Task1

a

The Hamiltonian for a classical particle in a 2D harmonic oscillator potential is given by:

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2). \tag{1}$$

The density of states, $g(E)$, is obtained from the phase space volume $\Omega(E)$, the inequality $H \leq E$ defines a 4D hyper-ellipsoid:

$$\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) \leq E. \tag{2}$$

let

$$q_x = \sqrt{m\omega}x, \quad q_y = \sqrt{m\omega}y, \quad p_x = \sqrt{m}v_x, \quad p_y = \sqrt{m}v_y. \tag{3}$$

Rewriting the inequality:

$$\frac{v_x^2 + v_y^2}{2} + \frac{\omega^2}{2}(q_x^2 + q_y^2) \leq E. \tag{4}$$

This describes a 4D ball of radius:

$$R^2 = \frac{2E}{\omega^2}. \tag{5}$$

The volume of a 4D ball of radius R is:

$$V_4 = \frac{\pi^2}{2}R^4. \tag{6}$$

Thus, the phase space volume is:

$$\Omega(E) = \frac{\pi^2}{2}\left(\frac{2E}{\omega^2}\right)^2. \tag{7}$$

So the density of states is given by:

$$g(E) = \frac{d\Omega(E)}{dE}. \tag{8}$$

Differentiating:

$$\begin{aligned} g(E) &= \frac{d}{dE}\left[\frac{\pi^2}{2}\left(\frac{2E}{\omega^2}\right)^2\right] && (9) \\ &= \frac{4\pi^2}{\omega^2}E. && (10) \end{aligned}$$

b

Canonical Partition Function: We define

$$Z(\beta) \;=\; \int_0^\infty g(E)\, e^{-\beta E}\, dE,$$

where $\beta = 1/(k_B T)$ in usual thermodynamic notation.

Substitute $g(E)$ into the integral:

$$Z(\beta) \;=\; \int_0^\infty \left(4\,\pi^2\,\frac{E}{\omega^2}\right) e^{-\beta E}\, dE.$$

Factor out constants:

$$Z(\beta) \;=\; 4\,\pi^2\,\frac{1}{\omega^2}\int_0^\infty E\, e^{-\beta E}\, dE.$$

We now use the standard integral

$$\int_0^\infty E\, e^{-\beta E}\, dE \;=\; \frac{1}{\beta^2}, \quad (\text{for } \beta > 0).$$

Thus,

$$Z(\beta) \;=\; 4\,\pi^2\,\frac{1}{\omega^2}\,\frac{1}{\beta^2}.$$

Hence the classical partition function is

$$\boxed{Z(\beta) \;=\; \frac{4\,\pi^2}{\omega^2\,\beta^2}.}$$

C

1. The Hamiltonian and basic idea. We have

$$H \;=\; \frac{p_x^2+p_y^2}{2\,m} \;+\; \frac{1}{2}\,m\,\omega^2\,(x^2+y^2) \;+\; \lambda\,(x^2+y^2)^2 \quad (\hbar=1).$$

In classical statistical mechanics, the phase-space volume up to energy E is

$$\Omega(E) \;=\; \int_{H\,\le\,E} dx\,dy\,dp_x\,dp_y.$$

The density of states,

$$g(E) \;=\; \frac{d\Omega}{dE},$$

is the number of microstates per unit energy at energy E.

2. Polar coordinates in position space. Because the potential depends only on $r^2 = x^2+y^2$, it is natural to switch to polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad d^2x = r\,dr\,d\theta.$$

Define

$$U(r) \;=\; \frac{1}{2}\,m\,\omega^2\,r^2 \;+\; \lambda\,r^4.$$

Then, the condition $H \le E$ becomes

$$\frac{p_x^2+p_y^2}{2m} \;\le\; E - U(r).$$

For fixed r, the momentum variables (p_x, p_y) lie in a *2D disk* of radius

$$p_{\max}(r) \;=\; \sqrt{2m\,\bigl[E-U(r)\bigr]}\,, \quad \text{provided} \quad E \;\ge\; U(r).$$

Hence the area in momentum space at fixed r is

$$\pi\,\Bigl[p_{\max}(r)\Bigr]^2 \;=\; \pi\,\bigl[2m\,\bigl(E-U(r)\bigr)\bigr].$$

3. The full phase-space volume $\Omega(E)$. Integrate over $\theta\in[0,2\pi]$ and $r\in[0,r_{\max}]$, where $r_{\max}$ is determined by $U(r_{\max}) = E$ (i.e. the largest r for which $E \ge U(r)$).

