Spectral Method for Henon-Heiles System

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1 Background

Consider the Heonon-Heiles System

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}({p_1}^2 + {p_2}^2 + {q_1}^2 + {q_2}^2) + {q_1}^2 q_2 - \frac{1}{3}{q_2}^3$$

The corresponding system of nonlinear ODE for this H is

$$\dot{p_1}(t) = -\frac{\partial H}{\partial q_1} = -q_1 - 2q_1 q_2$$

$$\dot{p_2}(t) = -\frac{\partial H}{\partial q_2} = -q_2 - {q_1}^2 + {q_2}^2$$

$$\dot{q_1}(t) = \frac{\partial H}{\partial p_1} = p_1$$

$$\dot{q_2}(t) = \frac{\partial H}{\partial p_1} = p_1$$

First we can do a little theoretic analysis about the system. There are 4 four equilibrium points for this system which are $E_1 = (0, 0, 0, 0)$, $E_2 = (0, 0, 0, 1)$, $E_3 = (0, 0, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{2}})$, $E_4 = (0, 0, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{2}})$

The Hessen Matrix of H is

$$D^2H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 + 2q_2 & 2q_1 \\ 0 & 0 & 2q_1 & 1 - 2q_2 \end{pmatrix}$$

Buy judging it is positive , negative or neither, we can know E_1 is a local maximal point and others are saddle points. With the help of graph of contour plot of energy H in [2] , we know there are 3 exits for the energy to escape according to the 3 saddle points. The total energy H=0 for E_1 and $H = \frac{1}{6}$ for E_1, E_2, E_3. If the initial energy is far beyond this H, the particles wander inside the region for a certain time in the scattering region until they cross one of the three energy line and escape to infinity. In other words, when the initial $H < \frac{1}{6}$, the solution is regular; when $H > \frac{1}{6}$, the solution is chaotic. Note that the time they spent in bounded region is named " escape time ". The higher the energy, the shorter escape times are found.

2 The Algorithm

We use Chebyshev-Guass-Lobatto Collocation Method to solve it. We solve the system on [0,1] first ,then use the obtained values (p(1),q(1)) as an initial condition to repeat the process on [1,2] and so on......

We map the interval [0,1] to [-1,1] through the coordinate transform: x=2t-1 and denote P(x)=p(t), Q(x)=q(t).

The transformed ODEs read

$$\dot{P}_1(t) = -\frac{1}{2}Q_1 - Q_1 Q_2$$

$$\dot{P}_2(t) = -\frac{1}{2}Q_2 - \frac{1}{2}{Q_1}^2 + \frac{1}{2}{Q_2}^2$$

$$\dot{Q}_1(t) = \frac{1}{2}P_1$$

$$\dot{Q}_2(t) = \frac{1}{2}P_1$$

Let $x_j = -\cos(\frac{j\pi}{N}) j = 0, 1, ..., N$. We interpolate P_1, P_2, Q_1, Q_2 as $P_{1N}(x) = \sum_{j=0}^{N} P_1(x_j)l_j(x)$,$P_{2N}(x) = \sum_{j=0}^{N} P_2(x_j)l_j(x)$,$Q_{1N}(x) = \sum_{j=0}^{N} Q_1(x_j)l_j(x)$,$Q_{2N}(x) = \sum_{j=0}^{N} Q_2(x_j)l_j(x)$,where l_ji the langrange nodal basis function.

We are seeking numerical approximations of $(P_1(x_j), P_2(x_j), Q_1(x_j), Q_2(x_j))$, denote as $(p_{1j}, p_{2j}, q_{1j}, q_{2j})$. The explicit form of the Chebyshev differentiation matrix$D = (d_{ij})_{i,j=0}^{N}$ is known in [1] with $d_{ij} = l'_j(x_i)$.But [1] uses $\cos(\frac{j\pi}{N})$ as its collocation points, we just need to add a negtive sign before D. Note that the rank of the $(N + 1) \times (N + 1)$ matrix D is N, we actually use the $D_N = (d_{ij})_{i,j=1}^{N}$.Therefore we solve the following system to obtain $\boldsymbol{p_1}$, $\boldsymbol{p_2}$,$\boldsymbol{q_1}$, $\boldsymbol{q_2}$

