Graph Theory Course

CMSC 27500

Meeting times:

Tuesday / Thursday:

1 p.m to 2.20 p.m.

Zoom info:

Meeting ID: 650 117 9646

Passcode: 082803

Instructor: Ketan Mulmuley

Office hour: Tuesday: 5 p.m.

6 p.m.

T.A : Goutham Rajendran

Tutorial : Wednesday : 5 to 6.pm

Office hour : [later]
info will be sent by
Goutham later.

Reference book for the course :
Diestel : Graph Theory.
I will send pdf of this book
in later email today.

The grading :
Will be based entirely on
homeworks .

There will be a homework every week.

The homework will be made available before Thursday class every week. [Gautham will send the homework to students by email].

The ~~b~~ solutions will be due before the Thursday class of the next week. They should be sent by email to Gautham directly.

A week later, Gautham will send by email graded homeworks & also homework (individually) solns all students

(Goutham will set up Gradescope website for this task & send students the link by email later).

Lecture 1

Basics:

Graph:

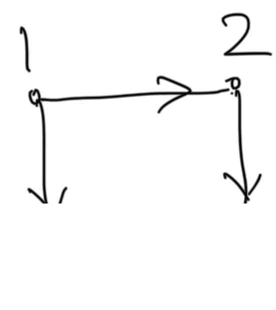
Directed graph:

$G = (V, E)$

V: set of vertices

E: the set of directed edges.

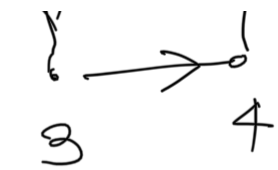
Example:

G: 

$V = \{1, 2, 3, 4\}$

not $(4,2)$

$E = \{(1,2), (2,4),$,



$(3,4)), (1,3)\}$

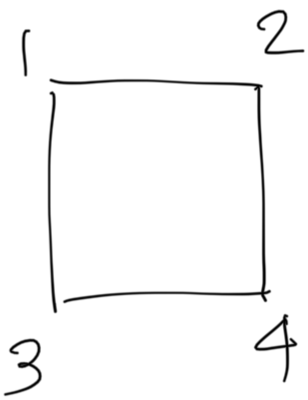
tail head.

Undirected graph

$G = (V, E)$ ← The set of undirected edges

The set of vertices

G :



$V = \{1, 2, 3, 4\}$

or $(2,1)$

$E = \{(1,2), (4,2), (4,3), (3,1)\}$.

The ordering does not matter.

$(1,2)$ & $(3,1)$ are adjacent.

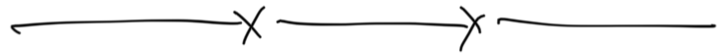
vertex 1 is adj to $(1,2)$.

- - 2 is also - - - - . . . .

Given a graph $G = (V, E)$, and $u, v \in V$, we say that u is <u>incident</u> to e if u is an endpoint of e.

We say two edges e_1 and $e_2 \in E$ are adjacent if they share an endpoint.

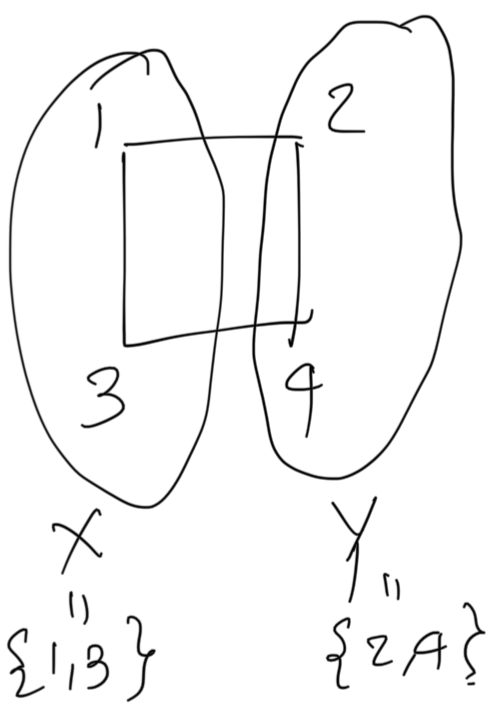
$G = (V, E)$



Given two subsets $X, Y \subseteq V$,

if $x \in X$, $y \in Y$ & $(x, y) \in E$,

then (x, y) is called an $X - Y$ edge.



then $(1,2)$ & $(3,4)$ are $X - Y$ edges.

