Lecture 7 (Graph Theory)

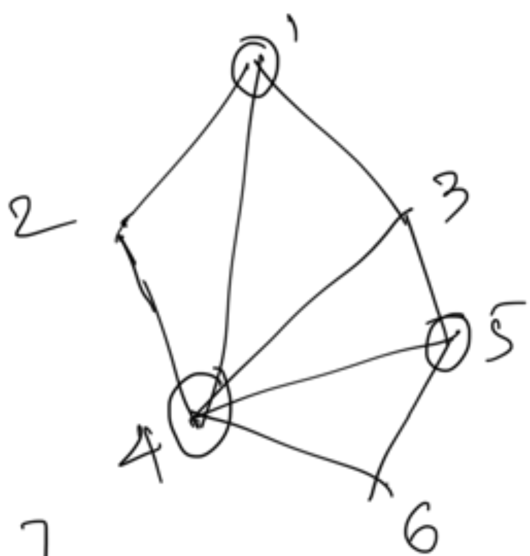
Matchings & Bipartite graphs.

König's theorem for bipartite graphs

$G = (V, E)$

Let us call a subset $\mathcal{U} \subseteq V$ a vertex cover of E (or of G) if every edge in E is incident to a vertex in $\mathcal{U}$.

G:



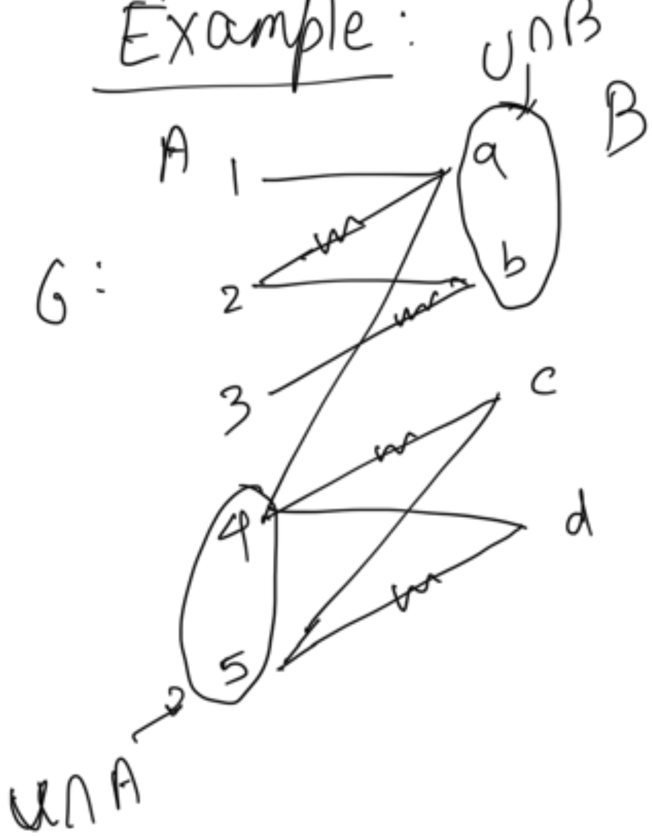
o: A vertex cover.

[Duality]

König's theorem: The maximum

cardinality of a matching in a bipartite graph G = minimum cardinality of a vertex cover of the edges of G.

Example:



maximum cardinality of matching $= 4$

$\sim$: Maximum matching

$\bigcirc$: vertex cover.

Proof of Kőnig's theorem:

Let $G = (V, E)$ be a bipartite graph. ($V = A \cup B$)

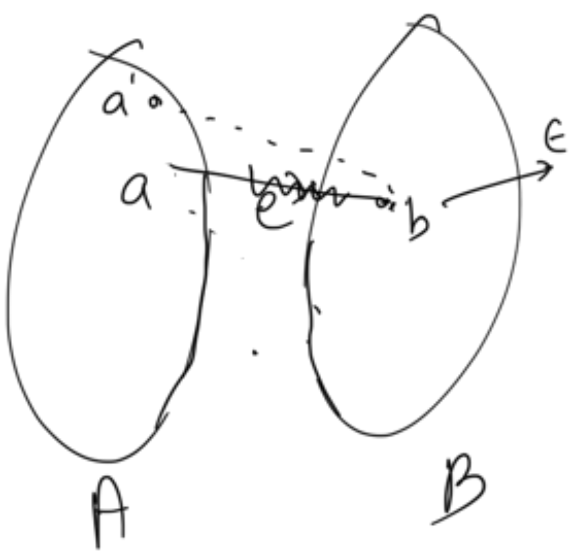
Let M be a maximum matching in G.

Goal: Construct a vertex cover $\mathcal{U} \subseteq V$ with $|\mathcal{U}| = |M|$. [& show that $\mathcal{U}$ is a minimum vertex cover].

Construction:

For every edge $\overset{(a,b)}{e} \in M$, choose one of its ends & put it in $\mathcal{U}$ as follows:

Case 1: Choose the end b of e in B & put it $\mathcal{U}$ if some alternating path ends at b.



$m: M$

Case 2: Choose the end a of e in A otherwise & put in $\mathcal{U}$.

Claim 1: The set $\mathcal{U}$ constructed as above covers the set E of the edges of G.

$\Longrightarrow$ theorem (why?)

