Spectral-Galerkin Method for n-Body Problem in 2D

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Considering 4n dimensional Hamiltonian $H(p,q) = \frac{1}{2}\sum_{i=1}^{n} m_i(p_{ix}^2 + p_{iy}^2) - \sum_{1\le i<j\le n} \frac{m_i m_j}{|q_i - q_j|}$, where $|q_i - q_j| = \sqrt{(q_{ix}-q_{jx})^2 + (q_{iy}-q_{jy})^2}$.

$$
\begin{cases}
\dot{p_{ix}} = \displaystyle\sum_{j\neq i} \frac{m_j(q_{jx}-q_{ix})}{|q_j - q_i|^3} \\
\dot{p_{iy}} = \displaystyle\sum_{j\neq i} \frac{m_j(q_{jy}-q_{iy})}{|q_j - q_i|^3} \\
\dot{q_{ix}} = m_i p_{ix} \\
\dot{q_{iy}} = m_i p_{iy}
\end{cases}
$$

1 Algorithm

1. Preprocess of the Equations
   (a) Zero initialization : Let $\tilde{p} = p - p(0)$, $\tilde{q} = q - q(0)$.
   (b) Normalization: Let $t = \frac{x+1}{2}, x \in [-1,1]$,
2. Weak Form of Hamiltonian Systems

   We introduce the following usual Sobolev space:
   $$H_E^1(I) := \{u : u \in H^1(I), u(-1) = 0\},$$
   where $I = [-1,1]$. Let $X_N = P_N \cap H_E^1(I)$.
3. Iterative Scheme
4. Efficient Implementation of the Algorithm
   We start by constructing a set of basis functions for X_N. Let
   $$\varphi_i(x) = L_i(x) - L_{i+2}(x), i = 0 \cdots N-2, \varphi_{N-1} = \frac{1}{2}(x+1), \tag{1}$$
   where L_i is the Legendre polynomial of degree i.
   It is clear that
   $$X_N = \text{span}\{\varphi_0(x), \cdots, \varphi_{N-2}(x), \varphi_{N-1}(x)\}.$$

Let $s_{jk} = \int_{-1}^{1} \varphi'_k \varphi_j dx$, $m_{jk} = \int_{-1}^{1} \varphi_k \varphi_j dx$, then it is known that
(1) when $k, j = 0, 1, \cdots, N-2$,

$$s_{jk} := \begin{cases} -2, & j = k+1, \\ 2, & j = k-1, \\ 0, & \text{other.} \end{cases} \quad m_{jk} = m_{kj} := \begin{cases} \frac{2}{2k+1} + \frac{2}{2k+5}, & j = k, \\ -\frac{2}{2k+5}, & j = k+2, \\ 0, & \text{other.} \end{cases}$$

(2) when $k = N-1, j = 0, 1, \cdots, N-2$,

$$s_{jk} = -s_{kj} =: \begin{cases} 1, & j = 0, \\ 0, & \text{other.} \end{cases} \quad m_{jk} = m_{kj} := \begin{cases} 1, & j = 0, \\ \frac{1}{3}, & j = 1, \\ 0, & \text{other.} \end{cases}$$

(3) when $k = j = N-1$, $s_{kj} = \frac{1}{2}, m_{kj} = \frac{2}{3}$.

Let $p_N^{k+1} = \sum_{i=0}^{N-1} p_i^{k+1} \varphi_i$ and $q_N^{k+1} = \sum_{i=0}^{N-1} q_i^{k+1} \varphi_i$ and plugging in iterative scheme , and taking $\tilde{p}_N, \tilde{q}_N$ through all the basis functions in X_N, respectively, we will arrive at the following matrix form system:

$$\mathcal{A} U^{k+1} = F^k, \tag{2}$$

In practice, we calculate the numerical integration by $P = 4N+1$ Gauss points and tolerate condition as discrete norm $\|u_N^{k+1} - u_N^k\|_{L_2}^2 > 10^{-14}$ where $u_N = (p_N, q_N)$