$E(X, Y) = \{(1,2), (3,4)\}$

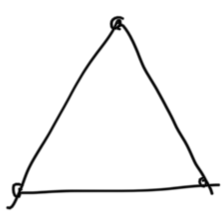
$E(X) = \{(1,2), (3,4)\}$

$E(X,Y) \subseteq E$: The set of X-Y edges in G (or E)

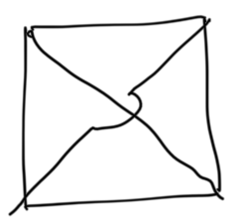
$U \subseteq V$:

$E(U)$: The set of all edges in E that are adjacent to vertices in U.

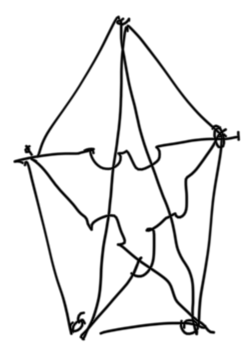
G (V,E) is <u>complete</u> if all pairs of distinct vertices in V are adjacent.



K_3



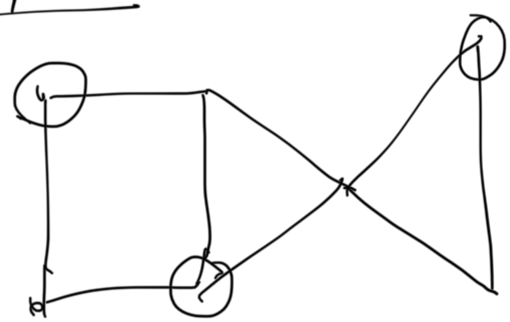
K_4



K_5

A set $\mathcal{U} \subseteq V$ of vertices in $G = (V, E)$ is called independent iff all pairs of vertices in $\mathcal{U}$ are non-adjacent.

<u>Example</u>:



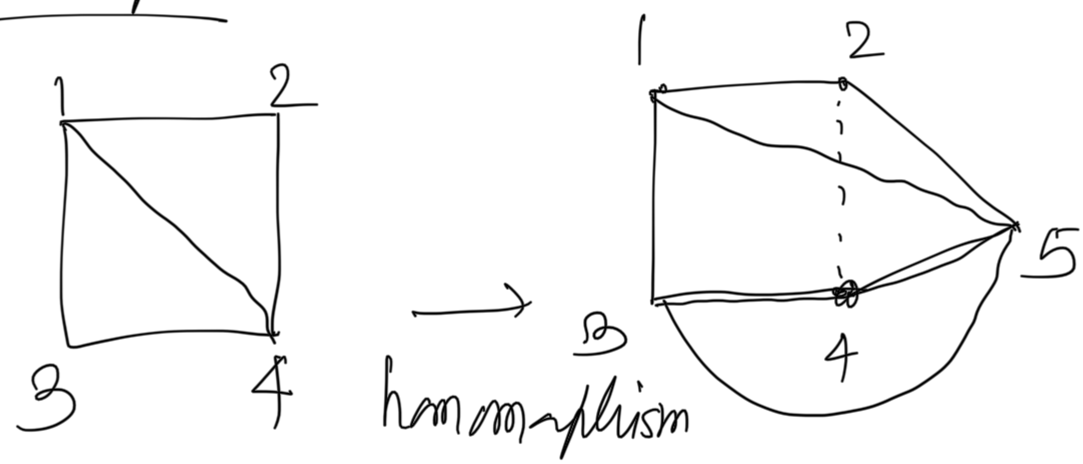
independent vertices

$G = (V, E)$, $G' = (V', E')$

A <u>homomorphism</u> from G to G' is a map $\varphi : V \longrightarrow V'$ which preserves adjacency of edges.

i.e. if $(v, w) \in E$ then $(\varphi(v), \varphi(w)) \in E'$.

