HW 8

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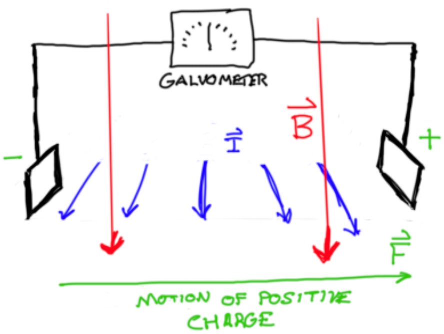
1 - Magneto-Hydrodynamic Generation

Waterloo Bridge Experiment



Faraday's configuration was meant to measure current generated from velocity of Thames cutting through earth's magnetic field. This was an attempt to illustrate electomagnetic induction. In 1938, funded by Westinghouse, Bela Karlovitz was the first to patent a Magneto-Hydrodynamic (MHD) generator. His MHD generator used hot moving gas (as opposed to a river).

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2 - Multipole Expansion of Magentic Vector Potential

r_o: distance to observervation point.

r_s: distance to source point (on the contour).

d_{os}: distance between r_o and r_s.

$\cos(\phi)$: angle between r_o and r_s.

Magentic Vector Potential Setup

$$A_\phi = \frac{\mu I}{4\pi} \oint_C \frac{1}{d_{os}}\, dr_o \qquad \textit{eq. 5.65 in 7th ed.}$$

Finding d_{os}

$$\begin{aligned} d_{os} &= |\vec{r_o} - \vec{r_s}| \\ &= \sqrt{(\vec{r_o} - \vec{r_s}) \cdot (\vec{r_o} - \vec{r_s})} \\ &= \sqrt{|\vec{r_o}|^2 + |\vec{r_s}|^2 - 2r_o r_s \cos(\phi)} \\ &= \boxed{\sqrt{r_o^2 + r_s^2 - 2r_o r_s \cos(\phi)}} \end{aligned}$$

Legendre Expansion of $1/d_{os}$

We need $1/d_{os}$ in the form of

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \qquad \text{Legendre generator}$$

where P_n is a polynomial of degree n.

So,

$$\begin{aligned} \frac{1}{d_{os}} &= (r_o^2 + r_s^2 - 2r_o r_s \cos(\phi))^{-1} \\ &= \left(r_o^2 \left(\frac{r_o^2}{r_o^2} + \frac{r_s^2}{r_o^2} - 2\frac{r_o r_s}{r_o^2} \cos(\phi) \right) \right)^{-1} && \text{extract } r_o^2 \\ &= \frac{1}{r_o} \left(1 + \left(\frac{r_s}{r_o}\right)^2 - 2\left(\frac{r_s}{r_o}\right) \cos(\phi) \right)^{-1} \end{aligned}$$

Let $x = \cos(\phi)$ and $t = \dfrac{r_s}{r_o}$. Then,

$$\begin{aligned} \frac{1}{d_{os}} &= \frac{1}{r_o} \frac{1}{\sqrt{1 + t^2 - 2at}} \\ &= \frac{1}{r_o} \sum_{n=0}^{\infty} P_n(x) t^n \\ &= \boxed{\frac{1}{r_o} \sum_{n=0}^{\infty} P_n(\cos(\phi)) \left(\frac{r_s}{r_o}\right)^n} && \text{replaced } x \text{ and } t \end{aligned}$$

Combining r_o from the Legendre function

Our contour integration is with respect to r_s and ϕ, but not r_o. Let's extract r_o which would be constant in our contour integration.

$$\begin{aligned} \frac{1}{d_{os}} &= \frac{1}{r_o} \sum_{n=0}^{\infty} P_n(\cos(\phi)) (r_s)^n \left(\frac{1}{r_o^n}\right) \\ &= \boxed{\sum_{n=0}^{\infty} \frac{1}{r_o^{n+1}} P_n(\cos(\phi)) (r_s)^n} && \text{since } r_o \cdot r_o^n = r_o^{n+1} \end{aligned}$$

Multipole Expansion of Magentic Vector Potential

A Legendre polynomial lookup table, to substitute $P_n(x)$, can be found here: `https://en.wikipedia.org/wiki/Legendre_polynomials#Legendre_polynomials_in_multipole_expansions`.