2 Numerical Result

2.1 3-Body

1. Equal-mass

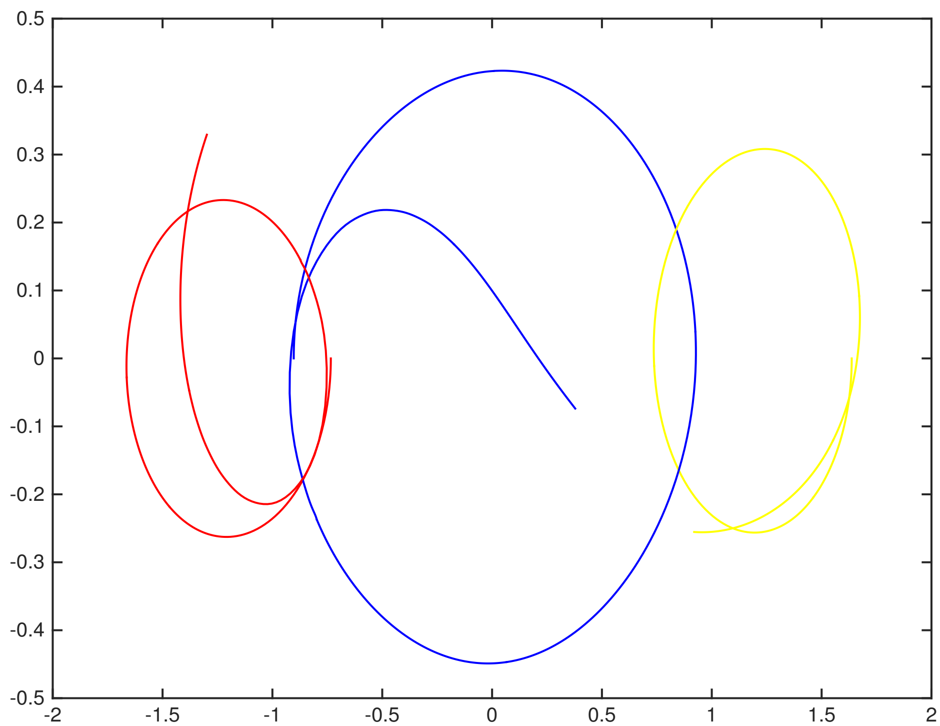

Figure 1: T=6 of 3-body Brocke-Hennon Orbit, error of Hamiltonian is 0.245 with $N = 20$

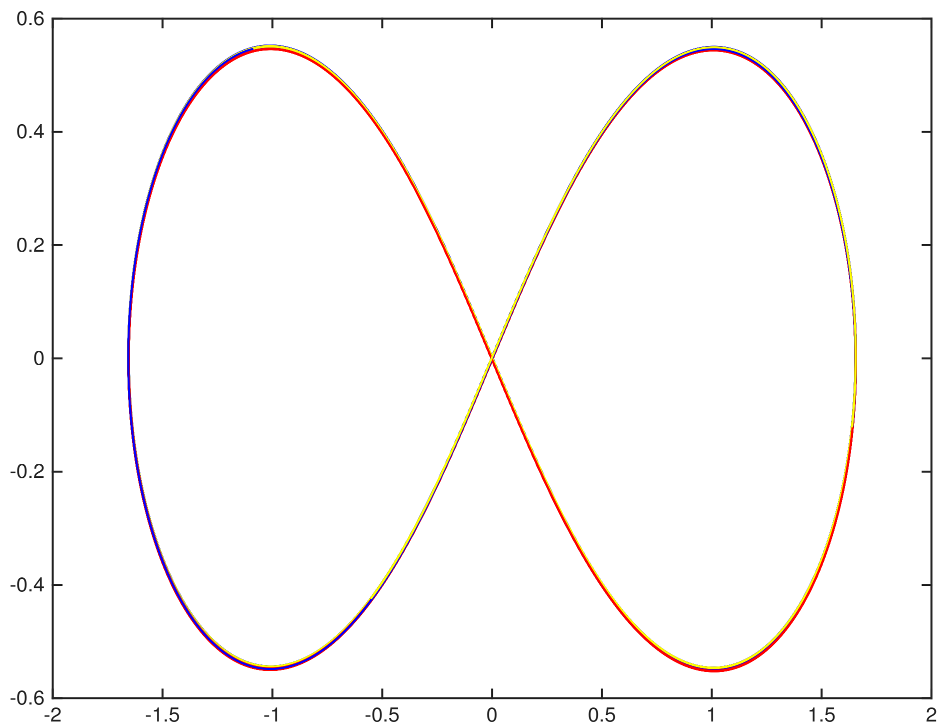

Figure 2: T=500 of 3-body Figure eight, error of Hamiltonian is 9.6916×10^{-12} with $N = 20$

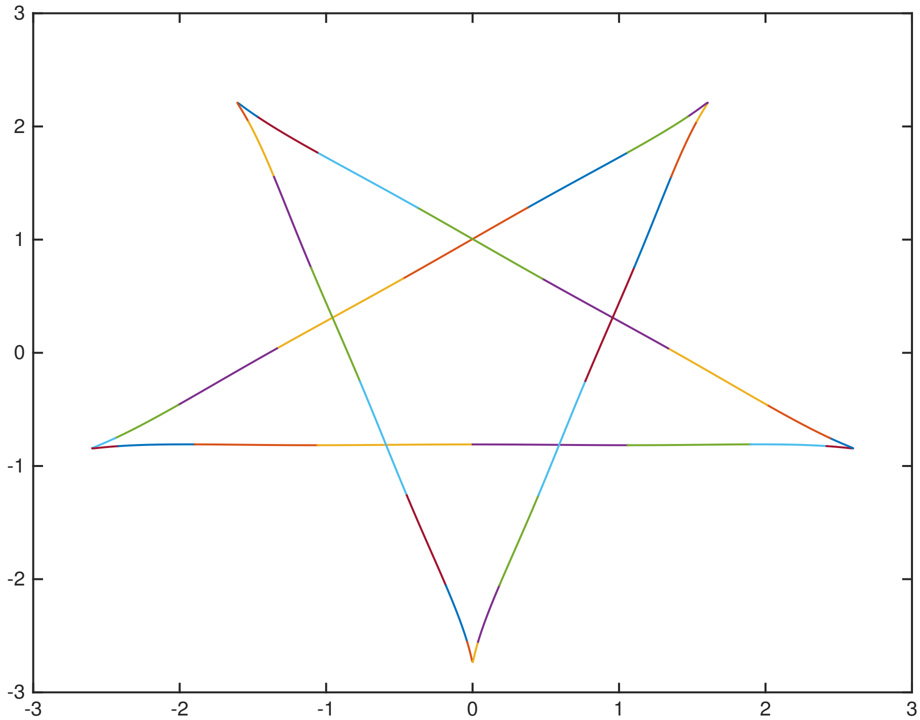

Figure 3: Minimum period T=40 of 1st body, error of Hamiltonian is 6.6613×10^{-16} with $N = 20$

