Task 1

Consider a system with two single-particle energy levels, $\epsilon_1 = \epsilon$ and $\epsilon_2 = 2\epsilon$. Because they are fermionic levels, each can be occupied by $n_i \in \{0, 1\}$. The grand canonical partition function is defined as:

$$\Xi \;=\; \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \exp\Big[\beta\mu \,(n_1 + n_2)\Big] \exp\Big[-\beta\big(\epsilon\, n_1 + 2\epsilon\, n_2\big)\Big].$$

Factor out the terms associated with each level:

$$\Xi \;=\; \left(\sum_{n_1=0}^{1} \exp\big[\beta\mu\, n_1 - \beta\,\epsilon\, n_1\big]\right) \left(\sum_{n_2=0}^{1} \exp\big[\beta\mu\, n_2 - 2\beta\,\epsilon\, n_2\big]\right).$$

Let $z \equiv e^{\beta\mu}$. Note that for each level i, the sum over $n_i = 0, 1$ is:

$$\sum_{n_i=0}^{1} \big(z\, e^{-\beta\epsilon_i}\big)^{n_i} \;=\; 1 \;+\; z\, e^{-\beta\epsilon_i}.$$

Hence, for our 2-level system:

$$\Xi \;=\; \Big(1 + z\, e^{-\beta\epsilon}\Big)\Big(1 + z\, e^{-2\beta\epsilon}\Big).$$

Task 2

a

In the two level BEC case, each pair of n_0 and n_1 is considered to be a microstate, where n_0 represents the number of particle in the ground state and n_0 for first excited state.

b

We have two single-particle energy levels:

$$\text{Ground state energy: } E_0 = 0, \quad \text{First excited state: } E_1 = \epsilon.$$

We assume there are N particles and, in a classical (Maxwell–Boltzmann) treatment, the particles are *effectively distinguishable* for counting microstates.

Let n_1 be the number of particles in the excited state (with energy ϵ), so the total energy is

$$E(n_1) \;=\; n_1\,\epsilon.$$

Since $n_0 + n_1 = N$, we have $n_0 = N - n_1$ in the ground state (energy 0).

Even though the particles are identical bosons physically, in the classical (MB) counting we introduce a degeneracy factor for which n_1 of the N particles occupy the excited state. This factor is given by the binomial coefficient

$$\binom{N}{n_1} \;=\; \frac{N!}{n_1!\,(N-n_1)!}.$$

It counts the number of ways of choosing which n_1 of the N *distinguishable* particles are in the excited state.

In the canonical ensemble, the classical partition function Z_C is the sum over all possible $n_1 = 0, 1, 2, \ldots, N$ of the Boltzmann factor times the degeneracy:

$$Z_C \;=\; \sum_{n_1=0}^{N} \binom{N}{n_1}\, e^{-\beta\, E(n_1)} \;=\; \sum_{n_1=0}^{N} \binom{N}{n_1} e^{-\beta\, n_1\, \epsilon},$$

where $\beta = \frac{1}{k_B\, T}$. This sum is a straightforward binomial expansion:

$$Z_C \;=\; \left(1 + e^{-\beta\,\epsilon}\right)^N.$$

The probability of finding exactly n_1 particles in the excited state (and $n_0 = N - n_1$ in the ground state) is

$$P(n_1) \;=\; \frac{\binom{N}{n_1}\, e^{-\beta\, n_1\, \epsilon}}{Z_C} \;=\; \frac{\binom{N}{n_1}\, e^{-\beta\, n_1\, \epsilon}}{\left(1 + e^{-\beta\,\epsilon}\right)^N}.$$

c

We have a 2-level system with:

$$\text{Ground state energy: } 0,$$
$$\text{Excited state energy: } \epsilon.$$

There are N bosons, and in a *classical* (Maxwell–Boltzmann) approach, the canonical partition function is

$$Z_C = \left(1 + e^{-\beta\epsilon}\right)^N, \quad \text{where } \beta = \frac{1}{k_B T}.$$

If we let n_1 be the number of particles in the excited state, then $n_0 = N - n_1$ is the number of particles in the ground state.

