Solutions to Sample Exam 1

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1. C - This is a simple application of the centripetal acceleration formula:
$$v = \sqrt{ar} = \sqrt{9 \cdot 4g} = 6\sqrt{g} \text{ m/s}.$$

2. E - Unfortunately, we have to calculate this one. The inelastic case simply gives a conservation of momentum equation. Neglecting all of the m's, we have $v = 4V$. The elastic case requires equations for momentum and energy conservation. These are
$$\begin{aligned} v &= 3V' + v' \\ v^2 &= 3V'^2 + v'^2. \end{aligned}$$
Using $v = 4V$, we get
$$\begin{aligned} 4V &= 3V' + v' \\ 16V^2 &= 3V'^2 + v'^2. \end{aligned}$$
Eliminating v', we get
$$16V^2 = 3V'^2 + 16V^2 - 24VV' + 9V'^2,$$
or simply
$$0 = 12V'^2 - 24VV',$$
whence $V'/V = 2$. This is a bit of a calculation, but it's simple and you should be fluent in solving kinematics problems quickly.

3. D - The oscillation frequency of an LC circuit is a good quantity to memorize
$$\omega = \frac{1}{\sqrt{LC}}.$$
If you (understandably) forget it, it is straightforward to reconstruct it from dimensional analysis by looking for combinations of L and C that give units of s^{-1}. The capacitance scales as $C \sim A/d$, so C increase by a factor of 2 under the doubling. Assuming that our inductor is a solenoid with a fixed number of loops, then the inductance scales just like the solenoid inductance (a 20 second derivation if you forget it) $L \sim A/\ell$. Overall, the frequency drops in half.

4. B - A dipole contains no net charge, and the charge on a conductor will rearrange itself to completely cancel out the dipole field.

5. A - The approximation of a product wavefunction comes from solving the Schrodinger equation by separation of variables, which is only possible when the Coulomb repulsion term is ignored.

6. E - Since the emitted photon carries momentum, the entire atom must recoil slightly to conserve linear momentum. The means that the total energy released in the transition is divided between the gamma and the recoil of the atom. As a result, the photon energy will be slightly less than the true transition energy of the atomic level. Choices A-C simply do not make sense in the context of the problem, and D appears to violate the conservation of energy, so E is the correct choice.

7. C - The Fermi energy is $E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$, where n is the density. Since we are at fixed volume, doubling the number of particles doubles the density, which multiplies E_F by a factor of $2^{2/3}$, choice C.

8. D - Since the gas is well above the Fermi temperature, it is essentially classical. By the equipartition theorem, each quadratic degree of freedom contributes specific heat per particle $k/2$. There are two quadratic degrees of freedom for the kinetic part of the Hamiltonian and two quadratic degrees of freedom for the potential part, so the specific heat is $2k$.

9. B - Recall that the effective potential for radial motion is $V_{eff}(r) = \frac{l^2}{2mr^2} + U(r)$. The radii of circular orbits are found by solving $\dfrac{dV_{eff}}{dr} = 0$, and stability is determined by the sign of $\dfrac{d^2V_{eff}}{dr^2}$. Here,
$$\frac{dV_{eff}}{dr} = -\frac{l^2}{mr^3} + \frac{k}{r^2},$$
and setting this equal to zero and solving for r gives $r = \frac{l^2}{mk}$, choice B. We could check that this is a minimum of V_{eff} by computing the second derivative, but it's easier to just think about the behavior of the potential at $r = 0$ and $r = \infty$. At $r = 0$, the repulsive centrifugal barrier dominates and $V_{eff} \to +\infty$. As r approaches infinity, the central potential $U = -k/r$ decays more slowly than the centrifugal term, so V_{eff} approaches zero from *below*. Since V_{eff} has only a single critical point, sketching the graph of V shows that this radius is indeed a minimum, and hence an allowed stable circular orbit.

10. C - This is a straightforward application of the parallel axis theorem. First calculate

the moment of inertia about the center of the disk:

$$\begin{aligned} I_{CM} &= \int r^2 dm \\ &= \int_0^R \rho(r) r^2 (2\pi r\, dr) \\ &= 2\pi \int_0^R A r^6 dr \\ &= \frac{2\pi A}{7} R^7. \end{aligned}$$

The mass is

$$M = \int dm = \int_0^R \rho(r)(2\pi r\, dr) = 2\pi \int_0^R A r^4 = \frac{2\pi A}{5} R^5,$$

so $A = \frac{5M}{2\pi R^5}$ and $I_{CM} = \frac{5}{7} M R^2$. The parallel axis theorem tells us $I = I_{CM} + Mr^2$, where here, $r = R$. Thus $I = \left(\frac{5}{7} + 1\right) MR^2 = \frac{12}{7} MR^2$, choice C.

11. E - Redshift z is defined by $1+z = \lambda_{obs}/\lambda_{emit}$. So a galaxy at redshift 2 has wavelengths expanded by a factor of 3, and the 21 cm hydrogen line gets redshifted to 63 cm, choice E. Beware the trap answer D: redshift 2 does *not* mean wavelengths are expanded by a factor of 2!

12. A - Recall that the wavefunction is always continuous, so $\psi(0_+) = \psi(0_-)$. This eliminates choice D. We can also elmiminate B by dimensional analysis, since the wavefunction has dimensions of 1/energy and hence must be present on the right-hand side. We can calculate the discontinuity in the derivative by integrating the Schrödinger equation

$$\begin{aligned} \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} dx \left(-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + A\delta(x)\psi(x) \right) &= \int_{-\epsilon}^{\epsilon} dx E\psi(x) \\ -\frac{\hbar^2}{2m} \left(\frac{d\psi}{dx}(0_+) - \frac{d\psi}{dx}(0_-) \right) + A\psi(0) &= 0 \\ \frac{d\psi}{dx}(0_+) - \frac{d\psi}{dx}(0_-) &= \frac{2mA\psi(0)}{\hbar^2}. \end{aligned}$$

13. E - The Planck mass is the unique combination of fundamental constants $\hbar$, c, and G_N which has units of mass. (The adjective "reduced" refers to a convenient normalization where this mass is divided by 8π in some applications, but we don't have to worry about this here.) Expressing all the constants in terms of mass, length, and time,

$$\begin{aligned} \hbar &= [M][L]^2[T]^{-1} \\ c &= [L][T]^{-1} \\ G_N &= [M]^{-1}[L]^3[T]^{-2}. \end{aligned}$$

For a combination $(\hbar)^p(c)^q(G_N)^r$ to have units of mass, or $[M]^1$, we need:

$$\begin{aligned} p - r &= 1 \\ 2p + q + 3r &= 0 \\ -p - q - 2r &= 0 \end{aligned}$$

Using your favorite method to solve systems of linear equations, we find $p = 1/2$, $q = 1/2$, $r = -1/2$, so $M_P = \sqrt{\frac{\hbar c}{G_N}}$. Plugging in the given values on the formula sheet, we get 2.176×10^{-8} kg, choice E.

14. B - This is a straightforward application of the relativistic Doppler shift formula:

$$\frac{\nu'}{\nu} = \sqrt{\frac{1+\beta}{1-\beta}},$$

where $\beta = v/c$ is the relative velocity of the source with respect to the observer. The easy way to keep track of the signs in the numerator and the denominator is to remember that the observed frequency *increases* when the source is moving towards the observer. Or, if you're partial to astrophysics, when the observer is receding, the signal is *redshifted* (in other words, the frequency decreases). Here the source is approaching, so $\beta = 0.6$ is positive and

$$\frac{\nu'}{\nu} = \sqrt{\frac{1.6}{0.4}} = \sqrt{4} = 2,$$

so the frequency doubles: $\nu' = 2$ GHz, choice B.

