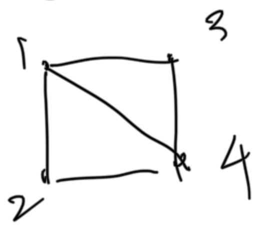
Lecture 2 (Graph Theory)

If $G=(V,E)$ & $G'=(V',E')$ then

$$G - G' := G - V(G')$$

Example:

G:



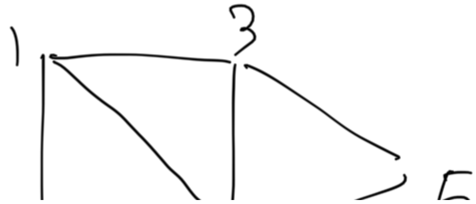
G':

$G - G'$:

Given $G=(V,E)$ & subset $F \subseteq V \times V$

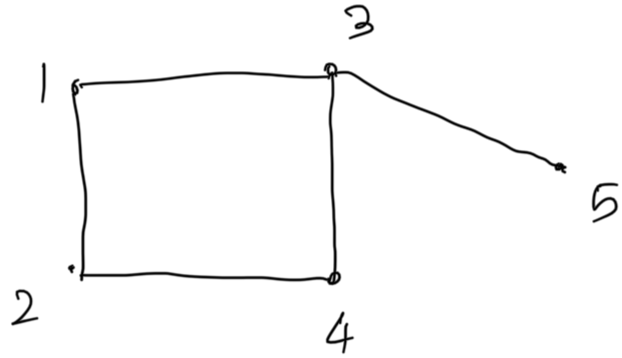
we write $G - F := (V, E \setminus F)$

$$G + F := (V, E \cup F)$$

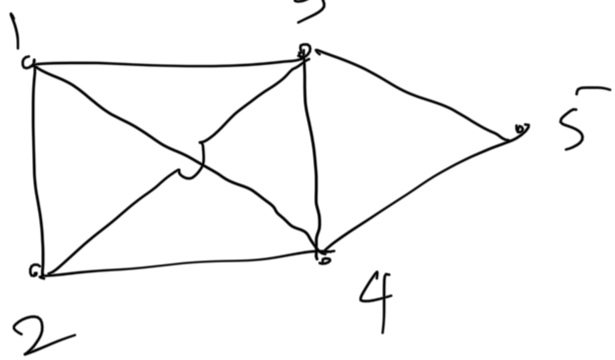


$F = \{(2,3),(1,4),(4,5)\}$

$G - F$:

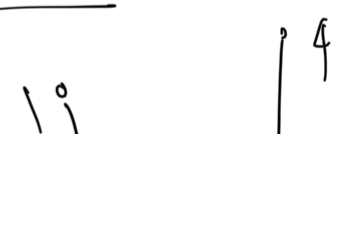


$G + F$:



We call $G=(V,E)$ <u>edge-maximal</u> with a given property if G has that property but no graph $G'=(V,F)$ (super) with $F \supsetneq E$ has that property.

<u>Example:</u>

 is edge-maximal has acyclicity-property.

We say that $G=(V,E)$ is <u>maximal</u> with respect to some property if no proper subgraph of G has that property.

<u>Example:</u>

is maximal with respect to non-acyclicity property.

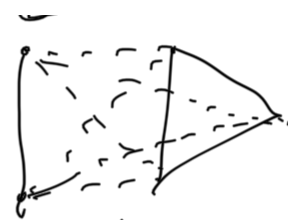
Suppose G and G' are disjoint graphs then $G * G'$ is the graph obtained by joining all vertices of G to all vertices of G'.

<u>Example:</u>

K_3 K_2

K_2

$K_2 * K_3 :=$

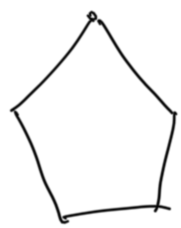


$\parallel$

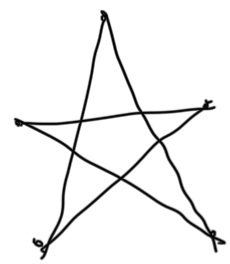
K_5.

