Lecture 3 (Graph Theory)

$G = (V, E)$

Defn: The <u>distance</u> $d_G(x,y)$ in G between two vertices x and y is the length of the shortest path in G connecting x & y.

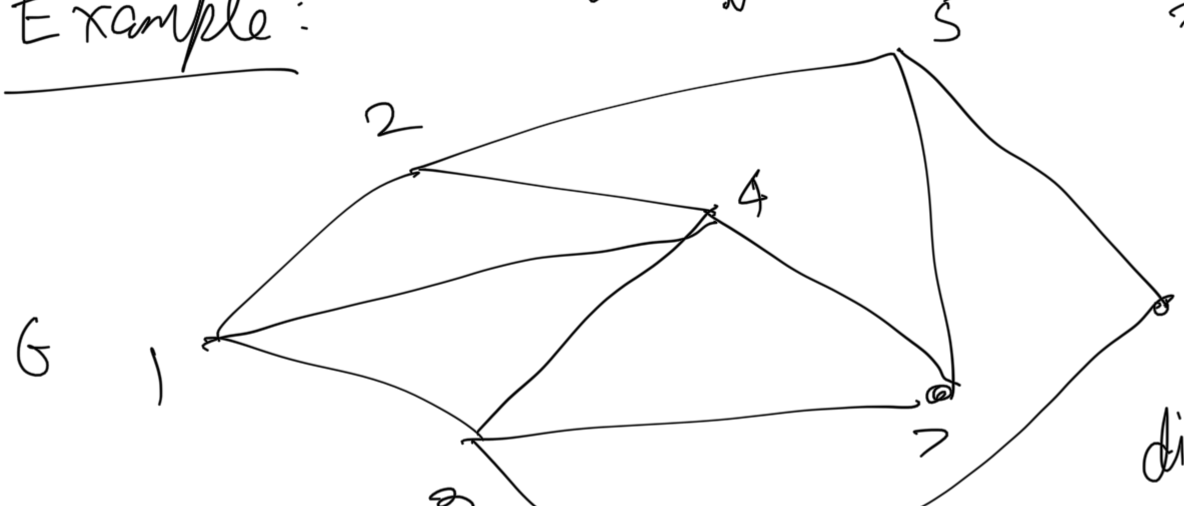
If no such path exists then

$$d_G(x,y) = \infty.$$

Every vertex is central (?)

rad(G) = 2

Example:



diam(G) = 2.

$d_G(1,7) = 2$ $\quad$ $d_G(3,5) = 2$

$d_G(1,5) = 2$

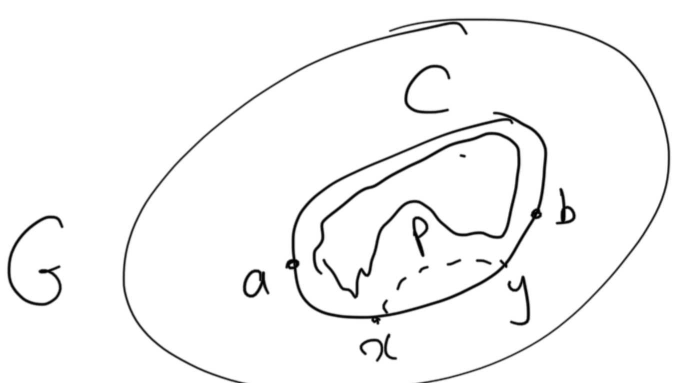
The greatest distance between any two vertices of G is called the <u>diameter</u> of G, denoted diam(G).

<u>Prop</u>: Every graph G containing a cycle satisfies:

$$g(G) \le 2\,\mathrm{diam}(G)+1.$$

<u>Proof</u>: Let C be a shortest cycle in G.

$\ell(C) = g(G)$.



If $\overset{\ell(C)}{\overset{\|}{g(G)}} \ge 2\,\mathrm{diam}(G)+2$ then C contains two vertices a and b whose distance <u>in C</u> $\ge \mathrm{diam}(G)+1$.

$P:(a,\dots,x,\dots,y,\dots b)$

$\ell(P) \le \mathrm{diam}(G)+1.$

In G, $dist_G(a,b) < diam(G)+1$

$\therefore$ any shortest path <u>in G</u> between a and b is not fully contained in C.

