Trajectory optimization using a novel barrier function

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1 Problem Statement

1.1 Objective

Formulate a barrier function for an autonomous vehicle which

- could describe the distances in latitude (y) and longitudinal (x) precisely
- deals with different direcions with different tolerance (allow the minimum allowable latitude distance smaller than the one in longitudinal direction)
- adjusts the different tolerances in different directions can be through changing the coefficient in the function
- is smooth and differentiable in both direction, so that an optimal trajectory can generated in the traffic flow with a safety distance held between other vehicles on the road

1.2 Assumptions

- Surrounding obstacles are circular and can be detected as x_O and y_O.
- $\mathbb{C}(S_i, t)$ is a point set of the 2-D contours of vehicle i based on their states at time t, consisting a rectangle.
- $d_{min,y}$ and $d_{min,x}$ are the minimum allowable distances with different setting in the direction of x and y, and it's not allowed to break them at the same time. They are given and fixed.

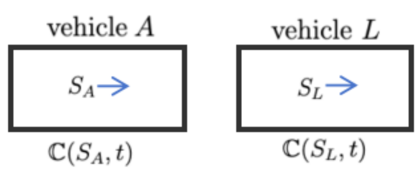

Figure 1: Get contour through $\mathbb{C}$ based on S

1.3 Inverse-proportional function

Find two points $\mathbf{m}, \mathbf{n}$ in the contours of vehicle A and obstacles respectively to minimize $(\|\mathbf{m} - \mathbf{n}\|)$. $d_i(S)$ is defined as the distance between these two closest points in two vehicles:

$$d_i(S) = \min_{\mathbf{m},\mathbf{n}}(\|\mathbf{m} - \mathbf{n}\|) \tag{1}$$

The constrain of dynamics function is $\dot{S}_t = f(S_t, u_t)$. The barrier function is defined as:

$$L(S, u, t) = \sum_i l_i(S, u, t) \tag{2}$$

with

$$l_i(S, u, t) = \begin{cases} \frac{1}{d_i(S)^2} - \frac{1}{d_{\min}^2}, & \text{if } d < d_{\min} \\ 0, & \text{if } d \geq d_{\min} \end{cases} \tag{3}$$

Vehicle A's trajectory optimal problem is defined to minimize a cost functional consisting of instantaneous cost and terminal cost

$$J(x, u) = \int_{T_0}^{T_f} L(S(t), u(t), t)\, dt + \Phi(x(T_f)) \tag{4}$$

1.4 Issues Anslysis

Let $\mathbb{O}$ denotes the set of $\mathbb{C}(O)$ for obstacles, and $\mathbb{A}$ denotes the set of $\mathbb{C}(S_A, t)$. The collision avoidance condition can be defined as

$$\mathbb{O}(t) \cap \mathbb{A}(t) = \emptyset \tag{5}$$

What issues can be caused?

- Single point distance can't describe the latitude and longitudinal distances precisely. For example, there exists such contour sets $\mathbb{O}$ and $\mathbb{A}$ that keep $\frac{d(x_O - S_{A,x})}{dt} = 0$, while $\Delta y = y_O - S_{A,y}$ can change arbitrarily..
- The defined d_i can be the distance in different directions, then in $L(S, u, t)$ the distances in latitude or longitudinal are calculated without difference.
- This definition of the barrier function can cause non-optimal control which is stuck in the local minimum.

2 Solution

2.1 A novel function

Define a barrier function $f_{barrier}(m, n)$, where m, n are variables of the function and K, k_1, k_2, a, b are adjustable coefficients and a, b are even numbers.

$$f_{barrier}(m, n) = K \frac{k_1}{e^{m^a} + k_1} \left(1 - \frac{k_2}{e^{-n^b} + k_2} \right). \tag{6}$$

If take $K = 100, k_1 = 1000, k_2 = 0.001, a = 4, b = 2$ (To make the function as a hard barrier, K shoud be as close as to infinity, $K = 100$ is set for easier computing), we can get the following surface plot which is continuous for all $m, n \in \mathbb{R}$.

