1. Find the vector **x** determined by the given coordinate vector $[\mathbf{x}]_B$ and the given basis B. Illustrate the answer with a figure.

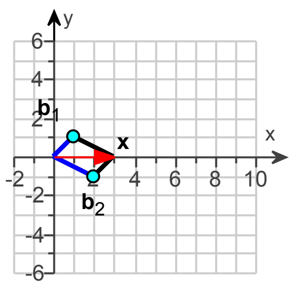
$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\},\ [\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the vector **x**.

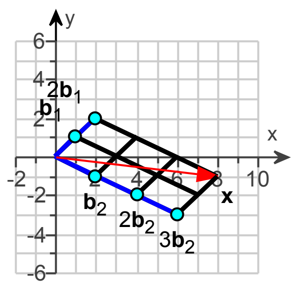
$$\mathbf{x} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$ (Type an integer or decimal for each matrix element.)

Let the basis vector $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the basis vector $\mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Choose the correct graph illustrating the vector **x**.

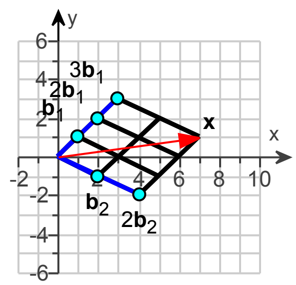
A.



B.



C. ✔



2. The vector x is in a subspace H with a basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the B-coordinate vector of **x**.

$$\mathbf{b}_1 = \begin{bmatrix} 4 \\ -5 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

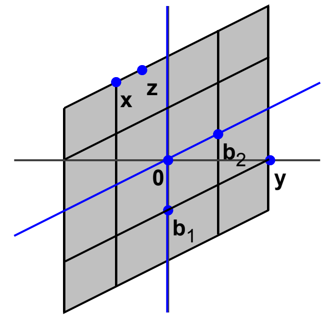
$$[\mathbf{x}]_B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

3. The vector **x** is in a subspace H with a basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the B-coordinate vector of **x**.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} -3 \\ -5 \\ 8 \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} -11 \\ -17 \\ 28 \end{bmatrix}$$

$$[\mathbf{x}]_B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

4. Let $\mathbf{b}_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -1 \\ 3.5 \end{bmatrix}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{x}]_B$, $[\mathbf{y}]_B$, and $[\mathbf{z}]_B$. Confirm your estimates of $[\mathbf{y}]_B$ and $[\mathbf{z}]_B$ by using them and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute **y** and **z**.



Use the figure to estimate $[\mathbf{x}]_B$. Choose the correct answer below.

A. $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ B. $[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ C. $[\mathbf{x}]_B = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$ D. $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Use the figure to estimate $[\mathbf{y}]_B$. Choose the correct answer below.

A. $[\mathbf{y}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ B. $[\mathbf{y}]_B = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ C. $[\mathbf{y}]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ D. $[\mathbf{y}]_B = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Use the figure to estimate $[\mathbf{z}]_B$. Choose the correct answer below.

A. $[\mathbf{z}]_B = \begin{bmatrix} -2 \\ -\frac{1}{2} \end{bmatrix}$ B. $[\mathbf{z}]_B = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$ C. $[\mathbf{z}]_B = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$ D. $[\mathbf{z}]_B = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$

5. Given below is a matrix A and an echelon form of A. Find bases for Col A and Nul A, and then state the dimensions of these subspaces.

$$A = \begin{bmatrix} 1 & 3 & 4 & -8 \\ 6 & 18 & 1 & 5 \\ 2 & 6 & -2 & 11 \\ 5 & 15 & 0 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Col A is given by $\left\{ \begin{bmatrix} 1 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 11 \\ 13 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

The dimension of Col A is 3 . (Type an integer.)

A basis for Nul A is given by $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

The dimension of Nul A is 1 . (Type an integer.)

6. Find the bases for Col A and Nul A, and then state the dimension of these subspaces for the matrix A and an echelon form of A below.

$$A = \begin{bmatrix} 1 & 2 & -5 & 2 & 0 \\ 2 & 5 & -8 & 7 & 2 \\ -3 & -9 & 9 & -10 & 4 \\ 3 & 10 & -7 & 13 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 2 & 0 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Col A is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -10 \\ 13 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

The dimension of Col A is 3.

A basis for Nul A is given by $\left\{ \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(Use a comma to separate vectors as needed.)

The dimension of Nul A is 2.

7. Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

$$\begin{bmatrix} 1 \\ -5 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 10 \\ -12 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -3 \\ 7 \end{bmatrix}$$

A basis for the subspace is given by $\left\{ \begin{bmatrix} 1 \\ -5 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ -3 \\ 7 \end{bmatrix} \right\}$.

(Use a comma to separate answers as needed.)

The dimension of this subspace is 3. (Type an integer.)

8. Suppose a 4×8 matrix A has four pivot columns. Is Col A = $\mathbb{R}^4$? Is Nul A = $\mathbb{R}^4$? Explain your answers.

Is Col A = $\mathbb{R}^4$? Explain your answer. Choose the correct answer and reasoning below.

- **A.** Yes, because the column space of a 4×8 matrix is a subspace of $\mathbb{R}^4$. There is a pivot in each row, so the column space is 4-dimensional. Since any 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$, Col A = $\mathbb{R}^4$.
- **B.**
- **C.** No, because a 4×8 matrix exists in $\mathbb{R}^8$. If its pivot columns form a 4-dimensional basis, then Col A is isomorphic to $\mathbb{R}^4$ but is not strictly equal to $\mathbb{R}^4$.
- **D.** No, Col A = $\mathbb{R}^4$. The number of pivot columns is equal to the dimension of the null space. Since the sum of the dimensions of the null space and column space equals the number of columns in the matrix, the dimension of the column space must be 4. Since any 4-dimensional basis is equal to $\mathbb{R}^4$, Col A = $\mathbb{R}^4$.

Is Nul A = $\mathbb{R}^4$? Explain your answer. Choose the correct answer and reasoning below.

- **A.** Yes, because a 4×8 matrix exists in $\mathbb{R}^4$. Therefore, if its null space is 4-dimensional and contained within $\mathbb{R}^4$, it must be equal to $\mathbb{R}^4$.
- **B.** No, because although the null space is 4-dimensional, its basis consists of four vectors and not four. Therefore, it cannot be equal to $\mathbb{R}^4$.
- **C.** No, because the null space of a 4×8 matrix is a subspace of $\mathbb{R}^8$. Although dim Nul A = 4, it is not strictly equal to $\mathbb{R}^4$ because each vector in Nul A has eight components. Each vector in $\mathbb{R}^4$ has four components. Therefore, Nul A is isomorphic to $\mathbb{R}^4$, but not equal.
- **D.** Yes, because the linearly dependent vectors in A form a basis in four dimensions. Any basis in four dimensions is also a basis for $\mathbb{R}^4$. Therefore, Nul A = $\mathbb{R}^4$.

9. For parts a through e, mark each statement True of False and justify each answer. Here, A is an $m \times n$ matrix.

a. If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for a subspace H and if $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$, then $c_1, \ldots, c_p$ are the coordinates of **x** relative to the basis *B*. Choose the correct answer below.

- **A.** True, because any coordinate in a subspace H, with basis *B*, can only be written in one way as a linear combination of basis vectors. The linear combination gives a unique coordinate vector $[\mathbf{x}]_B$ that is composed of the coordinates of **x** relative to *B*.
- **B.** False, because **x** is $\mathbf{v}_1$ coordinates in the direction of c_1, $\mathbf{v}_2$ coordinates in the direction of c_2, and so on.
- **C.** True, because the coordinates $c_1, \ldots, c_p$ are the same as the coordinates of **x** relative to the xy-plane.
- **D.** False, because the coordinate vector $[\mathbf{x}]_B$ is composed of the coordinates $c_1, \ldots, c_p$ only if the vector **x** in $\mathbb{R}^p$ is equal to

b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$. Choose the correct answer below.

- **A.** True, because any one-dimensional subspace of $\mathbb{R}^n$ must be a line.
- **B.** False, because any subspace of $\mathbb{R}^n$ must contain the zero-vector. Therefore, a line can only be a one-dimensional subspace of $\mathbb{R}^n$ if it passes through the origin.
- **C.** False, because any subspace of $\mathbb{R}^n$ must be at least n-dimensional.
- **D.** True, because any line in $\mathbb{R}^n$ satisfies all three requirements of a subspace.

c. The dimension of Col A is the number of pivot columns in A. Choose the correct answer below.

- **A.** True, because the pivot columns of A form a basis for Col A. Therefore, the number of pivot columns of A is the same as the dimension of Col A.
- **B.** False, because the dimension of Col A cannot be determined without the size of matrix A.
- **C.** False, because the number of pivot columns determines the dimension of the null space, not the column space.
- **D.** True, because the number of pivot columns is equal to the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$. The number of free variables in $A\mathbf{x} = \mathbf{0}$ is equal to the dimension of the column space.

d. The dimensions of Col A and Nul A add up to the total number of columns in A. Choose the correct answer below.