Example:



$$\left.\begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 2 \\ 4 \longrightarrow 5 \\ 3 \longrightarrow 3 \end{array}\right\} \text{homomorphism.}$$

$$\left.\begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 2 \\ 3 \longrightarrow 3 \\ 4 \longrightarrow 4 \end{array}\right\} \text{homomorphism?}$$

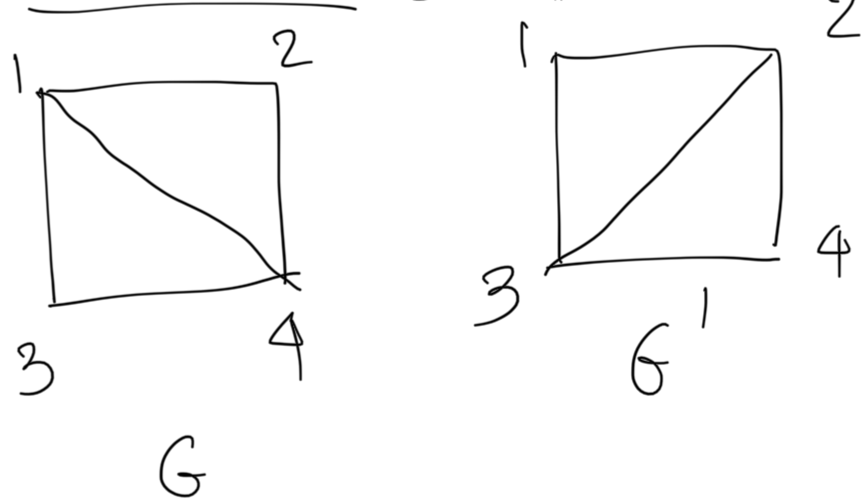
NO.

If the homomorphism $\varphi : V \longrightarrow V'$ is bijective & φ^{-1} is also a homomorphism then φ is called an isomorphism between

$G = (V, E)$ & $G' = (V', E')$.

If $G = G'$ then φ is called an automorphism.

Example: Isomorphic

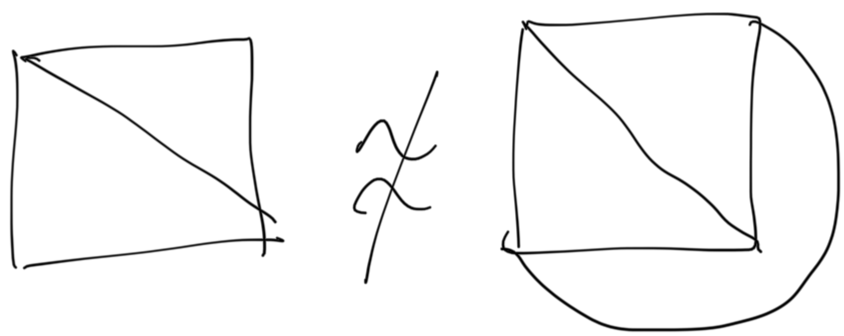


$\longrightarrow$ Isomorphism:

$$\left.\begin{array}{l} 1 \longrightarrow 2 \\ 2 \longrightarrow 4 \\ 4 \longrightarrow 3 \end{array}\right\} \text{Isomorphism.}$$

$3 \longrightarrow 1$

If there exists an isomorphism from G to G' then G & G' are called <u>isomorphic</u>

