Lab 3

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This lab will review the method of finding the pdf of a transformed random variable, and will generate samples from the transformed distribution.

Problem 1

Let the pdf $f(x)$ of a random variable X be given by $f(x) = \exp(-x)$ where $x > 0$. Furthermore, let $Y = \sqrt{2 * X}$. We are interested in finding the pdf of Y, f_Y, and then sampling from f_Y.

a) What distribution is this?
   It's clear that $X \sim \exp(\lambda = 1)$.

b) Find functions g and h such that $y = h(x)$ and $x = g(y)$. Is h monotone increasing?
   $y = h(x) = \sqrt{2x}$; h is monotone increasing.
   $x = g(y) = \frac{1}{2}y^2$.

c) Find the density of Y using the method of transformations.
   Recall the formula for finding the density f_Y given f_X and g:

$$f_Y(y) = f_X(g(y))g'(y). \tag{1}$$

Since $g'(y) = y$ and $f_X(g(y)) = \exp(-g(y)) = \exp(-\frac{1}{2}y^2)$ we see that

$$f_Y(y) = y\exp(-\frac{1}{2}y^2), \quad y > 0.$$

d) Find the cdf of Y
   $F_Y(t) = \int_0^t y\exp(-\frac{1}{2}y^2)dy = -\exp(-\frac{1}{2}y^2)|_0^t = 1 - \exp(-\frac{1}{2}t^2)$.

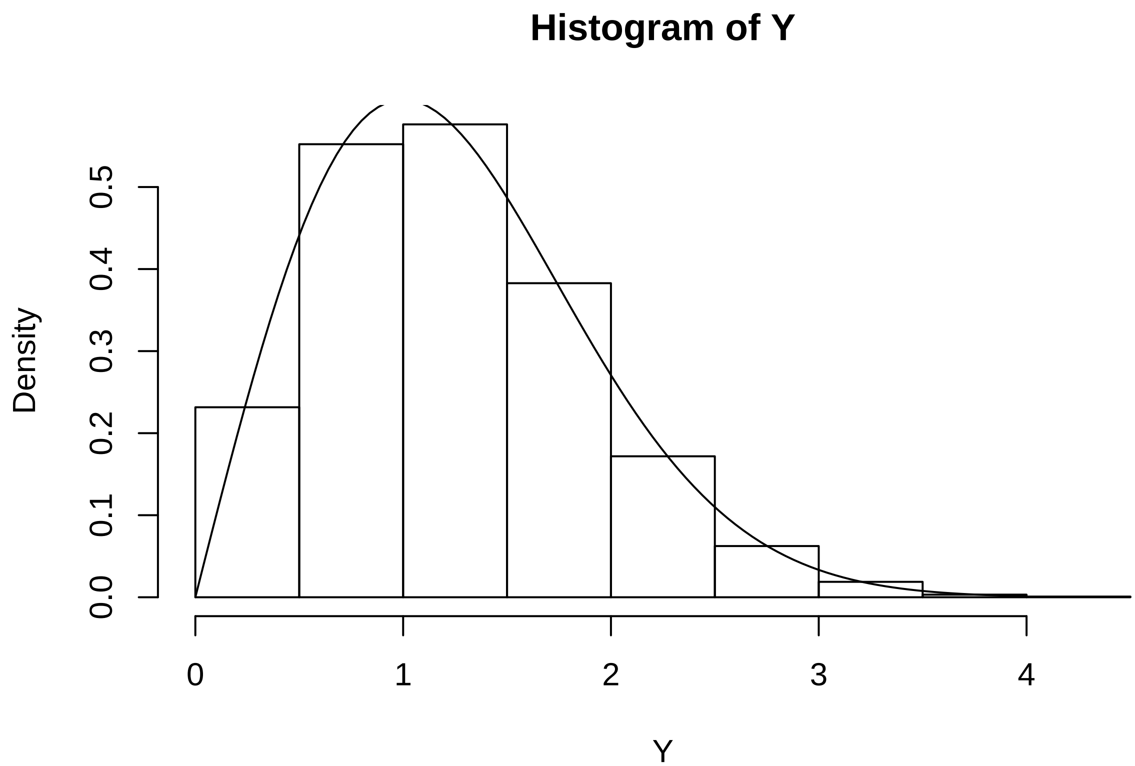
e) Find the inverse cdf of Y
   $F_Y^{-1}(t) = \sqrt{-2 * \log(t)}$.

f) Generate samples from the pdf of Y using the inversion method.
   All of the hard work has been done for us. We simply need to generate uniform(0,1) random variables and plug them into the inverse cdf of Y:

```
pdf_y <- function(y){y * exp(-.5 * y^2)}
my_inverse_cdf <- function(x){sqrt(-2*log(x))}

## Generate from fY
U <- runif(10000)
Y <- my_inverse_cdf(U)

## Plot our sample and overlay the true density
hist(Y, prob = TRUE)
curve(pdf_y, 0, 4, main = "PDF of Y", add = TRUE)
```



Problem 2

Let the pdf $f(x)$ of a random variable X be given by $f(x) = k * x^4$ where $0 \leq x \leq 1$. Furthermore, let $Y = \log(x)$. Note that the domain of Y is $(-\infty, 0)$.

a) What distribution is this? What is k?
This is a beta distribution with parameters $\alpha = 5$ and $\beta = 1$. The normalizing constant $k = 5$.

b) Find functions g and h such that $y = h(x)$ and $x = g(y)$. Is h monotone increasing?
$y = h(x) = \log x$; h is monotone increasing.
$x = g(y) = \exp(y)$.

c) Find the density of Y using the method of transformations.
From equation 1, we get

$$f_Y(y) = f_X(\exp(y))g'(y) = 5\exp(4y)\exp(y) = 5\exp(5y).$$

d) Find the cdf of Y.
$F_Y(t) = \int_{-\infty}^{t} 5\exp(5y)dy = \exp(5y)|_{-\infty}^{t} = \exp(5t)$.

e) Find the inverse cdf of Y.
$F_Y^{-1}(t) = \frac{1}{5}\log(t)$.

f) Generate samples from the pdf of Y using the inversion method.

```r
pdf_y <- function(y){5 * exp(5 * y)}
my_inverse_cdf <- function(y){0.2 * log(y)}

## Generate from fY
U <- runif(10000)
Y <- my_inverse_cdf(U)

## Plot our sample and overlay the true density
hist(Y, prob = TRUE)
curve(pdf_y, -5, 0, main = "PDF of Y", add = TRUE)
```