# H1.

- The assignment is due at Gradescope on Tuesday, January 19 at 12:00 noon.
- You can either type your homework using LaTex or scan your handwritten work. We will provide a LaTex template for each homework. If you writing by hand, please fill in the solutions in this template, inserting additional sheets as necessary. This will facilitate the grading.
- You are permitted to study with up to 2 other students in the class and discuss the problems; however, *you must write up your own solutions, in your own words*. Do not submit anything you cannot explain. If you do collaborate with any of the other students on any problem, please do list all your collaborators in your submission for each problem.
- Similarly, please list any other source you have used for each problem, including other textbooks or websites. *Consulting problem solutions on the web is not allowed.*
- *Show your work.* Answers without justification will be given little credit.

PROBLEM 1 (25 POINTS) *Answer the questions below using the following preference lists:*

*Group A's preference lists (from most preferred to least preferred):*

$a_1$: $b_1$, $b_4$, $b_3$, $b_2$

$a_2$: $b_4$, $b_3$, $b_1$, $b_2$

$a_3$: $b_1$, $b_2$, $b_4$, $b_3$

$a_4$: $b_3$, $b_1$, $b_2$, $b_4$

*Group B's preference lists (from most preferred to least preferred):*

$b_1$: $a_4$, $a_2$, $a_3$, $a_1$

$b_2$: $a_1$, $a_4$, $a_2$, $a_3$

$b_3$: $a_3$, $a_1$, $a_4$, $a_2$

$b_4$: $a_4$, $a_3$, $a_1$, $a_2$

(a) *Run the Gale-Shapley algorithm with group A making the offers to obtain a stable matching. For your answer (and for your answer to part b as well), at each step please give the offer that is made and whether or not this offer is accepted. You should also give the final matching you obtain.*

(b) *Now run the Gale-Shapley algorithm with group B making the offers to obtain another stable matching. Which people are happier in this new stable matching (compared to the stable matching found in part a)?*

(c) *What other stable matching(s) are there, if any? Note: For full credit you should show that you have indeed found all of the possible stable matchings.*

**Solution:** Your solution goes here.

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# Problem 1)

a) Group A making offers

$a_1$ proposes to $b_1 \rightarrow$ accepted $a_1b_1$
$a_2$ proposes to $b_4 \rightarrow$ accepted $a_2b_4$
$a_3$ proposes to $b_1 \rightarrow$ accepted $a_3b_1$, $a_1$ snubbed
$a_1$ proposes to $b_4 \rightarrow$ accepted $a_1b_4$, $a_2$ snubbed
$a_2$ proposes to $b_3 \rightarrow$ accepted $a_2b_3$
$a_4$ proposes to $b_3 \rightarrow$ accepted $a_4b_3$, $a_2$ snubbed
$a_2$ proposes to $b_1 \rightarrow$ accepted $a_2b_1$, $a_3$ snubbed
$a_3$ proposes to $b_2 \rightarrow$ accepted $a_3b_2$

Final matching: $(a_1b_4, a_4b_3, a_2b_1, a_3b_2)$

b) Group B Making offers

$b_1$ proposes to $a_4 \rightarrow$ accepted $b_1a_4$
$b_2$ proposes to $a_1 \rightarrow$ accepted $b_2a_1$
$b_3$ proposes to $a_3 \rightarrow$ accepted $b_3a_3$
$b_4$ proposes to $a_4 \rightarrow b_4$ snubbed
$b_4$ proposes to $a_3 \rightarrow$ accepted $b_4a_3$, $b_3$ snubbed
$b_3$ proposes to $a_1 \rightarrow$ accepted $b_3a_1$, $b_2$ snubbed
$b_2$ proposes to $a_4 \rightarrow b_2$ snubbed
$b_2$ proposes to $a_2 \rightarrow$ accepted $b_2a_2$

Final matching: $(b_1a_4, b_4a_3, b_3a_1, b_2a_2)$
Since G-S matches each $b \in B$ with best(b), all of group B is happier with this stable matching

c) When you run Gale-Shapley, every member of the proposing group ends up with their best possible valid partner. Thus 8 pairings are invalid: $a_1b_1$, $a_2b_4$, $a_2b_3$, $a_3b_1$, $b_2a_1$, $b_2a_4$, $b_3a_3$, $b_4a_4$. This leaves 8 pairings that can produce 2 possible stable matchings:

First stable matching: ($a_1b_4$, $a_2b_1$, $a_4b_3$, $a_3b_2$)

Second stable matching: ($a_1b_3$, $a_3b_4$, $a_2b_2$, $a_4b_1$)

Problem 2) exersize 8 chapter 1

There exists a preference list for which there is a switch ($m >_w m'$ but listed $m' >_w m$) that would improve the partner of woman w.