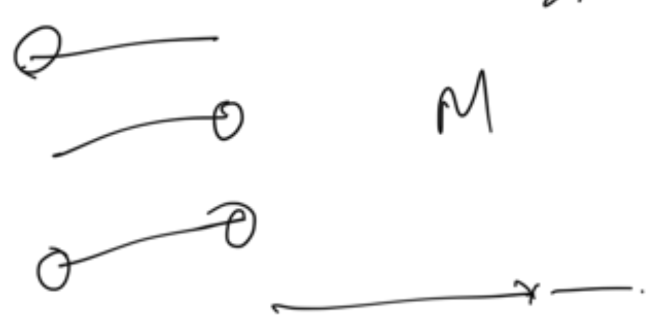
By construction: $|\mathcal{U}| = |M|$

If W is any vertex cover of E, then

$$|W| \geq |M| = |\mathcal{U}|. \quad \therefore \ \mathcal{U} \text{ has minimum cardinality.}$$

(Because W also covers $M \subseteq E$ & W must contain at least one vertex of every edges in M)

o: W :



Proof of ~~the~~ claim 1:

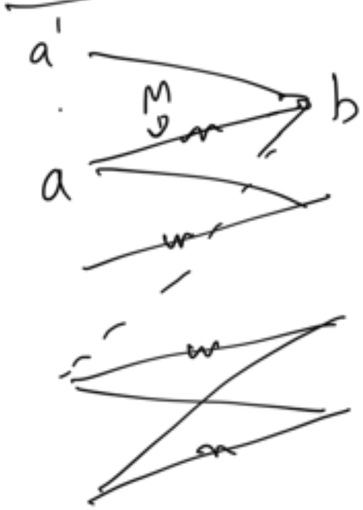
we will show:

step (1): If an alternating path P in G

Step (1) ... ends in a vertex $b \in B$, ... M)
then $b \in U$.

why?

<u>Example</u>:



$P = (a', b)$ (alternating) ends in b.

Proof of step (1):

Since M is a maximum matching, P is <u>not</u> an augmenting path.

(Otherwise $M' := M \oplus P$ will be a matching with $|M'| = |M| + 1 > |M|$, contradicting maximality of M.)

$\therefore$ b is not <u>unmatched</u>

$\therefore$ b is matched to some $a \in A$.

$\therefore$ b is put in U when we consider $(a, b) \in M$. Q.E.D. Step (1)

[Step(1)].

Step(2): $\mathcal{U}$ covers E:

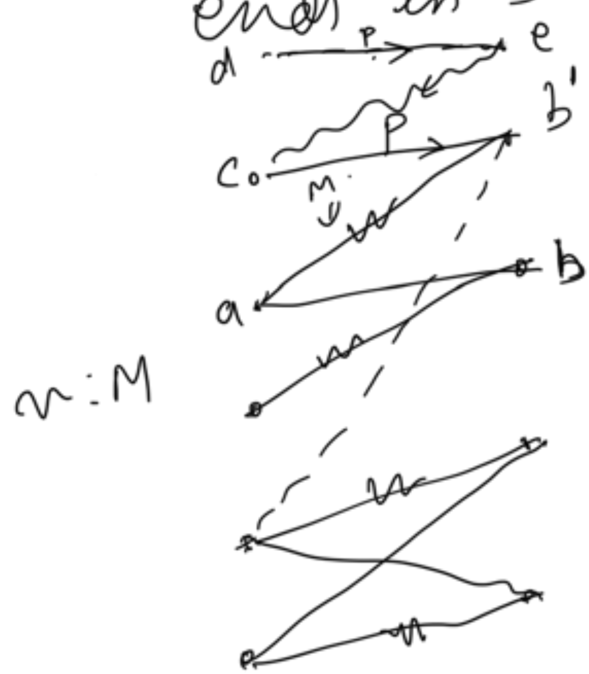
Claim // Consider any edge $(a,b) \in E$. We want to show that (a,b) is covered.

If $a \in \mathcal{U}$, we are done,

$\therefore$ assume that $a \notin \mathcal{U}$.

Then we want to show that $b \in \mathcal{U}$.

((By the step (1), for this it suffices to show that some alternating path ends in b.



Case 1: If a is unmatched: then (a,b) is an alternating path which ends in b.

Case 2: If not, (i.e. a is not unmatched), then $(a,b') \in M$ for some $b' \in B$

Since $a \notin \mathcal{U}$. (by our assumption)
there exists an alternating path
P ending in b' [by our construction of $\mathcal{U}$]

If $b \in P$ the
$\underline{\underline{Pb}}$ [the part of P upto $\underline{b}$]
is an alternating path ending
in b. (as we wanted to show]

If $b \notin P$ then.
$Pb'ab$ is an alternating
path ending in b.

$\therefore$ in either case, some alternating
path ends in b.

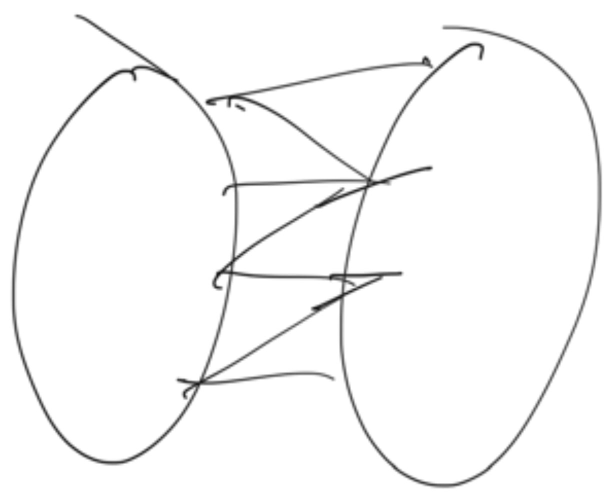
$\therefore$ $b \in \mathcal{U}$.

Q.E.D.

[König's thm].

Hall's theorem

$G = (V, E)$ be a bipartite graph.
$V = A \dot\cup B$



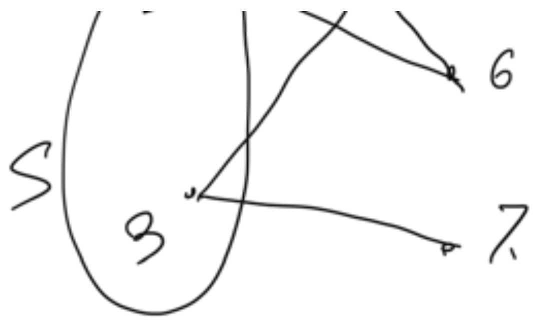
A B

Given any $S \subseteq A$, let $N(S) \subseteq B$ be the set of neighbours of S in B

Example:

B
A

$N(\{2,3\})$
$= \{4, 5, 6, 7\}$



$N(\{1,2\}) = \{4,5,6\}$

If M is a matching of A in a bipartite graph $G=(\underset{A\cup B}{U}, E)$,

then given any $S \subseteq A$, what is the reln between

Hall's marriage condition: $|N(S)| \geq |S|$.