15. A - This is the setup for the classic method of images problem. (If you don't know how this setup works, go back and review the relevant section of the Electricity and Magnetism chapter!) When the real charge is at height z above the plate, the image charge $-q$ is at height z below the plate, so the force the real charge feels (due entirely to the image charge) is $\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2z)^2}\hat{\mathbf{z}}$. Integrate this along the path from $z = d$ to $z = d/2$ to find the total work done on q:

$$\begin{aligned} W &= -\int_d^{d/2} \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2z)^2}\,dz \\ &= \frac{q^2}{16\pi\epsilon_0}\frac{1}{z}\bigg|_d^{d/2} \\ &= \frac{q^2}{16\pi\epsilon_0 d} \end{aligned}$$

This is equal to the kinetic energy gained by the particle, so

$$\frac{1}{2}mv^2 = \frac{q^2}{16\pi\epsilon_0 d} \implies v = \frac{q}{\sqrt{8\pi\epsilon_0 md}}.$$

Note that the fact that the image charge moves along with the real charge means that we can't naively apply conservation of energy, where the initial and final potential energies are those created by the image charge. This incorrect approach leads to the trap answer B.

16. D - The charge after one second is $Q = (10^{-3}\ \text{coulomb} \cdot \text{s}^{-1})(1\ \text{s}) = 10^{-3}$ coulomb. From $V = Q/C$, we get $V = 10^{-3}/10^{-5} = 100$ V.

17. E - The leading order perturbation is simply

$$\begin{aligned} \langle 2 |\, \delta V \,| 2 \rangle &= \frac{2}{L} \int_0^{L/2} V_0 \sin^2\left(\frac{2\pi x}{L}\right) dx \\ &= V_0/2. \end{aligned}$$

18. A - From $dE = T\,dS - P\,dV$, we do a Legendre transform to get $dF = -S\,dT - P\,dV$. Since the expansion is isothermal, it takes place at constant temperature, so the first term vanishes; the second term is positive because the volume increases, so dF is negative and the free energy decreases, rather than increases. This is consistent with the definition of "free" energy, which is essentially the available energy the gas has to do work on its surroundings. Since it does work in the isothermal expansion phase, at the end of the expansion it has less free energy available to do work.

19. E - The angle of the first diffraction minimum is $\sin\theta = \lambda/a$, where λ is the wavelength of the incident light. Since the screen is far away, we approximate $\sin\theta \approx \theta$, so the angular width of the central maximum is 2θ. For a screen a distance L away, the width of the maximum as seen on the screen is $L \tan\theta \approx L\theta$. So we want

$$2L\frac{\lambda}{a} = 100a \implies a^2 = 9 \times 10^{-8}\ \text{m}^2 \implies a = 3 \times 10^{-4}\ \text{m} = 0.3\ \text{mm}$$

20. B - From the definition of conjugate momentum,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m(a + b\cos\theta)^2 \dot{\phi},$$

choice B.

21. D - The Lagrangian isn't explicitly dependent on time, so the first two terms in L represent the kinetic energy T and the third represents the potential energy U. Since $L = T - U$, to get the total energy $T + U$ we have to take $L + 2U$, which is choice D.

22. A - The differential equation satisfied by the current $I(t)$ in this RL circuit, for $t > 0$, is

$$V - L\frac{dI}{dt} - IR = 0.$$

(Note that because there's no capacitor, it's not necessary to consider the differential equation for the charge $Q(t)$ – we can work with the current directly.) This is an inhomogeneous linear differential equation, with homogeneous solution $I_h(t) = Ae^{-Rt/L}$ for some constant A, and a particular solution which can be taken to be time-independent. But since all we care about is the voltage across the inductor, which is proportional to dI/dt, we can ignore the particular solution. Since $V_L = L\frac{dI}{dt} = -RI_h$, the inductor voltage drops to half its initial level when I_h itself drops to half its initial level:

$$\begin{aligned} Ae^{-Rt/L} &= \frac{A}{2} \\ -\frac{Rt}{L} + \ln A &= -\ln 2 + \ln A \\ t &= \frac{L \ln 2}{R} \end{aligned}$$

which is choice A.

23. E - The straight wire produces a magnetic field that circles azimuthally around in the $\hat{\phi}$-direction. The tension on the circular wire, if any, will be the result of the Lorentz force that the moving charges in the circular wire feel due to the magnetic field established by the straight wire. But the current in the circular wire is also flowing in the $\hat{\phi}$-direction, parallel to the magnetic field. The electrons flowing in the circular wire therefore feel no force and there is no tension. So the correct answer is E.

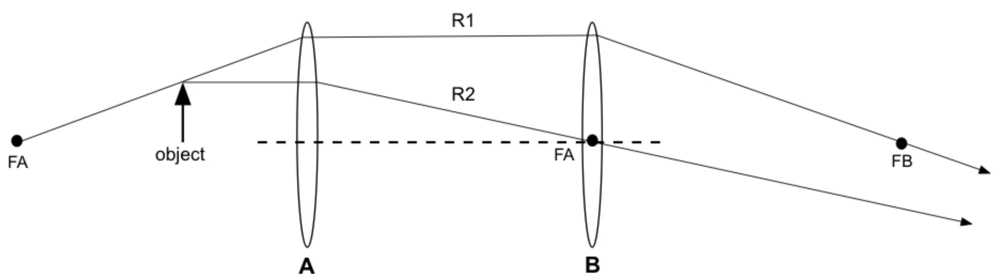
24. A - Considering the limit of $r \gg h$ eliminates C and D immediately, but we need to calculate to get the exact numerical factors. The cylinder will fall over when it is tipped just past the point where the center of mass is directly above the point of contact of the cylinder with the ground. The angle of the cylinder with the horizontal is then just A.

25. C - The γ factor is $\gamma = 3000\text{MeV}/100\text{MeV} = 30$. In the frame of the earth, the lifetime of the muon is $\tau = 30\tau_{rest} = 6 \times 10^{-5}$ s. Since the muons are close to the speed of light, the time required to arrive at the surface of the earth is approximately $t = 10^5\ \text{m}/c = 3.33 \times 10^{-4}$ s. The fraction of muons remaining is therefore

$$f = e^{-t/\tau} \approx e^{-5}.$$

26. B - This problem is a bit of trivia that is difficult to guess. The only way to guess it is to notice that the dispersion relation in B gives energy as proportional to momentum, which is ordinarily true for particles in the extreme relativistic limit (i.e. $E = pc$). Given that the problem refers to unusual electronic properties, it might seem reasonable that this unusual dispersion relation would produce unusual properties. This is indeed what happens in graphene, where electrons and holes behave like massless Dirac fermions, producing a semiconductor with no gap energy.

27. B - At first glance, this looks like an annoyingly complicated problem. The $n = 2$ states of the Coulomb potential are degenerate in l and m, so it seems we have to do degenerate perturbation theory and calculate the sixteen matrix elements $\langle l_1, m_1|\Delta H|l_2, m_2\rangle$ with $\Delta H = eE_0 z$. However, we don't actually need to do so much work: it's enough to observe that if a single one of these matrix elements is nonvanishing, the system will show a linear, rather than a quadratic, dependence on E_0. Indeed, the perturbation is of exactly the same form that appears in spontaneous emission problems, so we can apply the selection rules: we need $\Delta m = 0$ and $\Delta l = \pm 1$, so the candidate matrix element is $\langle l = 0, m = 0|\Delta H|l = 1, m = 0\rangle$. In slightly more detail, the $l = 1, m = 0$ state is proportional to $Y_1^0(\cos\theta) \propto \cos\theta$, the perturbation operator is $z = r\cos\theta$. The $l = 0, m = 0$ state is independent of θ, so multiplying by z makes it proportional to Y_1^0, and thus the matrix element is nonzero by orthonormality of the spherical harmonics.



28. D - We can use the lens equation twice. For the first lens, we have

$$\frac{1}{5\text{ cm}} = \frac{1}{2\text{ cm}} + \frac{1}{s_1'},$$

which implies that $s_1' = -10/3$ cm. The negative sign implies that the image is located to the left of lens A, or 25/3 cm to the left of lens B. Using the lens equation again for the second lens, we find that

$$\frac{1}{5\text{ cm}} = \frac{3}{25\text{ cm}} + \frac{1}{s_2'},$$

which implies that $s_2' = 12.5$ cm. Since the answer is positive, it is located to the right of lens B.

