Volatility Jumps and Pricing Kernel Features

Wolfgang Karl Härdle
Levin Schach

Ladislaus von Bortkiewicz Professor of Statistics
Humboldt-Universität zu Berlin
theIDA.net
ASE, ACI NUS, NYCU 玉山學者

Pricing Kernel

- Under the risk-neutral measure (RNM) Q, the current price p_t of an option with a payoff $\psi(S_T)$ at maturity T is given by

$$p_t = \mathsf{E}_Q\left\{\psi(S_T)\right\} = \int_0^\infty \psi(x)q(x)\,dx$$

- Relating the risk-neutral density (RND) to the historical density (HD) introduces the Pricing Kernel (PK) K, which adjusts the HD to reflect the RND

$$p_t = \int_0^\infty \psi(x)\frac{q(x)}{p(x)}p(x)dx = \int_0^\infty \psi(x)K(x)p(x)dx$$

- Therefore, we also express the price as an expectation under the physical measure (PM) P :

$$p_t = \mathsf{E}_P\left[\psi(S_T)K(S_T)\right]$$

Classical PK Estimator: Backward-looking return

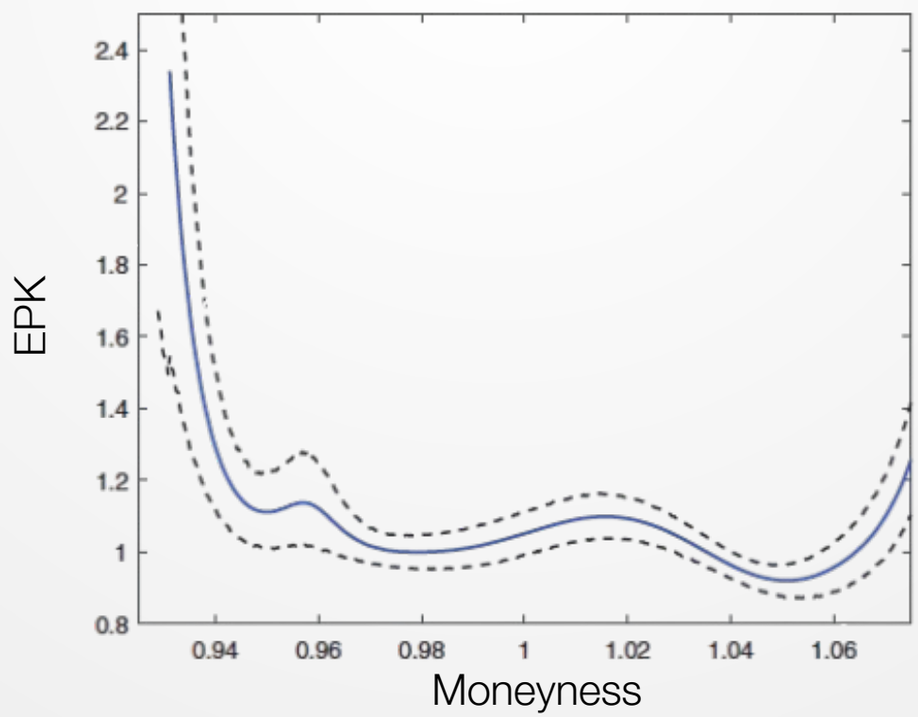
- RND and HD are estimated separately

$$\widehat{K}(S_T) = \frac{\widehat{q}(S_T)}{\widehat{p}(S_T)}$$

- $\widehat{q}$ estimated using option prices, while $\widehat{p}$ is estimated non parametrically (e.g. kernel density estimation)
- This PK represents the projection of any admissible stochastic discount factor (SDF) onto the space spanned by S_T
- P is estimated using backward-looking returns.

Pricing Kernel Puzzle

- According to economic theory the PK should be non-increasing in returns, reflecting the assumption that investors are risk averse.
- Empirically, it is often observed to be locally increasing in returns.
- This creates a "Pricing Puzzle", as it contradicts economic intuition:
  - Local "humps" in the PK suggest that investors prefer higher risk in certain regions of the return distribution.



Linn et al. Critique

- The Radon-Nikodym Theorem states that the SDF is the derivative that transforms the PM P into the RNM Q, both conditioned on the information set $\mathscr{F}_t$

$$m_t(x_{t+s}) = e^{-rs}\frac{dQ(x_{t+s} \mid \mathscr{F}_t)}{dP(x_{t+s} \mid \mathscr{F}_t)}$$

- Breeden and Litzenberger's results enables the forward-looking estimation of RND, conditional on all information used by investors.
- There is no method to estimate the forward-looking HD of returns from *t* to *t*+*s*.

$\succ$ The SDF is inconsistent with microeconomic theory.

Test the robustness of CDI in different volatility regimes

- The proposed Conditional Density Integration addresses inconsistencies by matching forward-looking option prices with realized returns.

- Claim: CDI estimates remain stable under conditions of stochastic volatility and sudden volatility jumps as it correctly matches conditional information.
  - Resolves PK-Puzzle

- We simulate option series and introduce abrupt volatility jumps to assess the behavior of the proposed pricing kernel.

Outline

1. Motivation✓
2. Conditional Density Integration (CDI) Approach
3. Challenges of averaging volatility regimes in PK estimation
4. Black-Scholes Pricing Model
5. Risk Neutral Density & Historical Density Calculation
6. Options simulation
7. Risk-free-density Data overview
   - Methodology
   - Experimental results

Conditional Density Integration (CDI) by Linn et al. (2018)

- Linn et al. integrate the RND, adjusted by the Radon-Nikodym derivative relative to the HD, up to the realized returns, ensuring alignment under the assumption that both densities should ideally yield a uniform distribution when correctly modeled.

$$\int_{-\infty}^{r_{t+s}} dP(r_{t+s}|\mathcal{F}_t) = \int_{-\infty}^{r_{t+s}} \frac{dP(r_{t+s}|\mathcal{F}_t)}{dQ(r_{t+s}|\mathcal{F}_t)} dQ(r_{t+s}|\mathcal{F}_t)$$

$$= \int_{-\infty}^{r_{t+s}} \left\{ \frac{dQ(r_{t+s}|\mathcal{F}_t)}{dP(r_{t+s}|\mathcal{F}_t)} \right\}^{-1} dQ(r_{t+s}|\mathcal{F}_t)$$

- $dP(r_{t+s}|\mathcal{F}_t)$ is the physical, $dQ(r_{t+s}|\mathcal{F}_t)$ the risk-neutral density and r_t is the return at time point t

Probability Integral Transformation

- Probability Integral Transformation (PIT)
  - $F(X)$, where F is the CDF of X, transforms any variable X into a uniformly distributed variable over [0,1].
- PIT validates the uniqueness

$$\int_{-\infty}^{r_{t+s}} \left\{ \frac{dQ(r_{t+s} \mid \mathscr{F}_t)}{dP(r_{t+s} \mid \mathscr{F}_t)} \right\}^{-1} dQ(r_{t+s} \mid \mathscr{F}_t) = \int_{-\infty}^{r_{t+s}} g_{t,s}(y) dQ(r_{t+s} \mid \mathscr{F}_t) \sim U(0,1)$$

Estimation

- Assume cubic B-spline

$$g(y) = \sum_{j=1}^{J} \theta_j B_j(y)$$

- Hence,

$$\int_{-\infty}^{r_{t+s}} g(y) dQ(y|\mathcal{F}_t) = \sum_{j=1}^{J} \theta_j \int_{-\infty}^{r_{t+s}} B_j(y) dQ(y|\mathcal{F}_t)$$

- Use GMM to get $\hat{\theta}$ (moments of the uniform distribution)

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^J}{\operatorname{argmin}} \sum_{i=1}^{m} \left[\frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{j=1}^{J} \theta_j \int_{-\infty}^{r_{t+s}} B_j(y) dQ(y|\mathcal{F}_t) \right\}^i - \frac{1}{i+1} \right]^2$$

Estimation

- It should be

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^J}{\operatorname{argmin}} \sum_{i=1}^{m} \left[\frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{j=1}^{J} \theta_j \int_{-\infty}^{R_{t+s}} B_j(y) dQ(y|\mathcal{F}_t) \right\}^i - \frac{1}{i+1} \right]^2$$

because

$$\sum_{j=1}^{J} \theta_j \int_{-\infty}^{R_{t+s}} B_j(y) dQ(y|\mathcal{F}_t) \approx \int_{-\infty}^{R_{t+s}} g_{t,s}(y) dQ(y|\mathcal{F}_t) \sim U(0,1)$$

$$\frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{j=1}^{J} \theta_j \int_{-\infty}^{R_{t+s}} B_j(y) dQ(y|\mathcal{F}_t) \right\}^i = \hat{E} \left[\left\{ \sum_{j=1}^{J} \theta_j \int_{-\infty}^{R_{t+s}} B_j(y) dQ(y|\mathcal{F}_t) \right\}^i \right]$$