Necessary condition for M to be a matching of A in G.

If $G=(\underset{A\cup B}{V}, E)$ has a matching, then $\underbrace{|N(S)| \geq |S|}_{\text{Hall's marriage condition.}}$ $\forall\, S \subseteq A$.

Trivial

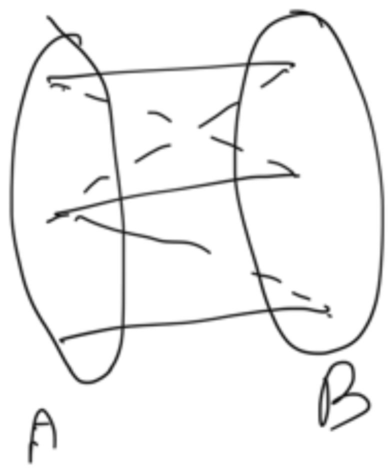
Thm [Hall]: A bipartite graph $G=(\underset{A\cup B}{V}, E)$ contains a matching of A iff $|N(S)| \geq |S|$ for all $S \subseteq A$.

Proof: $\longrightarrow$ (trivial)

$\longleftarrow$ (non-trivial) — In the next class.

<u>Examples</u>:

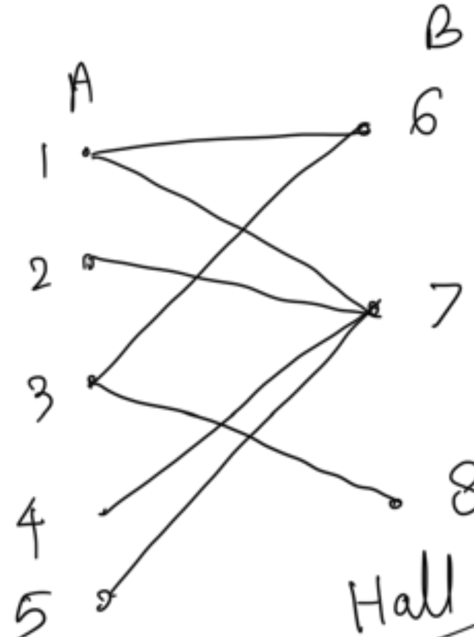
(1)



— : M.

(2)

G



Does G contain a matching of A?

No [Because $|A| > |B|$
$\quad\quad\quad\quad\quad\quad\quad\; \| \quad\; \|$
$\quad\quad\quad\quad\quad\quad\quad\; 5 \quad\; 3$

By Hall's theorem $\Rightarrow$ $\exists S \subseteq A$ for which the marriage condi…

<u>Hall violator</u>

Give one example

of S:

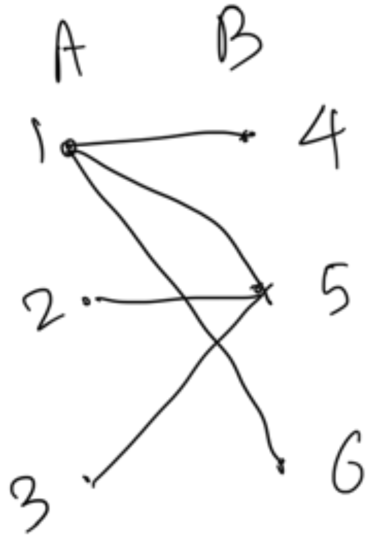
Condition $|N(S)| \geq |S|$ is violated.

$S = \{1,2,3,4,5\}$

$N(S) = \{6,7,8\}$

$\therefore \big| |N(S)| < |S| \big|$

(3)



Does this graph have a matching of A? No.

Hall violator

$S = \{2,3\}$

$N(S) = \{5\}$

$$\underbrace{|N(S)|}_{=1} < \underbrace{|S|}_{=2}$$

———x———

This class:

If Hall's marriage condition $|N(S)| \geq |S|$ $\forall S \subseteq A$.
holds for a bipartite
$G = (V, E)$ then A has a
$V = A \dot{\cup} B$ matching in G.

Hall's theorem

Proof:

Proof 1 [Nonconstructive] Let $G = (V, E)$ be a bipartite graph.
$V = A \dot{\cup} B$

Suppose $\forall S \subseteq A$, $|N(S)| \geq |S|$ — Marriage Condition.

We want to show that G contains a matching of A.

For this it suffices to show the following:

For every (partial) matching M in G

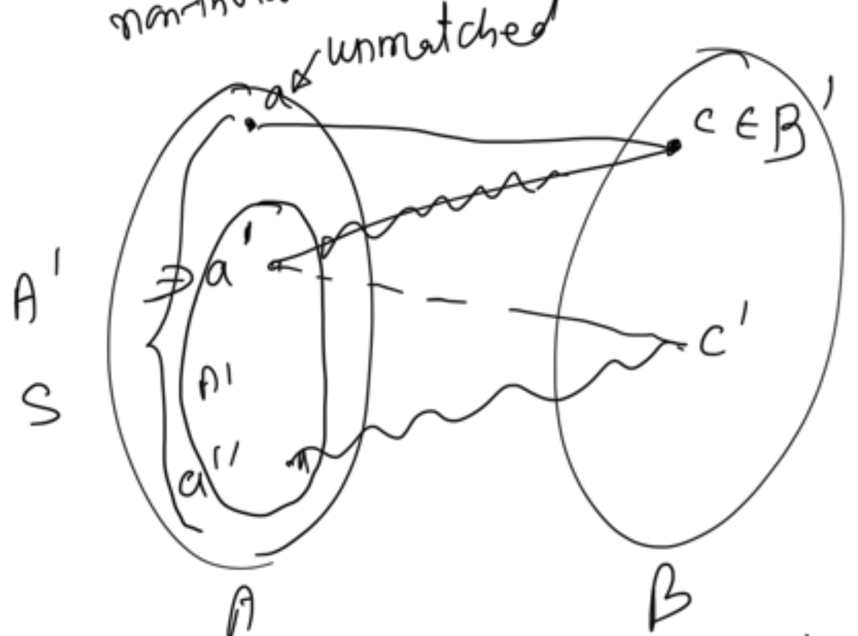
[For every ... fix ... [

(*) that leaves some $a \in A$ unmatched, $\exists$ an augmenting path P w.r.t. M.