2. Non-equal mass

   For $m_1 = 1$, $m_2 = 0.8$ and m_3=0.6

2.2 4-Body

Initial conditions given by [1] are

$$\begin{aligned} p_0 = [&-0.55391384867197 \quad -0.39895079845794 \quad 1.0936551555351 \quad 0 \\ &-0.55391558212647 \quad 0.39895379682134 \quad 0.01417427526245 \quad 0] \\ q_0 = [&1.0598738926379 \quad 1.7699901770118 \quad 0 \quad -0.80951135793043 \\ &-1.0598738926379 \quad 1.7699901770118 \quad 0 \quad -2.7304689960932] \end{aligned}$$

References

[1] Tiancheng Ouyang, Zhifu Xie, *Star pentagon and many stable choreographic solutions of the Newtonian 4-body problem*, Phyisca D.

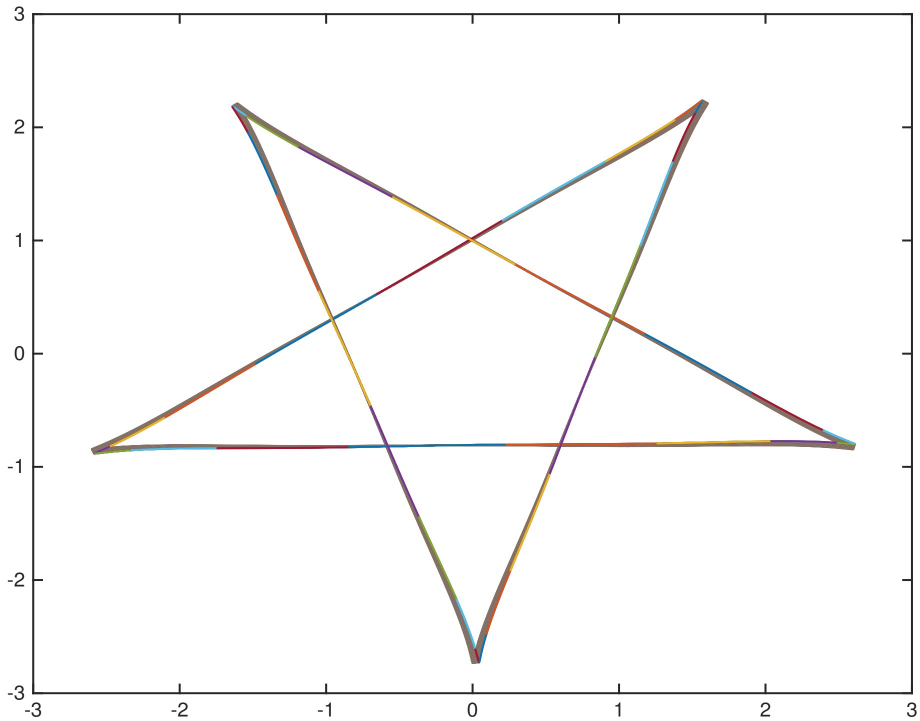

Figure 4: 100 periods T=4000 of 1st body, error of Hamiltonian is 4.4986×10^{-13} with $N = 20$

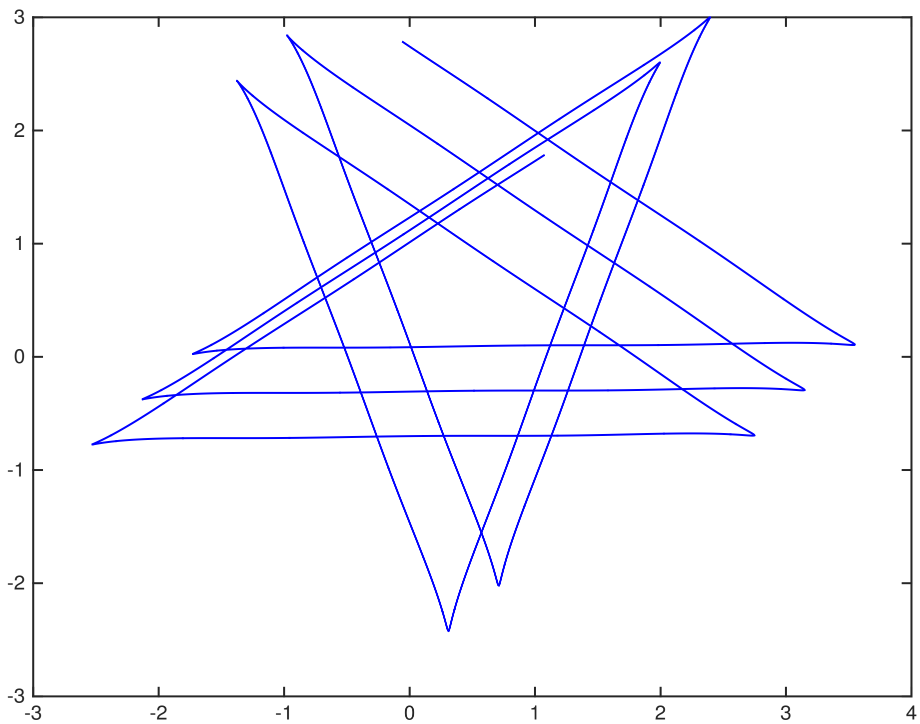

Figure 5: Perturbation of $p_0 + \epsilon$ and $q_0 + \epsilon$ with $\epsilon = 0.01$, T=100 of 1st body, error of Hamiltonian is 3.3307×10^{-15} with $N = 20$