Then: C': $a \dots x\ p\ y \dots b \dots a$

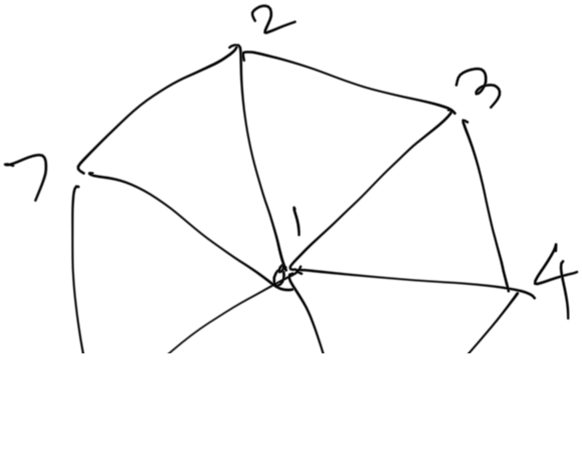
Then $l(C') < l(C)$.

$\therefore$ C' is shorter cycle in G

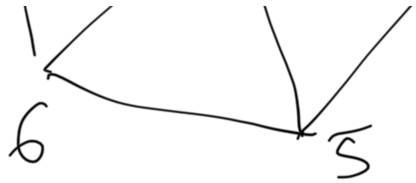
— Contradiction. Q.E.D.

A vertex x in a graph G is called <u>central</u> if its <u>greatest distance</u> from any other vertex is as small as possible.

$\to$ rad(G), the radius of G.



center: 1



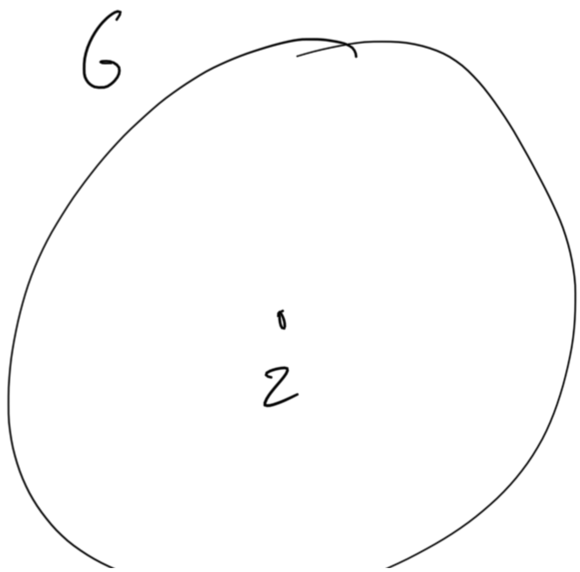
Exercise: $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$

Prop: A connected graph $G=(V,E)$ of radius at most k and maximum degree at most $d \ge 3$ has fewer than

$$\frac{d}{d-2}(d-1)^k$$

vertices.

Proof: Fix any centre z in G.



D_i: The set of vertices in G at distance i from z. $D_0 = \{z\}$

$|D_0| = 1$

So $V = \bigcup_{i=0}^{k} D_i$, $|D_1| \le d$.

Goal: Estimate $|D_i|$ by induction on i:

For $i \geq 1$:

$$|D_{i+1}| \leq (d-1)\, |D_i|.$$

[Because: (1) Every vertex in D_{i+1} is a neighbour of some vertex in D_i.
(2) Every vertex in D_i has at most $(d-1)$ neighbours in D_{i+1}, because it has at least one neighbour in D_{i-1}.]

$\therefore$ by induction: $|D_{i+1}| \leq d(d-1)^i$, for $i < k$

(Since $|D_0| = 1$, $|D_1| \leq d$)

$$\sum_{i=0}^{t} a^i = \frac{a^{t+1}-1}{a-1}$$

$$\therefore |G| = |V| = \left| \bigcup_{i=0}^{k} D_i \right|$$

$$\leq 1 + d \sum_{i=0}^{k-1} (d-1)^i$$

$$= 1 + d\,\frac{((d-1)^{r}-1)}{d-2} < \frac{d}{(d-2)}(d-1)^{r}.$$

Q.E.D.

<u>Thm 1</u>: If $\delta(G) \geq 3$ then

min-degree

$$g(G) \leq 2\log_2 |G|.$$

G is connected.

Thm 2 [Alon, Hoory, Linial] (without proof)

G: graph. Assume that

$d(G) \geq d \geq 2$ and

Average degree of G

$g(G) \geq g \in \mathbb{N}$.

girth of G

$r = \frac{(g-1)}{2}$.