(Why? Because then we get a larger matching $M' := M \oplus P$, with $|M'| = |M|+1$.)

Proof of (*):

Let $A' \subseteq A$ be the set of vertices in A that can be reached from a by some non-trivial alternating path.



& let $B' \subseteq B$ be the set of all penultimate vertices of such alternating paths

The last edges of all these alternating

paths lie in M

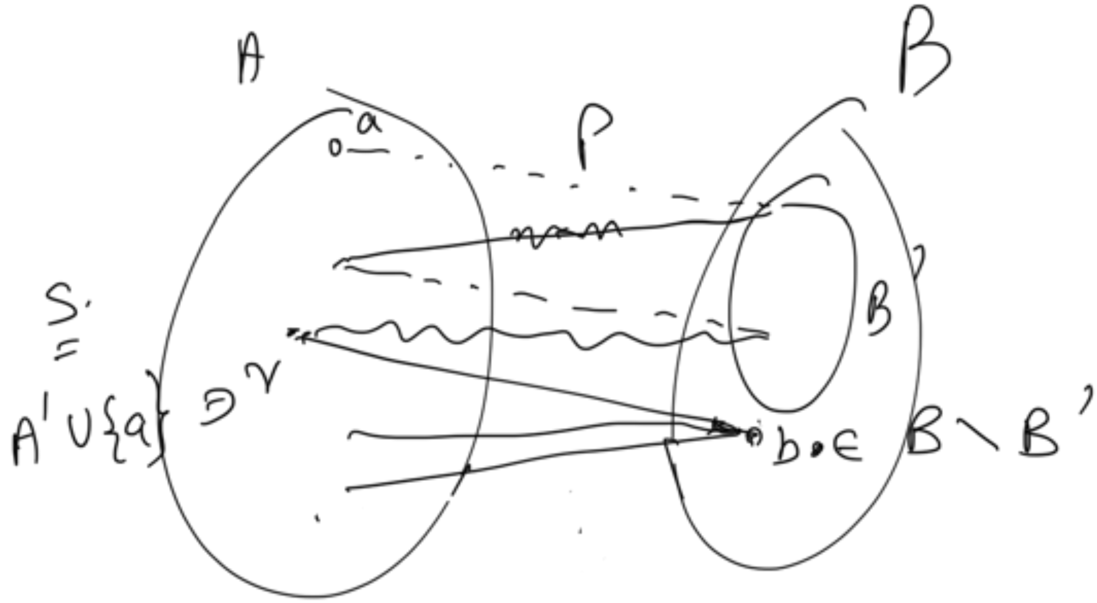
$\therefore |A'| = |B'|$

$\therefore$ By Hall's marriage condition:

For $S = A' \cup \{a\}$, $|N(S)| \geq |S|$.

$\therefore$ there is an edge from some vertex v in $S = A' \cup \{a\}$ to a vertex $b \in B \setminus B'$

Because
$|S| = |A'| + 1$
$= |B'| + 1$
$> |B'|$



Since $v \in A' \cup \{a\}$, by defn of A', there is an alternating path P from a to v (possibly trivial)

$P \cup \{b\}$ if $v =$

$\parallel$

Then Pvb is an alternating path from a to b.

The penultimate edge of Pv $\in M$.

Then $b \notin P$, since the vertices of P in B lie in B' (by defn B') & $b \in B \setminus B'$.

Two cases:

1) b is matched

by some $a'b \in M$, say,

Then $Pvba'$ would be an alternating path & hence by defn of A',

$a' \in A'$ & $b \in B'$ → contradiction.

$\therefore$ Case (1) cannot occur.

2) b is unmatched.

Then Pvb is an augmenting path.

[since it is alternating & ends on b, which is unmatched]

$\therefore$ We have found an augmenting path.

This proves $(*)$

$\therefore$ Hall's theorem is proved Q.E.D.

Second proof of Hall's theorem

[Also non-constructive].

By induction on $|A|$.

Assume that Hall's marriage condition holds for all $S \subseteq A$.

$|N(S)| \geq |S|$.

$S = A$.

$|N(S)| \geq |A|$

≥ 1.

Base case: $|A| = 1$. : trivial

$|N(S)| \geq 1$

So assume that $|A| \geq 2$.

By induction hypothesis:

... which satisfies

for any $A' \subsetneq A$, ...