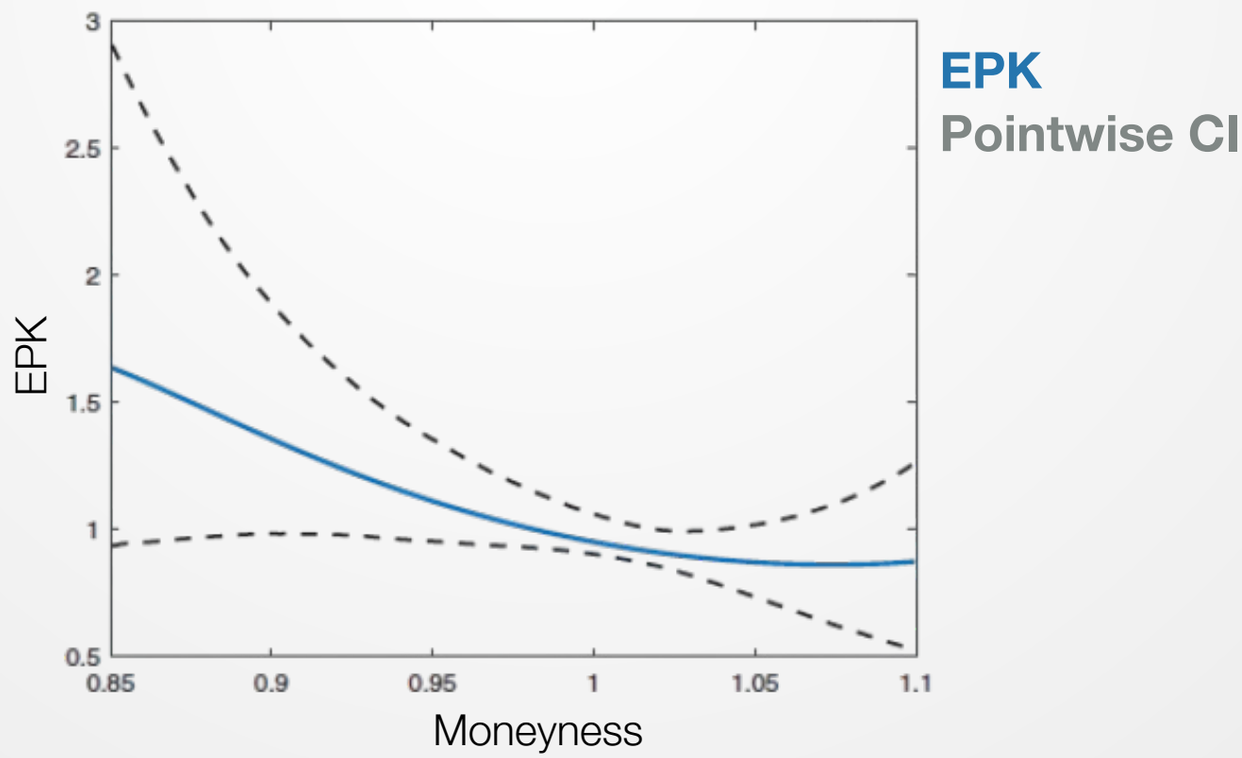
Should be close to $\dfrac{1}{i+1}$

Estimation II

- Plug-in estimate given by

$$\hat{g}(r_{t+s}) = \sum_{j=1}^{J} \hat{\theta}_j B_j(r_{t+s})$$

- EPK follows as $\widehat{K}(r_{t+s}) = \left\{\hat{g}(r_{t+s})\right\}^{-1}$



Physical density in the presence of volatility jumps

- The KDE ignores the volatility regimes the samples belong to, as it assumes a stationary distribution of returns.
  - In the presence of volatility jumps, this leads to a blending of characteristics from different regimes into a single density estimate.
- The averaged density under- or overestimates the probability of extreme values, as it doesn't account for different volatility levels separately.

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

CDI: Averaging across volatility regimes

- Averaging over all T results in an estimation of θ based on a mean value that includes data from different volatility regimes.
- Volatility jumps or regime shifts cause the specific characteristics of these regimes to blend into the estimate of θ.
- As a result, the estimated PK may not accurately reflect the true risk preferences of investors across different market phases.

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^J}{\operatorname{argmin}} \sum_{i=1}^{m} \left[\frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{j=1}^{J} \theta_j \int_{-\infty}^{r_{t+s}} B_j(y) dQ(y|\mathcal{F}_t) \right\}^i - \frac{1}{i+1} \right]^2$$

Change of Measure

- The stock price S is a geometric Brownian motion, represented by

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \quad W_t \text{ a } P\text{-Brownian motion}$$

- To price derivatives, we switch from the PM P to the RNM Q, ensuring all investors are risk-indifferent and discount future cash flows at the risk-free rate.
- The No-arbitrage rule, using the Cameron-Martin-Girsanov theorem, changes the probability measure to Q by adjusting the process drift to r, ensuring risk-neutral pricing.

$$S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^*\right\}, \quad W_t^* \text{ a } Q\text{-Brownian motion}$$

Black-Scholes Model - European Call

- The present value of the expected payoff under the RNM is given as

$$C_t = e^{-r\tau}\mathsf{E}_t^Q\left[(S_T - K)^+\right] = e^{-r\tau}\int_0^\infty (s-K)^+ q_t(s)\,ds,$$

- In the Black-Scholes world, the price is derived as

$$C_{BS} = S_t\Phi(d_1) - e^{-r\tau}K\Phi(d_2),$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}\,.$$

Risk Neutral Density

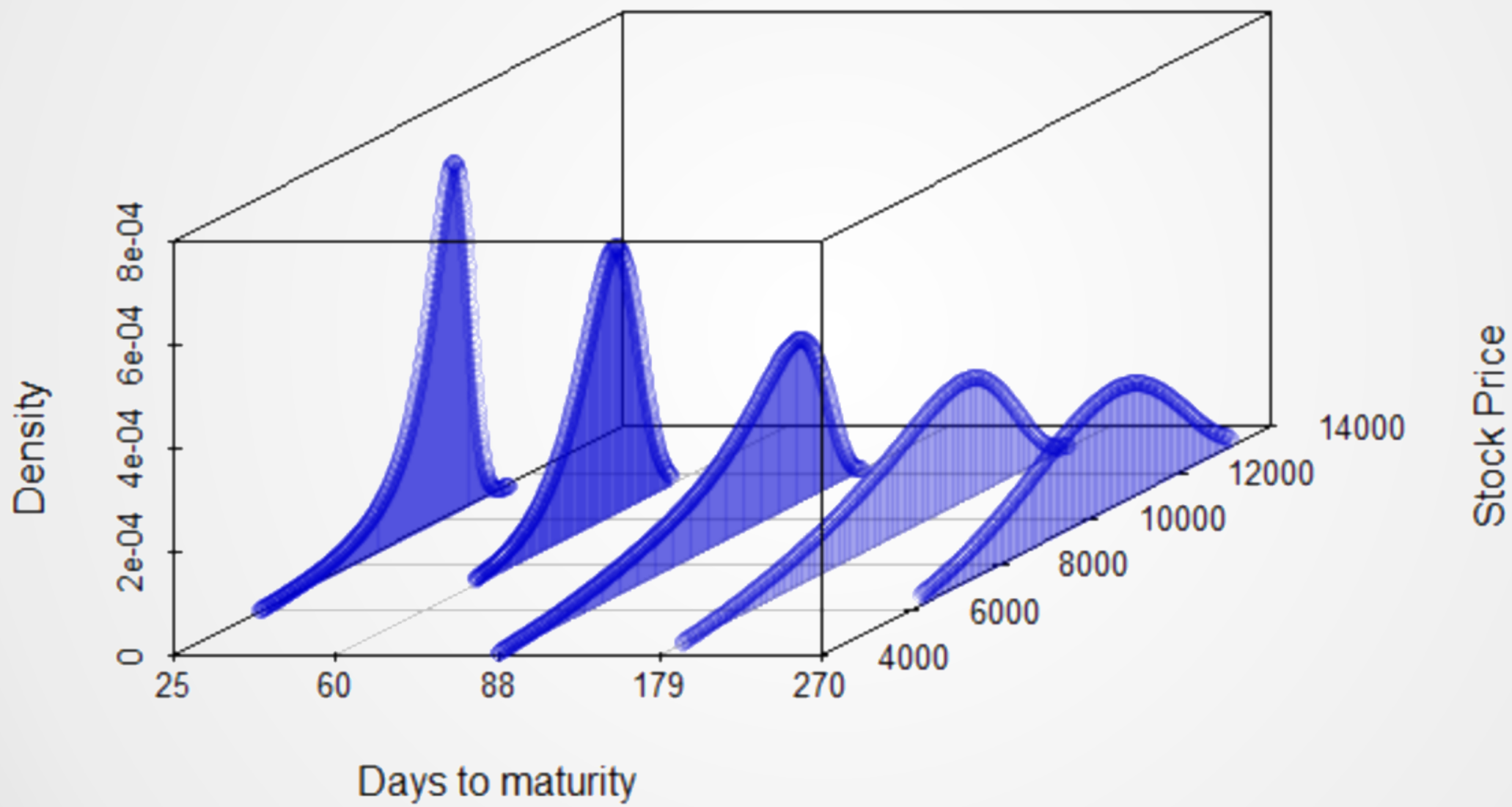
- Breeden and Litzenberger derived the RND as:

$$C_t = e^{-r\tau} \int_0^\infty (s - K)^+ q_t(s)\, ds,$$

$$q(\,\cdot\,) = e^{r\tau} \frac{\partial^2 C}{\partial K^2}$$

- To calculate the second derivative, we apply the Rookley-Method and estimate $\sigma_{IV}(M)$, $\sigma'_{IV}(M)$, $\sigma''_{IV}(M)$ by local polynomials, to smooth the implied volatility surface.
- We use the estimates to extract the risk neutral density based on the result of Breeden and Litzenberger.

Risk Neutral Density for the DAX



SFE_RND_HD

Physical Density

- Retrieve historical price data and calculate for each maturiy τ the returns over said time interval

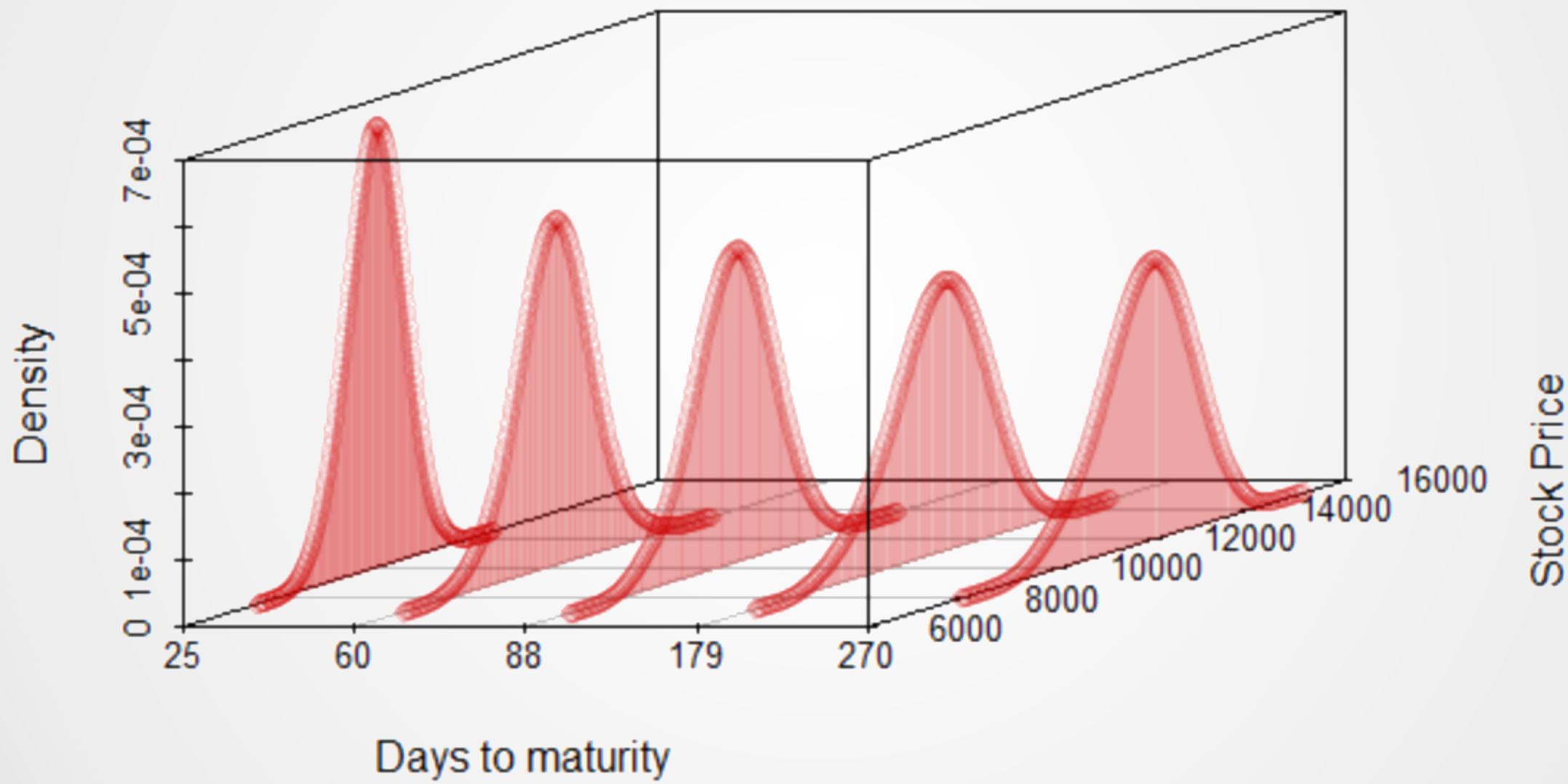
$$R_i^\tau = \frac{S_{t_0-i}}{S_{t_0-i-\tau}}, \quad \text{where } i = 1,2,\ldots$$

