Random Variable Functions

Los Angeles County
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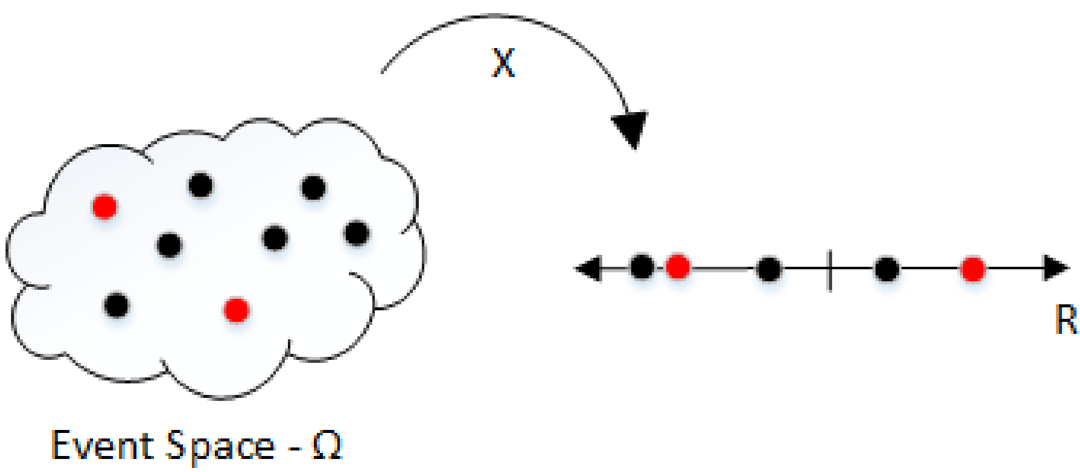
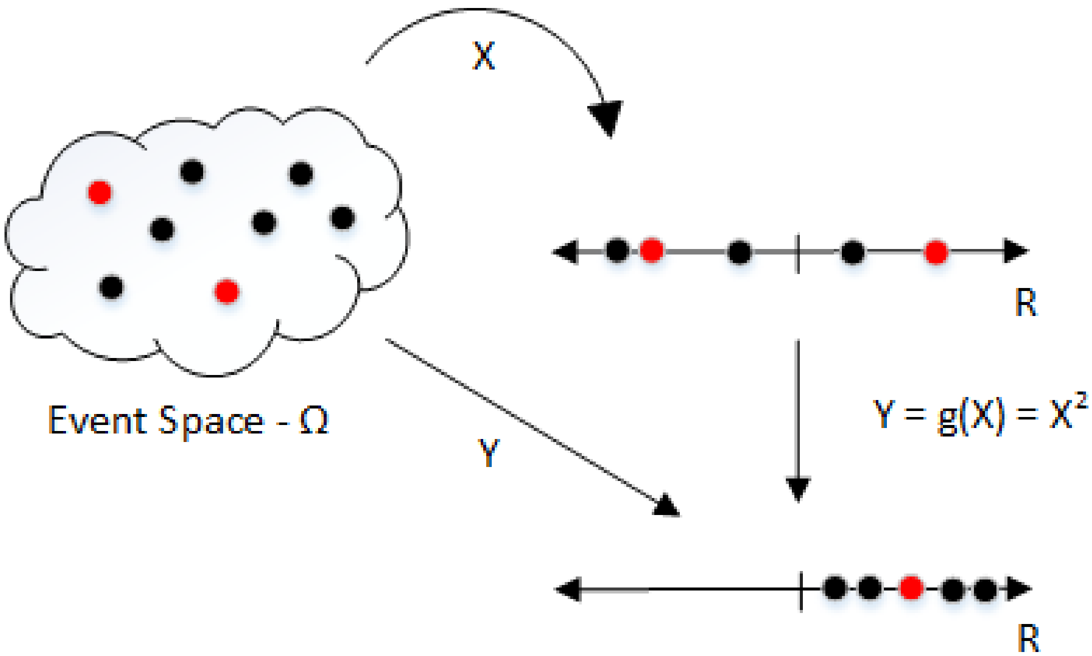
We've been discussing random variables, their CDF and PDF, and their moment generating functions. We've done a few exercises invovled with calculating their means and variance. The random variables we've discussed so far directly modeled certain phenomena, such as a coin flip, the number of heads in multiple coin flips, and selecting a number randomly from a set of numbers. This is all very nice; but it's time to step up our game and consider random variables that arise as a function of other random variables.