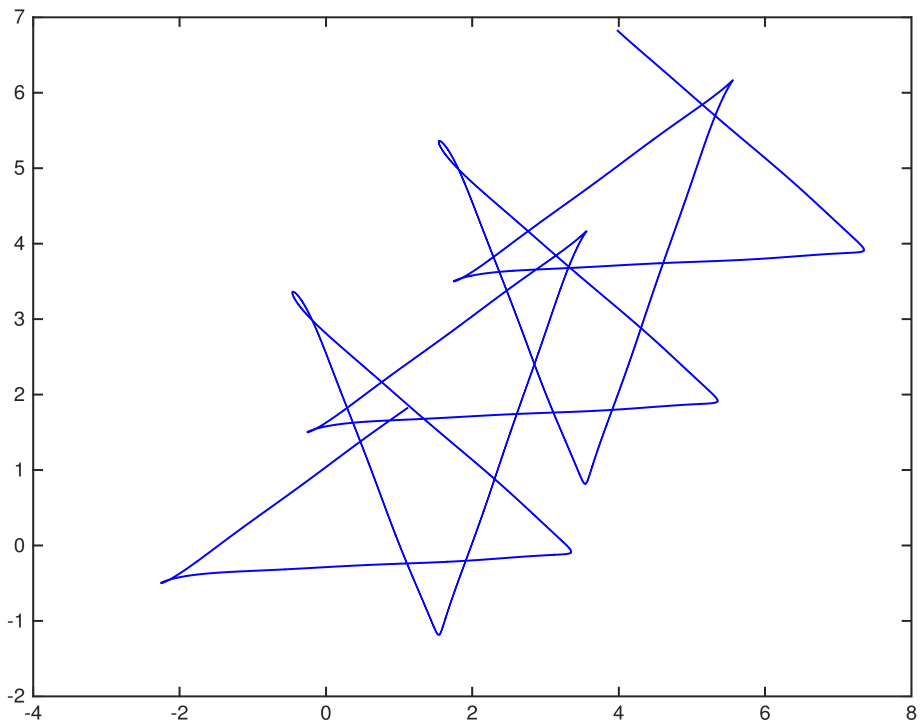

Figure 6: Perturbation of $p_0 + \epsilon$ and $q_0 + \epsilon$ with $\epsilon = 0.05$, T=100 of 1st body, error of Hamiltonian is 3.3307×10^{-15} with $N = 20$

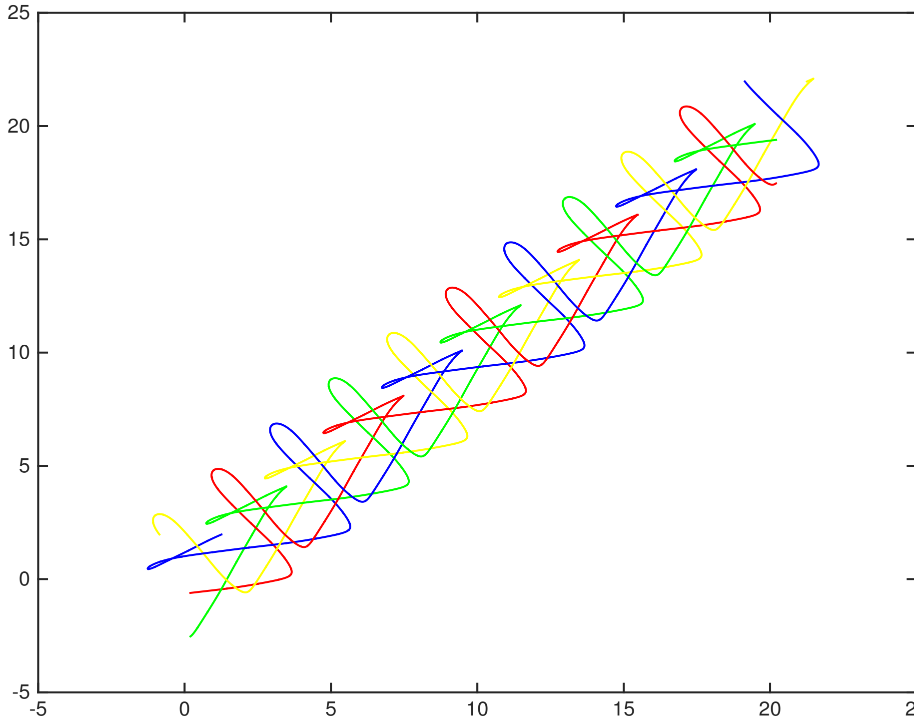

Figure 7: Perturbation of $p_0 + [0.2, 0.2]$ and $q_0 + [0.2, 0.2]$, T=100 of 4 bodies, error of Hamiltonian is 2.2204×10^{-15} with $N = 20$