$$G = (V, E)$$

Complement $\overline{G} := (V, V \times V \setminus E)$



G



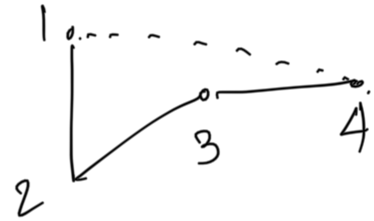
$\overline{G}$

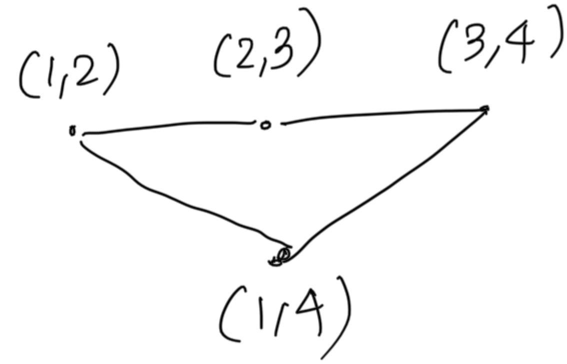
If $G = (V, E)$ then the <u>line graph</u> $L(G)$ of G is:

$$L(G) = (E, \underset{E \times E}{\underset{\cap}{W}})$$

where $x, y \in E$ are adjacent as vertices of $L(G)$ iff they are adjacent in G.

Example:



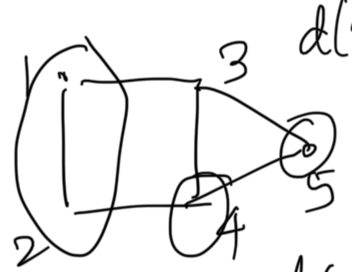


The degree:

$G = (V, E)$

If $U \subseteq V$, then $N(U) :=$ neighbours of U in $\underline{V \setminus U}$.

<u>Example:</u>



$d(5) = 2$

$N(\{1,2\}) = \{3,4\}$

$\delta(G) = 2$.

$\Delta(G) = 3$

$d(4) = 3$

G

$G = (V, E)$, $v \in V$:

degree of v, $d(v) :=$ # edges in E adjacent to v.

$\delta(G) := \min\{d(v) \mid v \in V\}$.

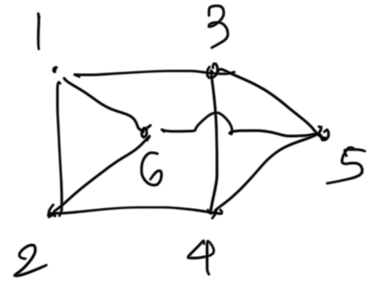
$\shortparallel$

minimum degree of G

$\Delta(G) := \max\{d(v) \mid v \in V\}$.

max-degree of G

3-regular



G

If $d(v) = k$ for all $v \in V$, we call G <u>k-regular</u>

3-regular graphs are also called <u>cubic</u> graphs.

Average degree of G, $d(G) := \frac{1}{|V|} \sum_{v \in V} d(v)$

Clearly:

$$\delta(G) \leq d(G) \leq \Delta(G).$$

min-degree

$$\varepsilon(G) := |E| / |V|$$

Then:

$$2|E| = \sum_{v \in V} d(v)$$

$$\varepsilon(G) = \frac{1}{2|V|} \sum_{v\in V} d(v)$$

$$= \frac{1}{2} d(G).$$

Prop: # vertices of odd degree in a graph is even.

Proof: $\sum_{v\in V} d(v) = 2E$: even.

$\therefore \sum_{\substack{v\in V \\ d(v)=\text{odd}}} d(v) = \text{even}$ Q.E.D.

Prop: Every G with at least one edge has a subgraph H with

$$\underbrace{\delta(H)}_{\text{min-degree}} > \varepsilon(H) \geq \varepsilon(G)$$

Proof: Construct a sequence $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq \overset{H}{\overset{\shortparallel}{G_i}}$ of induced subgraph of G as follows:

~~Pf~~ Inductively assume that G_i has been constructed.

If G has a vertex v_i of degree

$$d(v_i) \le \varepsilon(G_i),$$

$$\text{let } G_{i+1} := G_i - v_i.$$

otherwise stop & let $H = G_i$.

By choice of v_i:

$$\varepsilon(G_{i+1}) \ge \varepsilon(G_i) \quad \forall i$$

$$\therefore \varepsilon(H) \ge \varepsilon(G)$$

$\varepsilon(H)$ is positive.