- Construct sample sets S_T^τ, consisting of sample points reflecting the adjusted index level based on its return at time t_0

$$S_T^\tau = (S_{T,\tau}^1, S_{T,\tau}^2, \ldots)$$
$$S_T^i = S_{t_0} R_i^\tau, \; i = 1,2,\ldots$$

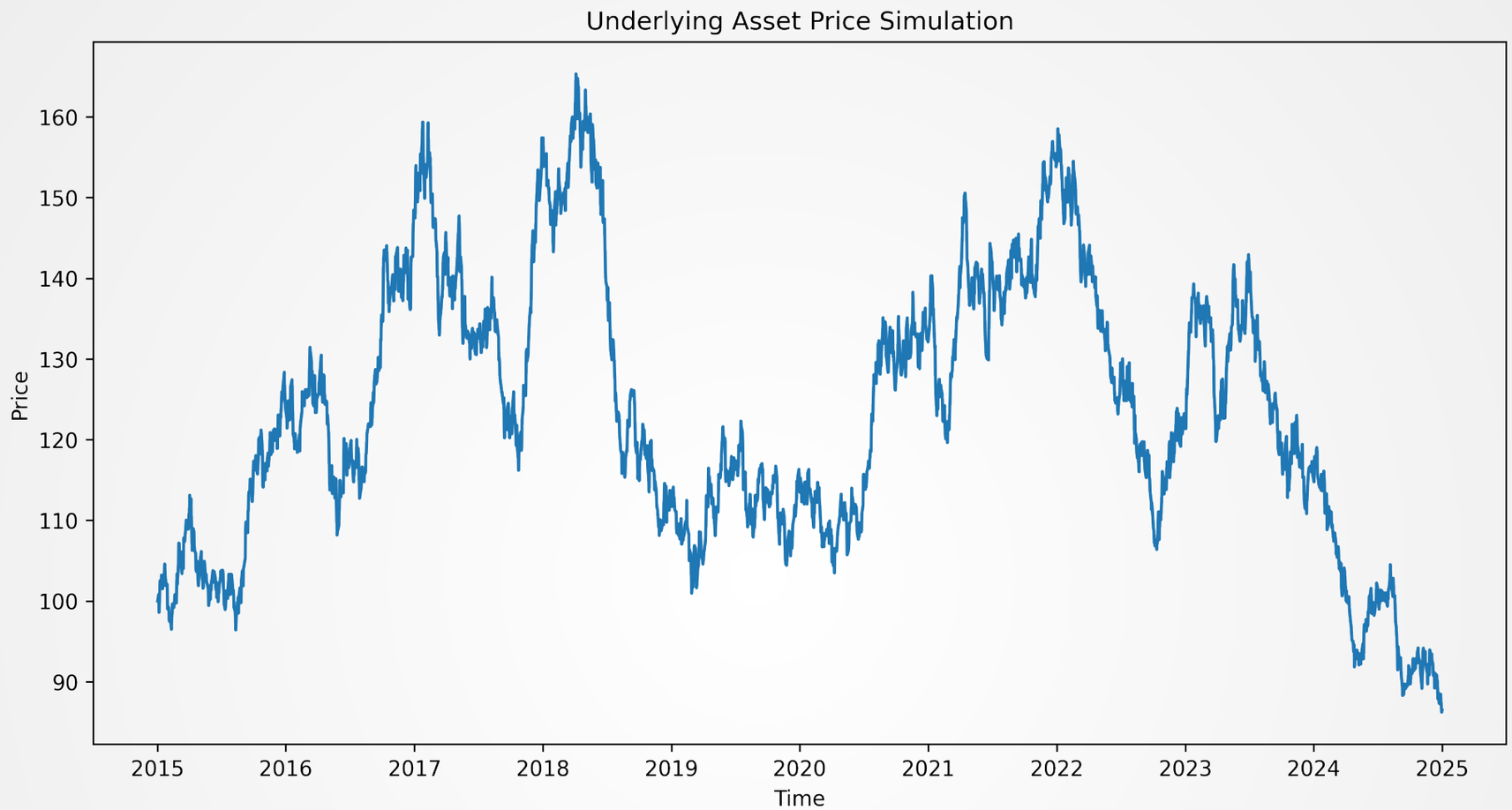
- Estimate physical density based on samples S_T^τ using kernel density estimation.

Physical Density for the DAX



QSFE_RND_HD

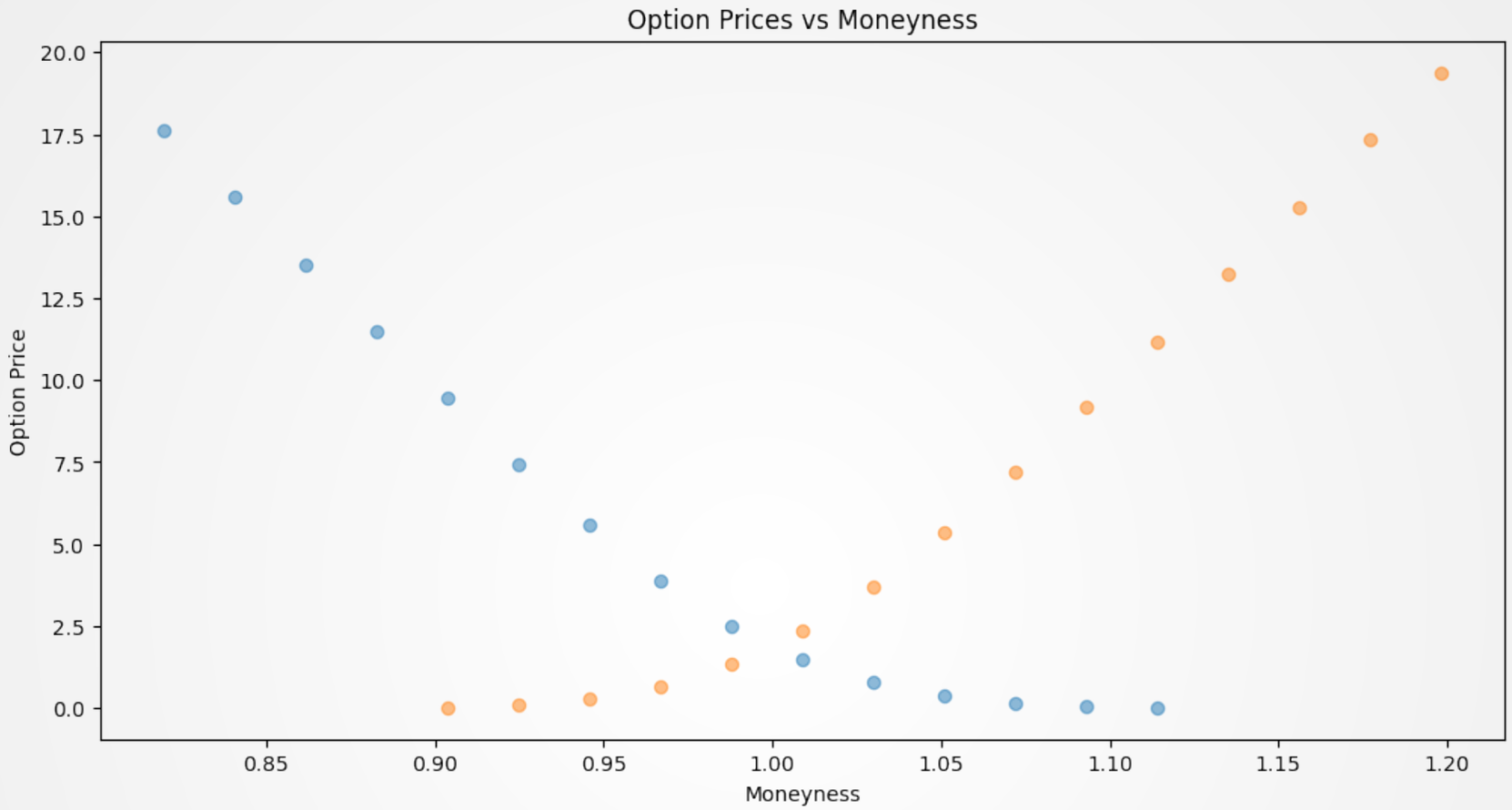
Price simulation



- The underlying price is simulated as a Q-Brownian motion, with
  - $S_0 = 100$
  - $r = 0$
  - $\sigma = 0.2$