$$\boldsymbol{F(p,q)} = \begin{pmatrix} \boldsymbol{D_N} & \boldsymbol{0} & \frac{1}{2}\boldsymbol{I_N} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D_N} & \boldsymbol{0} & \frac{1}{2}\boldsymbol{I_N} \\ -\frac{1}{2}\boldsymbol{I_N} & \boldsymbol{0} & \boldsymbol{D_N} & \boldsymbol{0} \\ \boldsymbol{0} & -\frac{1}{2}\boldsymbol{I_N} & \boldsymbol{0} & \boldsymbol{D_N} \end{pmatrix} \begin{pmatrix} \boldsymbol{p_1} \\ \boldsymbol{p_2} \\ \boldsymbol{q_1} \\ \boldsymbol{q_2} \end{pmatrix} + \begin{pmatrix} \boldsymbol{q_1 q_2} \\ \frac{1}{2}\boldsymbol{q_1}^2 - \frac{1}{2}\boldsymbol{q_2}^2 \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{p_1 0 (d_{i0})_{i=1}^{N}} \\ \boldsymbol{p_2 0 (d_{i0})_{i=1}^{N}} \\ \boldsymbol{q_1 0 (d_{i0})_{i=1}^{N}} \\ \boldsymbol{q_2 0 (d_{i0})_{i=1}^{N}} \end{pmatrix} \quad (1)$$

$$\begin{aligned} &\boldsymbol{p_1} = (\boldsymbol{p_{11}, p_{12}, ..., p_{1N}})^{\boldsymbol{T}}, \boldsymbol{p_2} = (\boldsymbol{p_{21}, p_{22}, ..., p_{2N}})^{\boldsymbol{T}} \\ &\boldsymbol{q_1} = (\boldsymbol{q_{11}, q_{12}, ..., q_{1N}})^{\boldsymbol{T}}, \boldsymbol{q_2} = (\boldsymbol{q_{21}, q_{22}, ..., q_{2N}})^{\boldsymbol{T}} \\ &\boldsymbol{q_1 q_2} = (\boldsymbol{q_{11}q_{21}, q_{12}q_{22}, ..., q_{1N}q_{2N}})^{\boldsymbol{T}} \\ &\boldsymbol{q_1}^2 = (\boldsymbol{q_{11}}^2, \boldsymbol{q_{12}}^2, ..., \boldsymbol{q_{1N}}^2)^{\boldsymbol{T}}, \boldsymbol{q_2}^2 = (\boldsymbol{q_{21}}^2, \boldsymbol{q_{22}}^2, ..., \boldsymbol{q_{2N}}^2)^{\boldsymbol{T}} \end{aligned}$$

We choose Newton Iteration Method to solve this system. Since Newton Method is sensitive to initial value,we use the explicit symplectic scheme of order 2 in [2] as following to get initial value.

$$\boldsymbol{p_1}^{k+1} = \boldsymbol{p_1}^{k} - \frac{h}{2}(\boldsymbol{q_1}^{k} + 2\boldsymbol{q_1}^{k}\boldsymbol{q_2}^{k}) \tag{2}$$

$$\boldsymbol{p_2}^{k+1} = \boldsymbol{p_2}^{k} - \frac{h}{2}(\boldsymbol{q_2}^{k} + \boldsymbol{q_1}^{k^2} - \boldsymbol{q_2}^{k^2}) \tag{3}$$

$$\boldsymbol{q_1}^{k+1} = \boldsymbol{q_1}^{k} - \frac{h}{2}\boldsymbol{p_1}^{k+1} \tag{4}$$

$$\boldsymbol{q_2}^{k+1} = \boldsymbol{q_2}^{k} - \frac{h}{2}\boldsymbol{p_2}^{k+1} \tag{5}$$

Note that the distribution of collocation points in [-1,1] is dense near the end and sparse in the centre. So we need to add some points in it. We select $\boldsymbol{\delta = \frac{1}{N}}$, if distance between two adjacent points $\boldsymbol{h < \delta}$, we do not add points. Otherwise we add $[\frac{h}{\delta}]$ equal-distance points. We denote the Jacobi matrix of F is J, the Newton iteration process is as following:

While $\boldsymbol{norm(dx) > 10^{-14}}$

$\boldsymbol{dx = -J^{-1}F;}$

$\boldsymbol{x_{n+1} = x_n + dx;}$

3 Numerical Experiment Result

We select two different sets of initial conditions. The first set represents a regular case with initial condition

$\boldsymbol{p_1(0) = 0.011, p_2(0) = 0, q_1(0) = 0.013, q_2(0) = -0.4;}$In this case, $\boldsymbol{H_0 = 0.101410733 < \frac{1}{6}}$

The second set is a chaotic case with

$\boldsymbol{p_1(0) = \sqrt{2 \times 0.15925}, p_2(0) = q_1(0) = q_2(0) = 0.12}$; In this case, $\boldsymbol{H_0 = 0.182002 > \frac{1}{6}}$

Figure 1a represents phase plots of a regular solution on [0,200] by the spectral collocation N=20