Since G has at least one edge $\varepsilon(G) = \frac{|E|}{|V|}$ is positive.

Clearly $\underline{\underline{H \neq \emptyset}}$,

Since H has no vertex v s.t.

$$d(v) \le \varepsilon(H)$$

$$\Rightarrow \delta(H) > \varepsilon(H) \ge \varepsilon(G). \quad \text{Q.E.D.}$$

Paths and cycles

$G = (V, E)$

Path P in G is a sequence of <u>disjoint</u> vertices in V $P := (x_0, \dots, x_k)$ s.t.

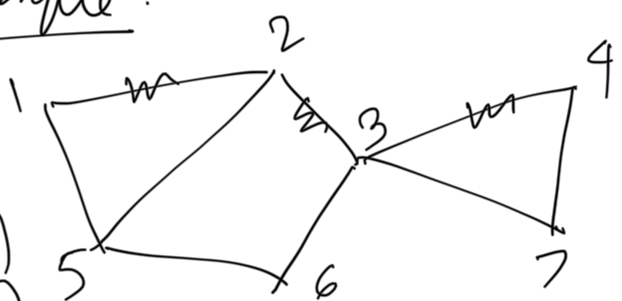
$$\forall i: (x_i, x_{i+1}) \in E.$$

<u>Example</u>:

$P\mathring{3} := (1,2)$

$\mathring{2}P := (3,4)$

$\mathring{1}P\mathring{4} := (2,3)$



P: ~

$l(P) = 3$

$P3 := (1,2,3)$

$3P := (3,4)$

$2P3 := (2,3)$

length of a path P := # edges in that path.

<u>$l(P)$</u>.

$$P: (x_0, \dots, x_k) \subseteq G.$$

Then

$$Px_i := (x_0, \dots, x_i)$$

$$x_i P := (x_i, \dots, x_k)$$

$i < j$ $\quad x_i P x_j := (x_i, \dots, x_j)$

$$P\mathring{x}_i := (x_0, \dots, x_{i-1})$$

$$\mathring{x}_i P := (x_{i+1}, \dots, x_k)$$

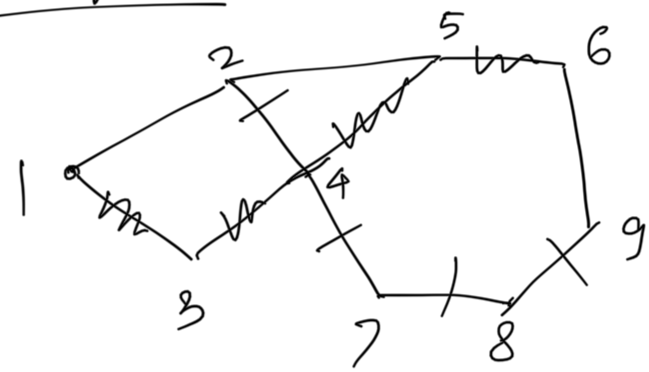
$x_i P$ ...

$$x_i^0 P x_j^0 := (x_{i+1}, \dots, x_{k-1})$$

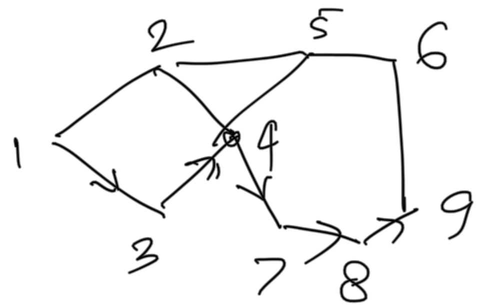
Example 2:

P : ~

Q : /



1P4Q9:

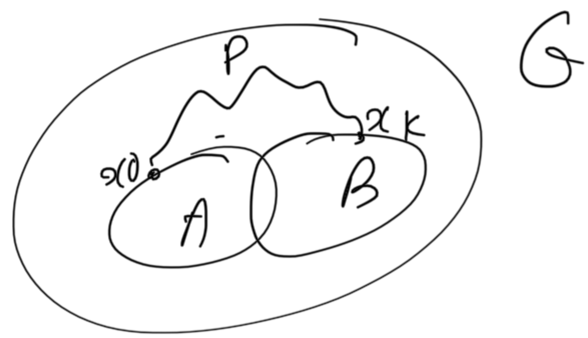


$$\underline{1 P 4 Q 9} := (1,3,4,7,8,9)$$

$G = (V, E)$. Given $A, B \subseteq V$

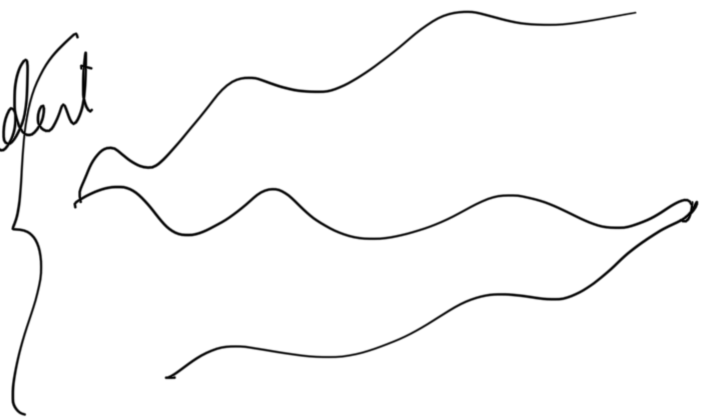
We call $P = (x_0, \dots, x_k)$ an <u>A–B path</u>

if $\underset{\substack{\uparrow \\ \text{vertices of } P}}{\underline{V(P)}} \cap A = x_0$, & $V(P) \cap B = x_k$.

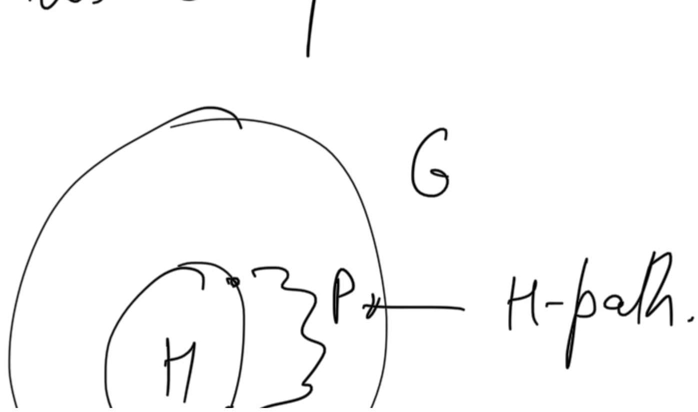


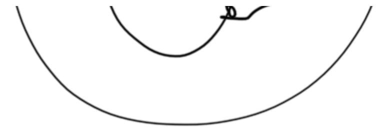
Two or more paths are called independent if none of them contains an inner vertex of another.
(not an endpoint)

3 independent paths.



Given subgraph $H \subseteq G$ we call a path P an H-path if P is non-trivial and meets H exactly in its endpoints.



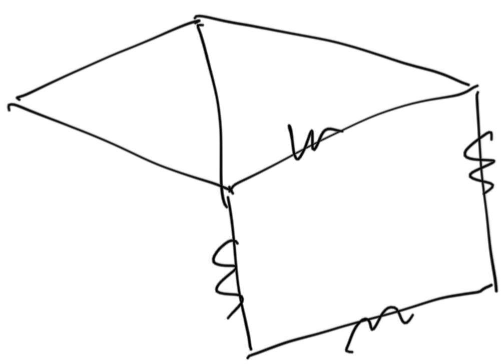


Cycle: If $P = (x_0, \dots, x_k)$ is a path in G (V, E)

all vertices are distinct & $\forall e : (x_i, x_{i+1}) \in E$

then $C := (x_0, \dots, x_k, x_0)$ is a cycle in G if $(x_k, x_0) \in E$.

Example:



ξ : cycle.