- **A.** True, because Col A and Nul A are both subspaces of $\mathbb{R}^n$, where n is the number of columns in matrix A.
- **B.** True, because the Rank Theorem states that if matrix A has n columns, then rank A + dim Nul A = n. Since rank A is the same as dim Col A, the dimensions of Col A and Nul A add up to the total number of columns in A.
- **C.** False, because the sum of dim Col A and dim Nul A is the number of rows of A.

- **D.** False, because the column space and null space sometimes intersect

Therefore, the sum of dim Col A and dim Nul A is the number of columns of A, only if the Col A and Nul A are disjoint.

e. If a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors form a basis of H. Choose the correct answer below.

- **A.** True, because if a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors must be linearly independent. Any linearly independent spanning set of p vectors forms a basis in p dimensions.
- **B.** True, because any spanning set in H will form a basis of H.
- **C.** False, because although the set of vectors spans H, there is not enough information to conclude that they form a basis of H.
- **D.** False, only vectors that span H and are linearly independent will form a basis of H. Since the set contains too many vectors, the spanning set cannot possibly be linearly independent.

10. Mark each statement True or False. Justify each answer. Here A is an $m \times n$ matrix. Complete parts (a) through (e) below.

a. If B is a basis for a subspace H, then each vector in H can be written in only one way as a linear combination of the vectors in B. Choose the correct answer below.

- **A.** The statement is true. All bases for a subspace H are linearly independent and therefore each vector in H can only be generated as one unique linear combination of the vectors in B.
- **B.** The statement is false. Suppose $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and $\mathbf{x}$ is a vector in H. The vector $\mathbf{x}$ can be generated in a multiple of ways based on the values of the vectors in the set $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.
- **C.** The statement is true. Suppose $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and $\mathbf{x}$ is a vector in H that can be generated two ways. Say, $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$ and $\mathbf{x} = d_1\mathbf{v}_1 + \ldots + d_p\mathbf{v}_p$, then $\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{v}_1 + \ldots + (c_p - d_p)\mathbf{v}_p$. Therefore, $c_p = d_p$ and $\mathbf{x}$ can only be generated in one way.
- **D.** The statement is false. Bases for a subspace H may be linear dependent and therefore there can be multiple solutions for the same vector $\mathbf{x}$ in H.

b. If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$, then the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_B$ makes H look and act the same as $\mathbb{R}^p$. Choose the correct answer below.

- **A.** The statement is false. The vectors in H may contain more than p entries and therefore the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_B$ does not make H look and act the same as $\mathbb{R}^p$.
- **B.** The statement is true. The correspondence of $\mathbf{x} \mapsto [\mathbf{x}]_B$ implies a one-to-one correspondence between H and $\mathbb{R}^p$ that preserves linear combinations.
- **C.** The statement is true. The fact that $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ implies that the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_B$ makes H look and act the same as $\mathbb{R}^p$.
- **D.** The statement is false. The correspondence of $\mathbf{x} \mapsto [\mathbf{x}]_B$ does not imply a one-to-one correspondence between H and $\mathbb{R}^p$ that preserves linear combinations.

c. The dimension of Nul A is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$. Choose the correct answer below.

- **A.** The statement is false. The dimension of Nul A is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$ minus the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
- **B.** The statement is true. The number of total variables involved in solving the equation $A\mathbf{x} = \mathbf{0}$ is the dimension of Nul A.
- **C.** The statement is false. The dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.
- **D.** The statement is true. The dimension of Nul A is the same as the amount of vectors in the set $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ that satisfy the equation $A\mathbf{x} = \mathbf{0}$.

d. The dimension of the column space of A is rank A. Choose the correct answer below.

- **A.** The statement is true. The rank of an $m \times n$ matrix A is equal to n, which is also equal to the dimension of the column space of A.

e. If H is a p-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of p vectors in H is a basis for H. Choose the correct answer below.

- **A.** The statement is true. Any set of p linearly independent vectors is a basis for H.
- **B.** The statement is false. This is only true if $n \neq p$.
- **C.** The statement is false. This is only true if $n = p$.
- **D.** The statement is false. It is possible for p vectors to be linearly independent without spanning H.

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11. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 4×6 matrix, what is the rank of A?

rank A = 3 (Type a whole number.)

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12. If the rank of a 4×7 matrix A is 2, what is the dimension of the solution space $A\mathbf{x} = \mathbf{0}$?

The dimension of the solution space is 5 .