. Campagnolo, and J. Buescu. "Robust Simulations of Turing Machines with Analytic Maps and Flows". In: *New Computational Paradigms (CiE 2005)*. Ed. by S. B. Cooper, B. Löwe, and L. Torenvliet. Vol. Lecture Notes in Computer Science, vol 3526. Berlin Heidelberg: Springer, 2005, pp. 169–179. DOI: https://doi.org/10.1007/11494645_21.

[53] G. A. Hedlund. "Endomorphisms and automorphisms of the shift dynamical system." In: *Mathematical Systems Theory* 3.4 (1969), pp. 320–375. DOI: https://doi.org/10.1007/BF01691062.

[54] M. Hochman. "A note on universality in multidimensional symbolic dynamics". In: *Discrete and Continuous Dynamical Systems - S* 2.2 (2009), pp. 301–314.

[55] M. Hochman. "Are there factor-universal CA?" ms. URL: http://math.huji.ac.il/~mhochman/open-problems.html.

[56] M. Hochman. "Genericity in topological dynamics". In: *Ergodic Theory and Dynamical Systems* 28.1 (02/2008), pp. 125–165. DOI: 10.1017/S0143385707000521.

[57] M. Hochman. "Rohlin properties for $\mathbb{Z}^d$ actions on the Cantor set". In: *Transactions of the American Mathematical Society* 364.3 (03/2012), pp. 1127–1143. DOI: 10.1090/S0002-9947-2011-05319-5.

[58] W. Hodges. *Model theory*. Cambridge: Cambridge University Press, 1993.

[59] K. Hornik. "Approximation capabilities of multilayer feedforward networks". In: *Neural Networks* 4.2 (1991), pp. 251–257.

[60] L. Hornischer. "Dynamical Systems via Domains: Toward a Unified Foundation of Symbolic and Non-symbolic Computation". https://www.illc.uva.nl/Research/Publications/Dissertations/DS-2021-10.text.pdf. PhD thesis. University of Amsterdam, Institute for Logic, Language and Computation, 2021.

[61] W. Ingram. *An Introduction to Inverse Limits with Set-valued Functions*. SpringerBriefs in Mathematics. New York: Springer, 2012. DOI: 10.1007/978-1-4614-4487-9.

[62] W. T. Ingram and W. S. Mahavier. *Inverse limits: From continua to Chaos*. Developments in Mathematics, vol 25. New York: Springer, 2012. DOI: `10.1007/978-1-4614-1797-2`.

[63] T. Irwin and S. Solecki. "Projective Fraïssé Limits and the Pseudo-Arc". In: *Transactions of the American Mathematical Society* 358.7 (07/2006), pp. 3077–3096.

[64] B. Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*. Cambridge: Cambridge University Press, 2016.

[65] A. Jacot, F. Gabriel, and C. Hongler. "Neural tangent kernel: Convergence and generalization in neural networks". In: *Advances in neural information processing systems* 31 (2018).

[66] H. Jaeger. *The "echostate" approach to analysing and training recurrent neural networks*. Tech. rep. GMD Report 148. German National Research Center for Information Technology, 2001. URL: `https://www.ai.rug.nl/minds/uploads/EchoStatesTechRep.pdf`.

[67] M. Kashiwara and P. Schapira. *Categories and Sheaves*. Vol. 332. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg: Springer-Verlag, 2006.

[68] K. Kawamura and J. Kennedy. "Shift maps and their variants on inverse limits with set-valued functions". In: *Topology and its Applications* 239 (04/2018), pp. 92–114. DOI: `https://doi.org/10.1016/j.topol.2018.02.015`.

[69] A. S. Kechris. "Descriptive Dynamics". In: *Descriptive Set Theory and Dynamical Systems*. Ed. by M. Foreman, A. S. Kechris, A. Louveau, and B. Weiss. 277. Cambridge: Cambridge University Press, 2000, pp. 231–258.