If for some reason you forgot the lens equation, you can still solve this problem using straightforward geometry. Doing ray-tracing as shown in the diagram, we can consider just two rays: ray $R1$, which passes through the focus of A, comes out parallel between A and B, and passes through the focus of B; and ray $R2$, parallel to the axis which passes through the focus of A, which is also the center of B. The location of the intersection of these two rays will give the location of the object. Using your favorite method to find the intersection of two straight lines, we get $x = 12.5$ cm, choice D.

29. C - We can sum the moments of inertia of each side of the square frame separately, using the parallel axis theorem. The moment of inertia about the center of each rod is given on the formula sheet, $I_{CM} = \frac{1}{12}ML^2$. The parallel axis theorem states that for each edge of the frame, $I = I_{CM} + MR^2$, where R is the distance between the nail and the center of mass of the edge. Using properties of right and equilateral triangles and the Law of Cosines (or doing a little coordinate geometry), we find $R_{top} = L\sqrt{3}/2$, $R_{bottom} = L + L\sqrt{3}/2$, and $R_{sides} = \frac{L}{2}\sqrt{5+2\sqrt{3}}$. Thus

$$\begin{aligned} I_{tot} &= 4I_{CM} + M(R_{top}^2 + R_{bottom}^2 + 2R_{sides}^2) \\ &= \frac{1}{3}ML^2 + ML^2\left(\frac{3}{4} + (1 + \frac{3}{4} + \sqrt{3}) + \frac{1}{2}(5+2\sqrt{3})\right) \\ &= \left(\frac{16}{3} + 2\sqrt{3}\right)ML^2. \end{aligned}$$

Note that to apply the parallel axis theorem correctly, we *must* take all moments of inertia from the centers of the various rods. Forgetting to do this at one point or another leads to many of the trap answers.

30. E - When combining spins l_1 and l_2 (where without loss of generality we can take $l_1 < l_2$), we can get all values of l between $l_2 + l_1$ and $l_2 - l_1$ in integer steps. In the present case, with $l_1 = 1$ and $l_2 = 2$, we can get $l = 3, 2, 1$. The only condition on m_l is $|m_l| \leq l$, so choices A-D are all fine. Since $l = 0$ is impossible, the answer is E.

31. D - The radial probability density for the $2p$ state is given by

$$r^2|R_{21}(r)|^2 = \frac{1}{24}a_0^{-5}r^4\exp(-r/a_0),$$

where the factor of r^2 is from the volume element in spherical coordinates, $dV = r^2 \sin\theta dr\, d\theta\, d\phi$. To find the most probable value, we take the derivative and set to zero (and in doing so, we can ignore all the annoying constants out front):

$$\begin{aligned} 4r^3e^{-r/a_0} - \frac{r^4}{a_0}e^{-r/a_0} &= 0 \\ \implies r &= 4a_0, \end{aligned}$$

choice D. Forgetting the factor of r^2 from the volume element leads to trap answer C, forgetting to square the wavefunction leads to E, and forgetting both also leads to C. These are all very common mistakes – don't make them!

32. C - This is almost a giveaway, since the "mass" in "mass spectrometry" does indeed refer to the mass of the particle involved. However, it is important to note that only the charge-to-mass ratio can be measured by electromagnetic fields, so the answer can't be D, which can change the mass without affecting the charge.

33. E - Beat frequencies are caused by destructive interference between two closely spaced frequencies, resulting in a modulation with a long enough period that the minimum of each cycle is heard independently. For this problem, all that is relevant is the sum-to-product identity

$$\cos 2\pi at + \cos 2\pi bt = 2\cos\left(2\pi\frac{a+b}{2}t\right)\cos\left(2\pi\frac{a-b}{2}t\right)$$

The second term in the product modulates the wave; we hear beats when its amplitude is zero, which occurs at frequency $a-b$. Note that this is *twice* the apparent frequency of the cosine! If all we know is that this frequency is 3 Hz, and that one of a or b is 440 Hz, it is impossible to determine whether the other frequency is 443 Hz or 437 Hz, since both would give the same beat frequency (the sign of $a-b$ is irrelevant because cosine is even). Hence the correct answer is E.

34. C - This problem involves the addition of velocities formula with a small twist. For our setting, the addition of velocities formula is

$$s = \frac{u+v}{1+\frac{uv}{c^2}},$$

where u is the speed of the warship in the Enterprise frame, v is the speed of the Enterprise in the planet frame, and s is the speed of the warship in the planet frame. We are given v in the problem, and we are solving for s. To determine u, we divide the distance Δx traveled by the photon torpedo in the Enterprise frame by the time Δt taken for the photon torpedo to contact the warship in the Enterprise frame. This gives

$$u = \frac{\Delta x}{\Delta t}.$$

On the other hand, we know that since the torpedo travels at c, we must have

$$\Delta t = \frac{\Delta x + x_0}{c}.$$

This implies that

$$\Delta x = c\Delta t - x_0,$$

and therefore that

$$u = \frac{c\Delta t - x_0}{\Delta t} = c - \frac{6\times 10^6\text{m}}{0.1\text{s}} = 2.4\times 10^8\text{ms}^{-1} = 0.8c,$$

with $c = 3\times 10^8\text{ms}^{-1}$. Plugging this result into our expression for s above, we find

$$s = \frac{0.5c + 0.8c}{1+(0.8c)(0.5c)/c^2} = \frac{1.3}{1.4}c = \frac{13}{14}c,$$

which is C.

35. D - There are no external torques here, so angular momentum is conserved. Thus $I_0\omega_0 = I_1\omega_1$, and plugging in numbers we arrive at D.

36. C - IV is obviously false since we use the Euler-Lagrange equations to construct the equations of motion for any system with a Lagrangian. III is not always true because we are always free to add a constant to the potential – the fact that L is independent of x only means that the potential must be constant in space, and we can set that constant to zero if we wish.

    On the other hand, homogeneity of time *does* imply conservation of energy, and homogeneity of space implies linear momentum conservation. If these statements are unfamiliar to you, then let's prove them. Since L depends on x and $\dot{x}$ only, the total time derivative of the Lagrangian is

$$\frac{dL}{dt} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{\partial L}{\partial t}$$

    Using the Euler-Lagrange equations and the fact that $\frac{\partial L}{\partial t} = 0$, we have

$$\begin{aligned}
\frac{dL}{dt} &= \dot{x}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial \dot{x}}\ddot{x} \\
&= \frac{d}{dt}\left(\dot{x}\frac{\partial L}{\partial \dot{x}}\right) \\
0 &= \frac{d}{dt}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} - L\right).
\end{aligned}$$

    But the quantity in parentheses is precisely the energy of the system that we obtain when we construct the Hamiltonian of the system from the Lagrangian via the Legendre transform.

    The situation for momentum is more simple. By Euler-Lagrange, we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0,$$

    and therefore

$$\frac{\partial L}{\partial \dot{x}} = p,$$

    where p is a constant of motion that is defined to be the linear momentum.

37. D - The formula we want is $N = \epsilon\mathcal{L}\sigma T$, where N is the number of events seen, ϵ is the detector efficiency, $\mathcal{L}$ is the luminosity, σ is the cross-section, and T is the running time. Note that this just comes straight from dimensional analysis – any possible numerical factors are all absorbed in the definitions of luminosity, cross-section, and efficiency. There are 8.64×10^4 seconds in a day, so $N = (0.5)(10^{22})(10^{-20})(8.64\times10^4) = 4.32\times10^6$, choice D.

38. B - Cyclotron motion is simple enough that it can be derived in a matter of seconds from the Lorentz force law and centripetal force. Rewriting a bit, we find that

$$r = \frac{p}{qB}.$$

Triple the field, and the momentum must also be tripled to maintain constant radius.