PK_VJUMP_SIM

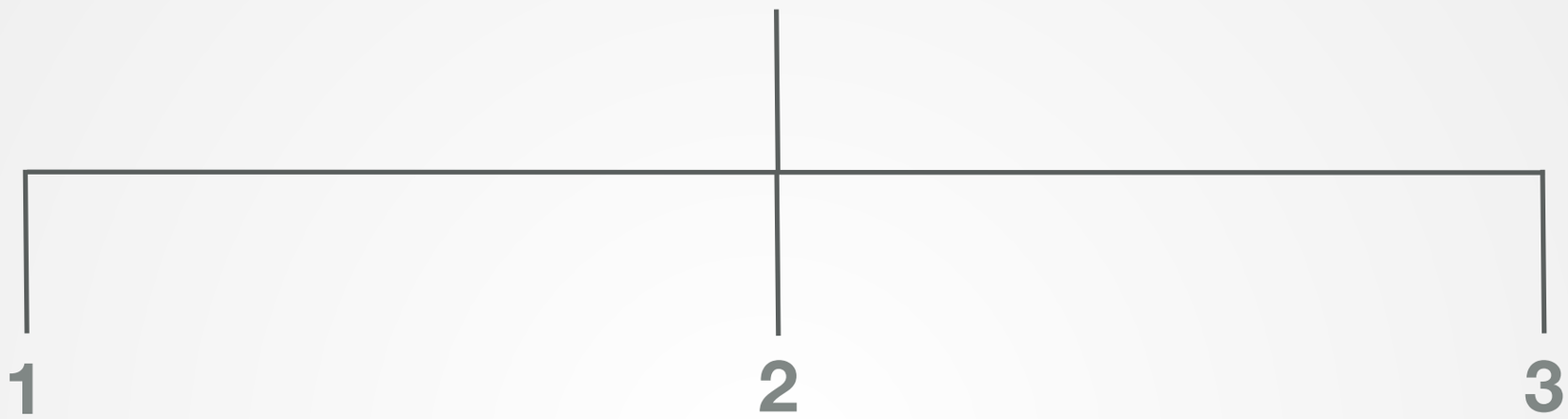
Options simulation



- On the first trading day of each expiry, I simulated 45 puts and calls between 0.8 and 1.2 moneyness.
- They are priced by the BS-model and afterwards filtered for outliers.

Volatility jumps

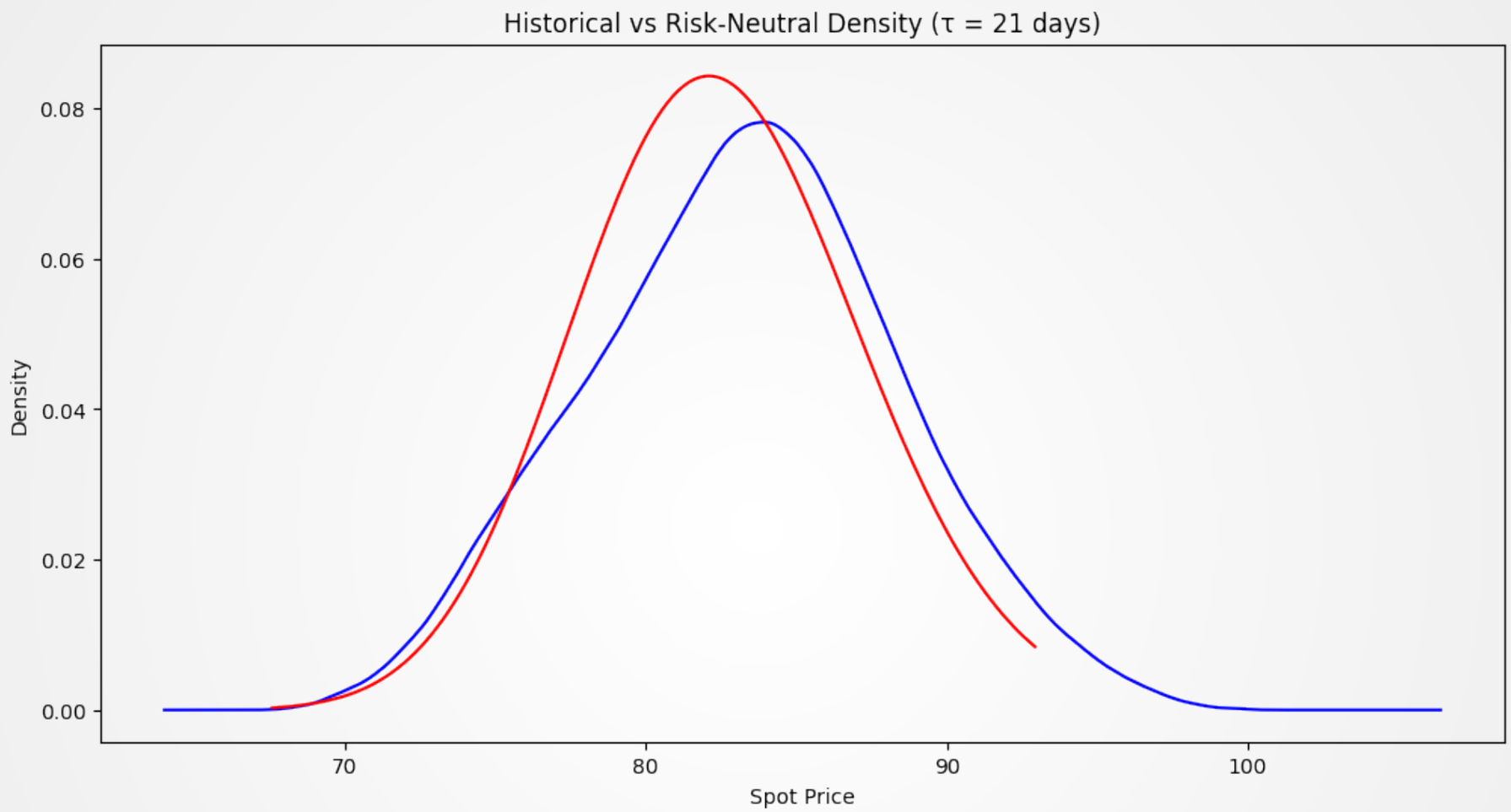
Simulated volatility regimes



Constant volatility | Volatility jump in the *past* | Volatility jump in the *future*

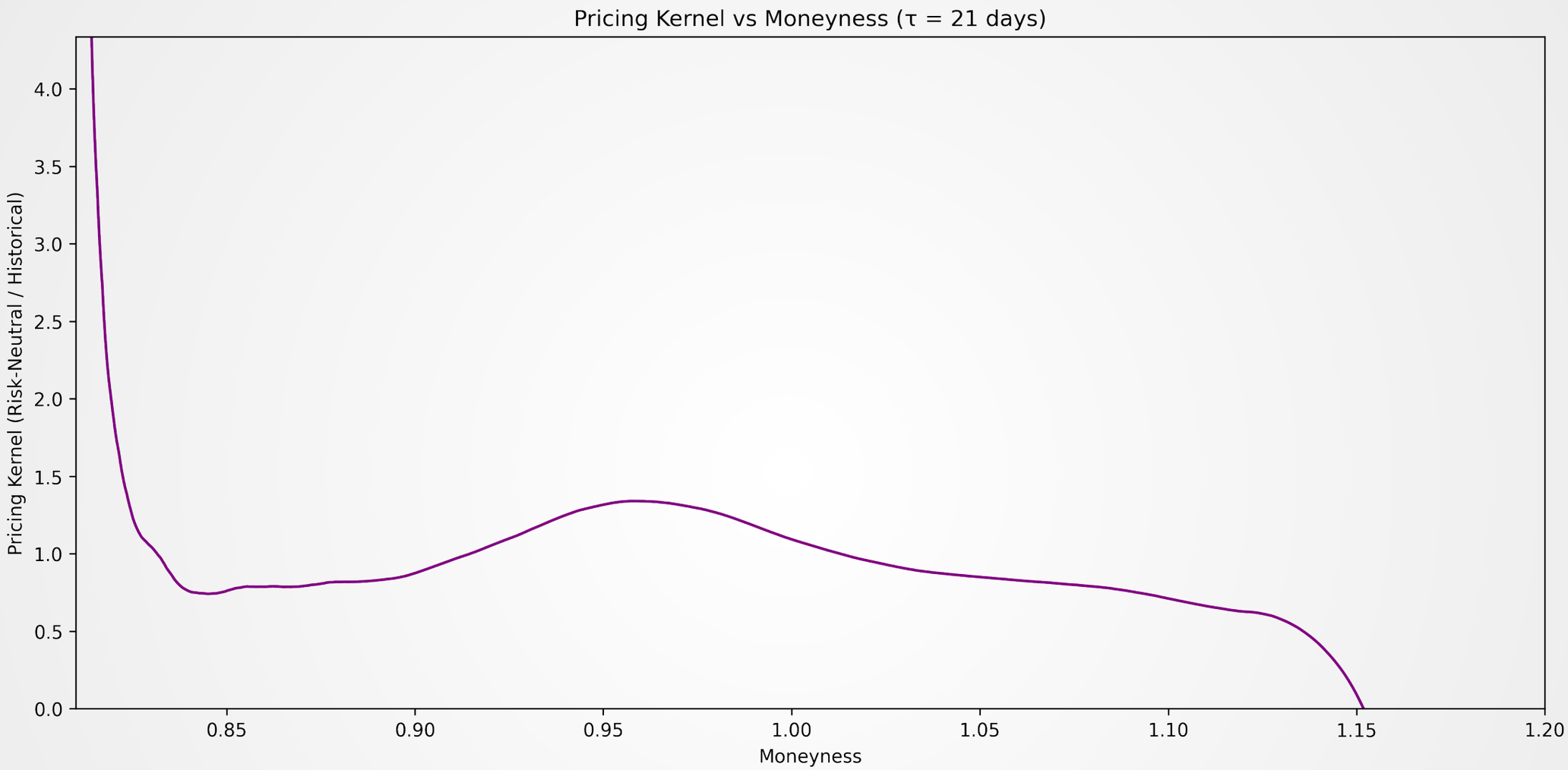
- To analyze the impact of volatility regime switches on the two proposed PK estimations, we examine the PKs at the same point in time under different overall volatility regime scenarios.
- The classical PK estimation relies on historical returns. It is to be assumed that a volatility jump in the past influences the kernel, while a volatility jump in the future does not.
- In contrast, the CDI PK estimation uses forward-looking returns. In theory past regime switches have no effect on the pricing kernel compared to future regime switches.