Hall's marriage condition,

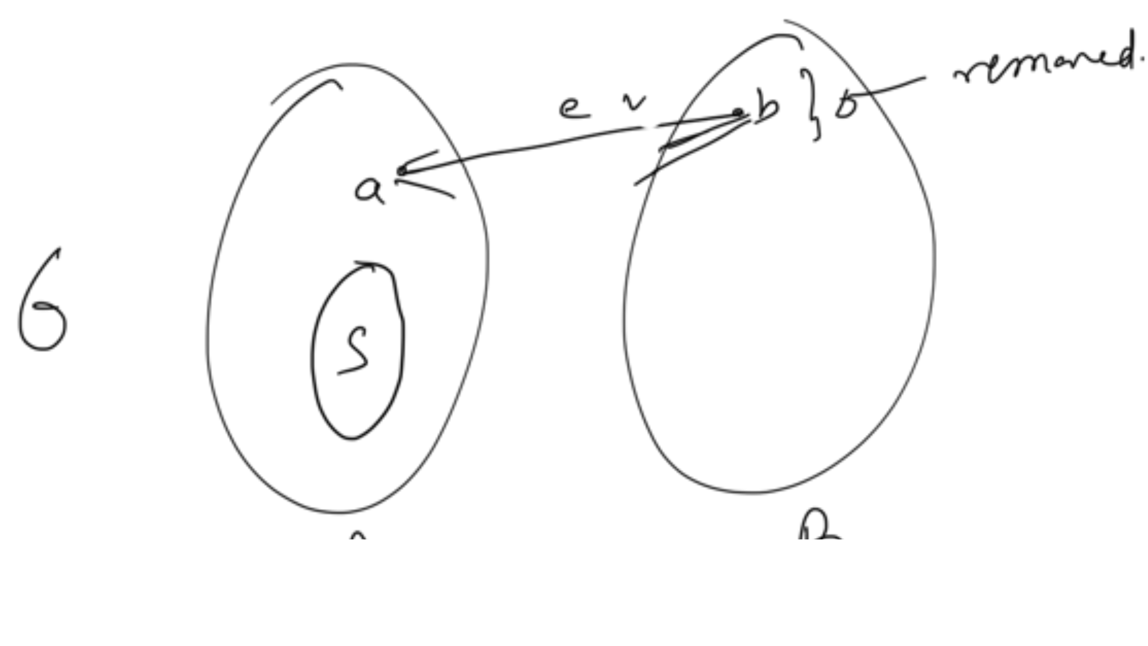
$\exists$ a matching of A'.

<u>Case 1</u> $|N(S)| \geq |S|+1$ for every non-empty proper subset $S \subsetneq A$.

Case 2: $|N(S)| = |S|$ for every non-empty proper subset $S \subsetneq A$.

<u>Case 1</u>: Goal: show that A has a matching.

Pick an edge $e = (a,b) \in G$

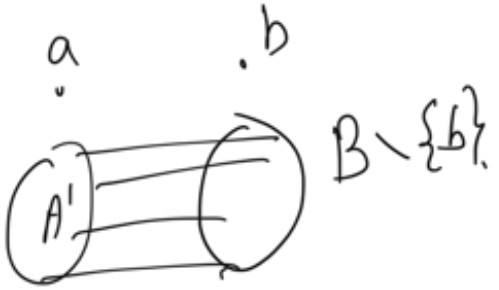


A B

Consider $G' := G - \{a, b\}$ [remove $e = (a,b)$ & all edges in G adjacent to a or b].

Then for every non-empty $S \subseteq A \setminus \{a\}$

$$|N_{G'}(S)| \geq \underbrace{|N_G(S)|}_{\text{could contain } b} - 1 \quad (\text{A})$$

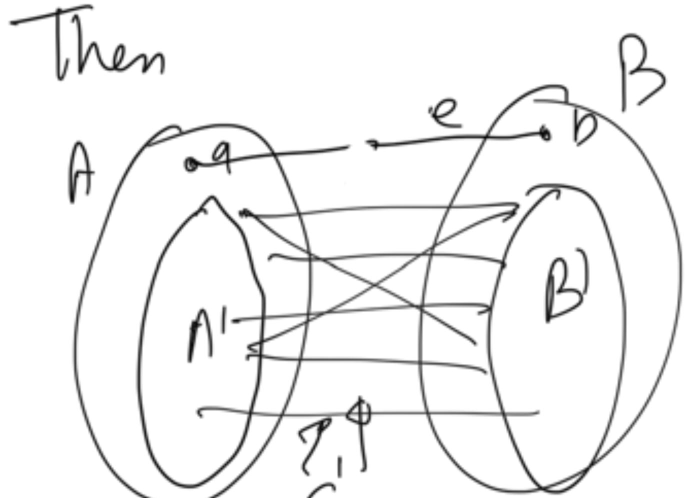


$$\geq \underline{|S| + 1} - 1 = |S|.$$

∴ G' satisfies Hall's marriage condition.

∴ by the induction hypothesis, applied to G' & A', G' contains a matching M of A'!

Then



G' contains a matching of A'.

$\underbrace{M \cup \{(a,b)\}}$ is a matching of A inside G.
as we wanted to show.

<u>Case 2</u>: $|N(S)| = |S|$ for every proper non-empty subset $S \subsetneq A$.

Goal: Show that A has matching in G assuming the induction hypothesis for smaller subsets of A.

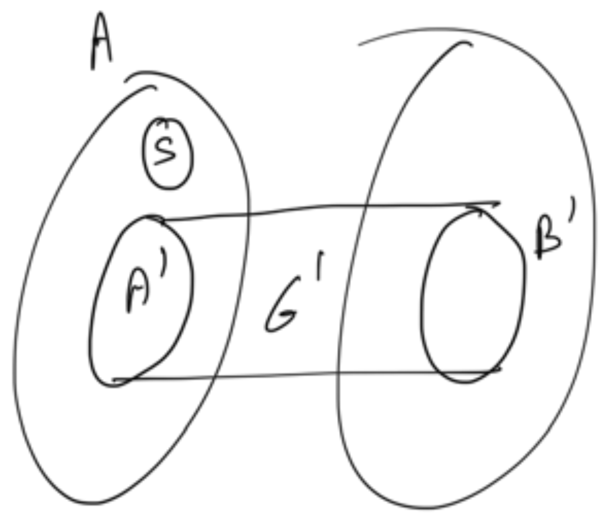
Let's choose any such proper subset $S \subsetneq A$.

Call S, A'.

$A' = \cancel{\emptyset}\, S$.

Then $B' := N(A')$

Then $|B'| = |A'|$.



By induction hypotheses, since $|A'| < |A|$,

$$G' := \underbrace{G[A' \cup B']}_{\text{Induced graph on } A' \cup B'}$$

contains a matching $\boxed{M' \text{ of } A'}$.

Goal: Extend $\underset{\substack{\cap \\ G'}}{M'}$ to a matching $M \subseteq G$, of A

$$\underline{\text{Claim}}: \underbrace{G - G'}_{\text{goes from } A-A' \text{ to } B-B'} := \underbrace{G - V(G')}_{\text{remove from } G \text{ all edges adjacent to } A' \text{ or } B'}.$$

satisfies the Hall's marriage condition.