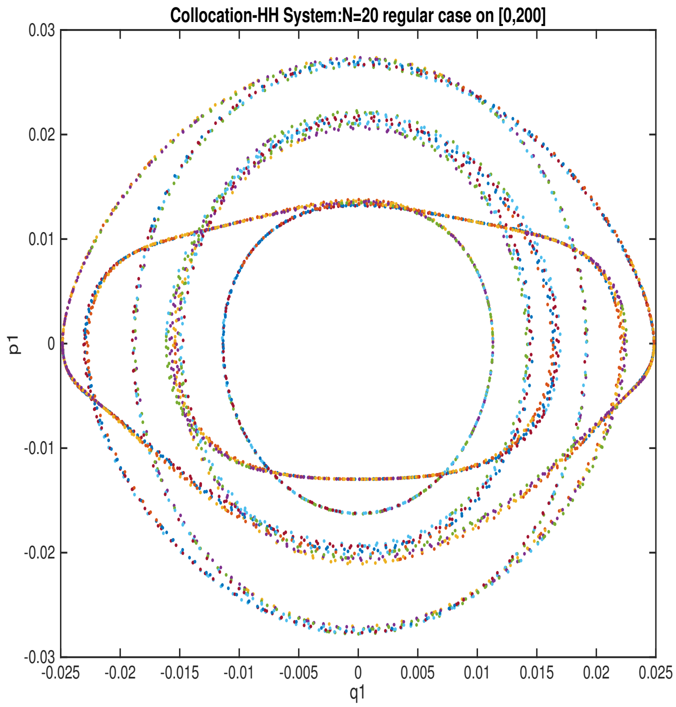


Figure 1b represents phase plots of a regular solution on [0,2000] by the spectral collocation N=20.

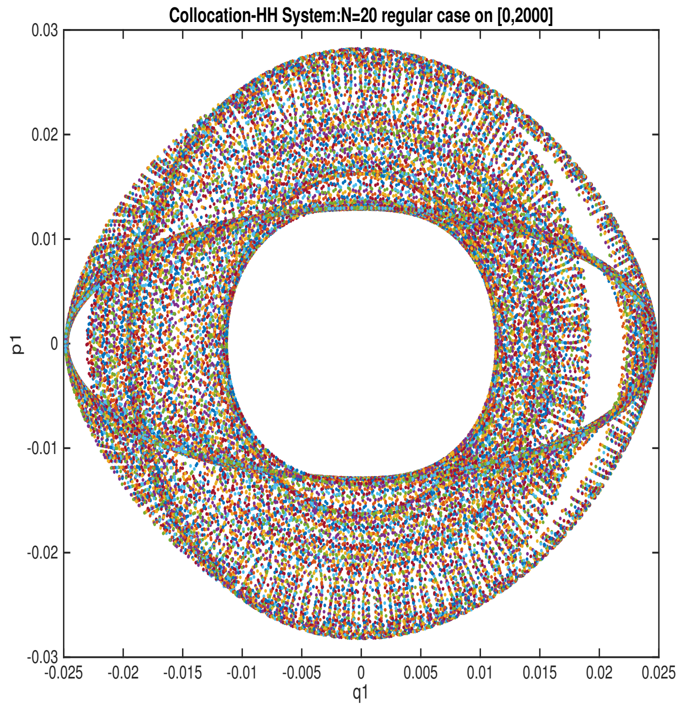
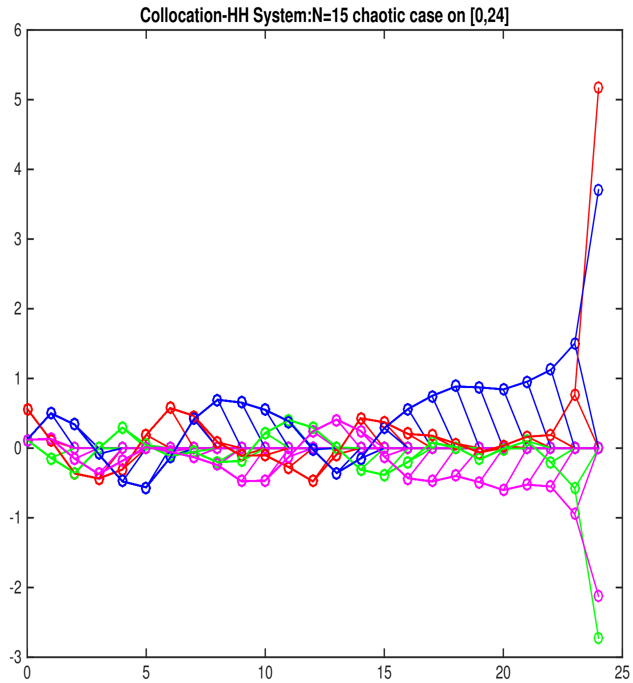
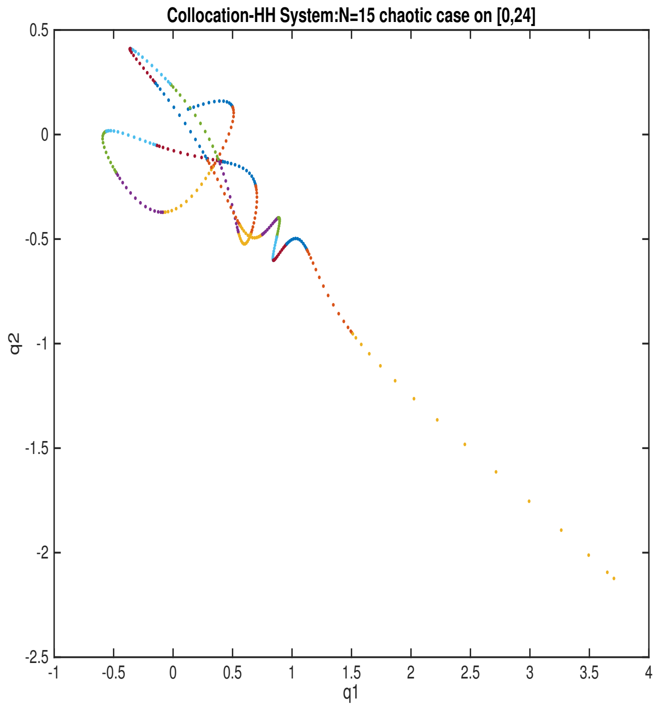