39. B - The key to this problem is to recognize that this configuration is a magnetic *quadrupole*, since each current loop by itself is a magnetic dipole, and we have two oppositely-oriented ones placed back to back. Just like in the electrical case, the field of a dipole goes like $1/z^3$ at distances far from the source, so the field of a quadrupole goes like $1/z^4$. Furthermore, recall that the magnetic dipole moment is proportional to the area of the loop, so it figures that the field should involve b^2, rather than just b. This leaves only choice B, so forget trying to compute the exact field – we're done.

    For completeness, here's how to calculate the answer from the beginning. The Biot-Savart tells us that the field due to the top loop is

$$\begin{aligned} B(z) &= \frac{\mu_0 I}{4\pi}\int \frac{\hat{\mathbf{r}} \times d\ell}{r^2} \\ &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{b d\theta}{r^2} \frac{b}{((z-a/2)^2+b^2)^{1/2}}\hat{\mathbf{z}} \\ &= \frac{\mu_0 I b^2}{2} \frac{1}{((z-a/2)^2+b^2)^{3/2}}\hat{\mathbf{z}}. \end{aligned}$$

    The magnitude of the field due to both loops is now

$$B(z) = \frac{\mu_0 I b^2}{2}\left[\frac{1}{((z-a/2)^2+b^2)^{3/2}} - \frac{1}{((z+a/2)^2+b^2)^{3/2}}\right].$$

    Since b is negligible[1], we have

$$B(z) = \frac{\mu_0 I b^2}{2}\left[(z-a/2)^{-3} - (z+a/2)^{-3}\right].$$

    The first nonvanishing term in the generalized binomial expansion is

$$B(z) = \frac{3\mu_0 I a b^2}{2z^4}.$$

    While admittedly a bit lengthy, the Biot-Savart calculation should be very familiar to you, the first approximation is easy, and the binomial approximation is something that you should learn for the exam. But remember, if you can take 30 seconds to stare at the answer choices and find the right solution right away, there's no point in calculating!

[1] If you don't believe this approximation, try including b by Taylor-expanding one more time: you'll find that it contributes a term proportional to $(z-a/2)^{-5}$, which is higher-order than the leading z^{-4} term.

40. D - Recall that L^2 commutes with each component of L because it commutes with L_z. For the same reason, L^2 commutes with $J^2 = L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S}$: $\mathbf{L}$ and $\mathbf{S}$ act on different parts of the wavefunction, so all components of $\mathbf{L}$ commute with all components of $\mathbf{S}$, and L^2 commutes with the last term because it commutes with $\mathbf{L}$ as well. However, L^2 only commutes with rotationally symmetric Hamiltonians, and a Hamiltonian need not have rotational symmetry: for example, an atom in a strong magnetic field which picks out a particular direction in space.

41. B - The possible energies are just the eigenvalues of the Hamiltonian matrix, which we can obtain by solving the equation

$$\begin{aligned} \det(H - \lambda I) &= 0 \\ (a-\lambda)(\lambda^2 - b^2) &= 0. \end{aligned}$$

The solutions are clearly $\lambda = a$ and $\lambda = \pm b$, choice B.

42. B - This problem is a little subtle, but can be solved either by guessing or by doing real physics. If you are in the mood the guess, then notice that you can eliminate E because it has nothing to do with magnetic charges or fields. We can also eliminate A because if there were magnetic charges, then presumably we would have a Maxwell equation $\nabla \cdot \mathbf{B} \propto \rho_m$ just like we have a Maxwell equation for electric (monopole) charges $\nabla \cdot \mathbf{E} \propto \rho_e$. This is clearly well-defined. The existence of monopoles does not changes dipole behavior, so C is eliminated. And D does not necessarily have anything to do with magnetic fields. So B is correct.

To see why B is the correct choice, consider that monopoles would give us a new Maxwell equation

$$\nabla \cdot \mathbf{B} = A\rho_m,$$

where A is some constant, and ρ_m is the monopole charge density. Integrating over a closed surface, and invoking the divergence and Stokes's theorems, we find that

$$\int \nabla \cdot \mathbf{B} = \int \mathbf{B} \cdot d\mathbf{S} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_0 \mathbf{A} \cdot d\mathbf{l} = 0.$$

The last equality comes from the fact that Stokes's theorem forces us to integrate over the boundary of a closed surface, which is, by definition, nothing. But this is awfully strange! The integrated monopole charge, should not be zero unless one of our steps was unjustified. Indeed, assuming that $\mathbf{B} = \nabla \times \mathbf{A}$ is only valid for a divergenceless field, and monopoles have nonzero divergence. Thus the vector potential as we normally understand it is not well-defined everywhere for monopoles. It turns out that it is possible to define a single *local* vector potential, but not one that is valid over an entire closed surface.

43. D - The perpendicular electric field only provides an acceleration perpendicular to the direction of motion, so we can solve this just like an analogous kinematics problem

where gravity provides the perpendicular force. The velocity in the direction of the beam stays v, so the time it takes to strike the target is $t = L/v$. On the other hand, the perpendicular acceleration is $a_\perp = qE/m$, so in time t the protons are deflected $\frac{1}{2}a_\perp t^2 = \frac{qE}{2m}t^2$ in the perpendicular direction. Setting this equal to R, plugging in $t = L/v$, and solving for E, we obtain $E = \dfrac{2mRv^2}{qL^2}$, choice D.

44. E - Obviously the big bang comes first. In the standard cosmology, the universe then undergoes inflation, at the end of which a quark-gluon plasma fills the universe. Once the universe cools enough to allow quarks to form hadrons, the only light particles left are free leptons, hence the lepton era. Even without knowing the exact timeline governing I and II, we can deduce immediately that II precedes I, since (re-)ionization refers to the stripping of an electron from a neutral atom, and to have atoms we must first have nuclei, which are formed during nucleosynthesis. So the correct order is III, IV, II, I, choice E.

45. C - For an adiabatic process, we have $PV^\gamma = \text{const.}$, where $\gamma \equiv c_P/c_V = (\alpha+1)/\alpha$ and 2α is the number of degrees of freedom for the system. For an ideal gas $\alpha = 3/2$ and $\gamma = 5/3$.

46. A - A solid cylinder of uniform mass density has moment of inertia $I = \frac{1}{2}MR^2$. The potential energy mgL of the weight when the rope is wound is entirely converted to kinetic energy once the rope is unwound. Note that the kinetic energy has contributions both from the rotational energy of the cylinder and the velocity of the weight. So by conservation of energy,

$$mgL = \frac{1}{2}mv^2 + \frac{1}{4}MR^2\frac{v^2}{R^2} \implies v = \sqrt{\frac{4mgL}{M+2m}},$$

choice A.

47. A - The force on an object is related to the potential energy by

$$\mathbf{F} = -\nabla U.$$

All we need is to take the gradient of the potential in the question

$$\begin{aligned}\mathbf{F} &= -\hat{\mathbf{x}}\frac{\partial}{\partial x}(x) - \hat{\mathbf{y}}\frac{\partial}{\partial y}\left(y^2\right) + \hat{\mathbf{z}}\frac{\partial}{\partial z}(\cos z) \\ &= -\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} - \sin z\hat{\mathbf{z}}\end{aligned}$$

48. B - The formula for the energy is

$$E = \frac{\sum_\epsilon \epsilon d(\epsilon)e^{-\epsilon/kT}}{\sum_\epsilon d(\epsilon)e^{-\epsilon/kT}}.$$

Plugging in the given energies and degeneracies,

$$E = \frac{(-2\epsilon)e^{\epsilon/kT} + (3\epsilon)e^{-\epsilon/kT}}{2e^{\epsilon/kT} + 1 + 3e^{-\epsilon/kT}}.$$

As $T \to \infty$, the exponent goes to zero, so each of the exponential factors collapses to 1. Thus the $T \to \infty$ limit of the energy is

$$E = \frac{-2\epsilon + 3\epsilon}{2 + 1 + 3} = \frac{\epsilon}{6},$$

choice B. This last step is equivalent to saying that at infinite temperature, each state becomes equally probable, so we can forget about the Boltzmann factor and just calculate a weighted mean.

