Week Twelve PHY-480

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1   Class 22

Q22.1

Show that the matrix operation $H0$ gives the uniform superposition state of a qubit, where H is the Hadamard matrix.

The Hadamard matrix for a single qubit is

$$H = \frac{1}{\sqrt{2}} \, (\, 1 \,) \, 11 - 1.$$

Applying H to the ground state $0 = (\, 1 \,)$
0 gives

$$H0 = \frac{1}{\sqrt{2}} \, (\, 1 \,) \, 11 - 1 \, (\, 1 \,) \, 0 = \frac{1}{\sqrt{2}} \, (\, 1 \,) \, 1 = \frac{1}{\sqrt{2}} (0 + 1).$$

Hence, $H0$ produces the **uniform superposition state**, where the qubit has equal probability of being in 0 or 1.

Q22.2

Give a concise statement of the amplitude amplification strategy used in designing quantum algorithms.

Amplitude amplification is a general quantum strategy for **increasing the probability amplitude** of target states within a quantum superposition while **suppressing all others**. It repeatedly applies a combination of:

- a **phase inversion** about the target state, and

- a **reflection** about the average amplitude,

thereby rotating the state vector toward the target in Hilbert space. Grover's algorithm is the canonical example, achieving a quadratic speedup over classical unstructured search.

Q22.3

Outline a Python algorithm for a simulation of the Grover algorithm on a classical computer using matrices of dimension 2^n, where n is the number of qubits.

1. Set the number of qubits n and compute $N = 2^n$.

2. Define the Hadamard matrix:

$$H = \frac{1}{\sqrt{2}} \, (1) \, 11 - 1, \quad H^{\otimes n} = Kroneckerproductof$$

   $n copies of H$.

2. Define the initial state $0^{\otimes n}$ and compute the uniform superposition:

$$\psi_0 = H^{\otimes n} 0^{\otimes n}.$$

3. Construct the target oracle:

$$U_{tar} = -I_N + 2E_{tar,tar},$$

   where $E_{tar,tar}$ is a matrix with a single 1 at the target index.

4. Construct the diffusion operator:

$$D = H^{\otimes n}(200 - I_N)H^{\otimes n}.$$

5. Define the Grover operator:

$$G = D\,U_{tar}.$$

6. Apply G repeatedly:
$$\psi_t = G^t \psi_0.$$

7. After each iteration, record the probability of the target state:

$$P_{tar}(t) = |\langle tar|\psi_t\rangle|^2.$$

8. Plot $P_{tar}(t)$ versus t to observe amplitude amplification.

Q22.4

Explain the expected behavior of the output-state probability of the target state as a function of t, the number of times the Grover operator is applied.

Each application of the Grover operator G performs a **rotation in a two dimensional subspace** spanned by the target state tar and the average of all non-target states. The probability of measuring the target state increases sinusoidally with the number of iterations t:

$$P_{tar}(t) = \sin^2\big((2t+1)\theta\big),$$

where $\sin^2(\theta) = 1/N$ and $N = 2^n$ is the total number of states. The probability reaches its maximum near

$$t_{opt} = \left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor,$$

after which additional iterations will begin to decrease the success probability again (overshooting the target).

Summary: The amplitude amplification process boosts the likelihood of finding the target state up to near certainty after $\sim \frac{\pi}{4}\sqrt{N}$ iterations.