Then

$$|G| \geq n_0(d,g) := \begin{cases} 1 + d\displaystyle\sum_{i=1}^{r-1}(d-1)^{i}, & \text{if } g := 2r+1 \text{ is odd} \\ \ldots g/2 \ldots \end{cases}$$

$\qquad\qquad\qquad\qquad\qquad \text{is odd},$

$$2\sum_{i=1}^{r-1}(d-1)^i \quad \text{if } g := 2r \text{ is even}$$

Proof

Thm 1 [Corollary of Thm 2]:

Let $d=3$. Then $d(G) \geq \underset{d}{3} \geq 2$.

By Thm 2:

$|G| \geq n_0(3,g)$. $[d=3]$

what is $n_0(3,g)$?

If $g := g(G) = 2r$, i.e. even, then

$$n_0(3,g) = 2\sum_{i=1}^{g/2-1} 2^i = 2\cdot\frac{2^{g/2}-1}{2-1}$$

$$= 2(2^{g/2}-1)$$

$$= 2^{g/2} + 2^{g/2} - 2 > 2^{g/2}$$

If $\overset{=g(G)}{g}$ is odd

then $g = 2r+1$

Then, by Thm 2,

$$n_0(3,g) = 1 + 3 \sum_{i=0}^{\frac{(g-1)}{2}-1} 2^i$$

$$= 1 + 3 \cdot \frac{2^{\frac{(g-1)}{2}} - 1}{2-1} = \frac{3}{\sqrt{2}} 2^{g/2} - 2 > 2^{g/2}.$$

$\therefore$ $n_0(3,g) > 2^{g/2}$ always.

$\therefore$ by Thm 2,

$$|G| \geq n_0(3,g) > 2^{g/2}.$$

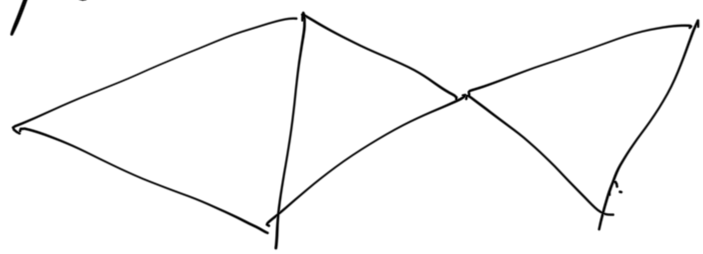
$$\therefore \quad g \leq 2 \log_2 |G|.$$

Q.E.D.

Connectedness:

A graph G is called <u>connected</u> if any two vertices in G are connected by a path.

Example:



$n = |G|$.

Prop: The vertices of a connected G can always be enumerated $v_1, \dots, v_n$ so that $G_i := G[v_1, \dots, v_i]$ is connected for all i.

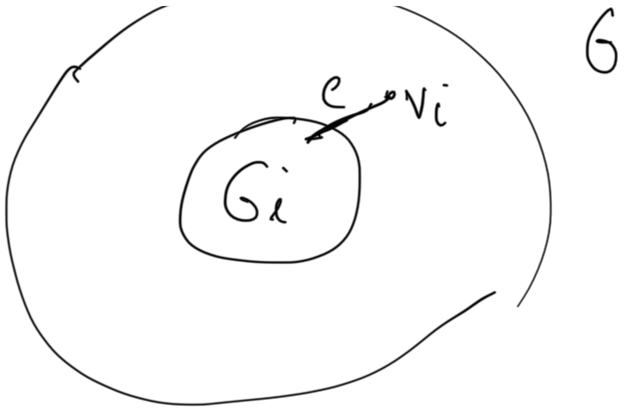
Proof: By induction on i.

$i=1$: trivially true. Choose v_1 to be any vertex of G.

By induction assume that $v_1, \dots, v_{i-1}$ are chosen. $\therefore G_{i-1} = G[v_1, \dots, v_{i-1}]$ is connected.

How do we choose v_i?