49. E - This quantity shows up so often that it may be useful simply to memorize the result. The derivation is rather straightforward, though. In the canonical ensemble, we calculate the partition function in a box of volume V. There is no potential energy, so the energy of a single particle is pure kinetic, $E = |p|c$. Thus the partition function is (up to factors of $\hbar$ which are only needed for dimensional reasons)

$$Z = \frac{V}{h^3} \int d^3p\, e^{-\beta |p| c},$$

where $\beta = 1/kT$ as usual. Going to spherical coordinates in momentum space and using the fact that the integrand is spherically symmetric,

$$Z = \frac{4\pi V}{h^3} \int_0^\infty p^2 e^{-\beta p c} dp = \frac{8\pi V}{(h\beta c)^3}.$$

(The integral can be done simply by repeated application of integration by parts.) Since we are calculating the heat capacity, we only care about the part of $\ln Z$ which depends on β:

$$\ln Z = -3 \ln \beta + \text{const.}$$

Continuing, $E = -\dfrac{\partial \ln Z}{\partial \beta} = \dfrac{3}{\beta} = 3kT$, so $C_V = \frac{dE}{dT} = 3k$, choice E.

50. D - The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}),$$

and if the electric field is $\mathbf{E} = E_0 \cos(kx - \omega t)\hat{\mathbf{z}}$, then the magnetic field is $\mathbf{B} = -(1/c)E_0 \cos(kx - \omega t)\hat{\mathbf{y}}$ for propagation in the $\hat{\mathbf{x}}$-direction. The average magnitude of the Poynting vector is simply

$$\langle \mathbf{S} \rangle = \frac{1}{\mu_0 c} \left\langle E_0^2 \cos^2(kx - \omega t) \right\rangle = \frac{E_0^2}{2\mu_0 c}.$$

(A very useful fact to remember is that the average of $\sin^2$ or $\cos^2$ over one period is 1/2.)

51. D - If we consider the collision in the CM frame of the e^+e^- system, then the γ must be at rest to conserve momentum. But γ always travels at the speed of light, so conservation of momentum is violated.

52. C - This is a simple application of the uncertainty principle. We know that

$$\Delta p \Delta x = \frac{\hbar}{2}.$$

So if r is the nuclear radius then

$$r = \frac{\Delta x}{2} = \frac{\hbar}{4\Delta p}.$$

Using our numerical values, we have

$$r = \frac{\hbar c}{4 \times 40\text{MeV}} = \frac{197}{180}\ \text{fm} \simeq 1.23\ \text{fm}.$$

53. A - Choices B-E all are spin-1 objects and therefore are bosons, which obey Bose-Einstein statistics. You should know that the photon has spin-1 from quantum mechanics. The statistics of helium nuclei and atoms follow from the rules of addition of angular momentum: the nucleus has 4 fermions (two protons and two neutrons), and the atom has two more fermions (the two atomic electrons), and an even number of spin-1/2 particles behaves as a boson with integer spin. Pions are a little tricky, but you might remember that in the quark model they are bound states of a quark and an antiquark, both fermions, so again an even number of fermions gives a boson. Only A, the neutrino, is a spin-1/2 particle and fermion that obeys Fermi statistics.

54. D - The mass of the electron is 511 keV/c^2, which is one of those ubiquitous numbers which should be memorized. So any time the energy scale 511 keV shows up, it must have something to do with electrons. In this case, electrons and positrons at rest in the galactic center can annihilate to two photons, each of energy 511 keV. (Incidentally, the fact that the line is "sharp" means that most of the electrons and positrons in the galactic center are moving slowly, since if annihilation took place between two highly energetic particles, the photons would be boosted in the galactic rest frame. The spectrum observed on Earth would then have a sharp "edge" at 511 keV and a long tail extending to higher energies.)

55. C - From basic kinematics, the cannonball falls a vertical distance of 100 m in time $\frac{1}{2}gt^2$; solving gives $t = \sqrt{200/g}$ seconds. In this time, the cannonball travels a horizontal distance of 300 m, so from $vt = 300m$, we plug in our previous result for t to get

$$v = 300\sqrt{g/200} \approx 300\sqrt{1/20} \approx 300(1/4.5) \approx 66.7\ \text{m/s}$$

This is closest to choice C, so we choose it and move on. This pattern of answer choices is typical of Physics GRE questions – since they're so widely spaced, it's not at all necessary to do arithmetic to three decimal places. The approximations $g \approx 10$ and $\sqrt{20} \approx 4.5$ were just fine here.

56. A - Choices C and D can be eliminated immediately since they imply that the spaceship exhausts more than its own mass in fuel, which is nonsensical. The remaining choices are trickier since they only differ by numerical factors in the exponent.

    To solve the problem, balance the momentum of the exhaust at an instant with the momentum of the spaceship

$$-dp_{exhaust} = dp_{spaceship}.$$

$$-v_0 dm = m_{spaceship} dv_{spaceship}.$$

    Rearranging and integrating, we find that

$$\begin{aligned} -\int_0^{v_f} \frac{1}{v_0} dv &= \int_{M_0}^{M_0 - M_{expelled}} \frac{1}{m_{spaceship}} dm_{spaceship} \\ -\frac{v_f}{v_0} &= \log\left(\frac{M_0 - M_{expelled}}{M_0}\right) \\ M_{expelled} &= M_0\left(1 - e^{-v_f/v_0}\right), \end{aligned}$$

    so A is the correct choice. The primary confusion that can arise in this problem is among the different masses and velocities. Take care to understand the subscripts included in the solution shown above.

57. A - Since all we care about is ratios, we can scale all elements of the circuit by some convenient numerical factor: let's divide all the resistances by 10 kΩ to make the numbers easier (so we're working in units of 10^{-4} A whenever we calculate currents). When S is open, the total resistance of the circuit is $4 + 1 = 5$, so the current is $I_1 = 1$. When S is closed, the three resistors in parallel have an effective resistance of $(1 + 1/2 + 1/3)^{-1} = 6/11$. So the total resistance is $4 + 6/11 = 50/11$, and the total current is $11/10$. The voltage across the 40 kΩ resistor is $IR = 22/5$, so the drop across the 10 kΩ resistor is $5 - 22/5 = 3/5$. Hence the current I_2 is $3/5$ in our units; since $I_1 = 1$ in these units, this is also the desired ratio.

58. C - This is a ballistic pendulum problem with two twists: the rod is not massless, and starts out an an angle with respect to the projectile so we have to be a little careful calculating the angular momentum. The initial angular momentum comes just from the clay and is $\mathcal{L} = m\mathbf{v} \times \mathbf{r} = mvL\sin(90^\circ - \alpha) = mvL\cos\alpha$, so by conservation of angular momentum, this is the angular momentum after the collision as well. We get the angular velocity from $\mathcal{L} = I\omega$, where the moment of inertia of the clay-rod system is $I = \frac{1}{3}ML^2 + mL^2$ (the first term from the moment of inertia of a rod about one end, and the second from the moment of inertia of a point mass). So

$$\omega = \frac{\mathcal{L}}{I} = \frac{3mv\cos\alpha}{(M + 3m)L},$$

    choice B.

59. E - Because the loop rotates, the angle the normal to the loop makes with the magnetic field oscillates sinusoidally, and so does the flux: $\Phi = B_0 A \sin \omega t$. Thus the EMF in the loop is $V = d\Phi/dt = B_0 A\omega \cos \omega t$, and the power dissipated in the resistor is $P = V^2/R = B_0^2 A^2 \omega^2 \cos^2 \omega t/R$. Solving for ω, and using the fact that the average of $\cos^2$ is $1/2$, we find

$$\omega = \frac{\sqrt{2PR}}{B_0 A} = 500\sqrt{2} \text{ rad/sec} \approx 707 \text{ rad/sec}.$$

60. B - If you do not recall the mathematical definition of chemical potential, you should at least remember that the chemical potential has something to do with the energy associated with changing the number of particles in a system. This reduces the options to A or B, but there is chemical potential for systems of bosons and fermions, so B is the correct choice.