Case 1: RND and HD at constant volatility

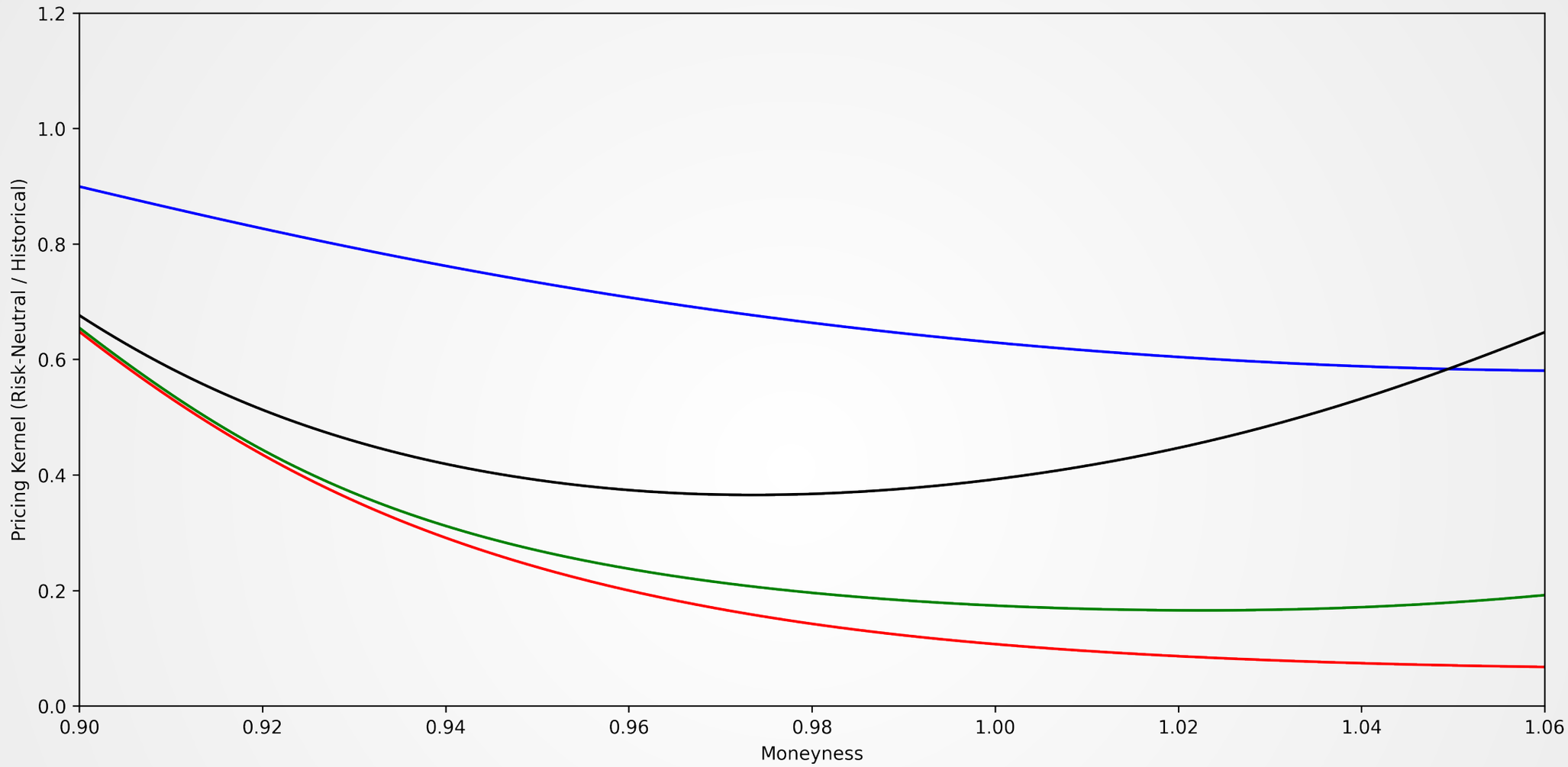


- RND and HD at 2020-01-10 with
  - $\sigma = 0.2$
  - $r = 0.00$
  - $\tau = 21$