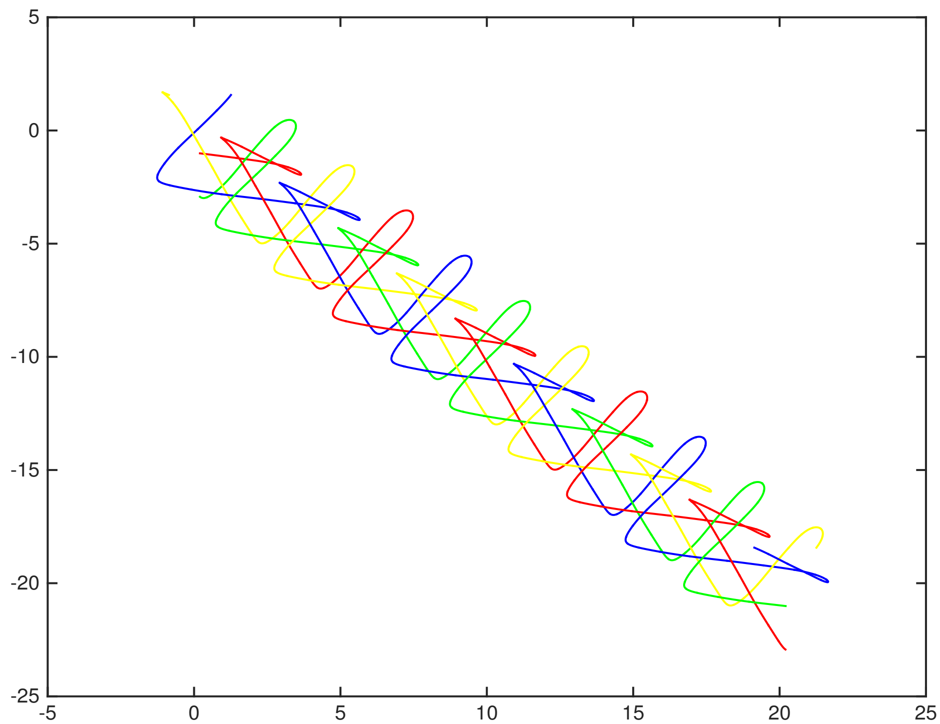

Figure 8: Perturbation of $p_0 + [0.2, -0.2]$ and $q_0 + [0.2, -0.2]$, T=100 of 4 bodies, error of Hamiltonian is 2.8866×10^{-15} with $N = 20$

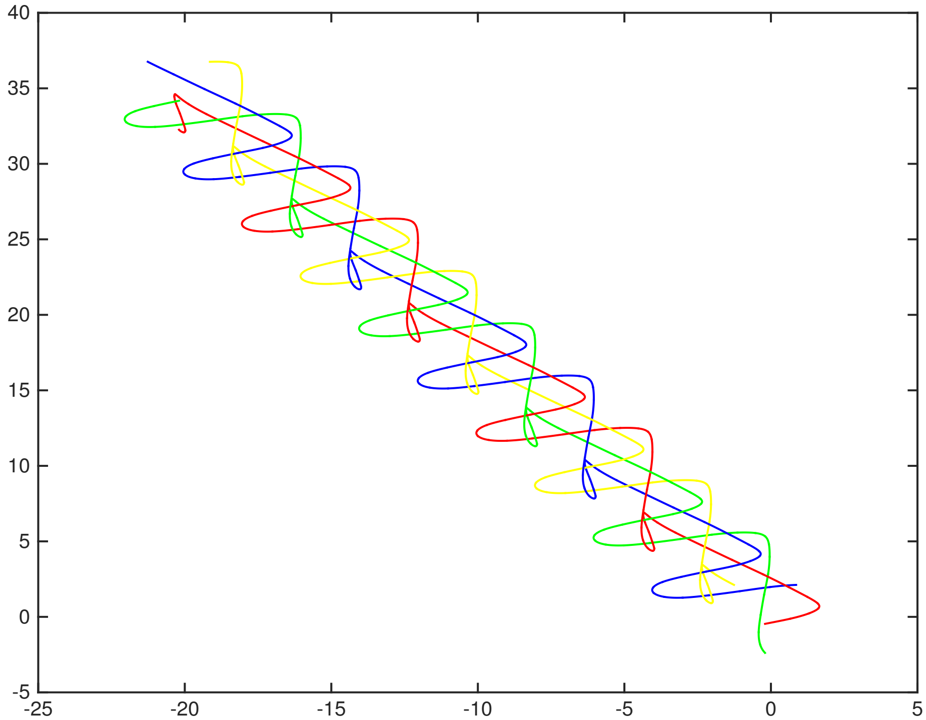

Figure 9: Perturbation of $p_0 + [-0.2, 0.2\sqrt{3}]$ and $q_0 + [-0.2, 0.2\sqrt{3}]$, T=100 of 4 bodies, error of Hamiltonian is 8.6597×10^{-15} with $N = 20$