<u>Proof of the claim:</u>

Suppose to contrary $\exists$ a set $S \subseteq A \setminus A'$ with $|N_{G-G'}(S)| < |S|$ [S violates Hall's marriage condition].

Then

$$\underbrace{|N_G(S \cup A')|}_{\substack{\rotatebox{90}{$\subseteq$} \\ N_G(S) \cup B'}} < |S| + |B'|$$

$$= |S| + |A'|$$

$$= |S \cup A'|$$

$\therefore$ $S \cup A'$ violates Hall's marriage condition in G — contradiction.

$\therefore$ it follows that $G - G'$

satisfies Hall's marriage Condition. Q.E.D [Claim]

Claim $\Rightarrow$ $G - G'$ contains a matching M'' of $A'' = A \setminus A'$.
+ Induction hypothesis

Then: $M = M' \cup M''$ is a matching of A.

This proves existence of a matching in case 2 as well.

Q.E.D [Hall's Theorem].

Third proof [Algorithmic proof] — Next class
+ deepest proof.

An application of Hall's theorem:

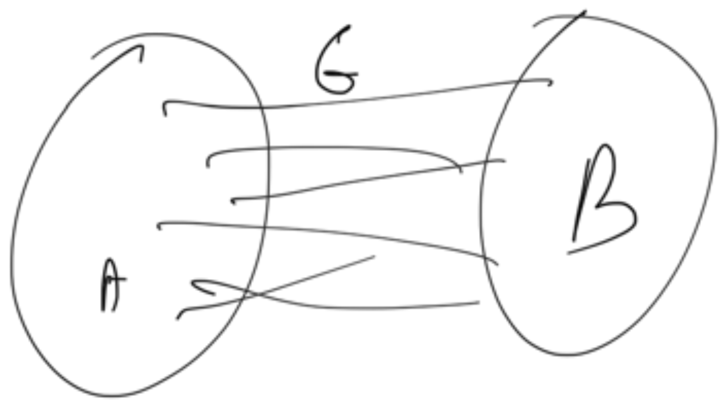
Corollary 1 [of Hall's theorem]:

Every k-regular ($k \geq 1$) bipartite graphs

G has a perfect matching.
(all vertices are matched)

Proof: W.l.g. we can assume that G is connected.

$$G = (V, E)$$
$$V = A \dot\cup B$$



Since G is k-regular,

$$|A| = |B|.$$

(Why? By k-regularity,

Both of these are exactly the edges of G:
$$\begin{cases} \#\text{ edges going out of } A = k|A| \\ \quad \text{-- -- -- -- -- -- -- -- } B = k|B|. \\ \quad \text{-- -- -- --} \end{cases}$$

$$k|A| = k|B| = |E|$$

$\therefore \underbrace{k|A| = k|B|}_{\Downarrow \; |A| = |B|}$)

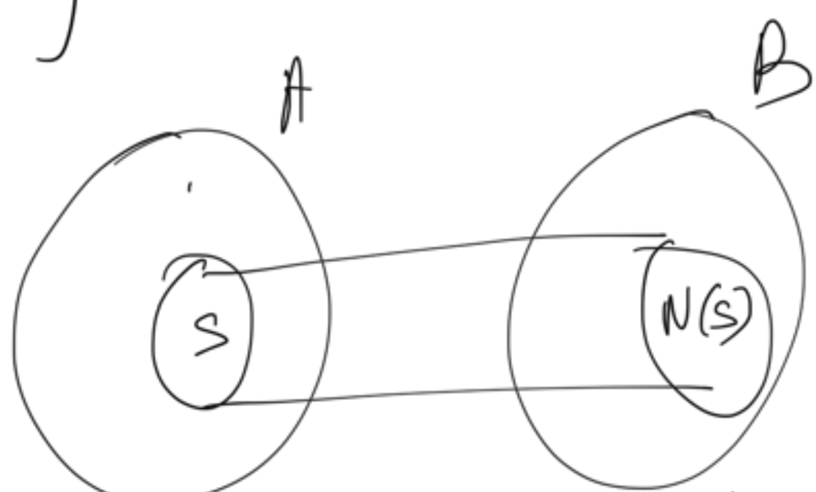
~~Goal~~ Goal: show that G has a matching.

By Hall's theorem, it suffices to show that G satisfies Hall's marriage condition.

That is: $\forall\ S \subseteq A$,

$$|N(S)| \geq |S|.$$

Fix any $S \subseteq A$:



... S is joined to $N(S)$ by

Then S is

by K-regularity

$K|S|$ edges

are among $K|N(S)|$ edges adjacent to $N(S)$.

$$\therefore \; K|S| \leq K|N(S)|$$

$$\Downarrow$$

$$|S| \leq |N(S)|$$ — $\therefore$ Hall's marriage condition is satisfied for any $S \subseteq A$.

$\therefore$ By Hall's theorem, G has a matching.

Q.E.D [Corollary]

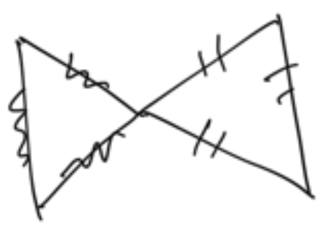
Given a graph $G = (V, E)$, $v \in V$, a neighbour of v, denoted $nbr(v)$, is any vertex $w \in V$ s.t. $(v, w) \in E$.

... neighbours of S, denoted

Given $S \subseteq V$, neighbours of S,
$N(S)$, is the set of all neighbours
of the vertices in S.