$$\begin{aligned}
\Omega(E) &= \int_{\theta=0}^{2\pi}\int_{r=0}^{r_{\max}} \Bigl[\underbrace{r\,dr\,d\theta}_{\text{real-space measure}}\Bigr]\Bigl[\underbrace{\pi\,\bigl(2m\,[\,E-U(r)\,]\bigr)}_{\text{area in momentum}}\Bigr] \\
&= \int_0^{2\pi} d\theta \int_0^{r_{\max}} \pi\,2m\,r\,\bigl[E-U(r)\bigr]\,dr \\
&= 2\pi\,\bigl[\pi\,2m\bigr]\int_0^{r_{\max}} r\,\bigl[E-U(r)\bigr]\,dr \\
&= 4\,\pi^2\,m \int_0^{r_{\max}} r\,\Bigl[E-\bigl(\tfrac{1}{2}m\,\omega^2\,r^2+\lambda\,r^4\bigr)\Bigr]\,dr.
\end{aligned}$$

Here, $r_{\max}$ solves

$$U(r_{\max}) \;=\; \frac{1}{2}\,m\,\omega^2\,r_{\max}^2 \;+\; \lambda\,r_{\max}^4 \;=\; E.$$

4. Differentiate w.r.t. E to get $g(E)$. Define the integrand

$$f\big(r,E\big) \;=\; r\,\big[E - U(r)\big].$$

Then

$$\Omega(E) \;=\; 4\,\pi^2\,m \int_0^{r_{\max}(E)} f(r,E)\,dr, \quad \text{where } U\big(r_{\max}(E)\big) = E.$$

By the Leibniz rule,

$$\frac{d\Omega}{dE} \;=\; 4\,\pi^2\,m\Big[\int_0^{r_{\max}(E)} \frac{\partial}{\partial E}\,f(r,E)\,dr \;+\; f\big(r_{\max}(E),E\big)\,\frac{d}{dE}\,r_{\max}(E)\Big].$$

But

$$\frac{\partial}{\partial E}\,\big[r\,(E - U(r))\big] \;=\; r, \quad \text{and} \quad f\big(r_{\max}(E),E\big) \;=\; r_{\max}\big[E - U(r_{\max})\big] \;=\; r_{\max}\,\big[E - E\big] \;=\; 0.$$

Hence the boundary term vanishes, and the derivative inside is just r. Therefore,

$$g(E) \;=\; \frac{d\Omega}{dE} \;=\; 4\,\pi^2\,m \int_0^{r_{\max}(E)} r\,dr \;=\; 4\,\pi^2\,m\,\frac{r_{\max}^2(E)}{2} \;=\; 2\,\pi^2\,m\,\big[r_{\max}(E)\big]^2.$$

$$\boxed{g(E) \;=\; 2\,\pi^2\,m\,\big[r_{\max}(E)\big]^2,}$$

where $r_{\max}(E)$ is the positive root of

$$\frac{1}{2}\,m\,\omega^2\,r_{\max}^2 \;+\; \lambda\,\big[r_{\max}^2\big]^2 \;=\; E.$$

5. Solving for $r_{\max}$ and writing $g(E)$ explicitly. To make this more explicit, let $R = r_{\max}^2$. Then

$$\frac{1}{2}\,m\,\omega^2\,R \;+\; \lambda\,R^2 \;=\; E.$$

This is a *quadratic* in R:

$$\lambda\,R^2 \;+\; \big(\tfrac{1}{2}\,m\,\omega^2\big)\,R \;-\; E \;=\; 0.$$

Hence

$$R_{\max} \;=\; \frac{-\frac{1}{2}m\,\omega^2 \;\pm\; \sqrt{\big(\frac{1}{2}m\,\omega^2\big)^2 \;+\; 4\,\lambda\,E}}{2\,\lambda}.$$

We want the positive root, so

$$R_{\max} \;=\; \frac{-\frac{1}{2}m\,\omega^2 \;+\; \sqrt{\frac{1}{4}\,m^2\,\omega^4 \;+\; 4\,\lambda\,E}}{2\,\lambda}.$$

Thus,

$$r_{\max}^2(E) \;=\; R_{\max}(E) \;=\; \frac{-\,\frac{1}{2}m\,\omega^2 \;+\; \sqrt{\frac{1}{4}\,m^2\,\omega^4 + 4\,\lambda\,E}}{2\,\lambda}.$$

Plugging back into $g(E) = 2\pi^2 m\,r_{\max}^2(E)$ gives a fully closed-form expression:

$$g(E) \;=\; 2\,\pi^2\,m \;\cdot\; \frac{-\frac{1}{2}m\,\omega^2 \;+\; \sqrt{\frac{1}{4}\,m^2\,\omega^4 + 4\,\lambda\,E}}{2\,\lambda}.$$

A slightly cleaner way is to factor out $\frac{1}{4}\, m^2\, \omega^4$ under the square root. One obtains, for $E > 0$,

$$\boxed{g(E) \;=\; \frac{\pi^2\, m}{2\,\lambda}\left[\sqrt{m^2\,\omega^4 \;+\; 16\,\lambda\, E} \;-\; m\,\omega^2\right].}$$