$\ell(\xi) : 4$

length of a cycle c,

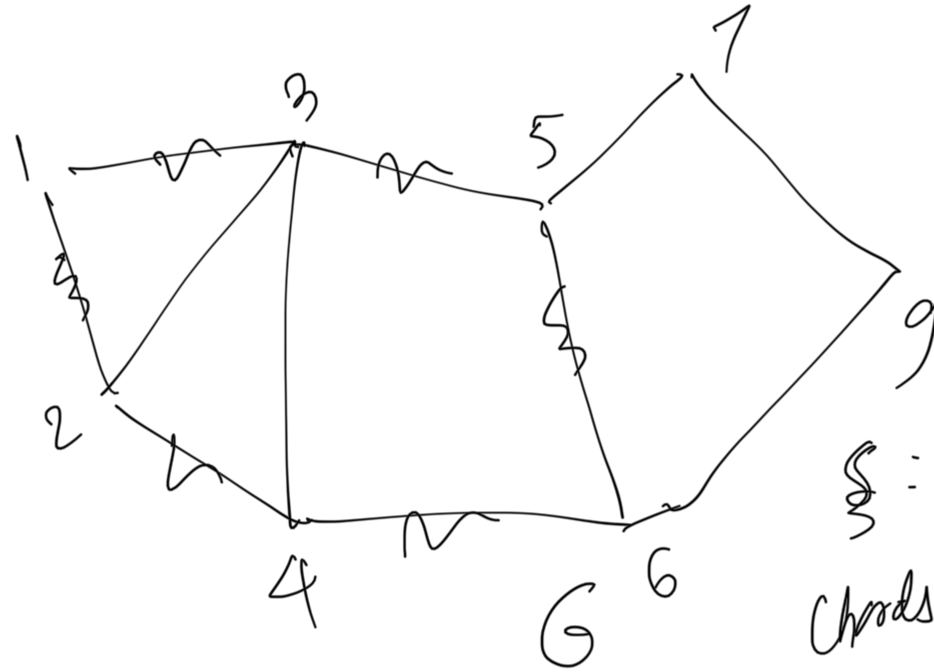
$\ell(c) := \#$ edges in c

Girth of a graph $G :=$

$g(G) :=$ min length of a cycle in G

a cycle in G.

Example:



$g(G) = 3$

$c(G) = 8$

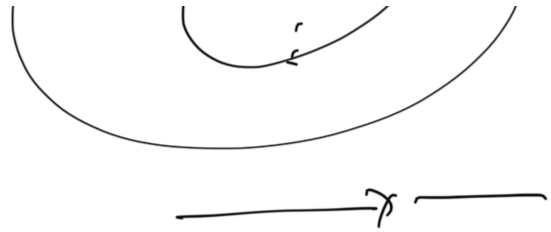
§ : cycle.

Chords: (2,3)
(3,4)

Circumference of a graph G,

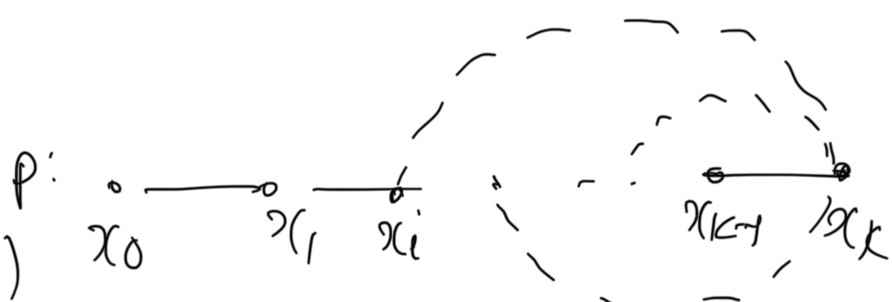
$c(G) :=$ max. length of a cycle in G.

A chord of a cycle := An edge which connects two vertices of a cycle but is not an edge of a cycle.



Prop: Every graph $G = (V, E)$ contains a path of length $\geq \underbrace{\delta(G)}_{\text{min-degree of } G}$ and a cycle of length $\geq \delta(G)+1$. (provided $\delta(G) \geq 2$)

Proof: Let $P = (x_0, \ldots, x_k)$ be a longest path in G. We want to show that $\ell(P) \geq \delta(G)$.



$\ell(P) \geq d(x_k)$ $\Leftarrow$ (All neighbours of x_k must lie in P [otherwise we can extend the path]

$\#$ nbr of x_k $= d(x_k) \geq \delta(G)$

$d(x_k) \geq \delta(G)$

$\ell(P) \geq d(x_k) \geq \delta(G).$

let $i<k$ be minimal s.t. x_i is a neighbour of x_k (i.e. $(x_i, x_k) \in E$)

Then $C := (x_i, \ldots x_k, x_i)$ is a cycle of length $\geq d(x_k)+1$
$\geq \delta(G)+1.$

Q.E.D.