Another application of Hall's theorem

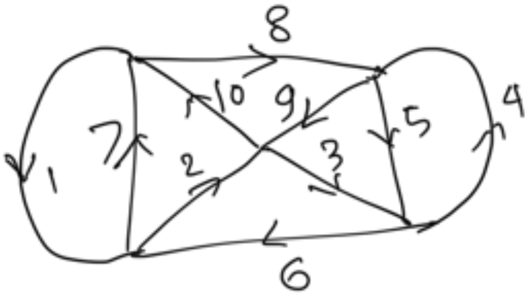
Theorem [Petersen]
Every regular graph of positive
even degree has a 2-factor.
$2k$
2-regular spanning
subgraph. edge
A spanning disjoint
collection of
cycles



For a proof, we need the notion of
an Eulerian tour.

Eulerian tour of a graph G: A tour
(a spanning cycle whose beginning & end points are
the same) which traverses every

edge of G exactly once.

G: 

A graph is called <u>Eulerian</u> if it admits an Eulerian tour.

Prop 1: A connected graph is Eulerian iff every vertex has an <u>even degree</u>.

Proof: $\longrightarrow$: Every vertex visited k times in an Eulerian tour must have degree $\underbrace{2k}_{\text{even}}$. (indegree = outdegree = k)

$\longleftarrow$: By induction on $\|G\|$. # edges

Base case: $\|G\| = 0$: trivial.

<u>Induction</u>: Assume that every graph ... even degree

with all vertices of even
and # edges $< \|G\|$ is Eulerian.

We want to show then that G
is also Eulerian.

Since every vertex of G has an
even degree, there exists a
non-trivial cycle C in G
(which visits any edge
at most one)

[why? start at any vertex,
keep moving until
we visit same earlier
visited vertex.
Then we have found
a cycle.]

Since G is finite this ~~degree~~ must happen.

Then $G \setminus C$ is Eulerian
(because every vertex of $G \setminus C$
also has an even degree)

∴ has Eulerian cycle D.

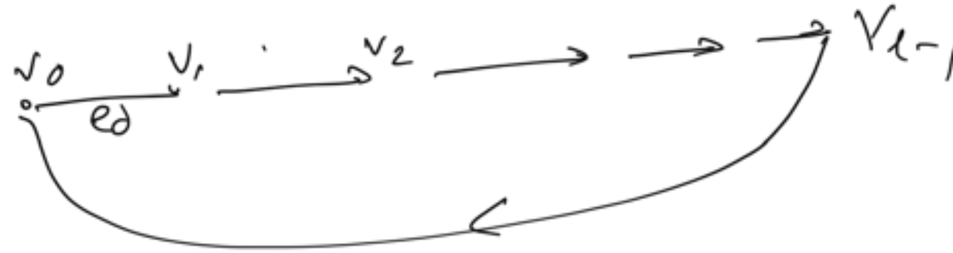
Then $C \cup D$ is an Eulerian cycle of G.

Q.E.D. (Prop)

By Prop 1 G is Eulerian

∴ G has an Eulerian tour. T.

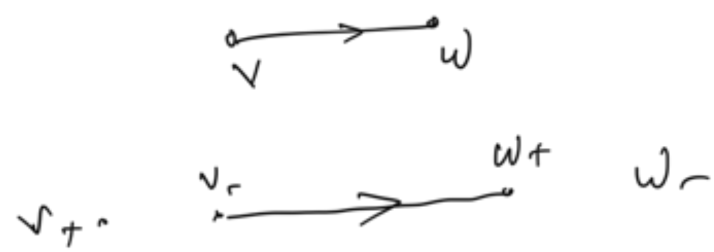
T:



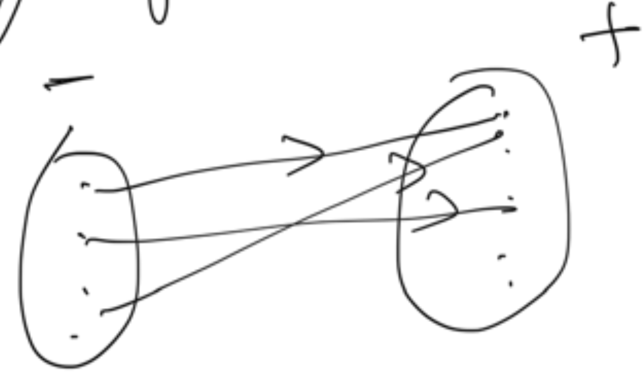
Replace every vertex $v \in G$ by a pair of vertices (v^-, v^+), & every edge $e_i = (v_i, v_{i+1})$ by an edge (v_i^+, v_{i+1}^-).

indegree(v) = out-degree(v)

(orientation as per the Eulerian tour)

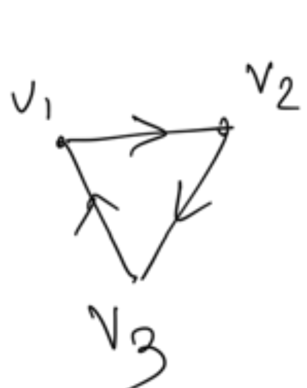


This procedure gives a bipartite graph G' where the edges go from − vertices to + vertices



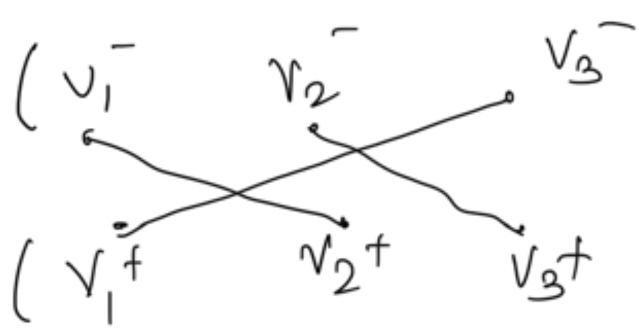
If G is $2k$-regular then G' is k-regular.

Example:



(2-regular)

$\rightsquigarrow$



1-regular.

… G' has

$\therefore$ By Hall's theorem, G has
(first application)
a perfect matching. $\underline{\underline{M}}$

Perfect matching $\bar{M}$ in G'
after we colease each pair (v_-, v_+)
~~vertic~~ to a vertex v,
will give rise to 2-factor
in G. [Exercise].