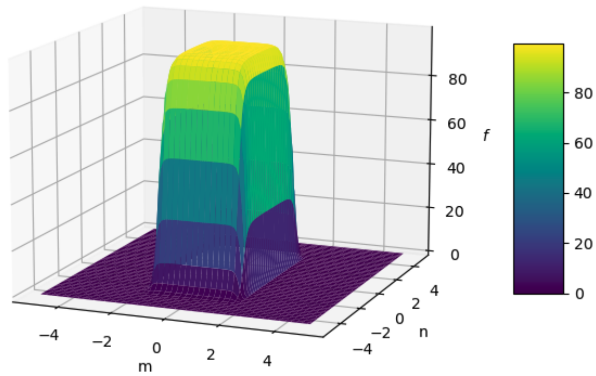

Figure 2: Surface of $f_{barrier}$ with $K = 100, k_1 = 1000, k_2 = 0.001, a = 4, b = 2$

To analyze the characteristics of the function in respect to variable m, n, firstly calculate its partial derivative on m, n:

$$\frac{\partial f}{\partial m} = K\left(1 - \frac{k_1}{e^{-n^b} + k_1}\right)\left(-\frac{k_2 a m^{a-1} e^{m^a}}{(e^{m^a} + k_2)^2}\right) \tag{7}$$

$$\frac{\partial f}{\partial n} = K\frac{k_2}{e^{m^a} + k_2}\left(-\frac{k_1 b n^{b-1} e^{-n^b}}{(e^{-n^b} + k_1)^2}\right) \tag{8}$$

The surface plots for $\frac{\partial f}{\partial m}$ and $\frac{\partial f}{\partial n}$ are:

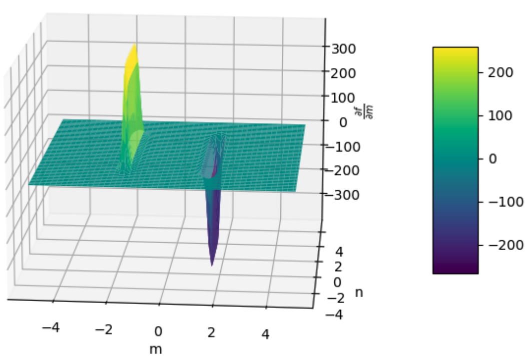

Figure 3: Surface of $\frac{\partial f}{\partial m}$

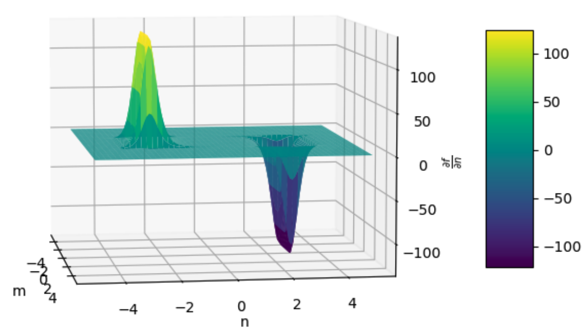

Figure 4: Surface of $\frac{\partial f}{\partial n}$

In the same way, calculate the second-order partial derivatives $\frac{\partial^2 f}{\partial m^2}$ and $\frac{\partial^2 f}{\partial n^2}$, it's continuous on m, n. We can get plots:

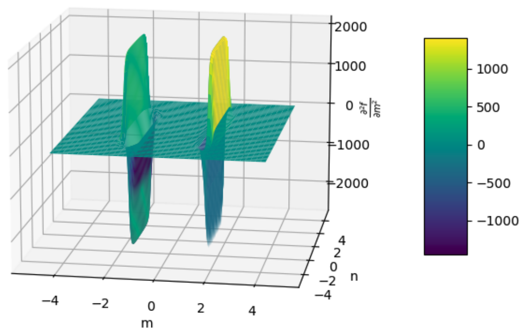

Figure 5: Surface of $\frac{\partial^2 f}{\partial m^2}$

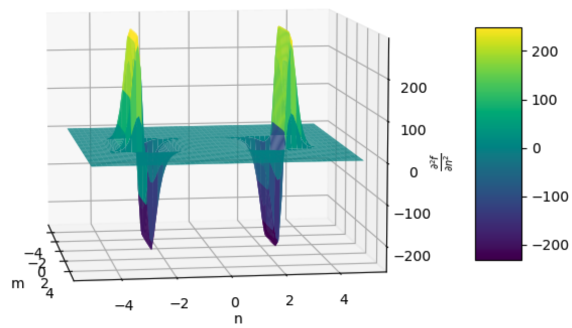

Figure 6: Surface of $\frac{\partial^2 f}{\partial n^2}$

Since a, b are even constants, $\frac{\partial f}{\partial m}$ and $\frac{\partial f}{\partial n}$ are odd (symmetric about the origin) on m and n respectively. So, $\frac{\partial^2 f}{\partial m^2}$ and $\frac{\partial^2 f}{\partial n^2}$ are symmetric about the y-axis.

With n fixed, we can get two opposite number through $m_{\max} = \text{argmax}(\frac{\partial^2 f}{\partial m^2})$ through selecting corresponding solution in $(\frac{\partial^3 f}{\partial m^3}) = 0$. Define the absolute value of this two m as m_0, so m_0 can also defined as where the value of function started to blow up when m shrinking from positive and negative infinity.

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2.2 How to satisfy the objective

Define following notations:

- $d_{min,y}$ and $d_{min,x}$: They are the minimum allowable distances with different setting in the direction of x and y, and it's not allowed to break them at the same time. Assume they are given.
- $S_{A,x}, S_{A,y}$ and x_O, y_O: They are the positions of vehicle A and L, taken from their states.

Considering vehicle L as an obstacle, we define the following function as the corresponding barrier function:

$$l_L(S, t) = f_{\text{barrier}}(S_{A,x} - x_O, S_{A,y} - y_O) \tag{9}$$

Selecting the coefficients K, k_1, k_2, a, b:

- Adjust a, b to change the steepness: setting greater value on a, the gradient of the function will be smaller as $(S_{A,x} - x_O) \longrightarrow m_0$ from positive infinity and greater as $(S_{A,x} - x_O) \longrightarrow (-m_0)$ from negative infinity, with $n \in [-n_0, n_0]$. It is the same for $S_{A,y} - y_O$. So we can conduct different penalties on different directions.

- Adjust k_1, k_2: With a, b selected, let $n_0 = d_{min,y}$ and $m_0 = d_{min,x}$, we can solve the k_1, k_2. This represents we can adjust the function to fit vehicles with different sizes to make sure that no collision happens. For example, setting $n_0 = d_{min,y} = \frac{\text{width of the vehicle}}{2} +$ safety distance to keep

- Setting K: K should be set close to infinity or very large to make this barrier function a hard barrier.

Considering the objective:

- This barrier function considers the distance in both latitude and longitudinal directions, and can be differentiated in both directions continuously.
- This barrier function has different gradients $\frac{\partial f}{\partial(S_{A,x}-x_O)}$ and $\frac{\partial f}{\partial(S_{A,y}-y_O)}$ on the distances in longitudinal and latitude directions.
- The gradient of barrier function and and m_0, n_0 can be tuned to conduct different penalties on the distances in latitude and longitudinal directions.
- This barrier function has smooth derivatives on $S_{A,x}, S_{A,y}$, so that it can be optimized to generate an optimal solution.

2.3 Discussion

- With above adaption of the barrier, the optimization problem can be defined to find S and u to minimize the function

$$J(x,u) = \int_{T_i}^{T_f} \sum_L l_L(S,t)\,dt + \Phi(x(T)) \tag{10}$$

  With the dynamic and bound constraints and L denotes all the surrounding vehicles can be detected, this cost function is continuous and differentiable.

- The method was proposed in a simplified dubins car model.

