Water competition among perennial plants

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April 5, 2022

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1 System with exponential growth and decay at packed equilibrium

$$\lambda(t) = \lambda_0 e^{\alpha t} \tag{1}$$

where $\alpha = f(a)$ and a is a plant's carbon accumulation rate. When $0 < t < t_i^*$:

$$\lambda(t) = \lambda_0 e^{f(g_i(W(t)))t}$$

when $t_i^* < t < t_e$

$$\lambda(t) = \lambda_m e^{f(r_i(t))t}$$

,

or equivalently:

$$\lambda_e e^{-f(r_i(t))(t_e - t_i^*)}$$

at equilibrium:

$$\lambda_e = \lambda_0 = \lambda^*$$

which implies:

$$\lambda_i(t) = \lambda_i^* e^{-f(r_i(t))(t_e - t_i^*)}$$

When species are packed and at equilibrium:

$$W_i^* - W_{i-1}^* = E\int_{t_{j-1}^*}^{t_j^*} \sum_{j=i}^{Q} \lambda_i^* e^{f(g_j(W(t)))t} dt$$

as the limit of $t_i^* - t_{i-1}^*$ goes to zero, we can approximate the integral as:

$$\frac{W_i^* - W_{i-1}^*}{t_i^* - t_{i-1}^*} = E\sum_{j=i}^{Q} \lambda_i^* e^{f(g_j(W(t_i^*)))t_i^*}$$

for species Q, this expression becomes:

$$\frac{W_Q^* - W_{Q-1}^*}{t_Q^* - t_{Q-1}^*} = E\lambda_Q^* e^{f(g_Q(W_Q^*))t_Q^*}$$

solving for λ^* we find:

$$\lambda_Q^* = \left(\frac{W_Q^* - W_{Q-1}^*}{t_Q^* - t_{Q-1}^*}\right)\frac{1}{Ee^{f(g_Q(W_Q^*))t_Q^*}}$$

which implies that species Q has positive leaf area at equilibrium when:

$$\left(\frac{W_Q^* - W_{Q-1}^*}{t_Q^* - t_{Q-1}^*}\right)\frac{1}{Ee^{f(g_Q(W_Q^*))t_Q^*}} > 0$$

which at the where species are infinitely packed becomes:

$$\left(\frac{dW^*(t)}{dt^*}|_{t^*=t_Q^*}\right)\frac{1}{Ee^{f(g_Q(W_Q^*))t_Q^*}} > 0$$

which is always true.

Plugging this expression into the expression for species $Q-1$ we find:

$$\frac{W_{Q-1}^* - W_{Q-2}^*}{t_{Q-1}^* - t_{Q-2}^*} = \left(\frac{W_Q^* - W_{Q-1}^*}{t_Q^* - t_{Q-1}^*}\right)\frac{e^{f(g_Q(W_{Q-1}^*))t_{Q-1}^*}}{e^{f(g_Q(W_Q^*))t_Q^*}} + E\lambda_{Q-1}^* e^{f(g_{Q-1}(W_{Q-1}^*))t_{Q-1}^*}$$

which is equivalent to:

$$\frac{\Delta W_{Q-1}^*}{\Delta t_{Q-1}^*} = \left(\frac{\Delta W_{Q-1}^*}{\Delta t_{Q-1}^*}\right) e^{f(g_Q(W_{Q-1}^*))t_{Q-1}^* - f(g_Q(W_Q^*))(t_Q^* - \Delta t_Q^*)} + E\lambda_{Q-1}^* e^{f(g_{Q-1}(W_{Q-1}^*))t_{Q-1}^*}$$

Taking the limit as the Δ's go to zero:

$$\frac{dW^*(t)}{dt^*}|_{t^*=t^*_{Q-1}} = \left(\frac{dW^*(t)}{dt^*}|_{t^*=t^*_Q}\right) e^{t^*_{Q-1}\frac{d}{dt^*}f(g_Q(W(t^*_{Q-1})))} + E\lambda^*_{Q-1}e^{f(g_{Q-1}(W(t^*_{Q-1})))}$$

and solving for λ^*_{Q-1}:

$$\lambda^*_{Q-1} = \left(\frac{dW^*(t)}{dt^*}|_{t^*=t^*_{Q-1}} - \left(\frac{dW^*(t)}{dt^*}|_{t^*=t^*_Q}\right) e^{t^*_{Q-1}\frac{d}{dt^*}f(g_Q(W(t^*_{Q-1})))}\right)\frac{1}{Ee^{f(g_{Q-1}(W(t^*_{Q-1})))}}$$

which implies that species $Q-1$ has positive leaf area at equilibrium when:

$$\frac{dW^*(t)}{dt^*}|_{t^*=t^*_{Q-1}} - \left(\frac{dW^*(t)}{dt^*}|_{t^*=t^*_Q}\right) e^{t^*_{Q-1}\frac{d}{dt^*}f(g_Q(W(t^*_{Q-1})))} > 0$$

which is true provided

$$\frac{d^2W^*(t)}{dt^{*2}}|_{t^*=t^*_{Q-1}} > 0$$

because $\frac{d}{dt^*}f(g_Q(W(t^*_{Q-1}))) < 0$

This holds for all species i when ordered by W^*_i, i.e. for species i to have a positive equilibrium density, the following expression must be satisfied

$$\frac{d^2W^*(t)}{dt^{*2}}|_{t^*=t^*_i} > 0$$

implying that if all species adhere to a tradeoff where $\frac{d^2W^*(t)}{dt^{*2}} > 0$ and $\frac{dW^*(t)}{dt^*} < 0$ they will all have feasible equilibria.

1.1 What is t^* in this context?

At equilibrium, the following expression must hold:

$$e^{f(g(W(t)))t^*} = e^{-f(r(t))(t_e - t^*_Q)}$$

which simplifying is equivalent to:

$$t^* = t_e\left(\frac{-f(r(t))}{f(g(W(t)))} + 1\right)$$

if $r(t)$ is constant then:

$$t^* = t_e \left(\frac{-f(r)}{f(g(W^*))} + 1 \right)$$

if $r(t)$ is not constant, the solution for t^* can still be found provided t^* can be pulled out of $f(r(t))$.