$$\begin{aligned} A_\phi &= \frac{\mu I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r_o^{n+1}} \oint_C P_n(\cos(\phi)) \cdot r_s^n \cdot dr_s \\ &= \frac{\mu I}{4\pi} \Bigg[\frac{1}{r_o} \oint_C dr_s + \\ &\qquad \frac{1}{r_o^2} \oint_C r_s \cos(\phi)\, dr_s + \\ &\qquad \frac{1}{r_o^3} \oint_C r_s^2 \left(\frac{1}{2}(3\cos(\phi) - 1) \right) dr_s + \ldots \Bigg] \end{aligned}$$

- $1/r_o$ term: Magnetic Monopole
- $1/r_o^2$ term: Magnetic Dipole
- $1/r_o^3$ term: Magnetic Quadrapole

We observe that the monopole term contour integration is zero around a closed contour. I.e.

$$\oint_C dr_s = 0.$$

This makes sense since monopoles do not evidently exist in nature. The dominating term, then, is the magnetic dipole term.

$$\begin{aligned}
A_\phi &= \frac{\mu I}{4\pi}\frac{1}{r_o^2}\oint_C r_s \cos(\phi)\, dr_s && \cos(\phi) = \hat{r_o}\cdot\hat{r_s} \\
&= \frac{\mu I}{4\pi}\frac{1}{r_o^2}\oint_C r_s(\hat{r_o}\cdot\hat{r_s})\, dr_s && r_s\cdot\hat{r_s} = \vec{r_s} \\
&= \frac{\mu I}{4\pi}\frac{1}{r_o^2}\oint_C \hat{r_o}\cdot\vec{r_s}\, dr_s \\
&= \frac{\mu I}{4\pi}\frac{1}{r_o^2}\int_S d\vec{a}\times\vec{r_o} && \text{Stokes Thereom} \\
&= \frac{\mu I}{4\pi}\frac{m\times\hat{r_o}}{r_o^2} && m = I\int_S d\vec{a} \text{ (magnetic dipole moment of loop)} \\
&= \boxed{\frac{\mu I m}{4\pi r_o^2}\sin(\theta)}
\end{aligned}$$

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4 - Longitudinal Polarization

Sound waves and water waves are longitudinal, and their vibration can potentially occur in all directions perpendicular to the direction of travel. Polarization can only occur if vibrations occur in one plane only.

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5 - Polarization with Electric Field of a Wave

$$\begin{aligned}
\vec{E} &= 5\sin\left(\omega t - kz + \tfrac{\pi}{6}\right)\hat{x} - 25\cos\left(\omega t - kz + \tfrac{\pi}{3}\right)\hat{y} \\
&= 5\cos\left(\omega t - kz + \tfrac{\pi}{6} - \tfrac{\pi}{2}\right)\hat{x} - 25\cos\left(\omega t - kz + \tfrac{\pi}{3}\right)\hat{y} \\
&= \hat{x}\, 5 e^{-jkz} e^{-j\frac{\pi}{3}} - \hat{y}\, 25 e^{-jkz} e^{j\frac{\pi}{3}} \\
&= \hat{x}\, 5 e^{-jkz} e^{-j\frac{\pi}{3}} + \hat{y}\, 25 e^{-jkz} e^{j\frac{4\pi}{3}}
\end{aligned}$$

$$\begin{aligned}
\psi_0 &= \tan^{-1}\left(\frac{a_y}{a_x}\right) \\
&= \tan^{-1}\left(\frac{25}{5}\right) \\
&\boxed{= 78.7^\circ}
\end{aligned}$$

$$\delta_x = -\frac{\pi}{3} \qquad \delta_y = \frac{4\pi}{3}$$

$$\begin{aligned}
\delta &= \delta_y - \delta_x \\
&= \frac{4\pi}{3} - \left(-\frac{\pi}{3}\right) \\
&\boxed{= \frac{5\pi}{3}}
\end{aligned}$$

$$\begin{aligned}
\sin 2\chi &= \sin 2\psi_0 \sin\delta \\
\sin 2\chi &= -0.33 \\
2\chi &= \sin^{-1}(-0.33) \\
2\chi &= -19.3^\circ \\
&\boxed{\chi = -9.65^\circ}
\end{aligned}$$

$$\begin{aligned}
\tan 2\gamma &= (\tan 2\psi_0)\cos\delta \\
&= \tan(2\cdot 78.7^\circ)\cos\left(\frac{5\pi}{3}\right) \\
&= -0.208
\end{aligned}$$

$$\rightarrow \gamma = \tan^{-1}(-0.208)/2 = -5.88^\circ$$

RIGHT ELLIPTICAL POLARIZATION

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