3 Results and Comparisons

3.1 Obstacles Setting

Set 10 different environments with 2 different sets of obstacle centers and 2 different radius, plotted as following:

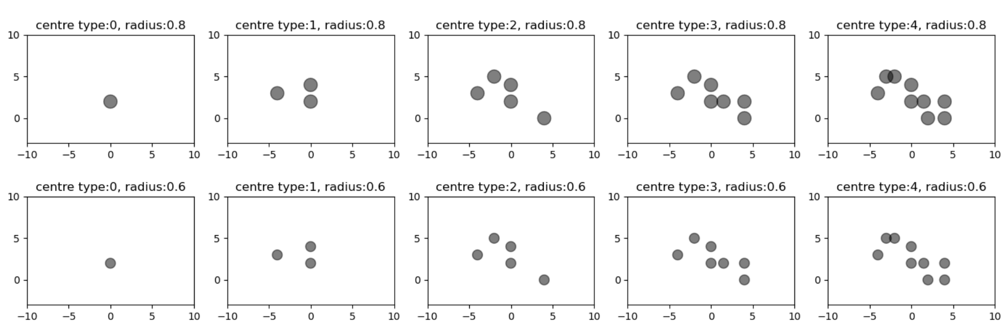

Figure 7: Plotted obstacles

3.2 Comparison between these two functions

Through comparing the trajectories generated, we can see the novel barrier function could keep a greater distance between the obstacles and generate a more optimized trajectory with less possibility of collision.