    To be more rigorous, recall that the entropy is $S = \sum_i p_i \log p_i$. We wish to maximize the entropy with respect to the constraints

$$\begin{aligned} 0 &= \sum_i \epsilon_i p_i - E \\ 0 &= \sum_i n_i p_i - N \\ 0 &= \sum_i p_i - 1. \end{aligned}$$

    To maximize S we introduce the Lagrange multipliers β, μ, and λ, we write

$$S = \sum_i p_i \log p_i - \beta \left(\sum_i \epsilon_i p_i - E \right) - \mu \left(\sum_i n_i p_i - N \right) - \lambda \left(\sum_i p_i - 1 \right),$$

    and solve for

$$\frac{\partial S}{\partial x} = 0.$$

    This μ turns out to be the chemical potential and is clearly the Lagrange multiplier that enforces the particle number constraint.

61. C - At first glance, choices D and E do not seem to make much sense. Choice B is a bit suspicious too because it does not contain a factor of 2, implying that the spin distributions are independent of the external magnetic field. Choice A does not make much sense either because the number of spin-up particles should increase, not decrease, with an increase in magnetic field: alignment with the magnetic field is energetically favorable. So C seems to be the best choice.

    To decide for sure, we can calculate. Since the eigenvalues of σ_z are ± 1, the possible energies are $E = \pm|H|$. The partition function is

$$Z = e^{E/kT} + e^{-E/kT}.$$

The ratio of spin up to spin down particles is just

$$A = \frac{e^{E/kT}}{e^{-E/kT}} = e^{2E/kT}.$$

If we double the magnetic field, we have $E \to 2E$, which implies that $A \to e^{4E/kT} = \left(e^{2E/kT}\right)^2 = A^2$.

62. A - While this is a good fact to memorize, we can get it quickly by recalling Poisson's equation in SI units:

$$\nabla^2 V = -\rho/\epsilon_0$$

Since the potential of a point charge at the origin is q is $V = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$, and its charge density is $\rho = q\delta^3(r)$, we can read off $\nabla^2 V = -4\pi\delta^3(r)$. Even without remembering this shortcut, it's not too bad to derive:

$$\begin{aligned} \int_V \nabla^2 \left(\frac{1}{r}\right) d^3\mathbf{r} &= \int \nabla \cdot \left(\nabla \left(\frac{1}{r}\right)\right) d^3\mathbf{r} \\ &= \int_S \nabla \left(\frac{1}{r}\right) \cdot d\mathbf{S} \\ &= \int_S \frac{\partial}{\partial r} \left(\frac{1}{r}\right) 4\pi R^2 dr \\ &= -4\pi \int_S \frac{R^2}{r^2} dr. \end{aligned}$$

When $r > 0$ and $R \to 0$, we obtain 0. When $r = R$ and $R \to 0$, we obtain $-4\pi\delta^3(0)$. Thus, $\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(r)$.

63. A - The formula for Compton wavelength is

$$\lambda = \frac{h}{mc},$$

and plugging in numbers gives 1.32×10^{-15} m, which is closest to choice A. If you didn't happen to remember the formula for Compton wavelength, you could get it by dimensional analysis. We know the Compton wavelength has something to do with quantum mechanics, so h or $\hbar$ must make an appearance, and the mass of a quantum particle is its only distinguishing characteristic (besides its spin of course, but that has the same units as $\hbar$). To get units of length, we need another dimensionful constant, and c fits the bill. So at worst, we would get $\lambda = \frac{\hbar}{mc}$ and be off by a factor of 2π, but happily the answer choices are widely-spaced enough that this isn't an issue. Alternatively, we could remember that the defining length scale for nuclear interactions is the fermi, 10^{-15} m, which is approximately the range of the strong force. So it makes sense that the quantum "size" of the proton is close to this value, but certainly not much larger.

64. B - Decay problems are usually easier in the center-of-mass frame, where the decaying particle is stationary, so let's start there. We'll work in units where $c = 1$ until the very end of the problem. In this frame, the K^0 has energy m_K, and since the decay products have equal mass, each gets energy $m_K/2$. Now, boosting to the lab frame must give the K^0 energy E; since $E = \gamma m_K$, we get the boost factor $\gamma_{Lab} = E/m_K$. From $\gamma = (1 - v^2)^{1/2}$, this corresponds to a velocity $v_{Lab} = (1 - m_K^2/E^2)$.

    Now, in the center-of-mass frame, a pion with energy $E_\pi = m_K/2$ has boost factor $\gamma_\pi = E_\pi/m_\pi = m_K/2m_\pi$, and momentum $p = (E_\pi^2 - m_\pi^2)^{1/2} = (m_K^2/4 - m_\pi^2)^{1/2}$. But $p = \gamma_\pi m_\pi v_\pi$, so the pion has velocity $v_\pi = \dfrac{2m_\pi(m_K^2/4 - m_\pi^2)^{1/2}}{m_K\, m_\pi} = (1 - 4m_\pi^2/m_K^2)^{1/2}$ in the center-of-mass frame. Tacking on the factors of c in the correct places gives choice B.

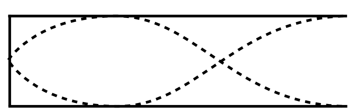
65. E - To boost from the rest frame of the K^0 before it decays to the lab frame, we require a boost factor

$$\gamma = \frac{E}{m_K c^2}.$$

    In the center of mass frame (the rest frame of the K^0, after it decays), the π^- has an energy $m_K c^2/2$. To find the energy of the π^- back in the lab frame, we use the γ that we found in the equation above to boost back. Since we want to know when the π^- is at rest, we want to set the energy after the boost equal to the rest energy of the π^-:

$$\begin{aligned} \gamma m_\pi c^2 &= \frac{m_K c^2}{2} \\ \Longrightarrow E &= \frac{m_K^2 c^2}{2m_\pi}. \end{aligned}$$

    We conclude that E is correct.



66. C - For a half-open pipe, the open end must be a pressure node, because the air inside and outside the pipe is at atmospheric pressure, hence it is a displacement antinode. On the other hand, the air at the closed end cannot go anywhere, so it is a displacement node. The allowable wavelengths are then given by the constraint that $L = \lambda\,(1/4 + n/2)$. This implies that

$$\lambda = \frac{4L}{2n+1}.$$

    For $n = 1$, we have $\lambda = 4 \cdot 0.6/3 = 0.8\text{m}$.

67. C - The field inside the sphere is as if there was a bound charge density of

$$\rho_b = -\nabla \cdot \mathbf{P}.$$

Plugging in the expression for the polarization, we find that

$$\rho_b = -C\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\right) = -4Cr.$$

The electric field an easily be solved using Gauss's law

$$E(r) = \frac{1}{4\pi r^2 \epsilon_0}\int_0^r -4Cr' 4\pi r'^2 dr' = \frac{1}{r^2\epsilon_0}\int_0^r -4Cr'^3 dr' = \frac{-Cr^2}{\epsilon_0}.$$

68. B - Remember that perfect conductors give rise to a 180° phase shift for the reflected wave, which means the direction of $\mathbf{E}$ is reversed: $\mathbf{E} \propto -\hat{x}$. We also know that $\mathbf{B} \propto \mathbf{k} \times \mathbf{E}$, where $\mathbf{k}$ is the wavevector; since $\mathbf{k} = -\hat{z}$ for the reflected wave, we must have $\mathbf{B} \propto +\hat{y}$.

69. B - This just involves repeated application of the angular momentum identity

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k.$$

Proceeding step-by-step, we have

$$\begin{aligned} [[[L_x, L_y], L_x], L_x] &= [[i\hbar L_z, L_x], L_x] \\ &= \left[(i\hbar)^2 L_y, L_x\right] \\ &= -(i\hbar)^3 L_z \\ &= i\hbar^3 L_z \end{aligned}$$

70. C - The vibrational energies of diatomic molecules are approximately those of a harmonic oscillator, so we solve for T in $kT \simeq \hbar\omega = hf$. To do this quickly, it helps to take advantage of the fact that h is given in the formula sheet in units of eV, and use the mnemonic that kT at room temperature (300 K) is about 1/40 of an eV. We get about 2400 K, which is closest to choice C. This problem illustrates a fact worth remembering – the vibrational degrees of freedom of light diatomic molecules are "frozen out" at room temperature, and are only unfrozen at temperatures an order of magnitude larger.