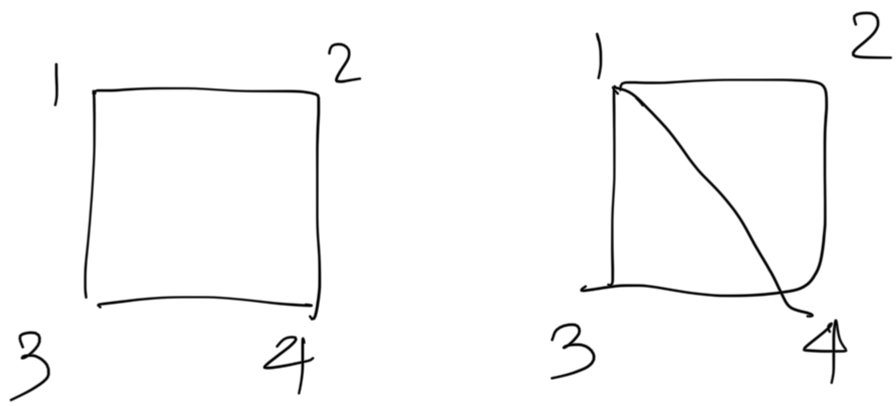


<u>Graph property:</u> Property of graphs that remains invariant under isomorphism.

Example of an homomorphism φ

whose inverse is not a homomorphism.

φ^{-1}



$$\varphi: \left.\begin{array}{l} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \\ 4 - 4 \end{array}\right\} \text{homomorphism}$$

but φ^{-1} is not a homomorphism.

A graph invariant:

A value on graphs that remains invariant under isomorphisms.

Example: 1) # edges

2) # vertices with a given degree

3) # edges in a maximum-size non-cyclic path.

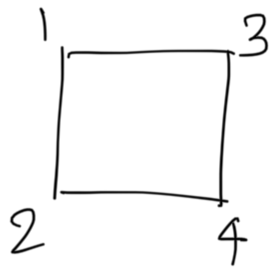
⋮

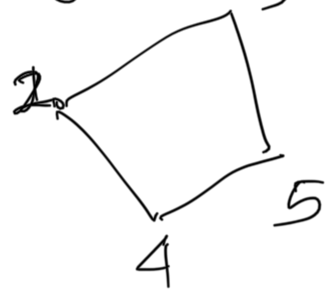
$G = (V, E)$ & $G' = (V', E')$

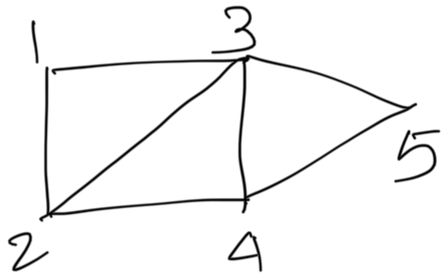
then we set $G \cup G' = (V \cup V', E \cup E')$

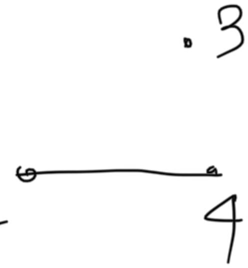
$G \cap G' = (V \cap V', E \cap E')$

Example:

G: 

G 

$G \cup G'$ 

$G \cap G'$ 

G is a subgraph of $G \cup G'$.

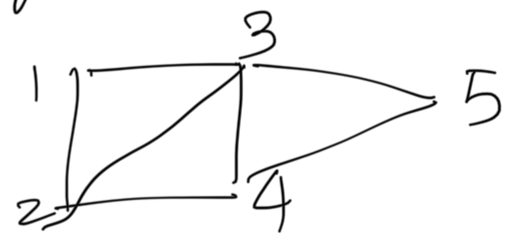
Given $G=(V,E)$ & $G'=(V',E')$
s.t. $V \subseteq V'$ & $E \subseteq E'$
we say that G is a <u>subgraph</u> of G'. (denoted $G \subseteq G'$) & G' is called supergraph of G. (denoted $G' \supseteq G$.)

<u>Proper subgraph</u>: $G \subsetneq G'$ (denoted)

if $G \subseteq G'$ & $G \neq G'$.