Case 1: Conventional Pricing Kernel



Case 1: CDI Pricing Kernel

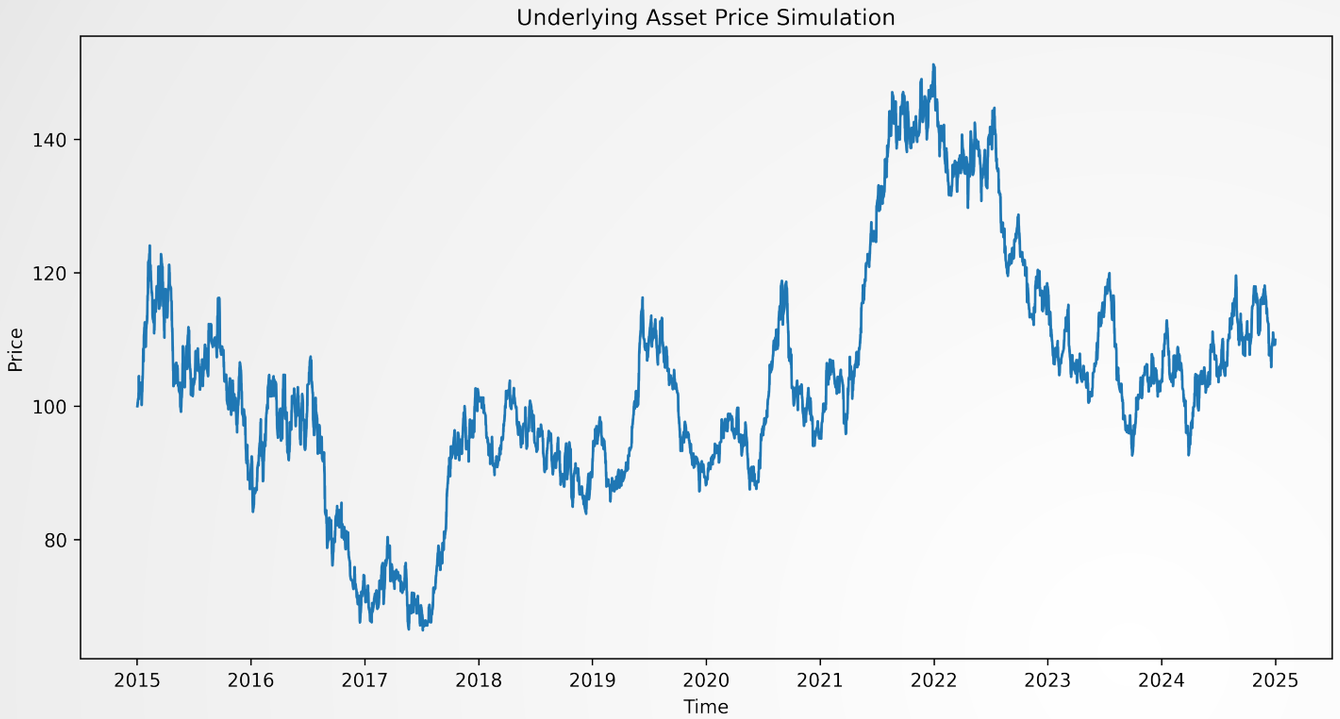


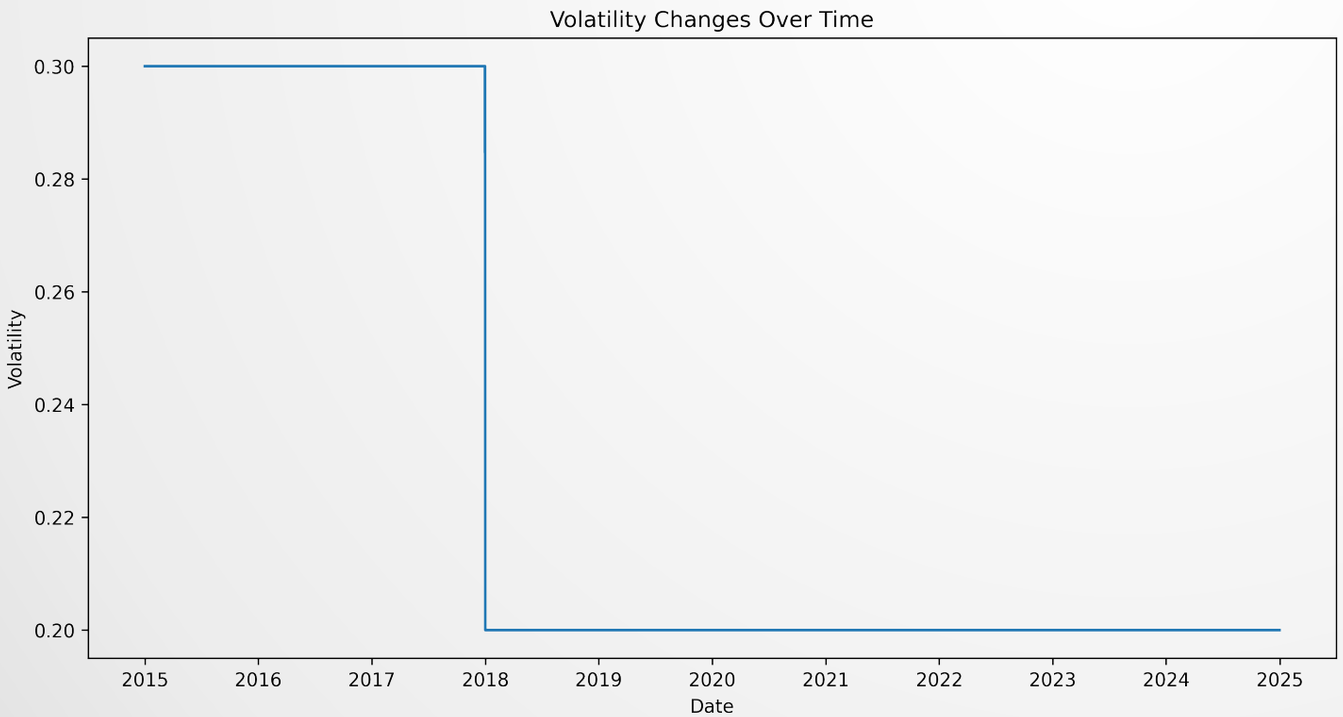
4 bases/moments, 5 bases/moments,

6 bases/moments, 7 bases/moments

Steps: 0.001

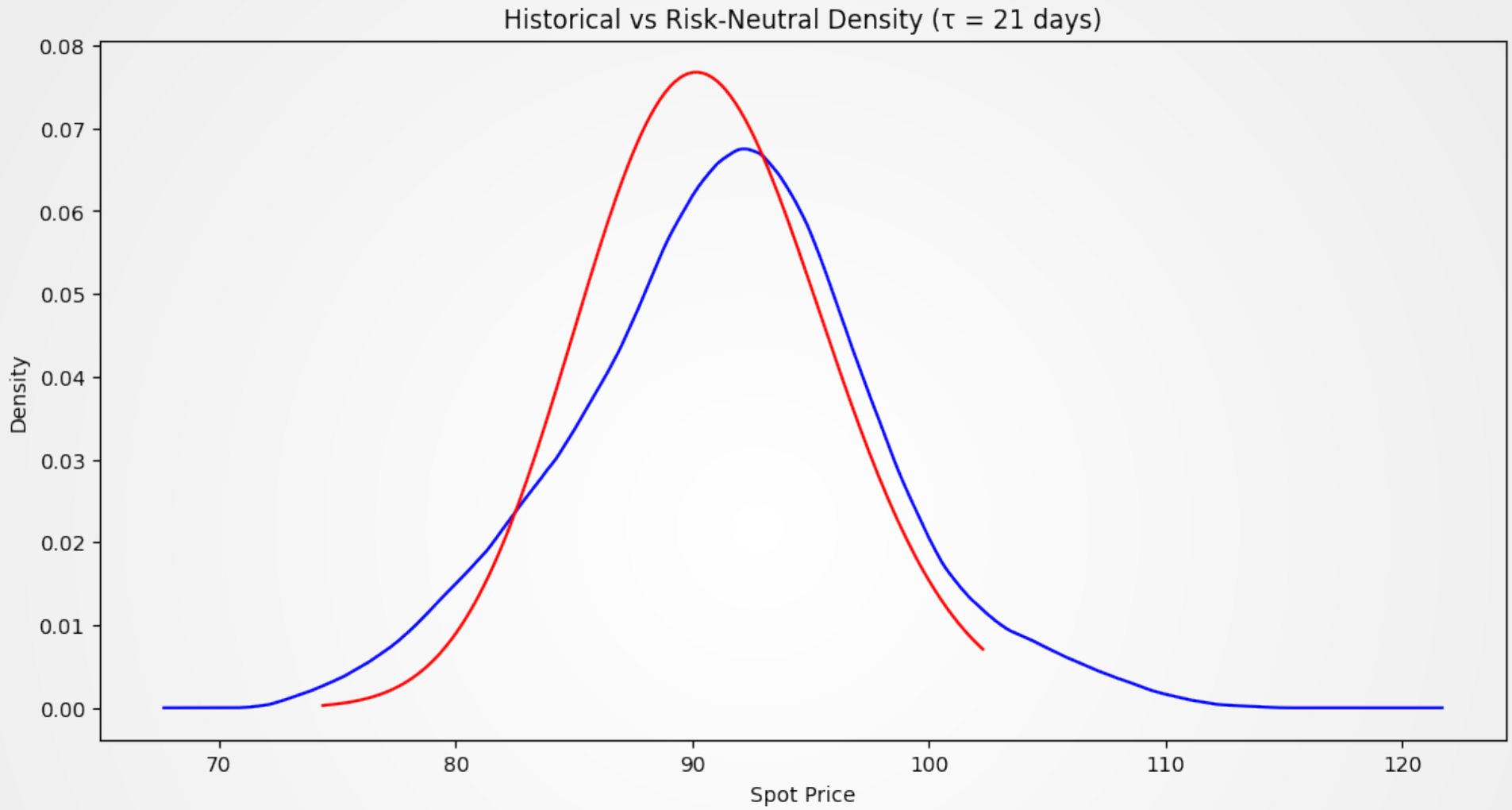
Case 2: Volatility jump in the *past*





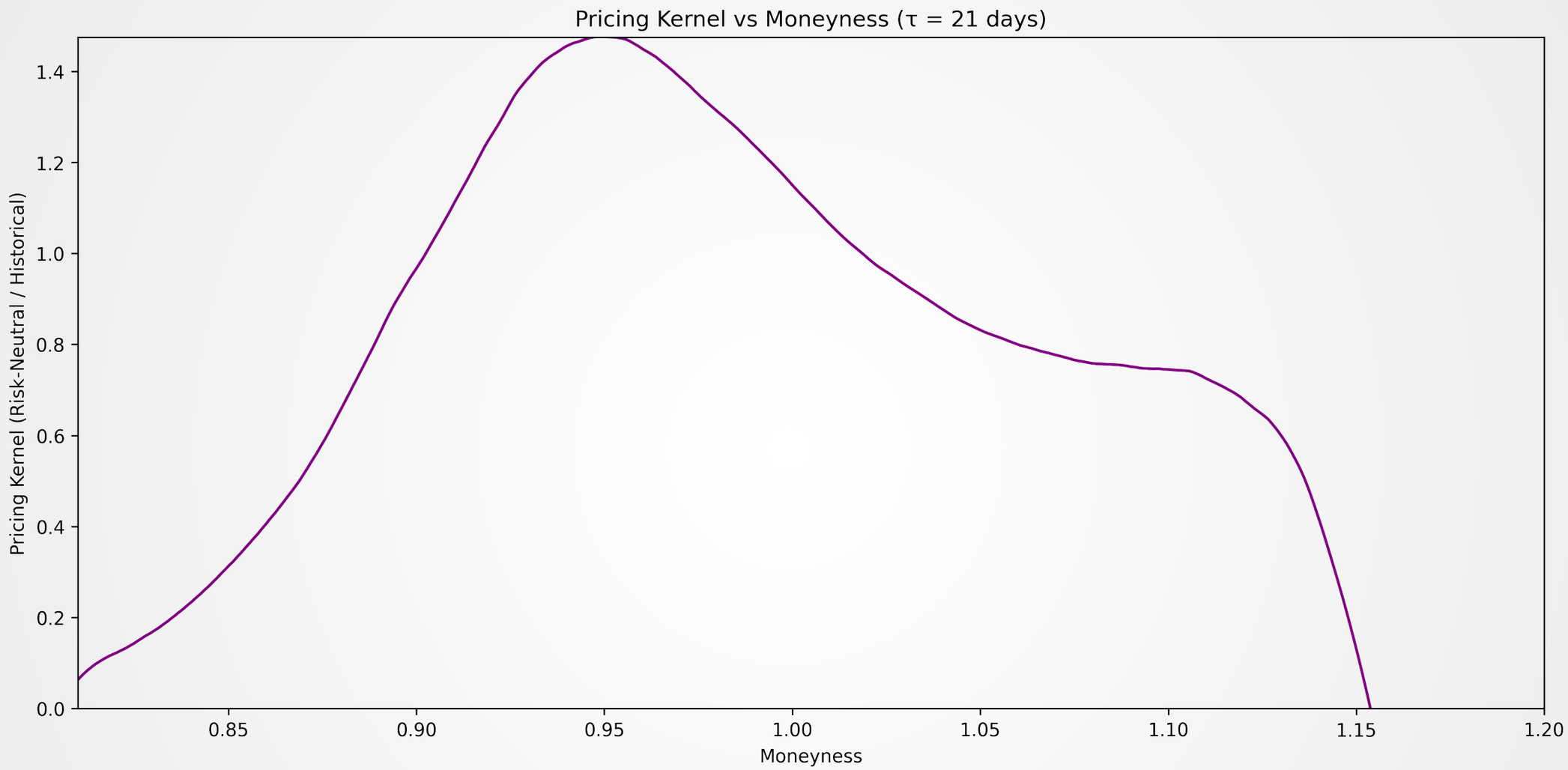
- Simulate a volatility regime shift at 2018-01-01
- $$\sigma(t) = \begin{cases} 0.3 & \text{für } t < 2018, \\ 0.2 & \text{für } t \geq 2018. \end{cases}$$