71. B - Observables must be Hermitian. The reason for the factor of i in the momentum operator $-i\hbar\nabla$ is precisely to make this single-derivative operator Hermitian – otherwise, we'd pick up an extraneous minus sign during integration by parts. So any operator involving only one derivative which does *not* have a factor of i can't be Hermitian.

72. B - Choices D and E are obviously incorrect. A is incorrect because the chief virtue of the BCS theory is that it *does* give the correct microscopic description of many superconductors. C is interesting, but B is a better answer because it not only escapes the violation of Pauli exclusion but also gives us a mechanism for superconductivity (bosons can occupy the same state, which produces superconductivity).

73. E - Analyze this system in a frame which rotates with the hoop, and consider only the equation of motion for the tangential component, since the bead is constrained to move on the hoop. There are no Coriolis or centrifugal force terms, so Newton's 2nd law just reads

$$F = -kR\omega = m\dot{\omega},$$

where ω here means the angular velocity *relative to the hoop.* The centrifugal force is irrelevant because the bead is constrained to move on the hoop. The Coriolis force is irrelevant because the motion is constrained to be in the same direction as the rotation of the hoop. This has solution $\omega(t) = Ce^{-kRt/m}$. Moving back to the stationary reference frame with the replacement $\omega \to \omega + \Omega$, we have $\omega(t) = \Omega + Ce^{-kRt/m}$. Imposing the initial condition $\omega(0) = \omega_0$ gives $C = \omega_0 - \Omega$, so

$$\omega(t) = \Omega - (\Omega - \omega_0)e^{-kRt/m},$$

which is choice E.

74. E - At the top of the hill, the cylinder has only potential energy. At the bottom, it will have purely kinetic energy, which is composed of both translational and rotational kinetic energy. So we first calculate the moment of inertia of the cylinder:

$$\begin{aligned} I &= \int r^2 dm \\ &= \int_0^R \rho(r) r^2 (2\pi r\, dr) \\ &= 2\pi \int_0^R A r^4 dr \\ &= \frac{2\pi A}{5} R^5 \end{aligned}$$

We can express this in terms of the mass of the cylinder,

$$M = \int dm = \int_0^R \rho(r)(2\pi r\, dr) = 2\pi \int_0^R A r^2 = \frac{2\pi A}{3} R^3,$$

so $A = \frac{3M}{2\pi R^3}$ and $I = \frac{3}{5} M R^2$.

Since the cylinder rolls without slipping, we have $v = R\omega$, so the total kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{3}{5}MR^2\right)\left(\frac{v}{R}\right)^2 \\ &= \frac{4}{5}Mv^2. \end{aligned}$$

Equating this to the total potential energy, Mgh, we have

$$Mgh = \frac{4}{5}Mv^2 \implies v = \sqrt{5gh}/2,$$

which is choice E.

75. D - The spin-3/2 states are the states with the highest total spin that can be formed from 3 spin-1/2 particles. The maximal-spin states are always totally symmetric under exchange of the particles. D is the only choice that is totally symmetric. We could also derive this quickly by starting $|\Psi\rangle$ and applying the lowering operator twice; or even more quickly, by starting with the $m = -3/2$ state $|\downarrow\downarrow\downarrow\rangle$ and applying the *raising* operator *once*.

76. A - A p-type semiconductor has an excess of positive charge carriers or holes, which are empty states in the valence band that electrons can fill. An n-type semiconductor has an excess of electrons. When p- and n-type materials are brought together, the electrons from the n-type material diffuse into the p-type material. This leaves a slight negative charge on the edge of the p-type material and a slight positive charge on the edge of the n-type material. The electric field thus points from n-type to p-type material.

77. C - The two reference frames S and S' must be related by a Lorentz transformation, so with the $(+,-,-,-)$ metric signature, the invariant interval of the position four-vector in both reference frames must be equal. The separation of E_1 and E_2 in the original frame is (2,1,1,0), which has invariant interval 2. Checking each choice, the invariant interval for A is -1.25, B is 3, C 2, and D is 1. Choice C is the same as in the original frame, and so is the correct answer.

78. C - With these answer choices, one can almost get by using pure dimensional analysis: the answer must have units of $(\text{time})^{-1}$, and must be non-negative. This leaves only B, C and E, and E seems rather unreasonable. More formally, assuming that dark matter detection follows a Poisson process with mean rate λ, the probability of seeing zero events is $e^{-\lambda T}$. The 90% confidence level upper limit is the mean rate such that 90% of the time we would see in our experiment a number of events that is inconsistent with our measurement of zero events. Practically, this is means that we want to find

the mean rate that would produce more than 0 events 90% of the time. In other words, we want to find the rate that gives 0 events only 10% of the time. This is just $0.1 = \exp(-\lambda)$, or $\lambda = -(1/T)\log 0.1$.

79. C - Remember that A and B go by the more familiar name "fine structure," while the third is "hyperfine structure." The name itself suggests C is the smallest. Indeed, fine-structure perturbations are smaller than the Bohr energies by a factor $\alpha^2 \approx (1/137)^2$, whereas hyperfine structure are smaller still by a factor $m_e/m_p \approx 5 \times 10^{-4}$. Choice D turns out to be a factor of α smaller than fine-structure, so it's still an order of magnitude larger than hyperfine structure; unfortunately, it takes quantum field theory to calculate this carefully, so better just to memorize it. For choice E, we need to be a little careful. The reduced mass is

$$\mu = \frac{m_e m_p}{m_e + m_p} = m_e(1 + m_e/m_p)^{-1} \approx m_e(1 - 5 \times 10^{-4}).$$

Since the Bohr energies are proportional to m_e, this gives a correction of 5×10^{-4}, which is about the same size as α^2, and hence still bigger than C.

80. D - Choices A and E are obvious incorrect. B seems initially plausible, but this process would happen just as often with tau neutrinos, so it far from clear that it would make up the discrepancy. C also seems vaguely plausible, but it does not give an obvious mechanism for production of tau neutrinos. D is the correct answer, with the difference between neutrino mass and flavor eigenstates being the basis of the famous neutrino oscillations. This is the same oscillation effect that takes place in 2- and 3-state quantum systems.

81. E - By rescaling energies, the partition function can be written as

$$Z = e^{\epsilon/2kT} + e^{-\epsilon/2kT} + 2.$$

The probability of being the triplet state is just

$$P = \frac{e^{\epsilon/2kT} + e^{-\epsilon/2kT} + 1}{e^{\epsilon/2kT} + e^{-\epsilon/2kT} + 2}.$$

82. A - Recalling that $H = T + U$, we can solve this problem by a careful consideration of signs in the answer choices. The spring potential energy $U_s = \frac{1}{2}kx^2$, where x is some relative displacement, is always non-negative, so we have only choices A and B. With coordinates y_1 and y_2 as defined in the problem, positive displacements correspond to downward motion, so gravitational potential energy is actually *negative.* Thus we are left with choice A.

83. B - The expectation value of energy is

$$E = \int_0^a \psi^* H \psi.$$

After a sudden expansion, ψ stays constant. H also stays constant on the interval $[0, a]$, but changes from ∞ to 0 on the interval $[0, 2a]$. The expectation value of energy after the expansion is therefore

$$\begin{aligned} E' &= \int_0^{2a} \psi^* H' \psi \\ &= \int_0^a \psi^* H \psi + \int_a^{2a} \psi^* H' \psi \\ &= \int_0^a \psi^* H \psi + 0 \\ &= E. \end{aligned}$$

By the way, this is an application of conservation of energy within the formalism of quantum mechanics - just as the temperature of an ideal gas remains constant during free expansion, the energy of a quantum system remains constant after a sudden change of potential.