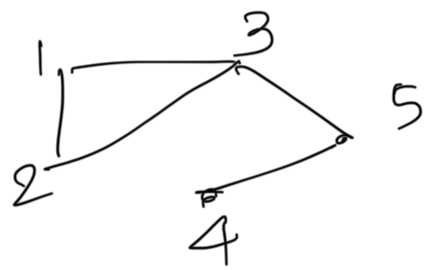
Example:

G'



$\cup|$ subgraph $G \subseteq G'$

G



$G := (V, E)$, $U \subseteq V$

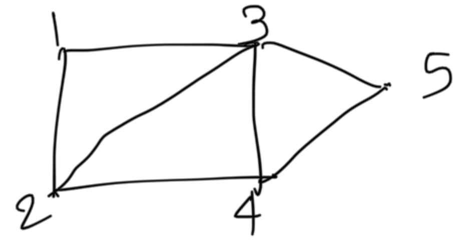
Then the induced graph

$$G[U] := (U, E')$$

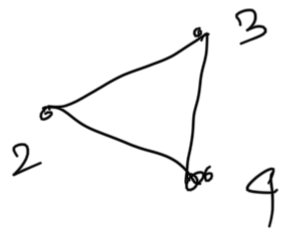
where $E' \subseteq E$ consists of precisely those edges in E whose both endpoints are in U.

Example:

G

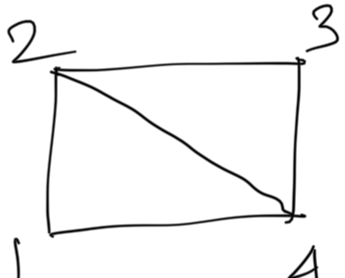


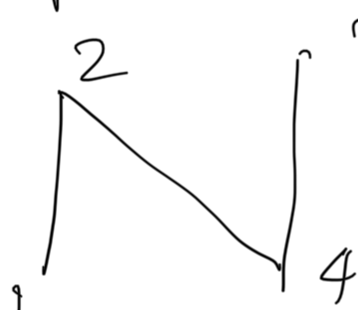
$U = \{2,3,4\}$

$G[U] =$ 

$\underset{G'}{\underset{\shortparallel}{(V', E')}}$ $\underset{G}{\underset{\shortparallel}{(V, E)}}$

$G' \subseteq G$ is called a <u>spanning subgraph</u> of G if $V' = V$.



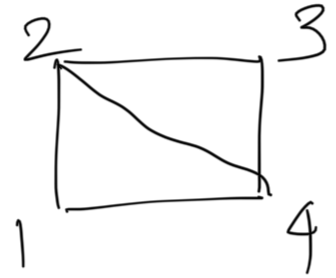


G $\quad$ G': spanning subgraph of G.

$G = (V, E)$ & $U \subseteq V$

then $G - U := G[V \setminus U]$

Example:



$U = \{1, 2\}$

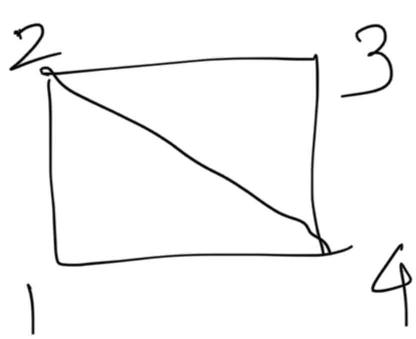
$G - U =$

singleton set

If $U = \{u\}$

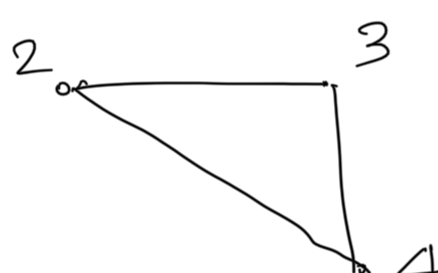
then $G - U$ is also denoted

as $G - u$.



$u = 1$.

$G - u$:



If $G = (V, E)$