[70] A. S. Kechris, V. G. Pestov, and S. Todorcevic. "Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups". In: *Geometric & Functional Analysis GAFA* 15 (2005), pp. 106–189.

[71] A. Kechris. "Dynamics of non-archimedean Polish groups". In: *European Congress of Mathematics Kraków, 2–7 July, 2012*. Ed. by R. Latała, A. Ruciński, P. Strzelecki, J. Świątkowski, D. Wrzosek, and P. Zakrzewski. European Mathematical Society Publishing House, 2014, pp. 375–397. DOI: `10.4171/120-1/22`.

[72] A. S. Kechris. *Classical Descriptive Set Theory*. New York: Springer, 1995.

[73] A. S. Kechris and C. Rosendal. "Turbulence, amalgamation, and generic automorphisms of homogeneous structures". In: *Proceedings of the London Mathematical Society* 94.2 (11/2006), pp. 302–350. DOI: `10.1112/plms/pdl007`.

[74] P. Koiran, M. Cosnard, and M. Garzon. "Computability with low-dimensional dynamical systems". In: *Theoretical Computer Science* 132.1-2 (09/1994), pp. 113–128. DOI: `https://doi.org/10.1016/0304-3975(94)90229-1`.

[75] W. Kubiś. "Fraïssé sequences: category-theoretic approach to universal homogeneous structures". In: *Annals of Pure and Applied Logic* 165.11 (2014), pp. 1755–1811.

[76] W. Kubiś. "Universal monoid actions: the power of freedom". 2023. arXiv: `2307.15937 [math.CT]`.

[77] D. Kudithipudi, C. Schuman, C. M. Vineyard, T. Pandit, C. Merkel, R. Kubendran, J. B. Aimone, G. Orchard, C. Mayr, R. Benosman, J. Hays, C. Young, C. Bartolozzi, A. Majumdar, S. G. Cardwell, M. Payvand, S. Buckley, S. Kulkarni, H. A. Gonzalez, G. Cauwenberghs, C. S. Thakur, A. Subramoney, and S. Furber. "Neuromorphic computing at scale". In: *Nature* 637.8047 (01/2025), pp. 801–812. DOI: 10.1038/s41586-024-08253-8.

[78] C. Kupke, A. Kurz, and Y. Venema. "Stone Coalgebras". In: *Electronic Notes in Theoretical Computer Science* 82.1 (2003), pp. 170–190.

[79] A. Kwiatkowska. "The group of homeomorphisms of the Cantor set has ample generics". In: *Bulletin of the London Mathematical Society* 44.6 (2012), pp. 1132–1146.

[80] F. W. Lawvere. "Taking categories seriously". In: *Revista Colombiana de Matematicas* XX (1986). Republished in: *Reprints in Theory and Applications of Categories*, No. 8 (2005) pp. 1-24, pp. 147–178.

[81] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge: Cambridge University Press, 1995.

[82] M. Lupini. "Fraïssé limits in functional analysis". In: *Advances in Mathematics* 338 (2018), pp. 93–174.

[83] W. Maass, T. Natschläger, and H. Markram. "Real-time computing without stable states: A new framework for neural computation based on perturbations". In: *Neural computation* 14.11 (2002), pp. 2531–2560.

[84] D. Macpherson. "A survey of homogeneous structures". In: *Discrete Mathematics* 311.15 (2011), pp. 1599–1634. DOI: https://doi.org/10.1016/j.disc.2011.01.024.

[85] W. S. Mahavier. "Inverse limits with subsets of $[0, 1] \times [0, 1]$". In: *Topology and its Applications* 141.1–3 (06/2004), pp. 225–231. DOI: 10.1016/j.topol.2003.12.008.

[86] D. Mašulović. "The Kechris–Pestov–Todorčević Correspondence from the Point of View of Category Theory". In: *Applied Categorical Structures* 29.1 (09/2021), pp. 141–169. DOI: 10.1007/s10485-020-09611-z.