Q.E.D. [Peterson's theorem].

—————x—————x—————

<u>The third algorithmic proof of Hall's theorem.</u> [deepest proof].

Recall:
Thm [Hall] A bipartite graph
$G = (V, E)$ has a matching of A
$V = A \dot{\cup} B$ iff $\underbrace{|N(S)| \geq |S| \ \forall S \subseteq A}$
condition

marriage condition.

$\Downarrow$

if G does not have a matching of A then

$\exists$ Hall-violator $S \subseteq A$

s.t. $|N(S)| < |S|$

Hall Thm:

Given a graph $G \underset{}{=} (V,E)$, $V = A \dot\cup B$, either G has

Existence

1) a matching of A or

2) a Hall violator $S \subseteq A$.

Stronger algorithmic version Hall's thm:

Thm1 [Hall – Algorithmic]: [Goal].

There exists a polynomial time algorithm which, given a bipartite graph $G = (V, E)$, $V = A \dot\cup B$, finds in $O(|V||E|)$, $|V| = n$, $|E| = m$

time either

$$\begin{cases} 1)\ \text{a matching } M \subseteq E \text{ of } A \text{ in } G, \text{ or} \\ 2)\ \text{a Hall violator } S \subseteq A \text{ s.t.} \\ \qquad |N(S)| < |S|. \end{cases}$$

A weaker form of Thm 1.

<u>Thm 2:</u> Given a bipartite graph $G = (V, E)$ a <u>maximum matching</u> in G (matching of maximum cardinality) can be found in $O(\text{poly}|G|)$ time.

$$O(\text{poly}|G|) = O(n^3), \quad n = |G|.$$

First we will prove Thm 2.

<u>Recall:</u>

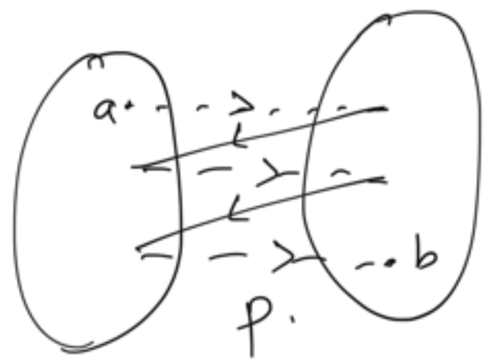
An augmenting path P in $G = (V, E)$, $V = A \cup B$, w.r.t a matching M (possibly partial)

is a path which

... an unmatched

1) begins & ends in an unmatched vertex [we do not require it to begin A in this class],

2) contains edges in $E \setminus M$ and M alternatively.



— : M.

bipartite graph

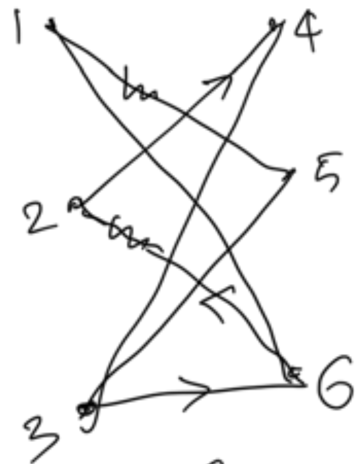
Prop: A matching $M \subseteq G$ is maximum iff G contains no augmenting path p w.r.t M.

Proof: $\longrightarrow$: Suppose to the contrary G contains an augmenting path P w.r.t. M, then

$M' := M \oplus P$ is matching

with $|M'| = |M|+1$.

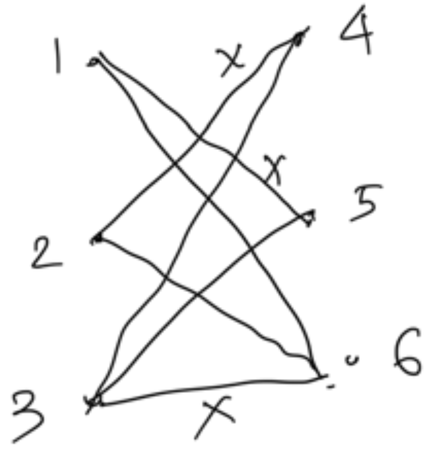
$\sim$: M
$\rightarrow$: P



G

$|M| = 2$

Then



G

M': X

$|M'| = 3 = |M|+1$

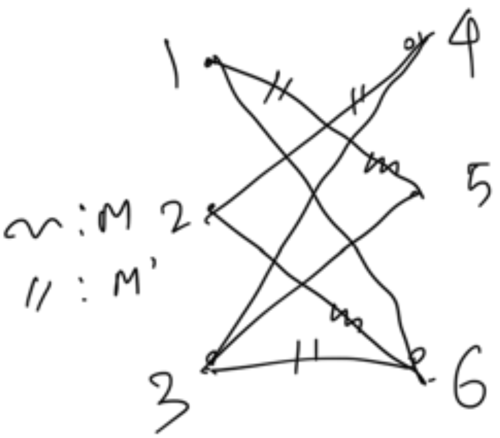
$\therefore$ M is not maximum.

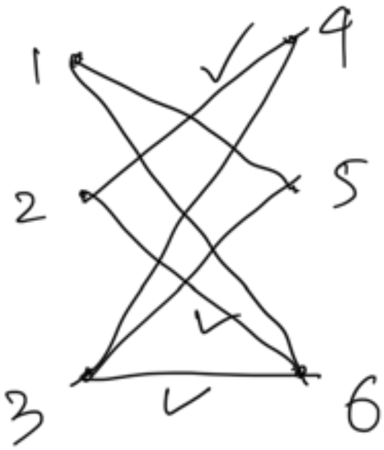
$\Leftarrow$ Suppose M to the contrary is not maximum then we want to show that $\exists$ an augmenting path P w.r.t. M in G.

Let M' $\neq M$ be any maximum matching in G.

Let $Q := M' \oplus M$.

In the example above:





$Q = M \oplus M'$

Q is a path (augmenting)

In general, what is the structure of Q?