Case 2: RND and HD

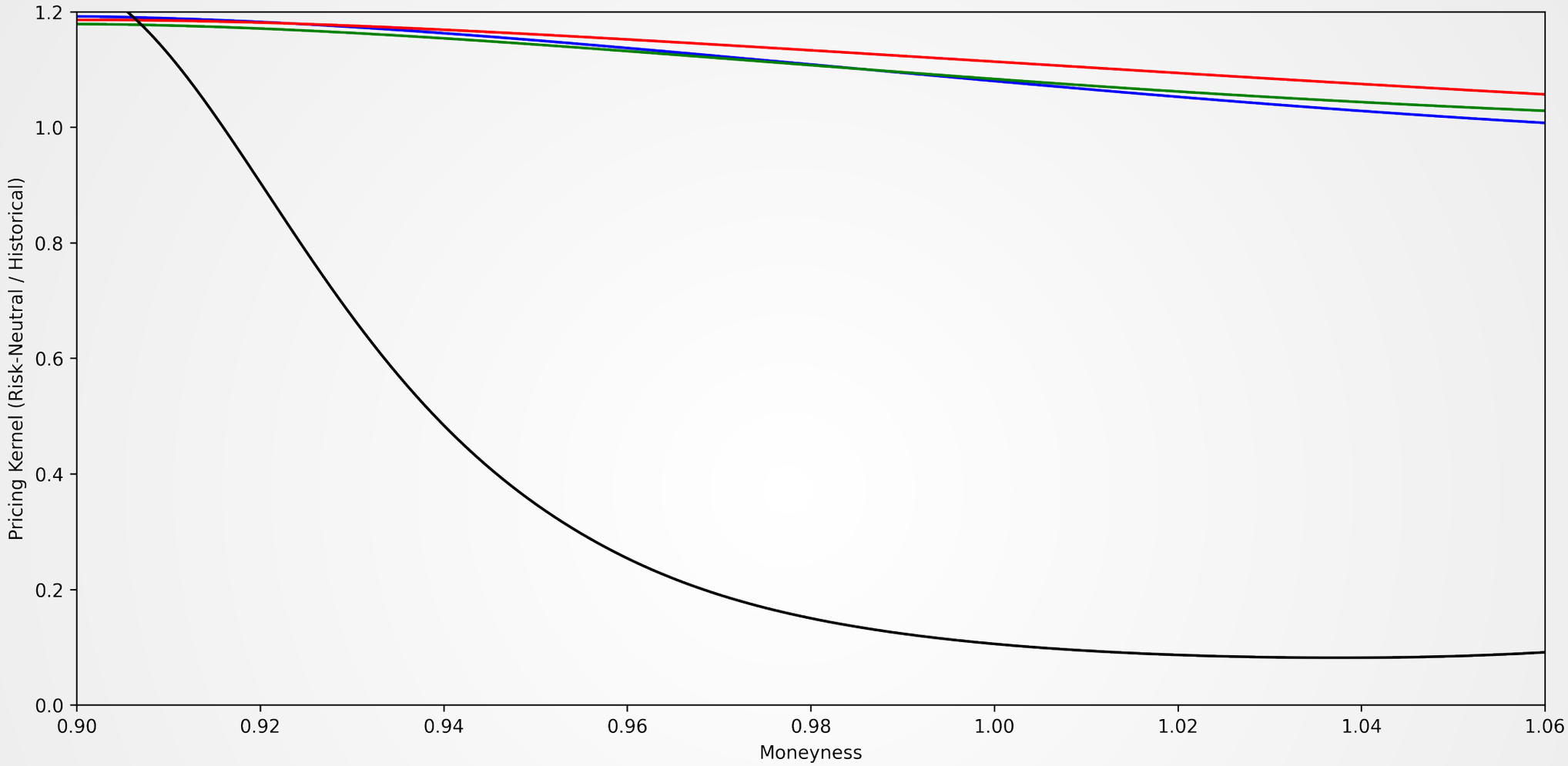


- RND and HD at 2020-01-10 with
  - $r = 0.00$
  - $\tau = 21$

Case 2: Conventional PK



Case 2: CDI Pricing Kernel

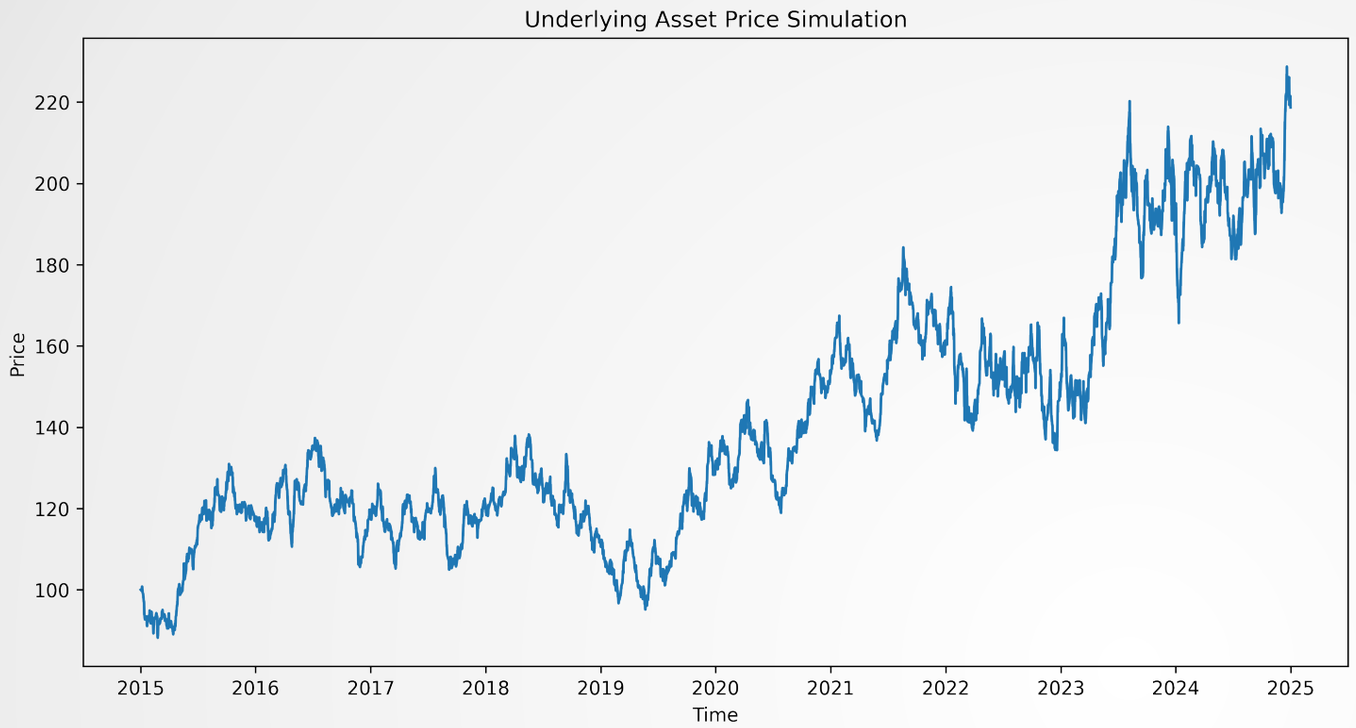


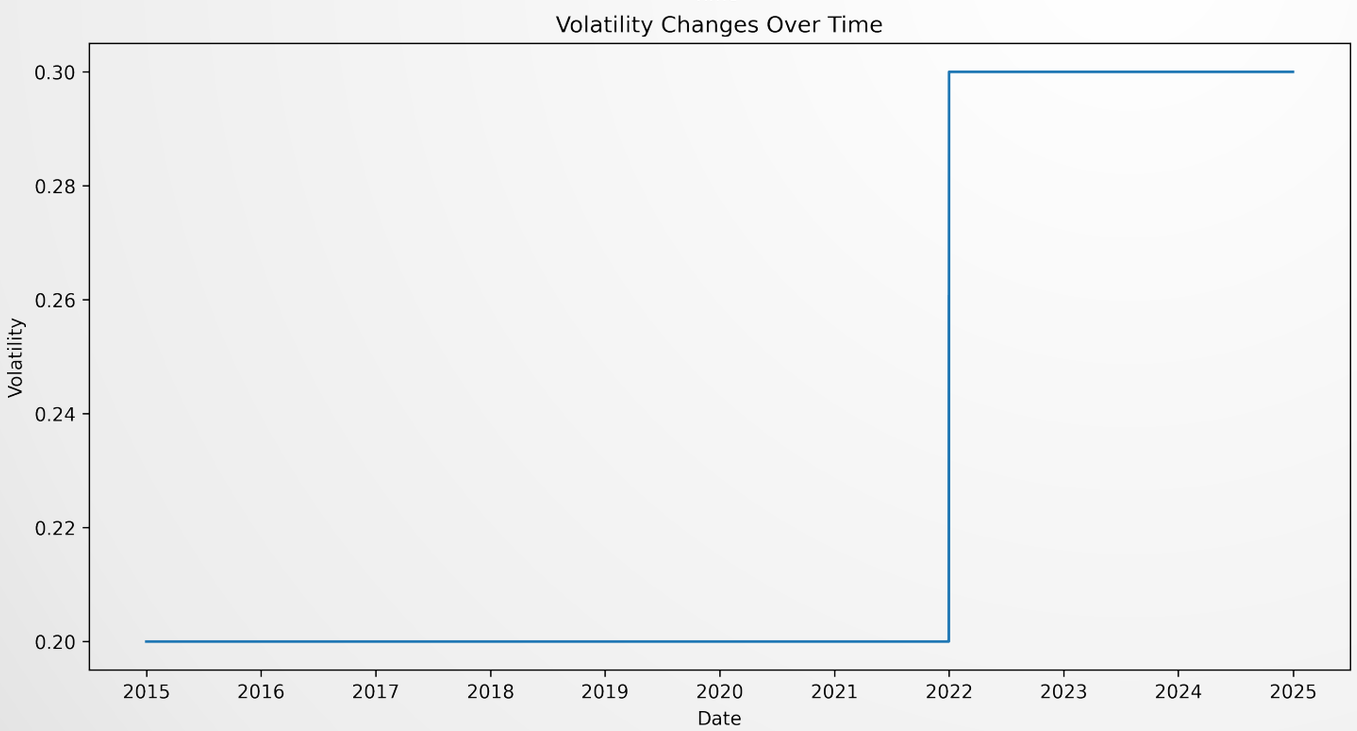
4 bases/moments, 5 bases/moments,

6 bases/moments, 7 bases/moments

Steps: 0.001

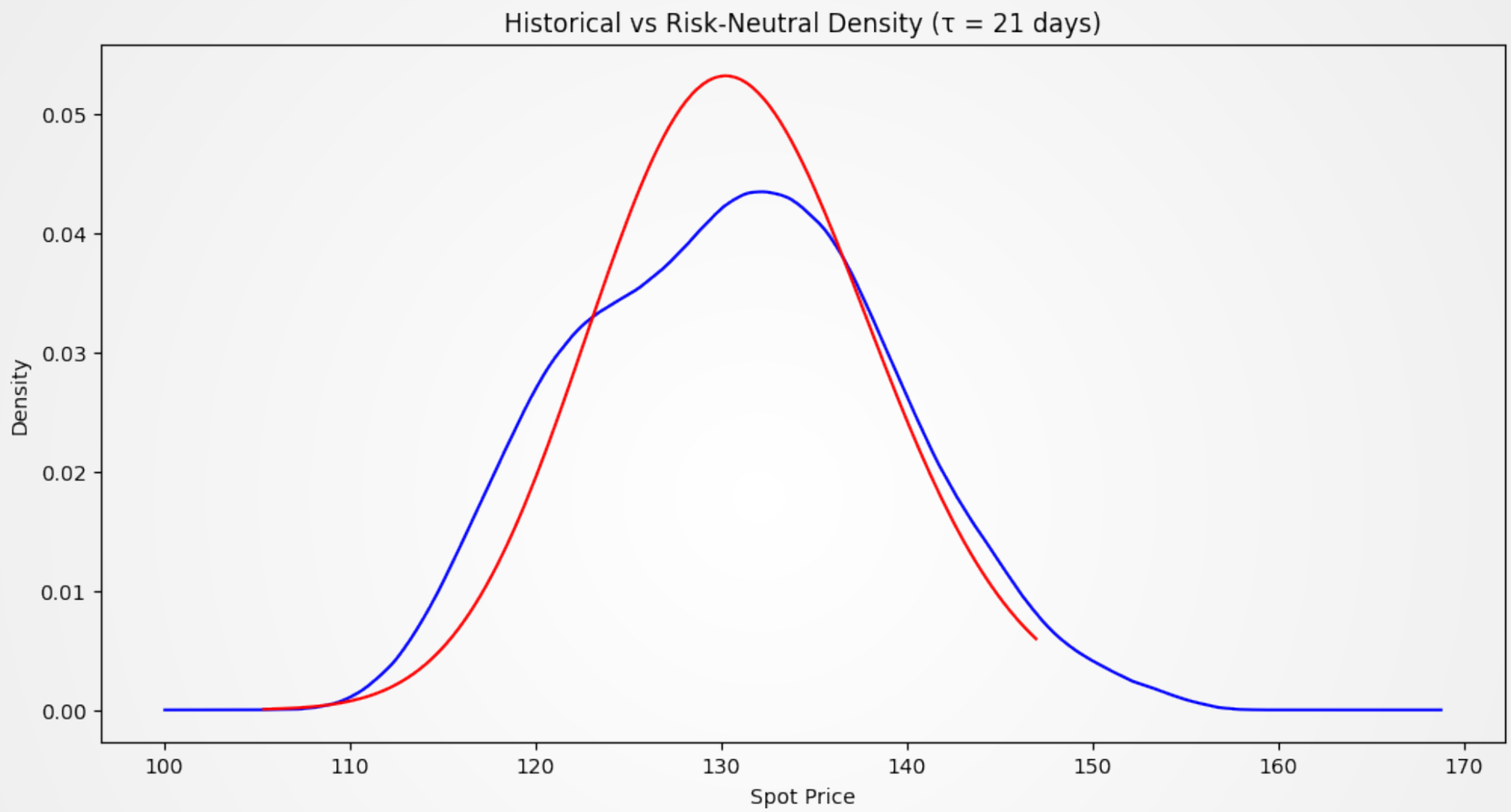
Case 3: Volatility jump in the "future"





