Workshop 6 Homework

Los Angeles County
ISAB

October 18, 2017

These exercises are intended to refresh your calculus brain cells for the upcoming statistics topics.

Print this out and take it with you to your meetings. Whenever the meeting becomes a waste of your time, discreetly take these pages out and start filling in the gaps of these derivations. Meeting hand-outs are a great source of scratch paper. Feign interest by asking for extra copies.

The gamma function and associated gamma distribution will be popping up here and there. It will help to become familiar with them.

Normal Distribution

Evaluate

$$R = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx$$

Trick: Polar Coordinates

Instead of computing R directly, compute R^2 as a double integral and convert to polar coordinates. The symmetry about the origin simplifies things.

$$
\begin{aligned}
R^2 &= R \cdot R \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dxdy
\end{aligned}
$$

The exponential in this integral is begging for polar coordinates.

$$
\begin{array}{rl}
x = \cos\theta & \quad y = \sin\theta \\
r^2 = x^2 + y^2 & \quad dxdy = J(r,\theta) dr d\theta
\end{array}
$$

$J(r,\theta)$ is the Jacobian of the transformation from (x,y) to (r,θ) coordinates. It accounts for how differential $dx \times dy$ area stretches to $dr \times d\theta$ area under the change of variables.

Problem 1 Verify the value of the Jacobian.

$$
J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r
$$

Substitute these back into the double integral.

$$
R^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr d\theta
$$

Problem 2 Compute this integral to determine the normalization constant of the standard normal distribution.

When you find the value of this integral, don't forget we're interested in R, not R^2.

Gamma Distribution

The Gamma function is defined as

$$
\Gamma(r) = \int_0^{\infty} x^{r-1} e^{-x} dx
$$

Problem 3 Calculate $\Gamma(1)$.

Problem 4 Calculate $\Gamma(2)$ using integration by parts.

Problem 5 Show for positive integers $n > 1$ that $\Gamma(n) = (n-1)\Gamma(n-1)$ using integration by parts.

The formula in Problem 5 means that for positive integers, $\Gamma(n) = (n-1)!$. But since there is no restriction on r to be an integer, this makes the gamma function a continuous version of the factorial function. Most non-integer values have to be computed numerically. But there is one more special value that we can compute directly.

Problem 6 Compute $\Gamma\left(\frac{1}{2}\right)$. Use the substitution

$$x = \frac{1}{2}u^2$$

and your result from Problem 2.

The Gamma distribution has the following density function.

$$\text{Gamma}(x; r, \lambda) = \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}$$

for $x \in [0, \infty)$.

Problem 7 Show that the integral of the density function is 1.

$$1 = \int_0^\infty \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}dx$$

Problem 8 Show that when $r = 1$, you get the exponential distribution.

Problem 9 Calculate the moment generating function for the Gamma distribution.

$$m_X(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx}\frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}dx$$

Hint: use substituion $u = (t - \lambda)x$.