The probability of having $n_1 = k$ particles in the excited state is

$$P(n_1 = k) \;=\; \frac{\binom{N}{k}\left(e^{-\beta\epsilon}\right)^k}{\left(1 + e^{-\beta\epsilon}\right)^N}.$$

(a) Average number in the excited state, $\langle n_\epsilon \rangle_C$ We define $n_\epsilon \equiv n_1$. Then

$$\langle n_\epsilon \rangle_C \;=\; \sum_{k=0}^{N} k\, P(n_1 = k).$$

Recognizing the binomial expansion identity $\sum_{k=0}^{N} k\binom{N}{k}x^k \;=\; N\,x\left(1+x\right)^{N-1}$, we set $x \;=\; e^{-\beta\epsilon}$. Therefore,

$$\langle n_\epsilon \rangle_C \;=\; \frac{1}{\left(1+x\right)^N}\sum_{k=0}^{N} k\binom{N}{k}x^k \;=\; \frac{N\,x\,(1+x)^{N-1}}{(1+x)^N} \;=\; \frac{N\,x}{1+x},$$

where $x = e^{-\beta\epsilon}$. Hence

$$\boxed{\langle n_\epsilon \rangle_C \;=\; \frac{N\,e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}}.}$$

(b) Average number in the ground state, $\langle n_0 \rangle_C$ Since $n_0 = N - n_1$, we have

$$\langle n_0 \rangle_C \;=\; N - \langle n_\epsilon \rangle_C \;=\; N \;-\; \frac{N\,e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}} \;=\; \frac{N}{1+e^{-\beta\epsilon}}.$$

Hence

$$\boxed{\langle n_0 \rangle_C \;=\; \frac{N}{1+e^{-\beta\epsilon}}.}$$

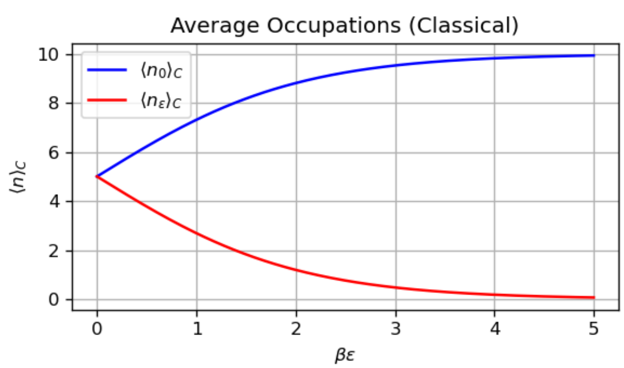

Figure 1: Schematic plot of $\langle n_0 \rangle_C$ and $\langle n_\epsilon \rangle_C$ *vs.* temperature (or equivalently vs. β). As $T \to 0$, $\langle n_0 \rangle_C \to N$ and $\langle n_\epsilon \rangle_C \to 0$. As $T \to \infty$, both approach $N/2$.

d

For N *indistinguishable* bosons in a 2-level system (energies 0 and ϵ), the allowed microstates are specified by the occupation numbers (n_0, n_1) such that $n_0 + n_1 \;=\; N$. Each distinct pair (n_0, n_1) corresponds to a unique many-body state with energy $E = n_1\,\epsilon$.

Partition function Z (no derivation shown):

$$Z \;=\; \sum_{n_1=0}^{N} e^{-\beta\,n_1\,\epsilon} \;=\; 1 + e^{-\beta\,\epsilon} + e^{-2\beta\,\epsilon} + \cdots + e^{-\beta\,N\,\epsilon} \;=\; \frac{1 - e^{-\beta\,(N+1)\,\epsilon}}{1 - e^{-\beta\,\epsilon}}.$$

Probability of microstate with $n_1 = n$ (hence $n_0 = N - n$) is

$$P(n_1 = n) \;=\; \frac{e^{-\beta\, n\, \epsilon}}{Z}.$$

e

For N indistinguishable bosons in a 2-level system with energies 0 and ϵ, the canonical partition function is

$$Z \;=\; \sum_{k=0}^{N} e^{-\beta k \epsilon}.$$

The probability of having k particles in the excited state is

$$P(k) \;=\; \frac{e^{-\beta k \epsilon}}{Z}.$$

Thus, the average number of particles in the excited state is

$$\langle n_\epsilon \rangle \;=\; \sum_{k=0}^{N} k \cdot P(k) \;=\; \frac{\sum_{k=0}^{N} k\, e^{-\beta k \epsilon}}{\sum_{k=0}^{N} e^{-\beta k \epsilon}},$$