Figure 2 shows the chaotic solution by spectral collocation N=15 and the phase plot when the particle wanders in the bounded region until it crosses the energy threshold line and escapes. The first one is on [0,24] and the other one is phase plot $\boldsymbol{q_2}$ versus $\boldsymbol{q_1}$ on [0,24]. When T¿24, it will convergent very slowly.

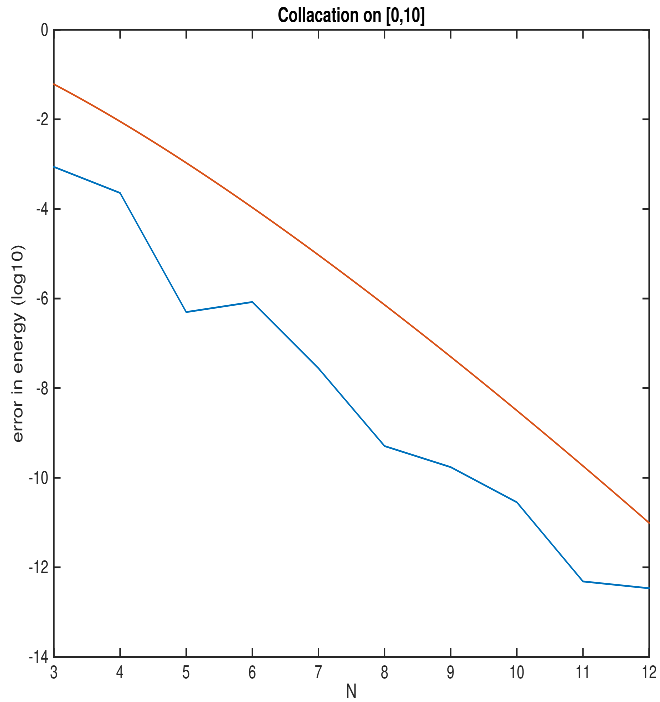

Collocation-HH System:N=15 chaotic case on [0,24]
6
5
4
3
2
1
0
-1
-2
-3
0
5
10
15
20
25



The error in energy H,and the CPU times are presented in the table. We choose initial conditions from the regular case when N=20. The CPU times used are much less than [2]. It seems that the energy loses as linear as losing $\mathbf{10^{-14}}$ every period t=1000.

T	time(secs)	Error in Energey
[0,1000]	3	$\mathbf{5.5345 \times 10^{-14}}$
[0,$\mathbf{10^4}$]	21	$\mathbf{5.6853 \times 10^{-13}}$
[0,$\mathbf{10^5}$]	232	$\mathbf{5.8277 \times 10^{-12}}$
[0,$\mathbf{10^6}$]	2234	$\mathbf{5.8806 \times 10^{-11}}$

The rate of convergence in energy on [0,10] using the regular initial values is shown in the following figure . Spectral Collocation gives the rate in the order of $\boldsymbol{(\frac{1}{N})^{0.85N}}$.



4 References

[1] Lloyd N.Trefethen, Spectral Methods in Matlab, Tsinghua University Press

[2] Nairat Kanyamee, Zhimin Zhang, Comparison of a Spectral Collocation Method and Symplectic Methods for Hamiltonian Systems, International Journal of Numerical Analysis and Modeling, Volume 8, No.1, Page 86-104, 2011 Institute for Scientific Computing and Information