Let's remind ourselves of what a random variable is. A random variable maps from an abstract event space to points on the real number line. Often the event space won't be so abstract and the mapping will be quite natural; so natural that we often just write X rather than $X(\omega) \in \mathcal{R}$ for $\omega \in \Omega$.



By mapping from "event spaces" to the concrete real number line, random variables allow us to directly employ analytic techniques that have long existed for the real number line. Our cumulative distribution functions and probability density functions are defined on the real number line by virtue of the random variable mapping events to the real number line.

A function of a random variable simply maps the points from one real number line to another real number line. In this way, a function of a random variable creates a new random variable on the same event space, i.e. a new way to map events to real numbers.



In the figure above, $g(x) = x^2$ is such a function. If we define Y as the result of mapping $\Omega \to \mathcal{R} \to \mathcal{R}$, then Y is also a random variable.

Assuming that such a mapping is useful (and we'll find out later that this $g(X) = X^2$ is), we'll want to know the CDF and PDF of the new random variable. Unfortunately, it's not as simply as plugging x^2 everywhere you see an x.

Using the figure above as an example, let's determine $F_Y(y)$ from basic principles. The basic principle is

$$F_Y(y) = P[Y \le y]$$

From this principle we plug in the function to derive the expression. Since $Y = g(X) = X^2$, we restrict our attention to $y > 0$ since $F_Y(y) = 0$ for $y \le 0$.

$$\begin{aligned} F_Y(y) &= P[Y \le y] \\ &= P[X^2 \le y] \end{aligned}$$

$$
\begin{aligned}
&= P\left[-\sqrt{y} \le X \le \sqrt{y}\right] \\
&= P\left[X \le \sqrt{y}\right] - P\left[X \le -\sqrt{y}\right]
\end{aligned}
$$

This gives us the CDF for Y. It's close to simply substituting $\sqrt{y}$ for x into the expression for F_X. Now that we have the CDF, we can derive the PDF.

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy}\left[F_x(\sqrt{y}) - F_X(-\sqrt{y})\right] \\
&= \left.\frac{dF_X}{dx}\right|_{x=\sqrt{y}} \frac{d(\sqrt{y})}{dy} - \left.\frac{dF_X}{dx}\right|_{x=-\sqrt{y}} \frac{d(-\sqrt{y})}{dy} \\
&= \frac{1}{2\sqrt{y}}\left[f_x(\sqrt{y}) + f_x(-\sqrt{y})\right]
\end{aligned}
$$

where

$$
\frac{dx}{dy} = \frac{dy^{\frac{1}{2}}}{dy} = \frac{1}{2} y^{-\frac{1}{2}}
$$

The expression for f_Y in terms of f_X in this specific case of $Y = X^2$ demonstrates two important complications.

1. If g is not 1-to-1, then we must account for all the points in $g^{-1}(y)$. In the case of a quadratic, there are usually two.

2. The f_Y expression introduced an additional multiplicative factor, known in calculus circles as *the Jacobian.* The Jacobian incorporates the stretch factor introduced by the transformation.

It's reasonable to ask why the CDF didn't need a stretch factor. It's because we don't integrate the CDF. We just evaluate it at various points. **The value of the CDF at any point represents a probability.**

This is not the case with a continuous density function. The value of f_X at point does not really mean anything **by itself. It certainly does not represent a probability.** We only get probabilities from f_X by integrating it over some interval or combination of intervals. In other words, it's only the area under f_X that has a probability interpretation. You can't say anything about an area with just the height. You need to also specify the width. So in the case of our transformation g, it's not enough to determine the height of f_Y at a certain point through corresponding values of f_X. We need to know how g is stretching the differential widths to get the full picture of how areas under f_X transform to *areas* under f_Y. The Jacobian factor provides this "width stretching" information.

We're still messing around in the realm of probability. In the next workshop, we'll consider some special functions of a random variable (like the average of a sample) and cross into the realm of *statistics* proper.