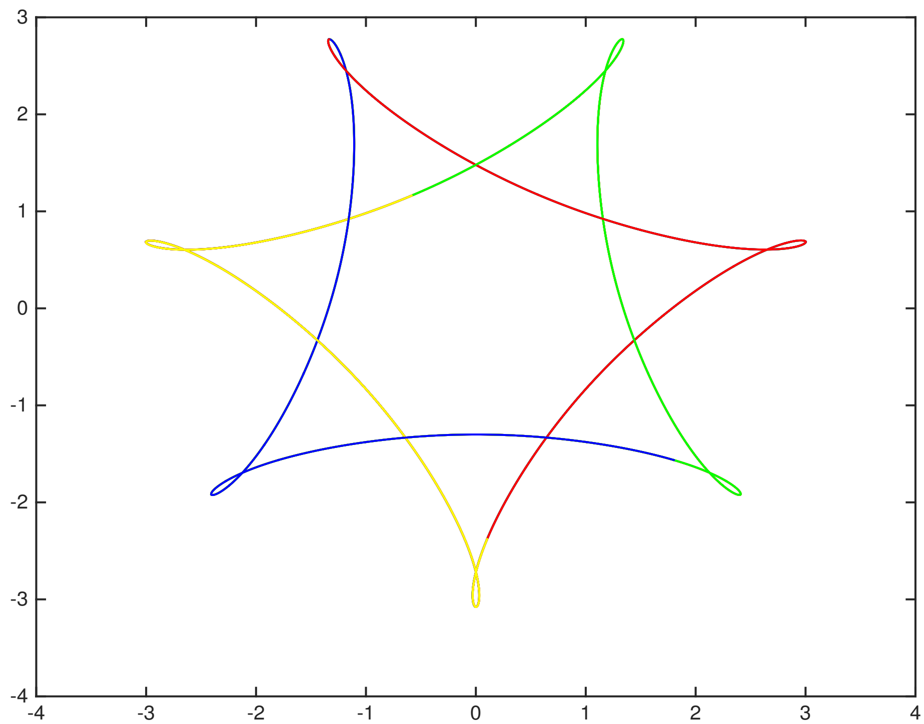

Figure 10: T=100 of simple choreographic of $\theta = \frac{3\pi}{7}$ in [1], 4 bodies chase each other, error of Hamiltonian is 2.6645×10^{-15} with $N = 20$

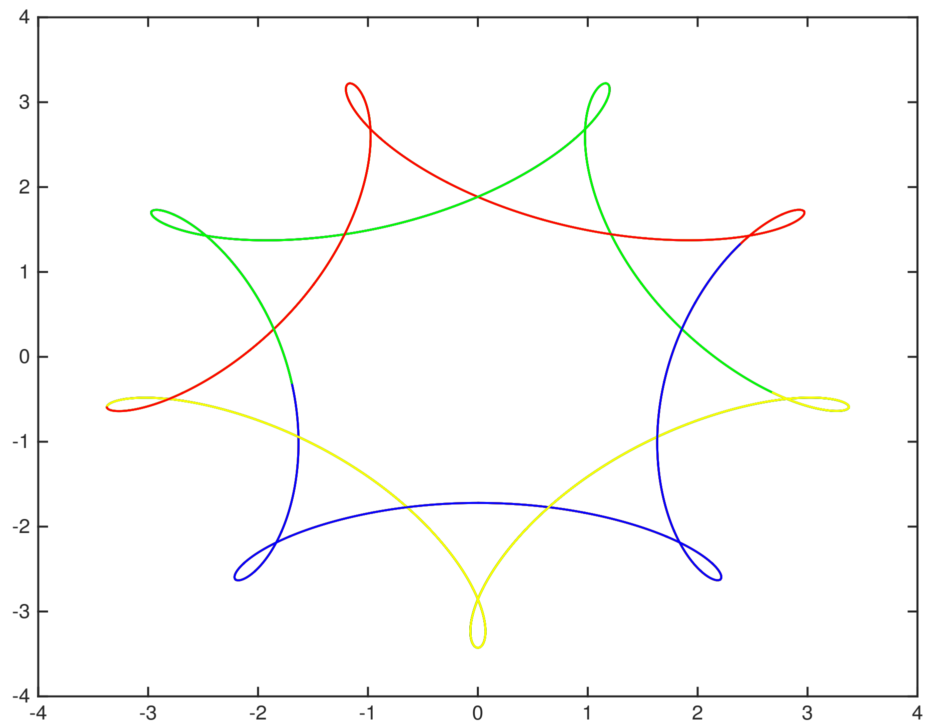

Figure 11: T=100 of simple choreographic of $\theta = \frac{4\pi}{9}$ in [1], 4 bodies chase each other, error of Hamiltonian is 1.9984×10^{-15} with $N = 20$

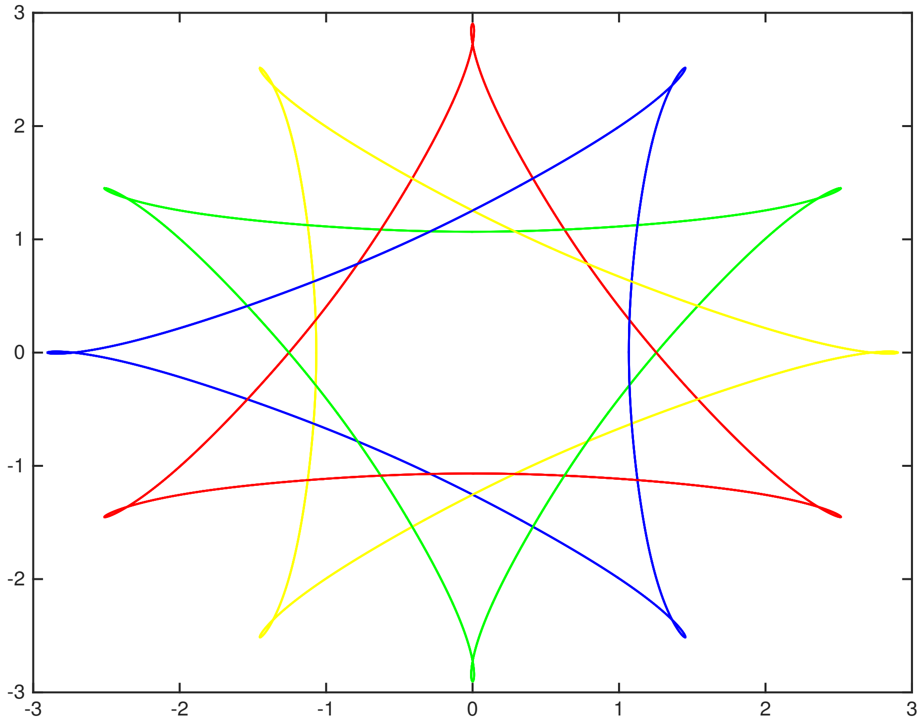

Figure 12: T=100 of non-choreographic periodic solutions of $\theta = \frac{5\pi}{12}$ in [1], error of Hamiltonian is 1.5543×10^{-15} with $N = 20$