Assume preference lists as follows: $M_2 = M''$, $M_1 = M$, $M_3 = M'$

$W_1: M_1\ M_2\ M_3$ $\qquad$ $M_1: W_3\ W_1\ W_2$

$W_2: M_1\ M_2\ M_3$ $\qquad$ $M_2: W_1\ W_3\ W_2$

True preferences $W_3: M_2\ M_1\ M_3$ $\qquad$ $M_3: W_3\ W_1\ W_2$

Fake preferences $W_3: M_2\ M_3\ M_1$

Executed With true preferences → ($M_1W_3$, $M_2W_1$, $M_3W_2$)

Executed with fake preferences →

$M_1$ proposes to $W_3 \to M_1W_3$

$M_2$ proposes to $W_1 \to M_2W_1$

$M_3$ proposes to $W_3 \to M_3W_3$, $M_1$ snub

$M_1$ proposes to $W_1 \to M_1W_1$, $M_2$ snub

$M_2$ proposes to $W_3 \to M_2W_3$, $M_3$ snub

$M_3$ proposes to $W_1 \to$ snub

$M_3$ proposes to $W_2 \to M_3W_2$

Final matching: ($M_1W_1$, $M_2W_3$, $M_3W_2$)

By lying about preferences a woman can get a better partner

PROBLEM 3 (30 POINTS) *Suppose you are given two distinct stable matchings $P$ and $Q$ of $n$ hospitals to $n$ students. Construct a new matching $R$ using the following rule: For each hospital $h$ who is matched to two different students $s$ and $s'$ in $P$ and $Q$, $h$ matches to its preferred student between $s$ and $s$ in $R$.*

1. *Show that $R$ is indeed a matching.*
2. *Show that $R$ is stable.*

**Solution:** Your solution goes here.

# Problem 3)

Given two distinct stable matchings $P$ and $Q$ of $n$ hospitals to $n$ students, $R$ is constructed by each hospital matching to their peferred student between $s$ in $P$ and $s'$ in $Q$.

1. It is not possible for 2 hospitals to have the same top choice student and be matched to the same student. If this happened the second matching would form an unstable pair in $P$ or $Q$. Thus all the hospitals are matched in $R$, and so are the students.

2. $R$ is stable because there is no hospital and student who prefer each other to their match. This pair would be an unstable matching in $P$ or $Q$.

PROBLEM 4 (15 POINTS) *Complete the following review exercises about big-O notation.*

(a) *Describe each of the following functions* $T : \mathbb{N} \to \mathbb{R}^+$ *using big O notation. For full credit, the big-O expression should be as simple as possible (to a reasonable observer). Note: For this problem it is sufficient to describe the upper bound for each of these functions. For example, if we had* $T(n) = 10000$*, we could say that* $T(n)$ *is* $\Theta(1)$ *but it is sufficient to say that* $T(n)$ *is* $O(1)$.

1. $T(n) = 5n\log_2(n) + 2n + 3$
2. $T(n) = 20n + n^2$
3. $T(n) = 100 + 2\sqrt{n} + 7\log_2(n)$
5. $T(n) = 10n^2 \cdot 4^n$

(b) *Prove that for functions* $f, g : \mathbb{N} \to \mathbb{R}^+$, $f(n)$ *is* $O(g(n))$ *if and only if*

$$\exists n_0, C : \forall n \geq n_0, \log(f(n)) \leq \log(g(n)) + C$$

(c) *Order the following bounds from smallest to largest:* $O(n^2)$, $O(2^n)$, $O(1)$, $O(\log_2(n))$, $O(n\log_2(n))$, $O(\sqrt{n})$, $O(n^{\log_2(n)})$, $O(n)$, $O(n^{\sqrt{n}})$, $O((\log_2(n))^{10})$. *[**Hint:** Use part (b)]*

# Problem 4)

a) 1. $T(n) = 5n\log_2(n) + 2n + 3 \longrightarrow O(n\log_2 n)$

2. $T(n) = 20n + n^2 \longrightarrow O(n^2)$

3. $T(n) = 100 + 2\sqrt{n} + 7\log_2(n) \longrightarrow O(\sqrt{n})$

4. $T(n) = 10n^2 \cdot 4^n$ $\quad \exists c, n_0 \;\; c>0, n_0>0$ s.t. $0 \le f(n) \le c \cdot g(n)$ $\forall n \ge n_0$

$\rightarrow T(n) \le c \cdot x^n$

$x > 4$

$\rightarrow O(5^n)$

b) By definition $f(n)$ is $O(g(n))$ if there exists constants $c>0$ and $n_0 > 0$ such that $0 \le f(n) \le c \cdot g(n)$ for all $n \ge n_0$

by taking the log of both sides

$\rightarrow \log f(n) \le \log(c \cdot g(n))$

$\rightarrow \log f(n) \le \log c + \log g(n)$

$\rightarrow \log f(n) \le \log g(n) + c'^{*}$ where $c' = \log c$

*if and only if proved by simplifying with $c = e^c$ back to definition

Thus $f(n)$ is $O(g(n))$ iff $\exists n_0, c: \forall n \ge n_0$
$\log f(n) \le \log g(n) + c$

c) 1. $O(1)$
2. $O(\log_2 n)$
3. $O((\log_2 n)^{10})$
4. $O(\sqrt{n})$
5. $O(n)$
6. $O(n\log_2 n)$
7. $O(n^2)$
8. $O(n^{\log_2 n})$
9. $O(n^{\sqrt{n}})$
10. $O(2^n)$