84. D - This looks long and complicated, but it's really just a matter of limiting cases. A and B are eliminated by dimensional analysis, since the reflection coefficient must be dimensionless. As $\alpha \to 0$, the coefficient of reflection must go to zero, since in that case the barrier disappears and the particle will continue to propagate to $x > 0$ with probability 1: this leaves only D.

85. C - The transition $2s \to 1s$ has $\Delta m = 0$ so it does not violate the dipole selection rule.

86. E - The electric dipole moment is a vector quantity, which changes sign under parity transformations, so a nonzero electric dipole moment violates parity. Interestingly, it *also* violates time-reversal invariance. To see this, recall that the neutron *does* have a nonzero magnetic dipole moment. Suppose the magnetic and electric dipole moments were parallel; then under time-reversal, the magnetic one would change sign but the electric one would remain the same, and the system would not be symmetric under time-reversal. So the *relative* orientations of the magnetic and electric dipole moments lead to a violation of time-reversal invariance.

87. C - The sharp drop of curve b is the signature of a process with an energy threshold around 1 MeV. Recalling that the electron mass is about 0.5 MeV, this must be the threshold for pair production.

88. D - The first coefficient in the Fourier sine series is given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \; dx = \frac{1}{\pi}(-\pi \cos(\pi) - \pi \cos(-\pi)) = 2.$$

89. E - We are looking for an $\mathbf{A}$ satisfying $\nabla \times \mathbf{A} = \mathbf{B}$. It's simplest to consider the components of $\mathbf{B}$ one by one. Since $B_x = (\nabla \times \mathbf{A})_x = \partial A_z / \partial y - \partial A_y / \partial z$, with an

eye on the answer choices we see that we can only satisfy this by taking $A_z = B_0 y$, since A_y is independent of z in all answer choices. This leaves only C and E. To get $B_z = 2xB_0$, we must have $2xB_0 = (\nabla \times \mathbf{A})_z = \partial A_y/\partial x - \partial A_x/\partial y$. Choice E has the correct sign, and we can check that it also satisfies $(\nabla \times \mathbf{A})_y = 0$.

90. D - The change in energy can be obtained from the change in the electric field energy density outside the sphere. There is no change in energy density inside the sphere, so we will neglect this contribution. Before the grounding, the field outside the sphere has energy

$$\begin{aligned} E_{before} &= \frac{\epsilon_0}{2}\int E^2 dV \\ &= \frac{\epsilon_0}{2}\frac{1}{16\pi^2\epsilon_0^2}\int \frac{Q^2}{r^4}4\pi r^2 dr \\ &= \frac{1}{8\pi\epsilon_0}\int_a^\infty \frac{Q^2}{r^2}dr \\ &= \frac{Q^2}{8\pi\epsilon_0 a}. \end{aligned}$$

Afterward, charge is induced on the conducting sphere to exactly cancel the electric field everywhere outside of the sphere of radius a, so $E_{after} = 0$. The change is therefore $\Delta E = -\frac{Q^2}{8\pi\epsilon_0 a}$, which is choice D.

91. C - The Lamb shift is due to the polarization of the vacuum due to the electric charge of the proton. Its magnitude depends on the charge radius of the proton, and thus it can be used as a probe of the charge radius. Using muonic hydrogen instead of ordinary hydrogen is an especially sensitive probe because the muon is much heavier than the electron and therefore is more likely to be close to the nucleus where the effect is stronger.

92. B - By momentum conservation, when a free nucleus emits a photon it must also recoil in the opposite direction, which causes the emitted photon to have a different energy (hence frequency) than it would if the nucleus were held stationary. You may have been thrown by choice C, since the Pound-Rebka experiment used a carefully contrived arrangement of absorbers and emitters in a vertical shaft so that the gravitational redshift was dominant. However, the adjective "tabletop" implies that the experiment takes place at (almost) constant gravitational potential, so gravitational redshift barely contributes.

93. E - It is a useful bit of trivia that a sequence of NAND gates can be combined to create any sequence of basic logical gates. (This is also true of NOR gates.) Even without knowing this, though, we can see fairly easily that A and C must be possible: since NAND is AND followed by NOT, to get an AND we just put two NAND gates in sequence and tie the output of the first to both inputs of the second, and to get NOT

we can tie both inputs of a single AND gate together. So if at least two of the answer choices are possible, the answer must be E.

94. E - Applying the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

to the given Lagrangian, we get $A\ddot{q} - (-2Bq) = 0$. Rearranging gives $\ddot{q} = -\frac{2B}{A}q$, choice E.

95. E - The field of a straight solenoid has uniform magnitude $\mu_0 n I$, where n is the number of turns per unit length; bending this solenoid around into a toroid of radius R sets $n = N/2\pi R$. (This is a hand-wavy argument, but it gives the correct answer, and it's an excellent way to remember the formula without having to rederive it from scratch.) The energy stored in the magnetic field is $U = \dfrac{1}{2\mu_0}\displaystyle\int \mathbf{B}^2\,dV$, where the integral is taken over all of space. Here the field is only nonzero inside the volume V, and inside this volume the field is *uniform*, so

$$U = \frac{V}{2\mu_0}|\mathbf{B}|^2 = \frac{V}{2\mu_0}\cdot\left(\frac{\mu_0 N I}{2\pi R}\right)^2 = \frac{\mu_0 N^2 I^2 V}{8\pi^2 R^2},$$

choice E.

96. A - Longitudinally polarized wave propagate in the same direction as the displacement of the wave medium. This means that the polarization vector is forced to be along the direction of propagation, excluding choices I. and II. This narrows the answer to A.

97. C - The phase velocity is

$$v_{ph} = \frac{\omega}{k} = Ak^{-1/2}.$$

The group velocity is

$$v_{gp} = \frac{d\omega}{dk} = \frac{1}{2}Ak^{-1/2}.$$

98. B - The setup $n_1 < n_2 < n_3$ occurs so often that it is probably useful to memorize the result. This is the configuration of an anti-reflective coating, and the 180° phase shift that occurs at *both* boundaries leads to the condition for destructive interference

$$2n_{film}t = \left(m - \frac{1}{2}\right)\lambda$$

where t is the thickness of the film. (Try to derive this formula if you forgot it.) We conclude that $t_{min} = \lambda/4n$, which can be memorized with the mnemonic that tn for an anti-reflective coating is a quarter-wavelength. For the numbers given in this problem, we get $t = 8.33 \times 10^{-8}$ m. So B is correct.

99. D - We first need to normalize the wavefunction. We thus require that

$$\begin{aligned} 1 &= A^2 \int_0^1 (1-x)^2 \, dx \\ 1 &= A^2 \left(1 - 2\frac{1}{2} + \frac{1}{3}\right) \\ A &= \sqrt{3}. \end{aligned}$$

We just have, $\langle x \rangle = \int |\Psi(x)|^2 x \, dx$, so

$$\begin{aligned} \langle x \rangle &= 3 \int_0^1 x(1-x)^2 \, dx \\ &= 3\left(\frac{1}{12}\right) \\ &= \frac{1}{4} \end{aligned}$$

100. A - A rigid rod has only rotational degrees of freedom, so its energy is determined by its rotational quantum numbers. The classical formula is $T = L^2/2I$ for the kinetic energy of a rotating body, where I is the moment of inertia about the center of mass, so in the quantum mechanics setting we get $E = \hbar^2 n(n+1)/2I$. The center of mass of this rod is at the center of the rod, so the moment of inertia is $I = 2 \cdot m \left(\frac{a}{2}\right)^2 = \frac{1}{2}ma^2$. Thus

$$E = \frac{\hbar^2 n(n+1)}{ma^2},$$

choice A.

Note that we could also have done the last step of the problem by calculating the reduced mass of the system, $\mu = \dfrac{m \cdot m}{m+m} = \dfrac{m}{2}$, and using the formula $I = \mu r^2$ for a single particle of mass μ.