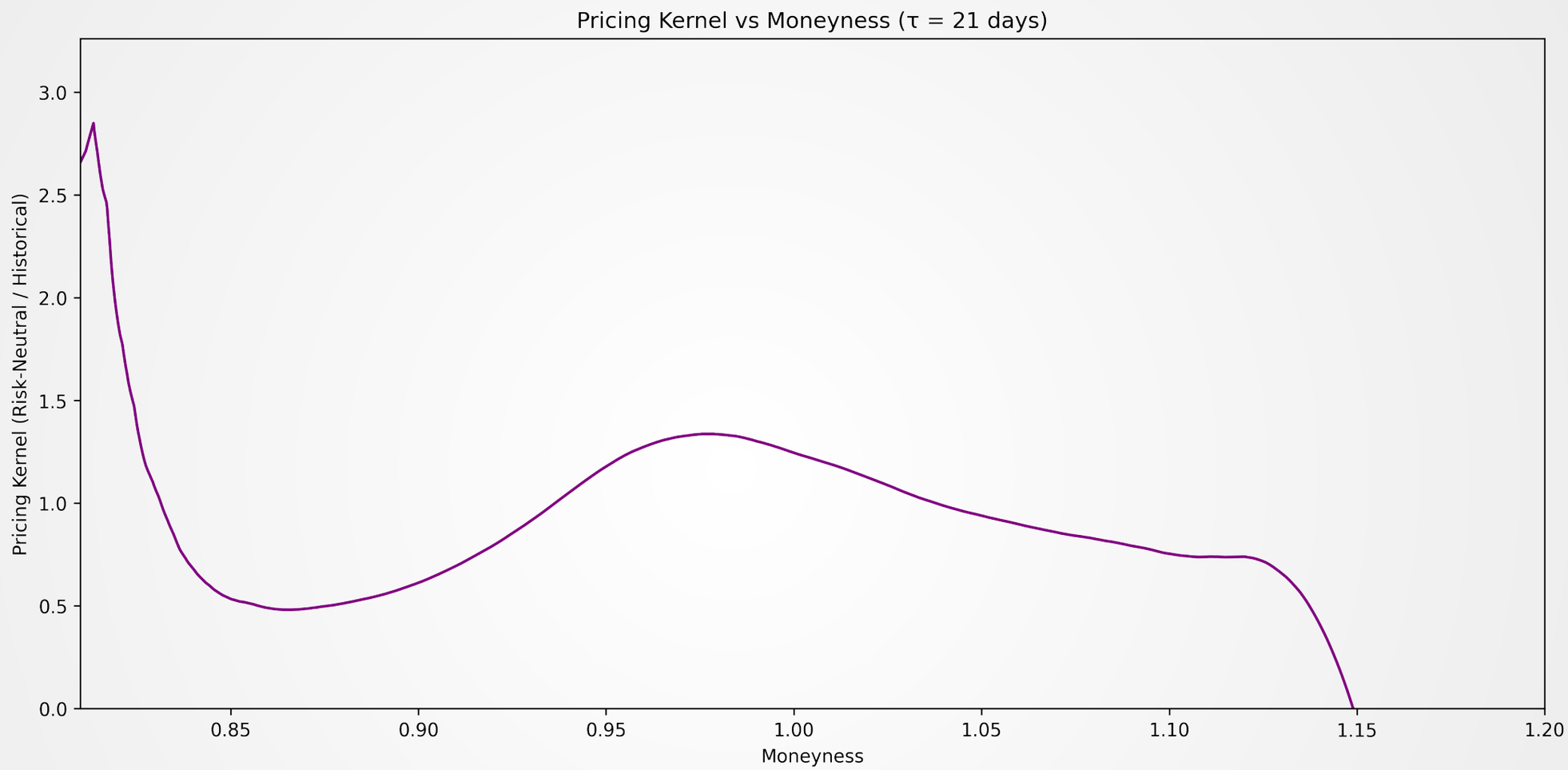
- We simulate a volatility regime shift at 2022-01-01
- $$\sigma(t) = \begin{cases} 0.2 & \text{für } t < 2022, \\ 0.3 & \text{für } t \geq 2022. \end{cases}$$

Case 3: RND and HD



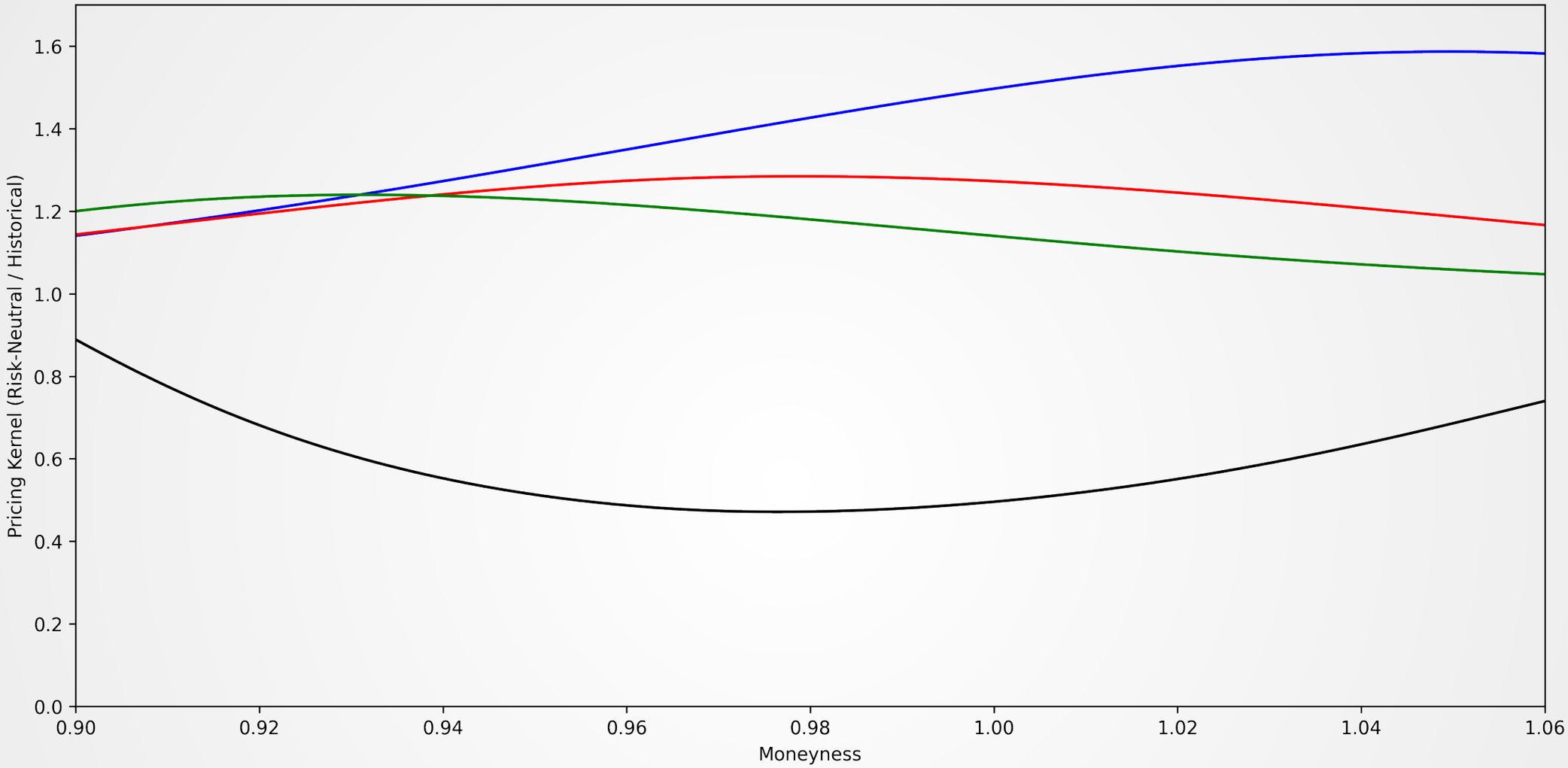
- RND and HD at 2020-01-10 with
  - $r = 0.00$
  - $\tau = 21$

Case 3: Conventional Pricing Kernel



- A volatility jump in the *future* has no effect on the characteristics of the PK, as only *past* data is being used for the estimation.
- The PK in case 1 & 3 are the same

Case 3: CDI Pricing Kernel



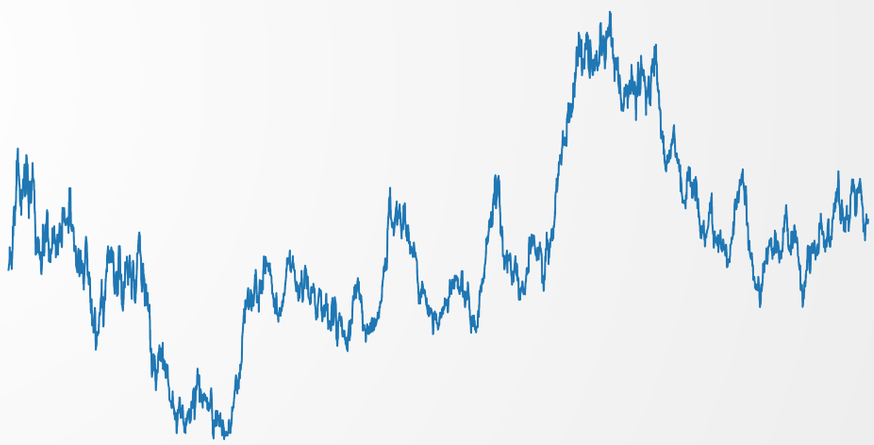
4 bases/moments, 5 bases/moments,

6 bases/moments, 7 bases/moments

Steps: 0.001

Volatility Jumps and Pricing Kernel Features

Wolfgang Karl Härdle
Levin Schach



Ladislaus von Bortkiewicz Professor of Statistics
Humboldt-Universität zu Berlin
theIDA.net
ASE, ACI NUS, NYCU 玉山學者