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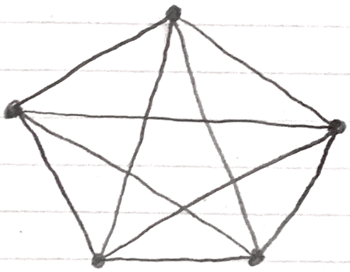
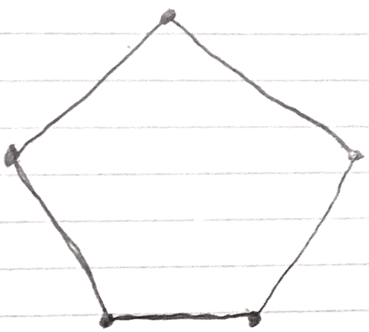
1) In a K_n graph, where n is the number of vertices, there are $\frac{n(n-1)}{2}$ edges



This K_5 has 10 edges

$\frac{n(n-1)}{2} \rightarrow \frac{5(5-1)}{2} = \frac{20}{2} = 10$ ✓

2)



This graph G has 5 vertices, contains a cycle, a diameter of 2 (the greatest distance between any two nodes is 2), and a girth (shortest cycle) of 5.

for $G \rightarrow 2\,\mathrm{diam}(G) + 1 = 2 \cdot 2 + 1 = 5 = \text{Girth}$

3) For a graph G, prove that $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$

The radius of a graph is the minimum of all the maximum distances between a vertex and all other vertices. The diameter of a graph is the maximum distance between any pair of vertices. So, clearly $\mathrm{rad}(G) \leq \mathrm{diam}(G)$.

Now, let u and v be vertices on G such that $d(u,v) = \mathrm{diam}(G)$, let c be a central vertex, so no vertex has a distance greater than $\mathrm{rad}(G)$ from c. Thus, $d(c,v) \leq \mathrm{rad}(G)$ and $d(c,u) \leq \mathrm{rad}(G)$. Therefore, $d(c,u) + d(c,v) \leq 2\,\mathrm{rad}(G)$ and by the triangle inequality $d(u,v) \leq d(c,u) + d(c,v)$ proving $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$

4) a) Since a d-dimensional hypercube has the set of vertices $\{0,1\}^d$, for any vertex there exists d edges connected to that vertex. Thus, this is true for all vertices and the average degree is d.

b) Each vertex is connected to d edges. There exist 2^d vertices, however $d \cdot 2^d$ counts edges twice, so the number of edges $= 2^d \cdot 2^{-1} \cdot d = 2^{d-1} \cdot d$.

c) A length 3 cycle is impossible as it would contain vertices V_1, V_2, and V_3, two of which must differ in more than one position. Thus, the girth is 4 for a $d \geq 2$ and infinity otherwise.

d) To show that the circumference is 2^d, we will prove the hypercube is hamiltonian by induction.
Base case: $d=2$, the length 4 cycle follows 00, 01, 11, 10.

Consider the d-1 dimensional hypercube. $V \in \{0,1\}^{d-1}$ is a vertex adjacent to a sequence of d-1 zeros in the hamiltonian cycle. Existence of hamiltonian cycle: we have a path that travels from the sequence of d zeros, to the sequence of all zeros except for in the V position, passing through all vertices containing a 1 in only one position.

Then we add an edge from the sequence containing all zeros except the v position to the sequence containing all 1s except the v position. Then we connect that sequence to the sequence containing a 1 in the first position and zeros in the other d-1. Finally we create an edge between that sequence and the all zeros sequence. This completes our hamiltonian cycle.

5. Show that if a grah G is connected, has a diameter k and a minimum degree d, then it has at least $\frac{kd+k+d+3}{3}$ vertices.

First, we will find two vertices on G, x and Z, such that the distance $d_G(x,Z)$ is k (diam(G)).
Now, let y be a vertex not on the shortest path P from x to z, but adjacent to a vertex on P.
Let there exist a vertex i on P which is the closest vertex to x that is adjacent to y. Skipping the next two vertices on path P, j and k, the following vertex l cannot be adjacent to y. Otherwise, there would necesarily exist a path from x to Z with length less than k $(x, \ldots, i, y, l, \ldots, Z)$. This is a contradiction.
So it must be true that any vertex adjacent to P (but not on P) can at most be adjacent to 3 vertices on P.

There are $2(d-1)+(k-1)(d-2) = kd-2k+d$ edges leaving P, as The start x and end z both have $d-1$ and all the other vertices $v_{\text{on } P}$ have $d-2$.

The path P must neighbor at least $\frac{kd-2k+d}{3}$ vertices since a vertex not on P can be adjacent to at most 3 vertices on P.

This results in a total number of at least $\frac{kd-2k+d}{3} + k+1 = \frac{kd-2k+3k+d+3}{3}$

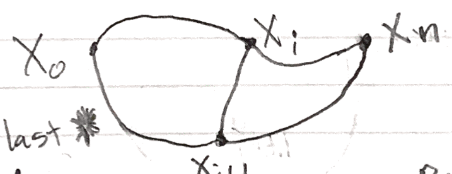
$= \frac{kd+k+d+3}{3}$ vertices

6) Show that if a graph G is connected, it contains either a path or a cycle of length at least $\min(2\delta(G), |G|)$

Consider the path P of maximum length (l) in G where $P = (x_0, x_1, \dots x_n)$. If $l \geq |G|$ the condition is met.

If $l < |G|$, then we will define a set of vertices $S = V(G) \setminus V(P)$. $S \neq \emptyset$ and there exists a $V(P)-S$ path $P' = (x'_0, x'_1 \dots x'_n)$.

We know that, if there exists a neighbor to x_0 or x_n, it must be on P. If (x_0, x_{i+1}) and (x_n, x_i) are edges on P, then there exists a cycle $C = (x_0, \dots, x_i, x_n, \dots x_{i+1}, x_0)$, (x_i, x_{i+1}) is an edge



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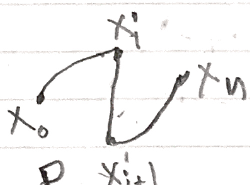
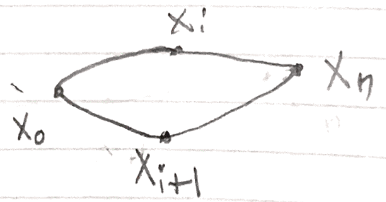


Path P



Cycle C

If $l < 2\delta(G)$:

The vertex x_0 has at least $\delta(G)-1$ neighbors in set $\{x_2, \dots x_{n-1}\}$. We know that any vertex x_i neighboring x_0 has a corresponding x_{i-1} not adjacent to x_n. x_n must also have at least $\delta(G)-1$ neighbors in $\{x_2, \dots, x_{n-1}\}$, and x_n cannot neighbor $\delta(G)-1$ vertices. However, we assumed $n < 2\delta(G)$ so there are fewer than $2\delta(G)-2$ possible neighbors. Thus, it is not possible for a neighbor of x_n not to neighbor a neighbor of x_0 (see graph*) Therefore, there must exist a cycle.

Finally, we can delete an edge on cycle C connected to X_0', which connects $V(P)$ to S. This creates a path extending the current path with P'. This path is longer than P, resulting in a contradiction.