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Homework 2

1) Show every tree T has at least $\Delta(T)$ leaves

First, let there exist a vertex V with degree $\Delta(t)$. Then, there exists vertices $X_1, \ldots, X_n$ ($n = \Delta(T)$) connected to V by an edge. Find a longest path starting with each edge $VX_1, VX_2, \ldots VX_n$. The final vertex of each of these paths must necesarily be a leaf. Thus, we have n paths that are disjoint exluding V. These paths must be disjoint otherwise there would exist a cycle. Since we have n paths, each providing a different leaf, we have n $(n = \Delta(T))$ leaves.

2) Prove or disprove: A connected graph is bipartide iff no two adjacent vertices have the same distance from any other vertex

Let there exist a bipartide graph G and two adjacent vertices $V, W \in G$. Then we can define two subgraphs in G, X and Y so there are no edges in X or $Y \Rightarrow E(X) = E(Y) = \emptyset$. There exists $V \in X$, $W \in Y$. Let there also exist $V' \in X$ and $W' \in Y$.

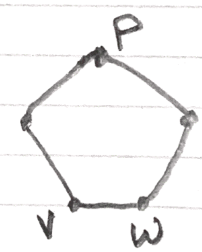
We know $E(X) = E(Y) = \emptyset$, so $d(v, v')$ and $d(w, w')$ are even. Also, $d(v, w')$ and $d(w, v')$ are odd. Thus adjacent vertices cannot have the same distance to another vertex. For, $\underbrace{d(v, v')}_{\text{even}} \neq \underbrace{d(w, v')}_{\text{odd}}$ and $\underbrace{d(w, w')}_{\text{even}} \neq \underbrace{d(v, w')}_{\text{odd}}$

Now we will consider what happens when G is not bipartide.

If G is not bipartide it must necesarily contain an odd cycle. Consider any two vertices v, w on the smallest such cycle X. The distance from v to w on G is equal to the distance from v to w on X. If this were not true there would necesarily exist a smaller cycle.

Thus, since X is an odd cycle, if v and w are adjacent, there exists a point p such that $d(p,v) = d(p,w)$.

Consider the following cycle:
$d(v,w)=1$, $d(p,v)=d(p,w)=2$



Generally, if $|X| = k$, $d(p,v)=d(p,w)=\lfloor \frac{k}{2} \rfloor$

Therefore, when a graph is not bipartide, there exist two adjacent vertices that have the same distance from another vertex. This is equivalent to: if no two adjacent edges have the same distance from another edge, then the graph is bipartide.

3. Suppose F, F' are forests on the same set of vertices such that F' has strictly more edges than F. Show F' must have an edge e such that $F+e$ is also a forest.

Connected components = #vertices - #edges

Proof by contradiction: If F' does not have an edge e such that $F+e$ is a forest, then $F+e$ must contain a cycle for all edges e in F'. Consider vertices v and w connected by an edge in F'. Since $F+e$ must contain a cycle for all edges e in F', v and w necesarily exist in the same connected component of F. This implies that v and w are in the same connected component of F' and the same connected component of F. We know that F' has at least as many connected components as F. Thus, the number of vertices (same for both forests) minus the number of connected components yields a result for F' that is less than or equal to the result for F. (This means #vertices − (#vertices − #edges) for $F' \leq$ #vertices − (#vertices − #edges) for F)

This is a contradiction as F' has strictly more edges than F. Thus, $F+e$ does not contain a cycle for all edges e in $F' \rightarrow \exists e$ s.t. $F+e$ is a forest.

4. Show that a graph is 2-edge-connected iff it has a strongly connected orientation, which means an orientation in which every vertex can reach any other vertex by a directed path.

First, we define a 2-edge-connected graph G. There must necesarily exist a cycle X in G. We then orient the edges in the same direction so X is strongly connected.

Consider a subgraph G', formed by these oriented edges, that has a strongly-connected orientation.

If $G' \neq G$: We know G is 2-edge-connected. Thus, there exists two disjoint paths from a vertex $v \notin G'$ to a set of vertices in G'. We can orient the edges in one of these paths towards v and the edges in the other path away from v. We repeat this process for every vertex $v \notin G'$ until $G = G'$.

Now, again consider a subgraph G' which is a strongly connected orientation of G. If G contains a cut edge (must not be in a cycle) it is not 2-edge-connected. Then G' is not strongly connected as there only exists a single directed edge between the two vertices connected by the cut edge. Therefore, there cannot exist a cut edge in G' and G' is 2-edge-connected.

5. ... Show the edges traversed form a normal spanning tree in G with root r.

The edges traversed form a spanning tree:

Let G' be a subgraph of G. To construct G' start at r and move along the edges of G, going whenever possible to a vertex not yet visited. If there is no such vertex, go back along the edge by which the current vertex was reached (stop if current vertex is r). G is connected, meaning G' spans G. The construction given above never revisits the same vertex, so G' contains no cycles. Thus, by definition G' is a spanning tree of G.

G' is normal in G:

let there exist two vertices $v, w \in G$ that are adjacent. We will assume v was visited first during the construction given above. When v is visited w has not yet been visited and it is adjacent to v, so the edge vw is added to G'. Thus, v will be on the unique rw path in G'. This means that $v \leq w$ in the tree order of G' (the two vertices are comparable) and G' is a normal spanning tree.

6. Show that for every positive integer k, every graph of minimum degree $2k$ has a $(k+1)$-edge-connected subgraph.

First, consider a graph G that is not $(k+1)$-edge-connected. Split the vertices of this graph G into two subsets, X and Y, such that there exists at most k edges connecting $V(X)$ to $V(Y)$. As assumed, G has a minimum degree $2k$. Thus the sum of the degrees of the vertices in G is at least $2nk$ where n is the number of vertices in G. Also, the sum of the degrees of the vertices in X is at least $2mk-k$ where m is the number of vertices in X.

Then, by removing the vertices in Y and the edges connected to those vertices, we are left with a subgraph X with m vertices such that the sum of their degrees is at least:

$$(2mk-k)-k=2mk-2k$$

We can continue to construct subgraphs in this way: Splitting the graph into subsets connected by at most k edges and then removing those edges until we get a $(k+1)$-edge-connected graph.

This will produce subgraphs whose sum of degrees of all n vertices equals:

$$2nk, 2nk-2k, 2nk-4k, 2nk-8k, \ldots$$

At some point one of these subgraphs must be $(k+1)$-edge-connected. For the case $k=1$, degree $=2$ holds true as the graph contains a cycle, and is thus 2-edge-connected.