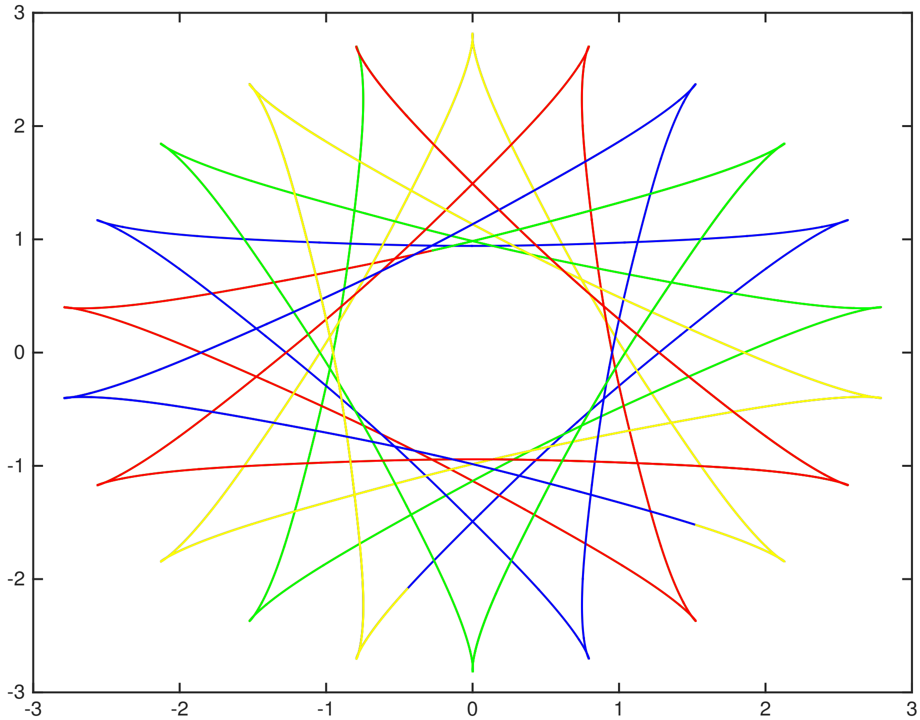

Figure 13: T=100 of double-choreographic periodic solutions of $\theta = \frac{5\pi}{22}$ in [1], error of Hamiltonian is 1.3323×10^{-15} with $N = 20$

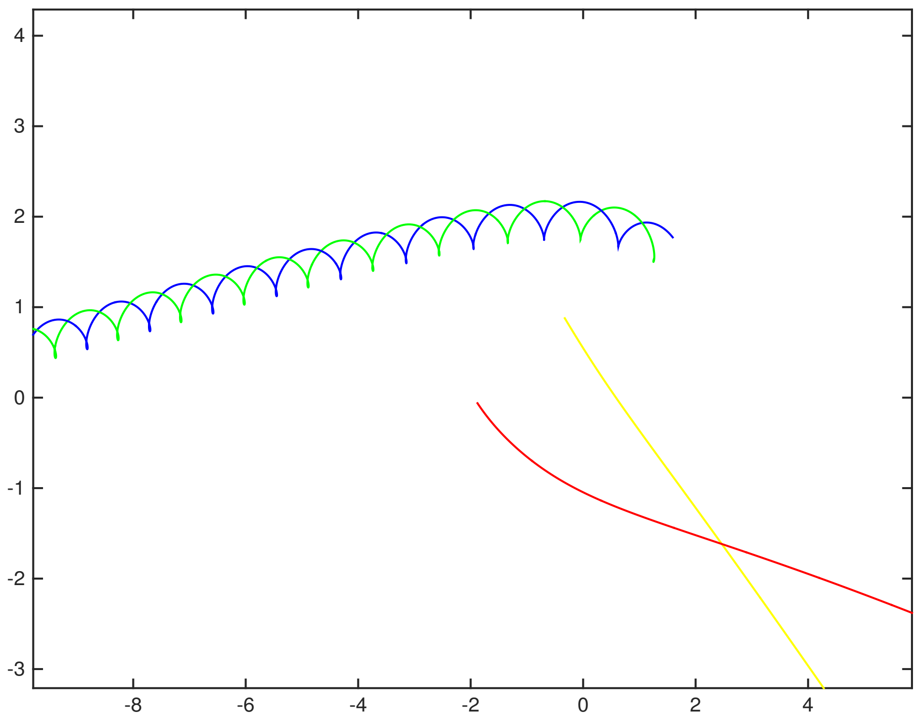

Figure 14: $p_0 = [-1.6115 \quad 1.2938 \quad 0.7793 \quad -0.7316 \quad 1.8009 \quad -1.8622 \quad -0.2450 \quad -0.4738]$, $q_0 = [1.5931 \quad 1.7712 \quad -1.8788 \quad -0.0614 \quad -0.3265 \quad 0.8779 \quad 1.2562 \quad 1.5281]$, error of Hamiltonian when T=100 is 1.088×10^{-8} with $N = 20$