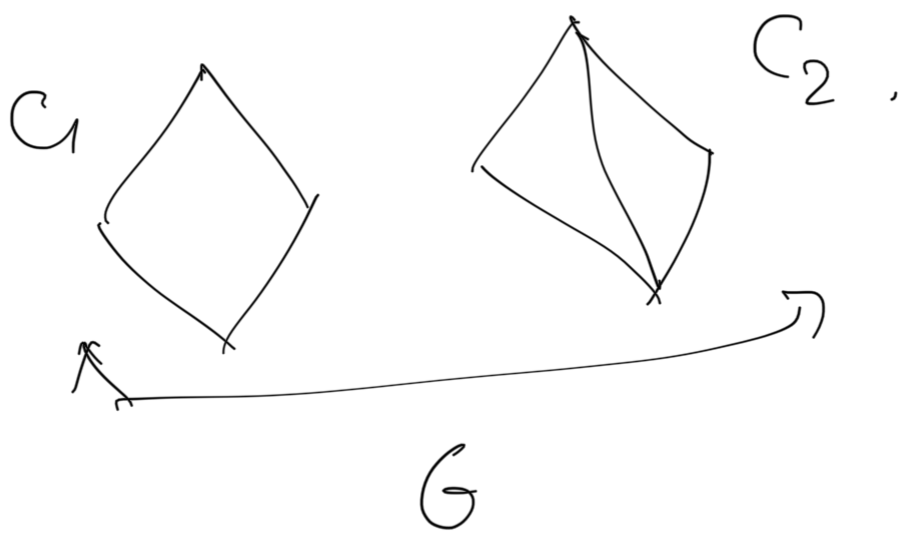
Since G is connected



$\exists$ an edge $e \in E$ which connects G_{i-1} to some vertex in $V \setminus \{v_1, \dots, v_{i-1}\}$.

Let v_i = the other end of e.
Then $G_i = G[v_1, \dots v_i]$ is connected. Q.E.D.

G: a graph.
A maximal connected subgraph of G is called a <u>component</u> of G.



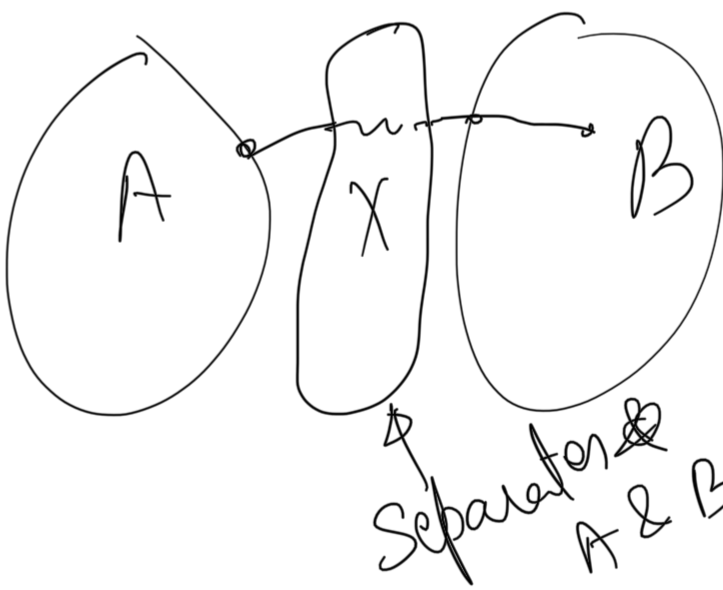
Let $G = (V, E)$ be a graph.

Let $G=(V,E)$ ...

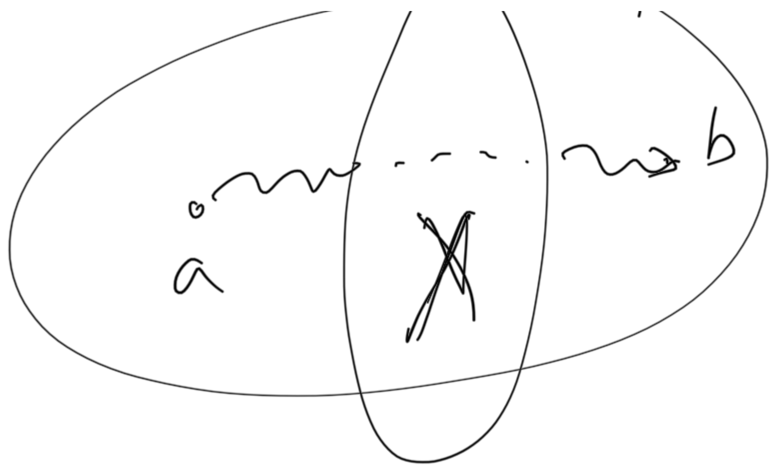
$A, B \subseteq V$ & $X \subseteq V \cup E$

such that every $A-B$ path in G contains a vertex or an edge in X, then we say that X separates A and B.