and

$$\langle n_0 \rangle \;=\; N - \langle n_\epsilon \rangle.$$

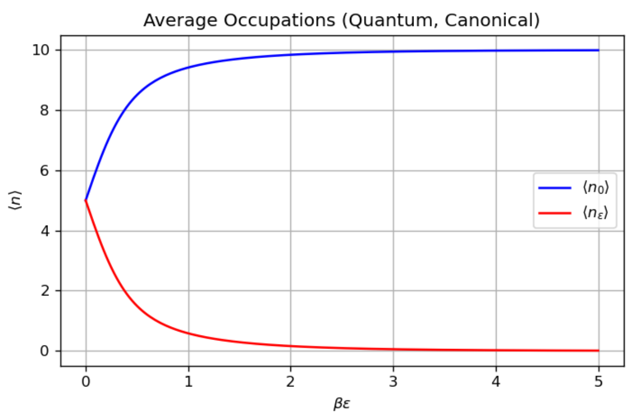

Figure 2: Plot of $langlen_0rangle$ (blue) and $langlen_{epsilon}rangle$ (red) vs. $betaepsilon$ for $N = 10$.

f

For a 2-level bosonic system with single-particle energies 0 and ϵ, the grand partition function Ω_G (sometimes denoted Ξ) is obtained by introducing the chemical potential μ via the fugacity $z = e^{\beta\mu}$. Each level can be occupied by $n = 0, 1, 2, \ldots$ bosons, so:

$$\Omega_G \;=\; \underbrace{\sum_{n=0}^{\infty} \left(z e^{-\beta \cdot 0}\right)^n}_{\text{ground state}} \times \underbrace{\sum_{m=0}^{\infty} \left(z e^{-\beta\epsilon}\right)^m}_{\text{excited state}} \;=\; \frac{1}{1-z}\,\frac{1}{1 - z\, e^{-\beta\epsilon}}.$$

The condition for the system to be normalizable (i.e. to ensure the sums converge) is that

$$e^{\beta\mu} < 1, \quad \text{which implies} \quad \mu < 0.$$

g

For a 2-level bosonic system (energies 0 and ϵ) with chemical potential μ, define the fugacity $z \equiv e^{\beta\mu}$. The grand partition function is

$$\Omega_G \;=\; \frac{1}{1-z}\,\frac{1}{1-z\,e^{-\beta\epsilon}}.$$

The *grand potential* is $\Phi = -k_B T\,\ln(\Omega_G)$, and by standard thermodynamic relations, the average total number of particles is

$$\langle N\rangle \;=\; k_B T\,\frac{\partial}{\partial\mu}\,\ln(\Omega_G).$$

Because the ground-state (energy 0) sector contributes a factor $\frac{1}{1-z}$, one finds that the **average number of particles in the ground state** is

$$\boxed{\langle n_0\rangle \;=\; \frac{z}{1-z} \;=\; \frac{e^{\beta\mu}}{1-e^{\beta\mu}},}$$

provided $|z| = e^{\beta\mu} < 1$ (i.e. $\mu < 0$ for convergence).

h

From the previous part, for a 2-level bosonic system (energies 0 and ϵ), the average *total* particle number in the grand canonical ensemble is

$$\langle N\rangle \;=\; \frac{z}{1-z} \;+\; \frac{z\,e^{-\beta\epsilon}}{1-z\,e^{-\beta\epsilon}}\,, \quad \text{where} \quad z = e^{\beta\mu}.$$

We impose $\langle N\rangle = 10^5$.

Final Numeric Example (Dimensionless Units) If we set $k_B T = \epsilon = 1$ (i.e. $\beta = 1$), then $x = e^{-\beta\epsilon} = e^{-1} \approx 0.3679$. Solving numerically for z such that $\langle N\rangle = 10^5$ gives

$$z \approx 0.99999,$$

and the average ground-state occupation is

$$\langle n_0\rangle \;=\; \frac{z}{1-z} \;\approx\; 9.999958\times 10^4,$$

while the excited state has

$$\langle n_\epsilon\rangle \;\approx\; 0.42,$$

so they sum to 10^5 as desired.