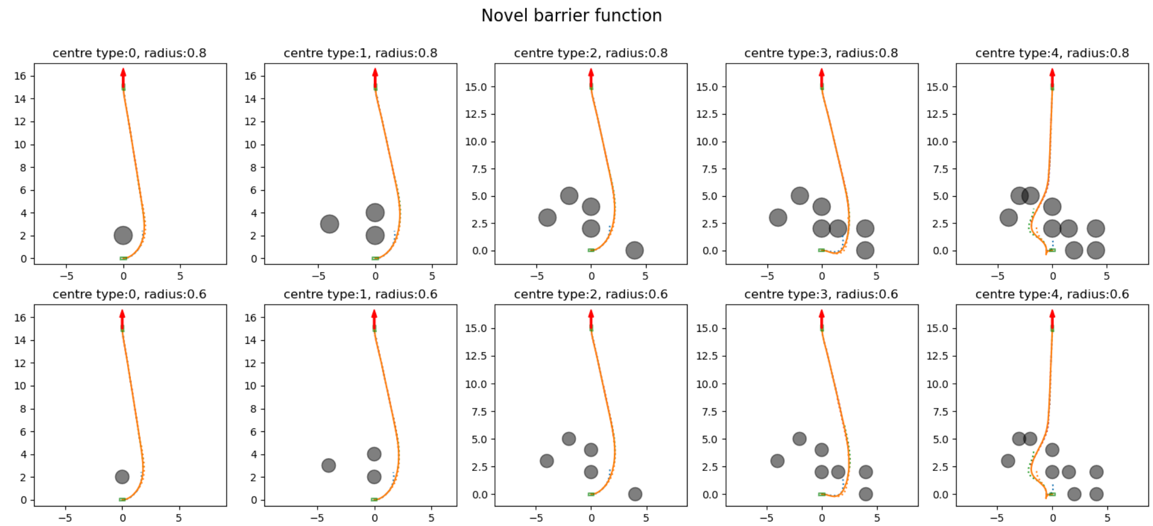

Figure 8: Trajectory generated by Novel function

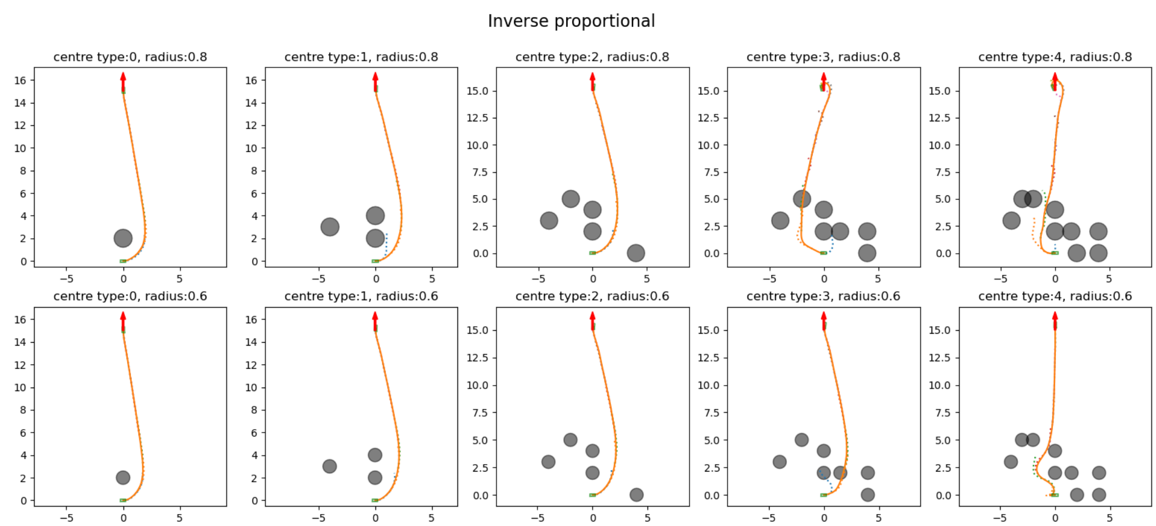

Figure 9: Trajectory generated by Inverse-proportional function

Considering the optimization time used, Novel function took less time in oprimization:

3.3 Further discussion

The novel function has some advantaged over traditional functions. However, it is disgusting to spend a lot of time tuning the paramaters.

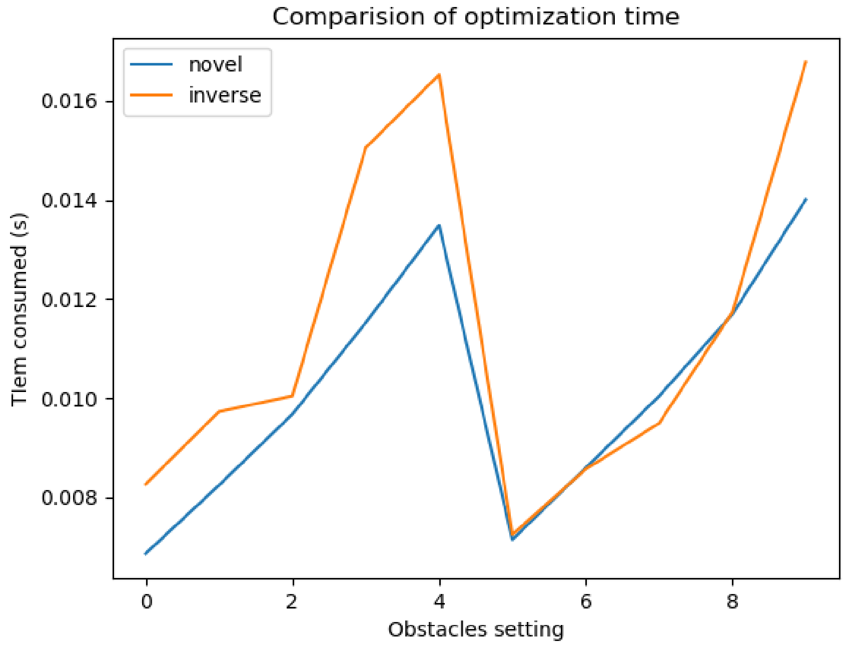

Figure 10: Optimization time

4 References

References

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[2] S. Sheng, P. Yu, D. Parker, M. Kwiatkowska and L. Feng, ”Safe POMDP Online Planning Among Dynamic Agents via Adaptive Conformal Prediction,” in IEEE Robotics and Automation Letters, vol. 9, no. 11, pp. 9946-9953, Nov. 2024, doi: 10.1109/LRA.2024.3468092.,