[87] L. Nguyen Van Thé. *A survey on structural Ramsey theory and topological dynamics with the Kechris-Pestov-Todorcevic correspondence in mind*. 2014. arXiv: 1412.3254 [math.CO]. URL: https://arxiv.org/abs/1412.3254.

[88] S. B. Niefield. "An Algebraic Approach to Chaos". In: *Applied Categorical Structures* 4 (1996), pp. 423–441.

[89] N. Ollinger. "Universalities in Cellular Automata". In: *Handbook of Natural Computing*. Ed. by G. Rozenberg, T. Bäck, and J. N. Kok. Berlin: Springer, 2012, pp. 189–229.

[90] P. Orponen and M. Matamala. "Universal computation by finite two-dimensional coupled map lattices". In: *PhysComp96: Proceedings of the 4th Workshop on Physics and Computation*. 1996.

[91] K. Palmer. *Shadowing in Dynamical Systems: Theory and Applications*. Mathematics and Its Applications. Dordrecht: Springer, 2000. DOI: 10.1007/978-1-4757-3210-8.

[92] G. Piccinini and C. Maley. "Computation in Physical Systems". In: *The Stanford Encyclopedia of Philosophy*. Ed. by E. N. Zalta. Summer 2021. Metaphysics Research Lab, Stanford University, 2021.

[93] I. Pitowsky. "The Physical Church Thesis and Physical Computational Complexity". In: *The Jerusalem Philosophical Quarterly* 39 (1990), pp. 81–99.

[94] M. B. Pour-El. "Abstract computability and its relation to the general purpose analog computer (some connections between logic, differential equations and analog computers)". In: *Transactions of the American Mathematical Society* 199 (1974), pp. 1–28.

[95] E. Rowland and R. Yassawi. "Profinite automata". In: *Advances in Applied Mathematics* 85 (2017), pp. 60–83. DOI: https://doi.org/10.1016/j.aam.2016.11.005.

[96] L. A. Rubel. "A universal differential equation". In: *Bulletin of the American Mathematical Society* 4 (1981), pp. 345–349.

[97] J. Rutten. "Universal coalgebra: a theory of systems". In: *Theoretical Computer Science* 249.1 (2000), pp. 3–80. DOI: https://doi.org/10.1016/S0304-3975(00)00056-6.

[98] A. M. Saxe, J. L. McClelland, and S. Ganguli. *Exact solutions to the nonlinear dynamics of learning in deep linear neural networks*. In the Conference Proceeding of the International Conference on Learning Representations (ICLR) 2014 https://iclr.cc/archive/2014/conference-proceedings/. 2014. arXiv: 1312.6120.

[99] P. Schultz, D. I. Spivak, and C. Vasilakopoulou. "Dynamical Systems and Sheaves". In: *Applied Categorical Structures* 28 (2020), pp. 1–57. DOI: https://doi.org/10.1007/s10485-019-09565-x.

[100] D. Scott. *Outline of a Mathematical Theory of Computation*. Tech. rep. PRG02. Available at https://www.cs.ox.ac.uk/files/3222/PRG02.pdf (last checked 24 January 2018). Oxford University Computing Laboratory, 1970.

[101] C. E. Shannon. "Mathematical theory of the differential analyser". In: *Journal of Mathematics and Physics MIT* 20 (1941), pp. 337–354.

[102] T. Shimomura. "The pseudo-orbit tracing property and expansiveness on the Cantor set". In: *Proceedings of the American Mathematical Society* 106.1 (1989), pp. 241–244. DOI: https://doi.org/10.1090/S0002-9939-1989-0942637-2.

[103] W. Sieg. "Calculations by man and machine: conceptual analysis". In: *Reflections on the foundations of mathematics: Essays in honor of Solomon Feferman*. Ed. by W. Sieg, R. Sommer, and C. Talcott. Vol. 15. Lecture Notes in Logic. Natick, Massachusetts: Association for Symbolic Logic, 2002, pp. 390–409.