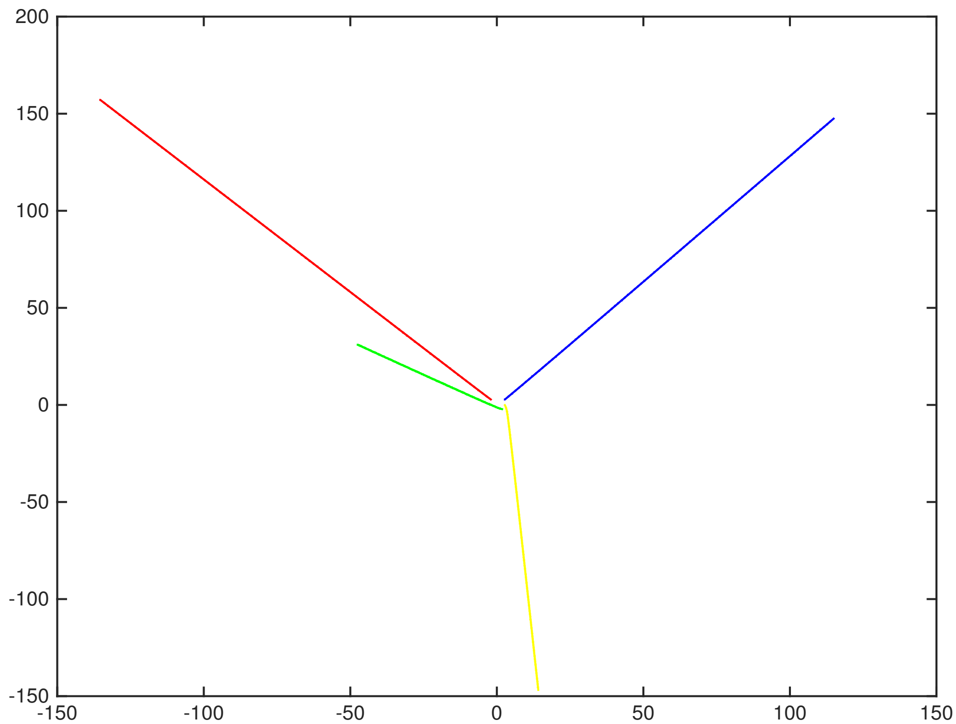

Figure 15: $p_0 = [1.2589 \quad 1.6232 \quad -1.4921 \quad 1.6535 \quad 0.5294 \quad -1.6098 \quad -0.8860 \quad 0.1875]$, $q_0 = [2.7450 \quad 2.7893 \quad -2.0543 \quad 2.8236 \quad 2.7430 \quad -0.0877 \quad 1.8017 \quad -2.1487]$, error of Hamiltonian when T=100 is 2.3093×10^{-13} with $N = 20$

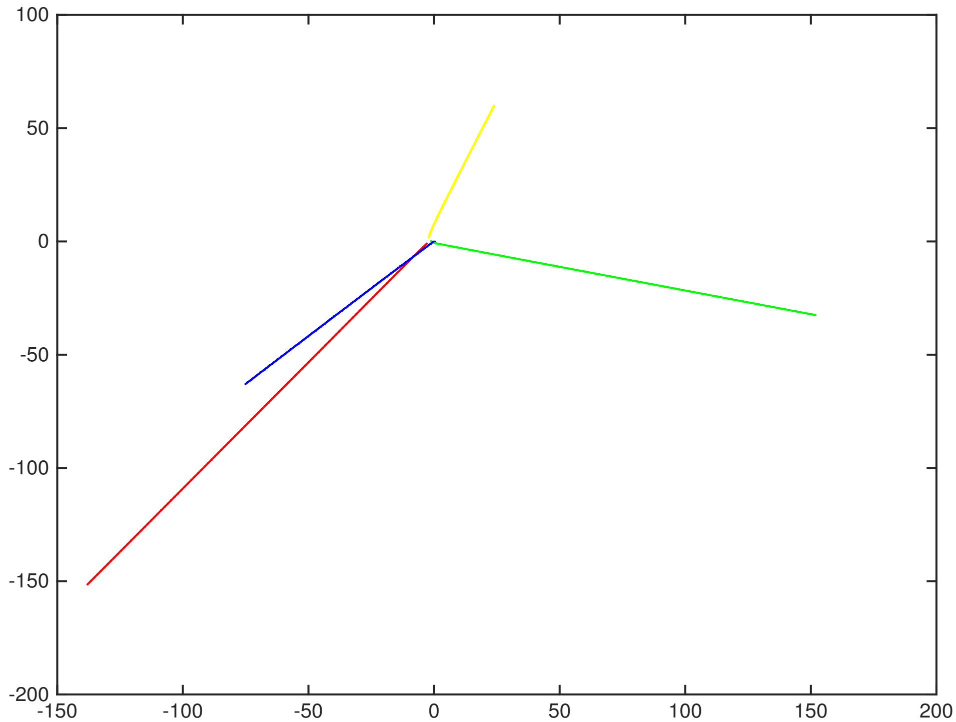

Figure 16: $p_0 = [-0.4782 \quad 0.2713 \quad -1.6966 \quad -1.7842 \quad 0.1232 \quad 1.1167 \quad 1.736 \quad -1.4804]$, $q_0 = [0.4129 \quad -0.1837 \quad -2.9286 \quad -0.9773 \quad -2.0269 \quad 1.7657 \quad -1.1327 \quad 0.1712]$, error of Hamiltonian when T=100 is 0.0012 with $N = 20$