G :



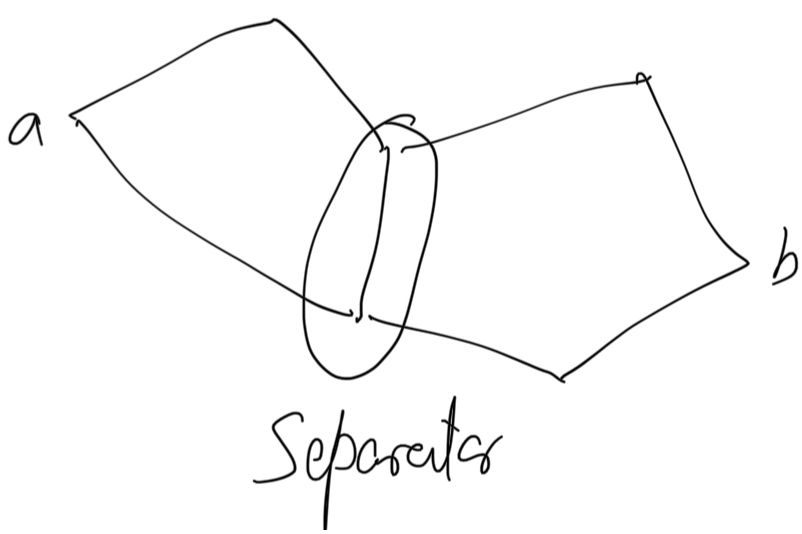
We say that X separates $a, b \in V$ if it separates $\{a\}$ and $\{b\}$ but $a, b \notin X$.



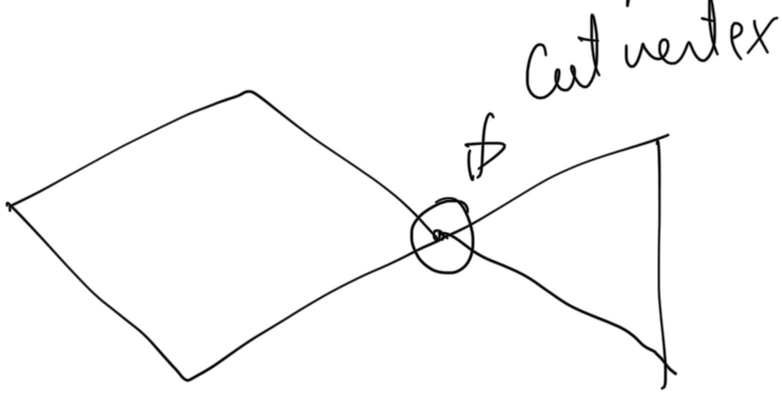
We say that X separates G if it separates some two vertices in G.

A separating sets of vertices is called a separator.

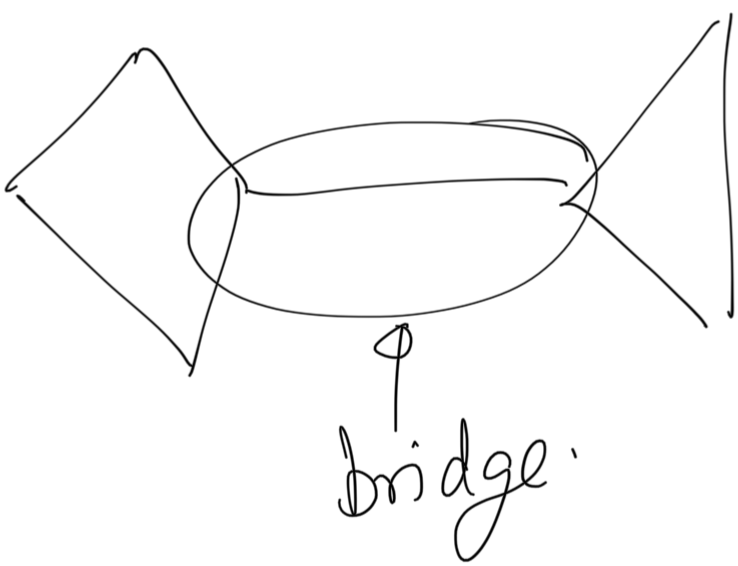
Example:



A cut-vertex is a vertex that separates two other vertices in the same component.



An edge separating its ends is called a <u>bridge</u>.