[104] H. Siegelmann and E. D. Sontag. "Analog computation via neural networks". In: *Theoretical Computer Science* 131.2 (1994), pp. 331–360.

[105] H. T. Siegelmann and S. Fishman. "Analog computation with dynamical systems". In: *Physica D: Nonlinear Phenomena* 120.1-2 (1998), pp. 214–235.

[106] J. Šíma and P. Orponen. "Continuous-Time Symmetric Hopfield Nets Are Computationally Universal". In: *Neural Computation* 15.3 (2003), pp. 693–733. DOI: 10.1162/089976603321192130.

[107] J. Šíma and P. Orponen. "General-Purpose Computation with Neural Networks: A Survey of Complexity Theoretic Results". In: *Neural Computation* 15 (2003), pp. 2727–2778.

[108] P. Smolensky. "On the proper treatment of connectionism". In: *Behavioral and brain sciences* 11.1 (1988), pp. 1–74. DOI: https://doi.org/10.1017/S0140525X00052791.

[109] M. B. Smyth and G. D. Plotkin. "The Category-Theoretic Solution of Recursive Domain Equations". In: *SIAM Journal on Computing* 11.4 (1982), pp. 761–783. DOI: 10.1137/0211062.

[110] S. Stepney. "Nonclassical Computation—A Dynamical Systems Perspective". In: *Handbook of Natural Computing*. Ed. by G. Rozenberg, T. Bäck, and J. N. Kok. Berlin Heidelberg: Springer-Verlag, 2012, pp. 1979–2025. DOI: 10.1007/978-3-540-92910-9_59.

[111] T. F. Sterkenburg. "Universal Prediction: A Philosophical Investigation". PhD thesis. Rijksuniversiteit Groningen, 2018. URL: https://ir.cwi.nl/pub/27326.

[112] G. Tanaka, T. Yamane, J. B. Héroux, R. Nakane, N. Kanazawa, S. Takeda, H. Numata, D. Nakano, and A. Hirose. "Recent advances in physical reservoir computing: A review". In: *Neural Networks* 115 (07/2019), pp. 100–123. DOI: 10.1016/j.neunet.2019.03.005.

[113] T. Tao. *On the universality of potential well dynamics*. 2020. arXiv: 1707.02389 [math.AP]. URL: https://arxiv.org/abs/1707.02389.

[114] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*. Rhode Island: American Mathematical Society, 2012.

[115] A. M. Turing. "On Computable Numbers, with an Application to the Entscheidungsproblem". In: *Proceedings of the London Mathematical Society* 42.1 (1936-7), pp. 230–265. DOI: https://doi.org/10.1112/plms/s2-42.1.230.

[116] B. Ulmann. *Analog Computing*. München: Oldenbourg Verlag München, 2013.

[117] J. de Vries. *Topological Dynamical Systems: An Introduction to the Dynamics of Continuous Mappings*. Vol. 59. De Gruyter Studies inMathematics. Berlin/Boston: de Gruyter, 2014.

[118] P. Walters. "On the pseudo orbit tracing property and its relationship to stability". In: *The Structure of Attractors in Dynamical Systems*. Ed. by N. Markley, J. Martin, and W. Perrizo. Lecture Notes in Mathematics, vol 668. Berlin, Heidelberg: Springer, 1978, pp. 231–244. DOI: 10.1007/BFb0101795.

[119] P. Weigert. "A Program Calculus for Dynamical Systems". MA thesis. University of Amsterdam, Institute for Logic, Language and Computation, 2023. URL: https://eprints.illc.uva.nl/id/eprint/2251/1/MoL-2023-09.text.pdf.

[120] G. Winskel and M. Nielsen. "Models for concurrency". In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum. Vol. 4. Oxford: Clarendon Press, 1995.

MUNICH CENTER FOR MATHEMATICAL PHILOSOPHY, LMU MUNICH
Email address: levin.hornischer@lmu.de