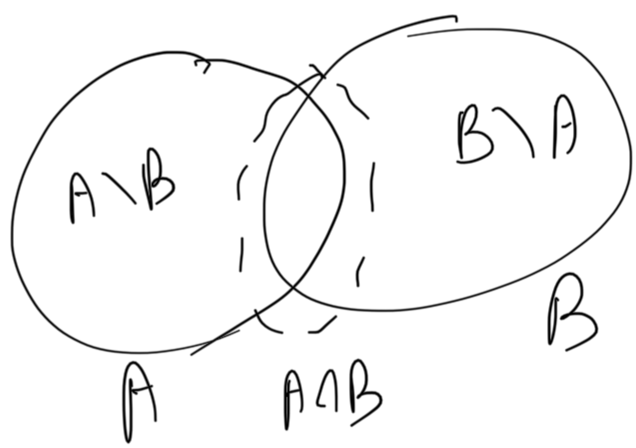
$G=(V,E)$

An unordered pair $\{A, B\}$ is called
$\quad\;\; \cap\!\mid \;\; \cap\!\mid$
$\quad\;\; V \;\;\; V$

a <u>separation</u> of G if $A \cup B = V$
& G has no edge between
$A \setminus B$ & $B \setminus A$.

SS

$A \cap B$ separates A from B



In this $|A \cap B|$ is called the <u>order</u> of separation.

We say that the seperation is <u>proper</u> if both $A \setminus B$ & $B \setminus A$ are <u>non-empty</u>.

Measures of connectivity of a graph.

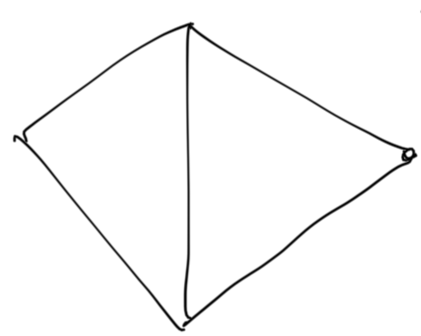
A graph $G=(V,E)$ is called <u>k-connected</u> if $|G|>k$ and $G-X$ is connected for every $X \subseteq V$ with $|X|<k$.

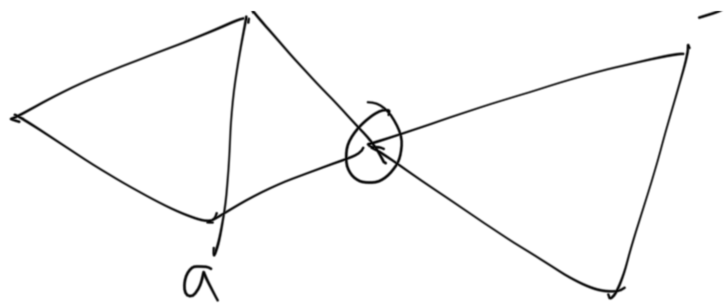
$$\Updownarrow$$

no two vertices in G are seperated by $<k$ other vertices.

1-connected = connected.

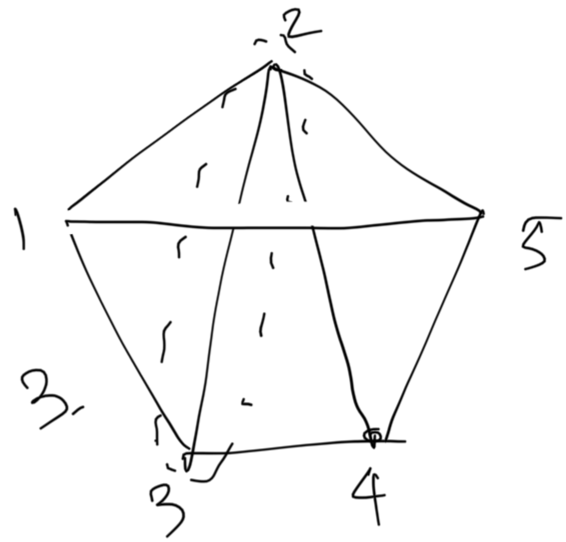
$\kappa(G)=2$

G  : 2-connected? yes.



$\kappa(G) = 1$

: not 2-connected



$\kappa(G) = 3.$

: 3-connected? yes?

The greatest integer k s.t. G is k-connected is called the connectivity $\kappa(G)$ of G.

If $\underset{|V|}{|G|} > 1$ & $G - F$ is connected for every $F \subseteq E$ with $|F| < l$ then we say that G

is l-edge-connected.

The greatest integer l such that G is l-connected is called the edge-connectivity $\underline{\underline{\lambda(G)}}$ of G.

Prop [Next class]

For any G, $\kappa(G) \leq \lambda(G) \leq \underset{\text{min-degree}}{\underset{\shortparallel}{\delta(G)}}$

[Try to prove this without looking at Diestel's book before the next class.]

——x——